2 Short-Answer

1. **[1]** What is the least integer greater than $0.1 + 0.2 + \cdots + 0.99 + 0.100$?

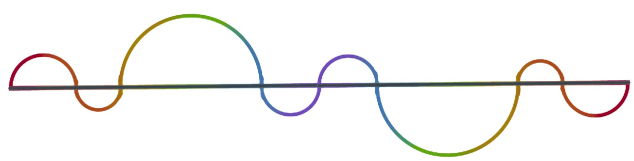
   Ryan Tang

   Solution. We take cases on the number of digits:

   - There are 9 "one-digit" terms $0.1, \ldots, 0.9$ whose average is $\frac{0.1+0.9}{2} = 0.5$, so their sum is $9 \cdot 0.5 = 4.5$.
   - There are 90 "two-digit" terms $0.10, \ldots, 0.99$ whose average is $\frac{0.10+0.99}{2} = 0.545$, so their sum is $90 \cdot 0.545 = 49.05$.
   - There is only 1 "three-digit" term, $0.100 = 0.1$.

   The final answer is $0.45 + 49.05 + 0.1 = 53.65 \implies \boxed{54}$.

2. **[1]** Shown below is a snake of length $n\pi$ that has coiled its body into semicircular arcs on a segment of length 2024. What is n?

   Jason Lee

   Solution. Recall that the length of a semicircular arc is π times half the length of the diameter. Hence, the sum of all the semicircular lengths is π times half the sum of all the diameters, which is given to be 2024. The answer is $\pi \cdot \frac{1}{2} \cdot 2024 = \boxed{1012}\pi$.

3. **[1]** Jason is coping with his subpar artistic capabilities by drawing a large plus sign. On day 1, he draws a plus sign with 5 squares, with 4 squares surrounding 1. Each day after the first, he lengthens each of the 4 "legs" by 1 square. At the end of the 10^{th} day, what is the perimeter of the whole figure?

   Alex Yang

   Solution. "Inflate" (i.e. project outward) the plus sign into a square of side length 21 and perimeter $4 \cdot 21 = \boxed{84}$.

4. **[2]** Five donors are fundraising for Laura's new phone. Laura receives a dollar from the first donor every 2 days, a dollar from the second donor every 3 days, and so on up to a dollar from the fifth donor every 6 days. On rare "lucky" days, Laura receives all five dollars. On how many days between two lucky days does she receive no money?

Ryan Chen

Solution 1 (Euler's Totient Function). Since $\text{lcm}(2, 3, \ldots, 6) = 2^2 \cdot 3 \cdot 5$, Euler's Totient Function gives $\phi(2^2 \cdot 3 \cdot 5) = 2 \cdot 2 \cdot 4 = \boxed{16}$.

Solution 2 (Inclusion-Exclusion). This problem is essentially asking for the number of positive integers less than $\text{lcm}(2, 3, \ldots, 6) = 60$ that are relatively prime to it. The prime factors of 60 are $2, 3, 5$, so applying the Inclusion-Exclusion Principle gives

$$60 - \frac{60}{2} - \frac{60}{3} - \frac{60}{5} + \frac{60}{2 \cdot 3} + \frac{60}{2 \cdot 5} + \frac{60}{3 \cdot 5} - \frac{60}{2 \cdot 3 \cdot 5} = 16.$$

5. **[2]** Ryan has 5 red, 5 green, and 5 blue balls, as well as a red, a green, and a blue box. In how many ways can he place 5 balls in each box so that the majority of the balls in each box matches the box's color?

   Jason Lee (inspired by Bruce Shu)

   Solution 1. For there to be a majority of the balls in a certain box match the box's color, there must be at least 3 balls of the box's color in that box. We can begin by having Ryan place 3 balls of each color into their respective box with the same color, which leaves 2 balls of each color. This guarantees that the majority condition is satisfied. Now, it suffices to find the number of ways to place 2 balls of each color into 3 different boxes such that each box contains 2 balls. Begin with casework.

   - The two red balls are placed into the same box. There are 3 ways to choose the box to place both the red balls. Then, there are 3 ways to place the green and blue balls, so there are $3 \cdot 3 = 9$ ways for this case.
   - The two red balls are placed into different boxes. There are 3 ways to choose two boxes to place the red balls. Then, enumerating, we find that there are 4 ways to place the remaining green and blue balls. This means that there are $3 \cdot 4 = 12$ ways in this case.

