Washington Math Olympiad

Short-Answer Section

May 30, 2024

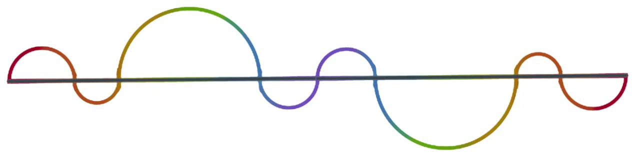
Welcome to the short-answer section of the Washington Math Olympiad (WAMO), a contest dedicated to serving and strengthening the Washington math community.

Instructions:

- You will have **75 minutes** to solve **15 integer-answer problems**.
- The total points for this section sum to **35 points**.
- There is **no penalty for guessing**.
- The only permitted aids are writing utensils, erasers, and blank scratch paper.
- Graph paper, calculators, and any other computing devices are **not** allowed.

1. **[1]** What is the least integer greater than $0.1 + 0.2 + \cdots + 0.99 + 0.100$?

2. **[1]** Shown below is a snake of length $n\pi$ that has coiled its body into semicircular arcs on a segment of length 2024. What is n?



3. **[1]** Jason is coping with his subpar artistic capabilities by drawing a large plus sign. On day 1, he draws a plus sign with 5 squares, with 4 squares surrounding 1. Each day after the first, he lengthens each of the 4 "legs" by 1 square. At the end of the 10^{th} day, what is the perimeter of the whole figure?

4. **[2]** Five donors are fundraising for Laura's new phone. Laura receives a dollar from the first donor every 2 days, a dollor from the second donor every 3 days, and so on up to a dollar from the fifth donor every 6 days. On rare "lucky" days, Laura receives all five dollars. On how many days between two lucky days does she receive no money?

5. **[2]** Ryan has 5 red, 5 green, and 5 blue balls, as well as a red, a green, and a blue box. In how many ways can he place 5 balls in each box so that the majority of the balls in each box matches the box's color?

6. **[2]** For each integer $n > 1$, let $S(n)$ denote the increasing sequence of integers that can be expressed as the sum of distinct perfect powers of n. Further, let $R(n)$ denote the remainder when the n^{th} term of $S(n)$ is divided by n^2. What is the value of $R(2) + R(3) + \cdots + R(100)$?

7. **[2]** Let a, b, and c be the roots of the cubic $x^3 - 6x^2 + 10x - 3$. In an alternate universe governed by Mathematician Kaiyuan Mao, a nuclear explosion occurs with probability $\frac{a}{a+b+c}$, a tornado with probability $\frac{b}{a+b+c}$, and a volcanic eruption with probability $\frac{c}{a+b+c}$. If these events occur independently, the probability that none of them occur can be expressed as the common fraction $\frac{m}{n}$. What is $m+n$?

8. **[2]** For how many 4-digit positive integers do any 2 adjacent digits differ by 1? For example, the integer 4345 has this property, but 9128 does not.

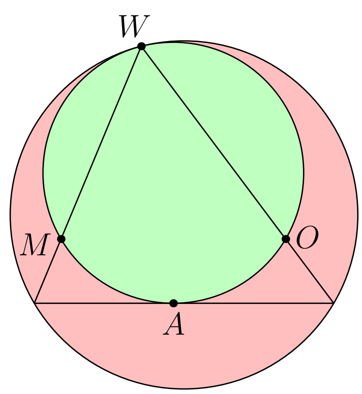
9. **[2]** For how many positive integers $n \neq 2024$ does $2024^2 - n^2$ divide $2024^3 - n^3$?

10. **[3]** A sphere of radius $2 + \sqrt{3}$ is dropped into the bottom of a cylinder with the same base radius. Then, three smaller, congruent spheres are dropped inside such that each is tangent to the other three spheres and to the cylinder's top base and lateral surface. The height of the cylinder can be expressed in the form $a + \sqrt{b} + \sqrt{c + \sqrt{d}}$, where a, b, c, d are positive integers. What is $a + b + c + d$?

11. **[3]** How many divisors of 1010045120210252210120045010001 end in the digit 1?

12. **[3]** Patrick is throwing darts at an infinite unit grid, earning 1 point for each dart. He has a $\frac{1}{5}$ chance of hitting the target square he aims at, a $\frac{1}{6}$ chance of hitting each square sharing an edge with his target, and a $\frac{1}{30}$ chance of hitting each square sharing only a corner with his target. The game ends once he hits a square that he has hit before, or once he has thrown 3 darts and none hit a previous square, in which case his total score is 4. If Patrick picks targets so that his total score is minimized, the expected value of his score can be expressed as the common fraction $\frac{m}{n}$. What is $m+n$?

13. **[3]** In the figure shown, if $\frac{WM}{WO} = \frac{\text{green area}}{\text{red area}} = \frac{4}{5}$ and $\frac{WA}{MO} = \frac{\sqrt{m}}{n}$ for coprime m, n, what is $m+n$?



14. **[3]** 495000 balls are dropped into a funnel, which channels each ball into one of the bins $a_1, a_2, a_3, \ldots, a_{99}$ with probabilities $p_1, p_2, p_3, \ldots, p_{99}$ respectively. Then, the balls fall again into bins $b_1, b_2, b_3, \ldots, b_{99}$, where the probability $P(i,j)$ that a ball in bin a_i falls into bin b_j is given by the following piece-wise function:

$$P(i,j) = \begin{cases} \frac{101-i-j}{4950} & \text{if } i+j \leq 100, \\ \frac{200-i-j}{4950} & \text{otherwise.} \end{cases}$$

Given that the expected number of balls in bin b_j is $j + 4950$ for all $1 \leq j \leq 99$, it follows that $p_1 + p_3 + p_5 + \cdots + p_{99} = \frac{m}{n}$ for coprime m, n. What is $m+n$?

15. **[5]** Alex, Bruce, and Ryan pass around a hot potato in a circle, starting with Alex. Each person holds the potato for random duration between 2 and 5 seconds before passing it to the next person, and whoever is holding the potato when it explodes loses the game. If the potato explodes after 10 seconds, the probability that Alex loses the game can be expressed as the common fraction $\frac{m}{n}$. What is $m+n$?

The 2024 Washington Math Olympiad was made possible by the contributions of the following problem writers and test solvers:

Ryan Chen	(BASIS)
John Clyde	(WARML)
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Bruce Shu	(TX)
Patrick Sun	(Tyee)
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