Model reduction for forward simulation and inverse problems: towards non-linear approaches

Agustín Somacal

Supervisors: Albert Cohen and Olga Mula

Introduction: motivation

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Modelling through a

parametric **PDE** (Partial differential equation)

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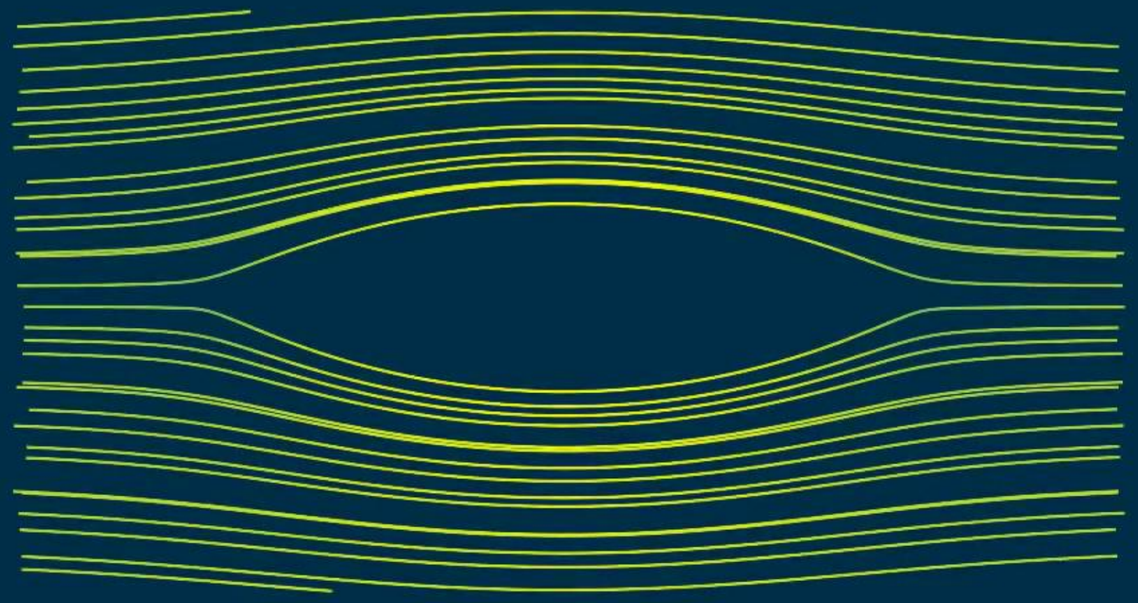

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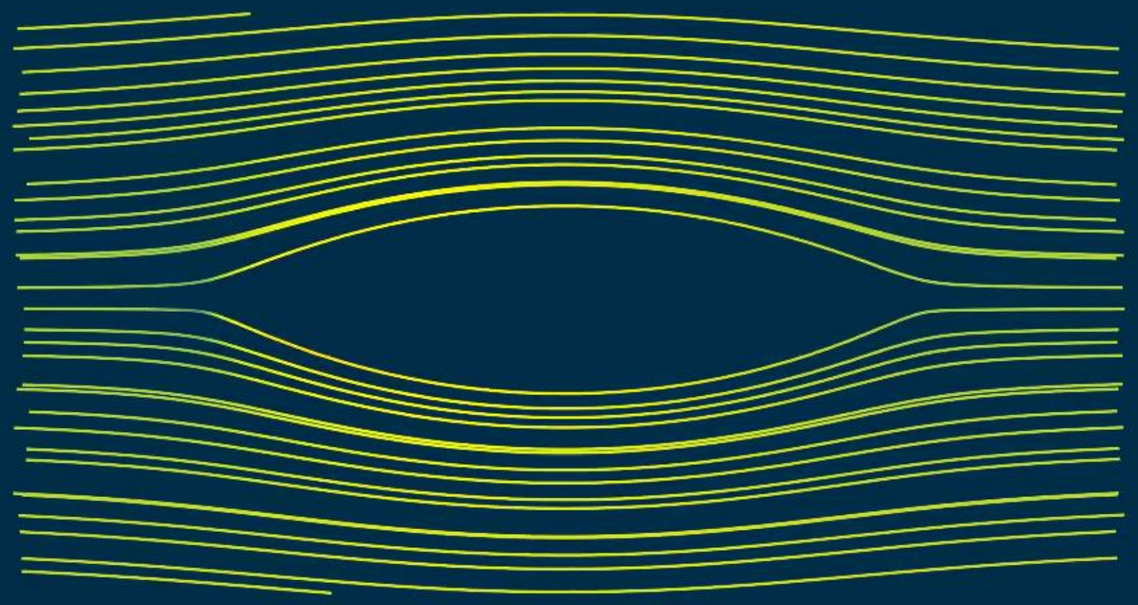

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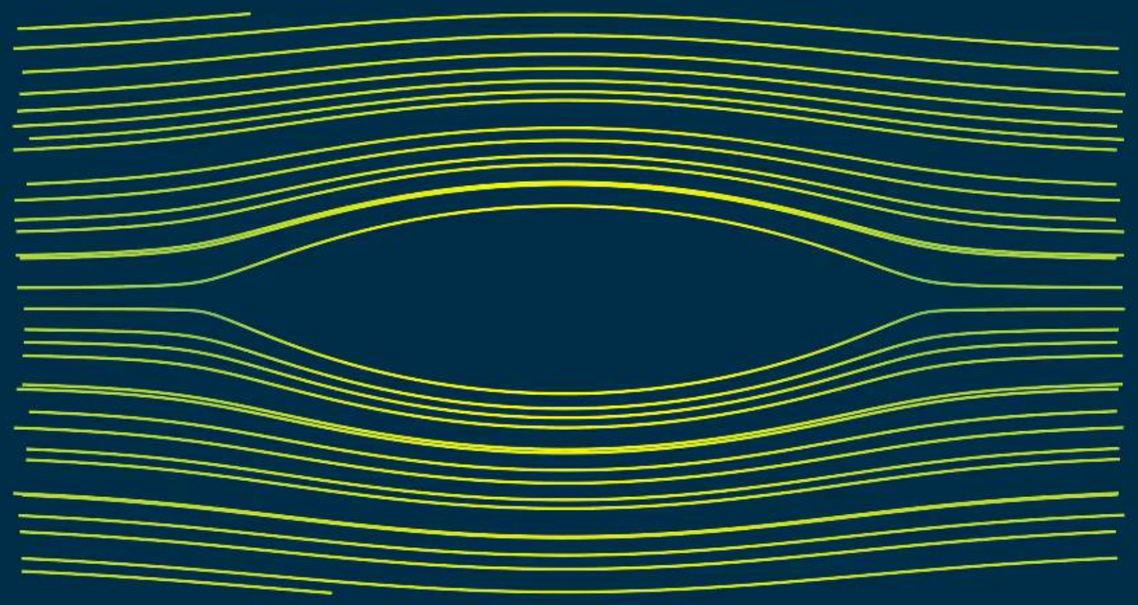

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$$\mathcal{P}(u, y) = 0$$

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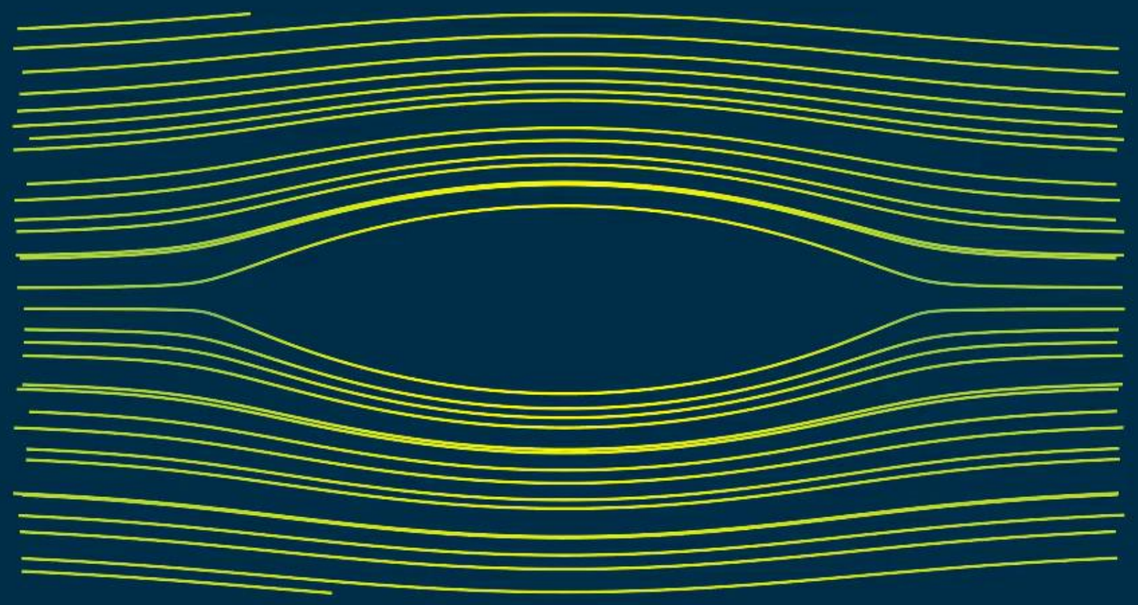

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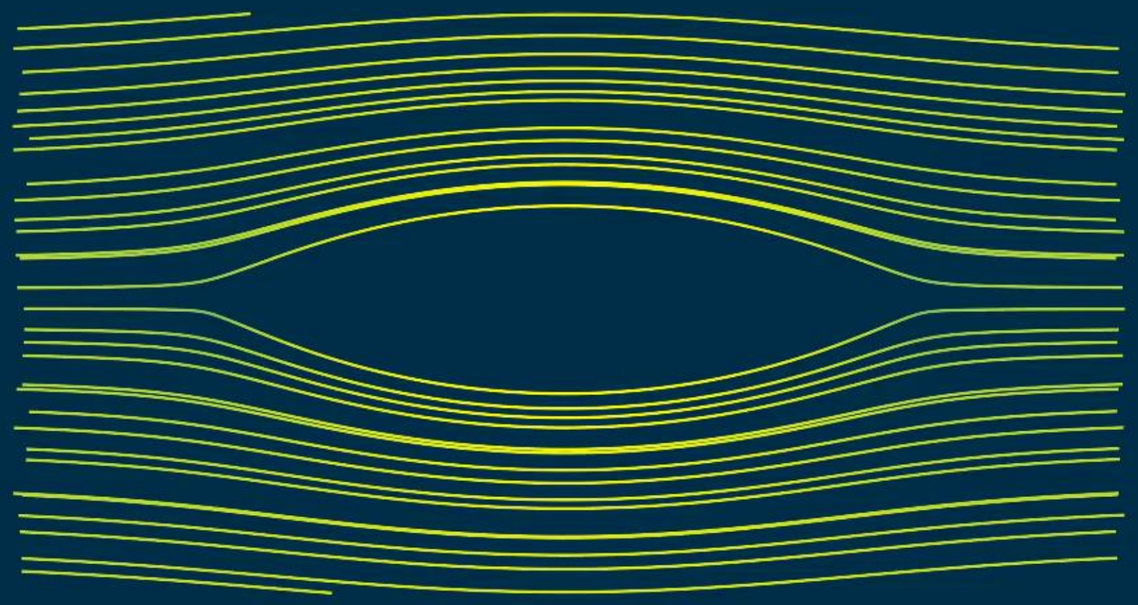

