Natural gradient descent with momentum

Séminaire de mathématiques appliquées du LMJL, 2026

Agustín Somacal

In collaboration with **Anthony Nouy** and **Joel Soffo**

École Centrale de Nantes

Laboratoire de Mathématiques Jean Leray

Outline

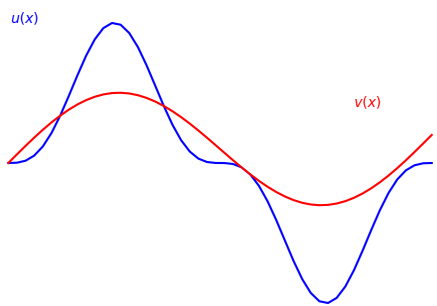
1. **What is natural gradient and why we may need it?**
   - Problem setting
   - From gradient descent to Newton's method.
   - From Newton's method to natural gradient.
   - Toy examples to gain intuition.
   - *Online learning* in Reduced Order Modelling.
2. **What is momentum and when we may need it?**
3. **How to combine momentum and natural gradient.**
   - A toy example
   - Some less toy examples

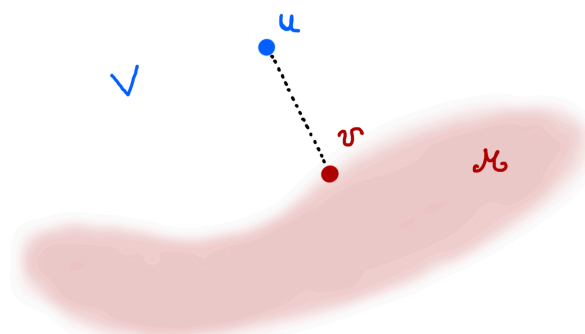
Problem formulation

Objective: approximate target function $u \in V$ by $v \in \mathcal{M} \subset V$

Target function $\quad u : \mathbb{R}^d \to \mathbb{R} \in V$

Hilbert space
$L^2(\Omega), H^1(\Omega), \ldots$





Problem formulation

Objective: approximate target function $u \in V$ by $v \in \mathcal{M} \subset V$

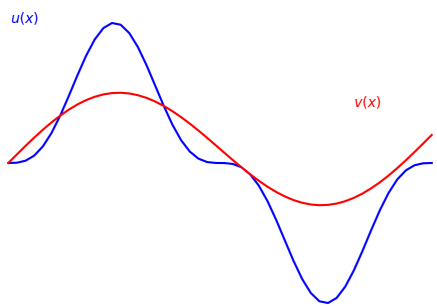
Target function $\quad u : \mathbb{R}^d \to \mathbb{R} \in V$

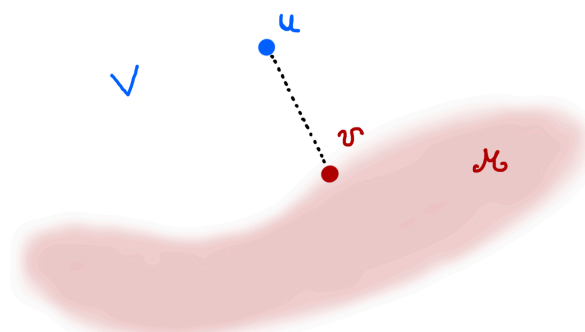
Hilbert space $L^2(\Omega), H^1(\Omega), \ldots$

Approximation manifold $\quad \mathcal{M} := \{v_\theta(x) = A(\theta)(x);\ \theta \in \mathbb{R}^p\}$

Linear model

$$A(\theta)(x) = \theta_1 \phi_1(x) + \cdots + \theta_p \phi_p(x)$$
$$= \theta^T \mathbf{\Phi}(x)$$

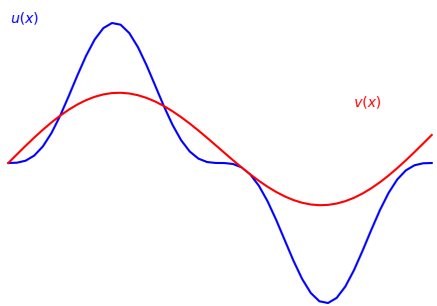


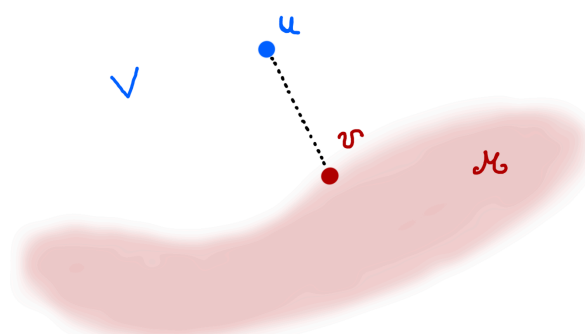


Problem formulation

Objective: approximate target function $u \in V$ by $v \in \mathcal{M} \subset V$

Target function	$u : \mathbb{R}^d \to \mathbb{R} \in V$	Hilbert space $L^2(\Omega), H^1(\Omega), \ldots$
Approximation manifold	$\mathcal{M} := \{v_\theta(x) = A(\theta)(x);\ \theta \in \mathbb{R}^p\}$	Linear model, Neural network, ...





Minimization problem

Objective: approximate $u \in V$ by $v \in \mathcal{M}$.

Continuous problem

$$
\begin{aligned}
\mathcal{L}_u(v) &= \frac{1}{2}\|u - v\|_V^2 \\
&= \frac{1}{2}\langle u - v, u - v\rangle_V \\
&= \frac{1}{2}\int (u(x) - v(x))^2 \mathrm{d}\mu(x)
\end{aligned}
$$

Minimization problem

Objective: approximate $u \in V$ by $v \in \mathcal{M}$.

Continuous problem

$$\begin{aligned}\mathcal{L}_u(v) &= \frac{1}{2}\|u - v\|_V^2 \\ &= \frac{1}{2}\langle u - v, u - v\rangle_V \\ &= \frac{1}{2}\int (u(x) - v(x))^2 \mathrm{d}\mu(x)\end{aligned}$$

Discrete problem

$$\begin{aligned}\mathcal{L}_u(v) &= \frac{1}{2}\|u - v\|_m^2 \\ &= \frac{1}{2}\langle u - v, u - v\rangle_m \\ &= \frac{1}{2m}\sum_{i=1}^{m}(u(x_i) - v(x_i))^2\end{aligned}$$

Minimization problem

Objective: approximate $u \in V$ by $v \in \mathcal{M}$.

Continuous problem

$$\begin{aligned}\mathcal{L}_u(v) &= \frac{1}{2}\|u - v\|_V^2 \\ &= \frac{1}{2}\langle u - v, u - v\rangle_V \\ &= \frac{1}{2}\int (u(x) - v(x))^2 \mathrm{d}\mu(x)\end{aligned}$$

Discrete problem

$$\begin{aligned}\mathcal{L}_u(v) &= \frac{1}{2}\|u - v\|_m^2 \\ &= \frac{1}{2}\langle u - v, u - v\rangle_m \\ &= \frac{1}{2m}\sum_{i=1}^{m}(u(x_i) - v(x_i))^2\end{aligned}$$

Functional perspective

$$v^* = \arg\min_{v \in \mathcal{M}} \mathcal{L}_u(v)$$

Parameter perspective

$$\theta^* = \arg\min_{\theta \in \mathbb{R}^p} \mathcal{L}_u(A(\theta))$$

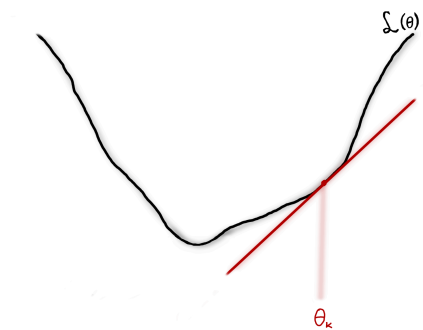
$$\mathcal{L}_u(A(\theta)) = (\mathcal{L}_u \circ A)(\theta) =: L(\theta)$$

Gradient descent

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k.

$$L(\theta) \approx L(\theta_k) + \langle \nabla_\theta L(\theta_k), \theta - \theta_k \rangle_{\mathbb{R}^p}$$

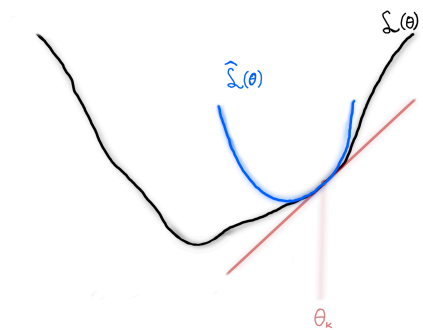


Gradient descent

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$L(\theta) \approx L(\theta_k) + \langle \nabla_\theta L(\theta_k), \theta - \theta_k \rangle_{\mathbb{R}^p} + \frac{1}{2s}\rho(\theta, \theta_k)$$

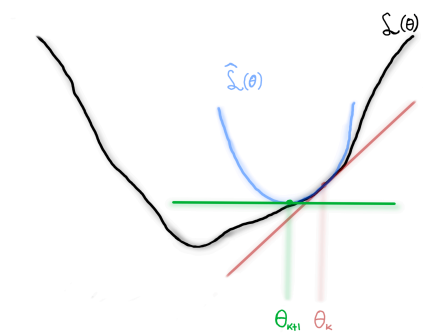


Gradient descent

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$0 = \nabla_\theta \left[L(\theta_k) + \langle \nabla_\theta L(\theta_k), \theta - \theta_k \rangle_{\mathbb{R}^p} + \frac{1}{2s} \rho(\theta, \theta_k) \right]$$



Gradient descent

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$\frac{1}{2}\nabla_\theta \rho(\theta, \theta_k) = -s\nabla_\theta L(\theta_k)$$

Gradient descent

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$\frac{1}{2}\nabla_\theta \rho(\theta, \theta_k) = -s\nabla_\theta L(\theta_k)$$

Gradient descent

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|^2_{\mathbb{R}^p}$$

$$\nabla_\theta \rho(\theta, \theta_k) = 2(\theta - \theta_k)$$

$$\theta = \theta_k - s\nabla_\theta L(\theta_k)$$

Preconditioned gradient

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$\frac{1}{2}\nabla_\theta \rho(\theta, \theta_k) = -s\nabla_\theta L(\theta_k)$$

Gradient descent	Preconditioned gradient
$\rho(\theta, \theta_k) = \|\theta - \theta_k\|^2_{\mathbb{R}^p}$	$\rho(\theta, \theta_k) = \|\theta - \theta_k\|^2_M$
$\nabla_\theta \rho(\theta, \theta_k) = 2(\theta - \theta_k)$	$\nabla_\theta \rho(\theta, \theta_k) = 2M(\theta - \theta_k)$
$\theta = \theta_k - s\nabla_\theta L(\theta_k)$	$\theta = \theta_k - sM^{-1}\nabla_\theta L(\theta_k)$

Newton's method

Iteratively improve approximation by minimizing $L(\theta_k)$.

