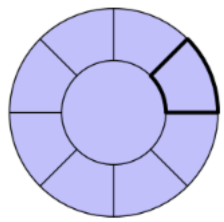
3 Proof

1. **[7]** For his friend's 10^{th} birthday, Wesley bakes a perfectly circular cake measuring 10 inches in diameter. He puts 10 candles in the cake. Prove that there exist 2 of these candles which are at most 4 inches apart.

   Alex Yang

   Solution. Cut the cake as follows, where the middle circle has diameter 4:

   By the Pigeonhole Principle, 2 of the candles must fall in the same slice. It suffices to show that the maximum distance between two points within each of these slices is ≤ 4. The result is trivial for the middle circle; and by Law of Cosines, the maximum distance within an outer slice is either

   $$\sqrt{5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cdot \cos 45^\circ} = \sqrt{50 - 25\sqrt{2}}$$

   or

   $$\sqrt{2^2 + 5^2 - 2 \cdot 2 \cdot 5 \cdot \cos 45^\circ} = \sqrt{29 - 10\sqrt{2}}.$$

   Since $\sqrt{2} \geq \frac{50-4^2}{25}, \frac{29-4^2}{10}$, both expressions are at most 4. $\square$

2. **[7]** Solve $2^a 3^b + 1 = n^2$ over the positive integers.

   Kaiyuan Mao

   Solution. Write $(n-1)(n+1) = 2^a 3^b$. Since $\gcd(n-1, n+1) = 2$, we have $(n-1, n+1) = (2^{a-1}, 2 \cdot 3^b)$ in arbitrary order; or in other words, $(2^{a-2}, 3^b)$ are consecutive. By Mihăilescu's theorem, only the following are possible:

   - $(2, 3) \implies (a, b, n) = (3, 1, 5)$,
   - $(3, 4) \implies (a, b, n) = (4, 1, 7)$,
   - $(8, 9) \implies (a, b, n) = (5, 2, 17)$. $\square$

3. **[7]** In equilateral triangle ABC, let D be a point on ray BA past A, and E a point on ray BC past C. Given that $CD = DE$, prove that $AD = BE$.

   Ryan Tang

Solution. Construct F on ray BC past C so that triangle ADF is equilateral. By symmetry across the perpendicular bisector of BF, $BC = EF$. Hence, if s, t are the side lengths of ABC and ADF, $AD = BE = t - s$, as desired. $\square$

4. **[7]** Determine, with proof, the largest constant c such that the following inequality holds for all distinct positive integers x, y, z:

$$\frac{x+y+z}{3} \geq \sqrt[3]{xyz+c}.$$

Kaiyuan Mao

Solution 1 ("Black Magic"). The answer is $c = 2$, tight at $(x, y, z) = (3, 2, 1)$ and permutations. WLOG let $x > y > z$, so that $(x, y, z) \succ (3, 2, 1)$ and $x+y+z \geq 6$. Denote $S = (x+y+z)^3 - 27xyz$ (!), whence

$$\begin{aligned}
S &= (x^3 + y^3 + z^3 + 3xy^2 + 3xz^2 + 3yz^2 + 3yx^2 + 3zx^2 + 3zy^2 + 6xyz) - 27xyz \\
&= (x^3 + y^3 + z^3 - 3xyz) + 3(x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2)) - 18xyz \\
&= \tfrac{1}{2}(x+y+z)((y-z)^2 + (z-x)^2 + (x-y)^2) + 3(x(y-z)^2 + y(z-x)^2 + z(x-y)^2) \\
&\geq \tfrac{1}{2} \cdot 6 \cdot (1^2 + 2^2 + 1^2) + 3 \cdot (3 \cdot 1^2 + 2 \cdot 2^2 + 1 \cdot 1^2) \\
&= \tfrac{1}{2} \cdot 6 \cdot 6 + 3 \cdot 12 \\
&= 54.
\end{aligned}$$

Substituting $S = (x+y+z)^3 - 27xyz$ and rearranging proves $c = 2$. $\square$

Solution 2 (Motivated by AM-GM). Recall that if x, y, z are relaxed from distinct positive integers to any positive reals, the *AM-GM Inequality* implies that $c = 0$ is the desired constant, with equality when $x = y = z$.

To "simulate" the aforementioned equality case, WLOG let $(x, y, z) = (y+a, y, y-b)$ for positive integers a, b. The idea is to take each pair of these three variables bring them as close together as possible, which preserves the LHS sum but increases the RHS product. Doing so brings the three variables as close together as possible under a fixed sum, with $(a, b) = (1, 1), (1, 2), (2, 1)$.