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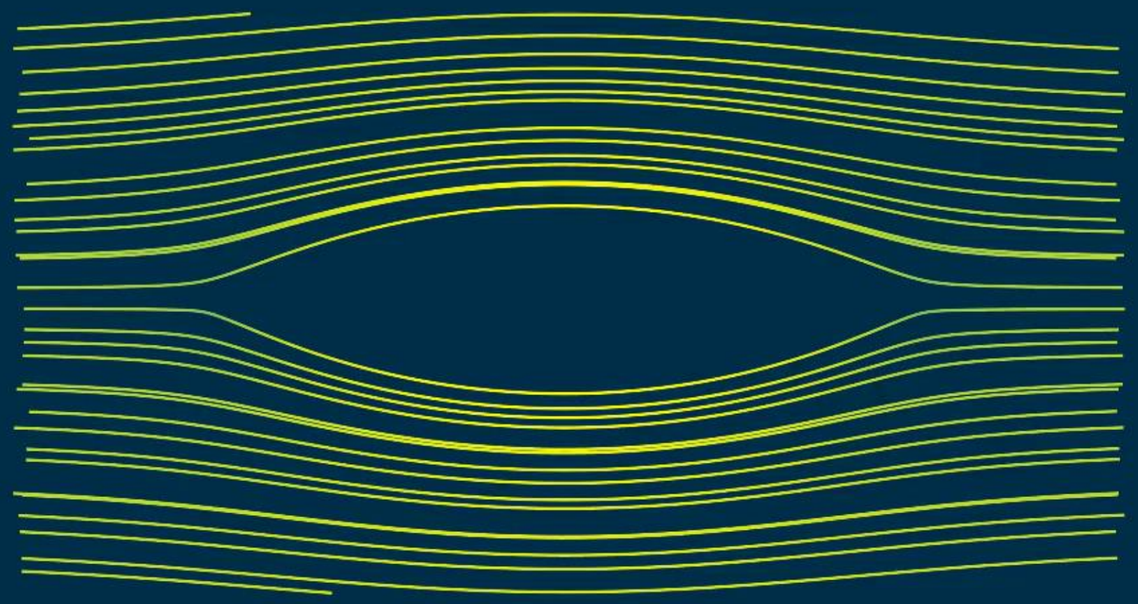

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$y := \{y_1, \ldots, y_d\} \in \mathbb{R}^d$

Modelling through a

parametric **PDE** (Partial differential equation)

Example PDEs

Transport PDE

$$\frac{\partial u}{\partial t} - v \cdot \nabla u = 0$$

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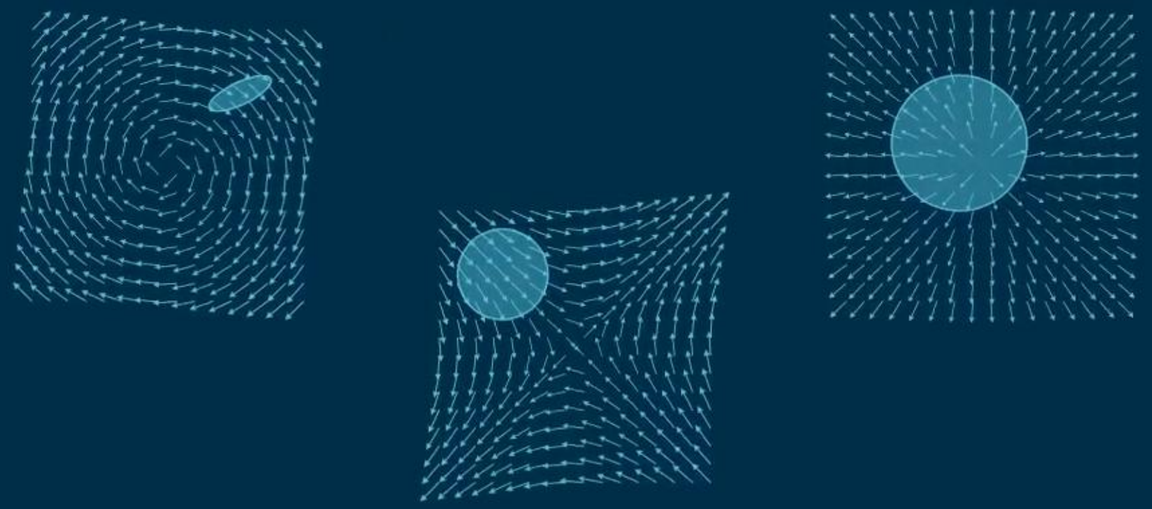

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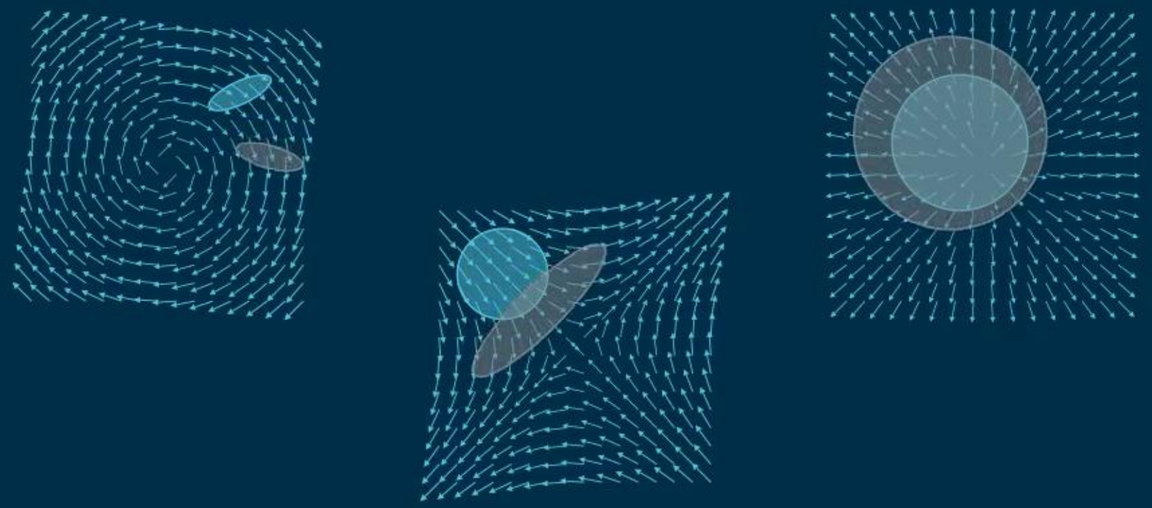

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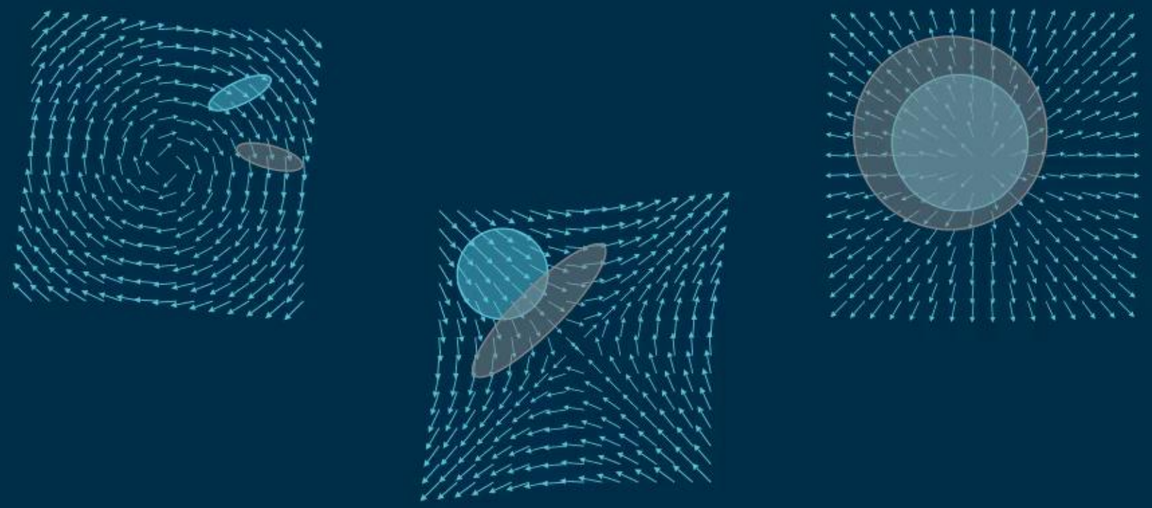

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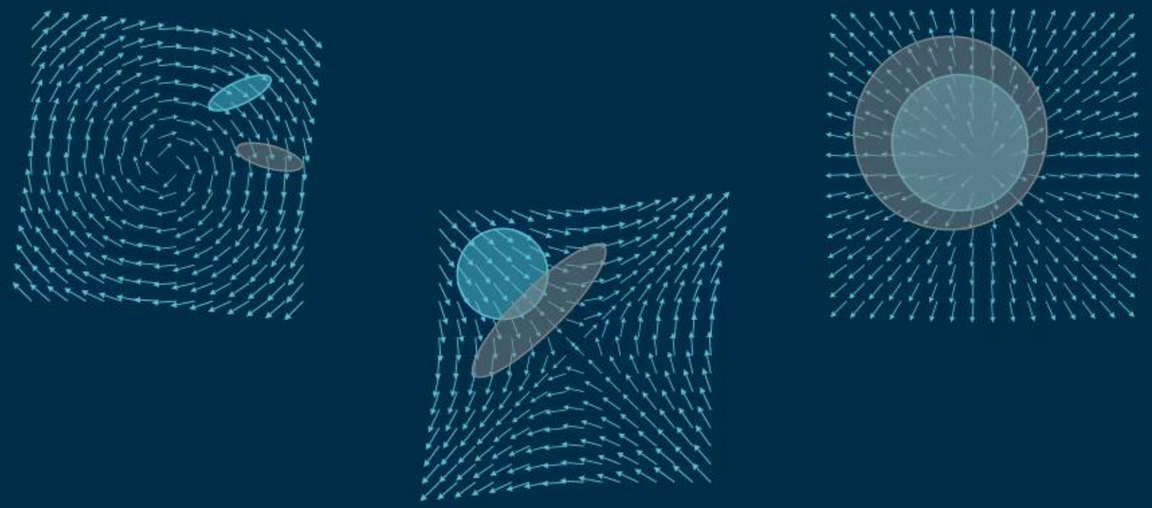

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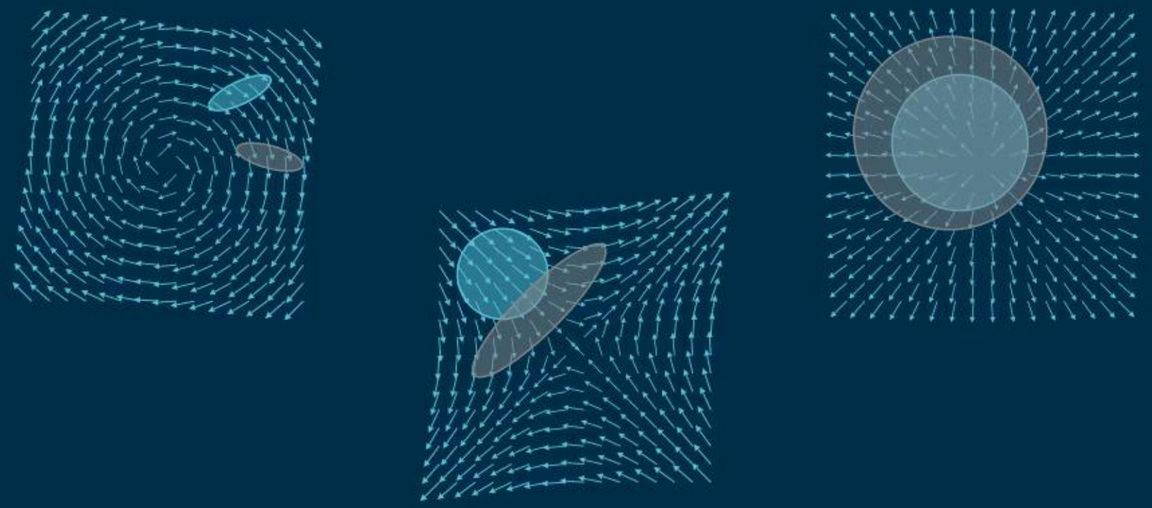

