Non-linear inverse problems applications to image reconstruction

Séminaire de Mathématiques Appliquées du LMJL

Agustín Somacal

Collaborators: Albert Cohen, Matthieu Dolbeault, Olga Mula

Inverse problem

$u \in V$ Banach

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Parametric PDEs

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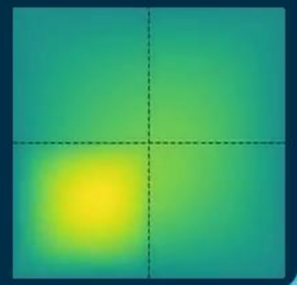

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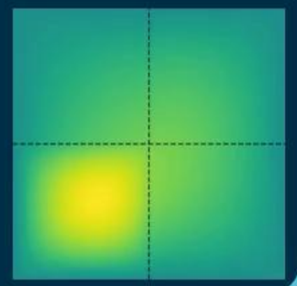

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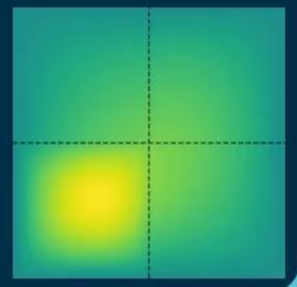

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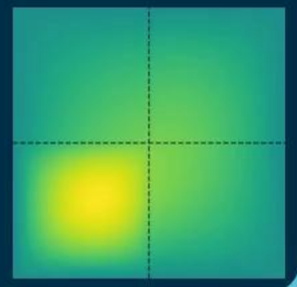

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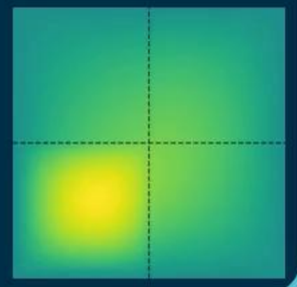

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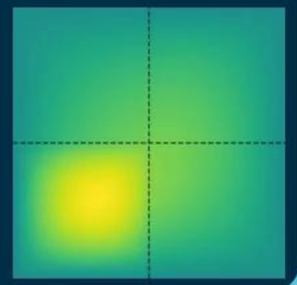

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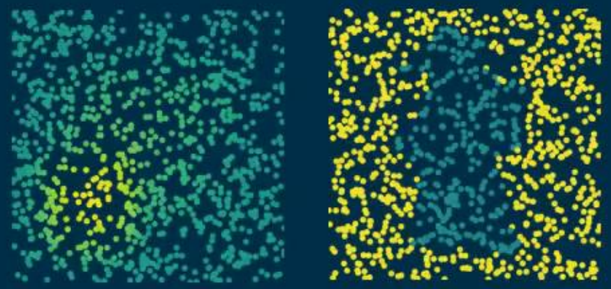

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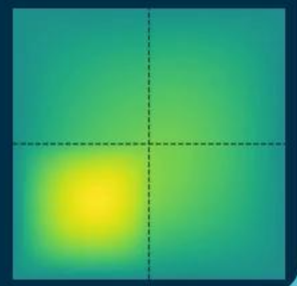

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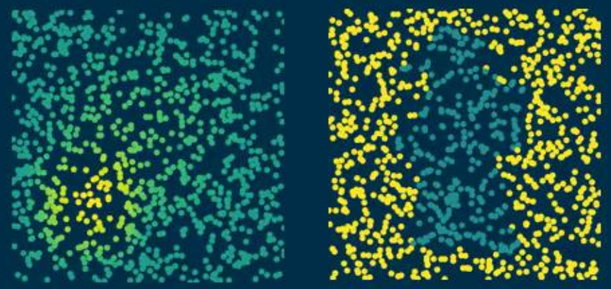

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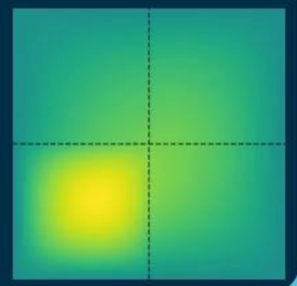

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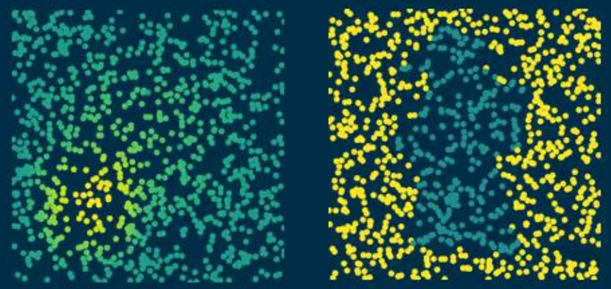

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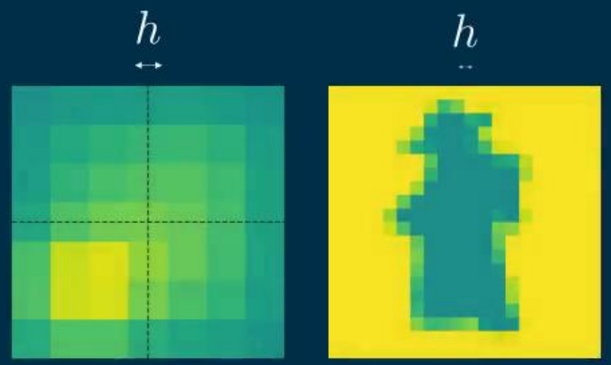

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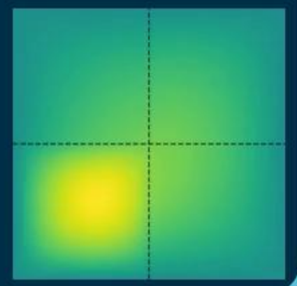

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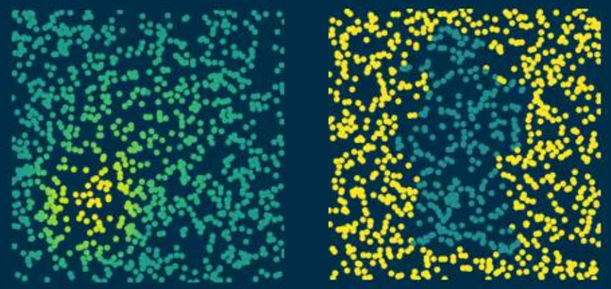

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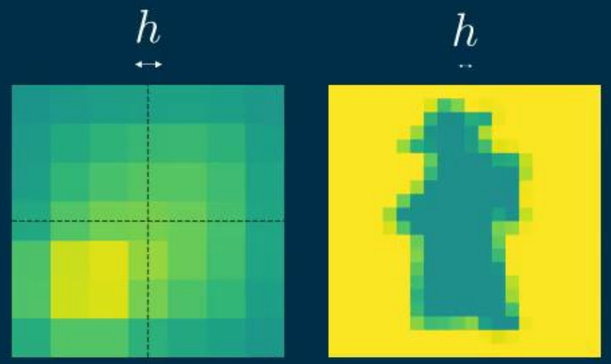

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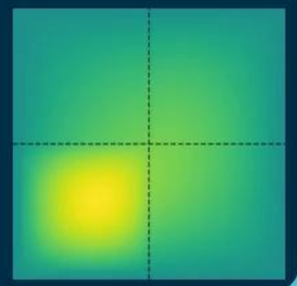

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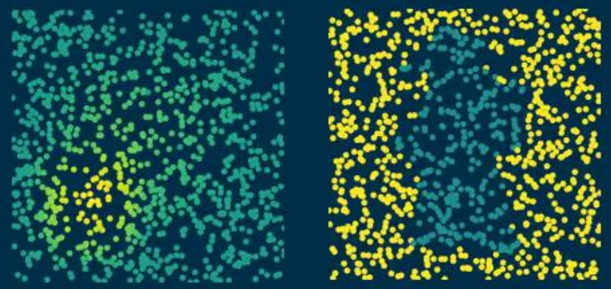

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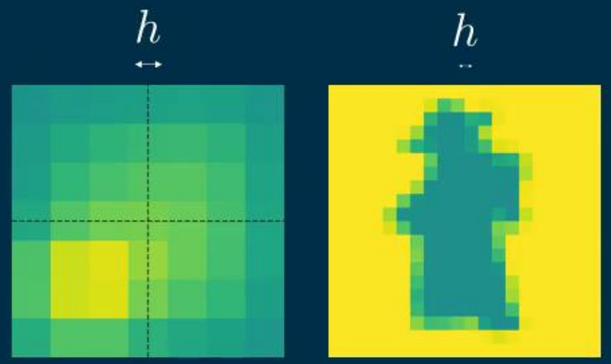

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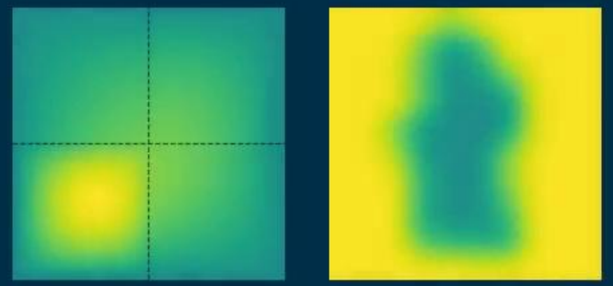

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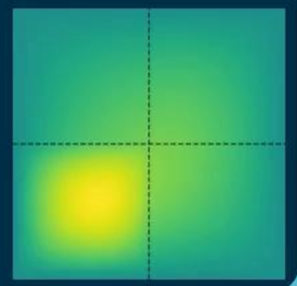

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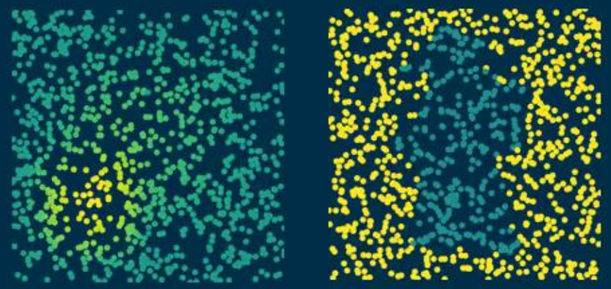

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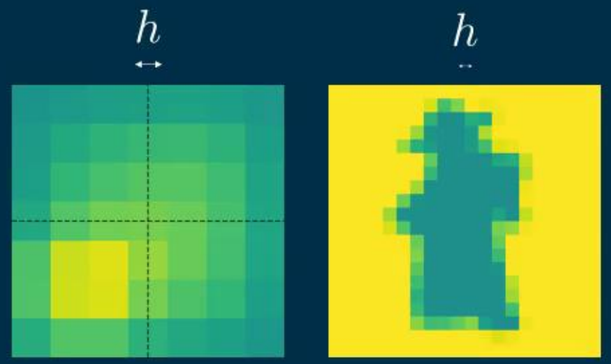

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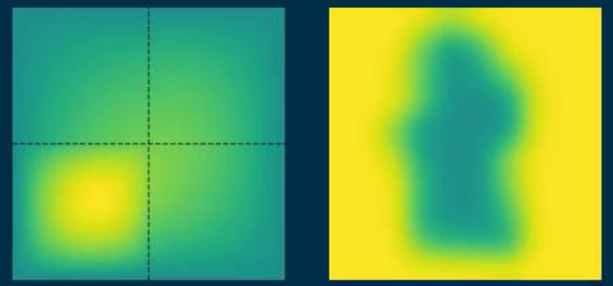

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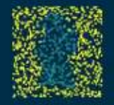

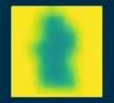

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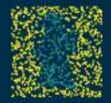

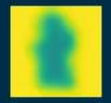

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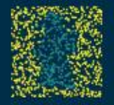

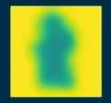

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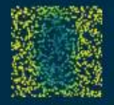

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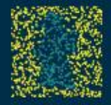

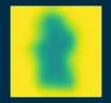

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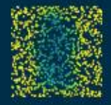

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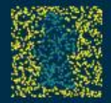

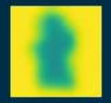

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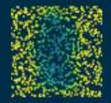

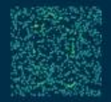

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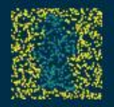

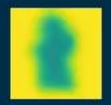

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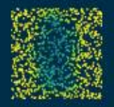
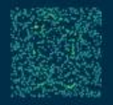
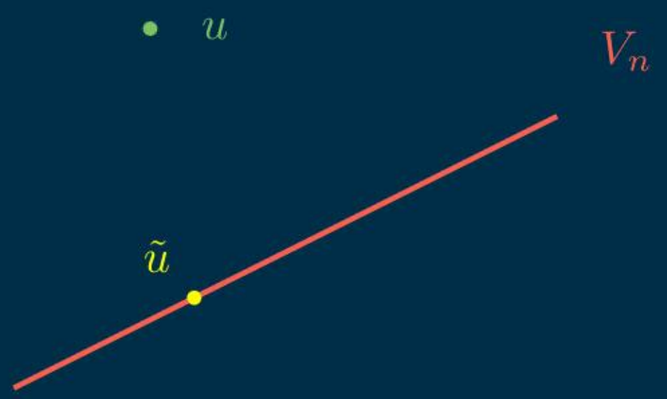
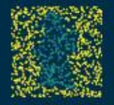




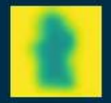


Near optimality property

$$\ell(u) + \eta = z \in \mathbb{R}^m$$

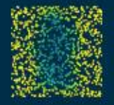


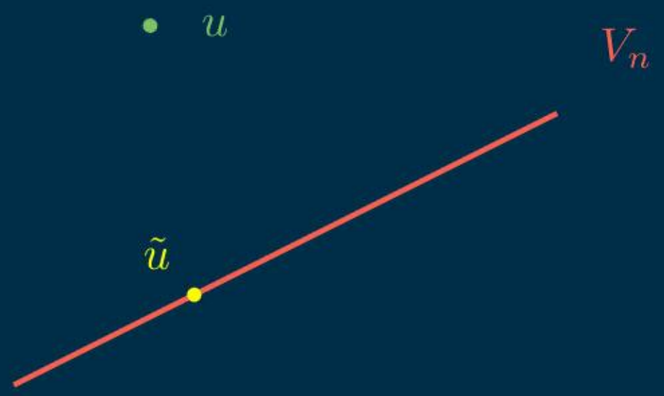
$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$



$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$





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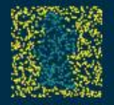
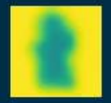


$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$



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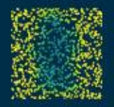

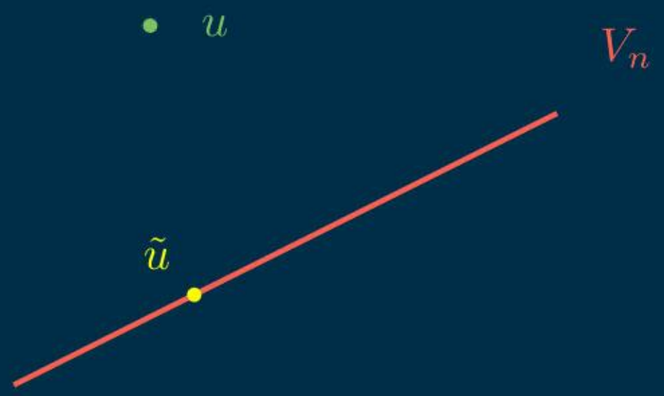

Near optimality property

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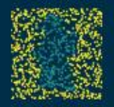

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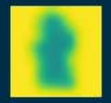

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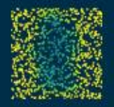

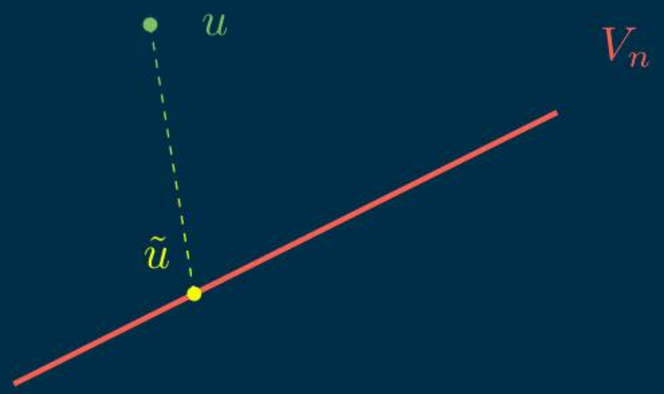

Near optimality property

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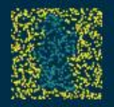

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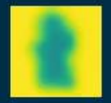

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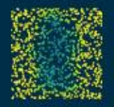

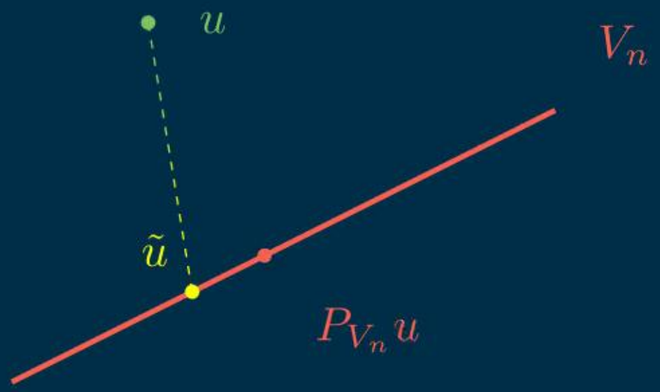

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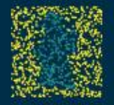

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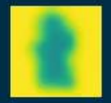

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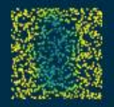

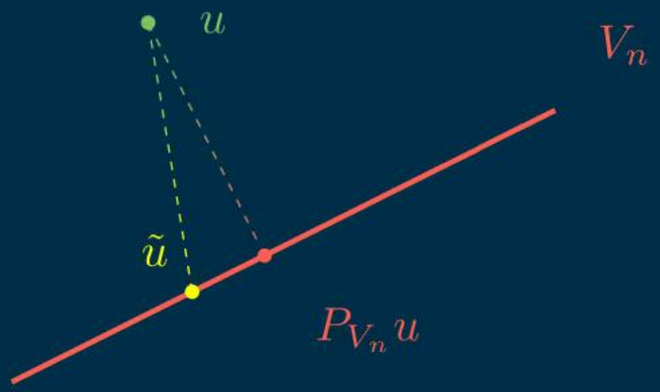

Near optimality property

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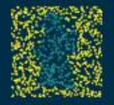

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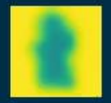

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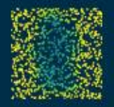
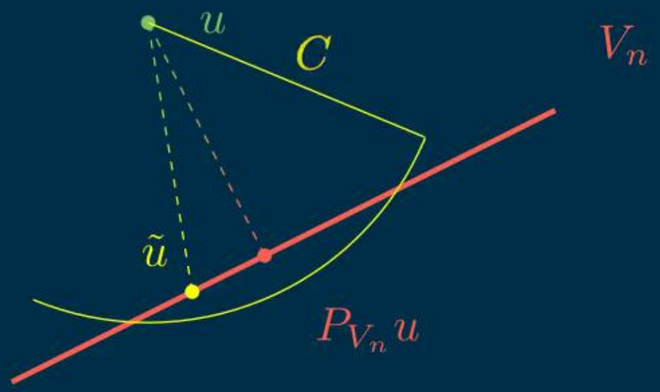
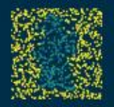




Parameterized Background Data Weak

$$\ell(u) + \eta = z \in \mathbb{R}^m$$



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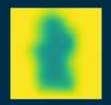

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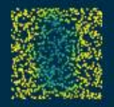
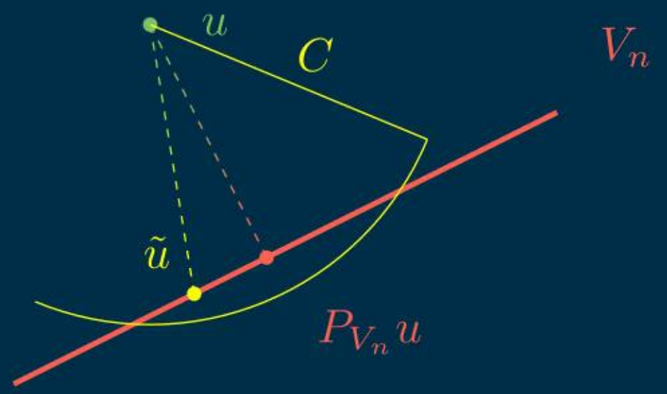
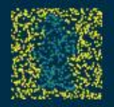


- V Hilbert space.



Parameterized Background Data Weak

$$\ell(u) + \eta = z \in \mathbb{R}^m$$



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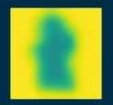

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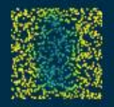

- V Hilbert space.
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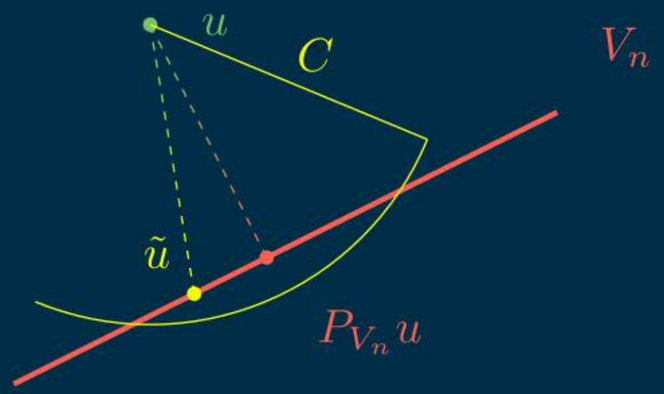

Parameterized Background Data Weak

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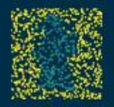

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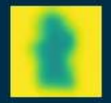

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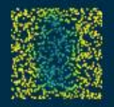

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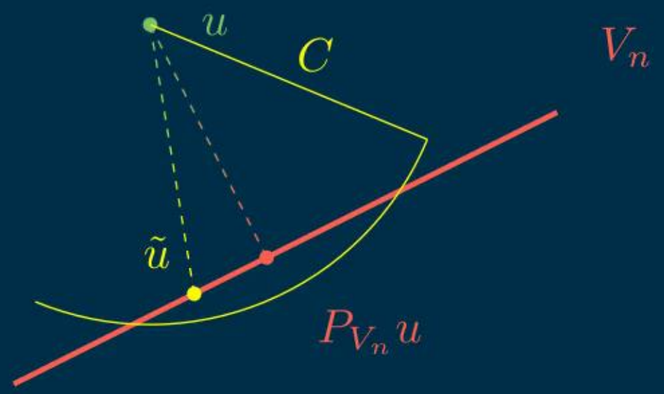

Parameterized Background Data Weak

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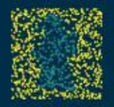

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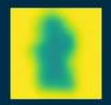

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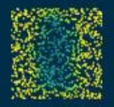

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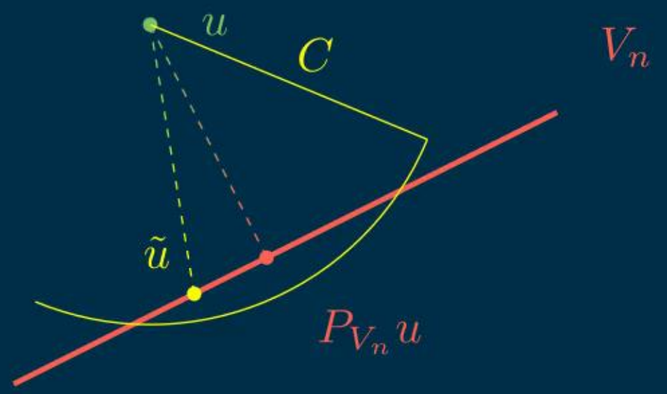
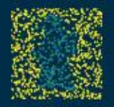


Parameterized Background Data Weak

$$\ell(u) + \eta = z \in \mathbb{R}^m$$



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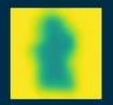

$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

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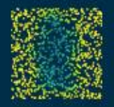

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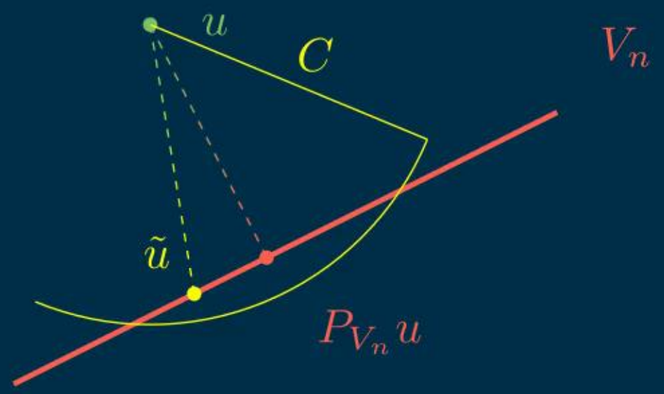
$w_j \in V$ the Riesz representers of ℓ_j

$$W := \operatorname{span}\{w_1, \ldots, w_m\}$$

$$w = P_W u$$



Parameterized Background Data Weak

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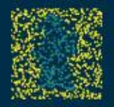

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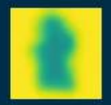

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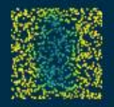

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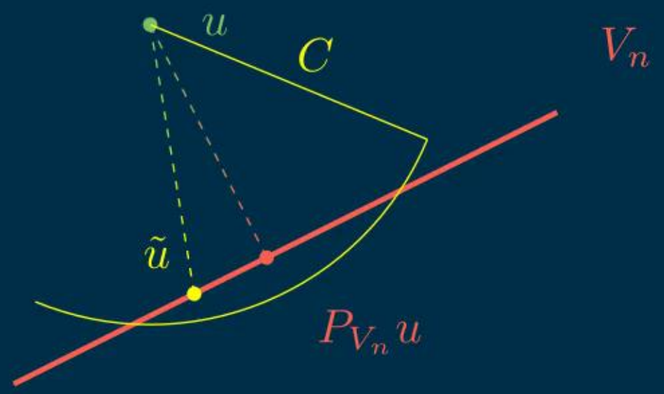

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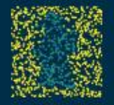

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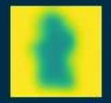

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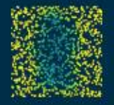

