Birefringence

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1 Why Birefringence

From WMAP Komatsu paper: Since the temperature and E-mode polarization are parity-even and the B-mode polarization is parity-odd, the TB and EB correlations should vanish in a universe that conserves parity. For this reason the TB and EB correlations are usually used to check for systematics, and not widely used as a cosmological probe. Polarization of photons offers a powerful way of probing the cosmological parity violation, or the "cosmological birefringence". Let us consider a parity-violating interaction term in the Lagrangian such as the Chern-Simons term $\mathcal{L}_{CS} = -\frac{1}{2}p_\alpha A_\beta \tilde{F}^{\alpha\beta}$ where $F^{\alpha\beta}$ and A_β are the usual electromagnetic tensor and vector potential, $\tilde{F}^{\alpha\beta}$ is the dual tensor and p_α is an arbitrary timelike vector. This Chern-Simons term makes two polarization states of photons propagate with different group velocities, causing the polarization plane to rotate by an angle α (Carroll 1990). p_α can be sources by a scalar field. Such a field might have something to do with dark energy, for example. We are therefore looking at a potential parity-violating interaction between the visible section (i.e., photons) and dark sector (i.e., dark energy)

2 Effects of Birefringence on the Power Spectra

(from 1904.12440) When polarisation angles are rotated uniformly over the sky by an angle α, spherical harmonics coefficients of the observed E- and B-mode polarisation, denoted by "o", are related to the intrinsic ones by

$$\begin{aligned} a_{\ell m}^{E,o} &= a_{\ell m}^{E}\cos 2\alpha - a_{\ell m}^{B}\sin 2\alpha \\ a_{\ell m}^{B,o} &= a_{\ell m}^{E}\sin 2\alpha + a_{\ell m}^{B}\cos 2\alpha \end{aligned} \tag{1}$$

For the power spectra, we obtained

$$\begin{aligned} C_\ell^{EE,o} &= C_\ell^{EE}\cos^2 2\alpha + C_\ell^{BB}\sin^2 2\alpha - C_\ell^{EB}\sin 4\alpha \\ C_\ell^{BB,o} &= C_\ell^{EE}\sin^2 2\alpha + C_\ell^{BB}\cos^2 2\alpha + C_\ell^{EB}\sin 4\alpha \\ C_\ell^{EB,o} &= C_\ell^{EE}\sin 2\alpha\cos 2\alpha. - C_\ell^{BB}\sin 2\alpha\cos 2\alpha + C_\ell^{EB}(\cos^2 2\alpha - \sin^2 2\alpha) \\ C_\ell^{EB,o} &= \frac{\sin 4\alpha}{2}(C_\ell^{EE}. - C_\ell^{BB}) + \cos 4\alpha\ C_\ell^{EB} \end{aligned} \tag{2}$$

We can relate the observed EB power spectrum to the observed EE and BB power spectrum

$$C_\ell^{EB,o} = \frac{1}{2}(C_\ell^{EE,o} - C_\ell^{BB,o})\tan 4\alpha + \frac{C_\ell^{EB}}{\cos 4\alpha} \tag{3}$$

indeed

$$\begin{aligned} C_\ell^{EB,o} &= \frac{1}{2}(C_\ell^{EE}\cos^2 2\alpha + C_\ell^{BB}\sin^2 2\alpha - C_\ell^{EB}\sin 4\alpha - C_\ell^{EE}\sin^2 2\alpha - C_\ell^{BB}\cos^2 2\alpha - C_\ell^{EB}\sin 4\alpha)\tan 4\alpha \\ &+ \frac{C_\ell^{EB}}{\cos 4\alpha} \\ &= \frac{1}{2}(C_\ell^{EE} - C_\ell^{BB})\sin 4\alpha + C_\ell^{EB}(\frac{1-\sin^2 4\alpha}{\cos 4\alpha}) = \frac{1}{2}(C_\ell^{EE} - C_\ell^{BB})\sin 4\alpha + C_\ell^{EB}\cos 4\alpha \end{aligned} \tag{4}$$