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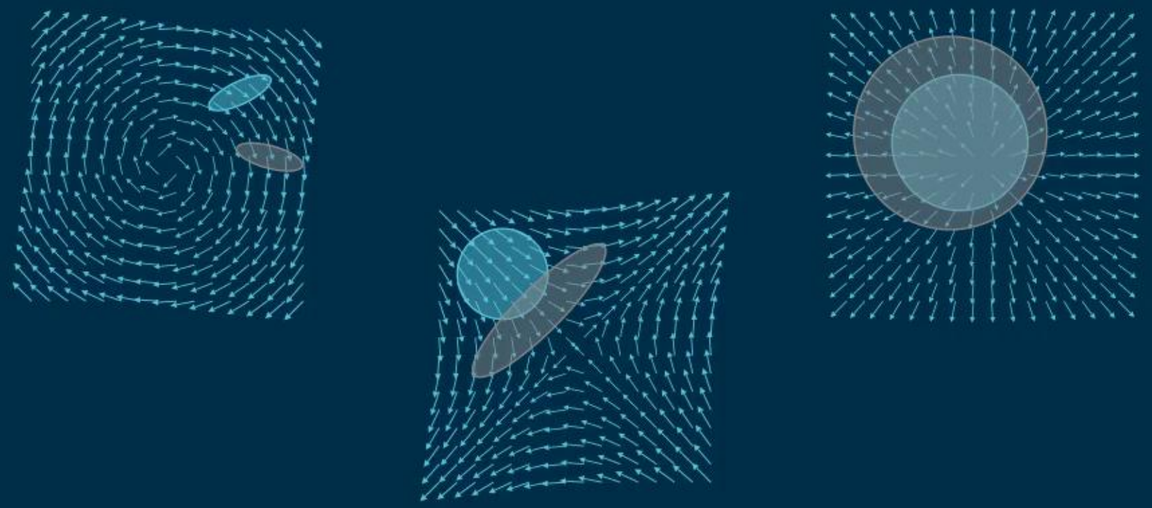

Diffusion PDE

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$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$$

8.0	6.0
1.0	2.0

6.0	2.0
8.0	1.0

1.0	8.0
4.0	1.0

Example PDEs

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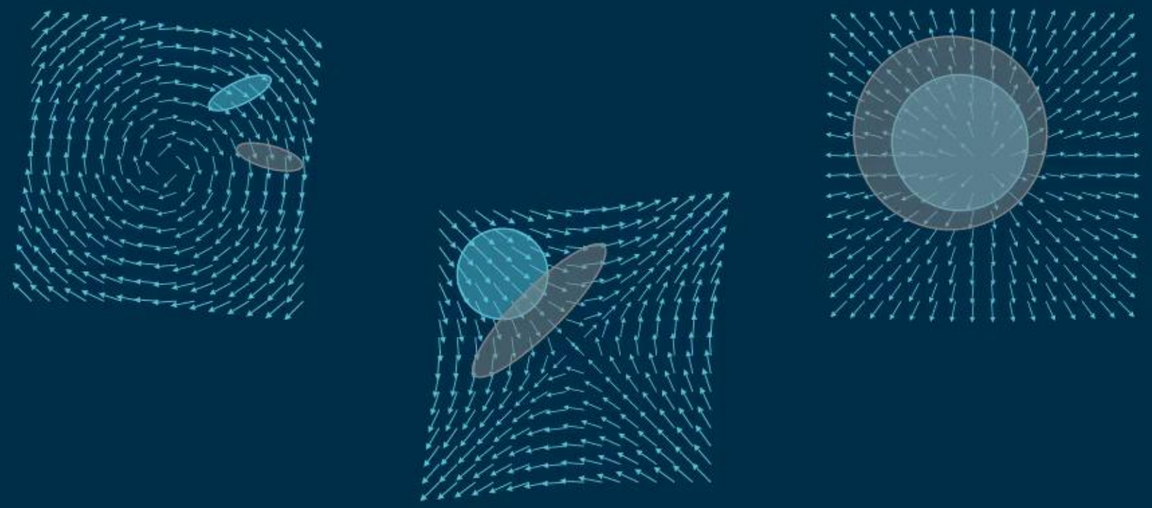

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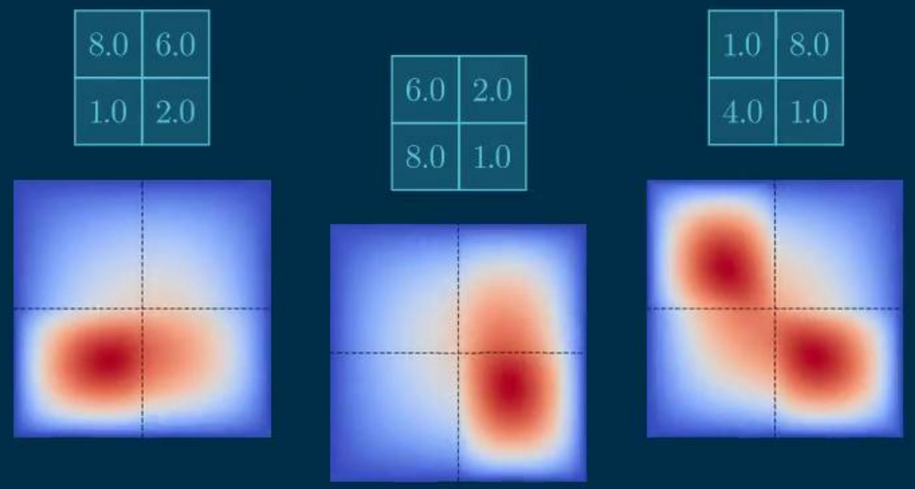

Two main problems

Forward modelling

Inverse problem

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$$y \in Y \mapsto u \in V$$

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3.0	7.0
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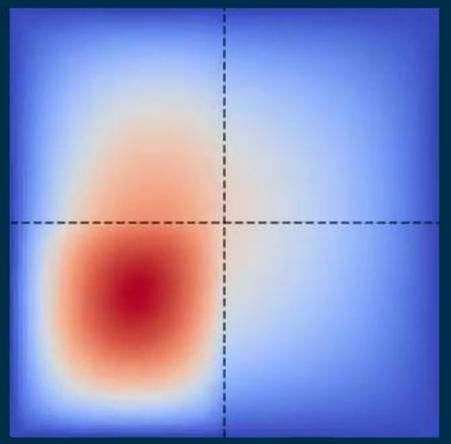


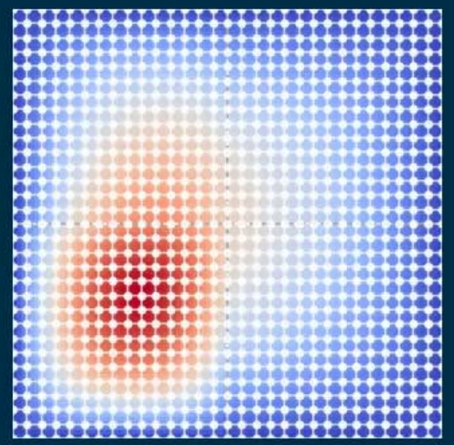
Inverse problem

Two main problems

Forward modelling

$$y \in Y \mapsto u \in V^N$$

FEM mesh



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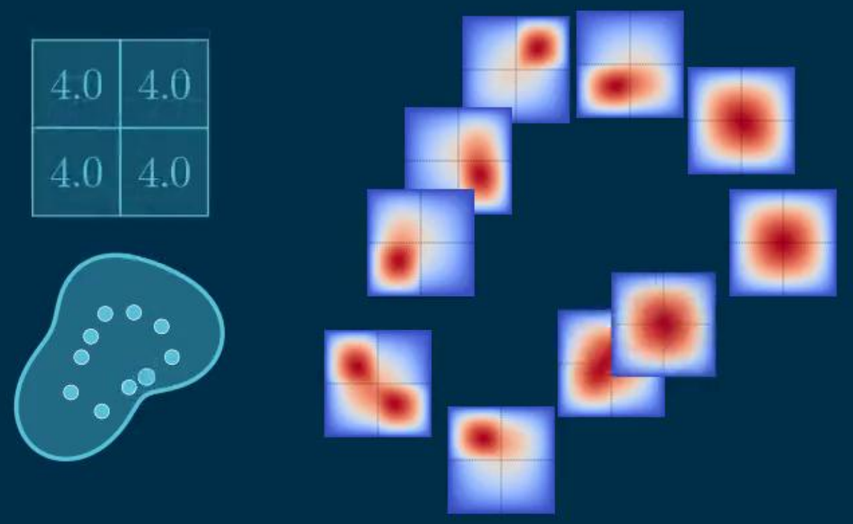

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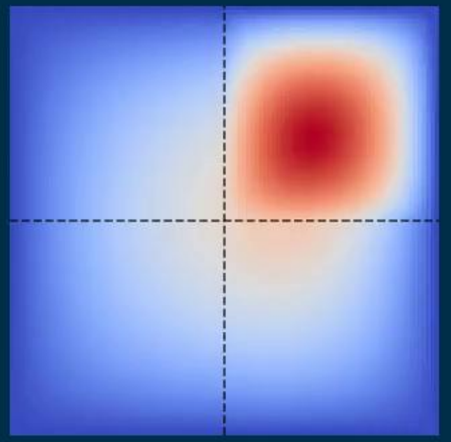

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$$\tilde{u} = S(y) \quad S : \mathbb{R}^d \to V$$

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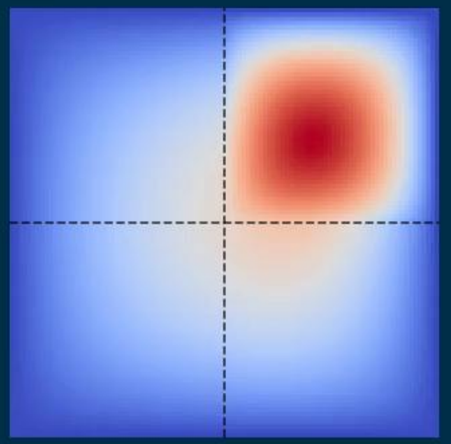

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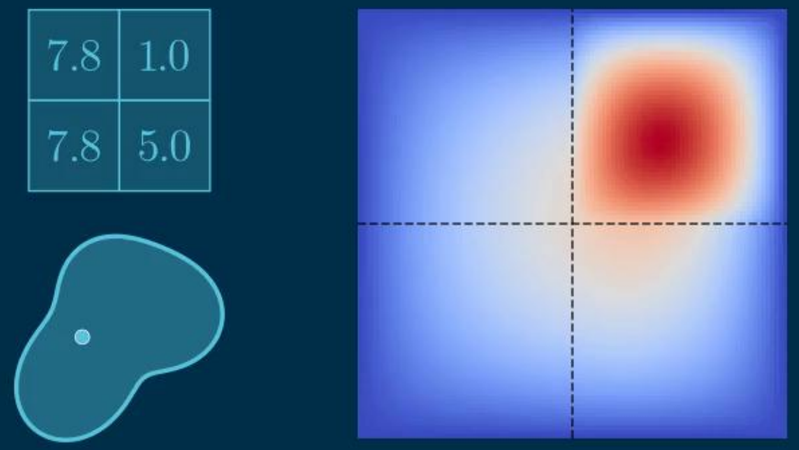