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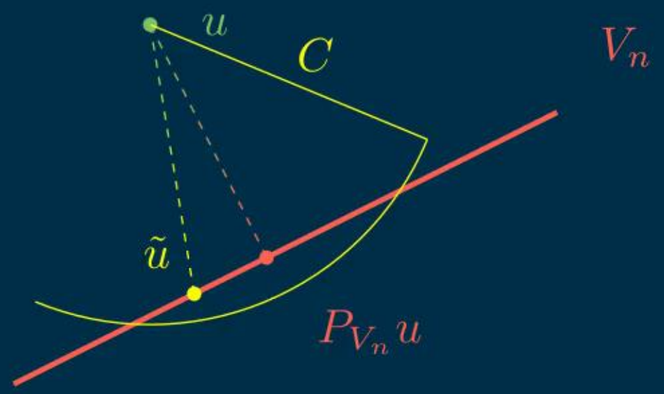

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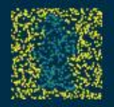

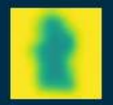

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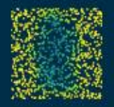

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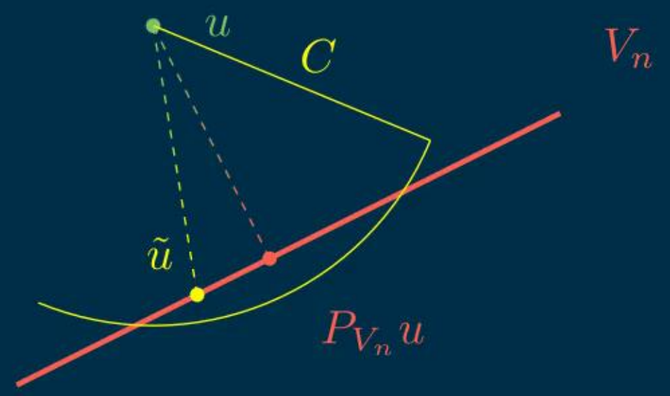

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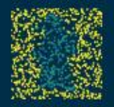

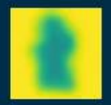

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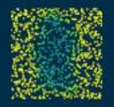

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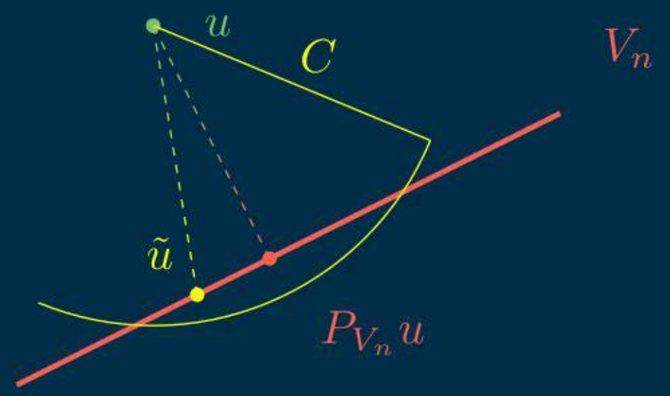

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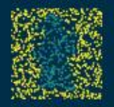

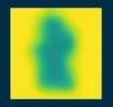

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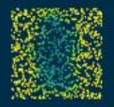

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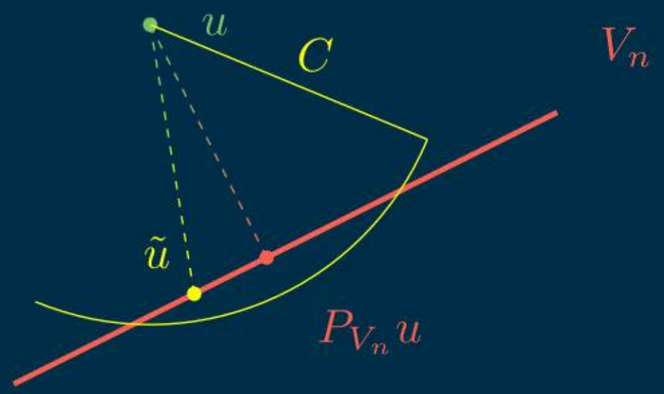

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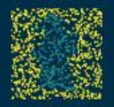

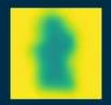

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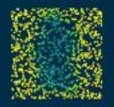

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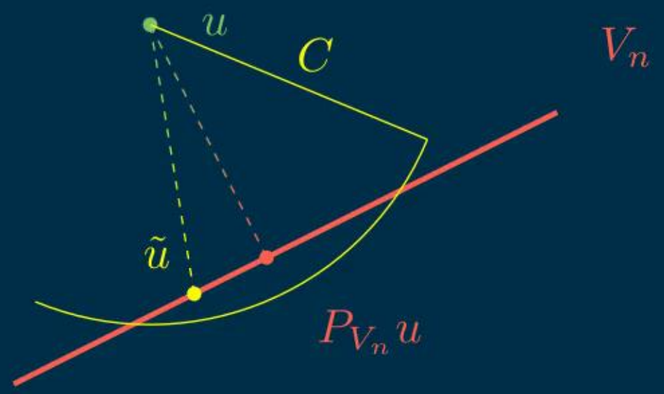

Towards non-linearity

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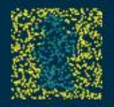

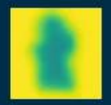

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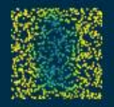

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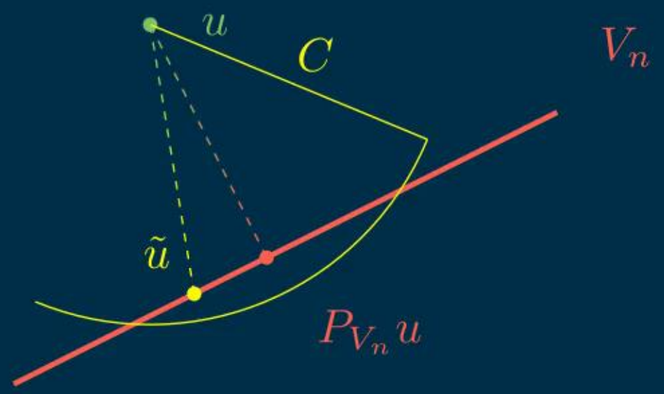

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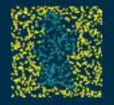

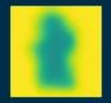

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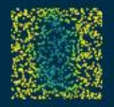

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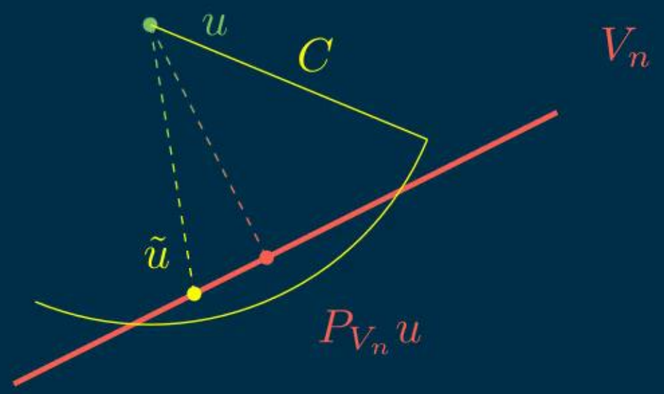

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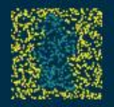

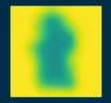

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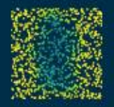

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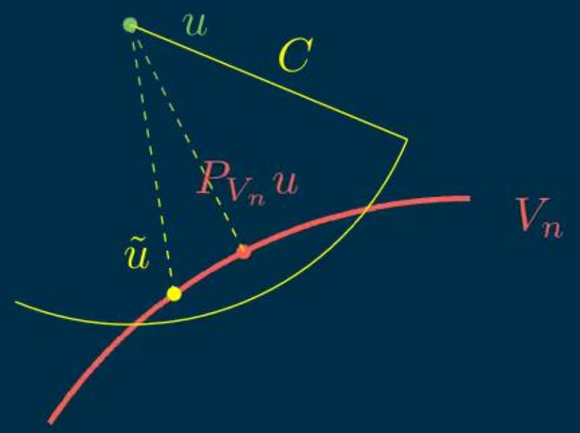

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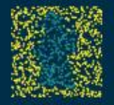

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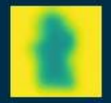

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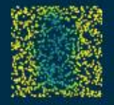

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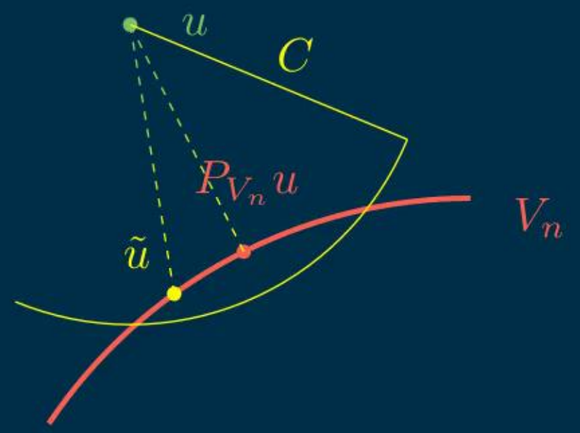
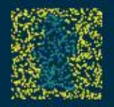
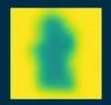


Sufficient conditions

$$\ell(u) + \eta = z \in \mathbb{R}^m$$



$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$



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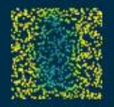

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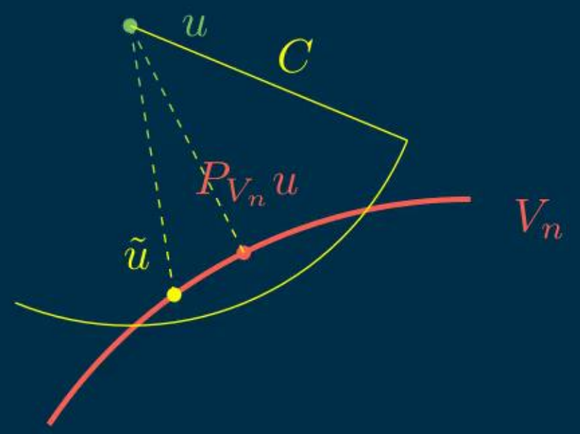
Lipschitz continuity

Inverse stability



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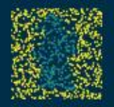

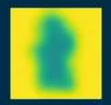

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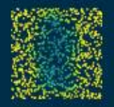

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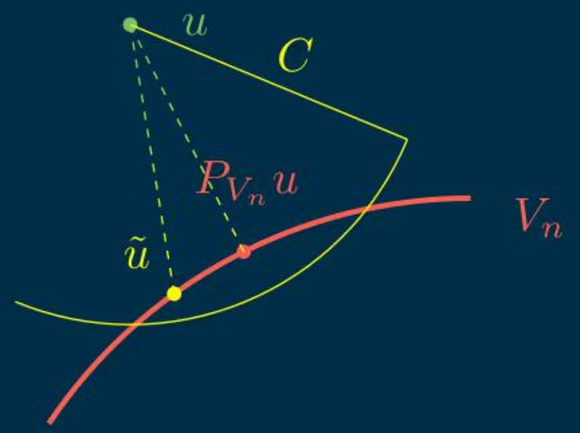

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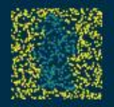

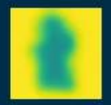

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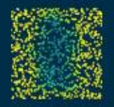

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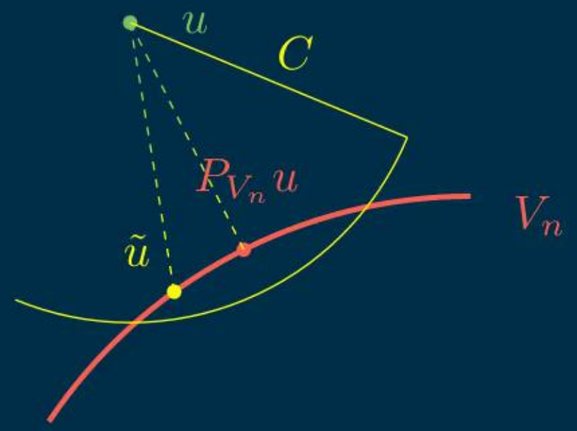

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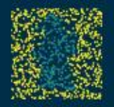

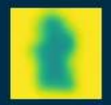

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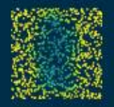

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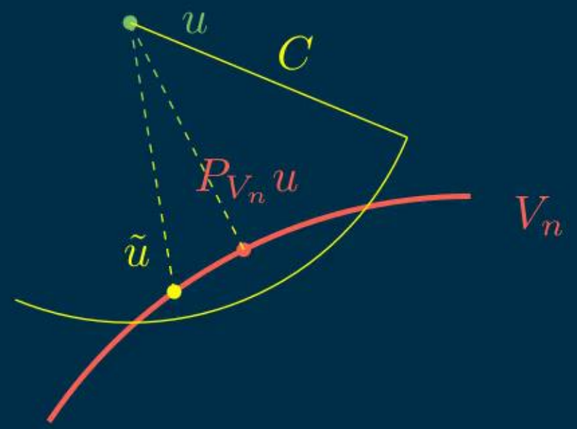

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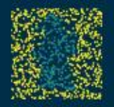

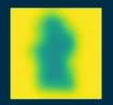

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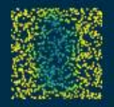

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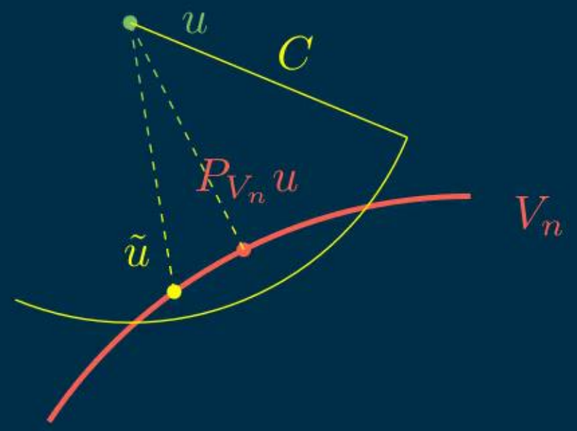

Near optimality property for non-linear spaces

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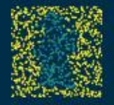

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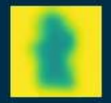

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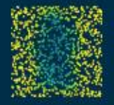

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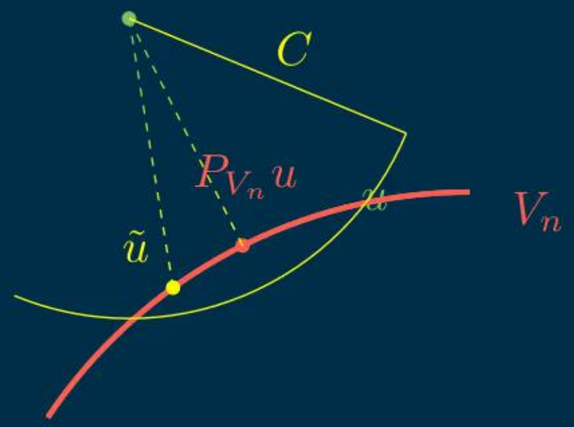

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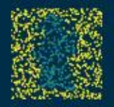

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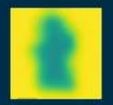

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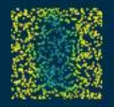

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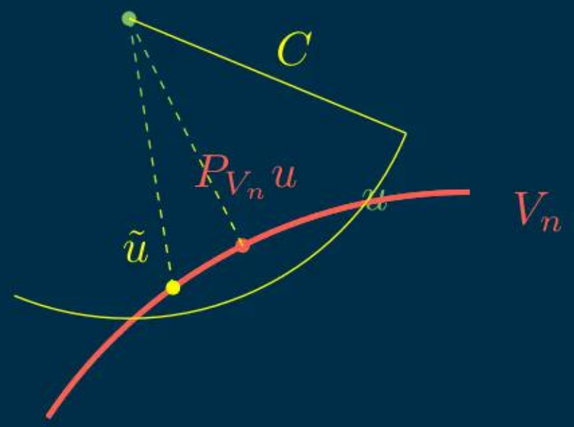

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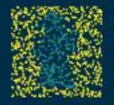

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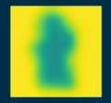

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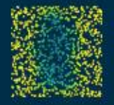

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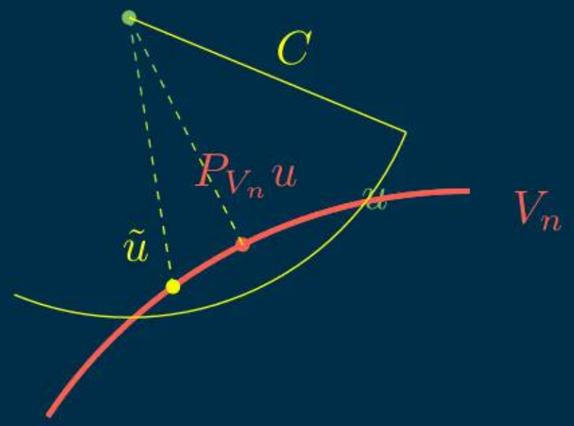

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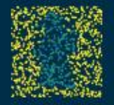

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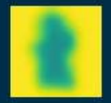

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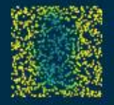

$$\overline{\|u - \tilde{u}\|_V} \leq \|u - v\| + \mu_Z \underline{\|\ell(v) - \ell(\tilde{u})\|_Z}$$

$$\overline{\|\ell(v) - \ell(\tilde{u})\|_Z} \leq \|z - \ell(v)\|_Z + \|z - \ell(\tilde{u})\|_Z$$

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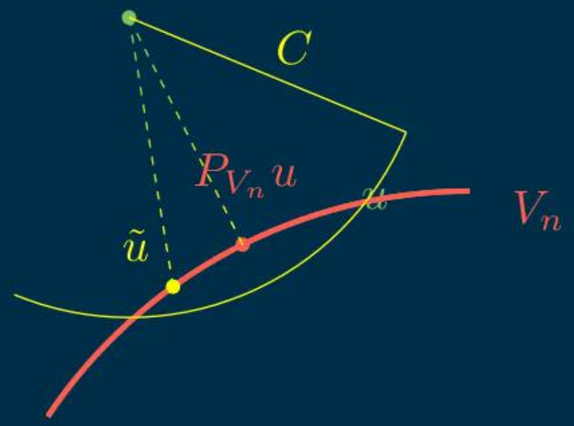

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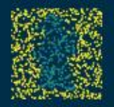
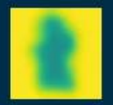


$$\underline{\|u - \tilde{u}\|_V} \leq C\|u - P_{V_n} u\|_V$$



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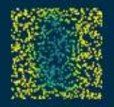

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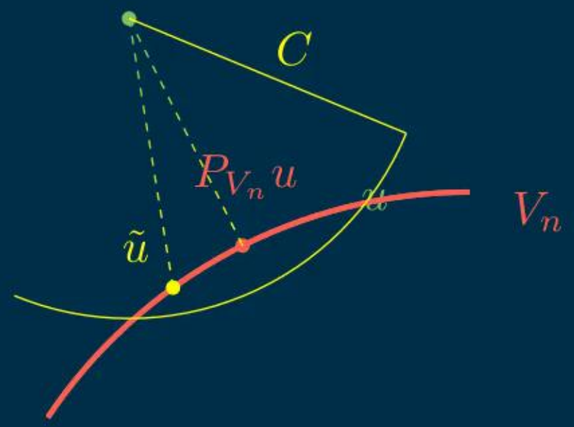

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Near optimality property for non-linear spaces

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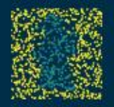

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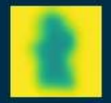

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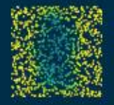

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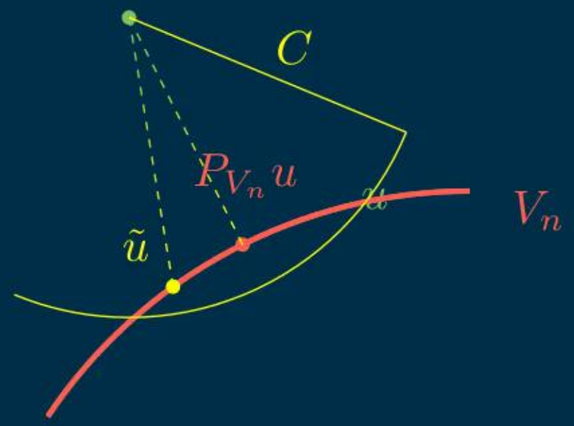

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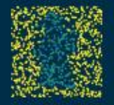

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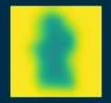

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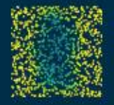

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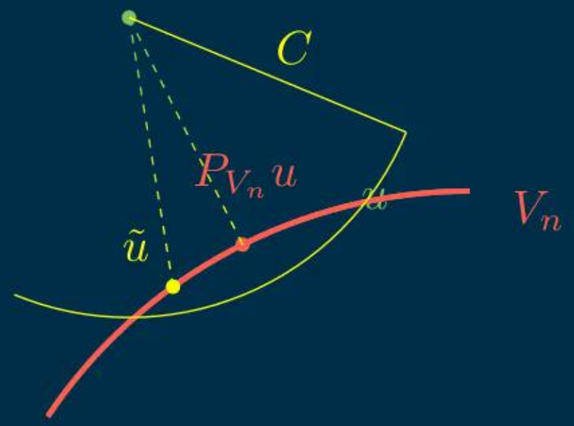

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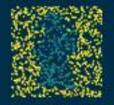
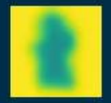


$$\underline{\|u - \tilde{u}\|_V} \leq (1 + 2\alpha_Z \mu_Z)\|u - P_{V_n} u\|_V + (1 + 2\beta\mu_Z)\|\eta\|_p$$

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]



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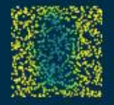

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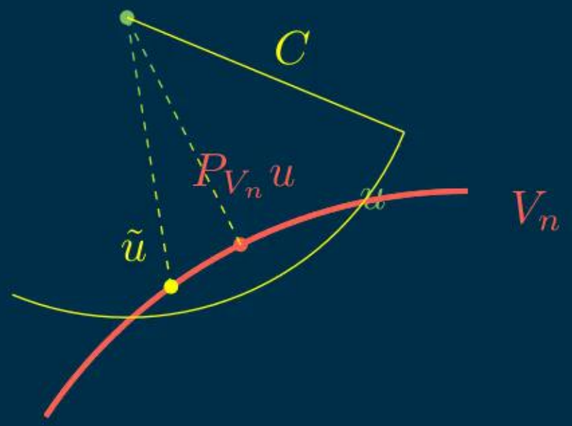

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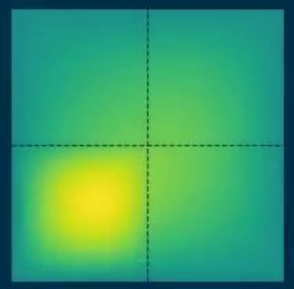
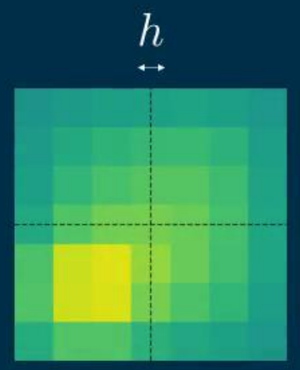
Interface reconstruction from cell averages



Interface reconstruction from cell averages

$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$



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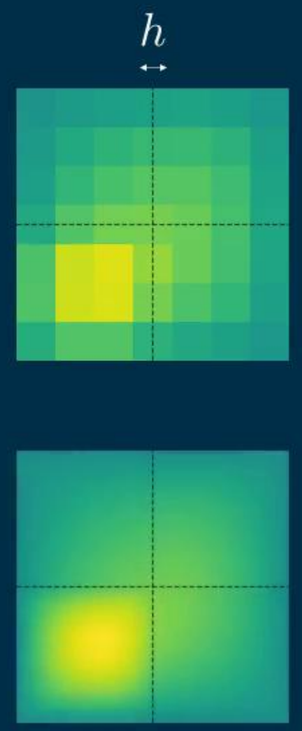

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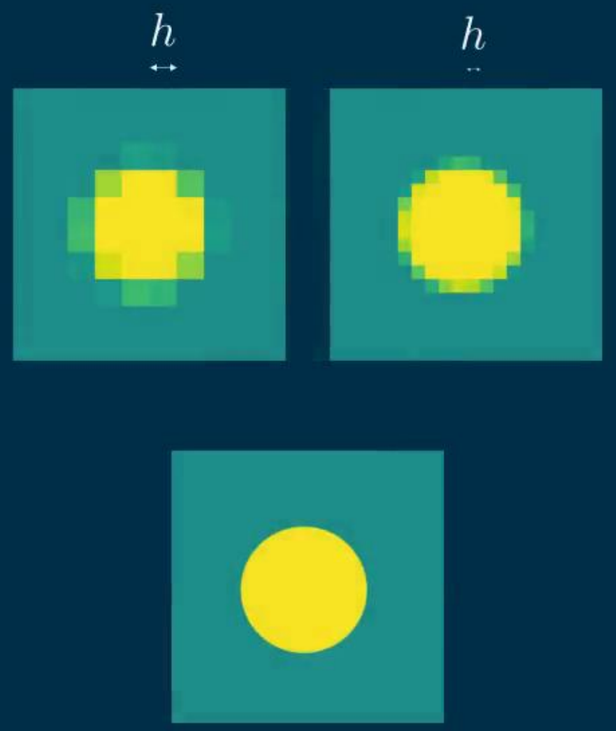

