AMES Week 12 class notes – Eigenvalues and Eigenvectors

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1 Introduction example

Consider the equation

$$\begin{aligned} N(t+1) &= N(t)e^{\lambda\epsilon} \\ \frac{\partial N(t)e^{\lambda\epsilon}}{\partial\lambda} &= \lambda N(t)e^{\lambda\epsilon} \\ \lim_{\epsilon\to 0}\lambda N(t)e^{\lambda\epsilon} &= \lambda N(t) \end{aligned}$$

by taking the derivative, we were able to linearize this equation so we can estimate the slope of this function (locally) in a linear manner.

2 Motivation of eigen values

The Eigen values will tell us about the dynamics around an equilibrium. They don't help us solve for the equilibrium, we do that by solving system of equations for the stock levels in the system that lead to it not changing, *i.e.* $\frac{dx}{dt} = 0 \implies x^*$. Instead, the eigen values will tell us if the equilibrium is stable, unstable, conditionally stable, as well as information about if the system approaches the equilibrium in a spiral or straight on.

3 Solving for Eigen values

3.1 Solving for equilibrium, (x^*, y^*)

Consider two stocks, x and y, that are changing through time. The equations for how they change through time are given,

$$\frac{\partial x}{\partial t} = \dot{x} = -0.1x + y \tag{1}$$

$$\frac{\partial y}{\partial t} = \dot{y} = \frac{-100}{x} + 0.1y = -100x^{-1} + 0.1y \tag{2}$$

What is a our Jacobian matrix? Remember, the Jacobian matrix tells us about the dynamics at an equilibrium.

$$J = \begin{bmatrix} \frac{\partial\dot{x}}{\partial x} & \frac{\partial\dot{x}}{\partial y} \\ \frac{\partial\dot{y}}{\partial x} & \frac{\partial t}{\partial y} \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} -0.1 & 1 \\ 100x^{-2} & 0.1 \end{bmatrix} \tag{4}$$

We want to know the behavior at the equilibrium, *i.e.* when the stocks x and y are not changing. Find the values of x and y that lead to an equilibrium and plug them into J. To do so, we want to find the x, y combination that lead to $\dot{x} = 0$ and $\dot{y} = 0$. This gives a system of 2 equations which we can solve for a the equilibrium values of x and y.

$$\dot{x} = -0.1x + y = 0 \tag{5}$$

$$\dot{y} = \frac{-100}{x} + 0.1y = -100x^{-1} + 0.1y = 0 \tag{6}$$

$$\implies x^* = 100 \tag{7}$$

$$\implies y^* = 10 \tag{8}$$

Plugging these equilibrium values back into the Jacobian yields

$$J = \begin{bmatrix} -0.1 & 1 \\ 0.01 & 0.1 \end{bmatrix} \tag{9}$$

3.2 Solving for eigen value

Let's now solve for our Eigen values. Consider how $\dot{N} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ changes. We can rewrite this as

$$\dot{N} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix} = \lambda q = \lambda \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{10}$$

where q are are stand ins for our equilibrium values of $x (= q_1)$ and $y (= q_2)$. λ will be our eigenvalues and each eigen values is a scalar. There are multiple.

From 10, we get

$$Jq = \lambda q \tag{11}$$

Just like the Jacobian matrix tells us about the behavior around the equilibrium (x^*, y^*), the eigen values tell us about the dynamics around the equilibrium as well.

Let's rename $J = A$.

$$Aq = \lambda q \tag{12}$$

plug this back $Jq = \lambda q$.

Do a bunch of linear algebra manipulation:

$$\lambda \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \implies \tag{13}$$

$$a_{11}q_1 + a_{12}q_2 = \lambda q_1 \implies (a_{11} - \lambda)q_1 + a_{12}q_2 = 0 \tag{14}$$

$$a_{21}q_1 + a_{22}q_2 = \lambda q_2 \implies (a_{21}q_1) + (a_{22} - \lambda)q_2 = 0 \tag{15}$$

$$\implies \left[\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right] \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \tag{16}$$

$$(A - \lambda I)q = 0 \tag{17}$$

$$(A - \lambda I) = 0 \tag{18}$$

If q is zero, x and y need to be zero and that's not interesting.

If we want 18 to be 0, that means we want the determinant of the LHS to equal zero

$$Det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \tag{19}$$

$$0 = \lambda^2 + \lambda(-a_{11} - a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) \tag{20}$$

$$\lambda = \frac{1}{2}\left[(a_{11} + a_{22}) \pm \sqrt{(-a_{11} - a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \right] \tag{21}$$

We now have both of our eigen values from 21.

3.3 Interpreting eigen values:

Again, Eigen values tell you about the behavior around the equilibrium point, so $q_1 = x^*$ and $q_2 = y^*$. So we can plug in our equilibrium values of x^* and y^* that we solved for when we solved our system of equations.

- If both eigen values are positive, it's unstable.
- If both eigen values are negative, it's stable.
- If one is positive, and the other is negative, it's conditionally stable.
- If in eigen value has an imaginary number in it $i = \sqrt{-1}$ such that $\lambda = R + Zi$ that means we have a spiraling equilibrium. If R is positive is spirals out, if R is negative it spirals in.

3.4 Returning to our number example

Now, let's plug in the numbers from our original problem and our Jacobian (Eqn 9) into our solution for the eigen values from Eqn 20. Remember that $A = J$, so the Jacobian values, j_{ij}, are plugged into a_{ij}.

$$\lambda^2 + \lambda(0.1 - 0.1) + (-0.1(0.1) - 0.01) = \lambda^2 + 0\lambda - 0.02 = 0 \tag{22}$$

$$\implies \lambda = \frac{-0 \pm \sqrt{0 - 4(0.02)}}{2} = \pm\sqrt{0.02} \tag{23}$$

Our eigen values are positive and negative, therefore our system is conditionally system.

3.5 A few notes on jargon

Eigen values are sometimes called *characteristic value*. Also, 20 will be called the *characteristic polynomial*.

You may see the *Routh-Horwitz Condition*, which means the system is stables.

May also see the *Jury Condition*, which has to do with systems where time is discrete. If eigen value is less than one in absolute terms then the system is stable and unstable otherwise.

If you end up with an eigen value that is Zi (instead of $R + Zi$) then your system will have an *orbit* instead of a spiral.

Hopf bifurcation: You can change a parameter so that there are multiple equilibrium in a system. Eli doesn't think much of these, but a lot of people tend to think they're clever. You can get deterministic chaos in these systems. You don't know where you'll go from the start point because the system keeps bifurcating.

3.6 Finding eigen values in R

Set up your matrix:
M = rbind(c(first row with commas), c(second row with commas), ...)
eigen(M) will give you the eigen values and the eigen vectors.

4 Solving for eigen vectors

Recall the q that dropped out of 17. The q is our eigen vector.

$$(A - \lambda I)q = 0 \tag{24}$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \tag{25}$$

$$\implies (a_{11} - \lambda)q_1 + a_{12}q_2 = 0 \tag{26}$$

$$a_{21}q_1 + (a_{22} - \lambda)q_2 = 0 \tag{27}$$

The q is only defined relative to another. Conditioned on a given λ, there is a q_2 for any q_1 that solves this system.

So we will say the vector length is 1 for convenience. This sets up the equation

$$q_1^2 + q_2^2 = 1 \tag{28}$$

So solve the system of equations in 26 then plug them into 28 to get your values of q.