Inverse problem

$$\ell(u) + \eta = z \in \mathbb{R}^m$$

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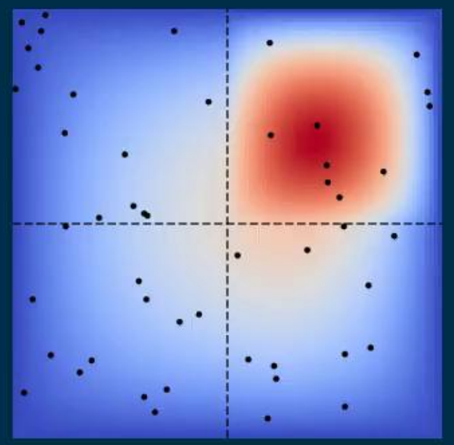

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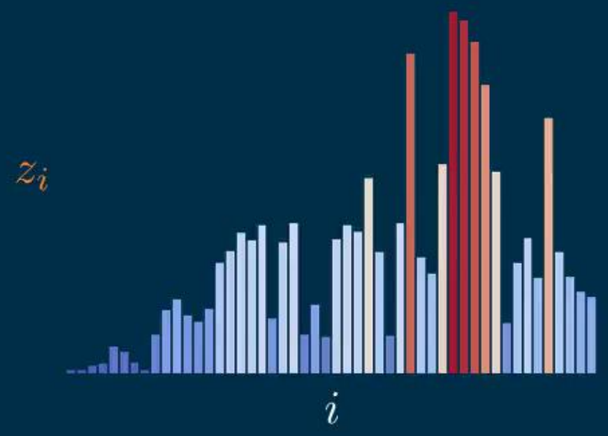

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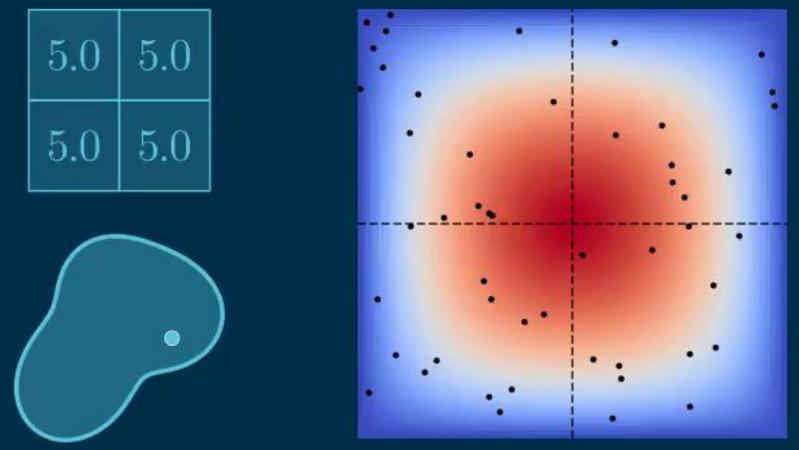

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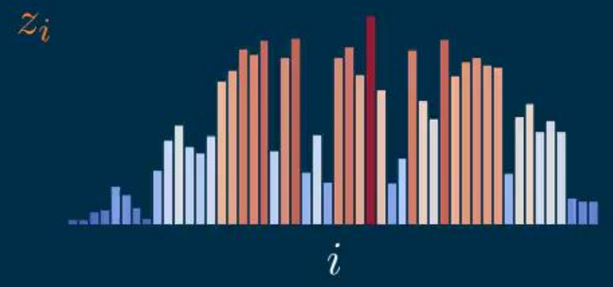

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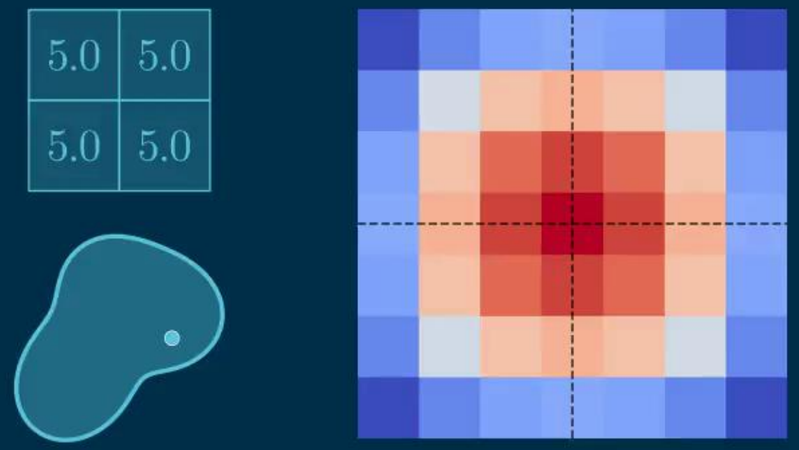

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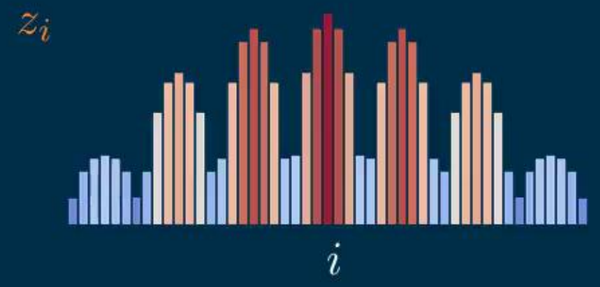
$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$



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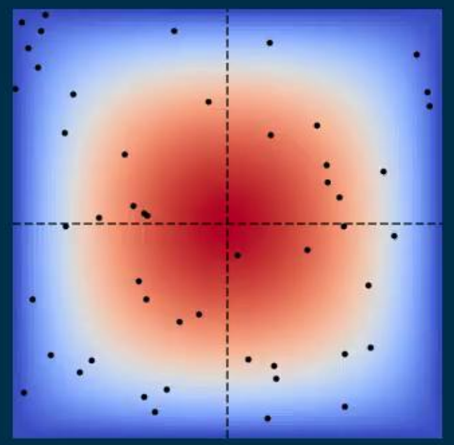

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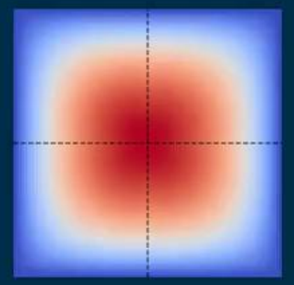

State estimation

$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

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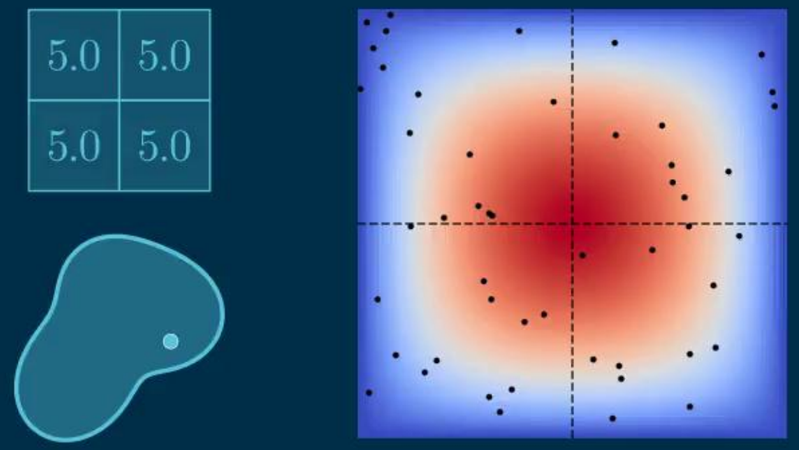

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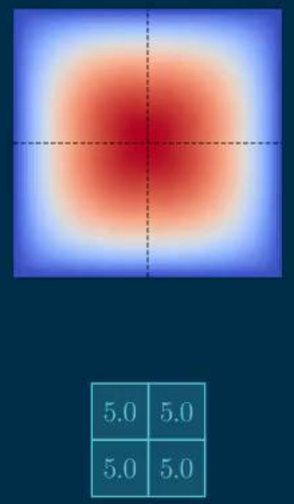

State estimation

$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

Parameter estimation

$$\tilde{y} = R(z) \quad R : \mathbb{R}^m \to \mathbb{R}^d$$

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$$y \in Y \mapsto u \in V$$

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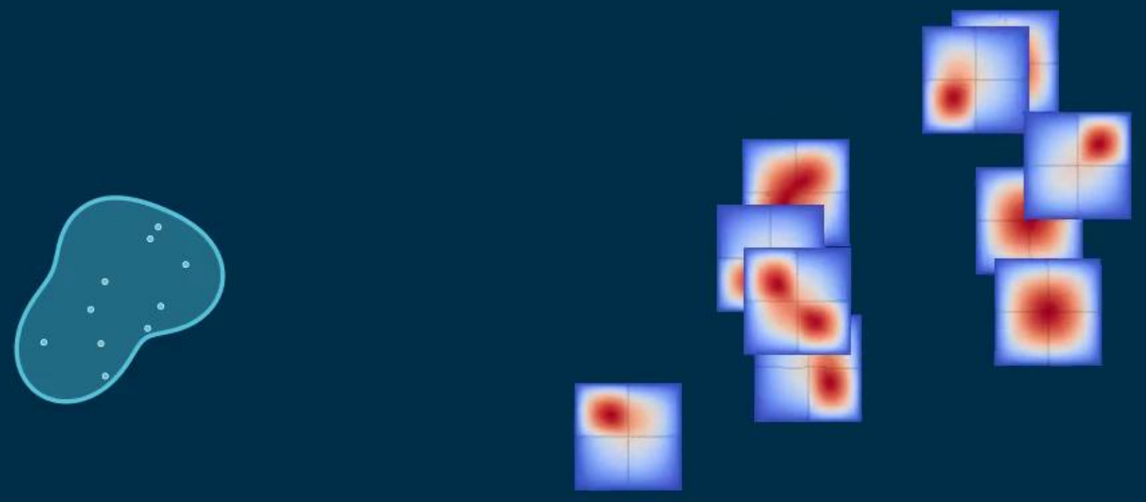

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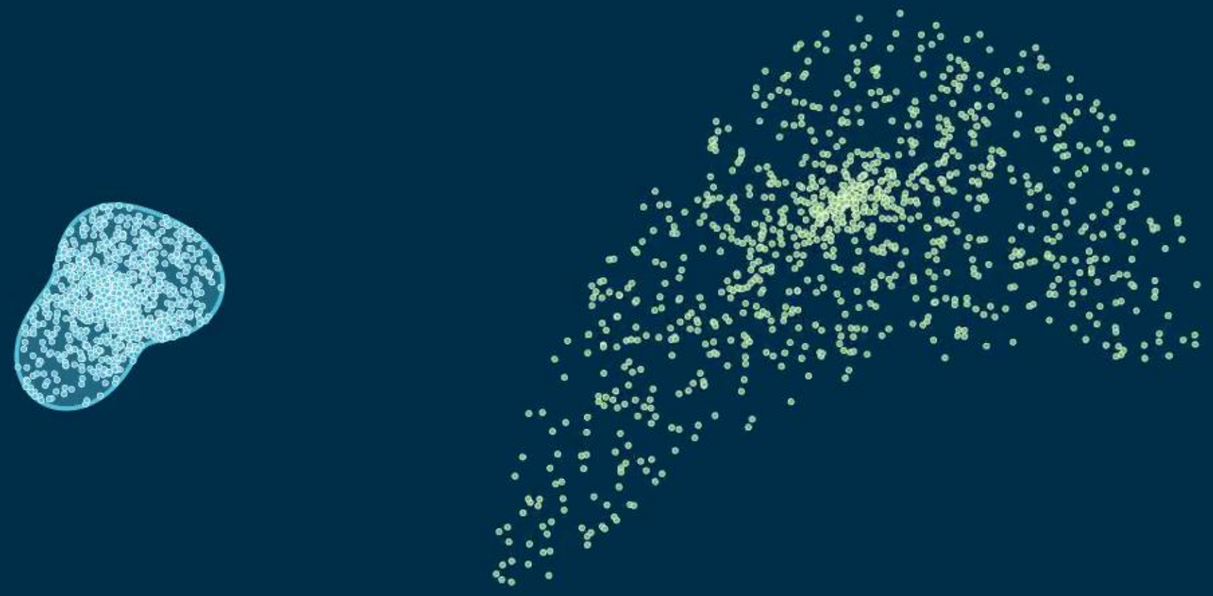

