AMES Week 13 class notes – Weds, Constrained Optimization

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1 Introduction

We learned how to find the maximums of a function *when the function has a maximum.* For example, consider a population of bears and we want to know the population level where their population is growing the fastest *i.e.* what's the population that leads to the maximum growth rate?

Let's work through this refresher example:

$$g = x(1-x) \tag{1}$$

where g is the growth rate and x is the population level.

Find the x that maximizes g.

$$\frac{dg}{dx} = 1 - 2x = 0 \tag{2}$$

$$\implies x^* = \frac{1}{2} \tag{3}$$

But what about if the function just keeps increasing, and it's your budget that constrains how much you can maximize?

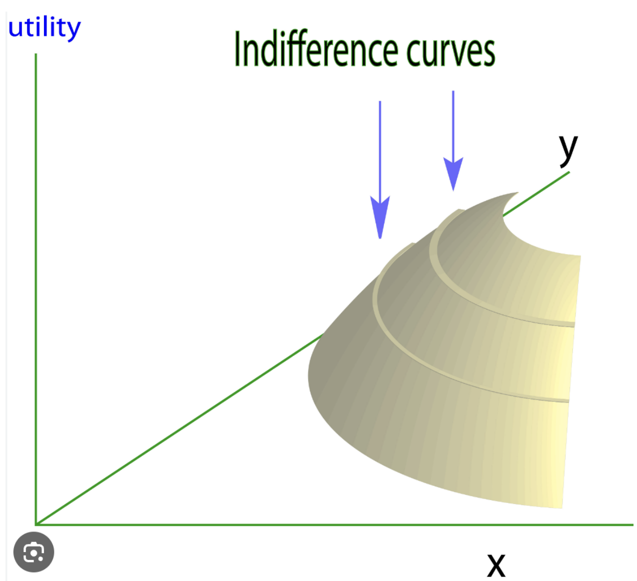

Figure 1: Constantly increasing function (no obvi maximum)

This is a function where U just keeps increasing as the (x, y) point gets further away from the origin. So if we want to maximize U but choosing x and y we'd choose infinite of each.

But let's say you can't afford infinite x and infinite y. You have a constraint on the amount of x and y you can have. This is what we call **constrained optimizations** and we use a **Lagrangian** to solve these problems.

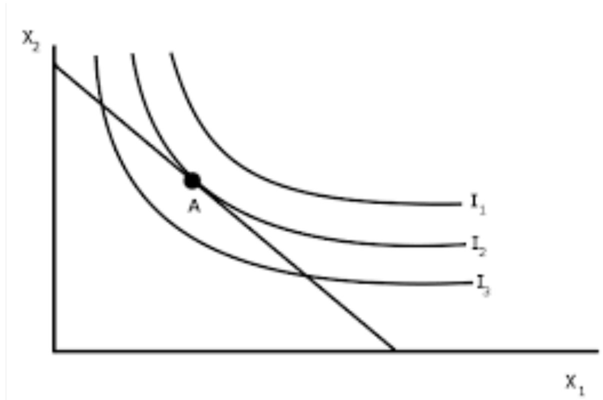

Figure 2: Birds eye view of same figure, straight line is the budget

2 Lagrangians

I think the most common example of constrained optimization people in this room will see is utility maximization with a budget constraint.

We use Lagrangians to solve for the optimal quantity of goods to consume in order to maximize a utility function, subject to a budget constraint.

A note on the word consume: traditionally, economists only thought people increased their utility by "consuming goods". In this class, we know people gain utility from flowers, park visits, or grizzly bear sightings. I still use the word consume for these non-consumptive goods for conciseness.

2.1 Utility function

A utility function measures the welfare we get from consuming goods (this use can be consumptive or non-consumptive). Utility functions can take different functional forms, a common one is Cobb-Douglas

$$U(x, y) = x^{\alpha} y^{1-\alpha}.$$

In this case, the individual only consumes two goods, good x and good y.

