# AMES Class Notes – Week Six, Day 2: More integrals

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## 1 The importance of $e^{-rt}$

Consider a funny integral:

$$\int_0^\infty e^{-rt} dt \tag{1}$$

Why is this weird? Because the support is 0 to $\infty$. Let's try to do this with a U-sub.

Let

$$U = -rt \tag{2}$$
$$\frac{dU}{dt} = -r \implies \tag{3}$$
$$dU = -rdt \implies \tag{4}$$
$$\frac{dU}{-r} = dt \tag{5}$$

plug in 2 and 5 into our original integral

$$= \int_0^\infty -\frac{1}{r} e^U du \tag{6}$$
$$= -\frac{1}{r} \int_0^\infty e^u du \tag{7}$$
$$= -\frac{1}{r} e^u \Big|_0^\infty \tag{8}$$
$$= -\frac{1}{r} e^{-rt} \Big|_0^\infty \tag{9}$$
$$= -\frac{1}{r} e^{-r\infty} - -\frac{1}{r} e^{-r0} \tag{10}$$
$$= \frac{1}{r}(-e^{-r\infty} + e^{-r0} \tag{11}$$
$$= \frac{1}{r}(0+1) \tag{12}$$
$$= \frac{1}{r} \tag{13}$$

Even though we integrated over an infinite support (0 to $\infty$) we have a finite area.

This is **exponential decay** or a **discounting process**. This is a very important integral and it's used in applied settings often.

This discount factor means that we can project a program into the future *infinitely* and still find a finite value for the impact of that program. We can compare a program from now to the end of the world to a another program for the same length of time and see how they differ.

For example:

$$X = \int_0^\infty M(t)e^{-rt}dt \tag{14}$$

$$Y = \int_0^\infty N(t)e^{-rt}dt \tag{15}$$

We can compare $X$ and $Y$ over an arbitrarily long period. If $M(t)$ represented a policy that affected wellbeing, and $N(t)$ was an alternative policy, we could use the discount rate $e^{-rt}$ to be able to evaluate which of those programs would achieve sustainable development goals over an arbitrarily long period.

### 1.1 Thin and Fat Tails

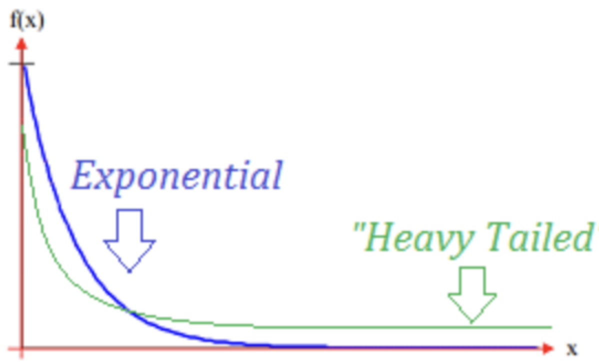


Figure 1: https://www.statisticshowto.com/fat-tail-distribution/

Above, if you were to integrate the "heavy tail" function it would not be finite! As x goes to $\infty$ the area under the function keeps getting bigger and bigger. If you were to integrate the "heavy tailed" function it would "blow up" aka go to infinity.

The if you were to integrate under the "exponential" function in the picture above, you would get a finite answer similar to the function $e^{-rt}$.

## 2 Integration by parts

It's not as bad as you think it is! Previously, integration by substitution is similar to the chain rule for derivatives. Integration by parts is *kind of* like the product rule for derivatives.

Integration by parts is not really an integral trick, it's more like an algebra trick.