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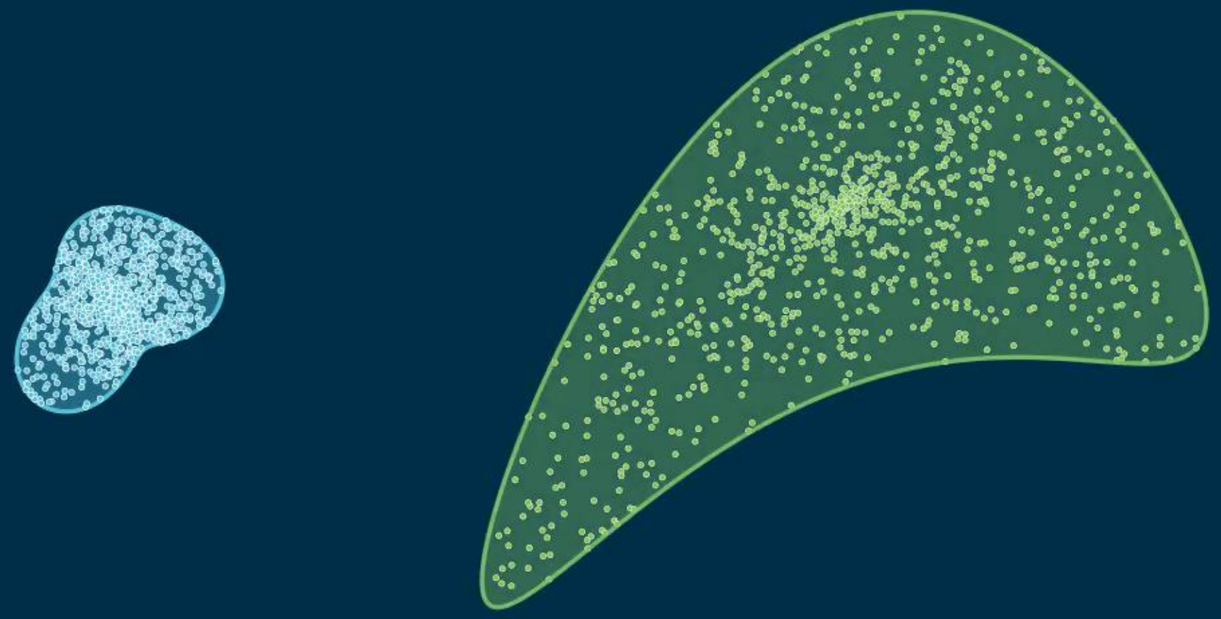
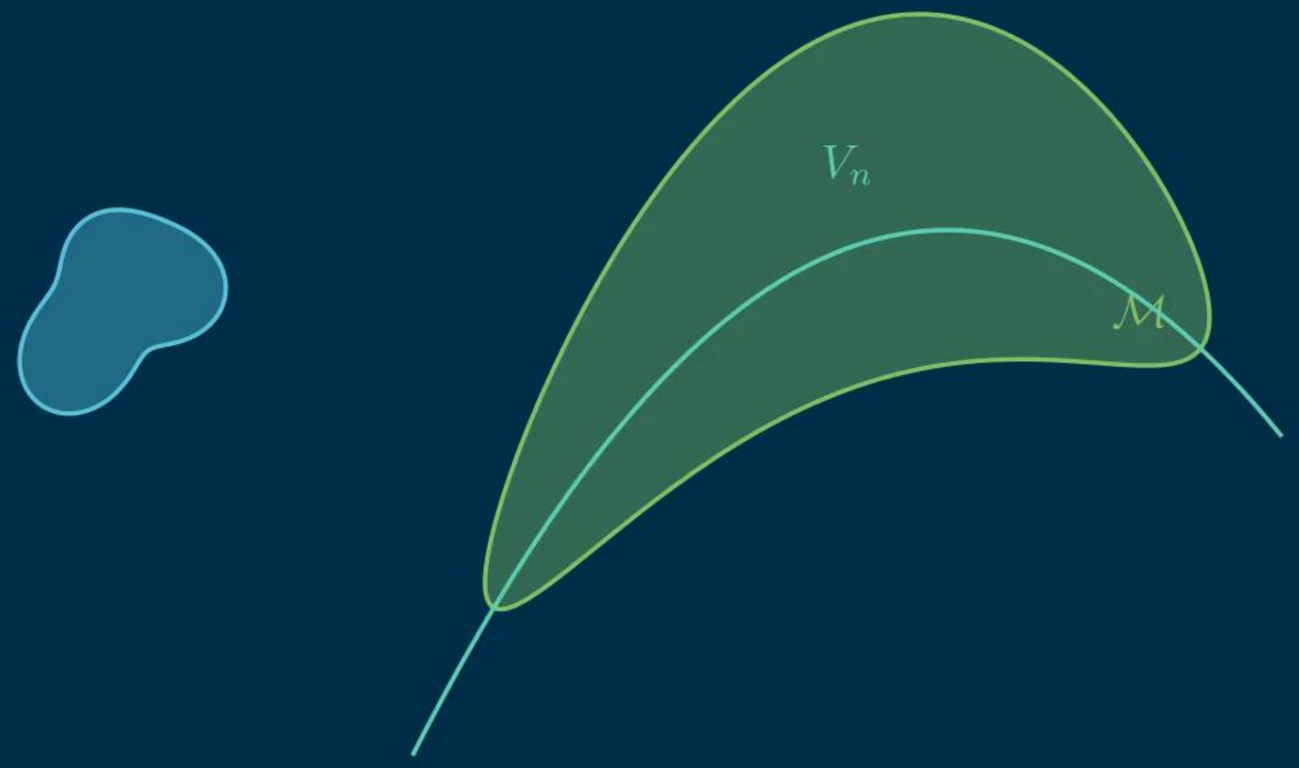


