AMES Class Notes – Week 8, Monday: Linear Algebra

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1 Intro

Consider the system of equations:

$$x = ay + bz + u \tag{1}$$
$$y = cx + dz + w \tag{2}$$
$$z = fx + gy + v \tag{3}$$

You could solve this system of three equations for the three unknowns x, y, z. But it would be tedious! Let's rearrange it so that we only have to solve for one unknown by using linear algebra.

$$1x - ay - bz = u \tag{4}$$
$$-cx + 1y - dz = w \tag{5}$$
$$-fx - gy + 1z = v \tag{6}$$

This can be rewritten as vectors and matrices.

$$\begin{bmatrix} u \\ w \\ v \end{bmatrix} = \begin{bmatrix} 1 & -a & -b \\ -c & 1 & -d \\ -f & -g & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{7}$$

Which we can rewrite with vectors and matrices as,

$$\vec{N} = \mathbf{M}\vec{X} \tag{8}$$

A reoccurring question: How can you write out a system of equations, and how can you rewrite it as a matrix?

2 Matrices

The first subscript i is the row and the second subscript j is the column.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} \\ \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} \end{bmatrix} \tag{9}$$

This is an $(i \times j)$ matrix. Always keep track of your dimensions.

3 Addition and Subtraction

Consider two (2×2) matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{10}$$

We can add these matrices because they're of the same dimensions.

$$A + C = \begin{bmatrix} a_{11} + c_{11} & a_{12} + c_{12} \\ a_{21} + c_{21} & a_{22} + c_{22} \end{bmatrix} \tag{11}$$

Subtraction works the same way.

Notes: Addition and subtraction are done element by element. You add all the elements in the [1,1] position, then all the elements in the [1, 2] position, so on and so forth. Notice that to add two matrices together, they must have the same dimensions.

4 Scalar Multiplication

You could "scale" a whole matrix with a scalar. Consider the 2×2 matrix, A.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{12}$$

You could scale the matrix A by 10,

$$10 \times A = \begin{bmatrix} 10a_{11} & 10a_{12} \\ 10a_{21} & 10a_{22} \end{bmatrix} \tag{13}$$

5 Matrix Multiplication

If we have matrix B that's $(m \times n)$ and matrix D that's $(n \times p)$ then the matrix $B \times M$ is going to be $(m \times p)$. **Order matters** when multiplying matrices. $M \times B$ is NOT conformable, you cannot compute that matrix!!

Consider the matrices $A(2 \times 2)$ and $C(2 \times 2)$ again

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{14}$$

Now, $A \times C$. Matrix multiplication requires row multiplied by columns

$$A \times B = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} \tag{15}$$

Consider a vector

$$x = \begin{bmatrix} M \\ N \end{bmatrix} \tag{16}$$

Pre-multiply x by A.

$$\underset{(2\times2)}{A}\underset{(2\times1)}{x} = \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} a_{11}M + a_{12}N \\ a_{21}M + a_{22}N \end{bmatrix} \tag{18}$$

There is no equivalent to division in matrix algebra. However, you can invert a matrix, which we will get to.

6 Transpose

When you transpose a matrix, the first row becomes the first column.

$$A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \tag{19}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ 7 & 3 \end{bmatrix} \tag{20}$$

Transpose is important if you're interested in a matrix multiplied by itself!

$$A^2 = A^T A \tag{21}$$

To square, you always pre-multiply the original by the transpose. This keeps the matrices conformable.

$$\underset{(7\times 3)}{B^2} = \underset{(3\times 7)}{B^T}\underset{(7\times 3)}{B} = \underset{(3\times 3)}{C} \tag{22}$$

Transpose is not the inverse!!! Which is a common mistake.

7 Names of elements of a matrix

- Diagonal: the elements all the diagonal (top left corner to the lower right)
- Lower Triangular matrix: elements under the diagonal
- Upper triangular matrix: elements above the diagonal

8 Identify Matrix

The identity matrix have similar properties to 1.

$$I = \begin{bmatrix} 1 & 0 & ... & \\ 0 & 1 & ... & \\ ... & & & \\ 0 & 0 & ... & 1 \end{bmatrix} \tag{23}$$

There are 1s along the diagonal. An important characteristic of the identity matrix is that for any A

$$\underset{(2\times 2)}{A}{}^{-1}\underset{(2\times 2)}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

Any matrix pre-multiplied by it's inverse will be an identity matrix.

9 Rank

Rank is the largest unique square matrix you can have inside a matrix.

The key here is knowing if a row is unique. For instance

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \tag{25}$$

Only has 1 unique row because the first row is the second row multiplied by 2.