AMES Class Notes – Week Five, Day 2

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1 Mean Value Theorem and Taylor Series

Remember the Taylor series. A **Taylor Series** can approximate *any continuous function* $f(x)$. The modeler decides if they wants to approximate $f(x)$ with a 0th order taylor series (mean), a 1st order taylor series (line), a 2nd order taylor series (parabola), a 3rd order taylor series (at this point, you better have a LOT of data!!). We can approximate $f(x)$ with our taylor series,

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^k(a)}{k!}(x-a)^k + R \tag{1}$$

where R is the residual.

Remember that what the **Mean Value Theorem** tells us: If you have two points, the slope between those two points will be parallel to the tangent line of the original function. So, the taylor series is a large application of the mean value theorem, where a taylor series approximates for the slope at a point many many times.

1.1 Discussion of models

Any Taylor Series approximation is a model. If someone says "I don't do models, let's only do means" they actually *are* suggesting a model. A mean is a model, it's a zero order Taylor Series approximation!

Higher order Taylor Series will approximate an underlying function better than a lower order. *However,* we do not always have enough data to fit a higher order Taylor series because it would require more estimating parameters and your data set may not have enough power (aka enough data) to do so.

1.2 Exercise: What is $\sqrt{10}$?

What's is $\sqrt{10}$? The function we are considering is

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}. \tag{2}$$

We're interested when $x = 10$. Let's do a **0th order taylor series approximation**:

$$f(x) = f(a) \tag{3}$$

what should we choose a to be? Let's choose $a = 9$ because 9 is close to 10.

$$\begin{aligned} f(10) &\approx f(a) && (4)\\ &\approx f(9) && (5)\\ &\approx \sqrt{9} && (6)\\ &\approx 3 && (7) \end{aligned}$$

So the 0th order approximation of $f(10) \approx f(9) = 3$. It's close!

Now, let's do a **first order taylor series approximation**

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)^1. \tag{8}$$

We need to find $f'(a)$,

$$f(a) = a^{1/2} \tag{9}$$

$$f'(a) = \frac{1}{2}a^{-1/2} \tag{10}$$

We can plug 10 back into 8 to get

$$f(x) \approx f(a) + \frac{1}{2}\frac{a^{-1/2}}{1}(x-a)^1 \tag{11}$$

Now, let's find $f(10)$ where $x = 10$ by using the taylor expansion around $f(9)$ where $a = 9$,

$$f(10) \approx f(9) + \frac{f'(9)}{1!}(10-9)^1 \tag{12}$$

$$\approx \sqrt{9} + \frac{1}{2}(9)^{-1/2} \times 1 \tag{13}$$

$$\approx 3 + \frac{1}{2}\frac{1}{(9)^{1/2}} \tag{14}$$

$$\approx 3\frac{1}{6} \tag{15}$$

So our first order approximation of $f(10) \approx 3\frac{1}{6}$. This is even closer than our original approximation, which was 3!

We can then do a **second order taylor series approximation** and get even closer.

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 \tag{16}$$

we already found what $f'(a)$ is. What's $f''(a)$?

$$f(a) = a^{1/2} \tag{17}$$

$$f'(a) = \frac{1}{2}a^{-1/2} \text{ (power rule)} \tag{18}$$

$$f''(a) = -\frac{1}{4}a^{-3/2} \text{ (power rule again)} \tag{19}$$

We can plug in 19 into 16 (along with what we found in our first order approximation) to get

$$f(x) \approx f(a) + \frac{1}{2}\frac{a^{-1/2}}{1}(x-a)^1 + \frac{1}{2}(-\frac{1}{4})a^{-3/2}(x-a)^2 \tag{20}$$

we know from our first order that the first two terms simplify to $3\frac{1}{6}$. Lets simplify this expression

$$f(10) \approx 3 + \frac{1}{6} - \frac{1}{8}\frac{1}{9^{3/2}}(10-9)^2 \tag{21}$$

$$= 3 + \frac{1}{6} - \frac{1}{8}\frac{1}{27} \tag{22}$$

$$= 3.16203703704 \tag{23}$$

So what is $f(10) = \sqrt{10}$? $\sqrt{10} = 3.16227766017$. So we can see that our second order approximation got pretty close!

