

AMES Class Notes – Week Six, Day 1: Integrals

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1 Reimann Sums

An integral is the area under a curve. A reimann sum approximates this area by using rectangles of different heights whose width gets smaller and smaller (*ie* width goes to zero).

Let $A(\cdot)$ be an area function. A returns the area under a curve.

Recall that small changes in x are equal to ϵ but also can be written as ∂x , $\Delta x = \epsilon = \partial x$.

$$f(a) = \frac{A(a + \epsilon) - A(a)}{\epsilon} \quad (1)$$

If we take the limit of this function as $\epsilon \rightarrow 0$, we get the definition of a derivative

$$\lim_{\epsilon \rightarrow 0} f(a) = \frac{A(a + \epsilon) - A(a)}{\epsilon} \implies \quad (2)$$

$$A'(x) = \frac{\partial A}{\partial x} \implies \quad (3)$$

$$A'(x)\partial x = \partial A \implies \quad (4)$$

$$A'(x)\epsilon = \partial A \quad (5)$$

where all of these steps come from knowing $\epsilon = \partial x$ and treating a derivative like a fraction. What is $A'(x)$? If $A(\cdot)$ is the area under the curve, then the derivative is the curve which is $f(x)$ therefore $A'(x) = f(x)$. Therefore, a change in area can be written as

$$\partial A = f(x)\epsilon \quad (6)$$

which is just a rectangle where $f(x)$ is the height and ϵ is the width.

We can rewrite this as

$$\partial A = \sum_{x=a}^b f(x)\epsilon. \quad (7)$$

if you want to want to find the area under the curve from the point where $x = a$ to the point where $x = b$. This is a sum of rectangles.

Integration is just fancy summation! So, as ϵ gets really small and we move from discrete changes to arbitrarily small changes (smooth). We can then write this as

$$\partial A = \int_a^b f(x)\partial x. \quad (8)$$

2 Additively separable

Consider a function

$$f(x) = h(x) + g(x) \quad (9)$$

where $f(x)$ can be separated into two simpler functions. For instance, any polynomial would take this form (e.g. $f(x) = 3x^2 + 6x$).

The function $A(\cdot)$ is still returning the area under the curve $f(x)$. Then, we can write the function as

$$dA = \int h(x) + g(x) dx \quad (10)$$

Because the integral is a linear operator, we do not need to worry about Jensen's inequality and we can pass the integral sign through the addition

$$dA = \int h(x) dx + \int g(x) dx \quad (11)$$

3 Considering constants

Remember that when you take the derivative of a constant, it's zero. When you integrate a function, you will not be able to get the constant back out (you don't know what the constant is),

$$F(x) + C = \int f(x) dx. \quad (12)$$

When ever you take an integral, remember to add on your unidentified constant C .

If you're doing a **definite** integral meaning you have the bounds a and b you do NOT need to worry about the unidentified constant C because it will difference out. However, if you are taking an **indefinite** integral then we don't know the bounds and so we do need to keep track of the unidentified constant.

4 Example

Consider this the function $f(x) = 3$,

$$F(x) = \int f(x) dx \quad (13)$$

$$= \int 3 dx \quad (14)$$

$$= 3 \int dx \quad (15)$$

$$= 3 \int 1 dx \quad (16)$$

$$= 3x + C \quad (17)$$

5 Derivative rules to remember

$$f(x) = X^n \implies f'(x) = Nx^{N-1} \quad (18)$$

$$f(x) = e^x \implies f'(x) = e^x \quad (19)$$