Project 3 - Stability of stratified shear flow

I. BACKGROUND

Although most natural flows are time-varying, and have complicated spatial structure, important insights can be gained by examining the stability of simple flows. For example, many identifiable features in the atmosphere and ocean, such as eddies and billow clouds (shown), are generated and derive their properties from fluid instabilities.

FIG. 1: Kelvin-Helmholtz billows developing in a cloud layer over Mount Shasta. Photo ©1999 Beverly Shannon.

II. INTRODUCTION

Here, we will examine the basic stability properties of a stratified shear flow, and will then use Diablo to examine the nonlinear evolution of the unstable state.

Start by considering a stratified shear flow of the form:

$$\mathbf{U} = \frac{S_0}{h} tanh\left(\frac{y - LY/2}{h}\right)\hat{\imath}, \tag{1}$$

$$B = \frac{N_0^2}{h} tanh\left(\frac{y - LY/2}{h}\right), \tag{2}$$

where $B = -g\rho/\rho_0$ is the buoyancy, h is the height of the shear layer, and S_0 and N_0 are the shear and buoyancy frequency at the center of the shear layer. Pick some parameters that permit shear instability by the Miles-Howard theorem (with $Ri = N^2/S^2 < 1/4$ somewhere in the flow). For example, $LY = 1$, $h = 1/10$, $S_0 = 10$, and $N_0 = 10$ work well.

III. LINEAR STABILITY ANALYSIS

Consider the stability of small perturbations the base state defined above: $\mathbf{u} = \mathbf{U} + \epsilon\mathbf{u}'$, $b = B + \epsilon b'$. We can then look for normal mode solutions to the linearised equations of the form

$$v' = \mathrm{Re}\left[\hat{v}(y)\exp(\sigma t + \imath(kx + lz))\right]. \tag{3}$$

The MATLAB script, *linstab.m*, which you can find inside the folder *project1*, solves the viscous linear stability problem for stratified shear flow, returning the vertical velocity and buoyancy eigenfunctions, $\hat{v}(y)$ and $\hat{b}(y)$, and the corresponding growth rates, σ. Specifically, the code solves the following equations for 2D perturbations ($l = 0$):

$$\sigma(\hat{v}_{yy} - k^2\hat{v}) = -\imath kU(y)\hat{v}_{yy} + \imath kU_{yy}\hat{v} + \nu(d_y^2 - k^2)^2\hat{v} - k^2\hat{b}, \tag{4}$$

$$\sigma\hat{b} = -B_y\hat{v} - \imath kU(y)\hat{b} + \kappa(d_y^2 - k^2)\hat{b}. \tag{5}$$

At the start of *linstab.m*, you can specify several parameters including the vertical domain size and number of gridpoints, h, and ν and κ. Optionally, edit these parameters to match your choice of parameters (the suggested ones listed above are in place now.)

Run the script *linstab.m* from within MATLAB, which will then plot the growth rate of the most unstable mode as a function of the horizontal wavenumber. How does the most unstable mode vary with the shear layer width, and the viscosity?

IV. NONLINEAR SIMULATIONS

Once you are happy with a choice of parameters, try to recreate the instability using Diablo. We will use periodic boundary conditions at both ends of the x-domain for all variables, and place free-slip, no normal flow walls at the top and bottom in the y-direction with gravity pointing down (in the negative y-direction). If we want to capture the most unstable mode, we need it to fit within our domain. Based on the results of your stability calculation from *linstab.m*, set the size of the domain in the x-direction to be some even multiple (say 3) of the wavelength of the most unstable mode.

Edit the following Diablo setup scripts:

set_params.m - A Reynolds number of 500 works reasonably well here. Match LY, the domain size in the y-direction with what you used in *linstab.m.* Start with $NY = 100$, and set NX so that the grid is close to isotropic $\Delta x \simeq \Delta y$ (within a factor of two should be fine).

set_bcs.m - Set up periodic boundary conditions at both ends of the $x - domain$. In the y-direction, let U2=0 at the edges (no normal flow), and let dU1/dy=0, dP/dy=0, and dTH/dy=0.

create_grid.m - Since most of the action will happen near the center of the domain, it is useful to cluster the grid in the y-direction to put more points there. Change GRID_TYPE_Y=2 to do this, and set the stretching factor to 1.0 (higher numbers will lead to more stretching.).

create_flow.m - Here, in the User input area, create an initial flow with a horizontal velocity (U1) and buoyancy (TH(:,:,1)) matching the profiles that you used in *linstab.m*, and uniform in the x-direction. Keep the random perturbation added to the velocity field (you can play around with this later).

