1. (24%) Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n distinct real numbers. Let $r_A(a)$ denote the rank of a in A, i.e. the number of elements in A whose value is no greater than a. We sample s_i for each $i \in [1, k]$ independently, uniformly at random from $\{a_1, a_2, \ldots, a_n\}$ with replacement. Let $S = \{s_1, s_2, \ldots, s_k\}$ be the set of random samples. Prove that if $k = \Omega(\sqrt{n})$, then for any given $t \in [1, n]$ we have

$$\Pr\left[\text{some } s_i \text{ in } S \text{ has rank } r_A(s_i) \in [t - \sqrt{n}, t + \sqrt{n}]\right] \geq \delta$$

for some constant $\delta > 0$.

2. (24%) Prove that every simple graph of average degree $n/3$ has $\Omega(n)$ nodes of degree at least $n/4$.

3. (24%) Let S_n be a random subset of $\{1, 2, \ldots, n\}$ so that each element $k \in [1, n]$ is included in S independently with probability

$$\Pr[k \in S] = \frac{1}{k}.$$

Prove that $|S_n| = \Theta(\log n)$ w.h.p.[1]

4. (12%) Let G be a random graph in $G(n, p)$, i.e. for every pair of nodes $u \neq v$, (u, v) is an edge in G with probability p. Prove that if $p = c/n$ for any constant $c > 0$ and n is sufficiently large, then G is triangle-free with probability δ for some constant $\delta > 0$.

5. (8+8%) Let (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n) be a sequence of points in $\mathbb{R}^2$, given one by one as the input. Let $\ell(k)$ denote the minimum length so that one can cover all the points by k axis-parallel squares whose four sides have length $\ell(k)$. Given $k \in \{2, 3\}$, devise an algorithm to find k axis-parallel squares whose four sides have length $\leq \ell(k)$ to cover at least $(1 - \varepsilon)n$ given points w.h.p. using $O(n + (1/\varepsilon)\log^2 n)$ time and $O((1/\varepsilon)\log n)$ space.

   (a) Solve the case of $k = 2$.

   (b) Solve the case of $k = 3$.

[1] w.h.p. means with probability $1 - 1/n^{\Omega(1)}$.