Taylor expansion around current iterate θ_k plus penalization on the distance traveled on each step.

$$\frac{1}{2}\nabla_\theta \rho(\theta, \theta_k) = -s\nabla_\theta L(\theta_k)$$

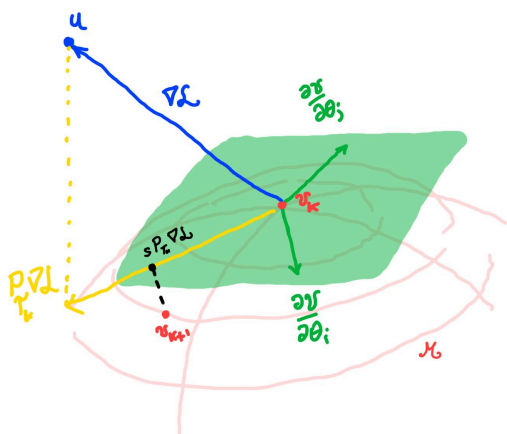
Gradient descent	Newton's method
$\rho(\theta, \theta_k) = \|\theta - \theta_k\|^2_{\mathbb{R}^p}$	$\rho(\theta, \theta_k) = \|\theta - \theta_k\|^2_H$
$\nabla_\theta \rho(\theta, \theta_k) = 2(\theta - \theta_k)$	$\nabla_\theta \rho(\theta, \theta_k) = 2H(\theta - \theta_k)$
$\theta = \theta_k - s\nabla_\theta L(\theta_k)$	$\theta = \theta_k - sH^{-1}\nabla_\theta L(\theta_k)$

Functional gradient and geometric intuition

Some properties [Gruhlke, Robert, Anthony Nouy, and Philipp Trunschke. 2024].

$$H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] = \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right]$$

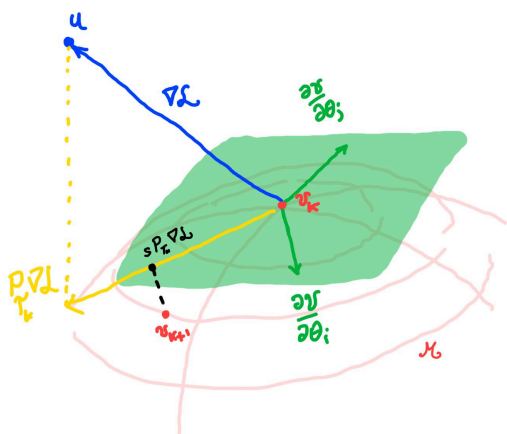
$$= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right]$$



Functional gradient and geometric intuition

Some properties [Gruhlke, Robert, Anthony Nouy, and Philipp Trunschke. 2024].

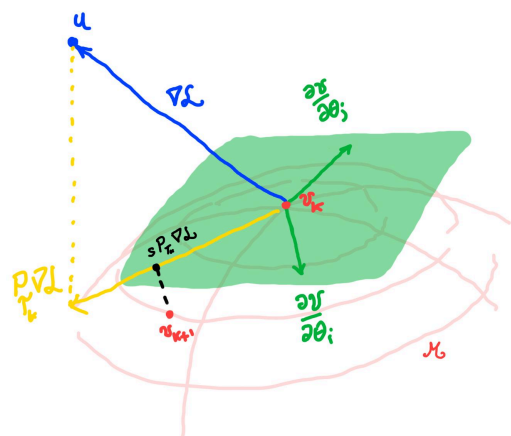
$$H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] = \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right]$$
$$= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right]$$



Functional gradient and geometric intuition

Some properties [Gruhlke, Robert, Anthony Nouy, and Philipp Trunschke. 2024].

$$H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] = \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right]$$
$$= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right]$$



$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2_{L^2(\Omega)}$$

$$V \ni \nabla\mathcal{L}_u = u - v$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} + \langle \nabla\mathcal{L}_u, H_A\rangle_V
\end{aligned}
$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} + \langle \nabla\mathcal{L}_u, H_A\rangle_V
\end{aligned}
$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} + \langle \nabla\mathcal{L}_u, H_A\rangle_V
\end{aligned}
$$

$$
G_{ij}^{H_\mathcal{L}} = \int \frac{\partial A}{\partial\theta_i}(x)[H_V\mathcal{L}(v)](x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x).
$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} + \langle \nabla\mathcal{L}_u, H_A\rangle_V
\end{aligned}
$$

$$
G_{ij}^{H_\mathcal{L}} = \int \frac{\partial A}{\partial\theta_i}(x)[H_V\mathcal{L}(v)](x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x).
$$

$$
\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2_{L^2(\Omega)} \qquad \longrightarrow \qquad H_V\mathcal{L}(v)(x) = 1
$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} + \langle \nabla\mathcal{L}_u, H_A\rangle_V
\end{aligned}
$$

$$
G_{ij}^{H_\mathcal{L}} = \int \frac{\partial A}{\partial\theta_i}(x)[H_V\mathcal{L}(v)](x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x).
$$

$$
\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2_{L^2(\Omega)} \qquad \longrightarrow \qquad H_V\mathcal{L}(v)(x) = 1
$$

$$
G_{ij} = \int \left[\frac{\partial A}{\partial\theta_i}\frac{\partial A}{\partial\theta_j}\right](x)\mathrm{d}x
$$

Natural gradient from Newton's method

[Amari, Shun-ichi. 1998] [Martens, James 2020].

$$
\begin{aligned}
H_{ij} = \frac{\partial^2 L}{\partial\theta_i \partial\theta_j} = \frac{\partial}{\partial\theta_i}\left[\frac{\partial L}{\partial\theta_j}\right] = \frac{\partial}{\partial\theta_i}\left[\frac{\partial}{\partial\theta_j}(\mathcal{L}_u \circ A)\right] &= \frac{\partial}{\partial\theta_i}\left[\int_\Omega \nabla\mathcal{L}_u(v_\theta)(x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x)\right] \\
&= \frac{\partial}{\partial\theta_i}\left[\left\langle \nabla\mathcal{L}_u, \frac{\partial A}{\partial\theta_j}\right\rangle_V\right] \\
&= \left\langle H_V\mathcal{L}\frac{\partial A}{\partial\theta_i}, \frac{\partial A}{\partial\theta_j}\right\rangle_V + \left\langle \nabla\mathcal{L}_u, \frac{\partial^2 A}{\partial\theta_i\partial\theta_j}\right\rangle_V \\
&= G^{H_\mathcal{L}} \cancel{+ \langle \nabla\mathcal{L}_u, H_A\rangle_V}
\end{aligned}
$$

Model linearization

$$
G^{H_\mathcal{L}}_{ij} = \int \frac{\partial A}{\partial\theta_i}(x)[H_V\mathcal{L}(v)](x)\frac{\partial A}{\partial\theta_j}(x)\mathrm{d}\mu(x).
$$

$$
\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2_{L^2(\Omega)} \qquad \longrightarrow \qquad H_V\mathcal{L}(v)(x) = 1
$$

$$
G_{ij} = \int \left[\frac{\partial A}{\partial\theta_i}\frac{\partial A}{\partial\theta_j}\right](x)\mathrm{d}x
$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla_\theta L(\theta_k)$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla_\theta L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla_\theta L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

$$\begin{aligned}
\frac{\partial v_\theta}{\partial s} = \frac{\partial v_\theta}{\partial \theta}\frac{\partial \theta}{\partial s} = \frac{\partial A}{\partial \theta}^T M^{-1}\nabla_\theta L &= -\frac{\partial A}{\partial \theta}^T M^{-1}\left\langle \nabla\mathcal{L}, \frac{\partial A}{\partial \theta}\right\rangle_V \\
&= -\left(G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right)^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \left\langle \nabla\mathcal{L}, G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right\rangle_V \\
&= -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \left\langle \nabla\mathcal{L}, E\right\rangle_V
\end{aligned}$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla_\theta L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

$$\begin{aligned}
\frac{\partial v_\theta}{\partial s} = \frac{\partial v_\theta}{\partial \theta}\frac{\partial \theta}{\partial s} = \frac{\partial A}{\partial \theta}^T M^{-1}\nabla_\theta L &= -\frac{\partial A}{\partial \theta}^T M^{-1}\left\langle \nabla\mathcal{L}, \frac{\partial A}{\partial \theta}\right\rangle_V \\
&= -\left(G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right)^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}}\left\langle \nabla\mathcal{L}, G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right\rangle_V \\
&= -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla\mathcal{L}, E\rangle_V
\end{aligned}$$

$$\langle E, E\rangle_V = \int_\Omega \left(G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right)^T \left(G^{-\frac{1}{2}}\frac{\partial A}{\partial \theta}\right) \mathrm{d}\mu(x) \quad = \int_\Omega \frac{\partial A}{\partial \theta}^T G^{-1}\frac{\partial A}{\partial \theta}\mathrm{d}\mu(x) = I$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Gradient descent $M = I$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} I^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Gradient descent $M = I$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Gradient descent $M = I$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient $M = G$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} G^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Gradient descent $M = I$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient $M = G$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient flow dynamics

$$\rho(\theta, \theta_k) = \|\theta - \theta_k\|_G^2 \qquad \longrightarrow \qquad \theta = \theta_k - sG^{-1}\nabla L(\theta_k)$$