3 Likelihood

Assuming the intrinsic EB is equal to zero, we can therefore form two different χ^2

$$\chi^2(\alpha)_{\rm data} = \sum_{bb'} \left[C_b^{EB,o} - \frac{1}{2}(C_b^{EE,o} - C_b^{BB,o}) \tan 4\alpha\right] \Xi^{-1}_{bb'} \left[C_{b'}^{EB,o} - \frac{1}{2}(C_{b'}^{EE,o} - C_{b'}^{BB,o}) \tan 4\alpha\right] \quad (5)$$

$$\chi^2(\alpha)_{\rm th} = \sum_{bb'} \left[C_b^{EB,o} - \frac{1}{2}(C_b^{EE} - C_b^{BB}) \sin 4\alpha\right] \Xi^{-1}_{bb'} \left[C_b^{EB,o} - \frac{1}{2}(C_b^{EE} - C_b^{BB}) \sin 4\alpha\right] \quad (6)$$

4 Fisher

$$F_{\alpha\alpha} = \left\langle \frac{\partial^2 \mathcal{L}}{\partial^2 \alpha} \right\rangle\Big|_{\alpha=0} = 4 \sum_{bb'} (C_b^{EE,o} - C_b^{BB,o}) \Xi^{-1}_{bb'} (C_{b'}^{EE,o} - C_{b'}^{BB,o}) \quad (7)$$

$$\sigma_\alpha = \frac{1}{2}(\sum_{bb'} (C_b^{EE,o} - C_b^{BB,o}) \Xi^{-1}_{bb'} (C_{b'}^{EE,o} - C_{b'}^{BB,o}))^{-1/2} \quad (8)$$

5 Separating CMB and foregrounds

The foregrounds are sensitive to the instrumental miscalibrated angle but is not sensitive to the cosmic birefringence (because fg is at redshift zero) we can therefore separate CMB and fg. Now let's call α the instrumental miscalibrated angle and β the birefringence angle

$$\begin{aligned}
a_{\ell m}^{E,o} &= a_{\ell m}^{\rm fg,E} \cos(2\alpha) - a_{\ell m}^{\rm fg,B} \sin(2\alpha) + a_{\ell m}^{\rm CMB,E} \cos(2\alpha + 2\beta) - a_{\ell m}^{\rm CMB,B} \sin(2\alpha + 2\beta) + a_{\ell m}^{E,N} \\
a_{\ell m}^{B,o} &= a_{\ell m}^{\rm fg,E} \sin(2\alpha) + a_{\ell m}^{\rm fg,B} \cos(2\alpha) + a_{\ell m}^{\rm CMB,E} \sin(2\alpha + 2\beta) + a_{\ell m}^{\rm CMB,B} \cos(2\alpha + 2\beta) + a_{\ell m}^{B,N} \quad (9)
\end{aligned}$$

Assuming we form cross spectra

$$\begin{aligned}
C_\ell^{EE,o} &= \langle (a_{\ell m}^{\rm fg,E} \cos(2\alpha) - a_{\ell m}^{\rm fg,B} \sin(2\alpha) + a_{\ell m}^{\rm CMB,E} \cos(2\alpha + 2\beta) - a_{\ell m}^{\rm CMB,B} \sin(2\alpha + 2\beta)) \\
&\times (a_{\ell m}^{\rm fg,E} \cos(2\alpha) - a_{\ell m}^{\rm fg,B} \sin(2\alpha) + a_{\ell m}^{\rm CMB,E} \cos(2\alpha + 2\beta) - a_{\ell m}^{\rm CMB,B} \sin(2\alpha + 2\beta)) \rangle \\
&= C_\ell^{\rm EE,fg} \cos^2(2\alpha) + C_\ell^{\rm BB,fg} \sin^2(2\alpha) + C_\ell^{\rm EE,CMB} \cos^2(2\alpha + 2\beta) + C_\ell^{\rm BB,CMB} \sin^2(2\alpha + 2\beta) \\
&- 2C_\ell^{\rm EB,fg} \cos(2\alpha) \sin(2\alpha) - 2C_\ell^{\rm EB,CMB} \cos(2\alpha + 2\beta) \sin(2\alpha + 2\beta) \\
C_\ell^{BB,o} &= C_\ell^{\rm EE,fg} \sin^2(2\alpha) + C_\ell^{\rm BB,fg} \cos^2(2\alpha) + C_\ell^{\rm EE,CMB} \sin^2(2\alpha + 2\beta) + C_\ell^{\rm BB,CMB} \cos^2(2\alpha + 2\beta) \\
&+ 2C_\ell^{\rm EB,fg} \cos(2\alpha) \sin(2\alpha) + 2C_\ell^{\rm EB,CMB} \cos(2\alpha + 2\beta) \sin(2\alpha + 2\beta)
\end{aligned} \quad (10)$$