Now, run Diablo, and see what happens. Do Kelvin-Helmholtz billows develop with a wavelength close to what you predicted? If not, what might be happening? (Hint: you might need to change the random perturbation added to the velocity field in *create_flow.m*)

Plot the *rms* and mean velocity and buoyancy as a function of y and t. Can you identify an exponential growth phase at the beginning of the simulation? Calculate and plot the gradient Richardson number: $Ri_g = \frac{\partial b/\partial y}{\partial u/\partial y}$.

V. SUGGESTED FURTHER INVESTIGATIONS

A. Mixing efficiency

In a stratified flows, kinetic energy can be converted to potential energy by mixing the stable density profile, raising the center of mass of the fluid. Some kinetic energy is also lost to viscous dissipation. The mixing efficiency, Λ,of great interest in the stratified turbulence literature, is the ratio of the kinetic energy used to mix the density profile to the loss to viscous dissipation. Set up and run a simulation using Diablo, and let the flow evolve long enough so that it settles back into a non-turbulent state (you may need to decrease the resolution for this, and you might also want to increase the size of the domain in y to minimise boundary effects). Calculate the kinetic and potential energy at the start and end of the simulation, and the change over the simulation, ΔKE and ΔPE. Then, calculate the *flux coefficient* $\Gamma \equiv B/\epsilon \simeq \Delta PE/(\Delta KE - \Delta PE)$, where B is the buoyancy flux, and ϵ is the kinetic energy dissipation. Then, use the flux coefficient to estimate the mixing efficiency, η, using the relation $\Gamma = \eta/(1-\eta)$. Many parameterizations for mixing in the ocean and atmosphere use a constant mixing efficiency with a value close to $\eta \simeq 0.2$. How does your result compare? *Advanced*: By saving the time series of the kinetic and potential energy and diagnosing the energy lost to viscous dissipation and the energy used for mixing, can you estimate the *instantaneous* mixing efficiency as a function of time?

B. Holmboe instability

When the profiles of shear and stratification are not identical, a second type of instability called 'Holmboe instability' can develop. Specifically, this instability develops when the width of the shear layer is larger than the width of the stratified layer. Holboe instability is characterized by disturbances that propagate relative to the mean flow, while the billows associated with Kelvin-Helmholtz instability remain nearly stationary. Repeat the process used to analyze K-H instability (starting from the linear stability analysis), but use a buoyancy profile where the width of the tanh is a factor of four smaller than for the velocity. Choose the stratification and shear profiles so that they are stable to K-H instability at the center of the interface ($Ri_g(y = LY/2) > 1/4$). You may need to increase the Reynolds number or Prandtl number in this simulation to prevent the density interface from smearing out too broadly before the simulation begins.

C. Stability in braid region

Run another simulation, and allow the billows to develop until the 'braid' region between the billows becomes thin. Stop the simulation (press Control-c in the command window). Extract the velocity and buoyancy profiles at a single point in x, corresponding to the braid region. Use these profiles as input in *linstab.m* (you will need to remove the definition of *vel* and *buoy*). How does the scale of the most unstable mode compare with the original instability? Which missing factors might modify the secondary instability, or prevent it from forming?

VI. REFERENCES

Several papers that might be helpful: Peltier and Caulfield[1] and Smyth and Moum[2] contain nice discussions of mixing in stratified shear flows. The appendix in Smyth et al.[3] describes the setup of the linear stability analysis in more detail. Smyth and Winters[4], and the references therein, describe Holmboe instability.

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[1] W. Peltier and C. Caulfield, Annual review of fluid mechanics **35**, 135 (2003).

[2] W. Smyth and J. Moum, Oceanography **25**, 140 (2012).

[3] W. Smyth, J. Moum, and J. Nash, Journal of Physical Oceanography **41**, 412 (2011).

[4] W. Smyth and K. Winters, Journal of physical oceanography **33**, 694 (2003).