Setting that aside momentarily, substituting $(x, y, z) = (y+a, y, y-b)$ into the original inequality gives

$$\begin{aligned}
&\tfrac{(y+a)+y+(y-b)}{3} \geq \sqrt[3]{(y+a)(y)(y-b)+c} \\
\iff \; &y + \tfrac{a-b}{3} \geq \sqrt[3]{y^3 + (a-b)y^2 - aby + c}.
\end{aligned}$$

Upon cubing and cancelling like terms, we find

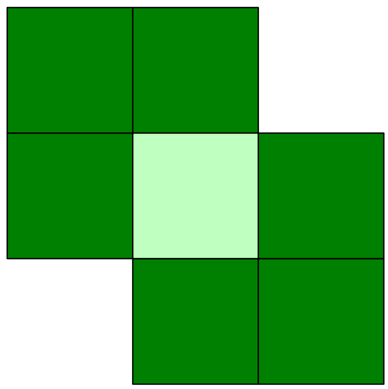
$$c \leq (ab + \tfrac{a-b}{3})y + \tfrac{(a-b)^3}{27}.$$

Here we take cases on (a, b) as found before:

- If $(a,b) = (1,1)$, the inequality becomes $c \le y$; taking $y = 2$ gives $c \le 2$ here. The corresponding equality case is $(x,y,z) = (3,2,1)$.
- If $(a,b) = (1,2),(2,1)$, the inequality becomes $c \le \frac{6\pm1}{3}y \pm \frac{1}{27}$; taking both minus signs and $y = 2$ gives $c \le \frac{89}{27}$, suboptimal.

In all, $c = 2$ is optimal at $(x,y,z) = (3,2,1)$. $\square$

5. **[7]** Kyle is coloring squares on an empty infinite grid. For each square he colors, his "score" is incremented by how many of its 8 neighbors were previously colored. For example, coloring the center square below would increment his score by 6.



Determine his largest possible score after coloring 2024 squares.

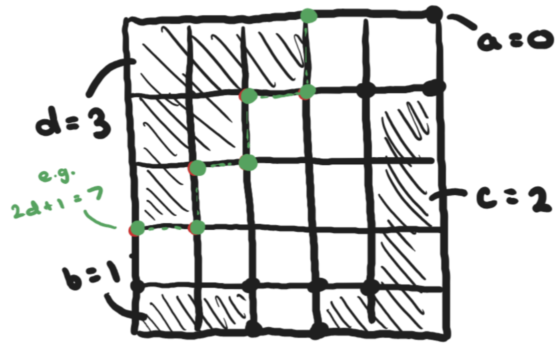
Jason Lee (inspired by Laura Wang)

Solution. The answer is 7857; the construction will only make sense later in the proof (and will therefore be presented then).

First things first, observe that the final score is simply the number of neighbor pairs ("NPs") in the final figure $\mathcal{F}$. We begin by limiting the possibilities for $\mathcal{F}$:

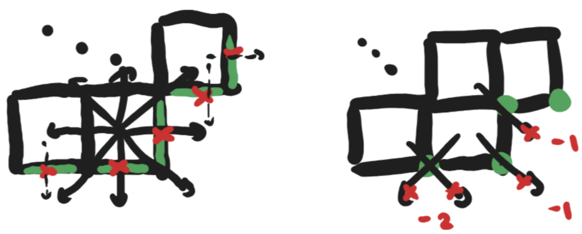
- Clearly, $\mathcal{F}$ should be one contiguous region (i.e. any two squares are connected by a sequence of shared edges); otherwise, take any two disjoint regions and connect them in any way, which increases the NPs.
- Further, $\mathcal{F}$ shouldn't contain any holes; otherwise, take any square with ≤ 4 neighbors (say, the leftmost topmost one) and move it to the hole(s), so that its NPs increase from ≤ 4 to 8.
- Similarly, $\mathcal{F}$ shouldn't contain two adjacent interior angles measuring $270°$ (i.e. a "concave dent"); otherwise, take any square with ≤ 4 neighbors and move it to the dent, so that its NPs increase from ≤ 4 to 5.

In other words, $\mathcal{F}$ is a rectangle ("bounding box") with a "corner triangle" cut out of each corner, and has the same perimeter as the bounding box. We label certain properties that will become useful later on: let the box have dimensions $l \leq w$, and let the corner triangles have $a \leq b \leq c \leq d$ "layers" or "stairs", so that $\mathcal{F}$ has $(2a+1)+(2b+1)+(2c+1)+(2d+1)$ vertices. An example with $l = w = 5$ and $(a,b,c,d) = (0,1,2,3)$ is shown below.



Next, we determine an explicit formula for the final score, the number of NPs. Observe that the number of NPs is half the sum of each square's number of neighbors, across all 2024 squares. Said sum is $8 \cdot 2024$, barring a few overcounts:

- Each of the p edges on the perimeter of $\mathcal{F}$ corresponds to an overcounted neighbor, as shown at left below, and so we should subtract p.