Solution Manifold

$$y \in Y \mapsto u \in \mathcal{M} \subset V$$

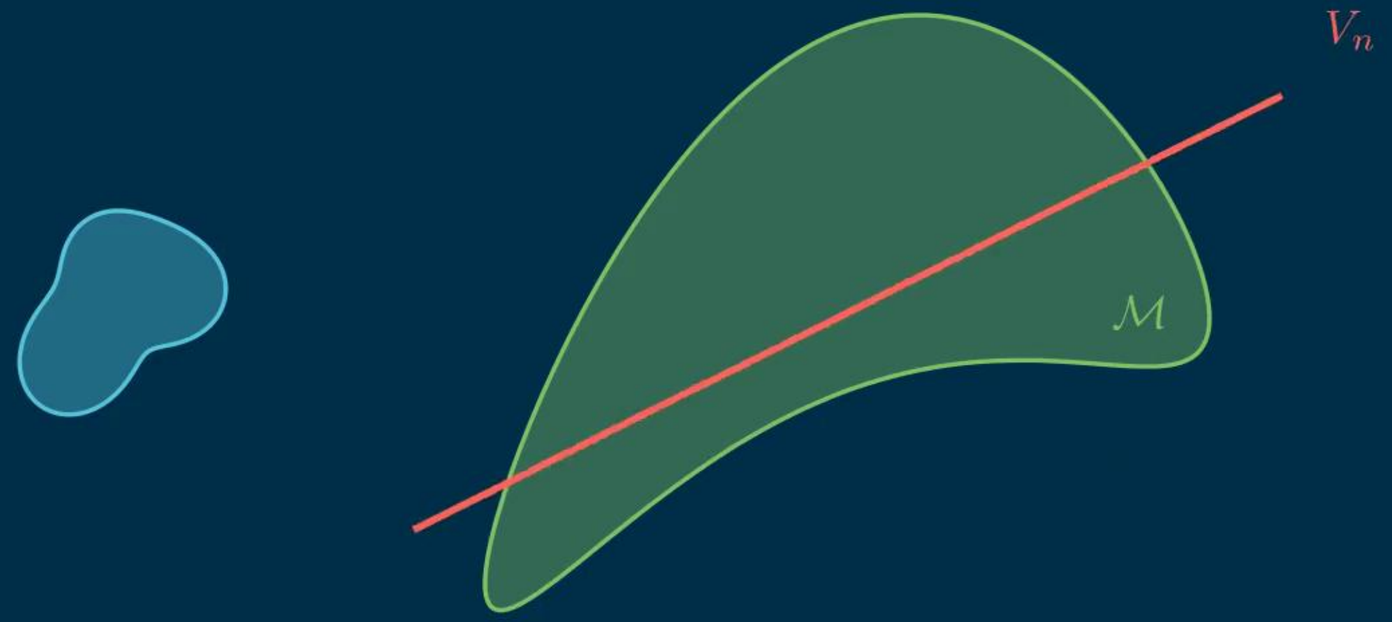
Reduce order models

$$y \in Y \mapsto u \in \mathcal{M} \subset V$$



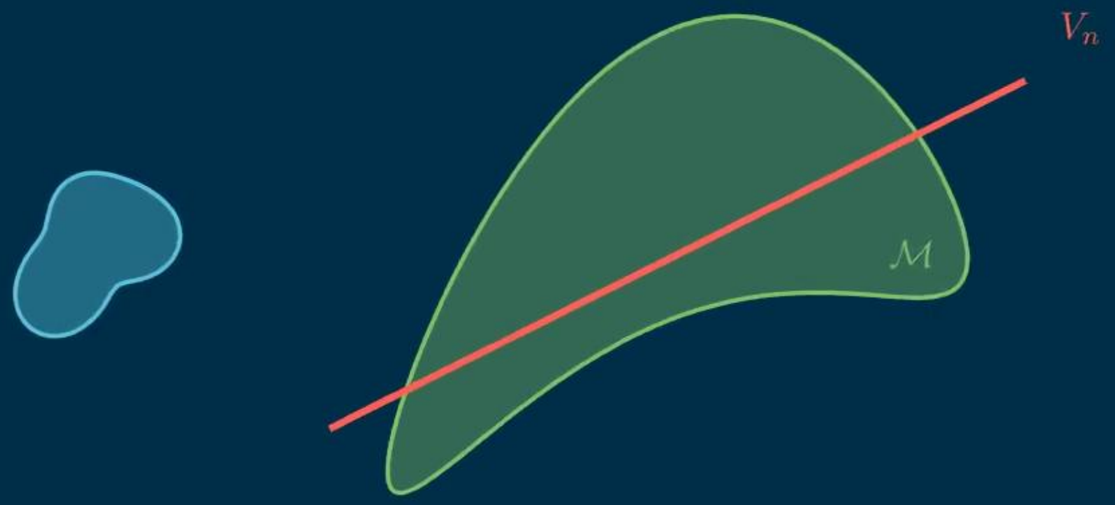
Linear reduce order models

$$y \in Y \mapsto u \in \mathcal{M} \subset V$$



Linear approaches

$y \in Y \mapsto u \in \mathcal{M} \subset V$



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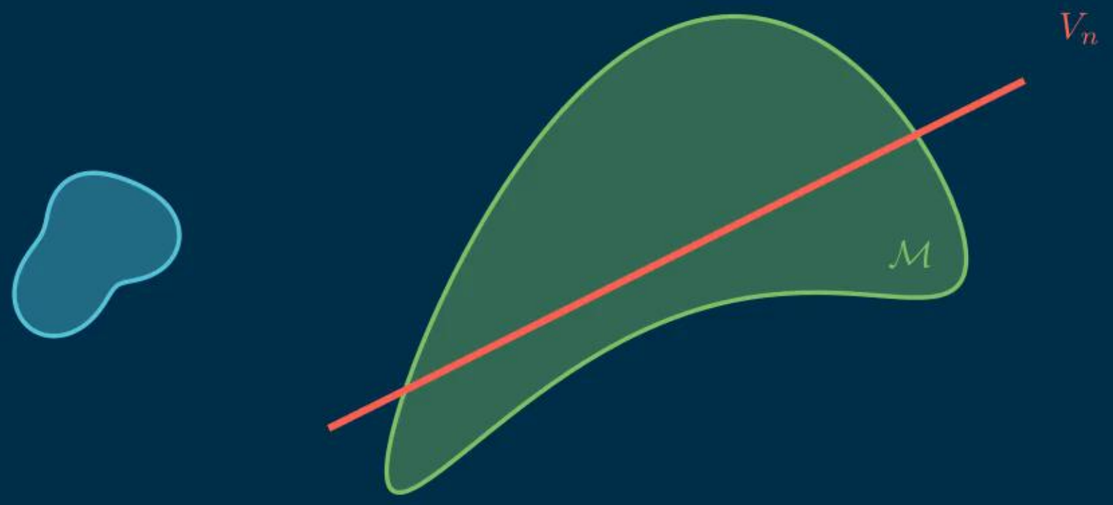
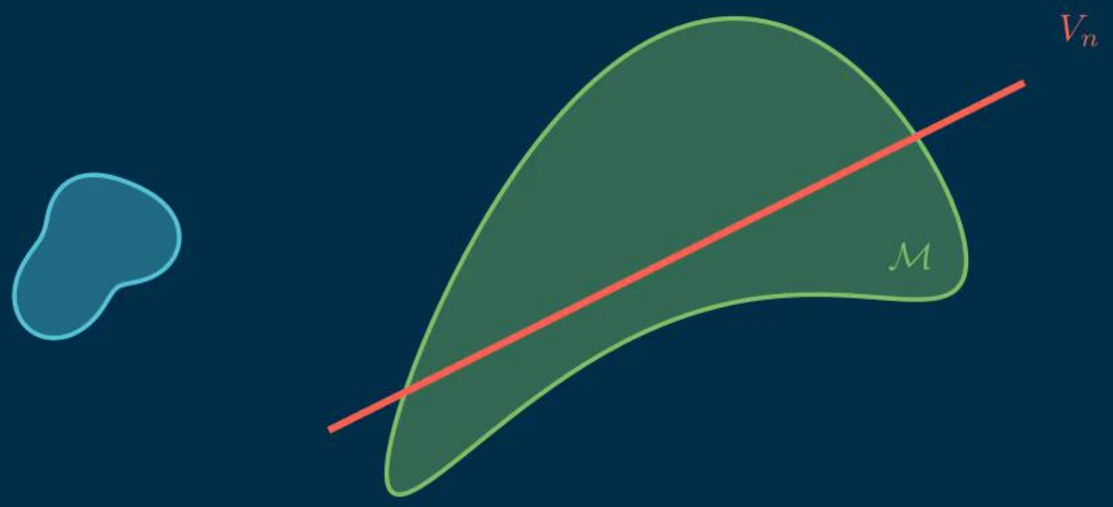


$V_n := \mathrm{span}\{v_1, \ldots, v_n\}$

$$\tilde{u} = \sum_{i=1}^{n} c_i v_i$$

Linear approaches

$y \in Y \mapsto u \in \mathcal{M} \subset V$



$V_n := \text{span}\{v_1, \ldots, v_n\}$

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Forward modelling:
Galerkin projection

$$\int_\Omega a(x, y) \nabla u \nabla v dx = \int_\Omega f v dx$$

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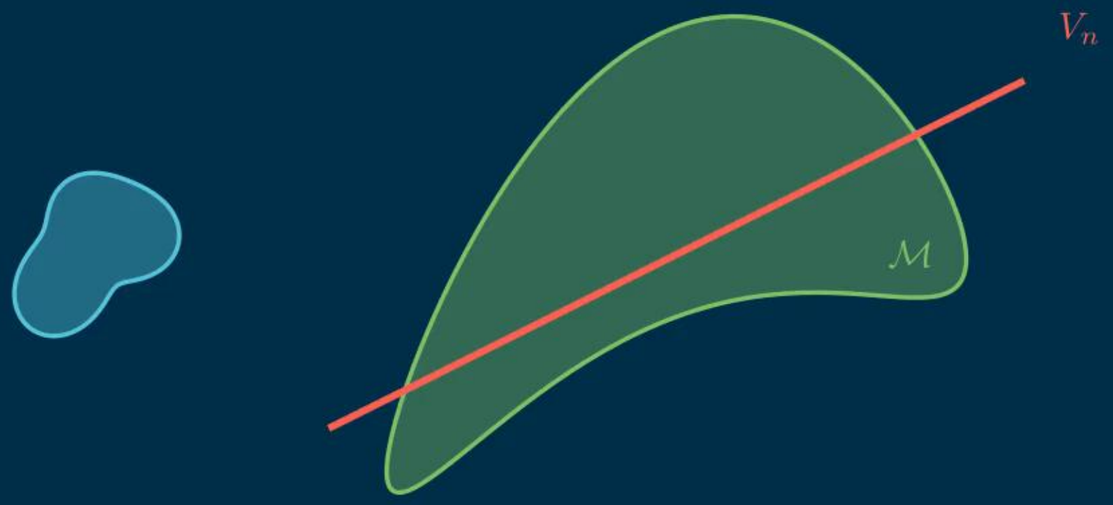

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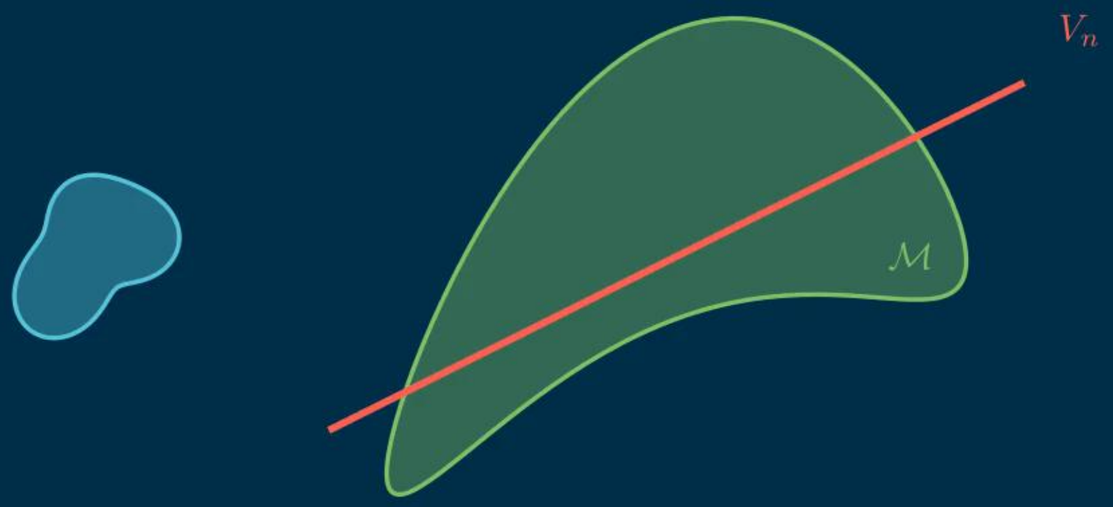

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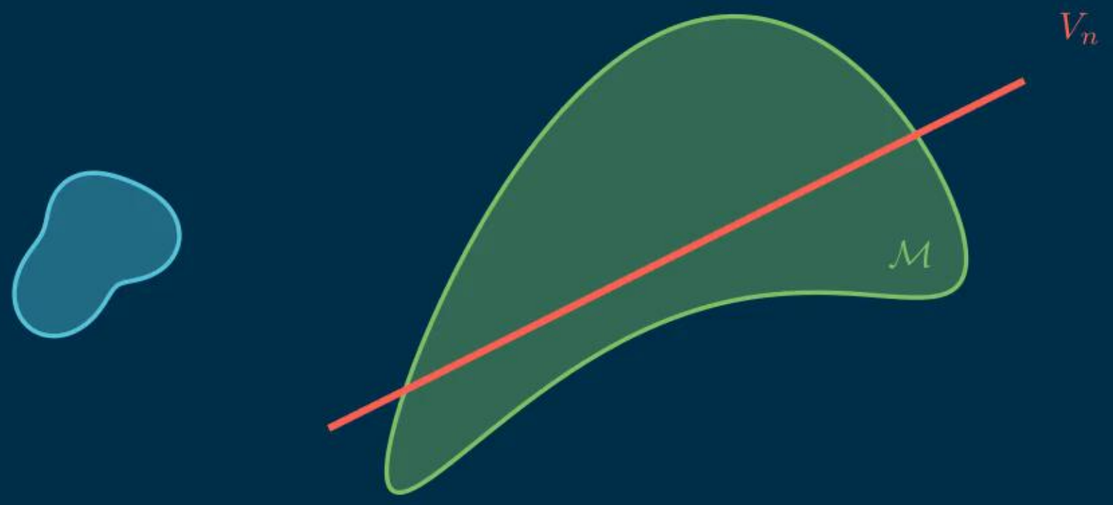

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$1 \ll N$ FEM

Linear approaches

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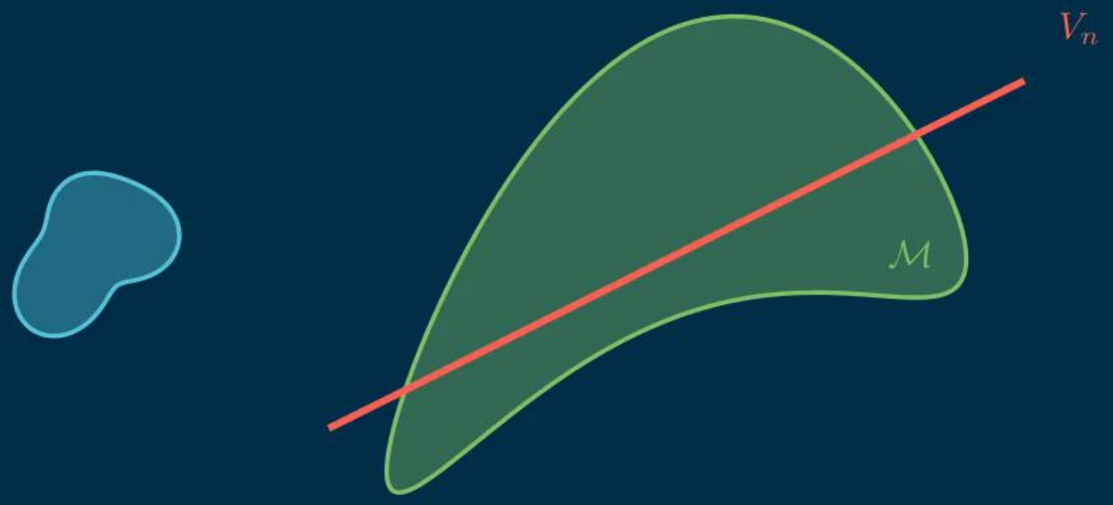