Application

1. Image superresolution.

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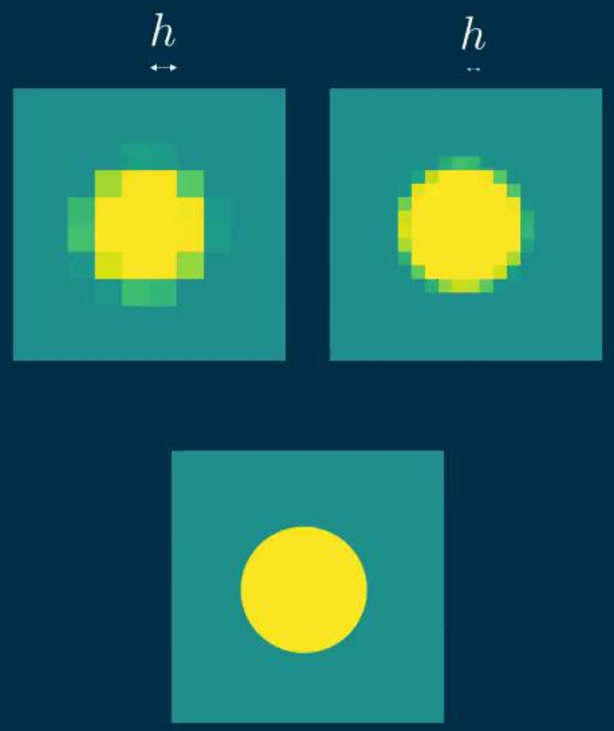

Application

1. Image superresolution.
2. Finite Volumes PDE Solvers

Regular case

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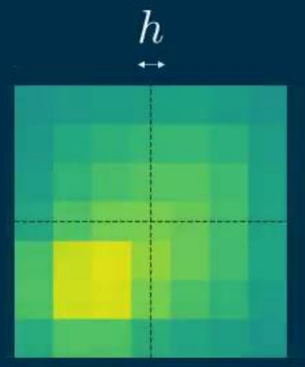

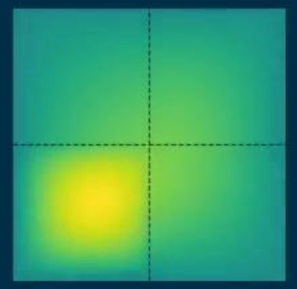

u smooth

Regular case

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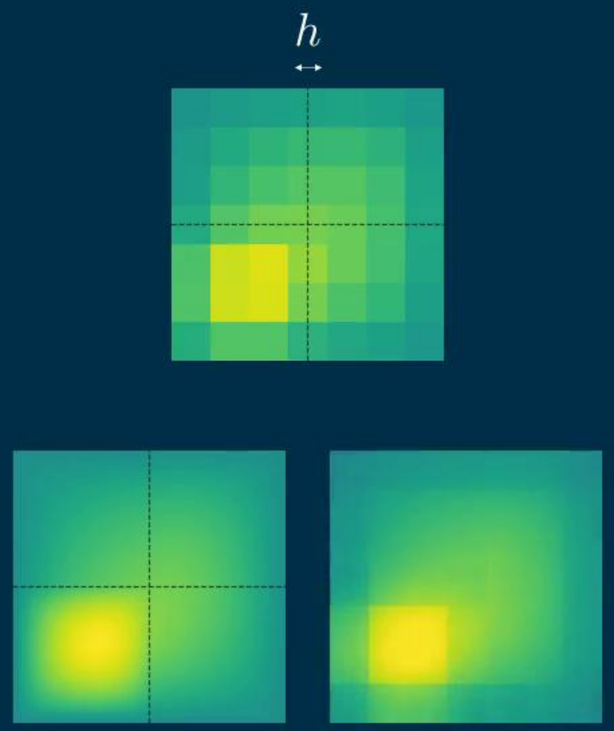

u smooth locally fit polynomials

Interface reconstruction

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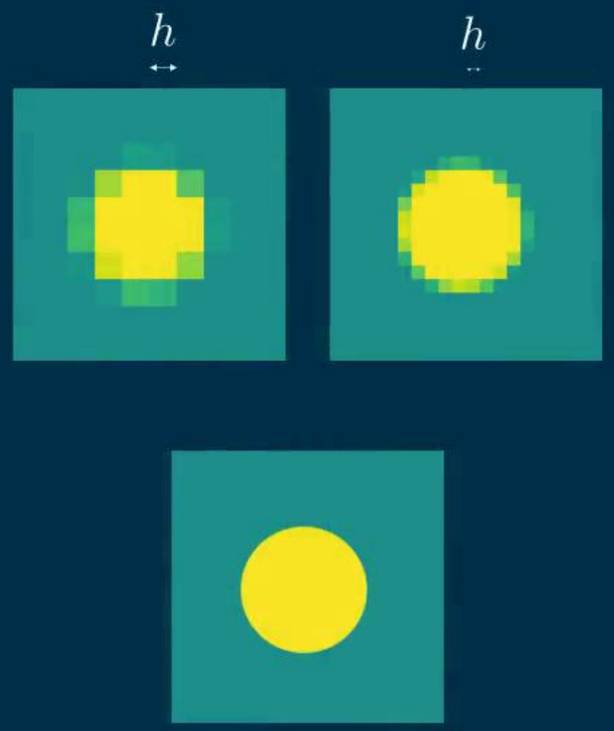

u presents jump discontinuities

Interface reconstruction: linear approach

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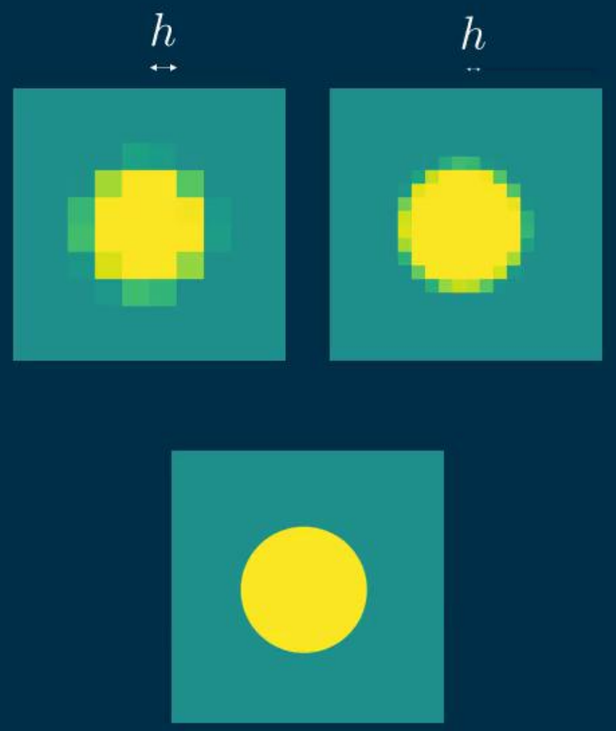

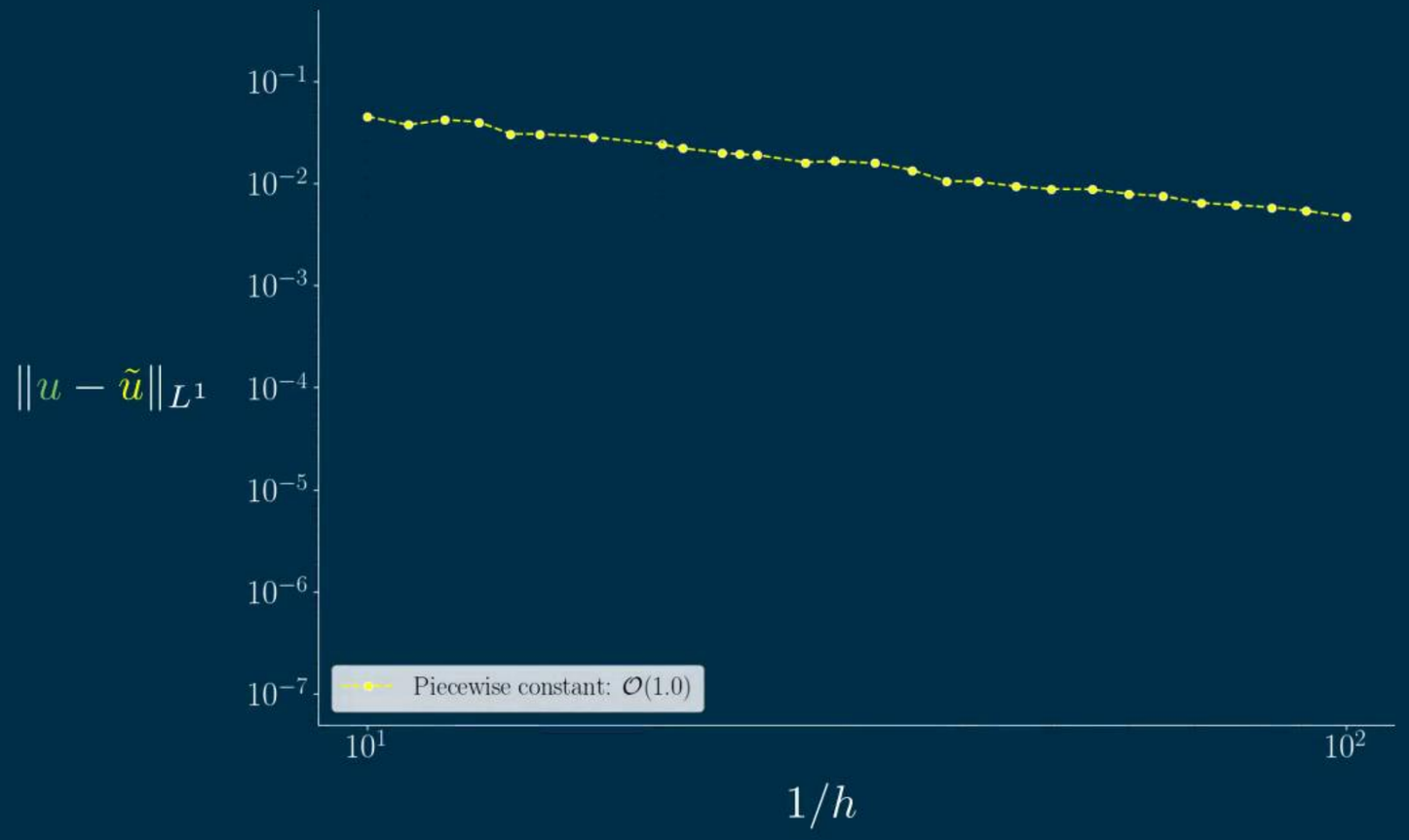

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Non-linear approximation

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Non-linear approximation: linear interface

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

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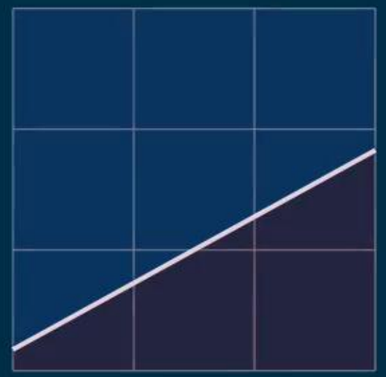
$v \in V_2$



0.0	0.0	0.0
0.0	0.1	0.6
0.4	0.9	1.0

$\ell(v) \in \mathbb{R}^m$

Non-linear approximation: linear interface

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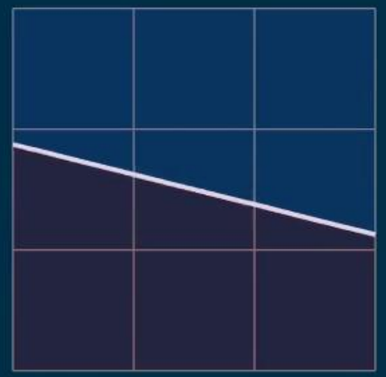

0.0	0.0	0.0
0.7	0.5	0.3
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Non-linear approximation: linear interface

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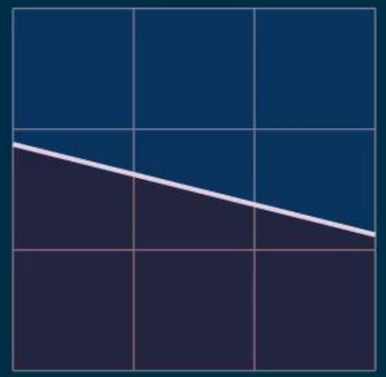
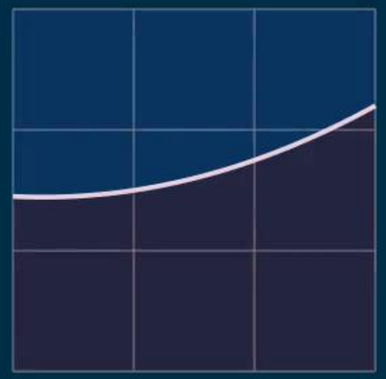


0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

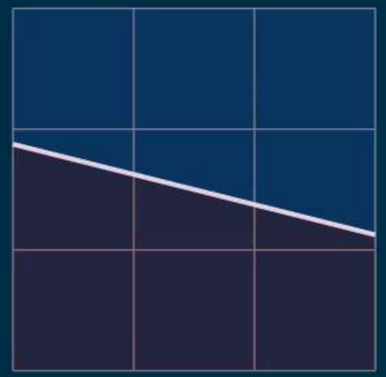
$z \in \mathbb{R}^m$
Observed averages

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$ LVIRA [E.G. Puckett, 1991]

$v \in V_2$

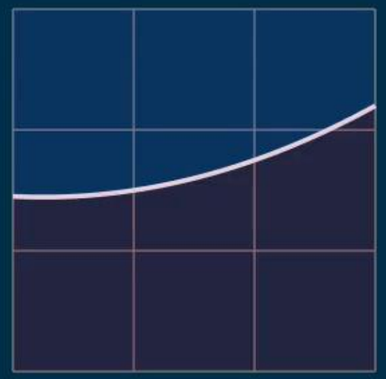


0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$

Observed averages

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

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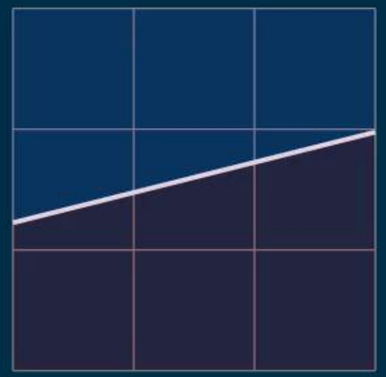
$v \in V_2$

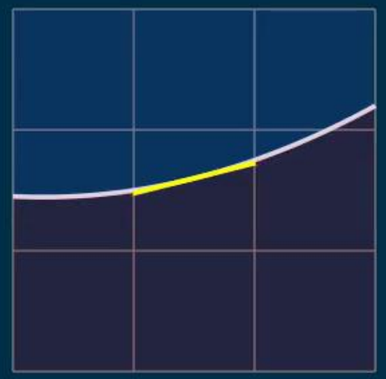


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$

Observed averages

Best fit estimator: LVIRA

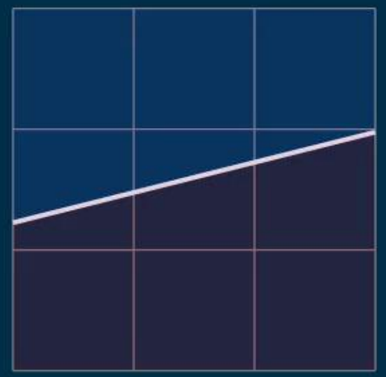
$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

$v \in V_2$

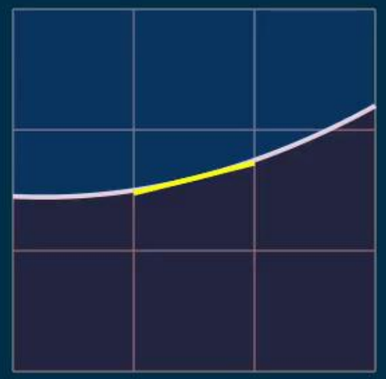


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$

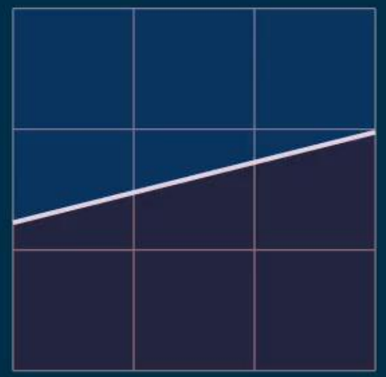
Observed averages

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$

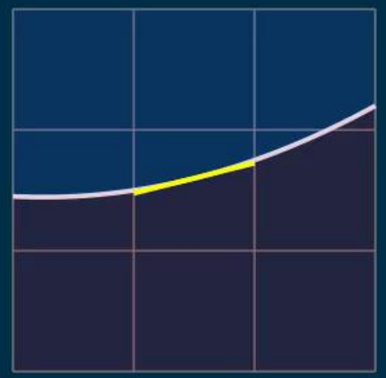
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

Lipschitz continuity

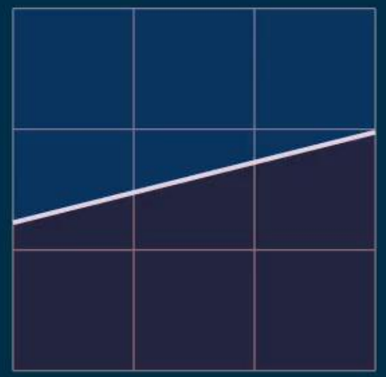
$$\|\ell(u) - \ell(v)\|_Z \leq \alpha_Z \|u - v\|_V, \quad u, v \in V$$

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$

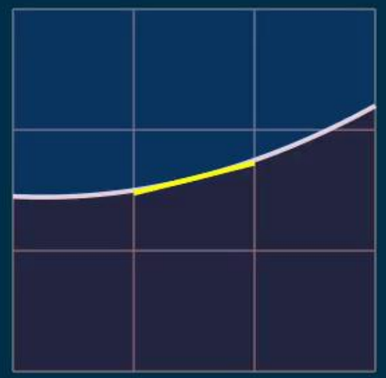
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

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$$\|\ell(u) - \ell(v)\|_Z \leq \alpha_Z \|u - v\|_V, \quad u, v \in V$$

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$$\|u - v\|_V \leq \mu_Z \|\ell(u) - \ell(v)\|_Z, \quad u, v \in V_n$$

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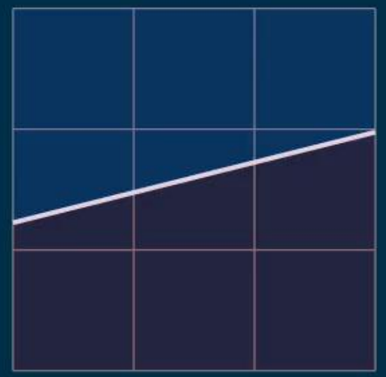
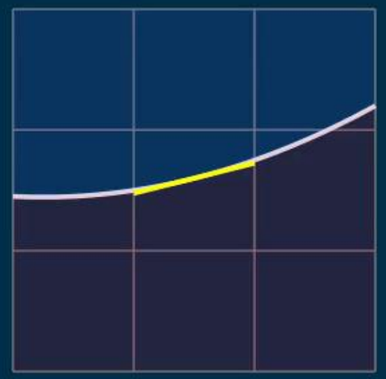
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

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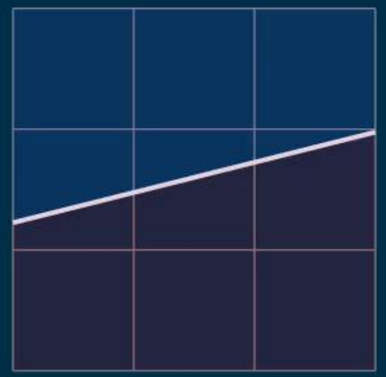
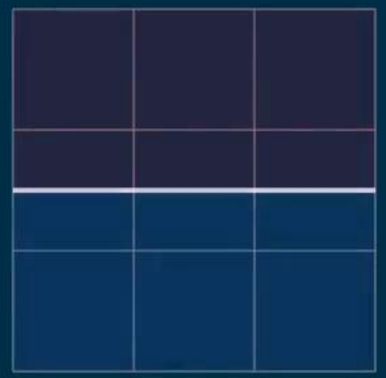
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

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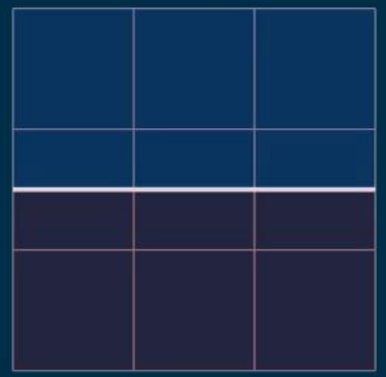
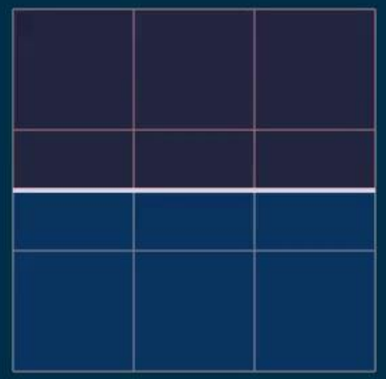


0.0	0.0	0.0
0.5	0.5	0.5
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

$z \in \mathbb{R}^m$
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[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

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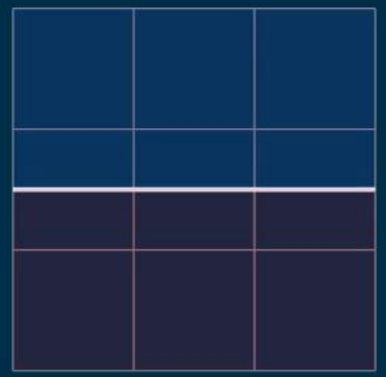
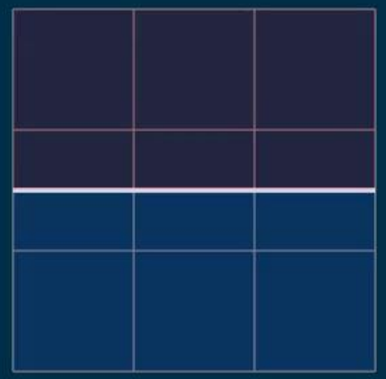
$v \in V_2$



0.0	0.0	0.0
0.5	0.5	0.5
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

$z \in \mathbb{R}^m$
Observed averages

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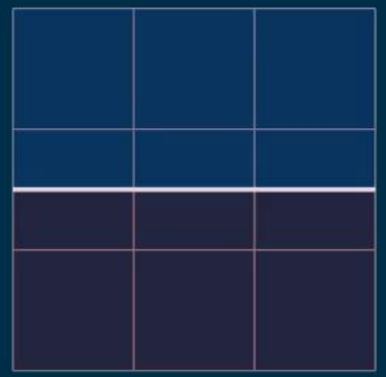
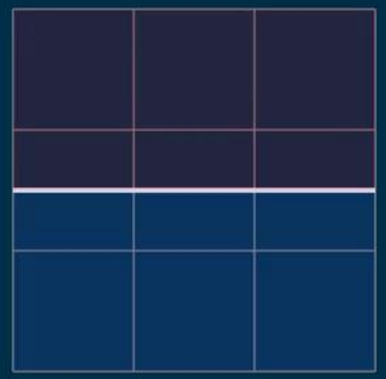
$v \in V_2$



0.0	0.0	0.0
0.5	0.5	0.5
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_{L^1} \leq C\|u - P_{V_n} u\|_{L^1}$$

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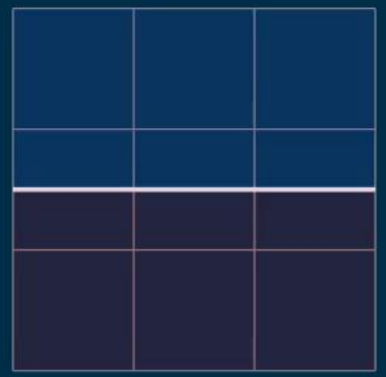
$v \in V_2$

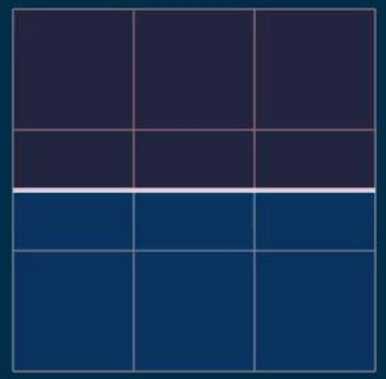


0.0	0.0	0.0
0.5	0.5	0.5
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_{L^1} \leq 10\|u - P_{V_n} u\|_{L^1}$$

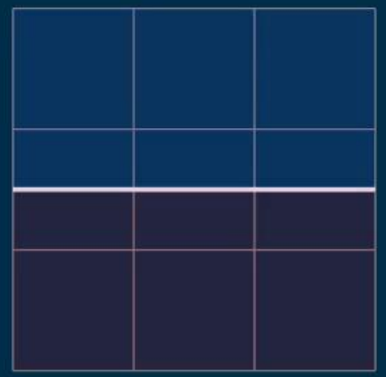
[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_{\ell^1}$$