Preconditioned gradient flow
in **parameter space**

$$\frac{\partial \theta}{\partial s} = -M^{-1}\nabla_\theta L$$

Preconditioned gradient flow
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G^{\frac{1}{2}} M^{-1} G^{\frac{1}{2}} \langle \nabla \mathcal{L}, E \rangle_V$$

Gradient descent $M = I$
in **functional space**

$$\frac{\partial v_\theta}{\partial s} = -E^T G \langle \nabla \mathcal{L}, E \rangle_V$$

Natural gradient $M = G$
in **functional space**

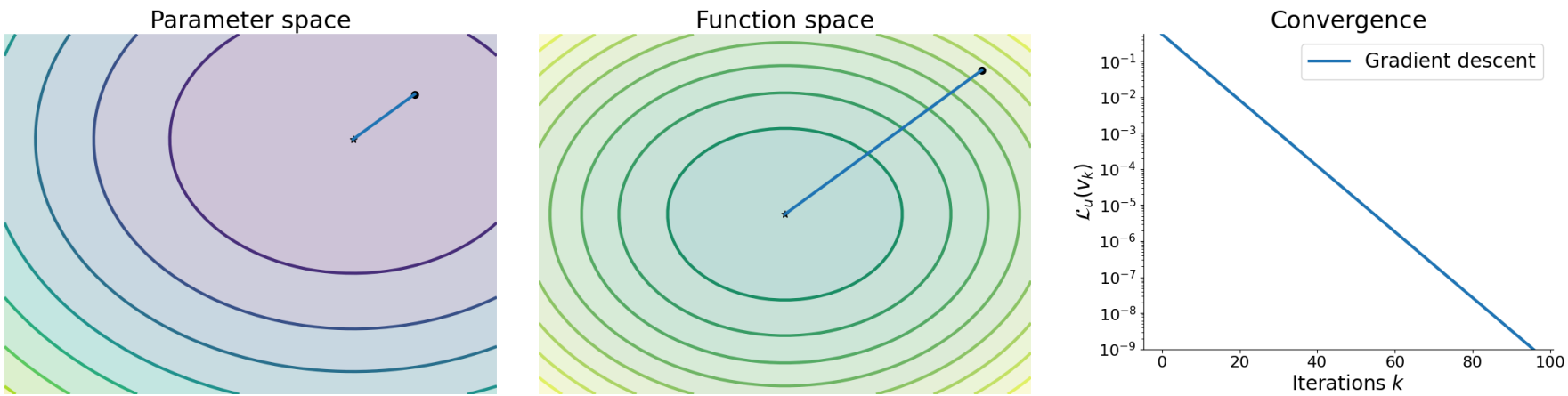
$$\frac{\partial v_\theta}{\partial s} = -P_{\mathcal{T}_k} \nabla \mathcal{L}$$

Toy example

Gradient descent trajectory.

$$u \in L^2([-1,1])$$
$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2$$
$$v_\theta(x) = A(\theta)(x) = \theta^T \Phi(x)$$
$$= \theta^T \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = \Phi_i(x)$$
$$G_{ij} = \int [\Phi_i \Phi_j](x)\mathrm{d}x = \delta_{ij}$$



Toy example

Gradient descent is biased in functional space.

$$u \in L^2([-1,1])$$

$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2$$

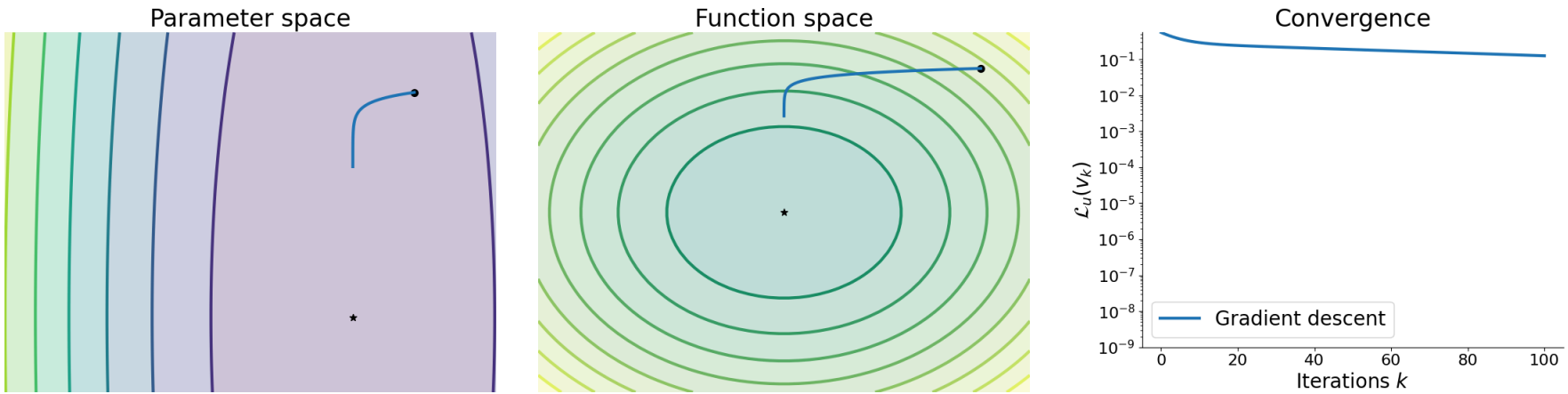
$$v_\theta(x) = A(\theta)(x) = \theta^T B \Phi(x)$$

$$= \theta^T B \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = B_i \Phi(x)$$

$$G_{ij} = \int \left[\Phi^T B_i^T B_j \Phi\right](x)\mathrm{d}x = B_i^T B_j$$

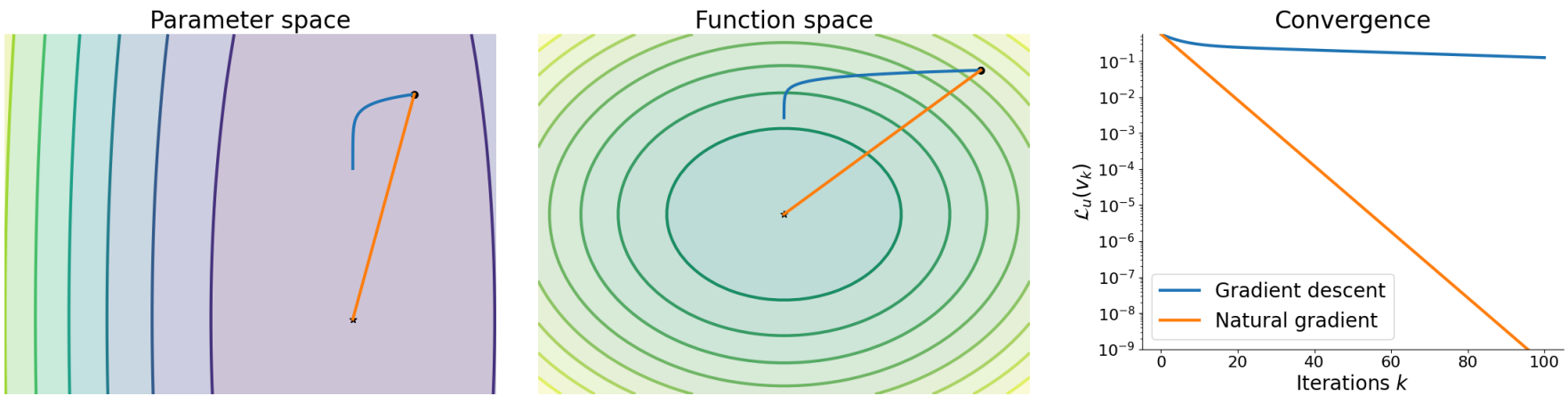


Toy example

Natural gradient descent.

$$u \in L^2([-1,1])$$
$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|^2$$
$$v_\theta(x) = A(\theta)(x) = \theta^T B \Phi(x)$$
$$= \theta^T B \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = B_i \Phi(x)$$
$$G_{ij} = \int \left[\Phi^T B_i^T B_j \Phi\right](x)\mathrm{d}x = B_i^T B_j$$

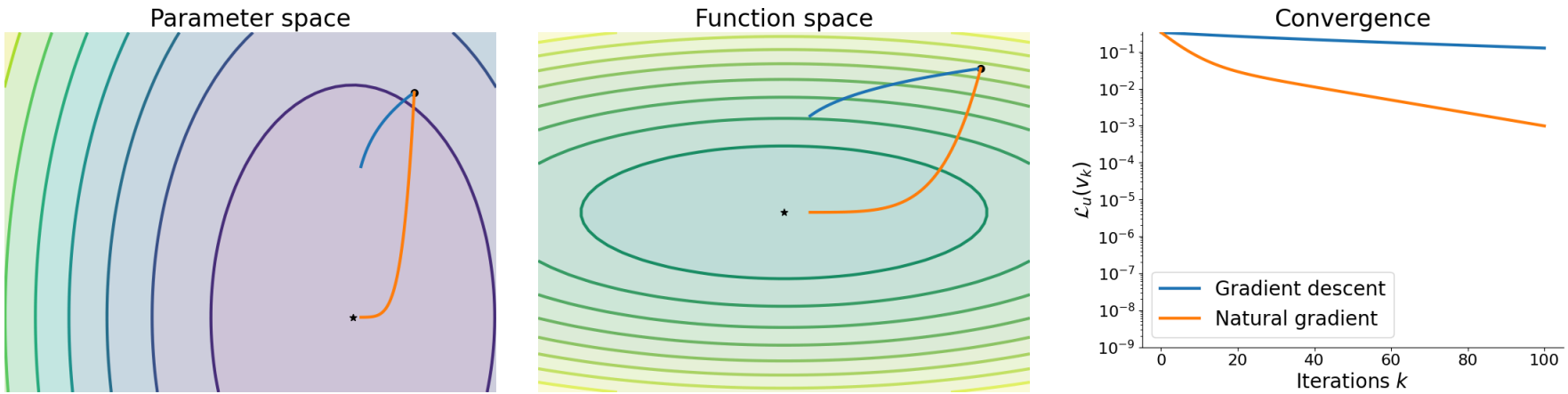


Toy example

Non isotropic loss.