- Most of the p points on the perimeter of $\mathcal{F}$ corresponds to two overcounts. However, each of the v vertices gives only one overcount, as shown at right above, meaning we should subtract $2p - v$.

Subtracting these overcounts and having produces a final score of $\frac{8 \cdot 2024 - p - (2p-v)}{2} = 4 \cdot 2024 - \frac{3p-v}{2}$. Hence, we have reduced the problem to the following:

$$\text{Minimize } S := 3p - v = 6(l+w) - 2(a+b+c+d) - 4.$$

In other words, we wish to minimize $l + w$ and maximize $a + b + c + d$. The constraint on S is the area:

$$\begin{aligned} 2024 &= lw - [\Delta_a] - [\Delta_b] - [\Delta_c] - [\Delta_d] \\ &\leq lw - \tfrac{a(a+1)}{2} - \tfrac{b(b+1)}{2} - \tfrac{c(c+1)}{2} - \tfrac{d(d+1)}{2} \end{aligned}$$

or by rearrangement

$$\tfrac{a(a+1)}{2} + \tfrac{b(b+1)}{2} + \tfrac{c(c+1)}{2} + \tfrac{d(d+1)}{2} \leq lw - 2024, \tag{$*$}$$

with equality when $[\Delta_i] = 1 + 2 + \cdots + i$ for $i = a, b, c, d$, i.e., each triangle has "stairs" of length 1. Intuit and prove the following:

- Under a fixed $l + w$: one should set l, w as close together as possible to maximize $lw - 2024$, which in turn increases $a + b + c + d$ and so decreases S. Rigorously, if $\lfloor \frac{w-l}{2} \rfloor = d \geq 1$, replacing $(l, w) \mapsto (l + d, w - d)$ preserves $l + w$ and allows *at least as large* $a + b + c + d$ since

  $$\Sigma \tfrac{i(i+1)}{2} \overset{(*)}{\leq} lw - 2024 \leq (l + d)(w - d) - 2024.$$

  In other words, an (l, w) of the form (n, n) or $(n, n+1)$ gives *at most as large* S over all pairs with the same sum.
- Under a fixed $lw-2024$ $(*)$: one should set a, b, c, d close together to maximize $a+b+c+d$ (e.g. $f(1, 1, 1, 10) = 58 \approx 60 = f(5, 5, 5, 5)$ but $1+1+1+10 \ll 5 + 5 + 5 + 5$). Rigorously, if any two of a, b, c, d are ≥ 2 apart (say a, d), replacing $(a, d) \mapsto (a+1, d-1)$ again allows at least as large $a+b+c+d$ as

  $$\tfrac{(a+1)(a+2)}{2} + \cdots + \tfrac{(d-1)d}{2} \leq \tfrac{a(a+1)}{2} + \cdots + \tfrac{d(d+1)}{2} \leq lw - 2024.$$

  One may repeat the replacement until all variables are equal or differ by 1. In other words, an (a, b, c, d) with range $d - a \leq 1$ gives at most as large S over all quadruplets with the same upper bound $lw - 2024$.

With these in mind, we proceed by an inductive argument on (l, w). Begin with the base case $(l, w) = (45, 45)$, which gives a maximal (a, b, c, d) of $(0, 0, 0, 1)$:

$$\tfrac{0\cdot1}{2} + \tfrac{0\cdot1}{2} + \tfrac{0\cdot1}{2} + \tfrac{1\cdot2}{2} \leq 45 \cdot 45 - 2024.$$

Stepping up to $(45, 46)$ increments S by $+6$ (from the $6(l + w)$ term), but increments the RHS above as well, by $+k = 45$ specifically. Hence, as long as the decrement from the $-2(a + b + c + d)$ term is at least -8, i.e., we can increment all four variables without overstepping the RHS's $+k$, the step-up is favorable. Of course, this early on, the decrement is an overwhelming $-2 \cdot 16$:

$$\tfrac{4\cdot5}{2} + \tfrac{4\cdot5}{2} + \tfrac{4\cdot5}{2} + \tfrac{5\cdot6}{2} \leq 45 \cdot 46 - 2024.$$

On the other extreme, the step-up $(60, 60) \mapsto (60, 61)$ is *un*favorable, since only two variables can be incremented without overstepping the RHS's $+60$, giving a net effect of $+6 - 2 \cdot 2 = +2$ on S (bad).

$$\begin{aligned} \tfrac{27\cdot28}{2} + \tfrac{27\cdot28}{2} + \tfrac{28\cdot29}{2} + \tfrac{28\cdot29}{2} &\leq 60 \cdot 60 - 2024 \\ \tfrac{28\cdot29}{2} + \tfrac{28\cdot29}{2} + \tfrac{28\cdot29}{2} + \tfrac{28\cdot29}{2} &\leq 60 \cdot 61 - 2024. \end{aligned}$$

It seems that from $(45, 45)$ to $(60, 60)$, the step-ups go from favorable, to no-effect, to unfavorable—in fact, *monotonically* so, i.e., no step-up can allow more incremented variables than the previous step-up. The proof is left as an exercise to the reader. (Seriously, it's actually easy—intuitively, the increment necessary for each LHS term steps up, but the RHS increment only steps up by 1? No shot.)

