AMES Class Notes – Week 7, Monday: Integrals and Probability

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1 Integration by parts

Recall from Monday, integration by parts is an application of the product rule.

Consider the function

$$z(x) = f(x)g(x) \tag{1}$$

We want the integral of $z(x)$

$$Z(x) = \int f(x)g(x)dx \tag{2}$$

Now, we know how to get $dz(x)/dx$ using the product rule

$$\frac{dz(x)}{dx} = f'(x)g(x) + f(x)g'(x) \implies \tag{3}$$

$$\int dz = \int (f'(x)g(x) + f(x)g'(x))dx \tag{4}$$

Remember integrals are linear operators

$$\int dz = \int f'(x)g(x)dx + \int f(x)g'(x)dx \tag{5}$$

$$\tag{6}$$

Note that $\int dz = z(x) = f(x)g(x)$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx \implies \tag{7}$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int f'(x)g(x)dx \tag{8}$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int g(x)f'(x)dx \tag{9}$$

$$\int g(x)f'(x)dx = f(x)g(x) - \int f(x)g'(x)dx \tag{10}$$

Let

$$U = g(x) \tag{11}$$

$$dV = f'(x)dx \tag{12}$$

$$V = f(x) \tag{13}$$

$$dU = g'(x)dx \tag{14}$$

Plug these back into our last equation

$$\int UdV = VU - \int VdU \tag{15}$$

which is the equation for integration by parts.

2 Tips

How can you remember the order of what should be U and what should be dV?
U → Log → Inverse trig → Algebraic → Trig → Exponential → dV

3 Example

Consider the function

$$\int xe^x dx \tag{16}$$

Whats's U and what's V?

$$\int UdV = vu - \int VdU \tag{17}$$
$$U = x \tag{18}$$
$$dU = dx \tag{19}$$
$$V = e^x \tag{20}$$
$$dV = e^x dx \tag{21}$$

Rewrite the whole thing! And use integration by parts

$$\int xe^x dx = e^x x - \int e^x dx \tag{22}$$
$$= xe^x - e^x \tag{23}$$
$$= (x-1)e^x \tag{24}$$

4 Integrating logs

Consider the integral of $1/x$, which we know from knowing the derivative of $ln(x)$,

$$\int \frac{dx}{x} = \int \frac{1}{x} dx = ln(x) \tag{25}$$

Cool! Wait, but what the integral of a log,

$$\int ln(x)dx = ?? \tag{26}$$

This was a really hard problem that existed for a very long time in mathematics. What's the trick? To use integration by parts, we consider this function instead

$$\int ln(x)1dx \tag{27}$$

Now we can use integration by parts and make the following variable assignments,

$$U = ln(x) \tag{28}$$
$$dU = \frac{1}{x} dx \tag{29}$$
$$V = x \tag{30}$$
$$dV = 1dx \tag{31}$$
$$\tag{32}$$

Do NOT forget the dx when you get dU!! Now, plug it all back into our integration by parts formal

$$\int UdV = VU - \int VdU \tag{33}$$

$$\int ln(x)1dx = xln(x) - \int x\frac{1}{x}dx \implies \tag{34}$$

$$= xln(x) - \int 1dx \tag{35}$$

$$= xln(x) - x \tag{36}$$

$$= x(ln(x) - 1) \tag{37}$$

You can take the derivative of this final equation to check and you'll find the derivative of that final equation does equal $ln(x)$.

5 Cross Partials Derivatives to Double Integrals

We're going to take a **cross partial** derivative of a function (meaning derivative with respect to one variable and then derivative wrt a different variable).

We then are going to to do a **double integral** to reverse these two derivatives.

Consider the function

$$F(x, y) = x^\alpha y^\beta \tag{38}$$

Take the derivative with respect to x,

$$\frac{\partial F}{\partial x} = \alpha x^{\alpha-1} y^\beta \tag{39}$$

Now, take the derivative of $\partial F/\partial x$ with respect to y,

$$\frac{\partial F}{\partial x \partial y} = F_{x,y}(x, y) = \alpha\beta x^{\alpha-1} y^{\beta-1} \tag{40}$$

Now, we can do a **double integral** to undo the two derivatives we've taken (cross partial).

$$\int_x \int_y \alpha\beta x^{\alpha-1} y^{\beta-1} dydx \tag{41}$$

First, integrate with respect to y. To do so, we can pull the constants out front of the integral that has to do with variable y,

$$= \int_x \alpha\beta x^{\alpha-1} \int_y y^{\beta-1} dydx \tag{42}$$

$$= \int_x \alpha \frac{\beta}{\beta} x^{\alpha-1} y^\beta dx \tag{43}$$

$$= \int_x \alpha x^{\alpha-1} y^\beta dx \tag{44}$$

We have undone the derivative wrt to y by integrating wrt y! Note that the term in the integral in equation 44 is the same as 39. We can now undo the integral wrt to x by doing the integral wrt to x.

$$= \frac{\alpha}{\alpha} x^\alpha y^\beta + C \tag{45}$$

$$= x^\alpha y^\beta + C \tag{46}$$

and we have successfully returned to $F(x, y) = x^\alpha y^\beta + C$ which is the same as our original equation 38 (plus an arbitrary constant).

6 Double Integral

We need to sum over one variable and then we sum over the other.

$$\int\int xydxdy = \int\left(\int xydx\right)dy \tag{47}$$

We can solve the integral in parentheses first while treating y like a constant

$$= \int\left(y\int xdx\right)dy \tag{48}$$

$$= \int\left(y\frac{1}{2}x^2\right)dy \tag{49}$$

Now we can treat x as a constant

$$= \frac{1}{2}x^2\int ydy \tag{50}$$

$$= \frac{1}{2}x^2 * \frac{1}{2}y^2 + C \tag{51}$$

$$= \frac{1}{4}x^2y^2 + C \tag{52}$$

7 Probability density function

Consider the random variable Y. Random variables aren't random and they aren't variables. They're really more like functions.

7.1 Normal Distribution

If Y is distributed normally it will look like

$$Y = f(y) = \int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y-\mu}{\sigma})^2}dy = 1 \tag{53}$$

the inside of the integral is the probability density function. The integral of a pdf must equal 1.