AMES Class Notes – Week 11, Wednesday: Differential Equations, Cont.

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1 Review from last week

Consider we're at a **steady state**

$$\dot{x} = \frac{dx}{dt} = 0$$

solve for x to find the stock value that create a system where the stock doesn't change through time. (*ie* a carrying capacity).

Jarogon note: stable equilibrium = global/local attractor ⊆ steady states = equilibrium points = fixed points

Consider a population of x that has a logistic growth rate. This growth rate is a second order Taylor series approximation to any arbitrary function. A second order taylor series approximation creates a "single global attractor" aka a unique stably steady state. You'll sometimes hear this called a fixed point, as well. A nice feature of second order taylor series is that they're defined over convex sets, and that gaurantee's the existence of a "single global attractor (a unique stable steady state), which is p fixed point theories.

$$\dot{x} = \frac{dx}{dt} = ax(1 - x)$$

Example
What are the levels of x that produce equilibria/steady states of the following and what are stable and which are not,

$$\dot{x} = \frac{dx}{dt} = ax(\frac{x}{b} - 1)(1 - x)?$$
$$x = 0$$
$$x = b$$
$$x = 1$$

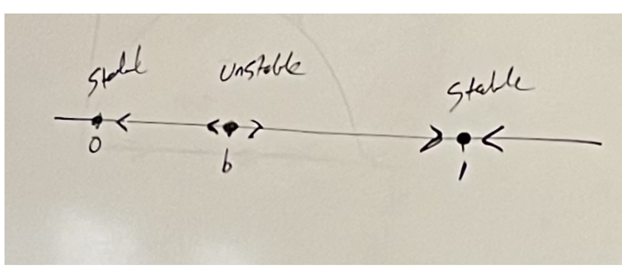

Figure 1: phase line

2 Lotka Voltera Model

Consider a population of deer x and a population of wolves y.
The growth rate of x through time is

$$\dot{x} = \frac{dx}{dt} = ax(1 - \frac{x + \alpha y}{J})$$
$$\dot{y} = \frac{dy}{dt} = by(1 - \frac{\beta x + y}{K})$$

What are the equilibrium / steady states of this system? *ie* what at the pairings of (x, y) that lead to $\dot{x} = 0$ and $\dot{y} = 0$.

From previous classes (problem set 1) we know it'll be $(0, 0)$, $(0, K)$, $(J, 0)$, (y^*, x^*)

How does the population of y change with respect to the population of x *around an equilibrium point*? AKA what is $\frac{dy}{dx}$, *around the equilibrium*?

Use the implicit function theorem because we're interested in the dynamics around the equilibrium aka we're staying around (x^*, y^*), which gives us:

$$\frac{dy}{dx} = -\frac{\frac{d\dot{x}}{dx}}{\frac{d\dot{x}}{dy}} \tag{1}$$

Notes on implicit function theorem: https://github.com/a5creel/AMES/blob/main/class_notes/5_weds/main.pdf

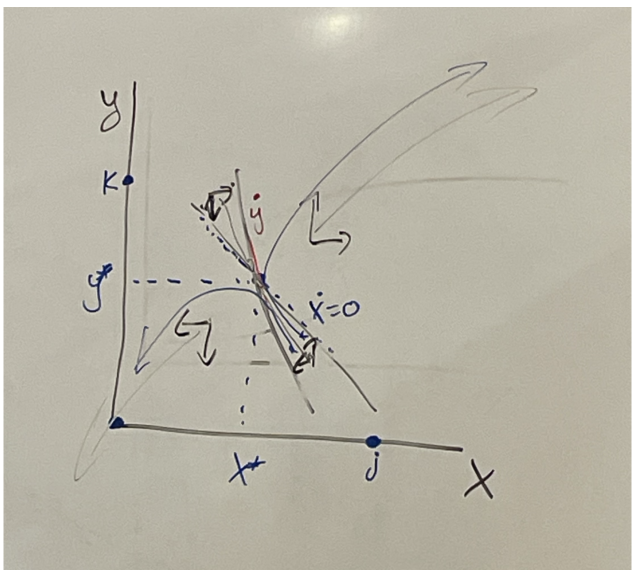

Figure 2: Phase plane around (x^*, y^*)

Now that we have dy/dx at the equilibrium (x^*, y^*), we can plug in different (x, y) pairs that are around the equilibrium to see if the system return to (x^*, y^*) or if would it travel away. This is how you draw a phase plane.

3 Bass and Crayfish Phase Plane

Let x be the population of crayfish. Let y be the population of bass.

$$\dot{x} = x(1 - x - \alpha y) - \frac{\delta - yx^2}{k^2 - x^2}$$
$$\dot{y} = ry(1 - \beta x - y) + \frac{\epsilon \delta y x^2}{k^2 - x^2}$$

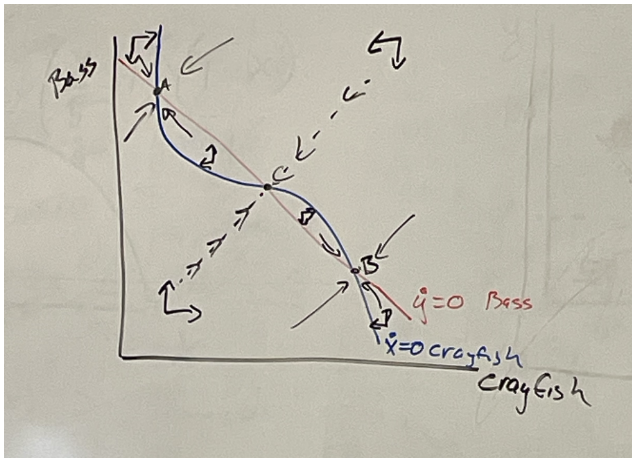

Figure 3: Phase plane for crayfish and bass

4 Different types of equilibrium in phase planes

There are stable, unstable, and conditionally stable equilibrium. There are different ways we can approach an equilibrium, like a spiral or a line. Sometimes, we have orbits aka stable limit cycles.

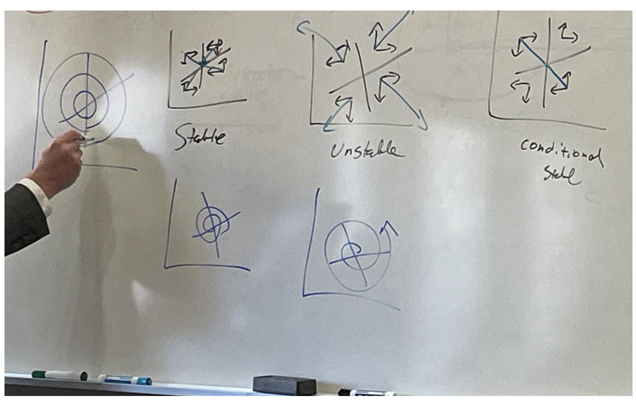

Figure 4: Types of equilibria

This is all really confusing! We're going to use eigen values and eigen vectors to determine what type of equilibrium a steady state is, and what the approach path to that steady state looks like (spiral vs line).

5 Tipping points

Tipping points can only be modeled with non convex sets. HOWEVER, we will not work with non-convex state spaces if we're using *linear* models and most *quadratic* models. Linear models and quadratic models are great approximations (shout out, taylor). HOWEVER, they will never produce scenarios with tipping points. And so, **we need to expand out standard statistical techniques to models with higher order terms to model scenarios with tipping points**.