AMES Class Notes – Week 9, Monday: Linear Algebra, cont

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1 Squaring matrices

Consider a matrix $\underset{N\times 1}{V}$ that you'd like to square.

$$V^2 = \underset{1\times N}{V}^T \underset{N\times 1}{V} \tag{1}$$
$$= \underset{1\times 1}{Z} \tag{2}$$

Sometimes you'll have a weighting matrix, $\underset{N\times N}{M}$

$$\underset{1\times N}{V}^T \underset{N\times N}{M} \underset{N\times 1}{V} = \underset{1\times 1}{Z} \tag{3}$$

2 Idempotent matrix

Defn: A matrix that when squared equals itself

$$Z^T Z = Z \tag{4}$$

3 Kronecker product

Kronecker products expand out. You see this is computer algorithms and data management problems. It's a common trick.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{5}$$
$$Z = \begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix} \tag{6}$$
$$A \otimes Z = \begin{bmatrix} a_1 1 z_1 & a_1 2 z_1 \\ a_1 1 z_2 & a_1 2 z_2 \\ a_2 1 z_1 & a_2 2 z_1 \\ a_2 1 z_2 & a_2 2 z_2 \end{bmatrix} \tag{7}$$

4 Trace

Defn: the product of all diagonal elements. The trace of A is $a_{11} * a_{22}$

5 Linear regression

Consider the equation estimating equation

$$y = a + bx + cz + \epsilon \tag{8}$$

We can rewrite this as

$$\underset{N\times 1}{Y} = \underset{N\times K}{X}\underset{K\times 1}{\beta} + \underset{N\times 1}{\epsilon} \tag{9}$$

$$\tag{10}$$

where

$$\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{11}$$

$$X = \begin{bmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ \vdots & \dots & \end{bmatrix} \tag{12}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} \tag{13}$$

Goal: Solve for β by minimizing the sum of square errors.

Solve for the error term:

$$\underset{N\times 1}{Y} - \underset{N\times K}{X\beta} = \underset{N\times 1}{\epsilon} \tag{14}$$

Square the error term:

$$\underset{1\times N}{\epsilon}^T \underset{N\times 1}{\epsilon} = \underset{1\times N}{(Y - X\beta)^T}\underset{N\times 1}{(Y - X\beta)} \tag{15}$$

$$= \underset{1\times N}{Y}^T \underset{N\times 1}{Y} - \underset{1\times K}{\beta}^T \underset{K\times N}{X}^T \underset{N\times 1}{Y} - \underset{1\times N}{Y}^T \underset{N\times K}{X}\underset{K\times 1}{\beta} + \underset{1\times K}{\beta}^T \underset{K\times N}{X}^T \underset{N\times K}{X}\underset{K\times 1}{\beta} \tag{16}$$

$$= \underset{1\times 1}{Y^T Y} - 2\underset{1\times 1}{\beta^T X^T Y} + \underset{1\times 1}{\beta^T X^T X\beta} \tag{17}$$

The squared error term is a scalar $\underset{(1\times 1)}{\epsilon^T \epsilon}$.

We want to minimize the squared error term by choosing β. To do so, take derivative of squared error and set equal to zero, solve for β.

$$\frac{\partial \epsilon^T \epsilon}{\partial \beta} = 0 - 2\underset{K\times 1}{X^T Y} + 2\underset{K\times 1}{X^T X\beta} = 0 \implies \tag{18}$$

$$2X^T Y = 2X^T X\beta \implies \tag{19}$$

$$\beta = (X^T X)^{-1} X^T Y \tag{20}$$

Where $(X^T X)^{-1}$ is an inversion (because we cannot divide matrices).

6 Life cycle assessment example

Let's consider and input output table (Ag, Transportation, Manufactured)

$$A = \underset{3\times 3}{\begin{bmatrix} & A & T & M \\ A & & & \\ T & & & \\ M & & & \end{bmatrix}} \tag{21}$$

And our final demand (the demand for goods by consumers)

$$d = \underset{3\times 1}{\begin{bmatrix} A \\ T \\ M \end{bmatrix}} \tag{22}$$

Total amount of goods X is,

$$\underset{3\times 1}{X} = \underset{3\times 3}{A}\underset{3\times 1}{X} + \underset{3\times 1}{d} \tag{23}$$

Solve for X by multiplying with the identity matrix

$$\underset{3\times 3}{I}X - AX = d \tag{24}$$

$$(I - A)X = d \tag{25}$$

$$\implies (I - A)^{-1}d = X \tag{26}$$

Total amount of goods is different than final demand because we need input good for the final good. A bunch of intermediate products are required to make a computer.

7 R and Excel

Watch the video! Couple of notes
- to invert a matrix in R you use the solve() command