$$u \in L^2([-1,1])$$
$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|_K^2$$
$$v_\theta(x) = A(\theta)(x) = \theta^T B \Phi(x)$$
$$= \theta^T B \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$
$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = B_i \Phi(x)$$
$$G_{ij} = \int \left[\Phi^T B_i^T K B_j \Phi\right](x)\mathrm{d}x = B_i^T K B_j$$



Toy example

Natural gradient descent with loss hessian.

$$u \in L^2([-1,1])$$

$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|_K^2$$

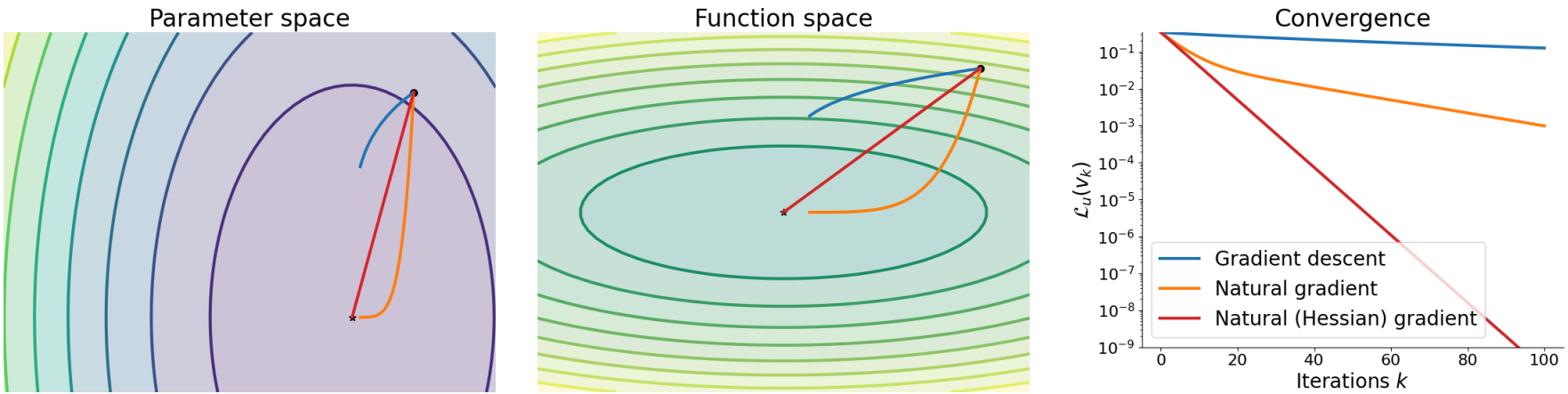
$$v_\theta(x) = A(\theta)(x) = \theta^T B \Phi(x)$$

$$= \theta^T B \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = B_i \Phi(x)$$

$$G_{ij} = \int \left[\Phi^T B_i^T K B_j \Phi\right](x)\mathrm{d}x = B_i^T K B_j$$



Toy example

Natural gradient and Newton method are equivalent for linear models.

$$u \in L^2([-1,1])$$

$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|_K^2$$

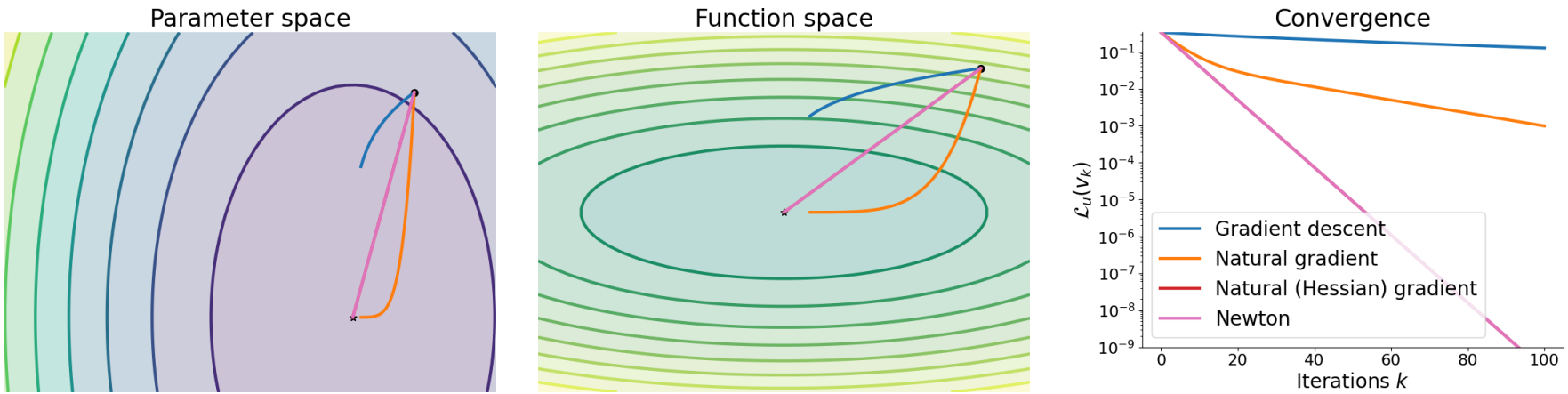
$$v_\theta(x) = A(\theta)(x) = \theta^T B \Phi(x)$$

$$= \theta^T B \begin{bmatrix} \sin(2\pi x) \\ \cos(2\pi x) \end{bmatrix}$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = B_i \Phi(x)$$

$$G_{ij} = \int \left[\Phi^T B_i^T K B_j \Phi\right](x)\mathrm{d}x = B_i^T K B_j$$

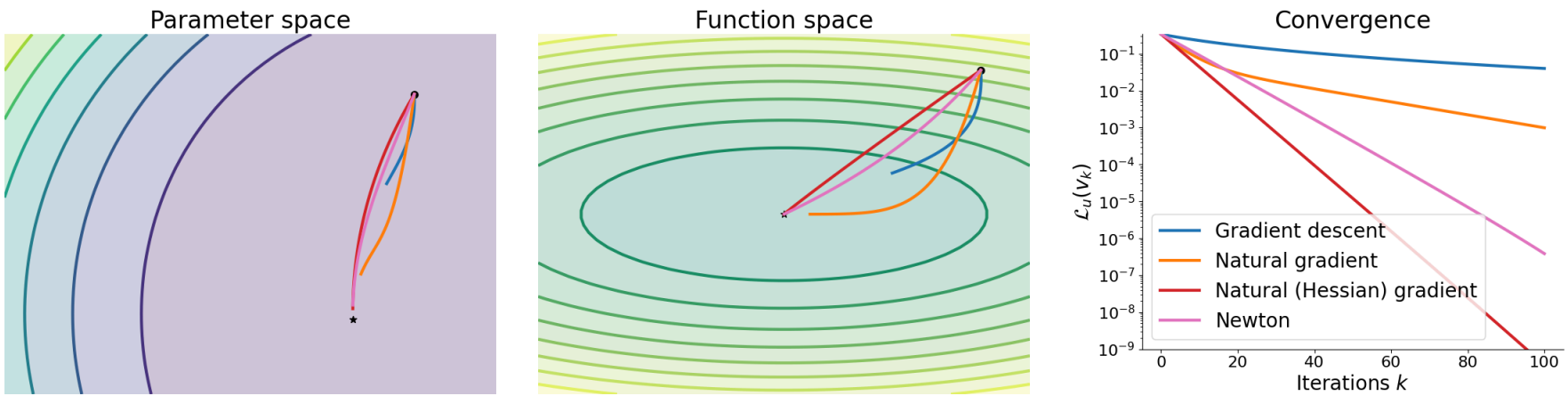


Toy example

Nonlinear manifold.

$$u \in L^2([-1,1])$$
$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|_K^2$$
$$v_\theta(x) = A(\theta)(x) = \theta^T(B + Q\theta)\Phi(x)$$

$$\mathbb{R}^{p\times p} \ni B = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}, Q \in \mathbb{R}^{p\times p\times p}$$
$$\frac{\partial A}{\partial \theta_i}(\theta)(x) = (B_i + Q_i\theta)\Phi_i(x)$$
$$G_{ij}(\theta) = [(B + Q_i\theta)^T K (B + Q_i\theta)]_{ij}$$