$v \in V_2$

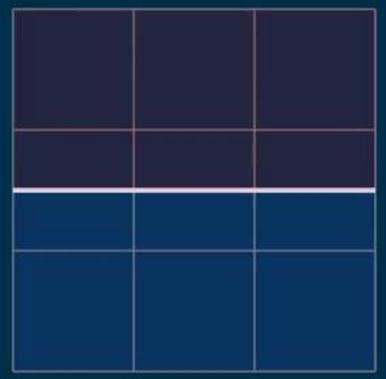


0.0	0.0	0.0
0.5	0.5	0.5
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



1.0	1.0	1.0
0.5	0.5	0.5
0.0	0.0	0.0

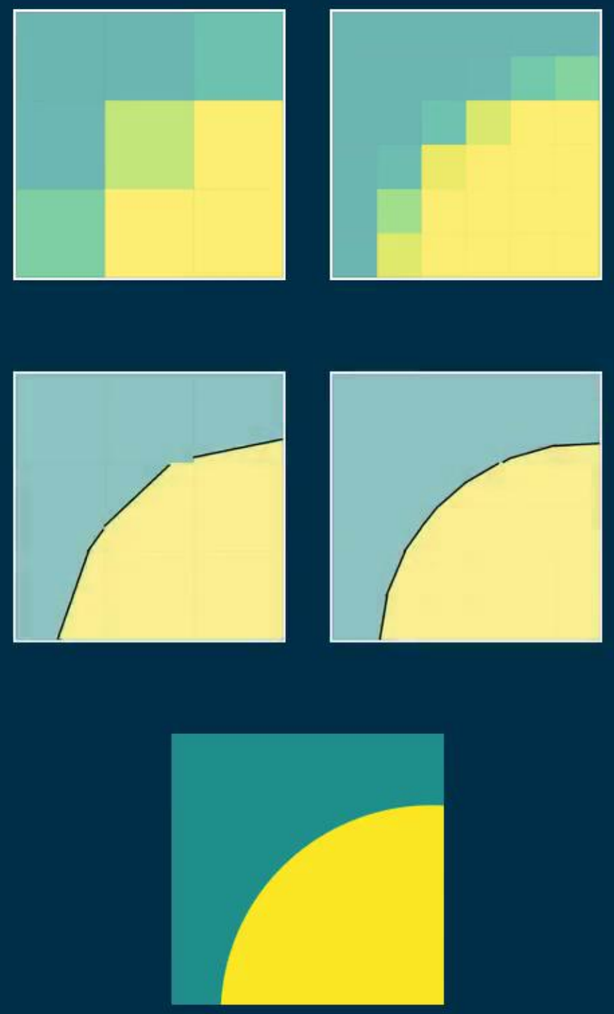
$z \in \mathbb{R}^m$
Observed averages

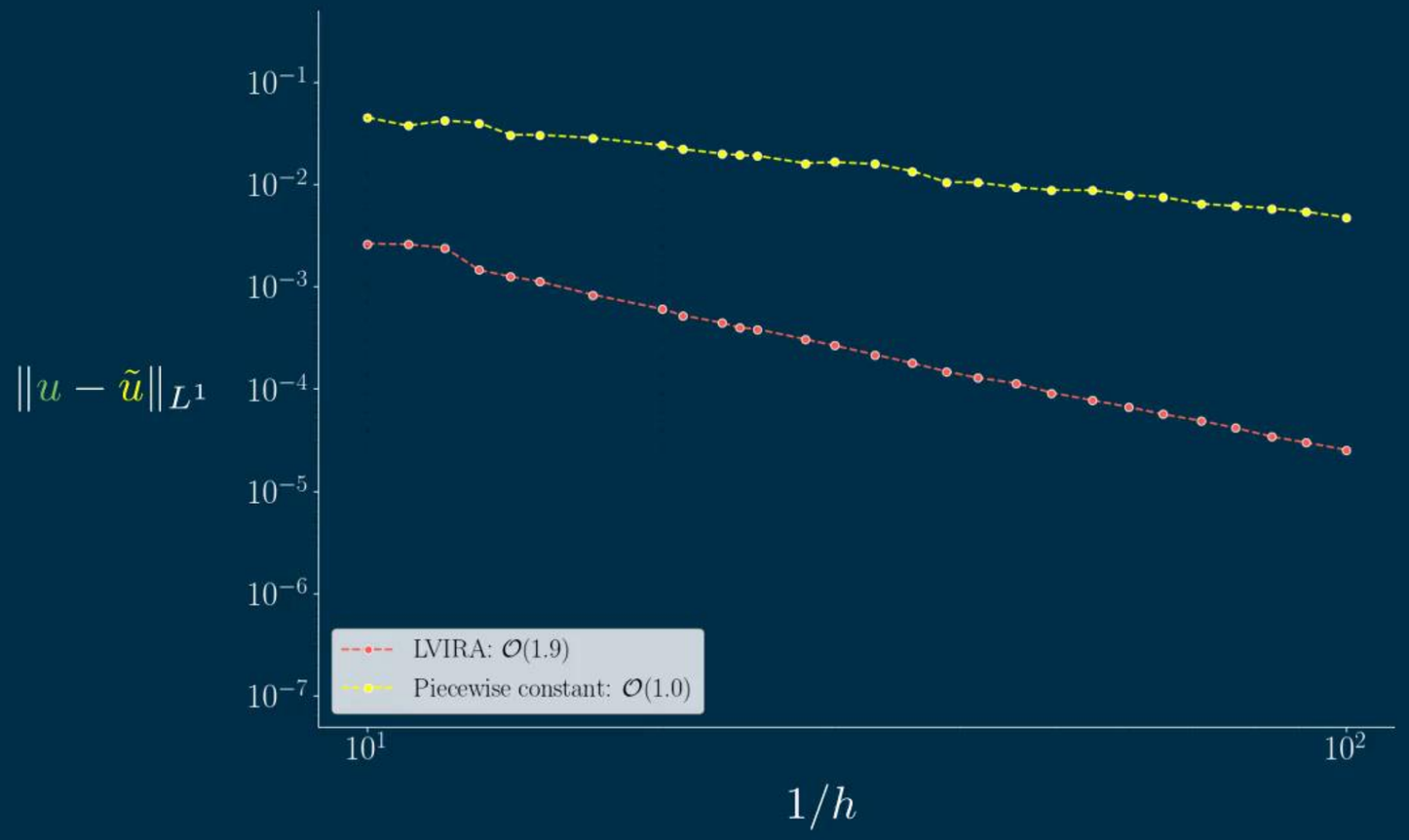
Global convergence

Corollary: If Ω is a $\mathcal{C}^s$ domain with $s \geq 2$, the LVIRA method converges in L^1 norm at rate h^2

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Best fit estimator: LVIRA





[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Higher order

Optimization Based Edge Reconstruction Algorithm

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

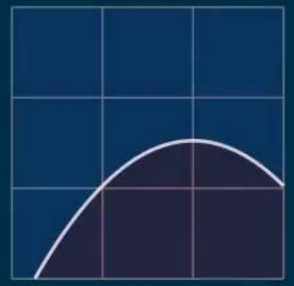
$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



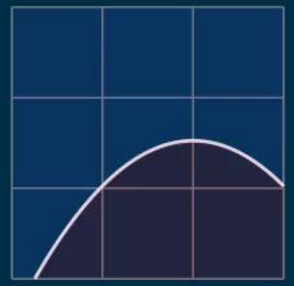
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

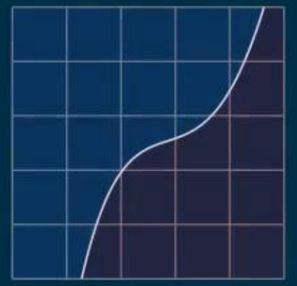
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



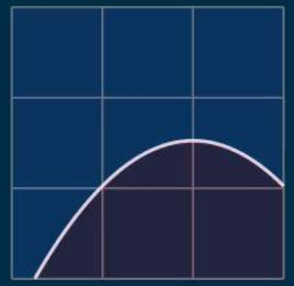
0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

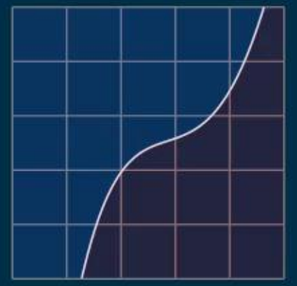
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



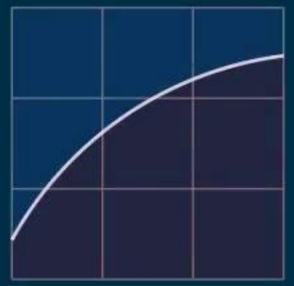
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Circle



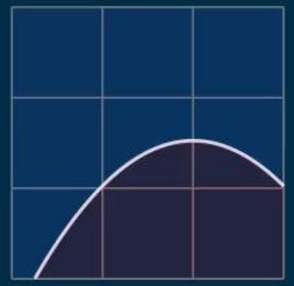
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

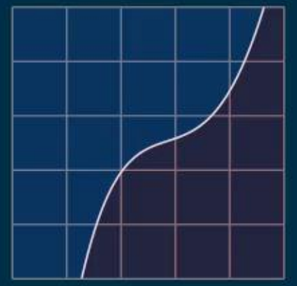
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

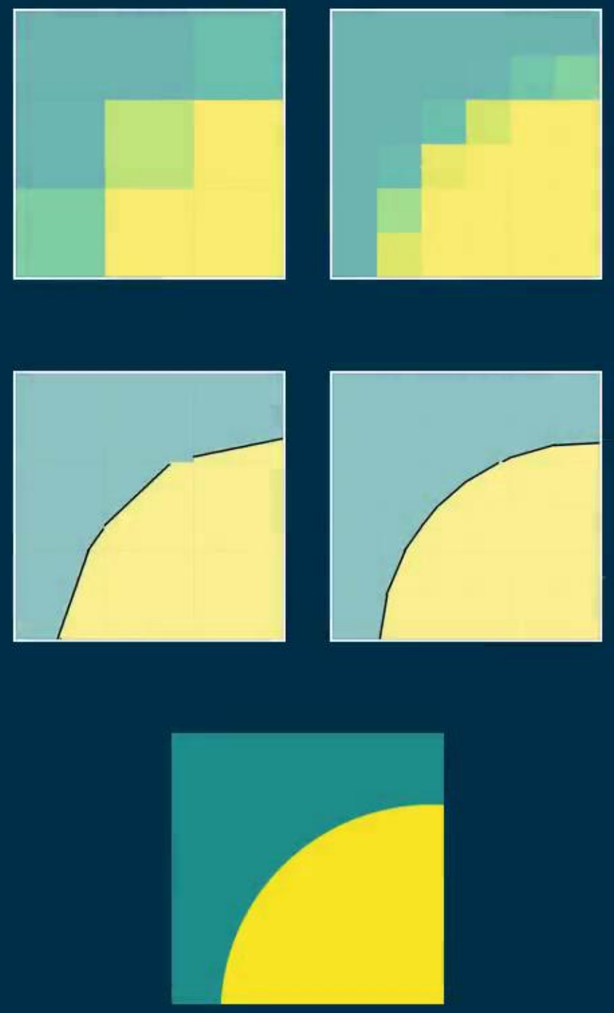
Circle

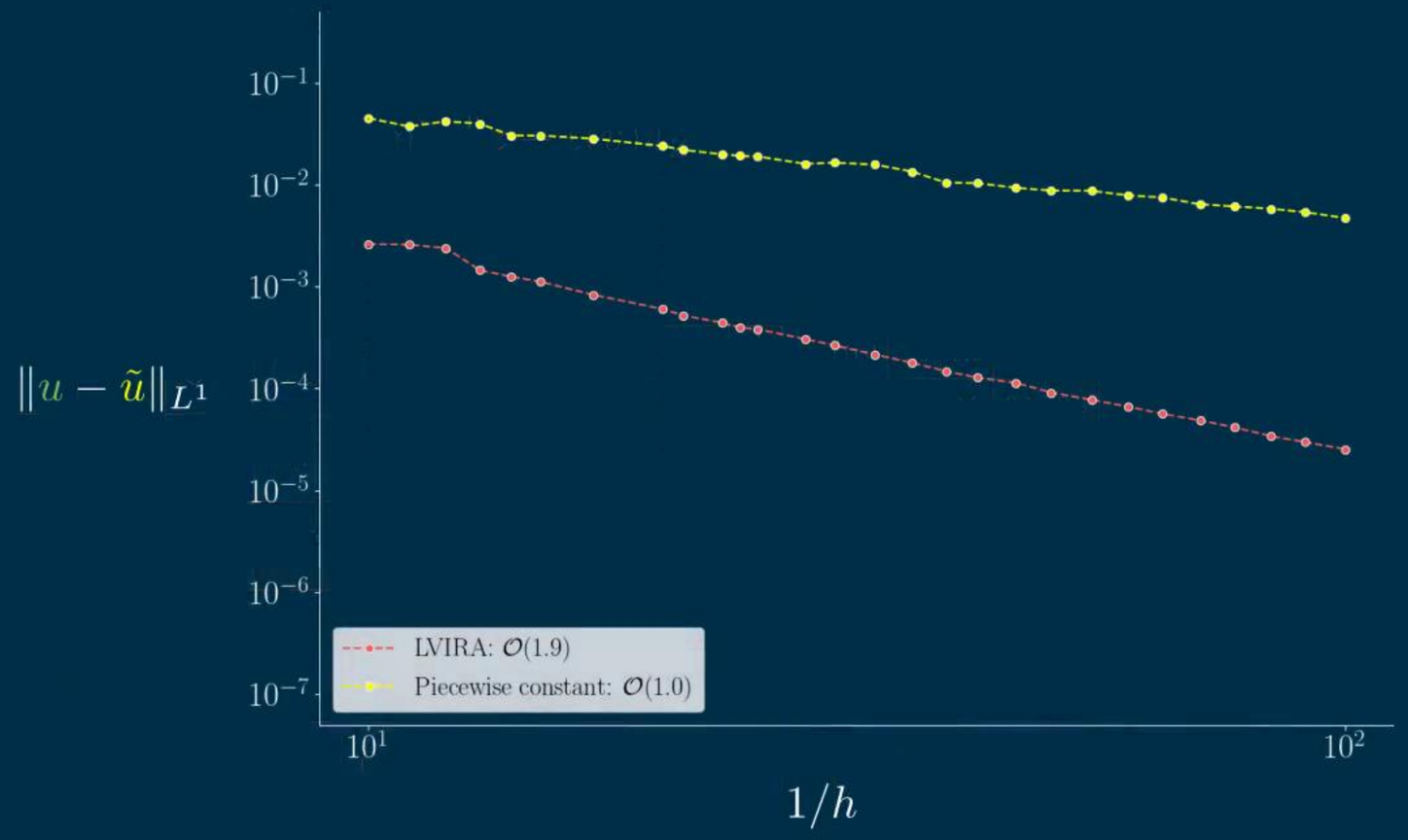
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

Vertex

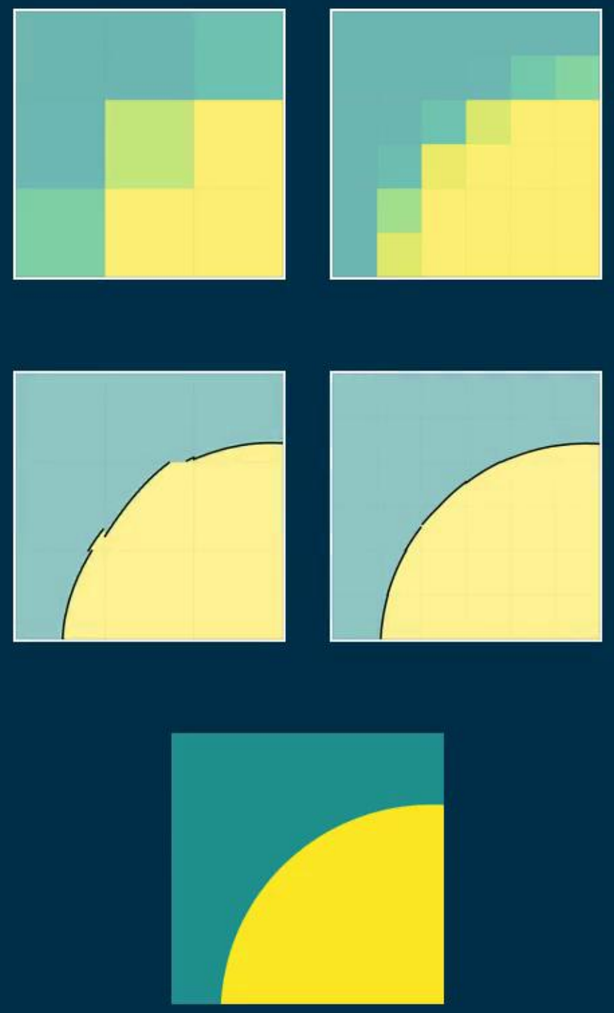
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.5	0.4	0.2
0.2	0.9	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

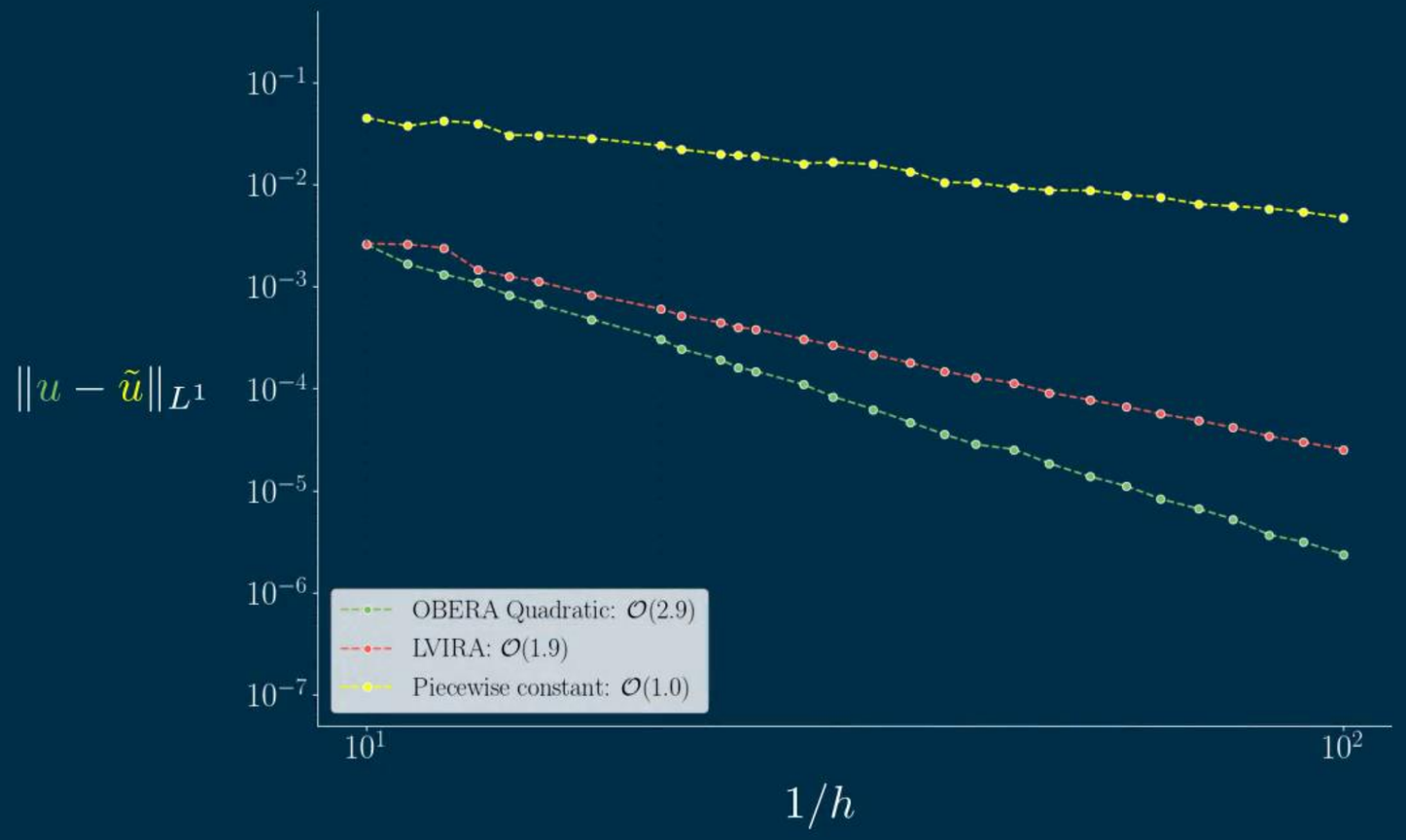
Higher order: OBERA





Higher order: OBERA





[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

Higher order: OBERA

Theoretical problem: difficult to prove $\mu_Z \leq \infty$

Higher order: OBERA

Practical problem: long computation time

Theoretical problem: difficult to prove $\mu_Z \leq \infty$

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Practical problem: long computation time

Theoretical problem: difficult to prove $\mu_Z \leq \infty$

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Practical problem: long computation time

Theoretical problem: difficult to prove $\mu_Z \leq \infty$

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

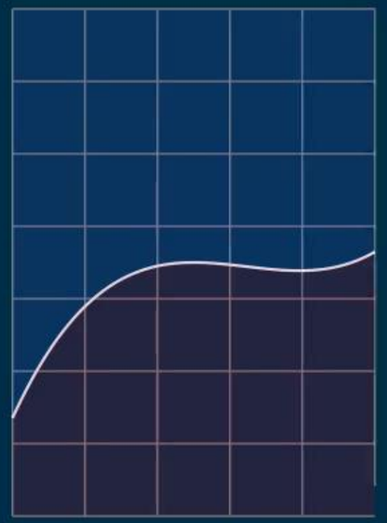
Practical problem: long computation time

Theoretical problem: difficult to prove $\mu_Z \leq \infty$

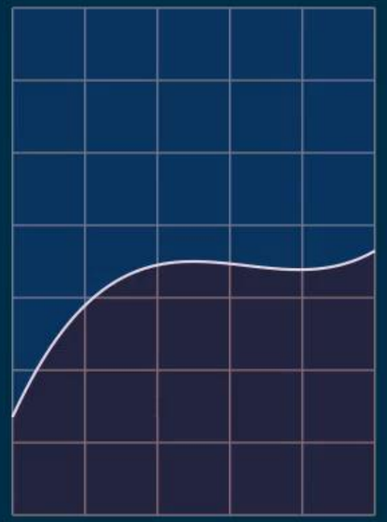
Fast higher order methods

Algorithm for Edge Reconstruction using Oriented Stencils

Higher order: AEROS

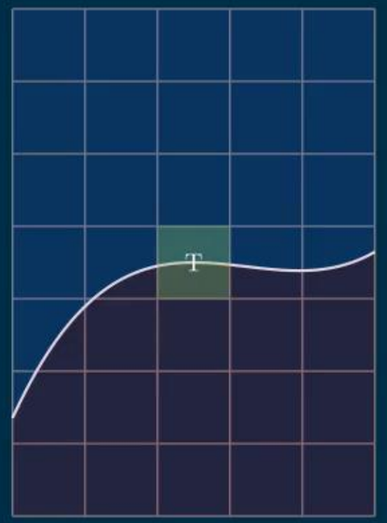


Higher order: AEROS



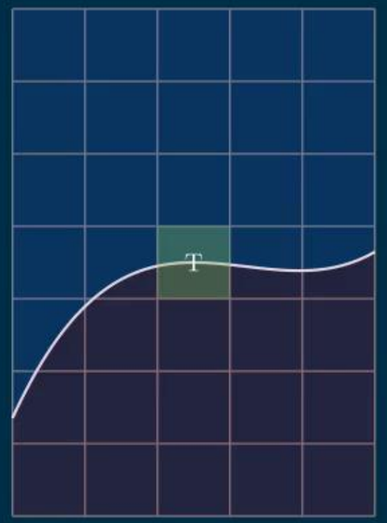
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

Higher order: AEROS

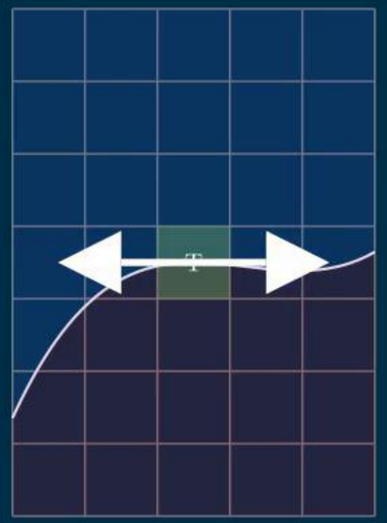


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Horizontal orientation

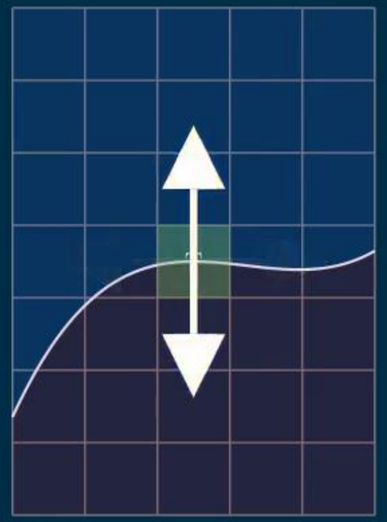


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation

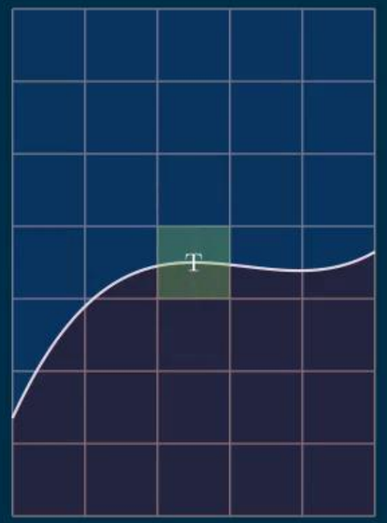


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation



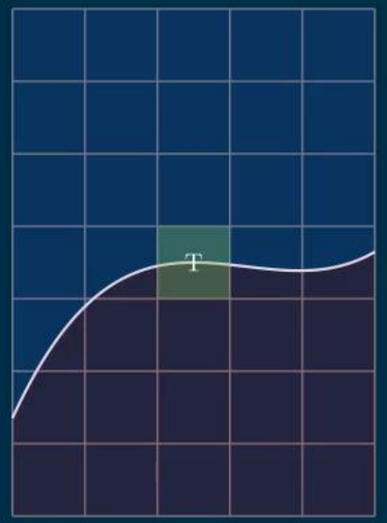
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

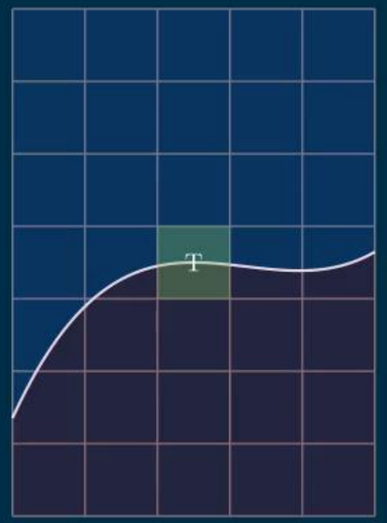
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

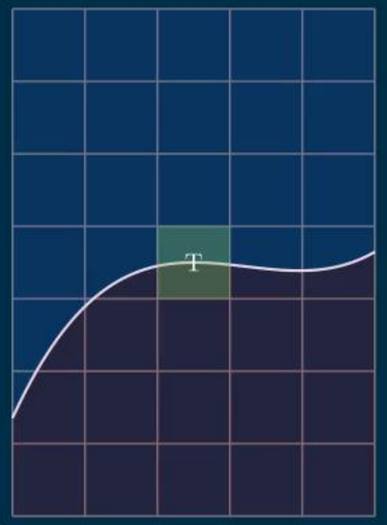
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