Our goal is to maximize utility subject to a constraint.

2.2 Constraint

If we could, we'd consume infinite x and infinite y because utility increases with the consumption of goods. However, we typically have a "budget" of how much x and how much y we can buy. Let B denote this budget, p_x denote the price of x and p_y denote the price of y. The money we spend on x and y needs to be less than our budget,

$$p_x x + p_y y \leq B.$$

Because we want consume as much x and y as possible, this will become an equality rather than a less than or equal sign,

$$p_x x + p_y y = B.$$

Note that we can rewrite our constraint as

$$0 = B - p_x x + p_y y \tag{4}$$

where we manipulate the constraint to *equal zero*.

2.3 Constrained optimization of utility

Our goal is to maximize utility subject to our constraint. We will write down a **Lagrangian**, which will become the function we want to maximize.

$$\begin{aligned} L &= U(x,y) + 0 &(5)\\ &= U(x,y) + \lambda(B - p_x x + p_y y) &(6)\end{aligned}$$

Our Lagrangian is our utility function *plus zero*, because λ multiplied by equation 4 is zero. So maximizing the Lagrangian is the same as maximizing our utility, but we're incorporating our constraint.

The term λ is referred to as the *marginal utility of money*. This is because $\frac{dL}{dB} = \lambda$. This interpretation is important for natural capital accounting, but is not important for this class.

We want to solve for the optimal level of consumption of goods x^* and y^* to maximize $U(x,y)$.

We take the derivatives and set them equal to zero to find the maximum. We call this **first order conditions (FOC)**, which are also known as the **optimality conditions**

FOCs:

$$\frac{\partial L}{\partial x} = ... = 0 \tag{7}$$

$$\frac{\partial L}{\partial y} = ... = 0 \tag{8}$$

$$\frac{\partial L}{\partial \lambda} = ... = 0 \tag{9}$$

The FOCs gives us three equations (7 - 9) and three unknowns (x, y, λ). We can solve for the optimal x^* and y^*. We can also solve for λ, which is the marginal utility of money in the Lagrangian case.

3 Example

Let's consider an explicit example.

$$\begin{aligned} U(x,y) &= x^{\frac{1}{2}} y^{\frac{1}{2}} \\ B &= 10 \\ p_x &= 1 \\ p_y &= 2 \end{aligned}$$

Our budget constraint will be

$$\begin{aligned} 10 &= x + 2y \\ 10 - x - 2y &= 0 \end{aligned}$$

Our Lagrangian will be

$$L = x^{\frac{1}{2}} y^{\frac{1}{2}} + \lambda(10 - x - 2y)$$

FOCs:

$$\frac{\partial L}{\partial y} = \frac{1}{2} x^{\frac{1}{2}} y^{-\frac{1}{2}} - \lambda 2 = 0 \tag{10}$$

$$\frac{\partial L}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{2}} - \lambda = 0 \tag{11}$$

$$\frac{\partial L}{\partial \lambda} = 10 - x - 2y = 0 \tag{12}$$

We now have 3 equations and 3 unknowns x, y, λ. We're not going to solve for λ today, but once you have x and y you can.

Solve 10 and 11 for λ and set these equations equal to one another to get:

$$\begin{aligned}
\frac{\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}}{2} &= \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} \\
\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} &= x^{-\frac{1}{2}}y^{\frac{1}{2}} \\
\frac{1}{2}x &= y
\end{aligned}$$

we can plug $\frac{1}{2}x = y$ back into 12

$$\begin{aligned}
10 &= x + 2\frac{1}{2}x \\
x^* &= 5
\end{aligned}$$

and we can plug this value of x^* into $\frac{1}{2}x = y$ to get

$$\begin{aligned}
y &= \frac{1}{2}x \\
y^* &= 2.5
\end{aligned}$$

so the value of (x, y) that maximized $U(x, y)$ subject to our budget constraint is $(5, 2.5)$.