Online learning in Reduced Order Modelling

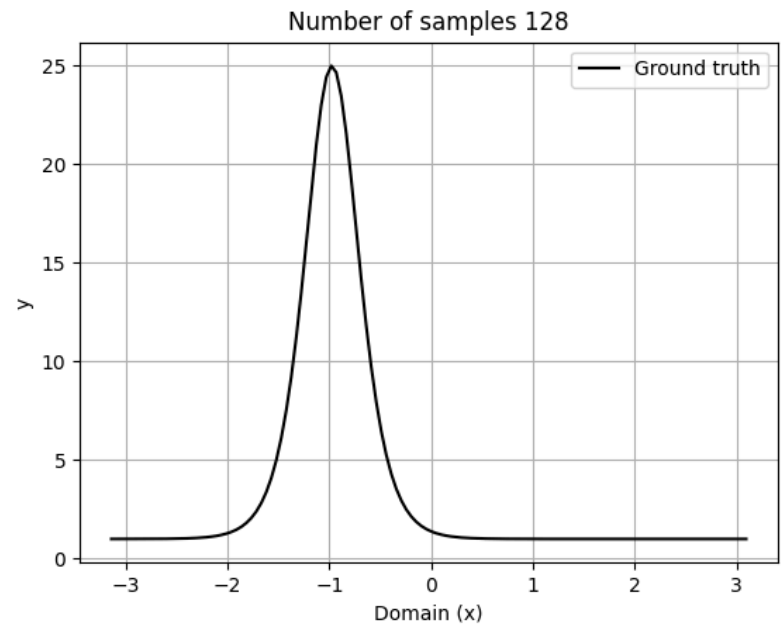
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

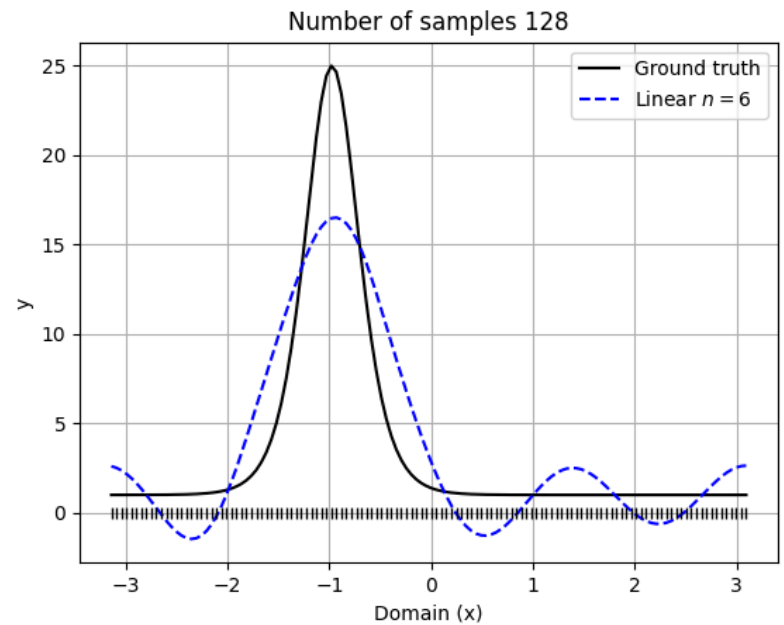
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

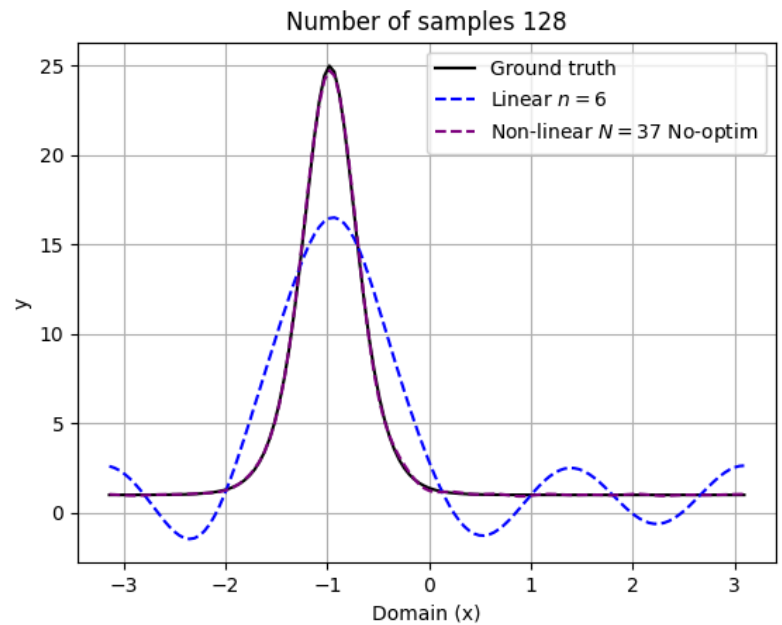
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

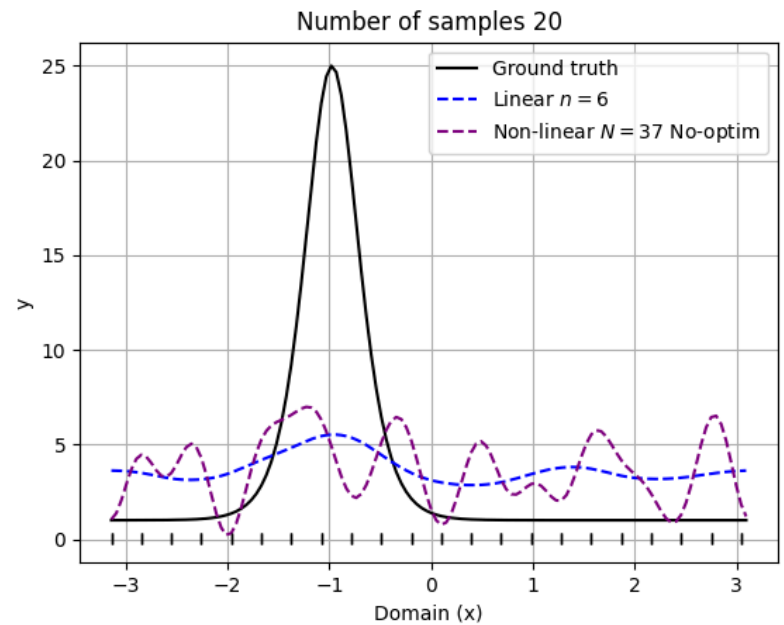
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

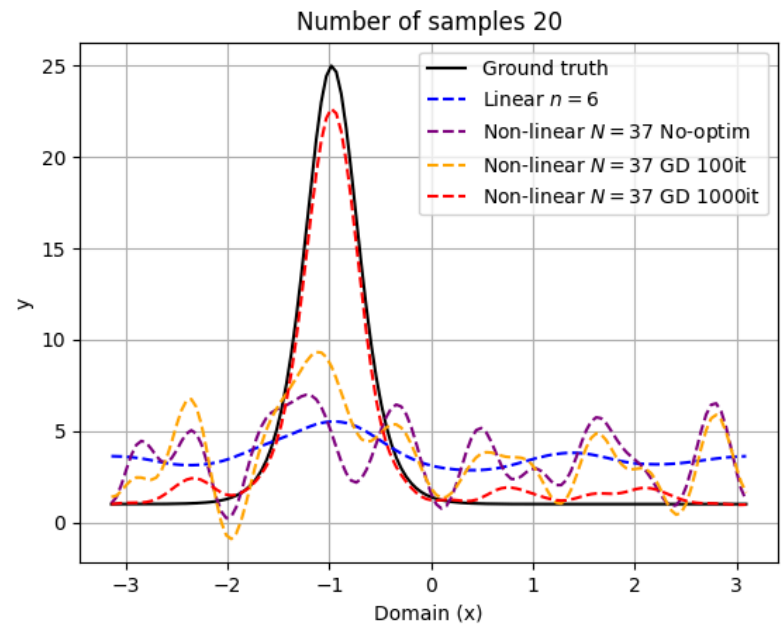
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

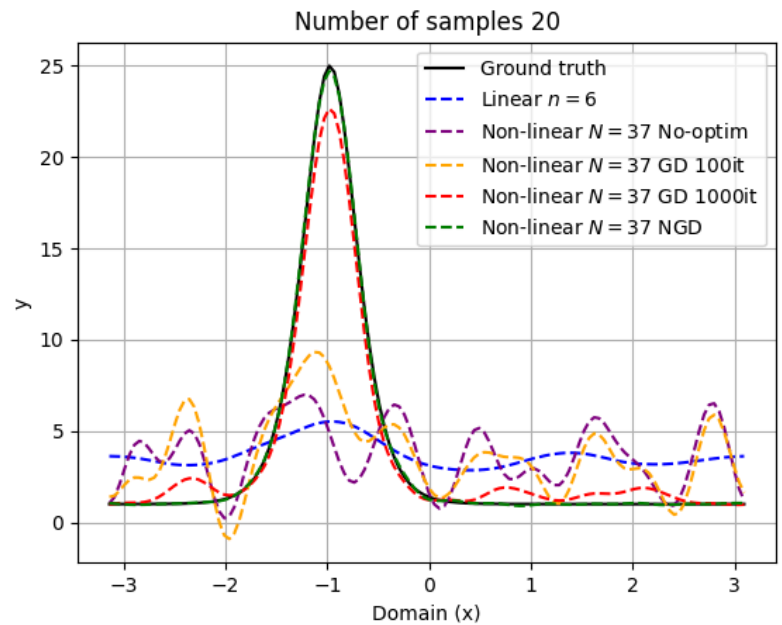
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

Korteweg-de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 4u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

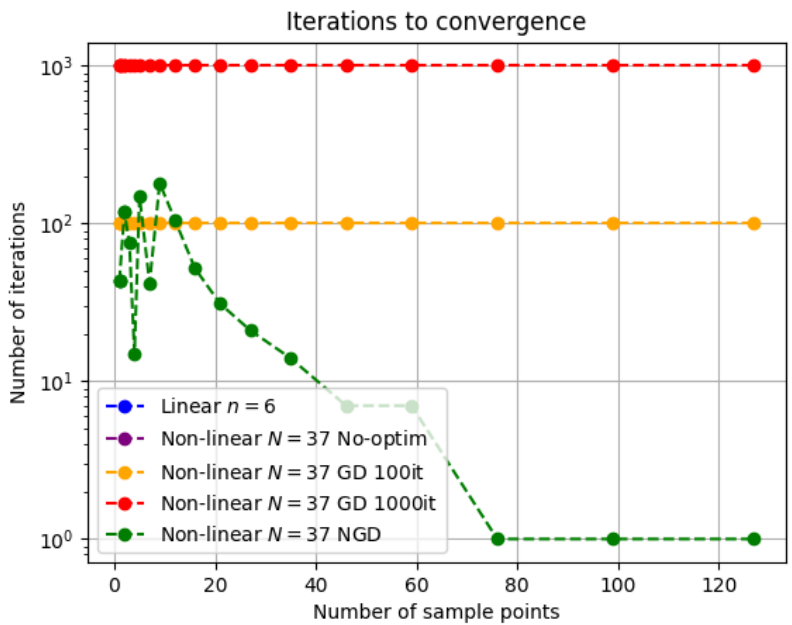
$$v_{\text{lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x)$$