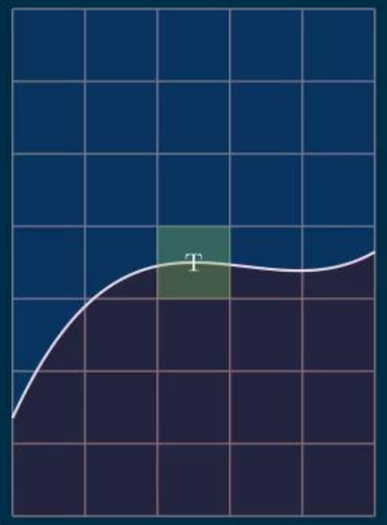
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

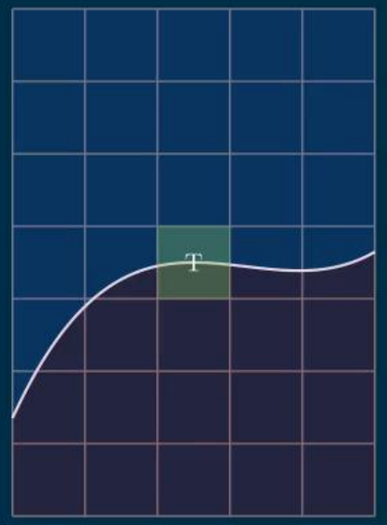
Orientation test based on Sobel filter

$$G_T = (-0.4, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

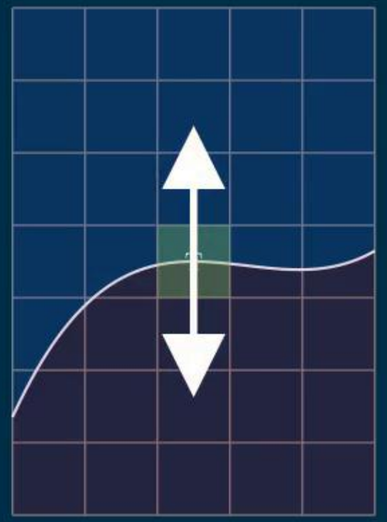
Orientation test based on Sobel filter

$$G_T = (-0.4, 4.0)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

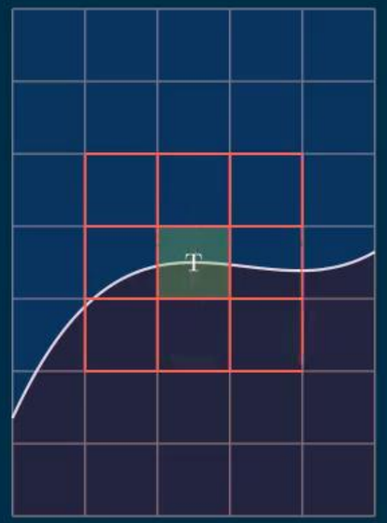
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$0.4 = |H_T| < |V_T| = 4.0$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

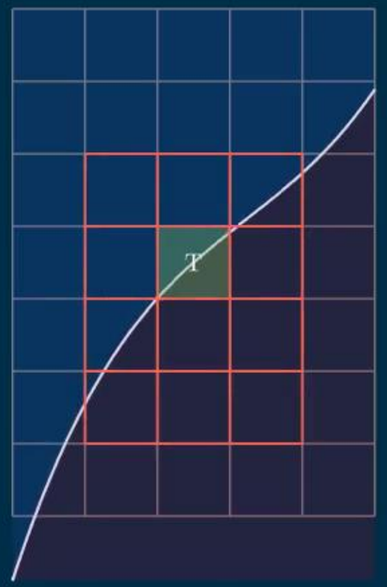
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
- Define an appropriate stencil S_T

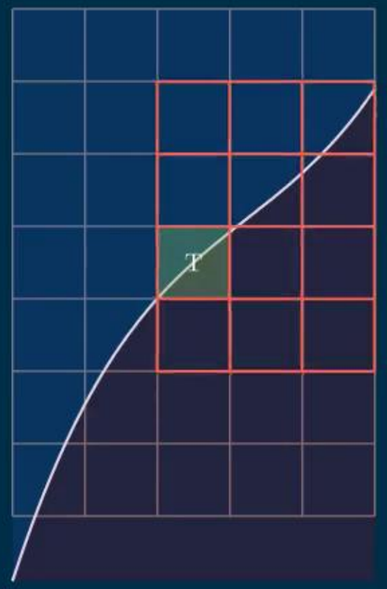
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
- Define an appropriate stencil S_T

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
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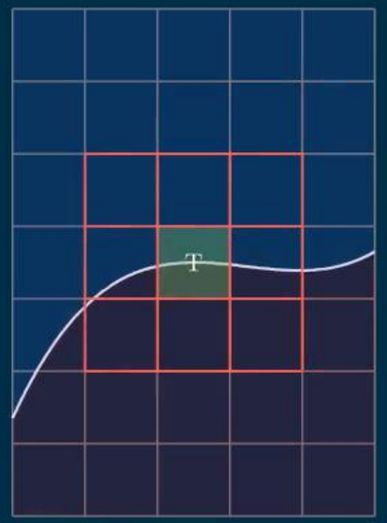

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
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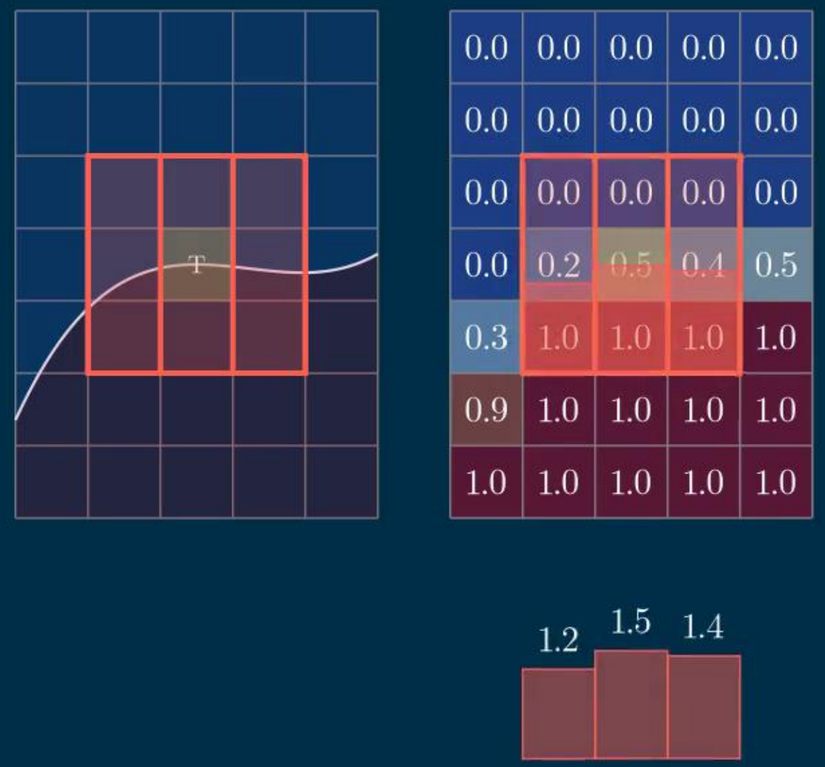

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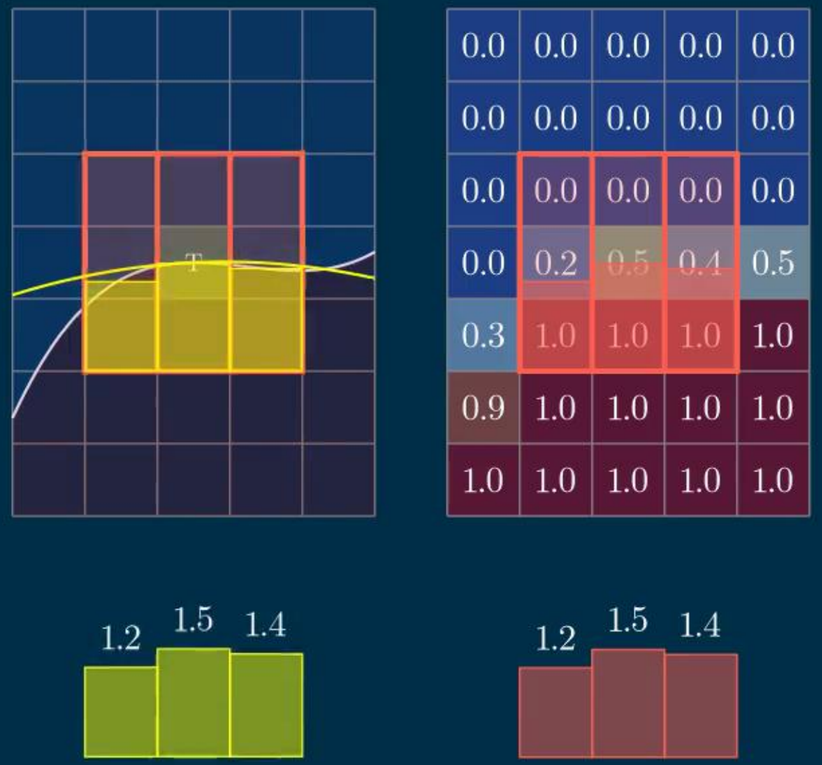
Higher order: AEROS



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$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

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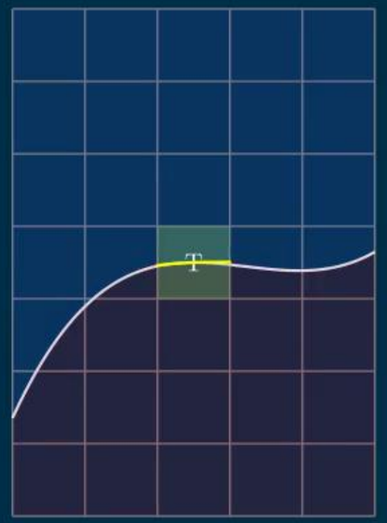

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0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
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AEROS theoretical results

Orientation test result

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AEROS global convergence theorem

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AEROS theoretical results

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The orientation given by the Sobel filter ensures that if the mesh size h is below a critical value h^* then there is a tall or wide enough stencil so that the interface croses the sides of it.

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AEROS global convergence theorem

If Ω is a $\mathcal{C}^s$ domain with $s \geq n$, the AEROS method based on polynomials of degree n has a global convergence rate in L^1 norm of h^{n+1}

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
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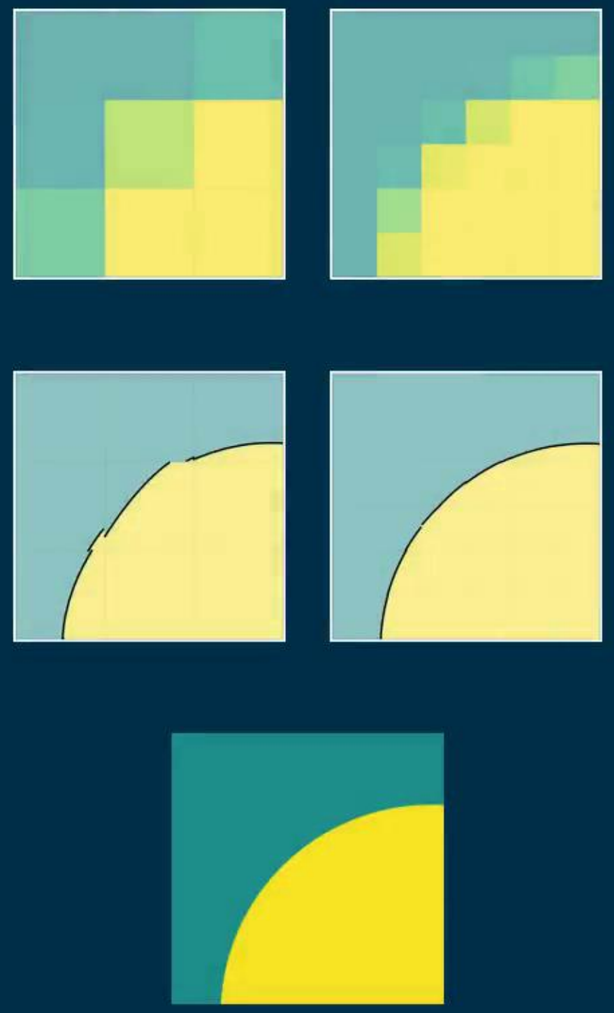
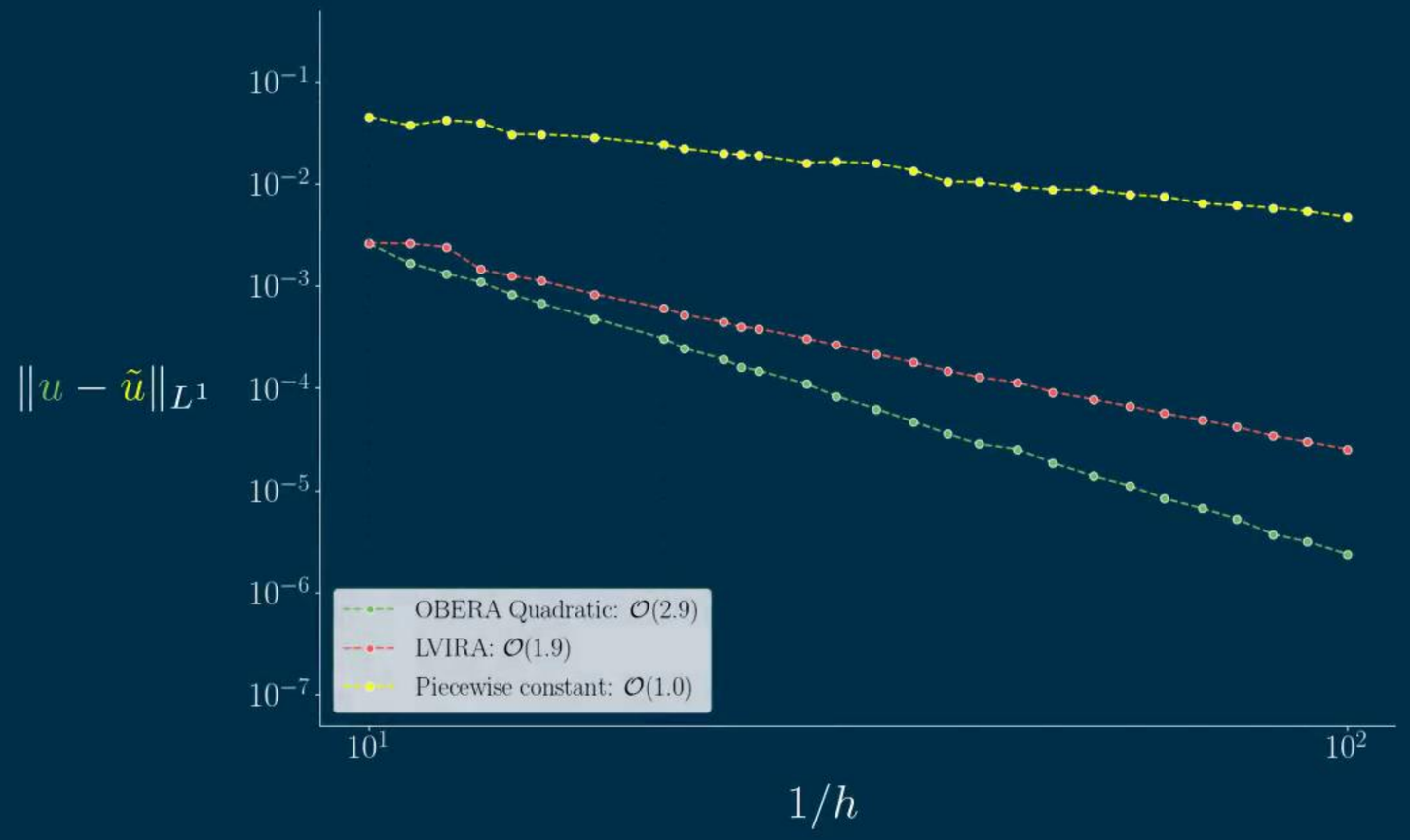
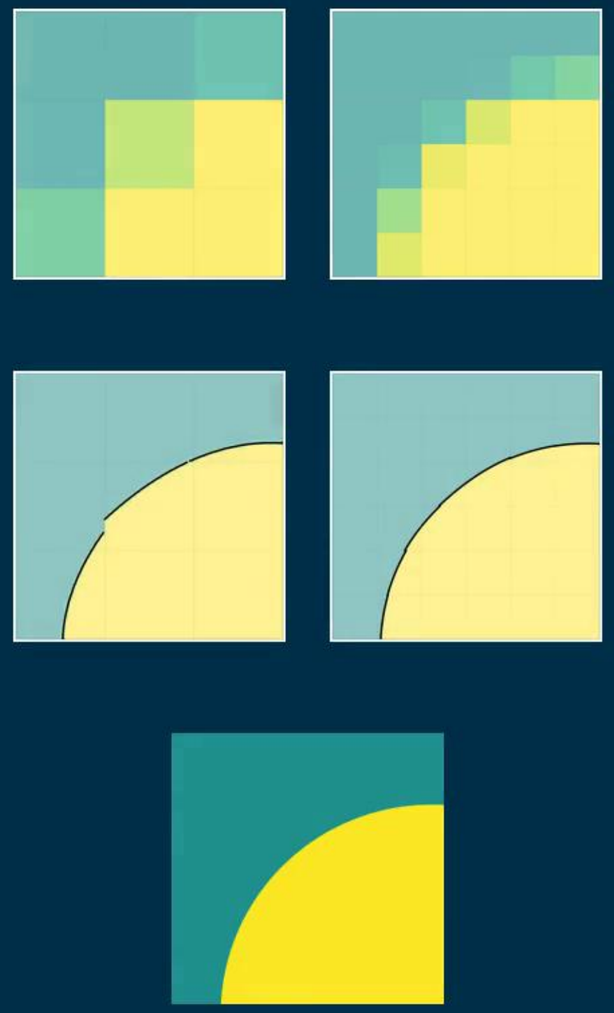
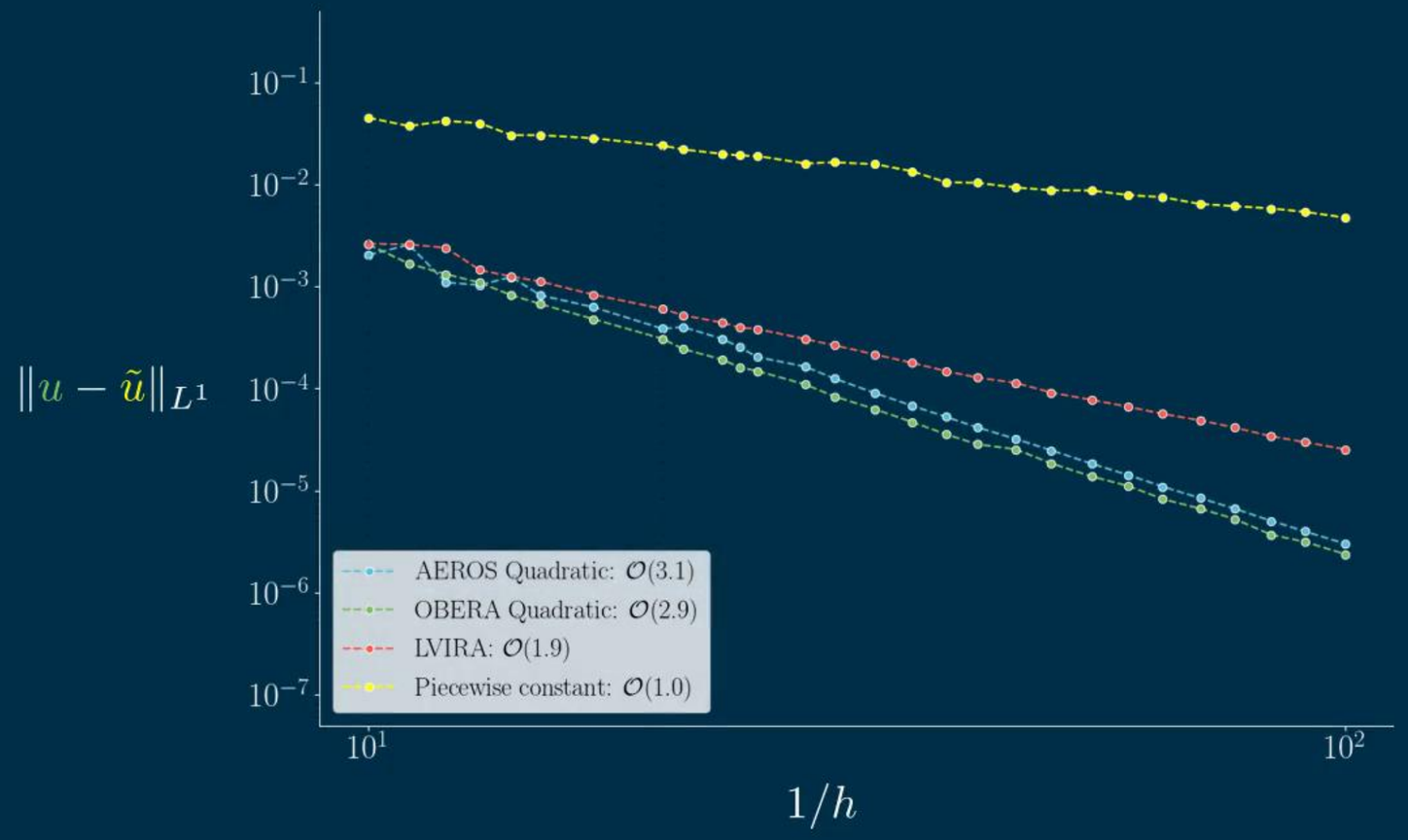
AEROS convergence





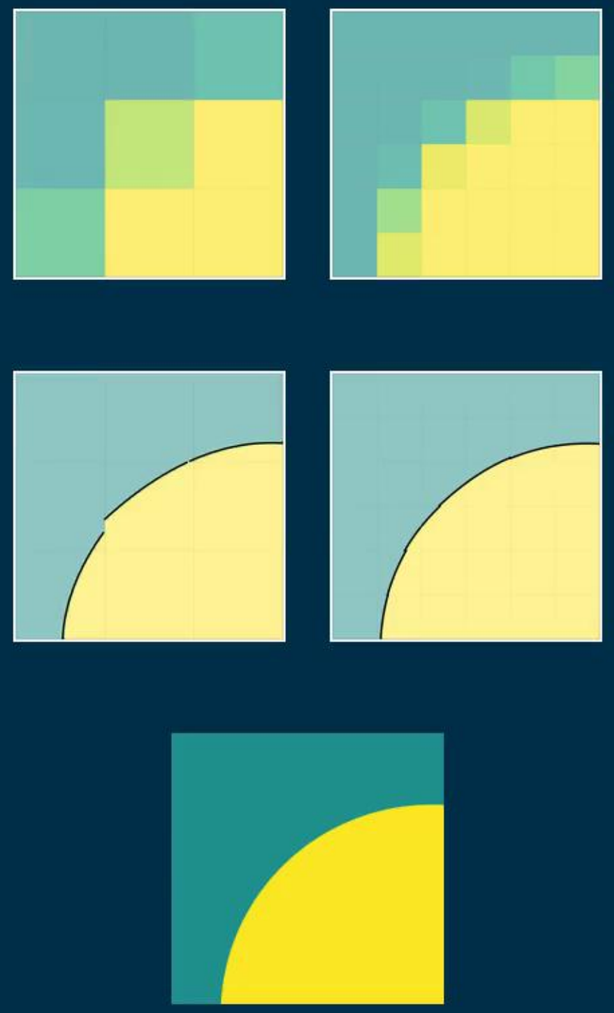
AEROS convergence: Quadratic

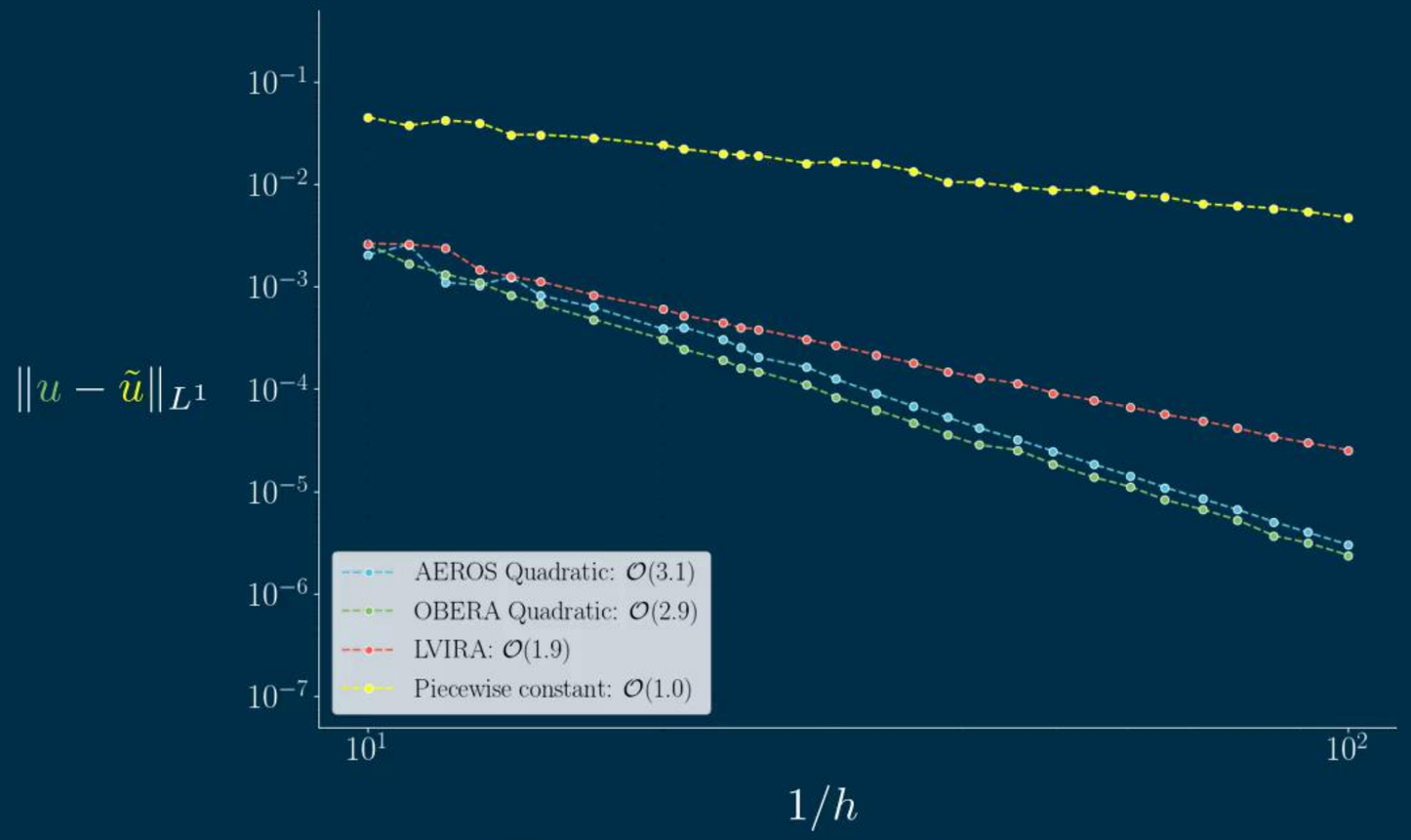




[A. Cohen, O. Mula, A. Somacal, ESAIM, 2024]