1.3 Example: Logistic growth

Consider the logistic growth function:

$$G(N) = rN(1 - \frac{N}{K}) \tag{24}$$

What would we get if we "taylor expanded" this function? Let's work with the per capital growth rate

$$\frac{G(N)}{N} = r(1 - \frac{N}{K}) \tag{25}$$

and see if we can use a Taylor series that would approximate this right hand side equation

$$\begin{align} \frac{G(N)}{N} &\approx G(a) + G'(a)(N - a) \tag{26} \\ &\approx G(a) + G'(a)N - G'(a)a \tag{27} \\ &\approx G(a) - G'(a)a + G'(a)N \tag{28} \end{align}$$

Let $r = -G'(a)a$.

$$\approx r + G(a) + G'(a)N \tag{29}$$

Let $G(a) = 0$ and rename $G'(a) = \frac{-r}{K}$

$$\begin{align} &\approx r + \frac{-r}{K}N \tag{30} \\ &\approx r(1 - \frac{N}{K}) \tag{31} \end{align}$$

And now we've shown that 31 is equivalent to 25. Which means we've shown that we can derive the logistic growth equation using a Taylor series approximation.

2 Total Derivatives

Total derivative in a function is equal to the sum of the partials multiplied by the change.
Consider the function $F = f(x, y)$. We can get our partial derivatives. Recall that when we take a partial derivative wrt x, we're holding y constant.

$$\begin{align} \frac{\partial f}{\partial x} &= f_x \tag{32} \\ \frac{\partial f}{\partial y} &= f_y \tag{33} \end{align}$$

Now consider how we would calculate the change in F if both x and y changed. We'd need to get a full derivative,

$$\begin{align} df &= f_x dx + f_y dy \tag{34} \\ &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{35} \\ \Delta f &= = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \tag{36} \end{align}$$

These are all equivalent ways of writing the full derivative.

2.1 Example of maximizing a multi-variate function

Consider the functions $F(x, y) = -x^2 - y^2$. Let's say that we want to maximize this function, and find the *global* maximum. To do so, we need to take the partial derivatives and set them equal to zero, then solve for x, y.

$$\frac{\partial F}{\partial x} = -2x = 0 \implies x = 0 \tag{37}$$

$$\frac{\partial F}{\partial y} = -2y = 0 \implies y = 0 \tag{38}$$

So the maximum point of $F(x, y) = -x^2 - y^2$ is $(0, 0)$, where $x = 0$ and $y = 0$.

2.2 Implicit Function Theorem (envelope theorem in econ)

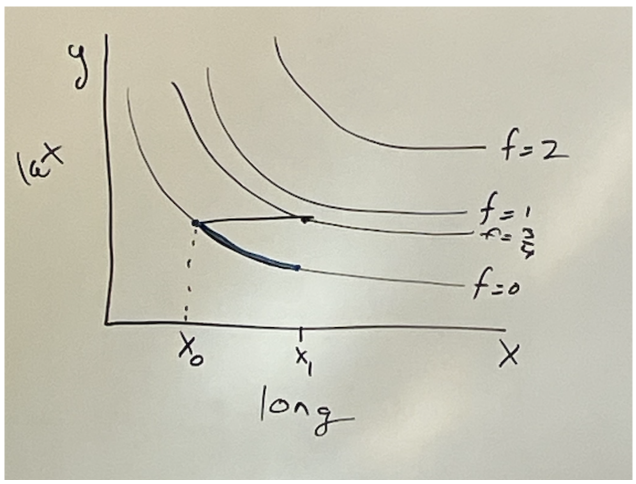

Figure 1: The f=0 represents an altitude

Now consider if you want to stay on one level set. For example, if you wan to stay on a contour line or on a utility level. That would mean we want the value of $f(x, y)$ to stay the same, $df = \Delta f = 0$, even when we change both the x and y. We can set the full deriative to 0 in order to acheive this.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \implies \tag{39}$$

$$dy\frac{\partial f}{\partial y}dy = -dx\frac{\partial f}{\partial x} \implies \tag{40}$$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{41}$$

Equation 41 is the implicit function theorem. What equation 41 tells us is "if I want to stay at the original value of $f(x, y)$, and I change x, how do I need to change y in order to stay at the original value of f". You could also think of this as you're hiking and you're at a certain altitude. The same questions would be "if I change my latitude, how do I need to change my longitude in order to stay at the same altitude?"

The envelope theorem is a special case of the implicit function theorem when first derivatives are set to zero. This leads to math simplifying because we can say that, at the optimum, a bunch of derivatives will equal zero.