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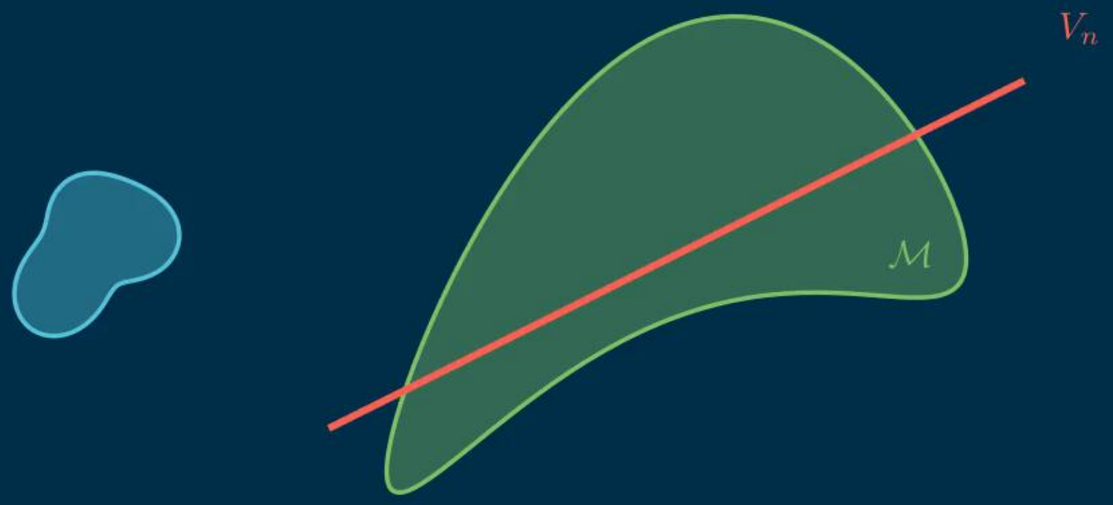

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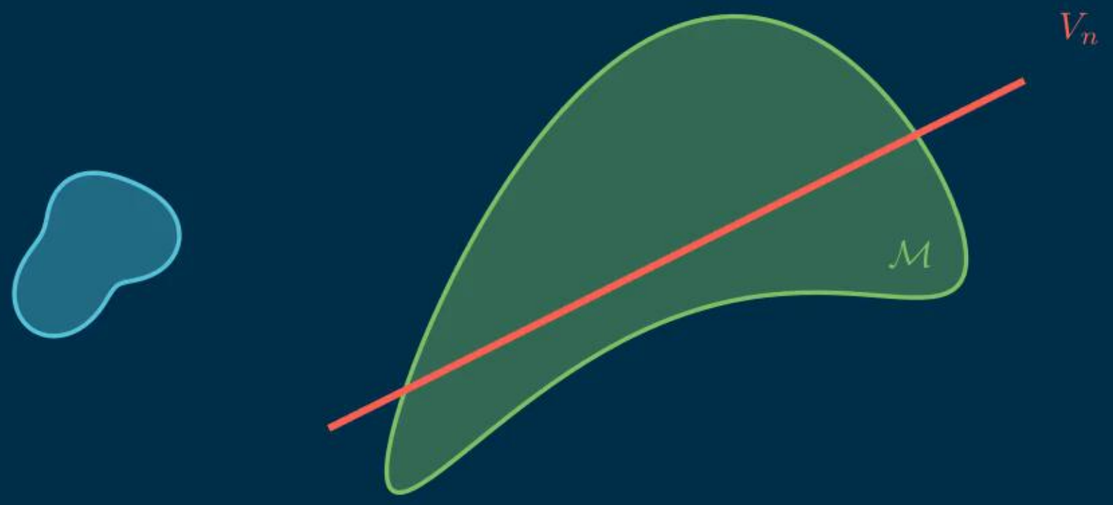
State estimation:
best fit estimator

$$\ell_j(\tilde{u}) \approx z_j$$

Linear approaches

$y \in Y \mapsto u \in \mathcal{M} \subset V$



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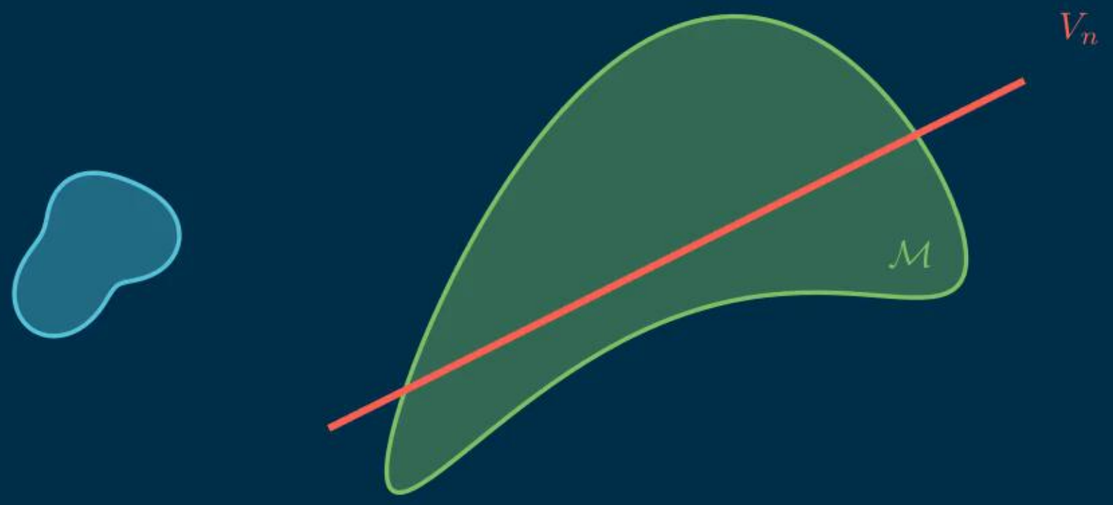

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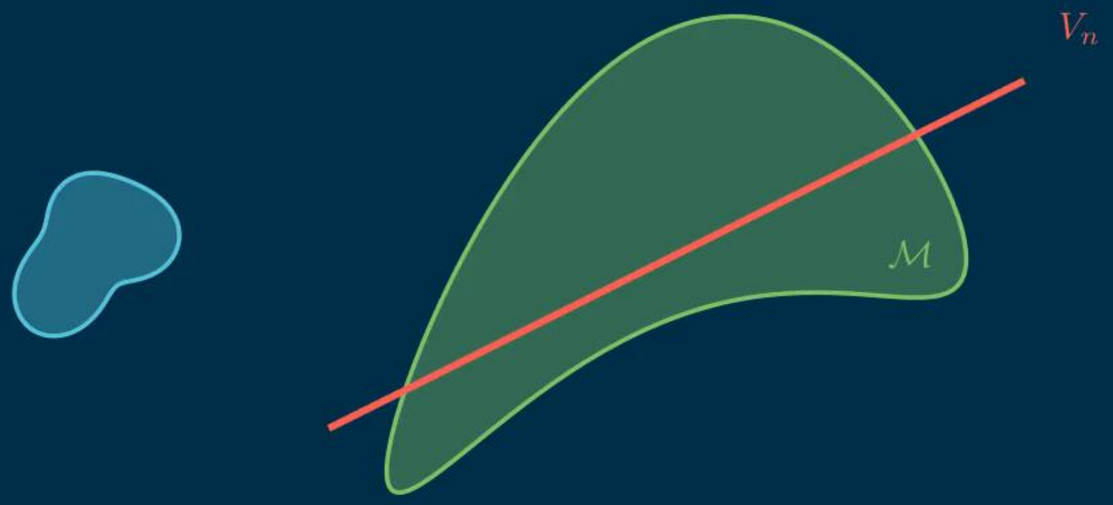

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$Lc \approx z, \quad L \in \mathbb{R}^{m \times n}$

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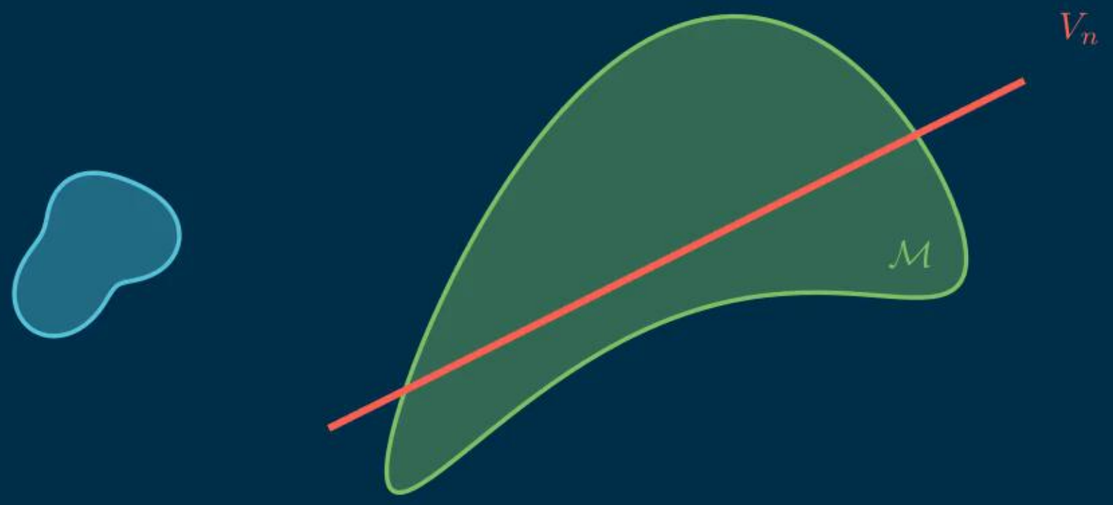

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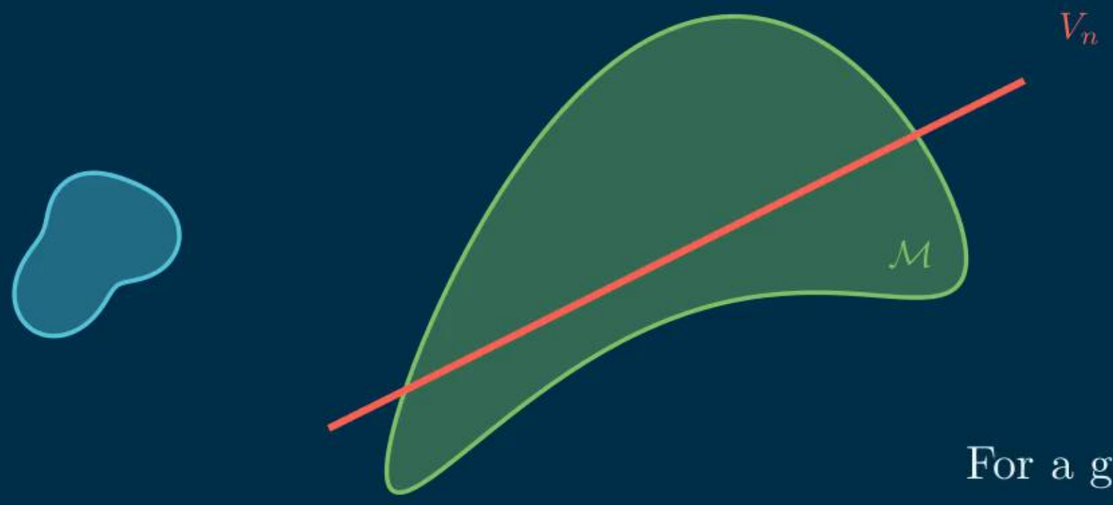
$$L^T L c = L^T z$$