AEROS convergence: Quadratic

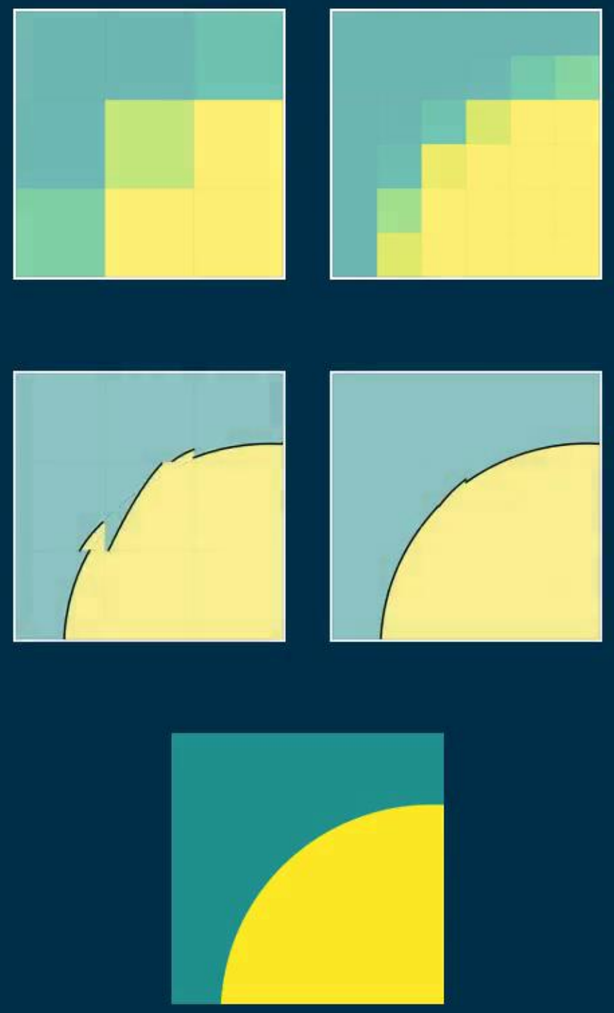


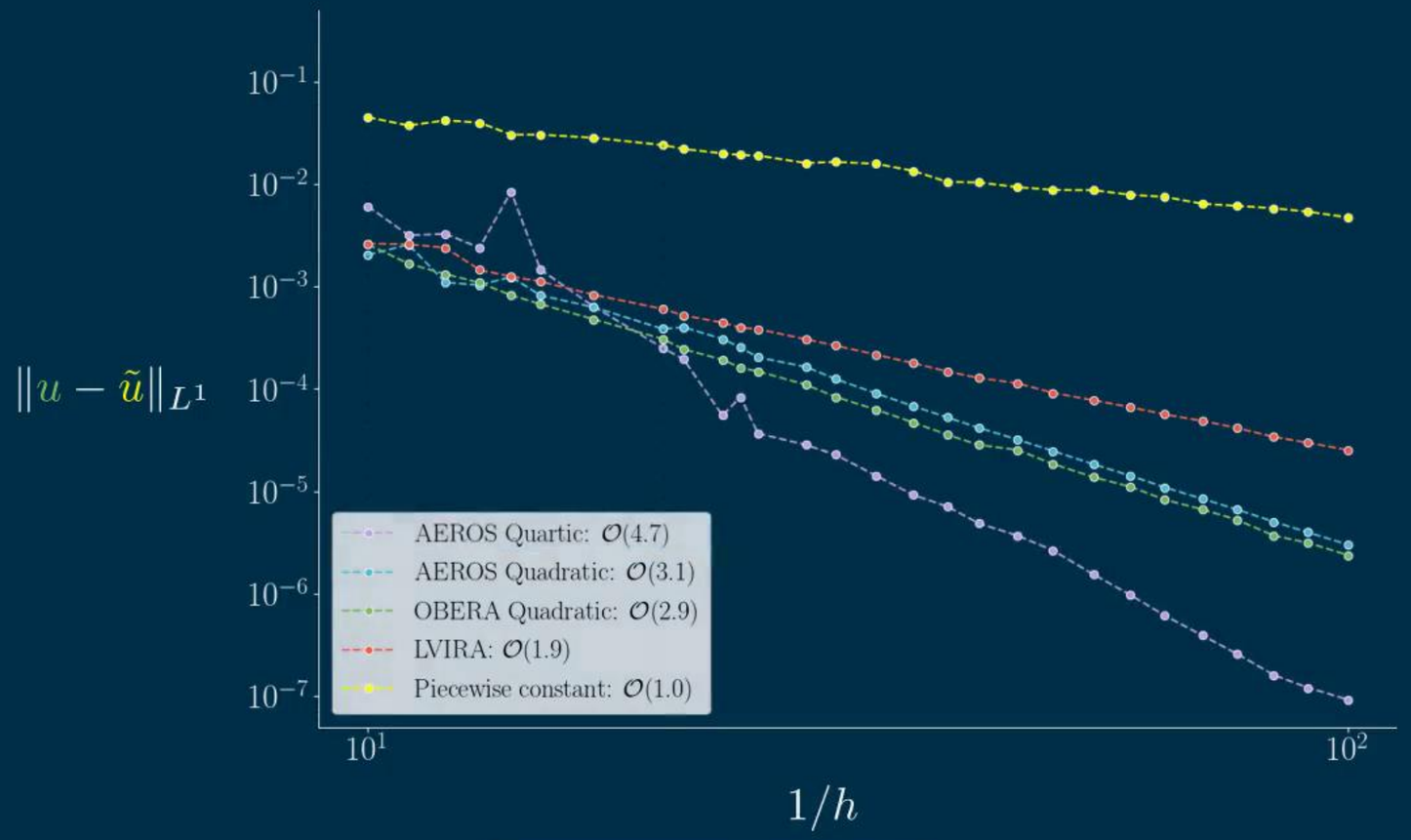


$$\tau_{AEROS} \lesssim \frac{\tau_{OBERA}}{100}$$

[A. Cohen, O. Mula, A. Somacal, ESAIM, 2024]

AEROS convergence: Quartic





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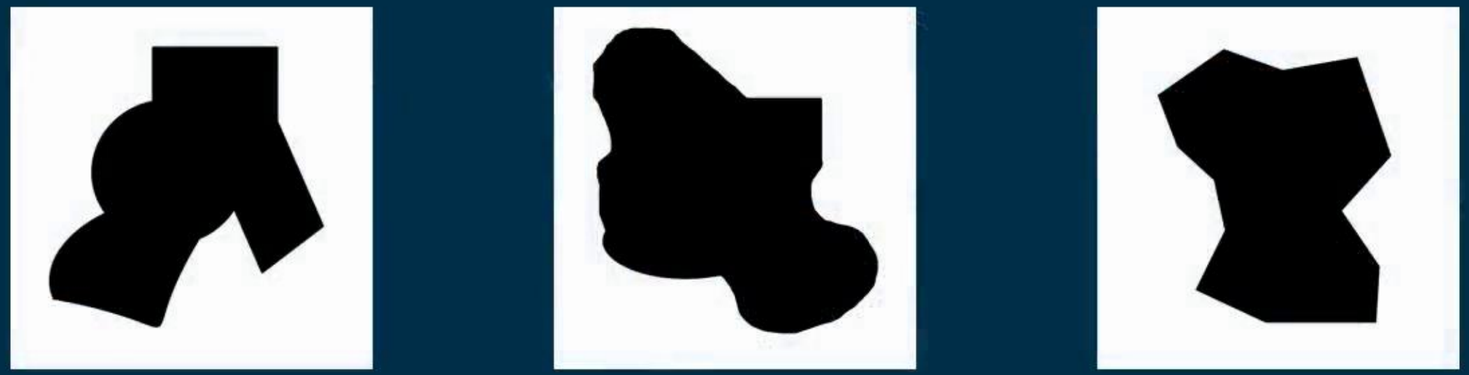
Corners

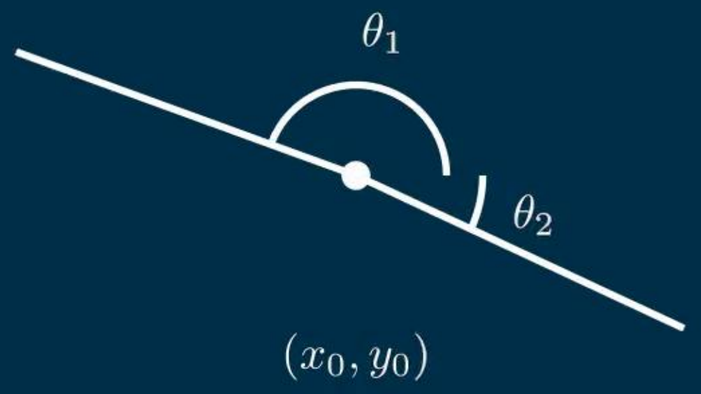


Corners

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Corners

$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$



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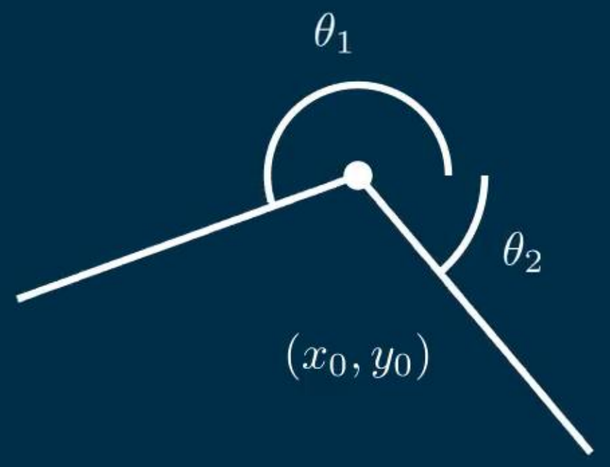

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AEROS Vertex

Tangent Extension Method (TEM)

Corners

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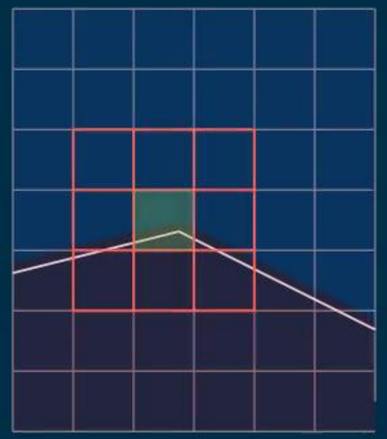
AEROS Vertex



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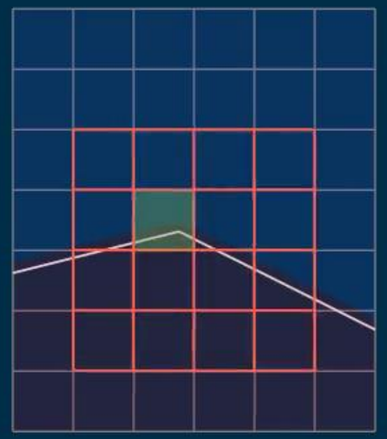

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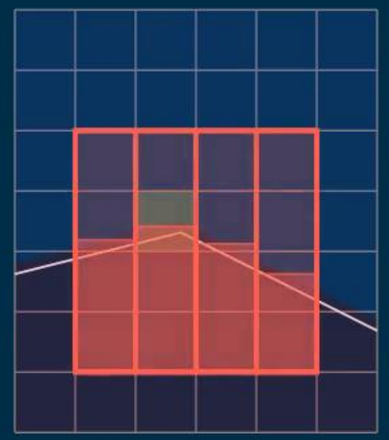

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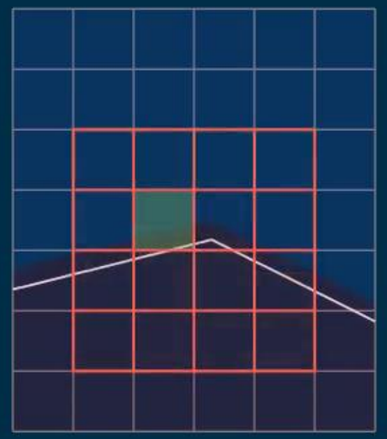

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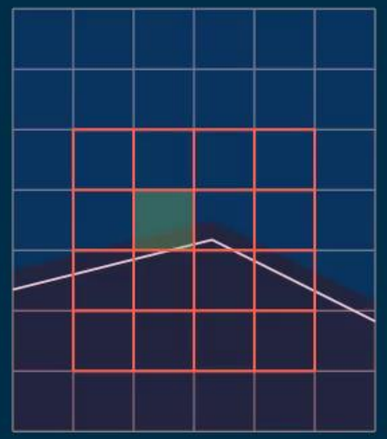

Tangent Extension Method (TEM)

- More than 1 posibility to test
- Can not deal with orientation change

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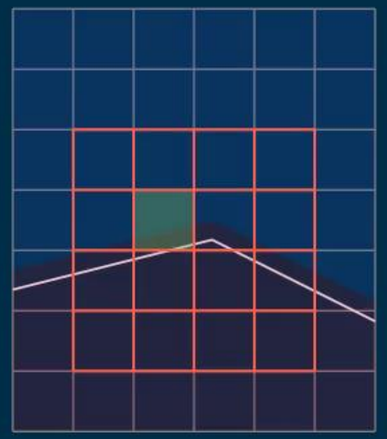

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[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

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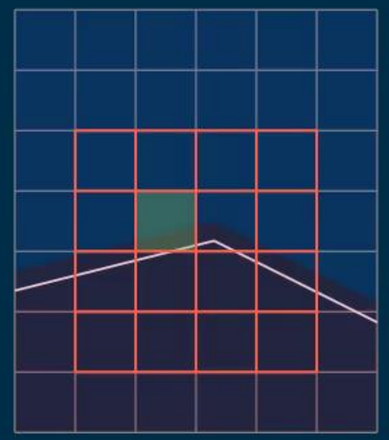

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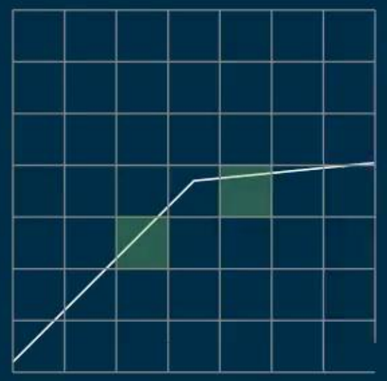

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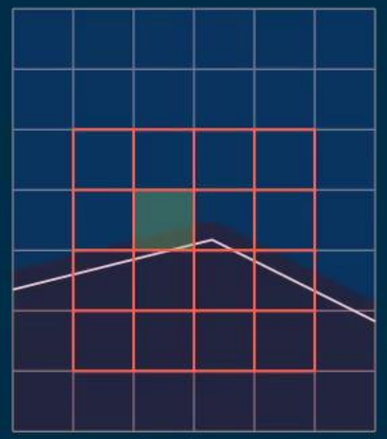

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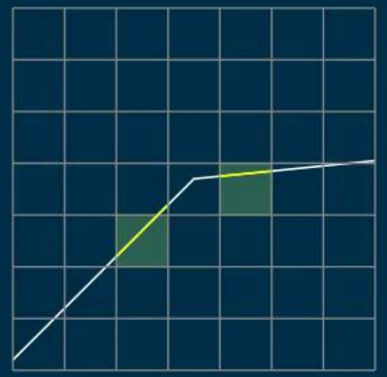

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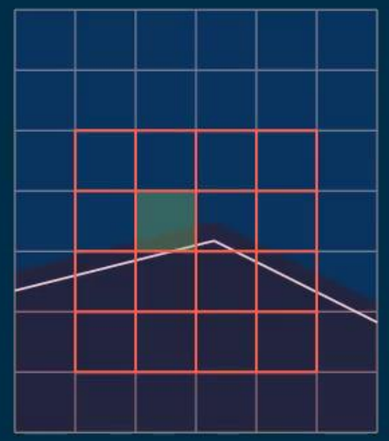

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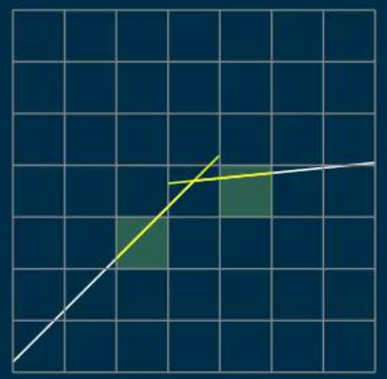

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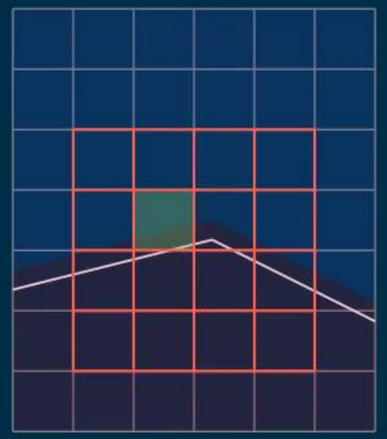

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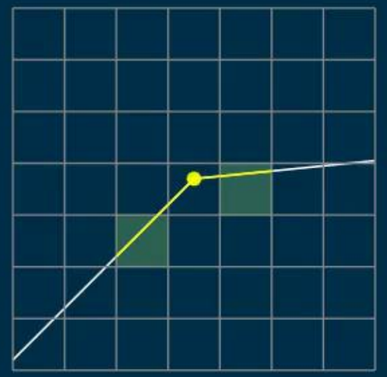

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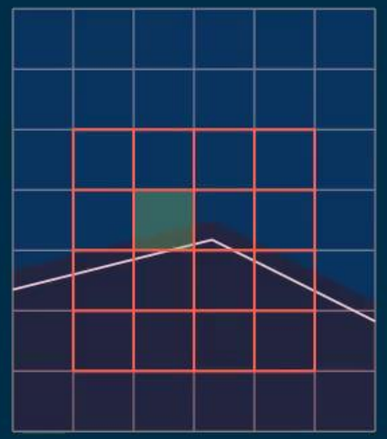

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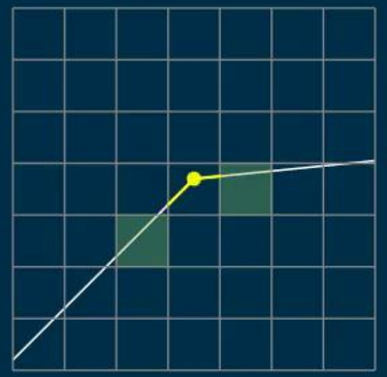

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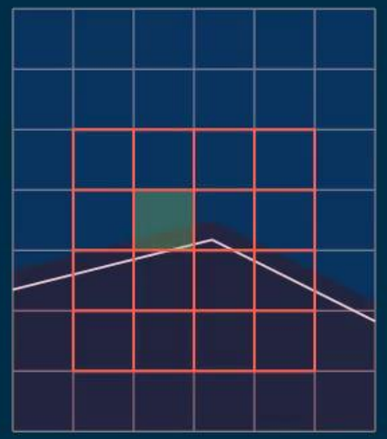

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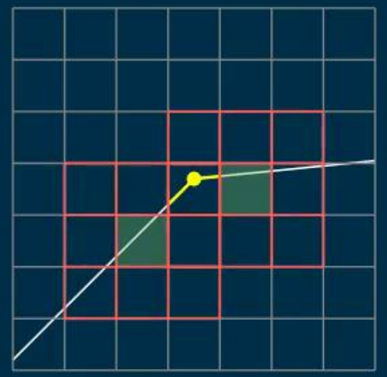

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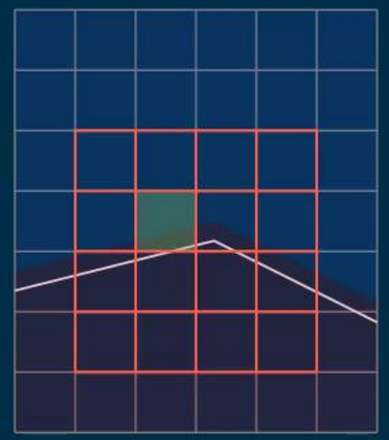
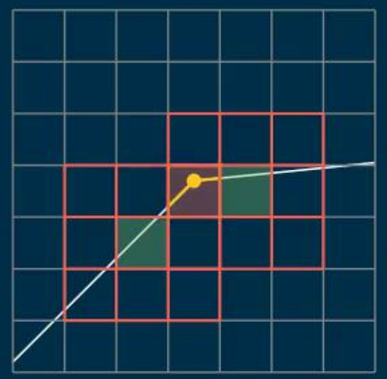


Tangent Extension Method (TEM)



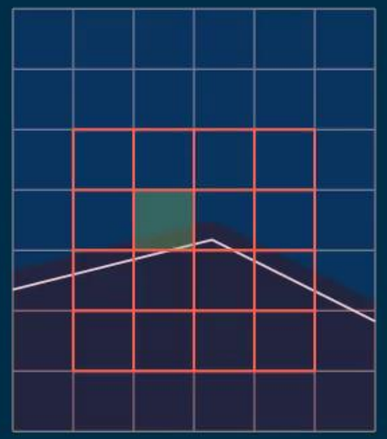
- More than 1 posibility to test
- Can not deal with orientation change

[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

Corners

$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$

AEROS Vertex



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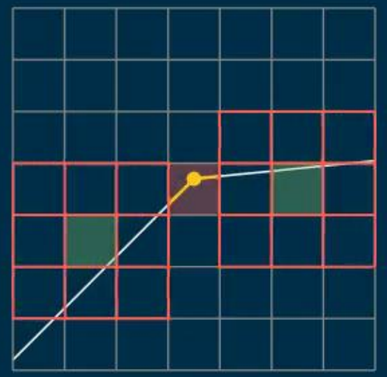

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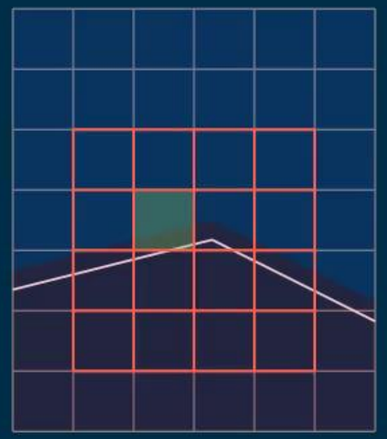

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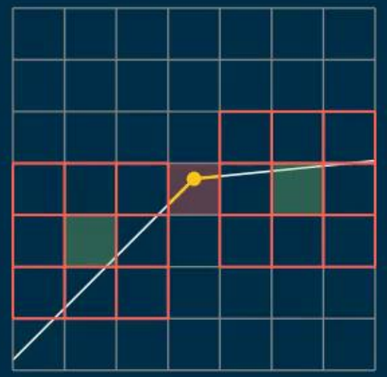

- Exact only if both sides of vertex are straight lines

[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

Reconstructing corners

$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

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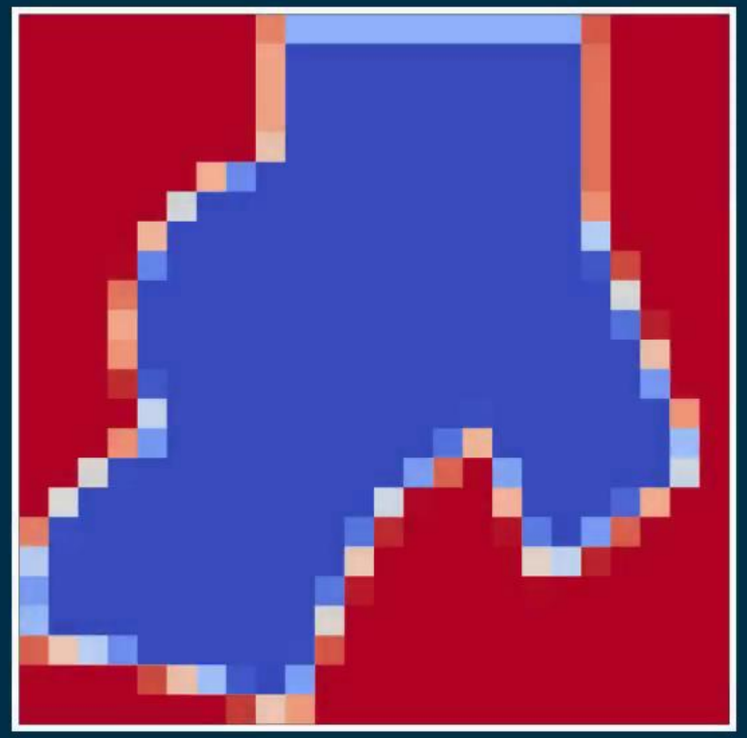

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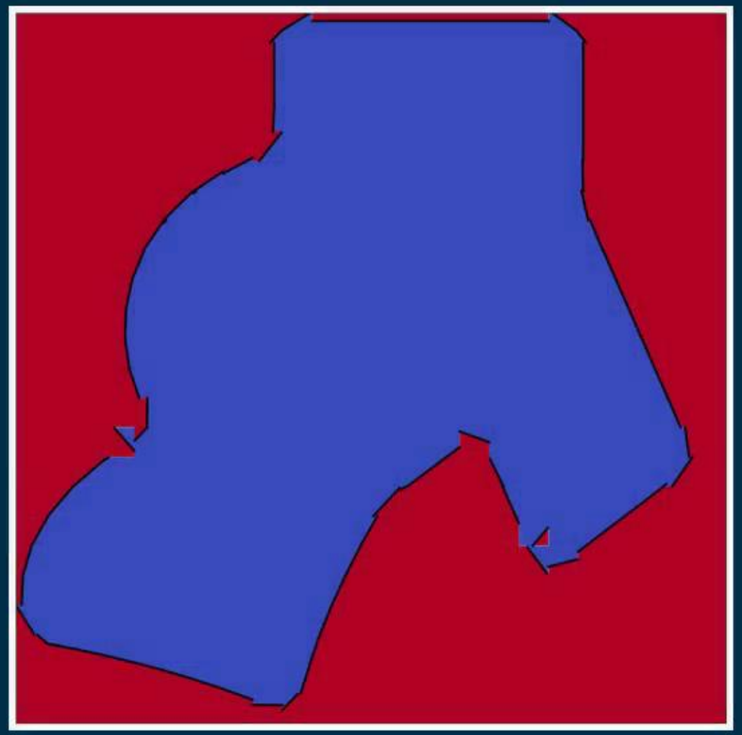

