AMES Class Notes – Week Six, Day 1

Andie Creel

2nd October, 2023

1 Integrals

1.1 Reimann Sums

An integral is the area under a curve. A reimann sum approximates this area by using rectangles of different heights whose width gets smaller and smaller (*ie* width goes to zero).

Let $A(\cdot)$ be an area function. A returns the area under a curve.

Recall that small changes in x are equal to ϵ but also can be written as ∂x, $\Delta x = \epsilon = \partial x$.

$$f(a) = \frac{A(a+\epsilon) - A(a)}{\epsilon} \tag{1}$$

If we take the limit of this function as $\epsilon \to 0$, we get the definition of a derivative

$$\lim_{\epsilon \to 0} f(a) = \frac{A(a+\epsilon) - A(a)}{\epsilon} \implies \tag{2}$$

$$A'(x) = \frac{\partial A}{\partial x} \implies \tag{3}$$

$$A'(x)\partial x = \partial A \implies \tag{4}$$

$$A'(x)\epsilon = \partial A \tag{5}$$

where all of these steps come from knowing $\epsilon = \partial x$ and treating a derivative like a fraction. What is $A'(x)$? If $A(\cdot)$ is the area under the curve, then the derivative is the curve which is $f(x)$ therefore $A'(x) = f(x)$. Therfore, a change in area can be written as

$$\partial A = f(x)\epsilon \tag{6}$$

which is just a rectangle where $f(x)$ is the height and ϵ is the width.

We can rewrite this as

$$\partial A = \sum_{x=a}^{b} f(x)\epsilon. \tag{7}$$

if you want to want to find the area under the curve from the point where $x = a$ to the point where $x = b$. This is a sum of rectangles.

Integration is just fancy summation! So, as ϵ gets really small and we move from discrete changes to arbitrarily small changes (smooth). We can then write this as

$$\partial A = \int_a^b f(x)\partial x. \tag{8}$$

1.2 Additively separable

Consider a function

$$f(x) = h(x) + g(x) \tag{9}$$

where $f(x)$ can be separated into two simpler functions. For instance, any polynomial would take this form (*e.g.* $f(x) = 3x^2 + 6x$).

The function $A(\cdot)$ is still returning the area under the curve $f(x)$. Then, we can write the function as

$$dA = \int h(x) + g(x)dx \tag{10}$$

Because the integral is a linear operator, we do not need to worry about Jensen's inequality and we can pass the integral sign through the addition

$$dA = \int h(x)dx + \int g(x)dx \tag{11}$$

1.3 Considering constants

Remember that when you take the derivative of a constant, it's zero. When you integrate a function, you will not be able to get the constant back out (you don't know what the constant is),

$$F(x) + C = \int f(x)dx. \tag{12}$$

When ever you take an integral, remember to add on your unidentified constant C.

If you're doing a **definite** integral meaning you have the bounds a and b you do NOT need to worry about the unidentified constant C because it will difference out. However, if you are taking an **indefinite** integral then we don't know the bounds and so we do need to keep track of the unidentified constant.

1.4 Example

Consider this the function $f(x) = 3$,

$$\begin{align} F(x) &= \int f(x)dx \tag{13}\\ &= \int 3dx \tag{14}\\ &= 3\int dx \tag{15}\\ &= 3\int 1dx \tag{16}\\ &= 3x + C \tag{17} \end{align}$$

2 Derivative rules

$$\begin{align} f(x) = X^n &\implies f'(x) = Nx^{N-1} \tag{18}\\ f(x) = e^x &\implies f'(x) = e^x \tag{19} \end{align}$$

3 Integral rules

Integration by U-substitution

$$\int f(u)\frac{du}{dx}dx = \int f(u)du \tag{20}$$

Example

$$\int xe^{x^2} \tag{21}$$

Let $u = x^2$, $du = 2xdx$, $\frac{du}{dx} = 2x$

$$= \int e^{x^2} xdx \tag{22}$$

$$= \int \frac{e^u}{2} du \tag{23}$$

$$= \frac{1}{2}\int e^u du \tag{24}$$

$$= \frac{1}{2}e^u \tag{25}$$

$$= \frac{1}{2}e^{x^2} \tag{26}$$