Hence, the minimal S occurs after all the favorable step-ups and before all the unfavorable step-ups; that is, during the no-effect step-ups, in the "critical strip". It suffices to find any point in the critical strip and compute S there. To do so, we approximate the following step-up:

$$\frac{(n-1)n}{2} + \frac{(n-1)n}{2} + \frac{(n-1)n}{2} + \frac{(n-1)n}{2} \leq 3n \cdot 3n - 2024$$
$$\frac{(n-1)n}{2} + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} + \frac{n(n+1)}{2} \leq 3n \cdot (3n+1) - 2024.$$

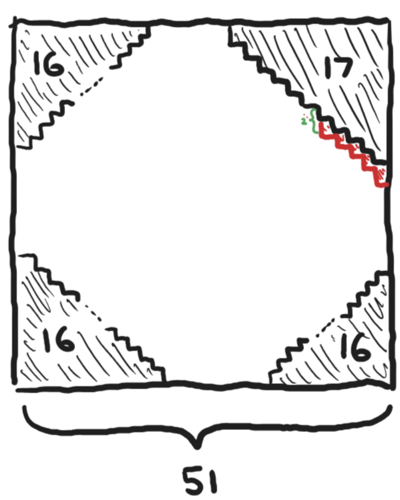
Above, both the LHS and RHS increment by exactly $+3n$, and exactly 3 variables incremented, hence an approximated no-effect step-up. Setting equality and solving gives $n \approx 16.9$, and testing $(l, w) = (3 \cdot 16.9, 3 \cdot 16.9) \approx (51, 51)$ produces

$$\frac{16 \cdot 17}{2} + \frac{16 \cdot 17}{2} + \frac{16 \cdot 17}{2} + \frac{17 \cdot 18}{2} \leq 51 \cdot 51 - 2024$$
$$\frac{17 \cdot 18}{2} + \frac{17 \cdot 18}{2} + \frac{17 \cdot 18}{2} + \frac{17 \cdot 18}{2} \leq 51 \cdot 52 - 2024,$$

which is indeed a no-effect step-up in the critical strip. Then $(l, w) = (51, 51)$ and $(a, b, c, d) = (16, 16, 16, 17)$ yields the following minimal S:

$$S = 6(51 + 51) - 2(16 + 16 + 16 + 17) - 4 = 478.$$

Finally, the maximum score is $4 \cdot 2024 - \frac{S \geq 478}{2} = 7857$, with one of the many circle-like equality cases in the critical strip shown. Remark that a 51×51 square with perfect $(16, 16, 16, 17)$ triangles cut out actually has an area of 2040, but one can arbitrarily remove squares (red below) from the triangles, making some "stairs" have length 2 rather than the optimal 1.



4 Grading Rubric

1. Hit-or-miss :P

2.
   - **0 pts** for guessing the answer
   - **1 pt** for factoring the difference-of-squares
   - **3 pts** for gcd in second sentence or similar
   - **-2 pt** for missing one or more solutions
   - **-0 pts** for not citing Mihăilescu's or similar

3.
   - **1 pt** for a "meaningful attempt" to apply Pigeonole
   - **3 pts** for cutting the cake as in the solution
   - **-1 pt** for checking only one length inside the slice

4.
   - **0 pts** for guessing the answer
   - **3 pts** for smoothing until $a + b \leq 3$
   - **-1 pt** for omitting equality case $(3, 2, 1)$

5.
   - **0 pts** for observing that score equals NPs
   - **+1 pts** for "limiting possibilities for $\mathcal{F}$"
   - **+1 pt** for expressing score in terms of p, v OR **+2 pts** for reformulating in terms of a, b, c, d
   - **5 pts** for "guessing" the answer
   - **-$\lceil \frac{\mathbf{k}}{\mathbf{2}} \rceil$ pts** for not addressing k of the following:
     - that $\mathcal{F}$ can't have concave dents, holes, etc.;
     - that l, w and a, b, c, d should be as close together as possible;
     - that step-up favorability is monotonically decreasing; and
     - that the construction deviates slightly from the bound.