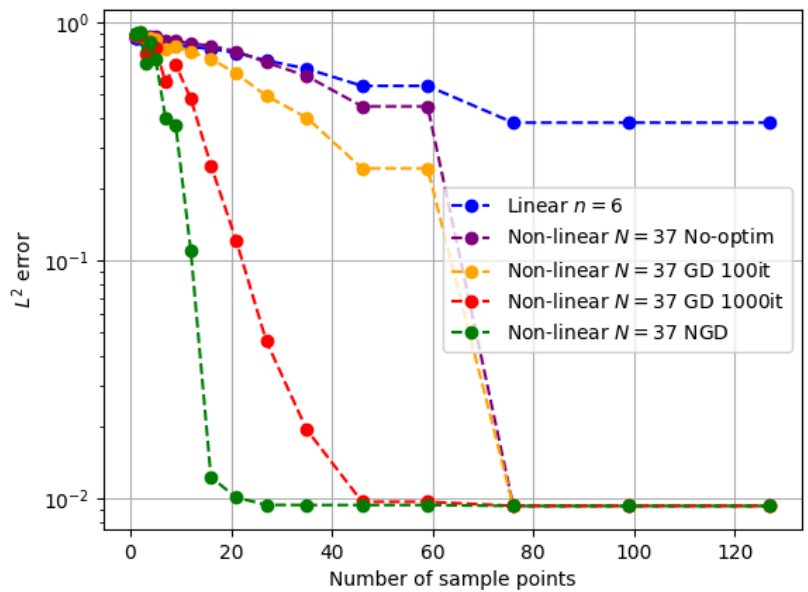
$$v_{\text{non-lin}}(x) = \sum_{i=1}^{n} \theta_i \phi_i(x) + \sum_{j=n+1}^{N} \mathcal{NN}(\theta)_j \phi_j(x)$$



Online learning in Reduced Order Modelling

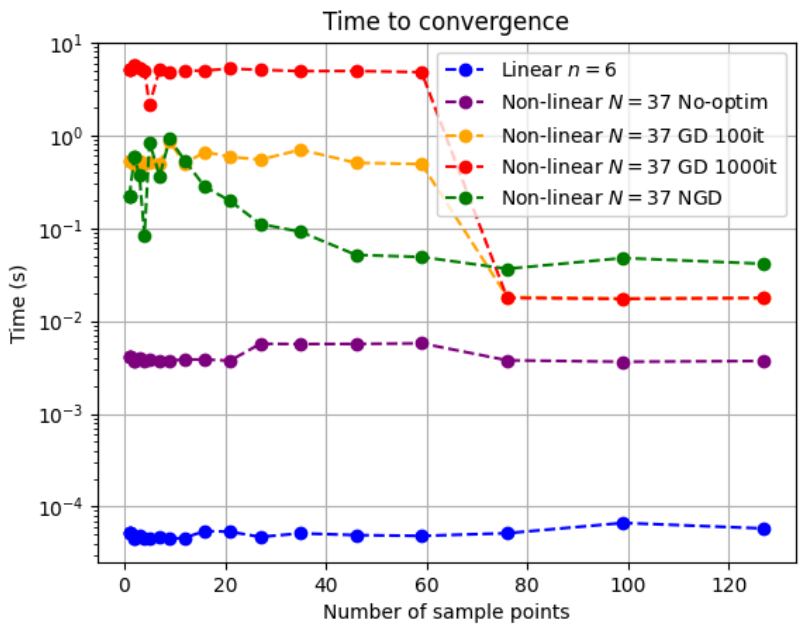
Natural Gradient for *online learning* in non-linear Reduced Order Modelling.

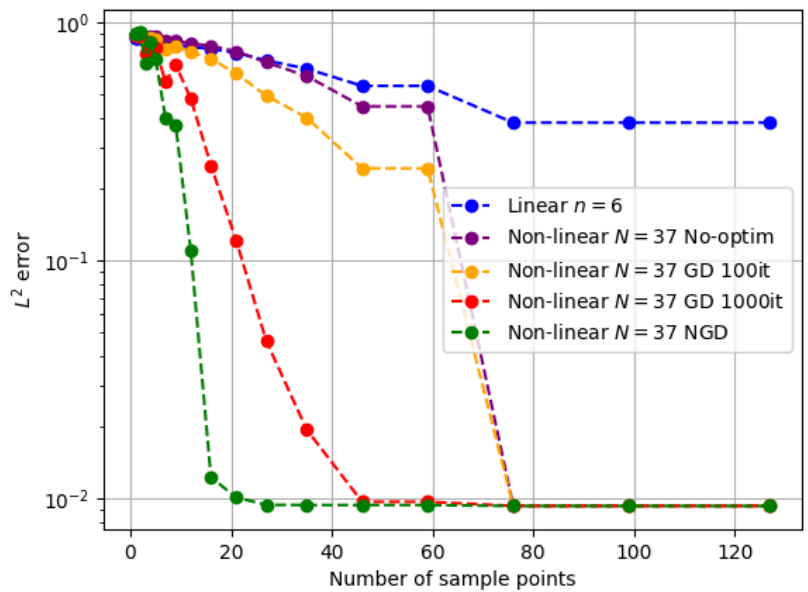




Online learning in Reduced Order Modelling

Natural Gradient for *online learning* in non-linear Reduced Order Modelling.





Why we need momentum

Using momentum to **scape local minima** and **correct the biased step** of NGD.

Beyond L^2 loss

Classification
KL-div

$$\int \left[v \log \frac{v}{u} \right](x)\mathrm{d}x$$

PDE residual

$$\begin{aligned}\mathcal{L}(v) &= \|R(v)\|^2 \\ &= \| -\epsilon \partial_{xx} v + \partial_x v - 1\|^2\end{aligned}$$

Non-linear model

$$\theta_{k+1} = \theta_k - s G_k^{-1} \nabla_\theta L(\theta_k)$$

$$\theta_s = \theta_k - \int_0^s G_k^{-1}(\theta_\ell) \nabla_\theta L(\theta_\ell) \mathrm{d}\ell$$

Inacurate estimation of update

Stochastic setting
empirical loss

$$\mathcal{L}_u(v_k) = \|u - v_k\|_m^2$$

Solving NGD system

$$G^{-1} \begin{cases} \xrightarrow{Penrose} V(\Sigma_{\sigma > \varepsilon})^{-1} V^T \\ \xrightarrow{Tikhonov} (G + \varepsilon I)^{-1} \end{cases}$$

Escape local minima

$$\frac{\partial v_\theta}{\partial s} = -P_{\mathcal{T}_k} \nabla \mathcal{L}$$

Why we need momentum

Using momentum to **scape local minima** and **correct the biased step** of NGD.

Beyond L^2 loss

Classification
KL-div

$$\int \left[v \log \frac{v}{u} \right] (x) \mathrm{d}x$$

PDE residual

$$\begin{aligned} \mathcal{L}(v) &= \|R(v)\|^2 \\ &= \| -\epsilon \partial_{xx} v + \partial_x v - 1 \|^2 \end{aligned}$$

Non-linear model

$$\theta_{k+1} = \theta_k - s G_k^{-1} \nabla_\theta L(\theta_k)$$

$$\theta_s = \theta_k - \int_0^s G_k^{-1}(\theta_\ell) \nabla_\theta L(\theta_\ell) \mathrm{d}\ell$$

Inacurate estimation of update

Stochastic setting
empirical loss

$$\mathcal{L}_u(v_k) = \|u - v_k\|_m^2$$

Solving NGD system

$$G^{-1} \begin{cases} \xrightarrow{Penrose} V(\Sigma_{\sigma > \varepsilon})^{-1} V^T \\ \xrightarrow{Tikhonov} (G + \varepsilon I)^{-1} \end{cases}$$

Escape local minima

$$\frac{\partial v_\theta}{\partial s} = -P_{\mathcal{T}_k} \nabla \mathcal{L}$$

Momentum dynamics

From gradient flow to momentum [Polyak, B.T. 1964] [Nesterov, Yurii. 1983].

$$\frac{\partial \theta}{\partial s} = -\nabla_\theta L \quad \longrightarrow \quad \frac{\partial^2 \theta}{\partial s^2} = -\gamma \frac{\partial \theta}{\partial s} - \nabla_\theta L$$

Momentum dynamics

From gradient flow to momentum [Polyak, B.T. 1964] [Nesterov, Yurii. 1983].

$$\frac{\partial \theta}{\partial s} = -\nabla_\theta L \longrightarrow \frac{\partial^2 \theta}{\partial s^2} = -\gamma \frac{\partial \theta}{\partial s} - \nabla_\theta L$$

Heavy-ball

$$\begin{aligned} \theta_{k+1} &= \theta_k + p_k \\ p_k &= \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

Momentum dynamics

From gradient flow to momentum [Polyak, B.T. 1964] [Nesterov, Yurii. 1983].

$$\frac{\partial \theta}{\partial s} = -\nabla_\theta L \longrightarrow \frac{\partial^2 \theta}{\partial s^2} = -\gamma \frac{\partial \theta}{\partial s} - \nabla_\theta L$$