   Summing the cases yields $9 + 12 = \boxed{21}$.

   Solution 2. As in Solution 1, reduce the problem to evenly distributing 2 balls of each color amongst the 3 boxes. Proceed with casework on the number of matching pairs which end up in the same box:

   - *Case 1:* 0 pairs. The boxes are RG, GB, and BR in one of $3! = 6$ orders.
   - *Case 2:* 1 pair. There are 3 ways to choose the matching pair and 3 ways to choose which box the pair goes in; the rest is determined, giving $3^2 = 9$.
   - *Case 3:* 3 pairs. The boxes are RR, GG, and BB in one of $3! = 6$ orders.

The final answer is $6 + 9 + 6 = 21$ as before.

6. **[2]** For each positive integer n, let $S(n)$ denote the increasing sequence of integers that can be expressed as the sum of distinct perfect powers of n. Further, let $R(n)$ denote the remainder when the n^{th} term of $S(n)$ is divided by n^2. What is $R(1) + R(2) + ... + R(100)$?

   Kaiyuan Mao

   Solution. Notice that there is a bijection between numbers in the sequence $S(x)$ and numbers written in binary. When we take the xth number in $S(x)$ mod x^2, the resulting value will be 0, 1, x, or $x+1$, depending on the remainder when x is divided by 4. Therefore, from x to $x+3$, the sum is $1+(x+1)+(x+2+1)+0 = 2x+5$, where x has a remainder of 1 when divided by 4. Clearly, this applies for all $x \equiv 1 \pmod 4$ less than 100, so the answer is $2(1+4+...+97)+5\cdot 25 = \boxed{2575}$.

7. **[2]** Let a, b, and c be the roots of the cubic $x^3 - 6x^2 + 10x - 3$. In an alternate universe governed by Mathematician Kaiyuan Mao, a nuclear explosion occurs with probability $\frac{a}{a+b+c}$, a tornado with probability $\frac{b}{a+b+c}$, and a volcanic eruption with probability $\frac{c}{a+b+c}$. If these events occur independently, the probability that none of them occur can be expressed as the common fraction $\frac{m}{n}$. What is $m+n$?

   Alex Yang & Ryan Tang

   Solution. Let $f(x)$ denote the cubic. Since $a+b+c = 6$ by Vieta's Formulas, the *complementary* probability that a nuclear explosion *doesn't* occur is $1 - \frac{a}{a+b+c} = 1 - \frac{a}{6} = \frac{6-a}{6}$. Similarly, the other two complementary probabilities are $\frac{6-b}{6}$ and $\frac{6-c}{6}$, which makes the answer

   $$\frac{6-a}{6} \cdot \frac{6-b}{6} \cdot \frac{6-c}{6} = \frac{f(6)}{6^3} = \frac{57}{216} = \frac{19}{72} \implies \boxed{91}.$$

8. **[2]** For how many 4-digit positive integers do any 2 adjacent digits differ by 1? For example, the integer 4345 has this property, but 9128 does not.

   Bruce Shu

   Solution. Let the 4-digit integer be $\overline{abcd}$. Take cases the first digit, a:

   - $a = 3, 4, 5, 6$. There are $2 \cdot 2 \cdot 2 = 8$ possible integers for each a because we can choose the current digit -1 or $+1$ for the next digit without worrying about going over 9 or under 0. Thus there are $4 \cdot 8 = 32$ integers here.
   - $a = 2, 7$. For $a = 2$, consider $b = 1$. Then, there are 3 cases by enumeration. If $b = 3$, there are 4 cases by enumeration. Thus there are $3 + 4 = 7$ cases for $a = 2$, and by symmetry, there are $7 \cdot 2 = 14$ cases.
   - $a = 1, 8$. Enumerating, there are 6 integers for $a = 1$ and by symmetry there are $6 \cdot 2 = 12$ cases.

- $a = 9$. Enumerating, there are 3 integers.

Summing all cases, the answer is $32 + 14 + 12 + 3 = \boxed{61}$.

9. **[2]** For how many positive integers $n \neq 2024$ does $2024^2 - n^2$ divide $2024^3 - n^3$?