And

$$\begin{aligned}
C_\ell^{EB,o} &= \langle (a_{\ell m}^{\rm fg,E} \cos(2\alpha) - a_{\ell m}^{\rm fg,B} \sin(2\alpha) + a_{\ell m}^{\rm CMB,E} \cos(2\alpha + 2\beta) - a_{\ell m}^{\rm CMB,B} \sin(2\alpha + 2\beta)) \\
&\times (a_{\ell m}^{\rm fg,E} \sin(2\alpha) + a_{\ell m}^{\rm fg,B} \cos(2\alpha) + a_{\ell m}^{\rm CMB,E} \sin(2\alpha + 2\beta) + a_{\ell m}^{\rm CMB,B} \cos(2\alpha + 2\beta)) \rangle \\
&= C_\ell^{\rm EE,fg} \cos(2\alpha) \sin(2\alpha) - C_\ell^{\rm BB,fg} \sin(2\alpha) \cos(2\alpha) + C_\ell^{\rm EE,CMB} \cos(2\alpha + 2\beta) \sin(2\alpha + 2\beta) \\
&- C_\ell^{\rm BB,CMB} \sin(2\alpha + 2\beta) \cos(2\alpha + 2\beta) + C_\ell^{\rm EB,fg} (\cos^2(2\alpha) - \sin^2(2\alpha)) \\
&+ C_\ell^{\rm EB,CMB} (\cos^2(2\alpha + 2\beta) - \sin^2(2\alpha + 2\beta)) \\
C_\ell^{EB,o} &= \frac{\sin(4\alpha)}{2} (C_\ell^{\rm EE,fg} - C_\ell^{\rm BB,fg}) + \frac{\sin(4\alpha + 4\beta)}{2} (C_\ell^{\rm EE,CMB} - C_\ell^{\rm BB,CMB}) \\
&+ \cos(4\alpha) C_\ell^{\rm EB,fg} + \cos(4\alpha + 4\beta) C_\ell^{\rm EB,CMB}
\end{aligned} \quad (11)$$

Let's demonstrate that we can refactorize $C_\ell^{EB,o}$ as

$$\begin{aligned}
C_\ell^{EB,o} &= \frac{\tan(4\alpha)}{2} \left(C_\ell^{EB,o} - C_\ell^{BB,o}\right) + \frac{\sin(4\beta)}{2\cos(4\alpha)} (C_\ell^{\rm EE,CMB} - C_\ell^{\rm BB,CMB}) \\
&+ \frac{1}{\cos(4\alpha)} C_\ell^{\rm EB,fg} + \frac{\cos(4\beta)}{\cos(4\alpha)} C_\ell^{\rm EB,CMB}
\end{aligned} \quad (12)$$