Heavy-ball

$$\begin{aligned} \theta_{k+1} &= \theta_k + p_k \\ p_k &= \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

Nesterov

$$\begin{aligned} y_k &= \theta_k + \beta(\theta_k - \theta_{k-1}) \\ \theta_{k+1} &= y_k - \alpha \nabla_\theta L(y_k) \end{aligned}$$

Momentum dynamics

From gradient flow to momentum [Polyak, B.T. 1964] [Nesterov, Yurii. 1983].

$$\frac{\partial \theta}{\partial s} = -\nabla_\theta L \quad \longrightarrow \quad \frac{\partial^2 \theta}{\partial s^2} = -\gamma \frac{\partial \theta}{\partial s} - \nabla_\theta L$$

Heavy-ball

$$\begin{aligned} \theta_{k+1} &= \theta_k + p_k \\ p_k &= \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

Nesterov

$$\begin{aligned} y_k &= \theta_k + \beta(\theta_k - \theta_{k-1}) \\ \theta_{k+1} &= y_k - \alpha \nabla_\theta L(y_k) \end{aligned}$$

$$\begin{aligned} \theta_{k+1} &= \theta_k + \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \\ &= \theta_k + \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

Momentum dynamics

From gradient flow to momentum [Polyak, B.T. 1964] [Nesterov, Yurii. 1983].

$$\frac{\partial \theta}{\partial s} = -\nabla_\theta L \longrightarrow \frac{\partial^2 \theta}{\partial s^2} = -\gamma \frac{\partial \theta}{\partial s} - \nabla_\theta L$$

Heavy-ball

$$\begin{aligned} \theta_{k+1} &= \theta_k + p_k \\ p_k &= \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

$$\begin{aligned} \theta_{k+1} &= \theta_k + \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k) \\ &= \theta_k + \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k) \end{aligned}$$

Nesterov

$$\begin{aligned} y_k &= \theta_k + \beta(\theta_k - \theta_{k-1}) \\ \theta_{k+1} &= y_k - \alpha \nabla_\theta L(y_k) \end{aligned}$$

$$\theta_{k+1} = \theta_k + \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(y_k)$$

Momentum dynamics in functional space

From momentum in parameter space to functional space.

Heavy-ball: "momentum variable"

$$\theta_{k+1} - \theta_k = \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k)$$

Heavy-ball: iterates difference

$$\theta_{k+1} - \theta_k = \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k)$$

Momentum dynamics in functional space

From momentum in parameter space to functional space.

Heavy-ball: "momentum variable"

$$\theta_{k+1} - \theta_k = \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta p_{k-1} - \alpha \nabla \mathcal{L}(v_k)]$$

Heavy-ball: iterates difference

$$\theta_{k+1} - \theta_k = \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta(v_k - v_{k-1}) - \alpha \nabla \mathcal{L}(v_k)]$$

Momentum dynamics in functional space

From momentum in parameter space to functional space.

Heavy-ball: "momentum variable"

$$\theta_{k+1} - \theta_k = \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta p_{k-1} - \alpha \nabla \mathcal{L}(v_k)]$$

$$P_{\mathcal{T}_k} \nabla \mathcal{L}(v_k)$$

$$P_{\mathcal{T}_k} p_{k-1}$$

Heavy-ball: iterates difference

$$\theta_{k+1} - \theta_k = \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta(v_k - v_{k-1}) - \alpha \nabla \mathcal{L}(v_k)]$$

$$P_{\mathcal{T}_k}(v_k - v_{k-1})$$

Momentum dynamics in functional space

From momentum in parameter space to functional space.

Heavy-ball: "momentum variable"

$$\theta_{k+1} - \theta_k = \beta p_{k-1} - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta p_{k-1} - \alpha \nabla \mathcal{L}(v_k)]$$

$$P_{\mathcal{T}_k} \nabla \mathcal{L}(v_k)$$

$$G_k^{-1} \nabla_\theta \mathcal{L}(\theta_k)$$

$$P_{\mathcal{T}_k} p_{k-1}$$

$$G_k^{-1} G_{k,k-1} p_{k-1}$$

$$G_{k,k-1} = \int \left[\left.\frac{\partial A}{\partial \theta}\right|_{\theta_k} \left.\frac{\partial A}{\partial \theta}\right|_{\theta_{k-1}} \right](x)\mathrm{d}x$$

Heavy-ball: iterates difference

$$\theta_{k+1} - \theta_k = \beta(\theta_k - \theta_{k-1}) - \alpha \nabla_\theta L(\theta_k)$$

$$\mathcal{T}_k \ni \mathrm{d}v_k = P_{\mathcal{T}_k}[\beta(v_k - v_{k-1}) - \alpha \nabla \mathcal{L}(v_k)]$$

$$P_{\mathcal{T}_k}(v_k - v_{k-1})$$

$$G_k^{-1} \int \left[\left.\frac{\partial A}{\partial \theta}\right|_{\theta_k} (v_k - v_{k-1}) \right](x)\mathrm{d}x$$

Toy example

Escaping local minima.

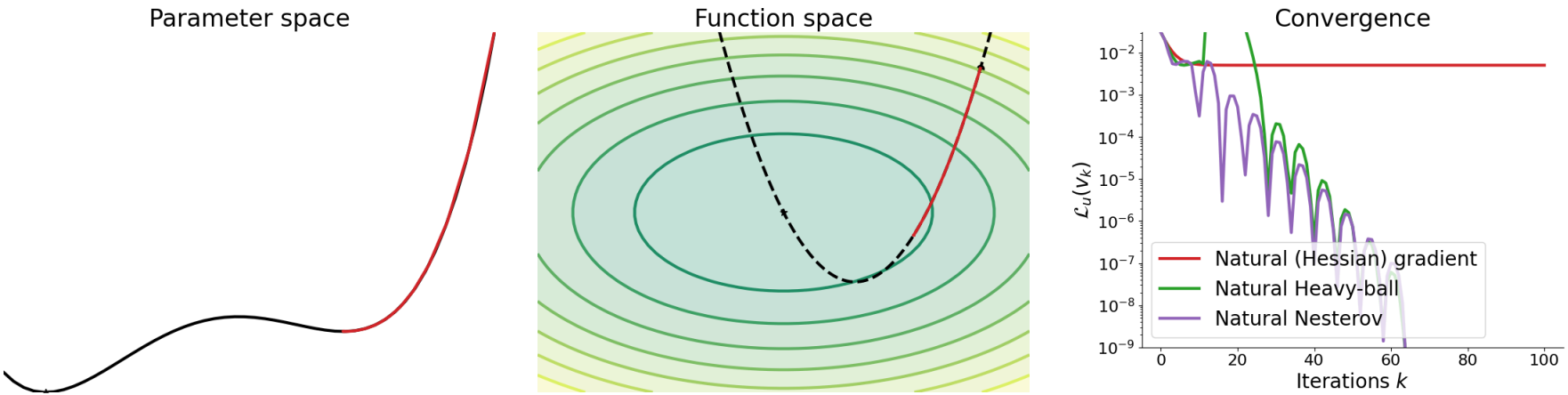
$$u \in L^2([0,1])$$

$$\mathcal{L}_u(v) = \frac{1}{2}\|u - v\|_K^2$$

$$v_\theta(x) = \theta_1 b^T \Phi(x) + \theta_1^2 b^{\perp T} \Phi(x)$$

$$\mathrm{d}v_k^{HB} = P_{\mathcal{T}_k}[\beta p_{k-1} - \alpha \nabla \mathcal{L}(v_k)]$$

$$\mathrm{d}v_k^{N} = P_{\mathcal{T}_k}[\beta(v_k - v_{k-1}) - \alpha \nabla \mathcal{L}(y_k)]$$



Physics informed learning.

Reaction diffusion (PINNs)
[Müller J., Zeinhofer M. (2023)]
[Schwencke N., Furtlehner C. (2024)].

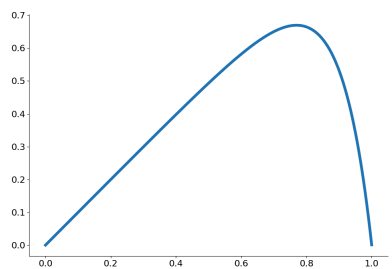
$$v(x) = x(1-x)\,\mathcal{NN}_\theta(x)$$

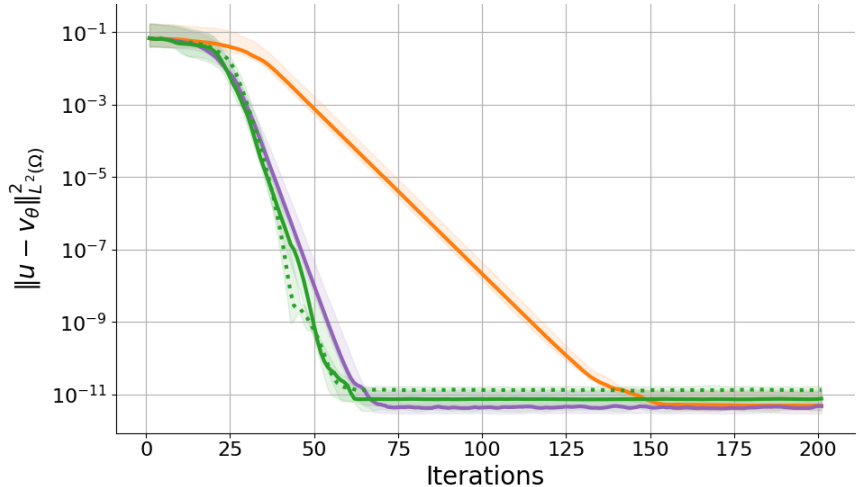
$$\mathcal{L}(v) = \| -\epsilon\partial_{xx}v + \partial_x v - 1\|^2_{L^2([0,1])}$$