   Bruce Shu & Jason Lee

   Solution. Since $\frac{2024^3-n^3}{2024^2-n^2} = \frac{2024^2+2024n+n^2}{2024+n} = n + \frac{2024^2}{2024+n}$, it is necessary and sufficient for $(2024 + n)$ to evenly divide 2024^2. That is, we seek the number of factors $2024 + n$ of $2024^2 = 2^6 11^2 23^2$, that are greater than 2024 since $n > 0$. There are $(6+1)(2+1)(2+1) = 63$ factors of 2024^2, and by symmetry, $\frac{63-1}{2} = 31$ of these exceed 2024. Subtracting one for the factor $2024 + n = 4048$ (which would imply $n = 2024$, impossible) yields $\boxed{30}$.

10. **[3]** A sphere of radius $2 + \sqrt{3}$ is dropped into the bottom of a cylinder with the same base radius. Then, three smaller, congruent spheres are dropped inside such that each is tangent to the other three spheres and to the cylinder's top base and lateral surface. The height of the cylinder can be expressed in the form $a + \sqrt{b} + \sqrt{c + \sqrt{d}}$, where a, b, c, d are positive integers. What is $a + b + c + d$?

    Laura Wang

    Solution. Take the cross section that intersects the centers of all three small spheres, which is a circle of radius $2 + \sqrt{3}$. Using $30 - 60 - 90$ triangles, we find that the radius of the cross section can also be expressed as $\frac{2r\sqrt{3}}{3} + r$, where r is the radius of the small spheres. Solving gives $r = \sqrt{3}$. Apply Pythagoras to find the distance from the center of the large sphere to the cross section, which gives $\sqrt{(2 + 2\sqrt{3})^2 - 2^2} = \sqrt{12 + 8\sqrt{3}}$. It is obvious that the height of the cylinder is equivalent to the sum of the radius of the small spheres, the radius of the large sphere, and the distance from the center of the large sphere to the cross section. We know all of these values, so summing them together, we get $\sqrt{12 + 8\sqrt{3}} + 2\sqrt{3} + 2$ which becomes $\sqrt{12 + \sqrt{192}} + \sqrt{12} + 2$ as specified by the problem statement, so the answer is $12 + 192 + 12 + 2 = \boxed{218}$.

11. **[3]** How many divisors of 1010045120210252210120045010001 end in the digit 1?

    Jason Lee (inspired by Patrick Sun)

    Solution. The given integer spells out the 10th row of Pascal's Triangle, making it equal to $1001^{10} = 7^{10}11^{10}13^{10}$. Clearly, we can find how many divisors $7^a 13^b$ of $7^{10}13^{10}$ end in the digit 1 and multiply by 11 (choices for the exponent of 11). By inspection, ending in 1 is equivalent to having $a \equiv b \pmod 4$, which can be selected in $3^2 + 3^2 + 3^2 + 2^2 = 31$ ways. Finally, $31 \cdot 11 = \boxed{341}$.

12. **[3]** Patrick is throwing darts at an infinite unit grid, earning 1 point for each dart. He has a $\frac{1}{5}$ chance of hitting the target square he aims at, a $\frac{1}{6}$ chance of hitting each square sharing an edge with his target, and a $\frac{1}{30}$ chance of hitting each square sharing only a corner with his target. The game ends once he hits a square that he has hit before, or once he has thrown 3 darts and none hit a previous square, in which case his total score is 4. If Patrick picks targets so that his total score is minimized, the expected value of his score can be expressed as the common fraction $\frac{m}{n}$. What is $m+n$?

    Laura Wang

    Solution. Clearly, Patrick can aim anywhere on his 1[st] throw, and for his 2[nd] throw he should aim where his 1[st] dart landed. Temporarily neglect the scores from these 2 throws and add 2 at the end.

    - With $\frac{1}{5}$ probability, his 2[nd] dart lands on the 1[st], whence his score is 0.
    - With $\frac{4}{6}$ probability, his 2[nd] dart lands adjacent to the 1[st]. Then for his 3[rd], he should target either of the previous squares, giving score 1 with probability $\frac{1}{5}+\frac{1}{6}$ and score 2 with probability $3\cdot\frac{1}{6}+4\cdot\frac{1}{30}$. His expected score is

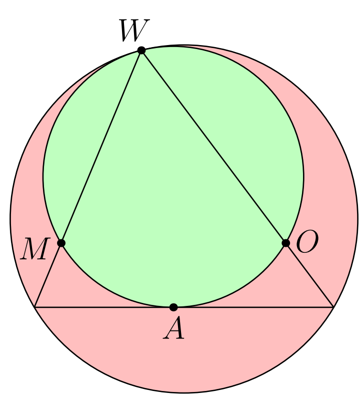
    $$\left(\tfrac{1}{5}+\tfrac{1}{6}\right)\cdot 1+\left(3\cdot\tfrac{1}{6}+4\cdot\tfrac{1}{30}\right)\cdot 2=\tfrac{49}{30}.$$