We can expand

$$
\begin{aligned}
\left(C_\ell^{EE,o} - C_\ell^{BB,o}\right) &= C_\ell^{\mathrm{EE,fg}} \cos^2(2\alpha) + C_\ell^{\mathrm{BB,fg}} \sin^2(2\alpha) + C_\ell^{\mathrm{EE,CMB}} \cos^2(2\alpha+2\beta) \\
&+ C_\ell^{\mathrm{BB,CMB}} \sin^2(2\alpha+2\beta) - 2C_\ell^{\mathrm{EB,fg}} \cos(2\alpha)\sin(2\alpha) - 2C_\ell^{\mathrm{EB,CMB}} \cos(2\alpha+2\beta)\sin(2\alpha+2\beta) \\
&- C_\ell^{\mathrm{EE,fg}} \sin^2(2\alpha) - C_\ell^{\mathrm{BB,fg}} \cos^2(2\alpha) - C_\ell^{\mathrm{EE,CMB}} \sin^2(2\alpha+2\beta) - C_\ell^{\mathrm{BB,CMB}} \cos^2(2\alpha+2\beta) \\
&- 2C_\ell^{\mathrm{EB,fg}} \cos(2\alpha)\sin(2\alpha) - 2C_\ell^{\mathrm{EB,CMB}} \cos(2\alpha+2\beta)\sin(2\alpha+2\beta) \\
&= \cos(4\alpha)(C_\ell^{\mathrm{EE,fg}} - C_\ell^{\mathrm{BB,fg}}) + \cos(4\alpha+4\beta)(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&- 2C_\ell^{\mathrm{EB,fg}} \sin(4\alpha) - 2C_\ell^{\mathrm{EB,CMB}} \sin(4\alpha+4\beta)
\end{aligned} \tag{13}
$$

so that

$$
\begin{aligned}
C_\ell^{EB,o} &= \frac{\sin(4\alpha)}{2}(C_\ell^{\mathrm{EE,fg}} - C_\ell^{\mathrm{BB,fg}}) + \frac{\tan(4\alpha)}{2}\cos(4\alpha+4\beta)(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&- C_\ell^{\mathrm{EB,fg}} \sin(4\alpha)\tan(4\alpha) - C_\ell^{\mathrm{EB,CMB}} \sin(4\alpha+4\beta)\tan(4\alpha) + \frac{\sin(4\beta)}{2\cos(4\alpha)}(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&+ \frac{1}{\cos(4\alpha)} C_\ell^{\mathrm{EB,fg}} + \frac{\cos(4\beta)}{\cos(4\alpha)} C_\ell^{\mathrm{EB,CMB}} \\
&= \frac{\sin(4\alpha)}{2}(C_\ell^{\mathrm{EE,fg}} - C_\ell^{\mathrm{BB,fg}}) + (C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}})\left(\frac{\tan(4\alpha)}{2}\cos(4\alpha+4\beta) + \frac{\sin(4\beta)}{2\cos(4\alpha)}\right) \\
&+ \left(\frac{1}{\cos(4\alpha)} - \sin(4\alpha)\tan(4\alpha)\right) C_\ell^{\mathrm{EB,fg}} + \left(\frac{\cos(4\beta)}{\cos(4\alpha)} - \sin(4\alpha+4\beta)\tan(4\alpha)\right) C_\ell^{\mathrm{EB,CMB}}
\end{aligned} \tag{14}
$$