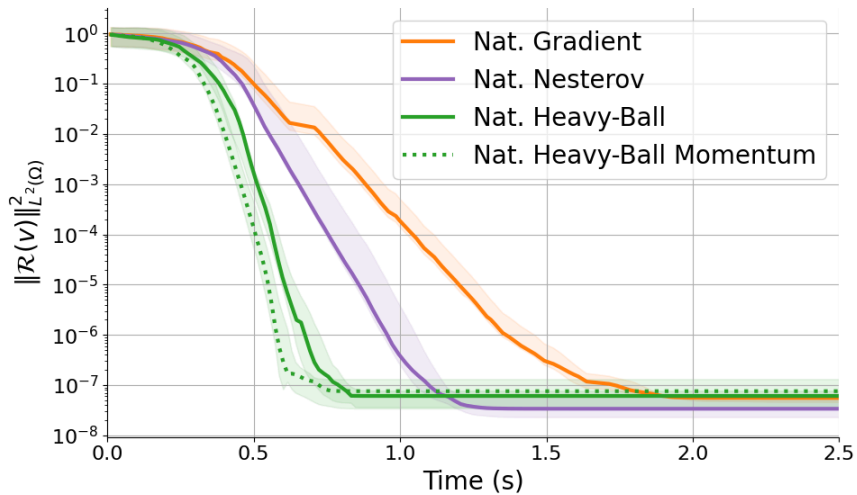
$$\mathcal{L}(v) = \|\mathcal{R}(v)\|^2_{L^2([0,1])}$$

$$\mathcal{L}(v') = \|v'\|^2_{L^2([0,1])}$$

$$v' \in \mathcal{M}' := \{v' = \mathcal{R}(v)\}$$







Physics informed learning.

Non-linear PDE (PINNs)
[Müller J., Zeinhofer M. (2023)]
[Schwencke N., Furtlehner C. (2024)].

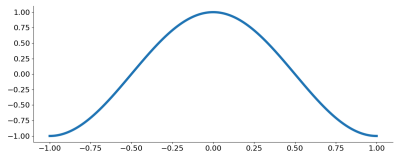
$$v(x) = (1-x)^2(x-1)^2\,\mathcal{NN}_\theta(x) + c + a\left(\frac{x^3}{3} - x\right)$$

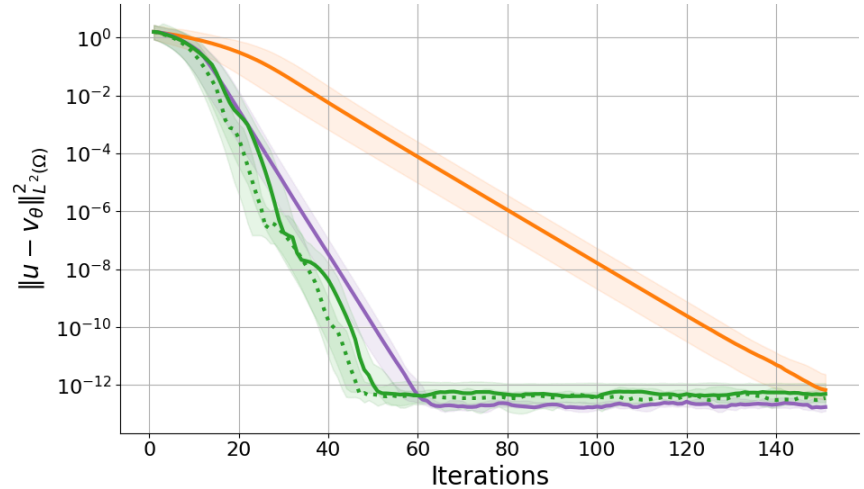
$$\mathcal{L}(v) = \| -\partial_{xx} v + v^3 - \pi^2\cos(\pi x) + \cos^3(\pi x)\|^2$$

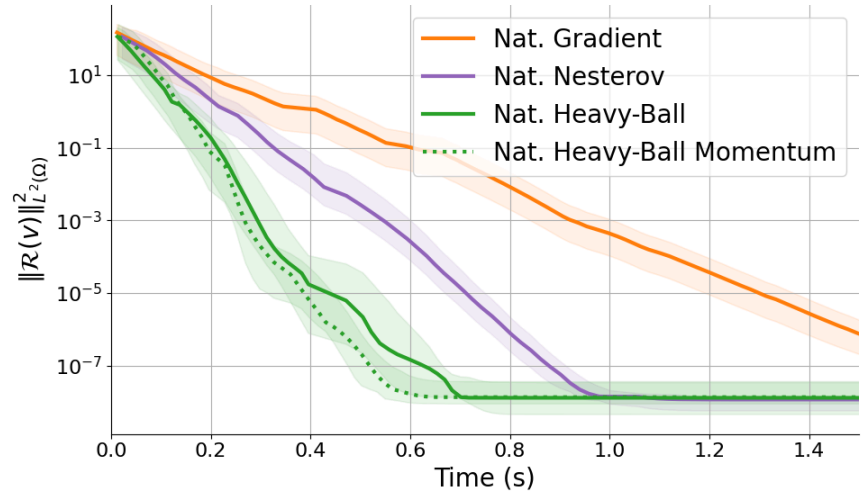
$$\mathcal{L}(v) = \|\mathcal{R}(v)\|^2_{L^2([-1,1])}$$

$$\mathcal{L}(v') = \|v'\|^2_{L^2([-1,1])}$$

$$v' \in \mathcal{M}' := \{v' = \mathcal{R}(v)\}$$







Classification task

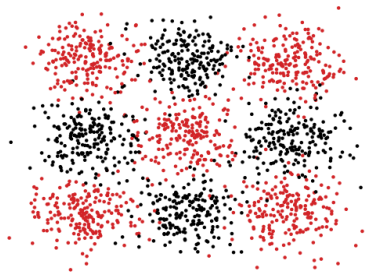
Another less toy example from [Park, H, S.-I Amari, and K Fukumizu (2000)].

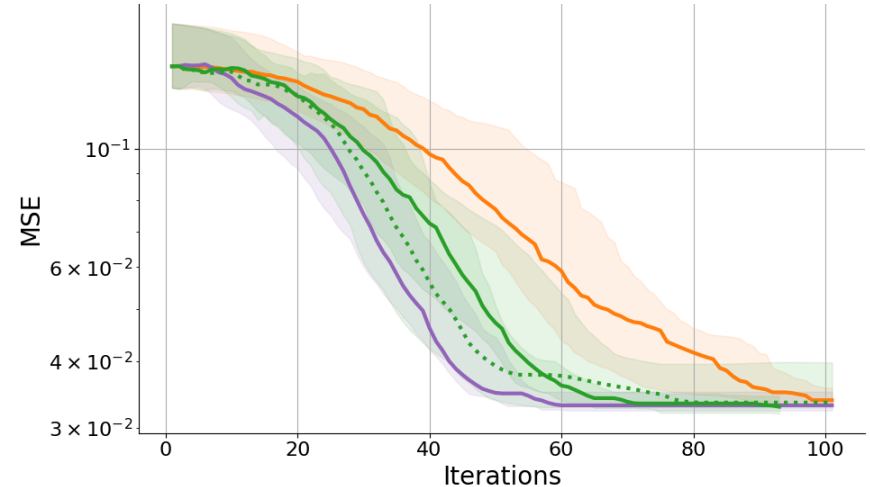
Mackey Glass caotic time series:

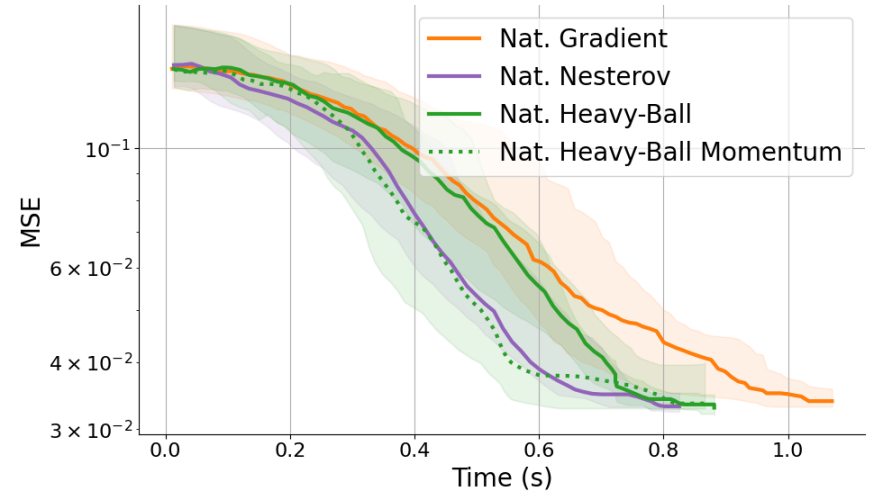
- Input: $z \in \mathbb{R}^2$
- Output: $c \in \{0, 1\}$

$$\mathcal{L}_u(v) = \int -u \log(v) - (1-u)\log(1-v)$$

Model (Shallow NN): $v_\theta : \mathbb{R}^2 \to [0, 1]$







Conclusion

What is done, what is still to be done.

We saw how to build a **Natural inertial dynamics** for gradient optimization on non linear approximation manifolds.

Some directions

- Step-size adaptivity
- Theoretical questions: convergence? optimality?

Thank you!

(2024) Schwencke N., Furtlehner C.	
(2023) Müller J., Zeinhofer M.	NGD and PINNs.
(2024) R. Gruhlke, A. Nouy, P. Trunschke.	NGD and optimal sampling.
(2020) J. Martens.	NGD Review.
(2000) H. Park, S. Amari, K. Fukumizu.	NGD Experiments.
(1998) S. Amari.	NGD Initial paper.
(1983) Y. Nesterov.	Nestorov acceleration.
(1964) B.T. Polyak.	Heavy Ball acceleration.

Powered by Slidev