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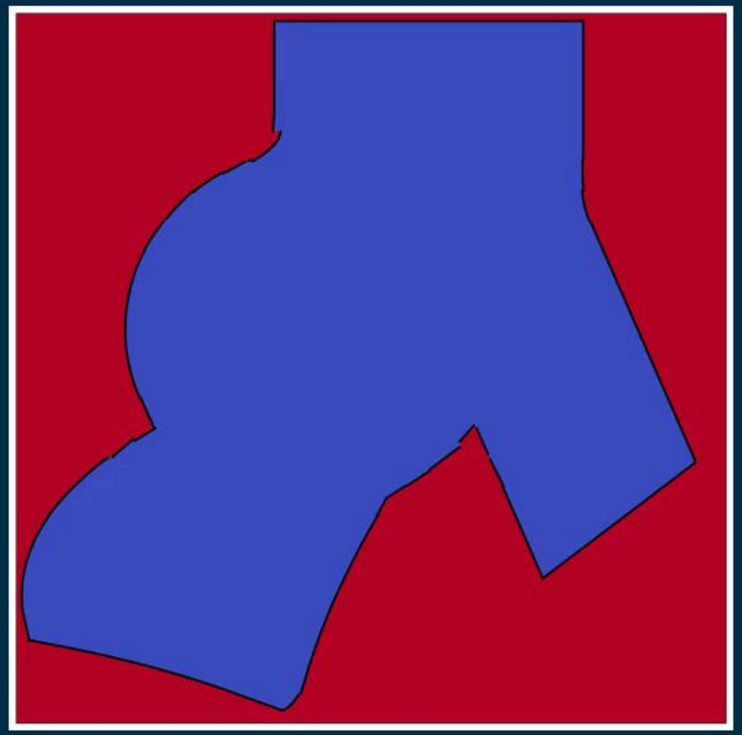
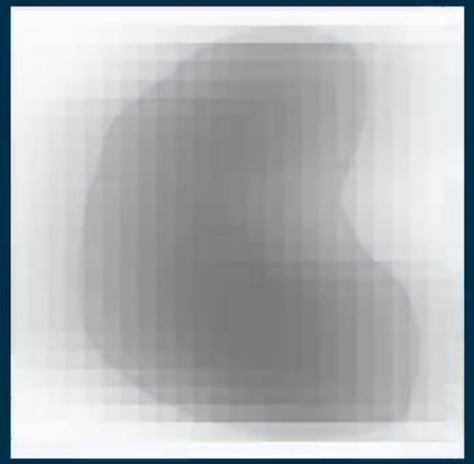
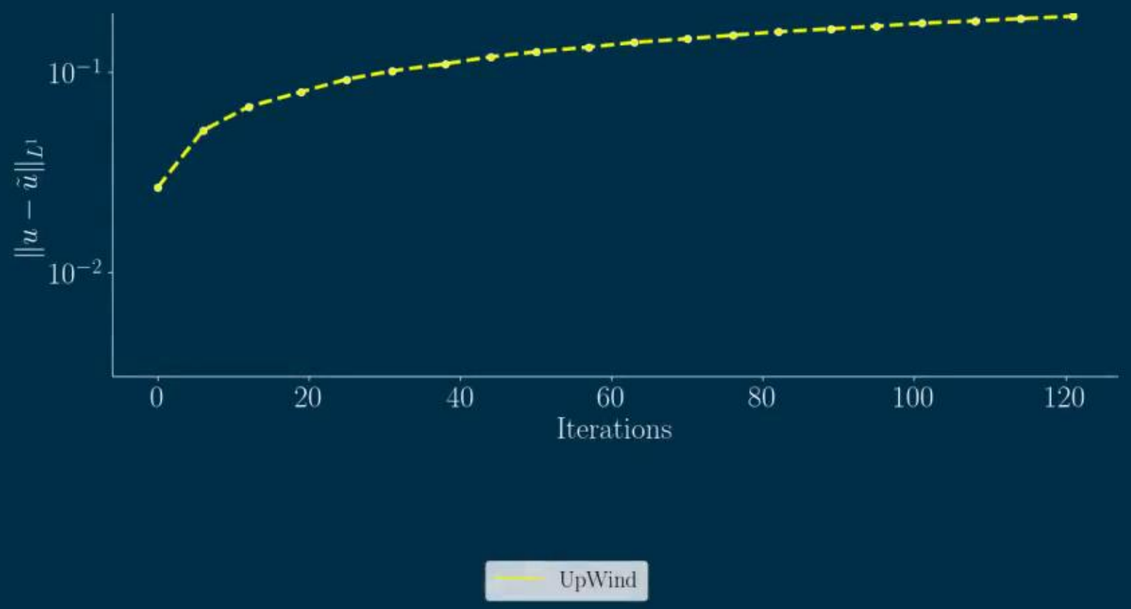


Finite volumes: smooth shape

UpWind





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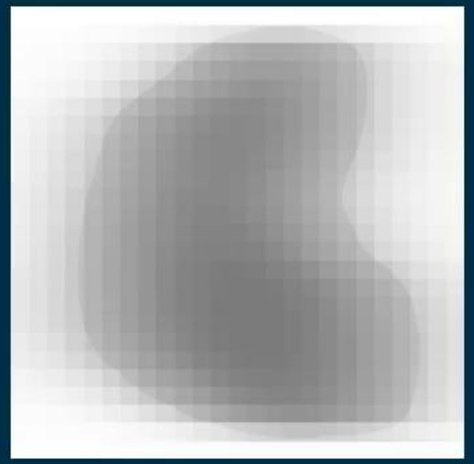
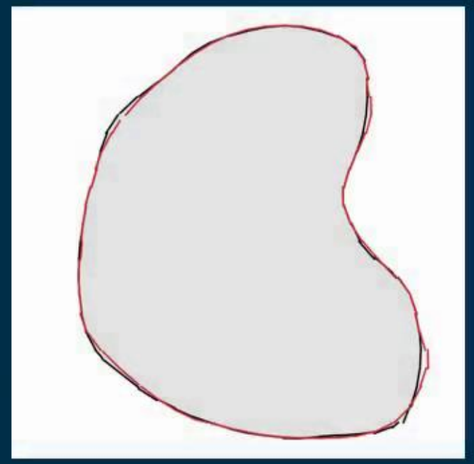
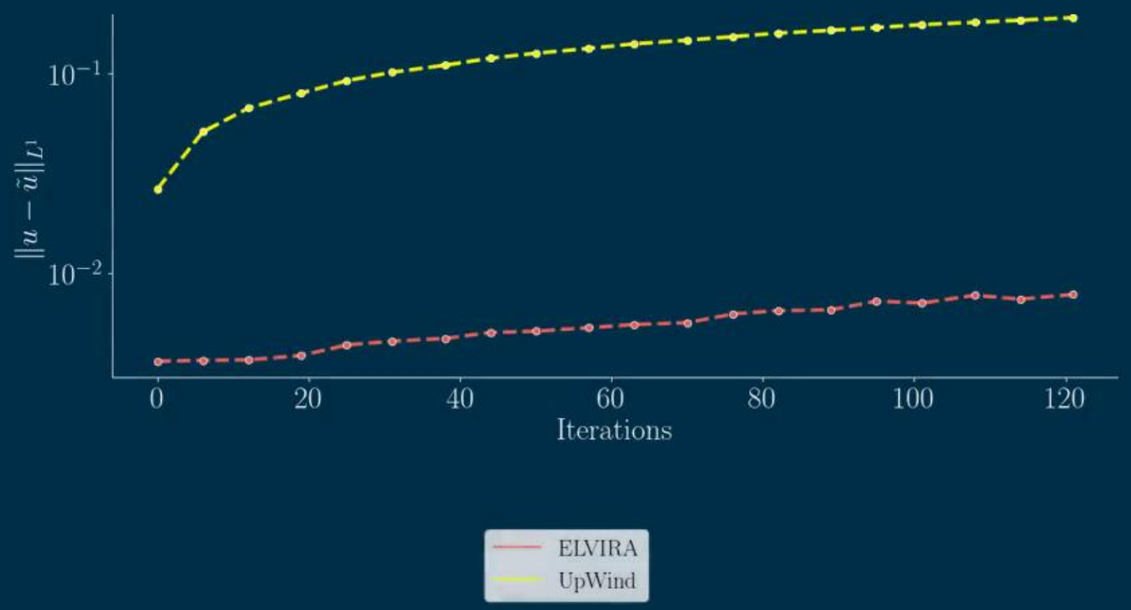


ELVIRA-WO





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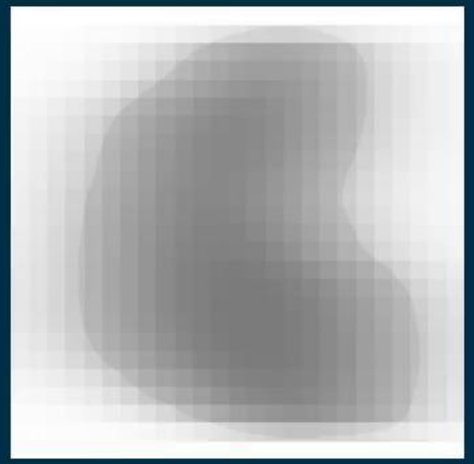

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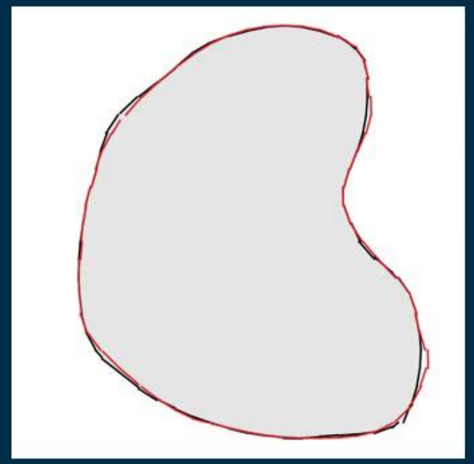
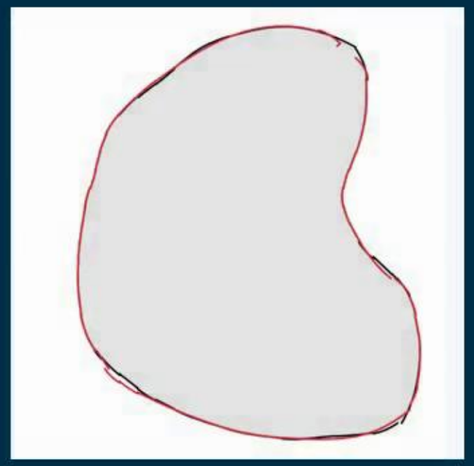
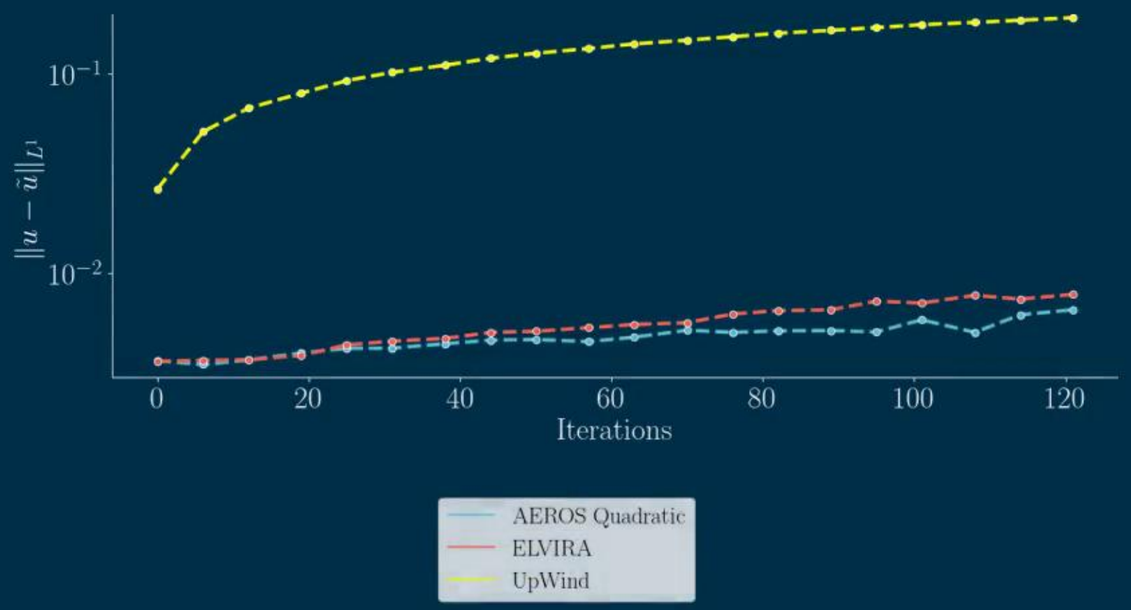


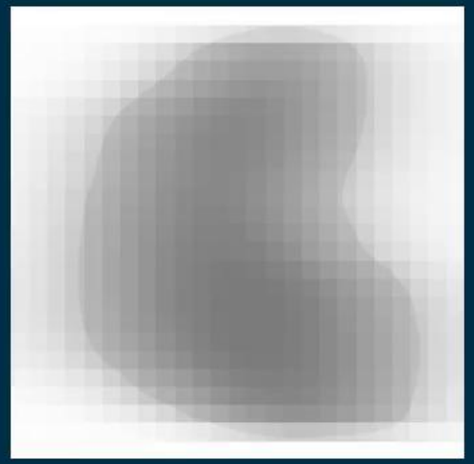


AEROS Quadratic



Finite volumes: smooth shape

UpWind



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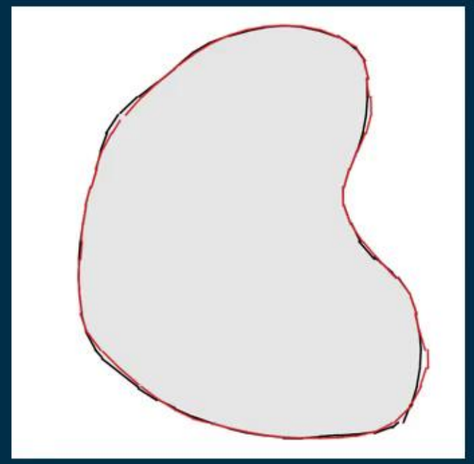

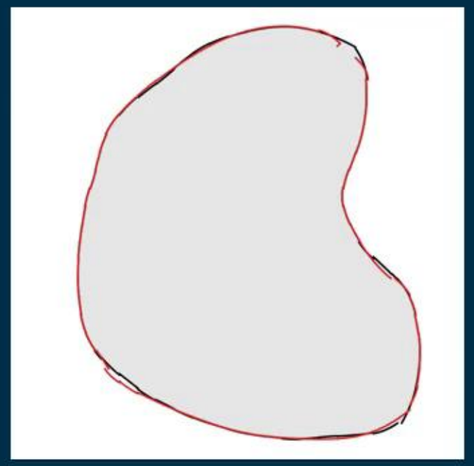

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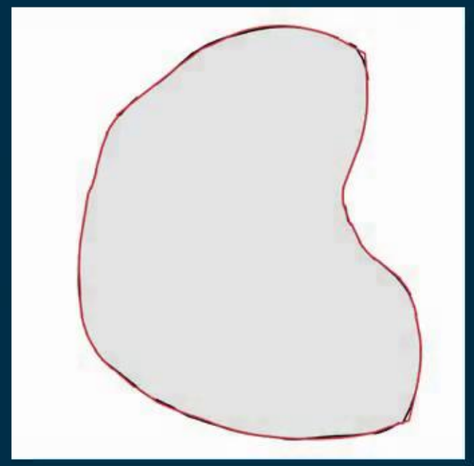
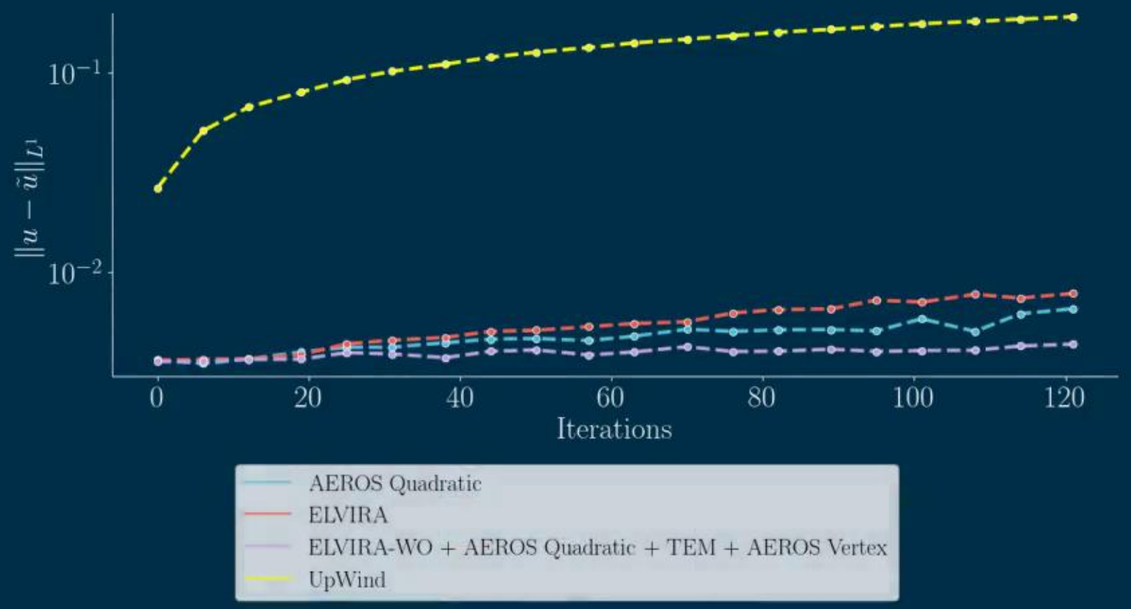
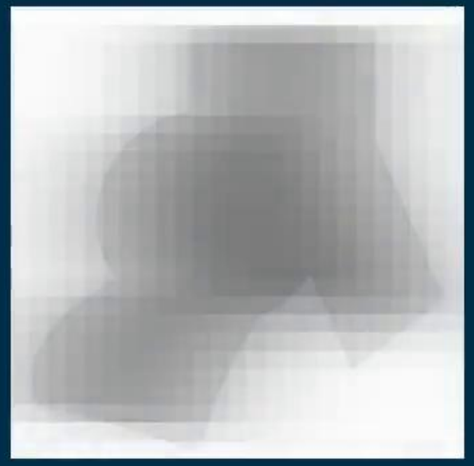


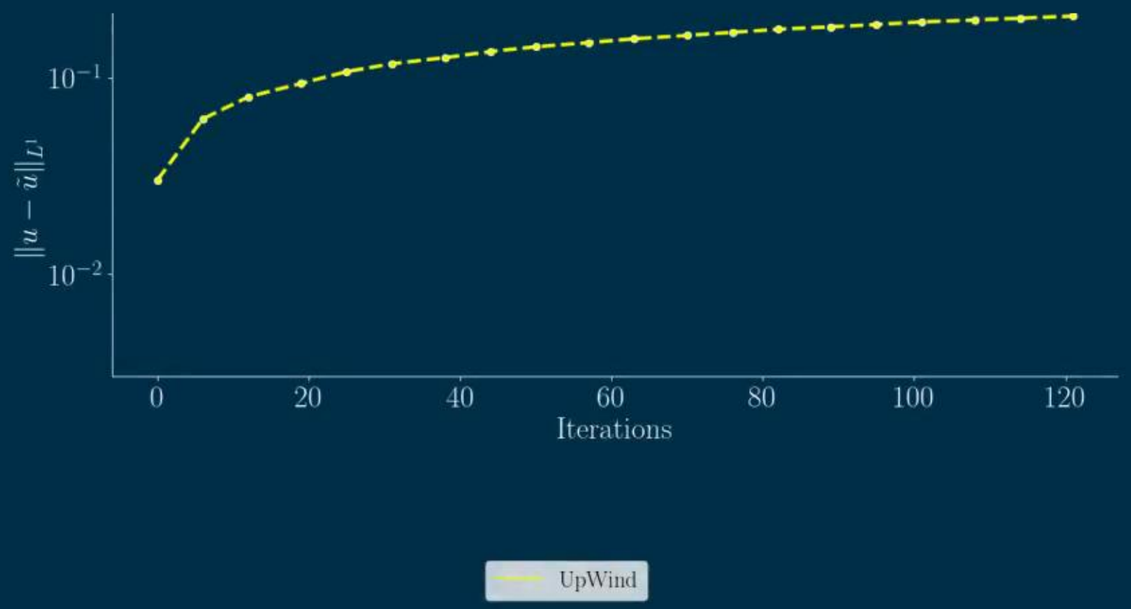
All together



Finite volumes: shape with corners

UpWind





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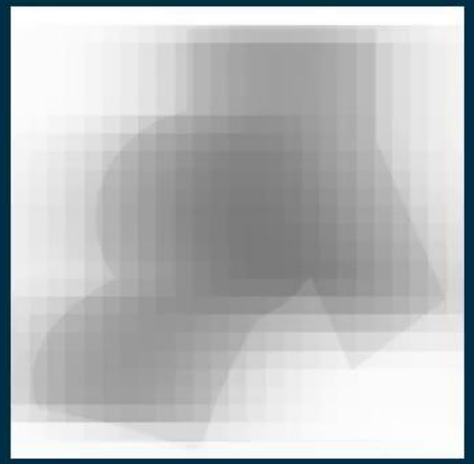

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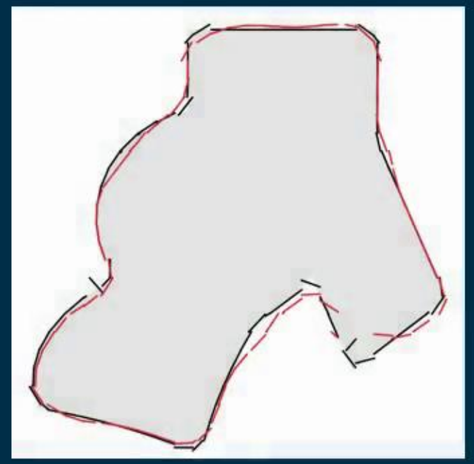

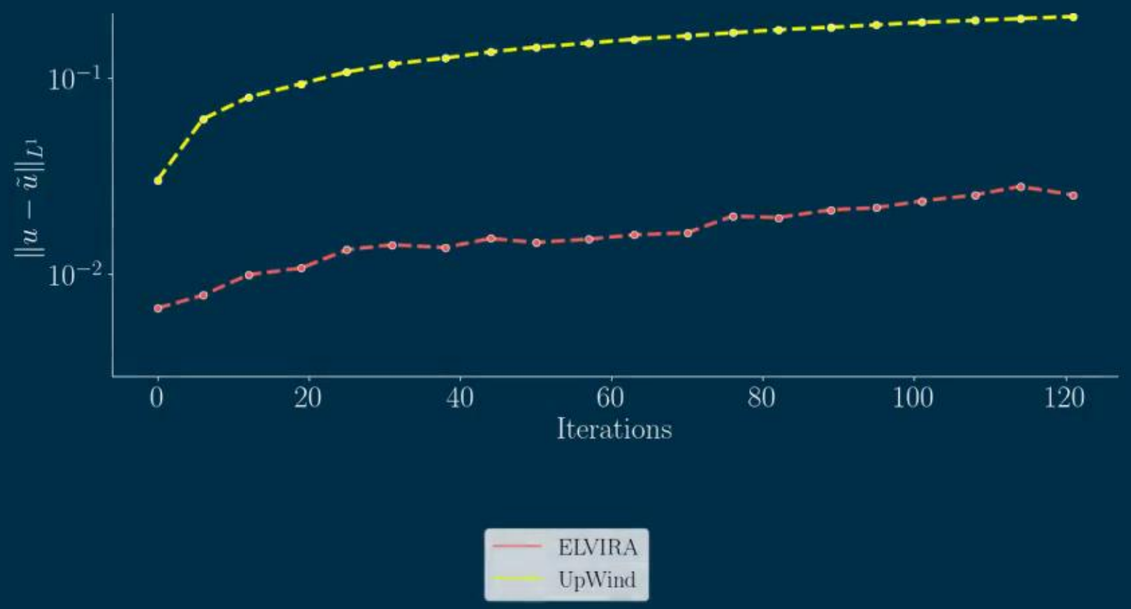

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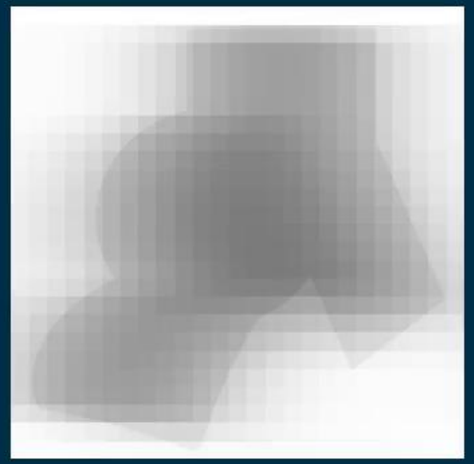

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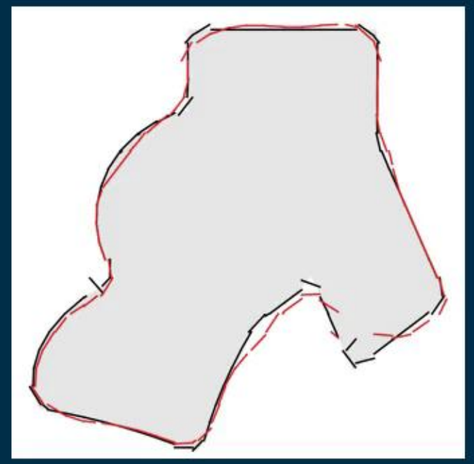

