AMES Class Notes – Week 9, Monday: Linear Algebra, cont

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30th October, 2024

1 Squaring matrices

Consider a matrix $\underset{N\times 1}{V}$ that you'd like to square.

$$V^2 = \underset{1\times N}{V}^T \underset{N\times 1}{V} \tag{1}$$

$$= \underset{1\times 1}{Z} \tag{2}$$

Sometimes you'll have a weighting matrix, $\underset{N\times N}{M}$

$$\underset{1\times N}{V}^T \underset{N\times N}{M} \underset{N\times 1}{V} = \underset{1\times 1}{Z} \tag{3}$$

2 Idempotent matrix

Defn: A matrix that when squared equals itself

$$Z^T Z = Z \tag{4}$$

This property means that applying the matrix operation twice (or any number of times) has the same effect as applying it once.

3 Kronecker product

You see this is computer algorithms and data management problems. It's a common trick. It's technically a "tenser" operation. A tenser is a 3^+D matrix object.

The **Kronecker product** is an operation that takes two matrices and produces a larger matrix by multiplying each element of the first matrix by the entire second matrix. If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is an $mp \times nq$ block matrix, where each entry a_{ij} in A is replaced by the block $a_{ij}B$.

Definition

For matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, the Kronecker product $A \otimes B$ is:

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{bmatrix}$$

The Kronecker product is widely used in signal processing, control theory, and other fields where constructing block matrices is essential.

4 Trace of a Matrix

The **trace** of a matrix is the sum of the elements along its *main diagonal* (the diagonal that runs from the top left to the bottom right of the matrix).

Definition

For a square matrix A of size $n \times n$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

the trace of A, denoted as $\text{tr}(A)$, is given by:

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}.$$

5 Rank of a Matrix

The **rank** of a matrix is the dimension of the *column space* (or *row space*) of the matrix. Intuitively, it represents the maximum number of linearly independent columns (or rows) in the matrix.
For a matrix A:

- The *column space* of A consists of all possible linear combinations of its columns.
- The rank tells us the number of independent directions in the space spanned by the columns.

Properties of Rank

1. **Range**: The rank of an $m \times n$ matrix A is at most $\min(m, n)$.
2. **Full Rank**:
   - If the rank of A is equal to n (for an $m \times n$ matrix), then A is said to have *full column rank*, meaning all columns are linearly independent.
   - Note: the dimensions of a matrix are always give rows by columns, which is why we know that n is referring to the number of columns.
   - If the rank of A is equal to m, then A has *full row rank*.

The rank is fundamental in linear algebra, especially for solving linear systems, determining invertibility, and performing dimensionality reduction in applications such as data science.

6 Linear regression

Consider the equation estimating equation

$$y = a + bx + cz + \epsilon \tag{5}$$

We can rewrite this as

$$\underset{N\times 1}{Y} = \underset{N\times K}{X}\underset{K\times 1}{\beta} + \underset{N\times 1}{\epsilon} \tag{6}$$

$$\tag{7}$$

where

$$\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{8}$$

$$X = \begin{bmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ \vdots & \dots & \end{bmatrix} \tag{9}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix} \tag{10}$$

Goal: Solve for β by minimizing the sum of square errors.

Solve for the error term:

$$\underset{N\times 1}{Y} - \underset{N\times K}{X\beta} = \underset{N\times 1}{\epsilon} \tag{11}$$

Square the error term:

$$\underset{1\times N}{\epsilon}^T \underset{N\times 1}{\epsilon} = \underset{1\times N}{(Y - X\beta)^T} \underset{N\times 1}{(Y - X\beta)} \tag{12}$$

$$= \underset{1\times N}{Y}^T \underset{N\times 1}{Y} - \underset{1\times K}{\beta}^T \underset{K\times N}{X}^T \underset{N\times 1}{Y} - \underset{1\times N}{Y}^T \underset{N\times K}{X} \underset{K\times 1}{\beta} + \underset{1\times K}{\beta}^T \underset{K\times N}{X}^T \underset{N\times K}{X} \underset{K\times 1}{\beta} \tag{13}$$

$$= \underset{1\times 1}{Y^T Y} - 2\underset{1\times 1}{\beta^T X^T Y} + \underset{1\times 1}{\beta^T X^T X\beta} \tag{14}$$

The squared error term is a scalar $\underset{(1\times 1)}{\epsilon^T \epsilon}$.

We want to find the β that minimizes the sum of squared errors. To do so, take derivative of squared error wrt to β^T and set equal to zero, then solve for β.

$$\frac{\partial \epsilon^T \epsilon}{\partial \beta} = 0 - 2\underset{K\times 1}{X^T Y} + 2\underset{K\times 1}{X^T X\beta} = 0 \implies \tag{15}$$

$$2X^T Y = 2X^T X\beta \implies \tag{16}$$

$$\beta = (X^T X)^{-1} X^T Y \tag{17}$$

Where $(X^T X)^{-1}$ is an inversion (because we cannot divide matrices).

We have solved for β, which is a vector of all the parameters that we saw in equation **??** (look at equation **??**). Remember, the first element of β is the intercept *because* the first column of $\mathbf{X}$ is all 1s (rather than any variable). The column of 1s is important and forces the first element of β to be the intercept.

7 Life cycle assessment example

Let's consider and input output table (Ag, Transportation, Manufactured)

$$A = \underset{3\times 3}{\begin{bmatrix} & A & T & M \\ A & & & \\ T & & & \\ M & & & \end{bmatrix}} \tag{18}$$

And our final demand (the demand for goods by consumers)

$$d = \underset{3\times 1}{\begin{bmatrix} A \\ T \\ M \end{bmatrix}} \tag{19}$$

Total amount of goods X is,

$$\underset{3\times 1}{X} = \underset{3\times 3}{A}\underset{3\times 1}{X} + \underset{3\times 1}{d} \tag{20}$$

Solve for X by multiplying with the identity matrix

$$\underset{3\times 3}{I}X - AX = d \tag{21}$$

$$(I - A)X = d \tag{22}$$

$$\implies (I - A)^{-1}d = X \tag{23}$$

Total amount of goods is different than final demand because we need input good for the final good. A bunch of intermediate products are required to make a computer.

8 R and Excel

Watch the video! Couple of notes
- to invert a matrix in R you use the solve() command