

AMES Week 12 class notes – Weds, Eigenvalues and Eigenvectors

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1 Summary

Eigenvalues and eigenvectors are fundamental concepts in linear algebra, with applications in various fields such as physics, computer science, and engineering.

1.1 Eigenvalues and Eigenvectors

Definition:

- **Eigenvalue (λ):** For a square matrix A , a scalar λ is considered an eigenvalue if there exists a non-zero vector q such that $Aq = \lambda q$.
- **Eigenvector (q):** A non-zero vector q is an eigenvector corresponding to the eigenvalue λ if $Aq = \lambda q$.

Eigenvalue Equation:

The relationship $Aq = \lambda q$ can be expressed as $(A - \lambda I)q = 0$, where I is the identity matrix.

Characteristic Polynomial:

The characteristic polynomial of A is given by $\det(A - \lambda I) = 0$. Solving this equation yields the eigenvalues of A .

1.2 Interpretation

The **eigenvalue** gives you "speed" of the dynamics around an equilibrium. Is a system going to approach equilibrium's, or move away? Is it happening fast or slow?

- If all eigen values are positive, the equilibrium is unstable. The system will move away from the equilibrium if perturbed.
- If all eigen values are negative, the equilibrium is stable. The system will move towards the equilibrium if perturbed.
- If one is positive, and the other is negative, it's conditionally stable.
- If an eigen value has an imaginary number in it $i = \sqrt{-1}$ such that $\lambda = R + Zi$ that means we have a spiraling equilibrium. If R is positive it spirals out, if R is negative it spirals in.

The **eigenvector** tells us about the "direction" of the dynamics around the equilibrium. So if you're thinking about populations, it'll tell you how many more or less of a different species you're getting. If you're thinking about the economy, it'll tell you how the share of the economy is going to be split across different industries.

2 Review of solving for eigenvectors

Recall from last class, when we consider how the stocks in a system are changing we end up with an equation

$$(A - \lambda I)q = 0 \quad (1)$$

where A is the the Jacobian matrix and q is our eigen vector.

$$(A - \lambda I)q = 0 \quad (2)$$

$$\begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 0 \quad (3)$$

$$\implies (a_{11} - \lambda)q_1 + a_{12}q_2 = 0 \quad (4)$$

$$a_{21}q_1 + (a_{22} - \lambda)q_2 = 0 \quad (5)$$

The q is only defined relative to another. Conditioned on a given λ , there is a q_2 for any q_1 that solves this system.

So we will say the vector length is 1 for convenience. This sets up the equation

$$q_1^2 + q_2^2 = 1 \quad (6)$$

So solve the system of equations in ?? then plug them into ?? to get your values of q .

3 Variance and covariance

Consider a matrix of data M with a column for x and column for y .

Our variance and covariance matrix would be

$$\begin{bmatrix} \sigma_x^2 & \sigma_x \sigma_y \\ \sigma_y \sigma_x & \sigma_y^2 \end{bmatrix} \quad (7)$$

where

$$\sigma_x^2 = Var_x(M) = \frac{1}{N} \sum (x_i - \bar{x})^2 \quad (8)$$

$$\sigma_y^2 = Var_y(M) = \frac{1}{N} \sum (y_i - \bar{y})^2 \quad (9)$$

$$\sigma_{xy} = \sigma_{yx} = Cov(x, y) = \frac{1}{N} \sum (x_i - \bar{x})(y_i - \bar{y}). \quad (10)$$

We could rescale our data so that the mean is zero, *i.e.* $\bar{x} = 0$ and $\bar{y} = 0$ and all our equations would simplify.

$$\sigma_x^2 = Var_x(M) = \frac{1}{N} \sum x_i^2 \quad (11)$$

$$\sigma_y^2 = Var_y(M) = \frac{1}{N} \sum y_i^2 \quad (12)$$

$$\sigma_{xy} = \sigma_{yx} = Cov(x, y) = \frac{1}{N} \sum x_i y_i. \quad (13)$$

We could use linear algebra to get the covariance matrix. Remember that x and y would be N and M would be $N \times 2$

$$Var(x) = x^T x \quad (14)$$

$$Var(y) = y^T y \quad (15)$$

$$Cov(M) = M^T M \quad (16)$$

where $Var(x)$ and $Var(y)$ are scalars and $cov(M)$ is a $k \times k$ matrix.

4 Principle components analysis

In the context of Principal Component Analysis (PCA) or covariance matrices, the eigenvectors represent the principal components or directions of maximum variance in the data. The eigenvalues indicate magnitude of variance along these directions.

In this course, you don't need to memorize how to do this. But it's good to understand what a principle components analysis is. A PCA determines what are the main variables (principle components) that drive results may be.

5 Graph analysis

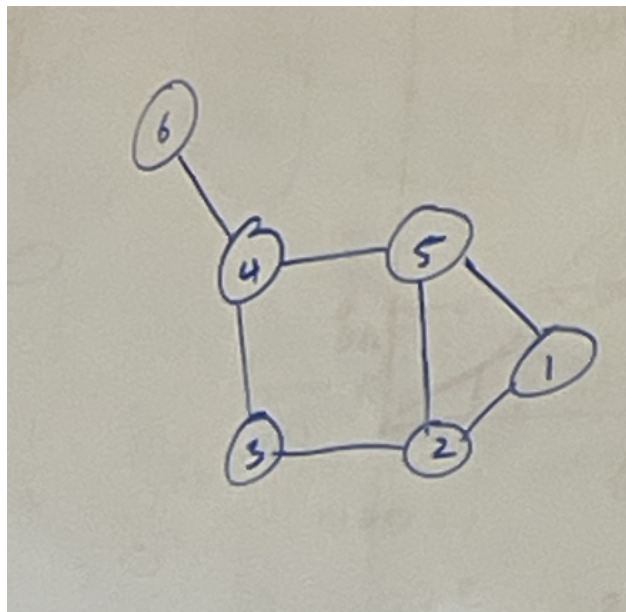


Figure 1: Graph

With a graph, we can make a **degree matrix**, which is the number of connections each node has. It's a diagonal matrix. We can make an **adjacency matrix** so we see which nodes are connected to which. And we could make a **laplacian matrix** which would be the degree matrix minus the adjacency which tells us about the "speed" of movement between nodes.

The adjacency matrix of a graph is a square matrix that represents the connections between vertices. For an undirected graph, the adjacency matrix is symmetric. The *eigenvalues and eigenvectors of the adjacency matrix* provide insights into the graph's connectivity. The *eigenvalues of the Laplacian matrix* are closely related to the number of connected components in a graph. The multiplicity of the eigenvalue in the Laplacian matrix is equal to the number of connected components in the graph. The corresponding eigenvectors provide information about the connectivity within each connected component.

Again, you don't need to know all this for this class, but know that it exists.

	Degree	Matrix	Adj Mat	Laplacian			
	1	2	3	4	5	6	
1	2	0	0	0	0	0	2 -1 0 0 -1 0
2	0	3	0	0	0	0	-1 3 -1 0 -1 0
3	0	0	2	0	0	0	
4	0	0	0	3	0	0	
5	0	0	0	0	3	0	
6	0	0	0	0	0	1	

Figure 2: The matrices