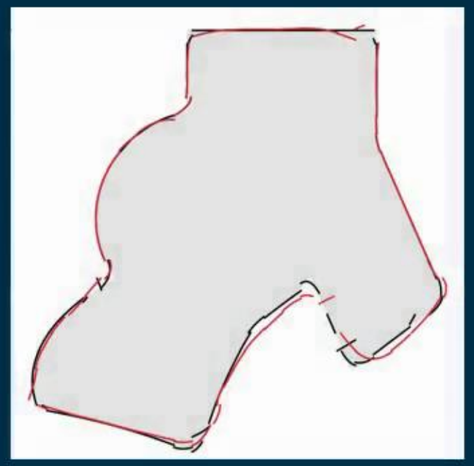

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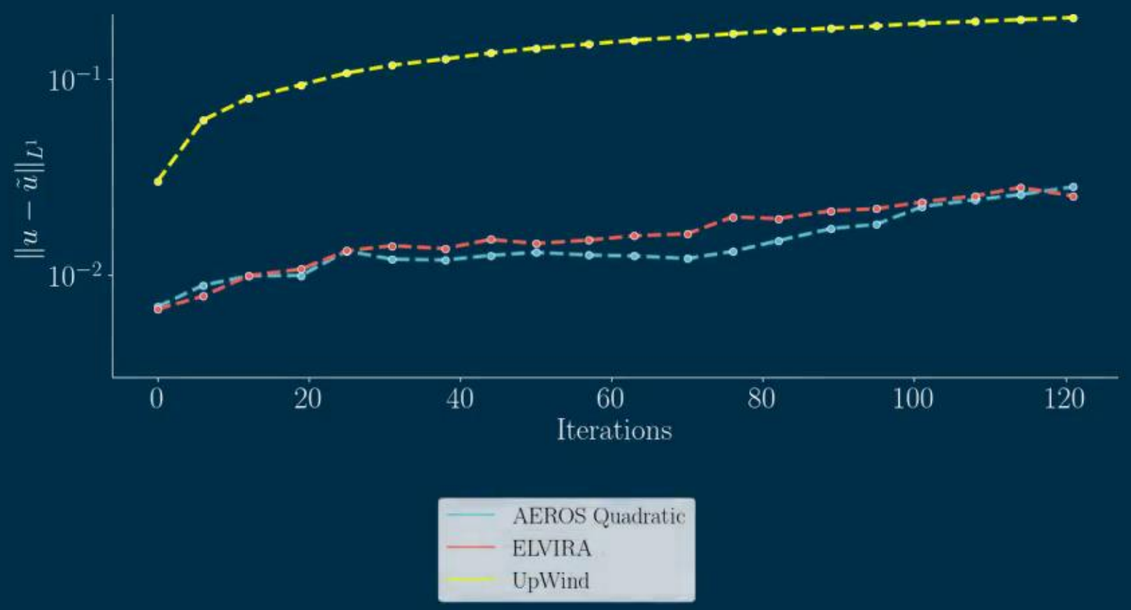

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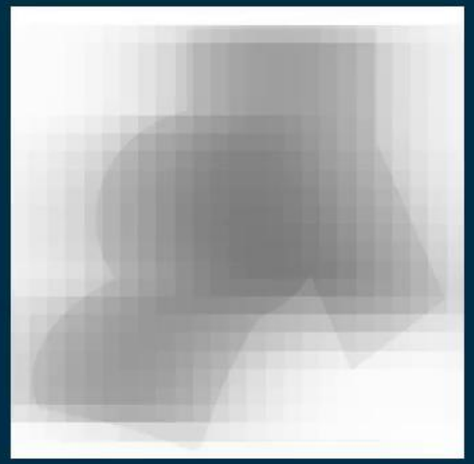
UpWind



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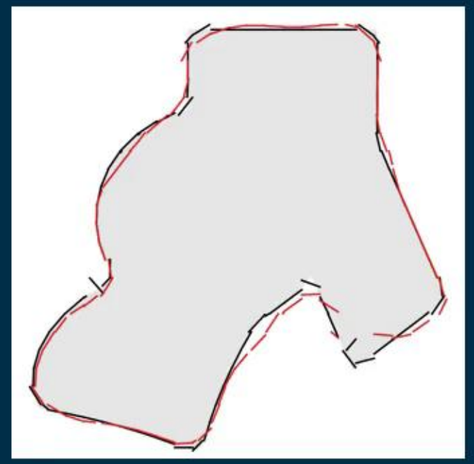

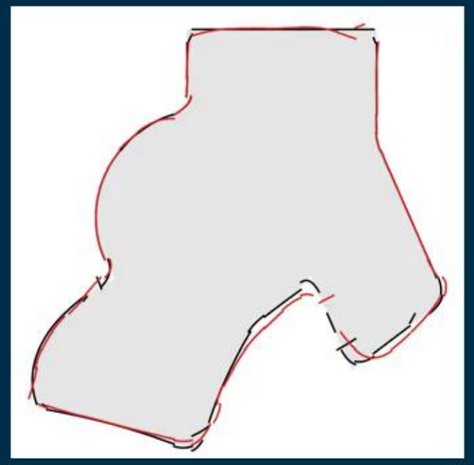

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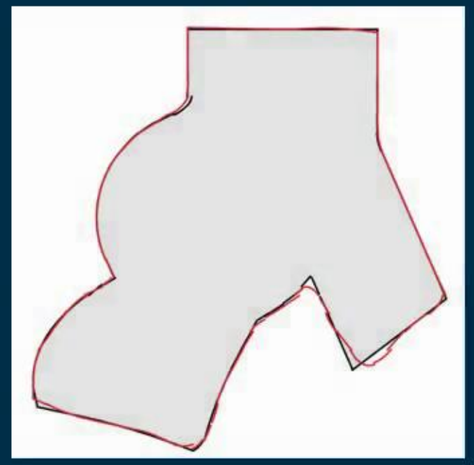

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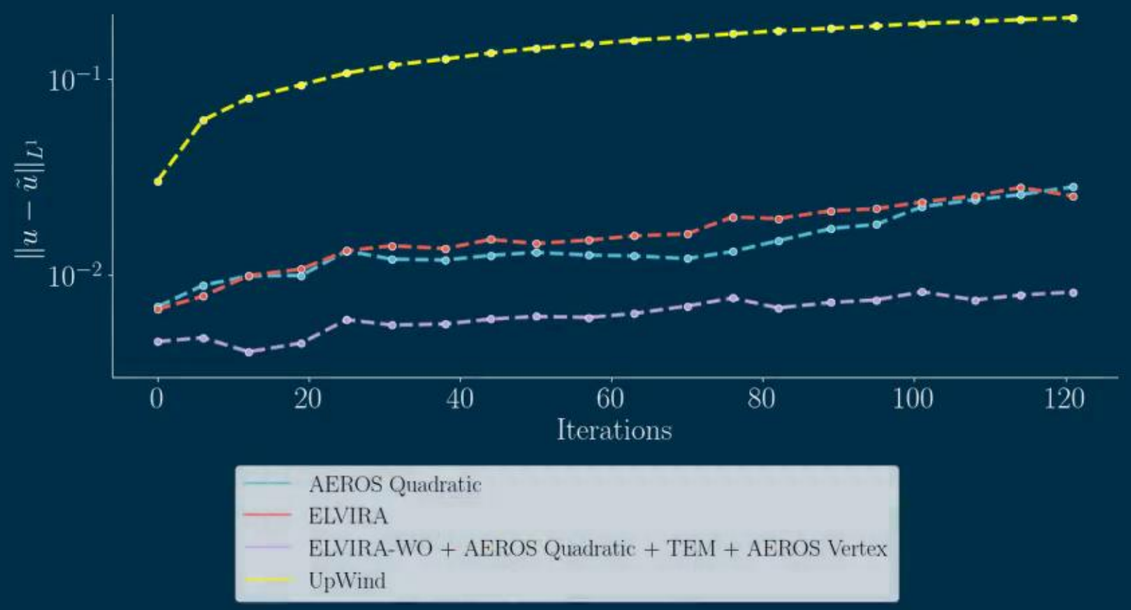
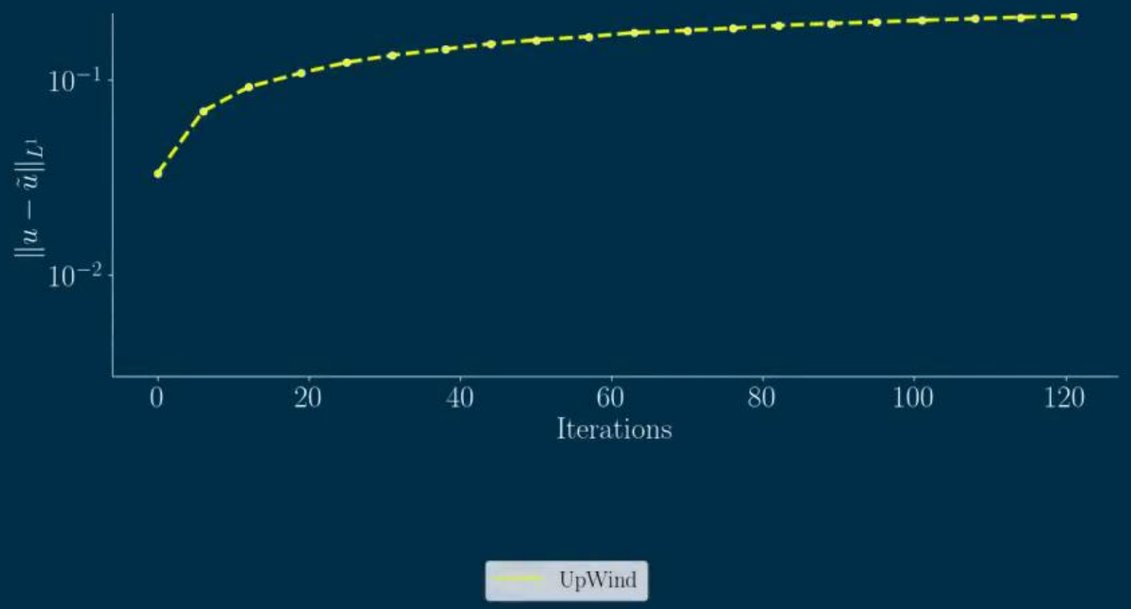


Finite volumes: Zalesak notched circle

UpWind



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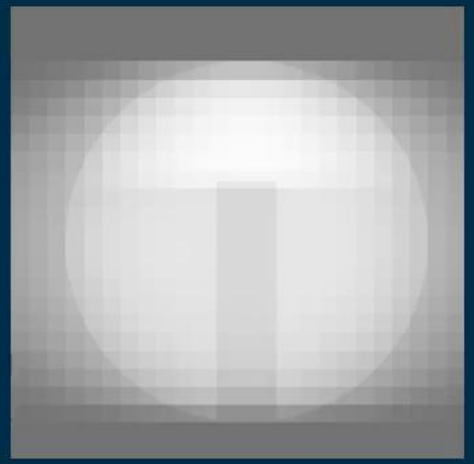

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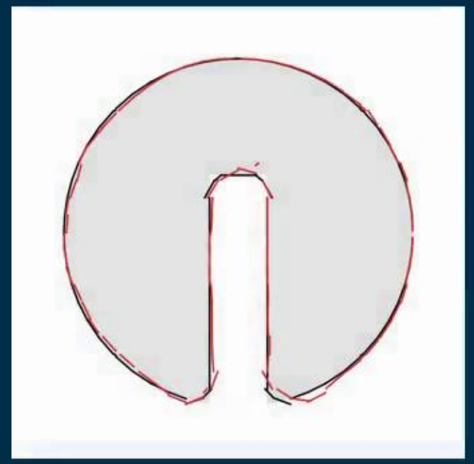

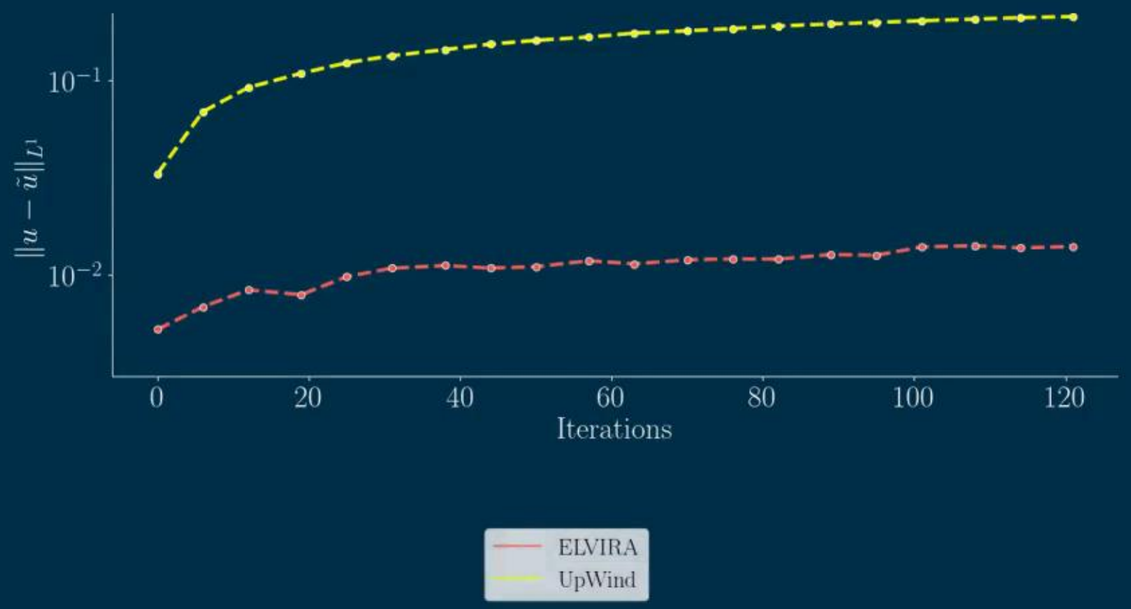

Finite volumes: Zalesak notched circle

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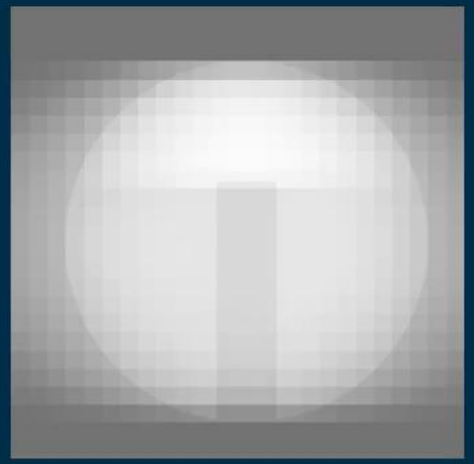

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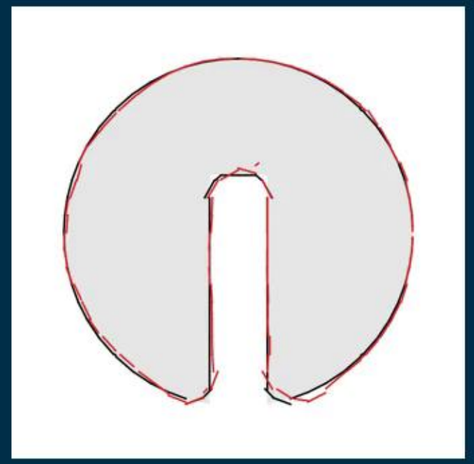
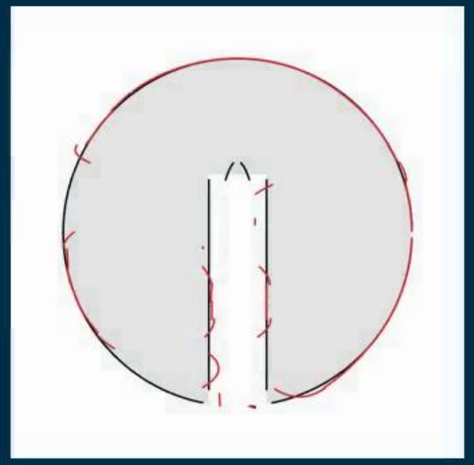
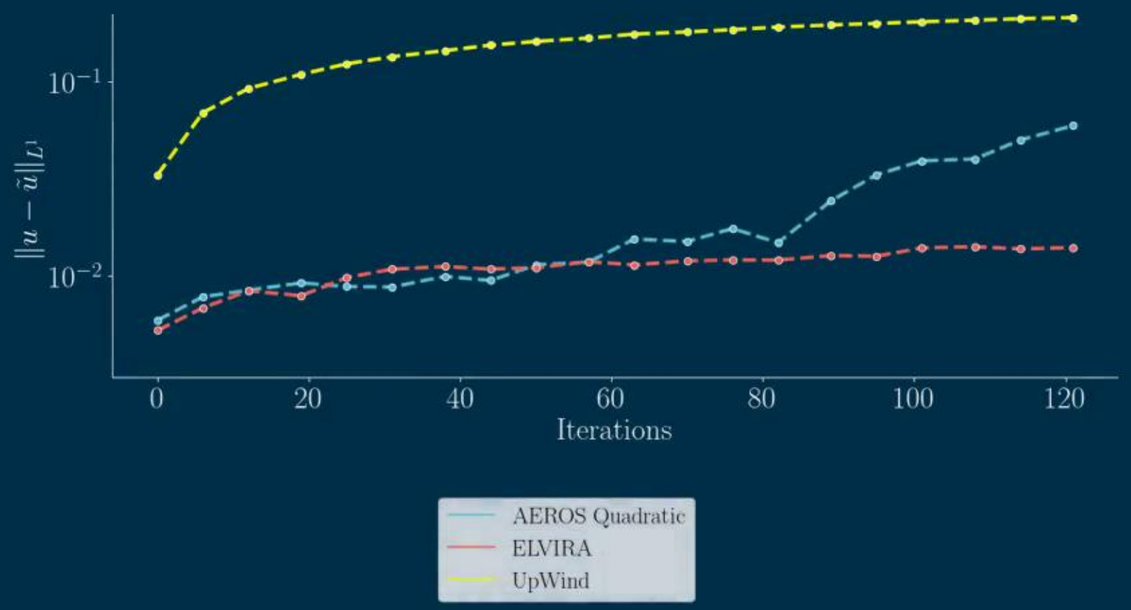


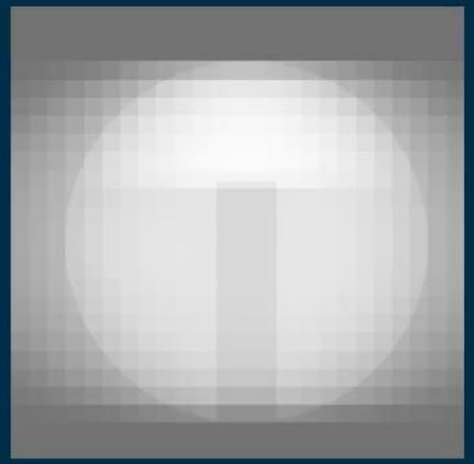


AEROS Quadratic

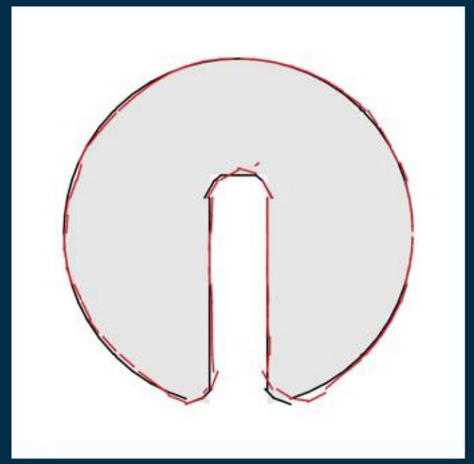


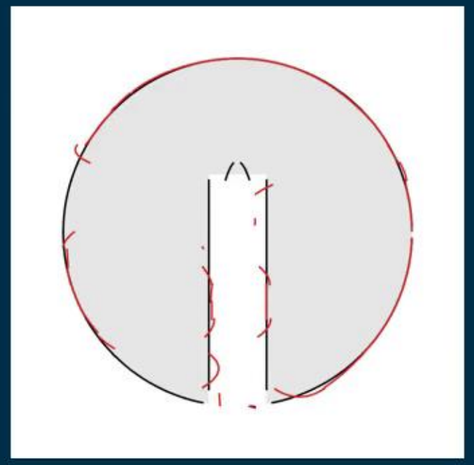
Finite volumes: Zalesak notched circle

UpWind



ELVIRA-WO





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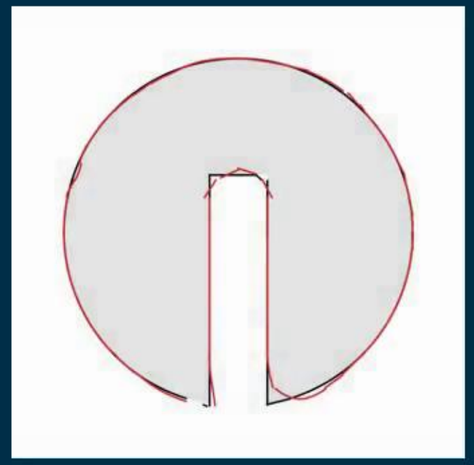
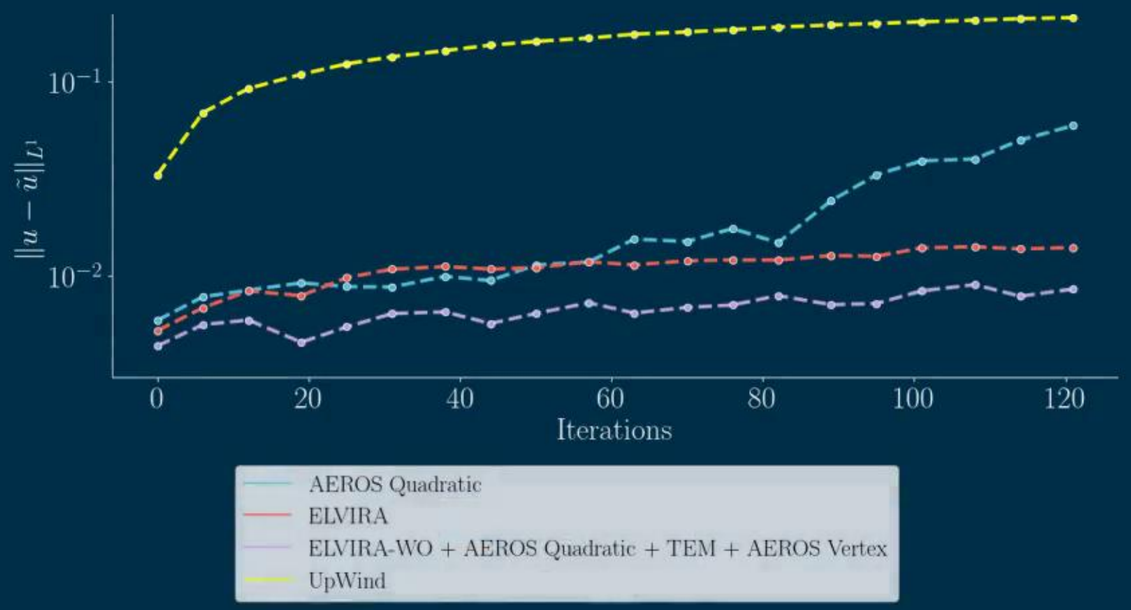


All together



Summary and perspectives

Summary and perspectives

Near optimality property for non-linear spaces and best-fit estimator

$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z \qquad \|u - \tilde{u}\|_V \leq (1 + 2\alpha_Z \mu_Z)\|u - P_{V_n} u\|_V + (1 + 2\beta\mu_Z)\|\eta\|_p$$

Summary and perspectives

Near optimality property for non-linear spaces and best-fit estimator

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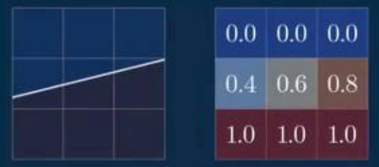
LVIRA: application to interface reconstruction



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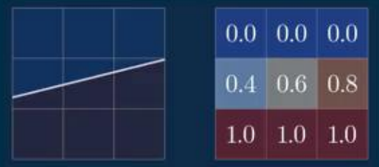
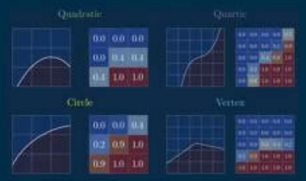


OBERA: more general interfaces



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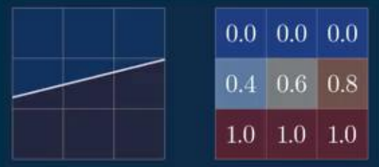

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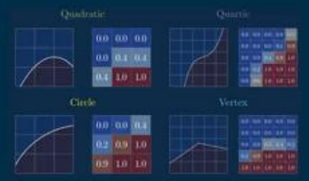
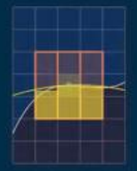


AEROS: addressing computational complexity problem



Summary and perspectives

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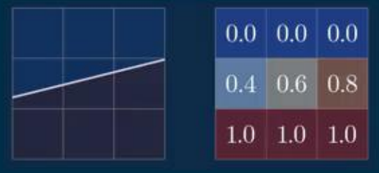
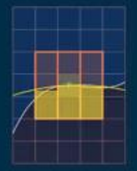


OBERA: more general interfaces

AEROS: addressing computational complexity problem



Vertices: when domains present corners

Summary and perspectives

Near optimality property for non-linear spaces and best-fit estimator

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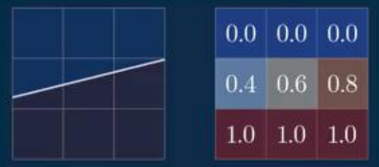

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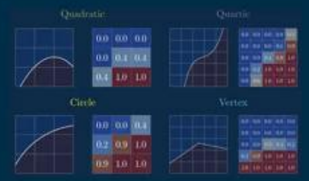

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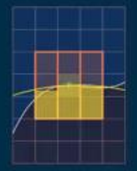
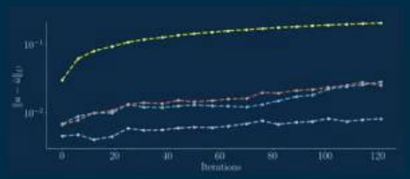


Vertices: when domains present corners

PDEs by Finite Volumes Schemes



All code available open source at:
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Code for presentation at:
https://github.com/agussomacal/presentations

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