Let's do some trigometry

$$
\begin{aligned}
\frac{1}{\cos(4\alpha)} - \sin(4\alpha)\tan(4\alpha) &= \cos(4\alpha) \\
\frac{\cos(4\beta)}{\cos(4\alpha)} - \sin(4\alpha+4\beta)\tan(4\alpha) &= \frac{\cos(4\beta)}{\cos(4\alpha)} - (\sin(4\alpha)\cos(4\beta) + \sin(4\beta)\cos(4\alpha))\tan(4\alpha) \\
&= \frac{\cos(4\beta)}{\cos(4\alpha)} - \left(\sin^2(4\alpha)\frac{\cos(4\beta)}{\cos(4\alpha)} + \sin(4\beta)\sin(4\alpha)\right) \\
&= \frac{\cos(4\beta)}{\cos(4\alpha)}(1 - \sin^2(4\alpha)) - \sin(4\beta)\sin(4\alpha)) \\
&= \cos(4\beta)\cos(4\alpha) - \sin(4\beta)\sin(4\alpha) = \cos(4\beta+4\alpha) \\
\frac{\tan(4\alpha)}{2}\cos(4\alpha+4\beta) + \frac{\sin(4\beta)}{2\cos(4\alpha)} &= \frac{\tan(4\alpha)}{2}[\cos(4\alpha)\cos(4\beta) - \sin(4\alpha)\sin(4\beta)] + \frac{\sin(4\beta)}{2\cos(4\alpha)} \\
&= \frac{\sin(4\alpha)\cos(4\beta)}{2} - \frac{\sin^2(4\alpha)\sin(4\beta)}{2\ \cos(4\alpha)} + \frac{\sin(4\beta)}{2\cos(4\alpha)} \\
&= \frac{\sin(4\alpha)\cos(4\beta)}{2} - \frac{\sin(4\beta)}{2\cos(4\alpha)}(\sin^2(4\alpha) - 1) \\
&= \frac{\sin(4\alpha)\cos(4\beta)}{2} + \frac{\cos(4\alpha)\sin(4\beta)}{2} = \frac{\sin(4\alpha+4\beta)}{2}
\end{aligned} \tag{15}
$$

With this, we demonstrate that

$$
\begin{aligned}
C_\ell^{EB,o} &= \frac{\tan(4\alpha)}{2}\left(C_\ell^{EB,o} - C_\ell^{BB,o}\right) + \frac{\sin(4\beta)}{2\cos(4\alpha)}(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&+ \frac{1}{\cos(4\alpha)} C_\ell^{\mathrm{EB,fg}} + \frac{\cos(4\beta)}{\cos(4\alpha)} C_\ell^{\mathrm{EB,CMB}}
\end{aligned} \tag{16}
$$

is indeed equal to

$$
\begin{aligned}
C_\ell^{EB,o} &= \frac{\sin(4\alpha)}{2}(C_\ell^{\mathrm{EE,fg}} - C_\ell^{\mathrm{BB,fg}}) + \frac{\sin(4\alpha+4\beta)}{2}(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&+ \cos(4\alpha) C_\ell^{\mathrm{EB,fg}} + \cos(4\alpha+4\beta) C_\ell^{\mathrm{EB,CMB}}
\end{aligned} \tag{17}
$$

which complete the proof.

6 A Multifrequency likelihood

We can rewrite

$$
\begin{aligned}
C_\ell^{EE,o} &= C_\ell^{\mathrm{EE,fg}}\cos^2(2\alpha) + C_\ell^{\mathrm{BB,fg}}\sin^2(2\alpha) + C_\ell^{\mathrm{EE,CMB}}\cos^2(2\alpha+2\beta) + C_\ell^{\mathrm{BB,CMB}}\sin^2(2\alpha+2\beta) \\
&- 2C_\ell^{\mathrm{EB,fg}}\cos(2\alpha)\sin(2\alpha) - 2C_\ell^{\mathrm{EB,CMB}}\cos(2\alpha+2\beta)\sin(2\alpha+2\beta) \\
C_\ell^{BB,o} &= C_\ell^{\mathrm{EE,fg}}\sin^2(2\alpha) + C_\ell^{\mathrm{BB,fg}}\cos^2(2\alpha) + C_\ell^{\mathrm{EE,CMB}}\sin^2(2\alpha+2\beta) + C_\ell^{\mathrm{BB,CMB}}\cos^2(2\alpha+2\beta) \\
&+ 2C_\ell^{\mathrm{EB,fg}}\cos(2\alpha)\sin(2\alpha) + 2C_\ell^{\mathrm{EB,CMB}}\cos(2\alpha+2\beta)\sin(2\alpha+2\beta) \\
C_\ell^{EB,o} &= \frac{\sin(4\alpha)}{2}(C_\ell^{\mathrm{EE,fg}} - C_\ell^{\mathrm{BB,fg}}) + \frac{\sin(4\alpha+4\beta)}{2}(C_\ell^{\mathrm{EE,CMB}} - C_\ell^{\mathrm{BB,CMB}}) \\
&+ \cos(4\alpha)C_\ell^{\mathrm{EB,fg}} + \cos(4\alpha+4\beta)C_\ell^{\mathrm{EB,CMB}}
\end{aligned} \tag{18}
$$

