AMES Class Notes – Week 8, Monday: Linear Algebra

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1 Intro

Consider the system of equations:

$$x = ay + bz + u \tag{1}$$
$$y = cx + dz + w \tag{2}$$
$$z = fx + gy + v \tag{3}$$

You could solve this system of three equations for the three unknowns x, y, z. But it would be tedious! Let's rearrange it so that we only have to solve for one unknown by using linear algebra.

$$1x - ay - bz = u \tag{4}$$
$$-cx + 1y - dz = w \tag{5}$$
$$-fx - gy + 1z = v \tag{6}$$

This can be rewritten as vectors and matrices.

$$\begin{bmatrix} u \\ w \\ v \end{bmatrix} = \begin{bmatrix} 1 & -a & -b \\ -c & 1 & -d \\ -f & -g & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{7}$$

A reoccurring question: How can you write out a system of equations, and how can you rewrite it as a matrix?

2 Matrices

The first subscript i is the row and the second subscript j is the column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} \end{bmatrix} \tag{8}$$

This is an $(i \times j)$ matrix. Always keep track of your dimensions.

3 Addition and Subtraction

Consider two (2×2) matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{9}$$

We can add these matrices because they're of the same dimensions.

$$A + C = \begin{bmatrix} a_{11} + c_{11} & a_{12} + c_{12} \\ a_{21} + c_{21} & a_{22} + c_{22} \end{bmatrix} \tag{10}$$

Subtraction works the same way.

4 Multiplication

If we have matrix B that's $(m \times n)$ and matrix D that's $(n \times p)$ then the matrix $B \times M$ is going to be $(m \times p)$. **Order matters** when multiplying matrices. $M \times B$ is NOT conformable, you cannot compute that matrix!!

Consider the matrices $A(2 \times 2)$ and $B(2 \times 2)$ again

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{11}$$

Now, $A \times B$. Matrix multiplication requires row multiplied by columns

$$A \times B = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} \tag{12}$$

Consider a vector

$$x = \begin{bmatrix} M \\ N \end{bmatrix} \tag{13}$$

Pre-multiply x by A.

$$\underset{(2\times2)}{A}\underset{(2\times1)}{x} = \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} a_{11}M + a_{12}N \\ a_{21}M + a_{22}N \end{bmatrix} \tag{15}$$

There is no equivalent to division in matrix algebra. However, you can invert a matrix, which we will get to.

5 Identify Matrix

The identity matrix have similar priorities to 1.

$$I = \begin{bmatrix} 1 & 0 & ... & \\ 0 & 1 & ... & \\ ... & & & \\ 0 & 0 & ... & 1 \end{bmatrix} \tag{16}$$

There are 1s along the diagonal. An important characteristic of the identity matrix is

$$\underset{(2\times2)}{A}^{-1}\underset{(2\times2)}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

Any matrix pre-multiplied by it's inverse is the identify matrix.

6 Rank

Rank is the largest unique square matrix you can have inside a matrix.

The key here is knowing if a row is unique. For instance

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \tag{18}$$

Only has 1 unique row because the first row is the second row multiplied by 2.

7 Transpose

When you transpose a matrix, the first row becomes the first column.

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \tag{19}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix} \tag{20}$$

Transpose is important if you're interested in a matrix multiplied by itself!

$$A^2 = A^T A \tag{21}$$

To square, you always pre-multiply the original by the transpose. This keeps the matrices conformable.

$$\underset{(7\times3)}{B^2} = \underset{(3\times7)}{B^T}\underset{(7\times3)}{B} = \underset{(3\times3)}{C} \tag{22}$$