    - With $\frac{4}{30}$ probability, his 2[nd] dart lands "cornering" 1[st]. Here he should target not either of the previous 2 squares but one neighboring both, since $2\cdot\frac{1}{6}>\frac{1}{5}+\frac{1}{30}$. This gives

    $$\left(2\cdot\tfrac{1}{6}\right)\cdot 1+\left(\tfrac{1}{5}+2\cdot\tfrac{1}{6}+3\cdot\tfrac{1}{30}\right)\cdot 2=\tfrac{5}{3}.$$

    Then the overall expected score is $\frac{1}{5}\cdot 0+\frac{4}{6}\cdot\frac{49}{30}+\frac{4}{30}\cdot\frac{5}{3}=\frac{59}{45}$, and adding 2 gives a final answer of $\frac{149}{45} \implies \boxed{194}$.

13. **[3]** In the figure shown, if $\frac{WM}{WO}=\frac{\text{green area}}{\text{red area}}=\frac{4}{5}$ and $\frac{WA}{MO}=\frac{\sqrt{m}}{n}$ for coprime m,n, what is $m+n$?



Jason Lee

Solution. We begin with a couple definitions: let C be the center of the small circle, and let P be the intersection of WA and MO. Additionally, let $\mathcal{D}$ denote the dilation that sends the small circle to the large one, and let X' denote the image of each point $X \in \{W, A, M, O, C, P\}$ under $\mathcal{D}$. Hence, W, M', O' are the vertices of the triangle, and $P' = A$.

Observe that $\triangle WCA \sim \triangle WC'A'$, and so $C'A' \parallel CA \perp M'O'$, i.e., A' is the midpoint of minor arc $M'O'$. In other words, WA' bisects $\angle M'WO'$. Specifically, it follows by the Angle Bisector Theorem that $\frac{WM}{WO} = \frac{MP}{OP} = \frac{4}{5}$, so that $(MP, OP, MO) = (4a, 5a, 9a)$ for some constant a.

Additionally, the condition $\frac{\text{green area}}{\text{red area}} = \frac{4}{5}$ implies that $\mathcal{D}$ has scale factor $\sqrt{\frac{4}{9}} = \frac{2}{3}$. Hence, $\frac{WA}{WA'} = \frac{WP}{WA} = \frac{2}{3}$, so that $(WP, PA, WA) = (2b, b, 3b)$ for some constant b.

Finally, by Power of a Point on P with respect to the small circle, we have

$$\begin{aligned} WP \cdot PA &= MP \cdot PO \\ 4a \cdot 5a &= 2b \cdot b \\ \tfrac{b}{a} &= \sqrt{10}. \end{aligned}$$

Then $\frac{WA}{MO} = \frac{3b}{9a} = \frac{\sqrt{10}}{3}$, giving $m + n = \boxed{13}$.

14. **[3]** 495000 balls are dropped into a funnel, which channels each ball into one of the bins $a_1, a_2, a_3, \ldots, a_{99}$ with probabilities $p_1, p_2, p_3, \ldots, p_{99}$ respectively. Then, the balls fall again into bins $b_1, b_2, b_3, \ldots, b_{99}$, where the probability $P(i, j)$ that a ball in bin a_i falls into bin b_j is given by the following piece-wise function:

$$P(i,j) = \begin{cases} \frac{101-i-j}{4950} & \text{if } i + j \leq 100, \\ \frac{200-i-j}{4950} & \text{otherwise.} \end{cases}$$

Given that the expected number of balls in bin b_j is $j + 4950$ for all $1 \leq j \leq 99$, it follows that $p_1 + p_3 + p_5 + \cdots + p_{99} = \frac{m}{n}$ for coprime m, n. What is $m + n$?