Consider the function

$$z(x) = f(x)g(x) \tag{16}$$

We want the intgral of $z(x)$

$$\int f(x)g(x)dx \tag{17}$$

Now, we know how to get $dzdx$

$$\frac{dz}{dx} = f'(x)g(x) + f(x)g'(x) \implies \tag{18}$$

$$\int dz = \int (f'(x)g(x) + f(x)g'(x))dx \tag{19}$$

Remember integrals are linear operators

$$\int dz = \int f'(x)g(x)dx + \int f(x)g'(x)dx \tag{20}$$

$$\tag{21}$$

Note that $\int dz = z = f(x)g(x)$

$$f(x)g(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx \implies \tag{22}$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int f'(x)g(x)dx \tag{23}$$

$$f(x)g(x) - \int f(x)g'(x)dx = \int g(x)f'(x)dx \tag{24}$$

$$\int g(x)f'(x)dx = f(x)g(x) - \int f(x)g'(x)dx \tag{25}$$

Let

$$U = g(x) \tag{26}$$

$$dV = f'(x) \implies \tag{27}$$

$$V = f(x) \tag{28}$$

$$dU = g'(x) \tag{29}$$

Plug these back into our last equation

$$\int UdV = VU - \int VdU \tag{30}$$

which is the equation for integration by parts.

Here's an example. Consider the function $A(x)B(x)$

$$\int A(x)B(x) = A(x)\int B(x)dx - \int A(x)\frac{dB(x)}{dx}dx \tag{31}$$

We've set:

$$dV = B(x) \implies \tag{32}$$

$$V = \int B(x)dx \tag{33}$$

which is why we have $\int B(x)x$ is in the first term of the right hand side.

## 2.1 Tips

How can you remamber the order of what should be U and what should be dV?
U $\rightarrow$ Log $\rightarrow$ Inverse trig $\rightarrow$ Algebraic $\rightarrow$ Trig $\rightarrow$ Exponential $\rightarrow$ dV

### 2.2 Example

Consider the function

$$\int xe^x dx \tag{34}$$

Whats's U and what's V?

$$\int UdV = vu - \int VdU \tag{35}$$
$$U = x \tag{36}$$
$$dU = dx \tag{37}$$
$$V = e^x \tag{38}$$
$$dV = e^x dx \tag{39}$$

Rewrite the whole thing! And use integration by parts

$$\int xe^x dx = e^x x - \int e^x dx \tag{40}$$
$$= ee^x - e^x \tag{41}$$
$$= (x-1)e^x \tag{42}$$

## 3 Integrating logs

Consider

$$\int \frac{dx}{x} = \int \frac{1}{x} dx = ln(x) \tag{43}$$

Cool! Wait, but what about

$$\int ln(x)dx = ?? \tag{44}$$

This was a really hard problem that existed for a very long time in mathematics. What's the trick? Consider this function instead

$$\int ln(x)1dx \tag{45}$$

Now we can use integration by parts.

$$U = ln(x) \tag{46}$$
$$dU = \frac{1}{x} dx \tag{47}$$
$$V = x \tag{48}$$
$$dV = 1dx \tag{49}$$
$$\tag{50}$$

Do NOT forget the $dx$ when you get $dU$!! Now, plug it all back into our integration by parts formal

$$\int UdV = VU - \int VdU \tag{51}$$

$$\int ln(x)1dx = xln(x) - \int x\frac{1}{x}dx \implies \tag{52}$$

$$= xln(x) - \int 1dx \tag{53}$$

$$= xln(x) - x \tag{54}$$

$$= x(ln(x) - 1) \tag{55}$$

You can take the derivative of this final equation to check and you'll find the derivative of that final equation does equal $ln(x)$.

# 4 Double Integral

We need to sum over one variable and then we sum over the other.

$$\int\int xydxdy = \int\left(\int xydx\right)dy \tag{56}$$

We can solve the integral in parentheses first while treating $y$ like a constant

$$= \int\left(y\int xdx\right)dy \tag{57}$$

$$= \int\left(y\frac{1}{2}x^2\right)dy \tag{58}$$

Now we can treat $x$ as a constant

$$= \frac{1}{2}x^2\int ydy \tag{59}$$

$$= \frac{1}{2}x^2 * \frac{1}{2}y^2 + C \tag{60}$$

$$= \frac{1}{4}x^2y^2 + C \tag{61}$$