AMES Class Notes – Week Five, Day 2

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27th September, 2023

1 Total Derivatives

Total derivative in a function is equal to the sum of the partials multiplied by the change.
Consider the function $F = f(x, y)$. We can get our partial derivatives. Recall that when we take a partial derivative wrt x, we're holding y constant.

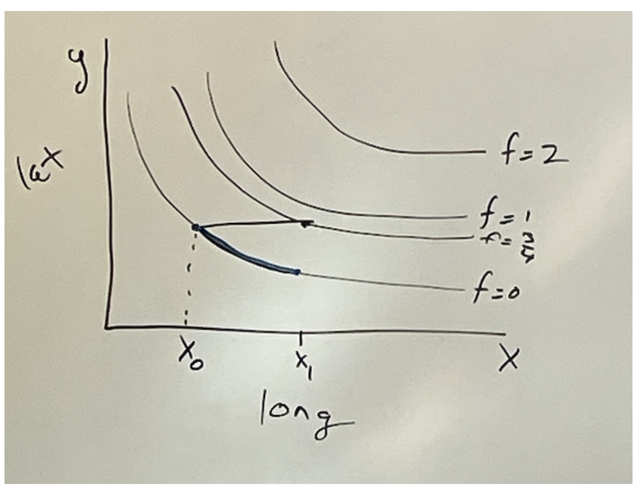
$$\frac{\partial f}{\partial x} = f_x \tag{1}$$
$$\frac{\partial f}{\partial y} = f_y \tag{2}$$

Now consider how we would calculate the change in F if both x and y changed. We'd need to get a full derivative,

$$df = f_x dx + f_y dy \tag{3}$$
$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{4}$$
$$\Delta f == \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \tag{5}$$

These are all equivalent ways of writing the full derivative.

1.1 Implicit Function Theorem (envelope theorem in econ)



Now consider if you want to stay on one level set. For example, if you wan to stay on a contour line or on a utility level. That would mean we want F to stay the same, $df = \Delta f = 0$.

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \implies \tag{6}$$
$$dy\frac{\partial f}{\partial y}dy = -dx\frac{\partial f}{\partial x} \implies \tag{7}$$
$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{8}$$

Equation **??** is the implicit function theorem. What equation **??** tells us is "if I want to stay on the same level set, and I change x, how do I need to change y in order to stay on that level set".

The envelope theorem is a special case of the implicit function theorem when first derivatives are set to zero. This leads to math simplifying because we can say that, at the optimum, a bunch of derivatives will equal zero.

2 Taylor Series

We know that straight lines are good approximators because a line is a conditional mean, and means are good at minimizing error.

Series are good approximators of sequences of numbers, because series are functions.

A **Taylor Series** is an extremely useful series for approximating the relationship of sequence of numbers (data is a sequence of numbers). Taylor series underpins a lot of the math we do today, particularly for any relationship that isn't linear.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^k(a)}{k!}(x-a)^k + R \tag{9}$$

where R is the residual.

2.1 0th order Taylor Series

Consider a 0th order taylor series: it would be a mean aka just a flat line equal to $f(a)$.

If you're only looking at the means of a dataset, that means you're doing a 0th order taylor approximation.

2.2 First order Taylor Series

Consider a first order taylor series:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 \tag{10}$$
$$= A + B(x-a) \tag{11}$$
$$= A - aB + Bx \tag{12}$$
$$= m + Bx \tag{13}$$

It's a line with a slope. A first order Taylor series is a linear approximation. Linear regression is a first order Taylor approximation.

2.3 Discussion of models

Any Taylor Series approximation is a model. If someone says "I don't do models, let's only do means" they actually *are* suggesting a model. A mean is a model, it's a zero order Taylor Series approximation!
Higher order Taylor Series will approximate an underlying function better than a lower order. *However,* we do not always have enough data to fit a higher order Taylor series because it would require more estimating parameters and your data set may not have enough power (aka enough data) to do so.

2.4 Example: Logistic growth

Consider the logistic growth function:

$$G(N) = rN(1 - \frac{N}{K}) \tag{14}$$

What would we get if we "taylor expanded" this function? Let's work with the per capital growth rate

$$\frac{G(N)}{N} = r(1 - \frac{N}{K}) \tag{15}$$

and see if we can use a Taylor series that would approximate this right hand side equation

$$\begin{align}
\frac{G(N)}{N} &\approx G(a) + G'(a)(N - a) \tag{16}\\
&\approx G(a) + G'(a)N - G'(a)a \tag{17}\\
&\approx G(a) - G'(a)a + G'(a)N \tag{18}
\end{align}$$

Let $r = -G'(a)a$.

$$\approx r + G(a) + G'(a)N \tag{19}$$

Let $G(a) = 0$ and rename $G'(a) = \frac{-r}{K}$

$$\begin{align}
&\approx r + \frac{-r}{K}N \tag{20}\\
&\approx r(1 - \frac{N}{K}) \tag{21}
\end{align}$$

And now we've shown that **??** is equivalent to **??**. Which means we've shown that we can derive the logistic growth equation using a Taylor series approximation.

2.5 Exercise

What's is $\sqrt{10}$? The function of consideration is

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}. \tag{22}$$

We're interested when $x = 10$. Let's do a **0th order taylor series approximation**:

$$f(x) = f(a) \tag{23}$$

what should we choose a to be? Let's choose $a = 9$ because 9 is close to 10.

$$\begin{align}
f(10) &\approx f(a) \tag{24}\\
&\approx f(9) \tag{25}\\
&\approx \sqrt{9} \tag{26}\\
&\approx 3 \tag{27}
\end{align}$$

Let's do a **first order taylor series approximation**

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a)^1 \tag{28}$$

$$f(10) \approx f(9) + \frac{f'(9)}{1!}(10-9)^1 \tag{29}$$

$$\approx \sqrt{9} + \frac{1}{2}(9)^{-1/2} \tag{30}$$

$$\approx 3 + \frac{1}{2}\frac{1}{(9)^{1/2}} \tag{31}$$

$$\approx 3\frac{1}{6} \tag{32}$$

You could then do a second order taylor series approximation and get even closer.