Ryan Tang

Solution. Begin with considering the expected number of balls in b_1. By directly calculating $f(i, 1)$ for $1 \leq i \leq 99$ and summing and multiplying, the expected number of balls in b_1 can be expressed as

$$495000 \cdot \tfrac{99p_1 + 98p_2 + \ldots + 2p_{98} + p_{99}}{4950} = 4951,$$

which simplifies to

$$99p_1 + 98p_2 + \ldots + 2p_{98} + p_{99} = \tfrac{4951}{100}.$$

Similarly, by considering the expected number of balls in $b_2, \ldots, b_{99}$, we have

$$\begin{aligned} 99p_1 + 98p_2 + \ldots + 2p_{98} + p_{99} &= \tfrac{4951}{100}, \\ 98p_1 + 97p_2 + \ldots + p_{98} + 99p_{99} &= \tfrac{4952}{100}, \\ 97p_1 + 96p_2 + \ldots + 99p_{98} + 98p_{99} &= \tfrac{4953}{100}, \\ &\vdots \\ p_1 + 99p_2 + \ldots + 3p_{98} + 2p_{99} &= \tfrac{5049}{100}. \end{aligned}$$

Additionally, the problem states that each ball must fall into $a_1, a_2, \ldots, a_{99}$, which implies $p_1 + p_2 + \ldots + p_{99} = 1$.

Start with finding p_1. Subtracting the first equation from the last yields

$$-98p_1 + p_2 + p_3 + ... + p_{99} = \tfrac{98}{100},$$

which turns into

$$-99p_1 + p_1 + p_2 + ... + p_{99} = -99p_1 + 1 = \tfrac{98}{100},$$

so $p_1 = \frac{2}{9900}$.

We can also generalize a formula for p_k for $k \geq 2$. Notice that for $k \geq 2$, the equation the $(101-k)^{\text{th}}$ equation contains $99p_k$, and the equation containing p_k is the $(100-k)^{\text{th}}$ equation. When we subtract these equations, we get

$$-99p_k + 1 = -\tfrac{1}{100} \implies p_k = \tfrac{101}{9900}.$$

Thus, $\sum_{k=1}^{50} p_{2k-1} = \frac{2}{9900} + 49 \cdot \frac{101}{9900} = \frac{4951}{9900} \implies \boxed{14851}$.

15. **[5]** Alex, Bruce, and Ryan pass around a hot potato in a circle, starting with Alex. Each person holds the potato for random duration between 2 and 5 seconds before passing it to the next person, and whoever is holding the potato when it explodes loses the game. If the potato explodes after 10 seconds, the probability that Alex loses the game can be expressed as the common fraction $\frac{m}{n}$. What is $m + n$?

Ryan Shin (Guest) & Jason Lee

Solution. Let a_i be the duration that the ith person holds the potato (where $4 \equiv 1$, etc.). Since $a_1 + a_2 < 2 \cdot 5 = 10$ and $a_1 + a_2 + a_3 + a_4 + a_5 > 5 \cdot 2 = 10$, it will explode during a_3, a_4, or a_5.

- The probability of explosion during a_3, i.e., $a_1 + a_2 + a_3 \geq 10$, equals that of $a_1' + a_2' + a_3' \geq 10 - 3 \cdot 2 = 4$, where we shifted $(a_1, a_2, a_3) \in [2,5]^3$ to $(a_1', a_2', a_3') \in [0,3]^3$. When graphed in the $a_1'a_2'a_3'$-plane, it is apparent that the complementary volume $V(a_1' + a_2' + a_3' \leq 4)$ is easier: it is a right tetrahedron of side length 4, minus tetrahedrons of side length 1. Hence, $V(a_1' + a_2' + a_3' \leq 4) = \frac{4^3}{6} - 3 \cdot \frac{1^3}{6} = \frac{61}{6}$; the desired probability is $1 - \frac{61/6}{3^3} = \frac{101}{162}$.

- Analogously, the probability of explosion during a_5 is that of $a_1'+a_2'+a_3'+a_4' \leq 10 - 4 \cdot 2 = 2$, where $(a_1', a_2', a_3', a_4') \in [0,3]^4$. Now, the area of an isosceles right triangle (2D) is $\frac{s^2}{2}$; and the volume of a right tetrahedron (3D) is $\frac{s^3}{6}$; so we expect that the hypervolume of a right hypertetrahedron (4D) is $\frac{s^4}{24}$ (we can prove this using calculus). Thus, we have $\frac{2^4/24}{3^4} = \frac{2}{243}$.

All in all, the probability of explosion during a_4 is the complement of the two probabilities above, $1 - \frac{101}{162} - \frac{2}{243} = \frac{179}{486} \implies \boxed{665}$.