# AMES Class Notes – Week Five, Day 2

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## 1 Total Derivatives

Total derivative in a function is equal to the sum of the partials multiplied by the change.
Consider the function $F = f(x, y)$. We can get our partial derivatives. Recall that when we take a partial derivative wrt $x$, we're holding $y$ constant.

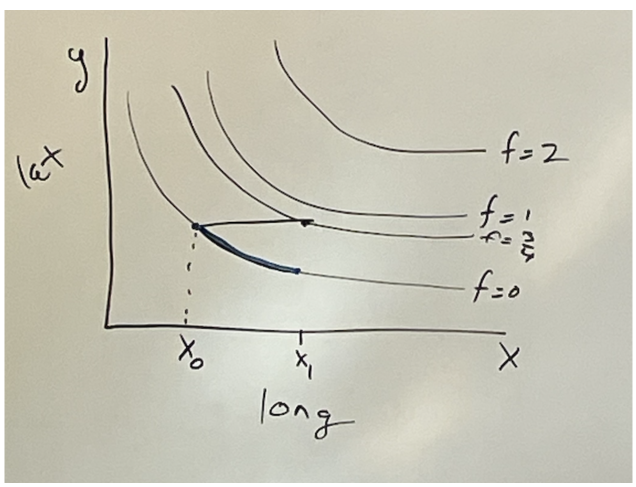
$$\frac{\partial f}{\partial x} = f_x \tag{1}$$
$$\frac{\partial f}{\partial y} = f_y \tag{2}$$

Now consider how we would calculate the change in $F$ if both $x$ and $y$ changed. We'd need to get a full derivative,

$$df = f_x dx + f_y dy \tag{3}$$
$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \tag{4}$$
$$\Delta f == \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y \tag{5}$$

These are all equivalent ways of writing the full derivative.

### 1.1 Implicit Function Theorem (envelope theorem in econ)




Now consider if you want to stay on one level set. For example, if you wan to stay on a contour line or on a utility level. That would mean we want $F$ to stay the same, $df = \Delta f = 0$.

$$
\begin{align}
df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \implies \tag{6}\\
dy\frac{\partial f}{\partial y}dy = -dx\frac{\partial f}{\partial x} \implies \tag{7}\\
\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \tag{8}
\end{align}
$$

Equation 8 is the implicit function theorem. What equation 8 tells us is "if I want to stay on the same level set, and I change $x$, how do I need to change $y$ in order to stay on that level set".

**The envelope theorem** is a special case of the implicit function theorem when first derivatives are set to zero. This leads to math simplifying because we can say that, at the optimum, a bunch of derivatives will equal zero.

# 2 Taylor Series

We know that straight lines are good approximators because a line is a conditional mean, and means are good at minimizing error.

Series are good approximators of sequences of numbers, because series are functions.

A **Taylor Series** is an extremely useful series for approximating the relationship of sequence of numbers (data is a sequence of numbers). Taylor series underpins a lot of the math we do today, particularly for any relationship that isn't linear.

$$
f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^k(a)}{k!}(x-a)^k + R \tag{9}
$$

where $R$ is the residual.

## 2.1 0th order Taylor Series

Consider a 0th order taylor series: it would be a mean aka just a flat line equal to $f(a)$.

If you're only looking at the means of a dataset, that means you're doing a 0th order taylor approximation.

## 2.2 First order Taylor Series

Consider a first order taylor series:

$$
\begin{align}
f(x) &= f(a) + \frac{f'(a)}{1!}(x-a)^1 \tag{10}\\
&= A + B(x-a) \tag{11}\\
&= A - aB + Bx \tag{12}\\
&= m + Bx \tag{13}
\end{align}
$$

It's a line with a slope. A first order Taylor series is a linear approximation. Linear regression is a first order Taylor approximation.

## 2.3 Discussion of models

Any Taylor Series approximation is a model. If someone says "I don't do models, let's only do means" they actually *are* suggesting a model. A mean is a model, it's a zero order Taylor Series approximation!
Higher order Taylor Series will approximate an underlying function better than a lower order. *However,* we do not always have enough data to fit a higher order Taylor series because it would require more estimating parameters and your data set may not have enough power (aka enough data) to do so.

## 2.4 Example: Logistic growth

Consider the logistic growth function:

$$G(N) = rN(1 - \frac{N}{K}) \tag{14}$$

What would we get if we "taylor expanded" this function? Let's work with the per capital growth rate

$$\frac{G(N)}{N} = r(1 - \frac{N}{K}) \tag{15}$$

and see if we can use a Taylor series that would approximate this right hand side equation

$$\frac{G(N)}{N} \approx G(a) + G'(a)(N - a) \tag{16}$$

$$\approx G(a) + G'(a)N - G'(a)a \tag{17}$$

$$\approx G(a) - G'(a)a + G'(a)N \tag{18}$$

Let $r = -G'(a)a$.

$$\approx r + G(a) + G'(a)N \tag{19}$$

Let $G(a) = 0$ and rename $G'(a) = \frac{-r}{K}$

$$\approx r + \frac{-r}{K}N \tag{20}$$

$$\approx r(1 - \frac{N}{K}) \tag{21}$$

And now we've shown that 21 is equivalent to 15. Which means we've shown that we can derive the logistic growth equation using a Taylor series approximation.

## 2.5 Exercise

What's is $\sqrt{10}$? The function of consideration is

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}. \tag{22}$$

We're interested when $x = 10$. Let's do a **0th order taylor series approximation**:

$$f(x) = f(a) \tag{23}$$

what should we choose $a$ to be? Let's choose $a = 9$ because 9 is close to 10.

$$f(10) \approx f(a) \tag{24}$$

$$\approx f(9) \tag{25}$$

$$\approx \sqrt{9} \tag{26}$$

$$\approx 3 \tag{27}$$

Let's do a **first order taylor series approximation**

$$
\begin{align}
f(x) &\approx f(a) + \frac{f'(a)}{1!}(x-a)^1 \tag{28}\\
f(10) &\approx f(9) + \frac{f'(9)}{1!}(10-9)^1 \tag{29}\\
&\approx \sqrt{9} + \frac{1}{2}(9)^{-1/2} \tag{30}\\
&\approx 3 + \frac{1}{2}\frac{1}{(9)^{1/2}} \tag{31}\\
&\approx 3\frac{1}{6} \tag{32}
\end{align}
$$

You could then do a second order taylor series approximation and get even closer.