In a matricial form, let's neglect intrinsic EB correlation

$$
\begin{aligned}
\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,o}} \\ C_\ell^{\mathrm{B_iB_j,o}} \end{pmatrix} &= \boldsymbol{R}(\alpha_i,\alpha_j)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,fg}} \\ C_\ell^{\mathrm{B_iB_j,fg}} \end{pmatrix} + \boldsymbol{R}(\alpha_i+\beta,\alpha_j+\beta)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix} \\
C_\ell^{\mathrm{E_iB_j,o}} &= \vec{R}^T(\alpha_i,\alpha_j)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,fg}} \\ C_\ell^{\mathrm{B_iB_j,fg}} \end{pmatrix} + \vec{R}^T(\alpha_i+\beta,\alpha_j+\beta)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix}
\end{aligned} \tag{19}
$$

With $\boldsymbol{R}$ a rotation matrix and $\vec{R}$ a rotation vector

$$
\boldsymbol{R}(\alpha_i,\alpha_j) = \begin{pmatrix} \cos(2\alpha_i)\cos(2\alpha_j) & \sin(2\alpha_i)\sin(2\alpha_j) \\ \sin(2\alpha_i)\sin(2\alpha_j) & \cos(2\alpha_i)\cos(2\alpha_j) \end{pmatrix} \tag{20}
$$

$$
\vec{R}(\alpha_i,\alpha_j) = \begin{pmatrix} \cos(2\alpha_i)\sin(2\alpha_j) \\ -\sin(2\alpha_i)\cos(2\alpha_j) \end{pmatrix} \tag{21}
$$

We can simply re-express $C_\ell^{EB,o}$ as a function of $C_\ell^{EE,o}$ and $C_\ell^{BB,o}$ using the equation above

$$
\begin{aligned}
C_\ell^{\mathrm{E_iB_j,o}} &= \vec{R}^T(\alpha_i,\alpha_j)\left[\boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,o}} \\ C_\ell^{\mathrm{B_iB_j,o}} \end{pmatrix} - \boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\boldsymbol{R}(\alpha_i+\beta,\alpha_j+\beta)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix}\right] \\
&+ \vec{R}^T(\alpha_i+\beta,\alpha_j+\beta)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix} \\
&= \vec{R}^T(\alpha_i,\alpha_j)\boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,o}} \\ C_\ell^{\mathrm{B_iB_j,o}} \end{pmatrix} \\
&+ \left[\vec{R}^T(\alpha_i+\beta,\alpha_j+\beta) - \vec{R}^T(\alpha_i,\alpha_j)\boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\boldsymbol{R}(\alpha_i+\beta,\alpha_j+\beta)\right]\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix}
\end{aligned} \tag{22}
$$

This can finally be rewritten

$$
C_\ell^{\mathrm{E_iB_j,o}} - \boldsymbol{A}\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,o}} \\ C_\ell^{\mathrm{B_iB_j,o}} \end{pmatrix} - \boldsymbol{B}\begin{pmatrix} C_\ell^{\mathrm{E_iE_j,CMB}} \\ C_\ell^{\mathrm{B_iB_j,CMB}} \end{pmatrix} = 0 \tag{23}
$$

with

$$
\boldsymbol{A} = \left(\vec{R}^T(\alpha_i,\alpha_j)\boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\right) \tag{24}
$$

and

$$
\boldsymbol{B} = \vec{R}^T(\alpha_i+\beta,\alpha_j+\beta) - \vec{R}^T(\alpha_i,\alpha_j)\boldsymbol{R}^{-1}(\alpha_i,\alpha_j)\boldsymbol{R}(\alpha_i+\beta,\alpha_j+\beta) \tag{25}
$$