Main questions

$y \in Y \mapsto u \in \mathcal{M} \subset V$



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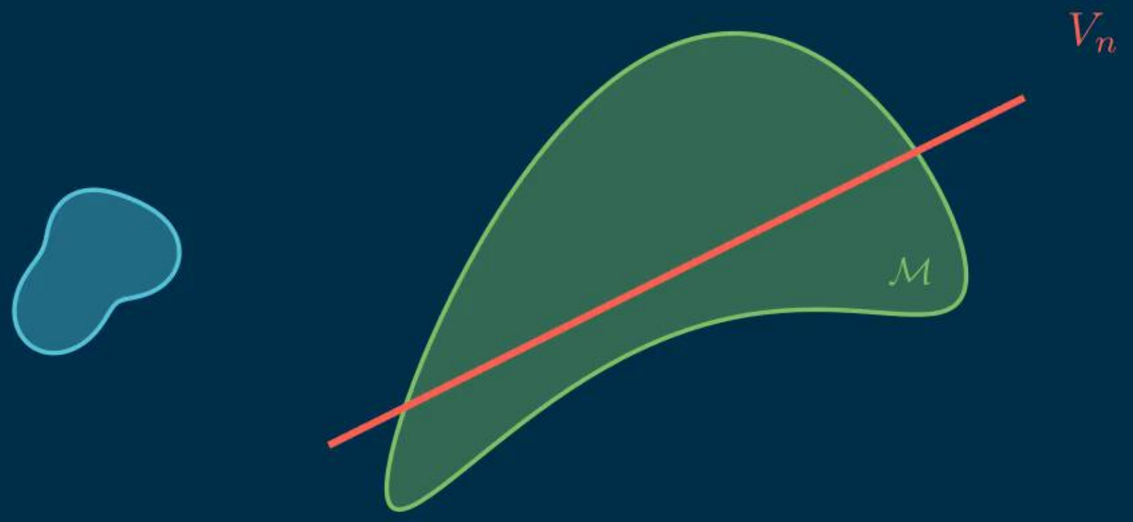
For a given problem (parametric PDE):

1. What can we achieve with optimally chosen linear spaces?
2. Is it enough?
3. Can we do better with non-linear spaces?

Distance to manifold

$y \in Y \mapsto u \in \mathcal{M} \subset V$



$$\|u - P_{V_n} u\|_V = \min_{v \in V_n} \|u - v\|_V$$

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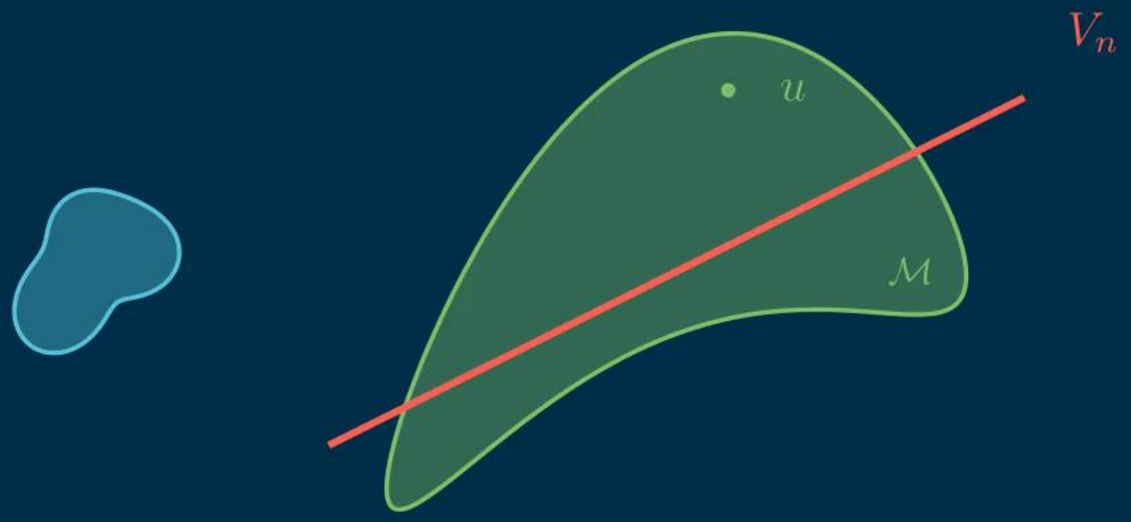

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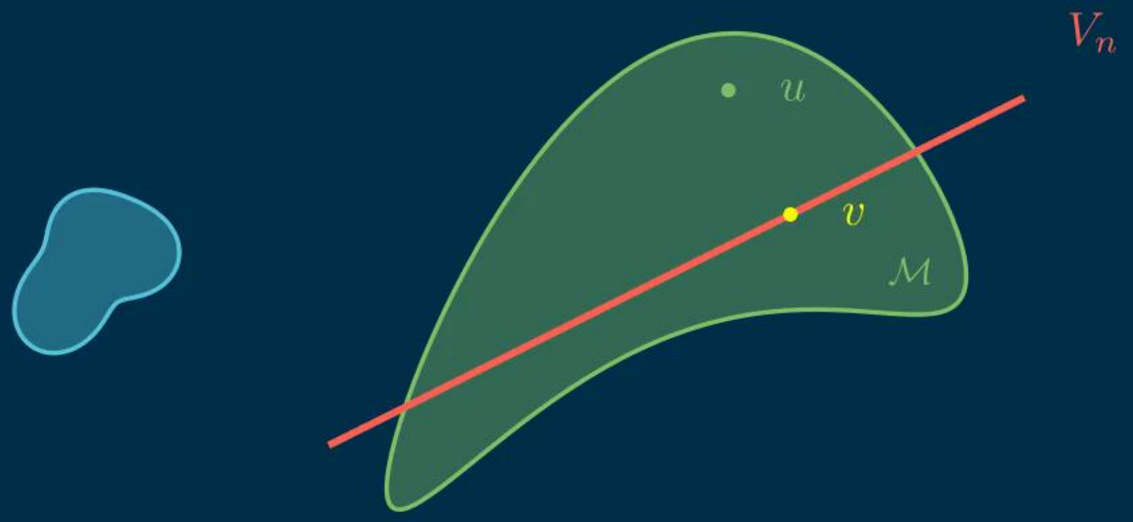

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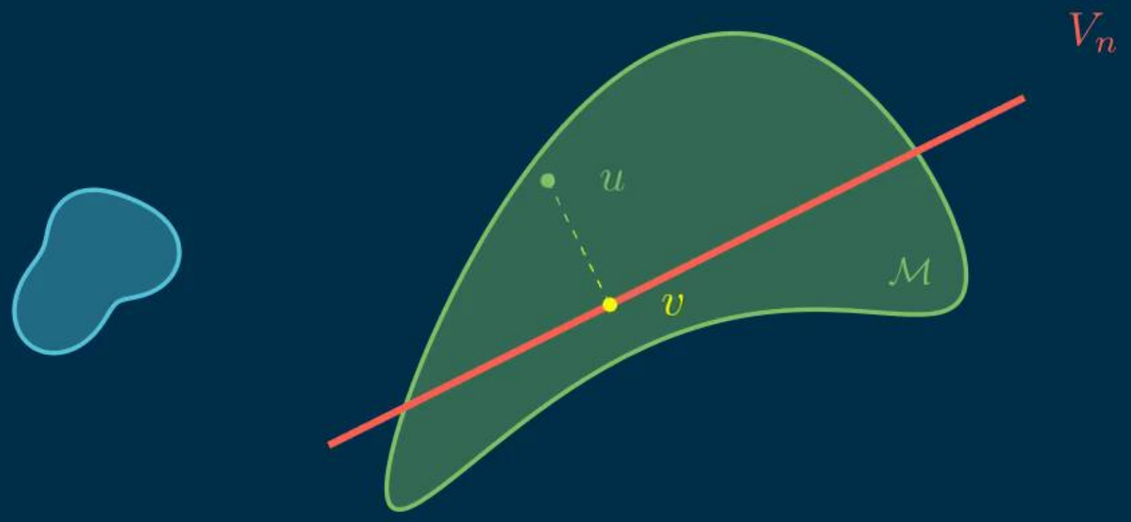

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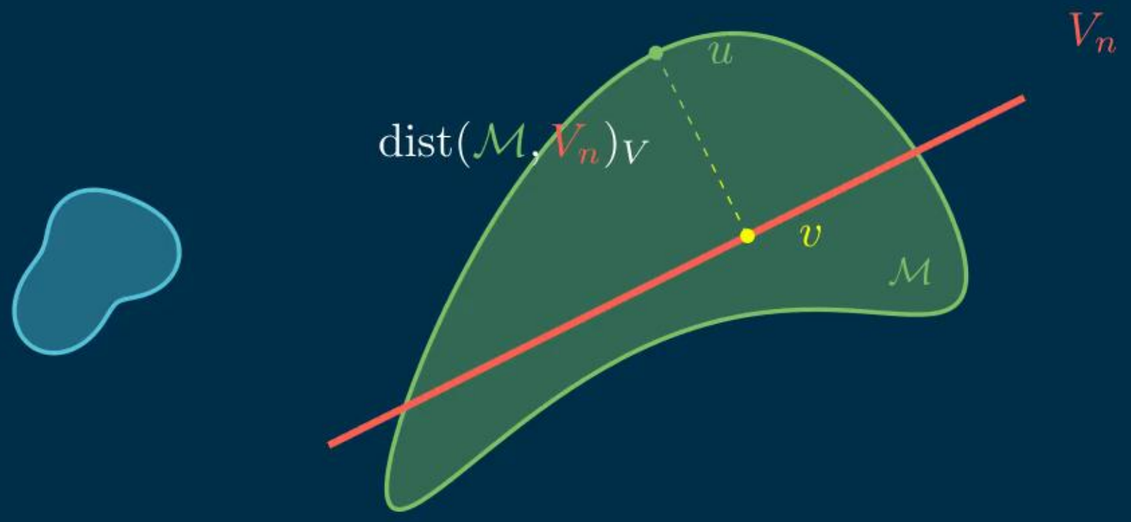

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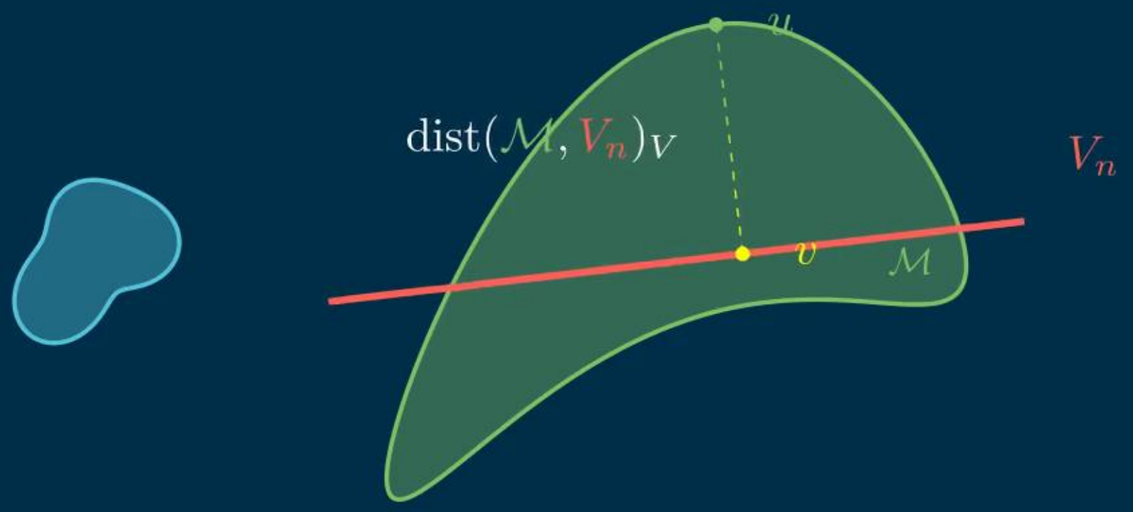

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Kolmogorov width

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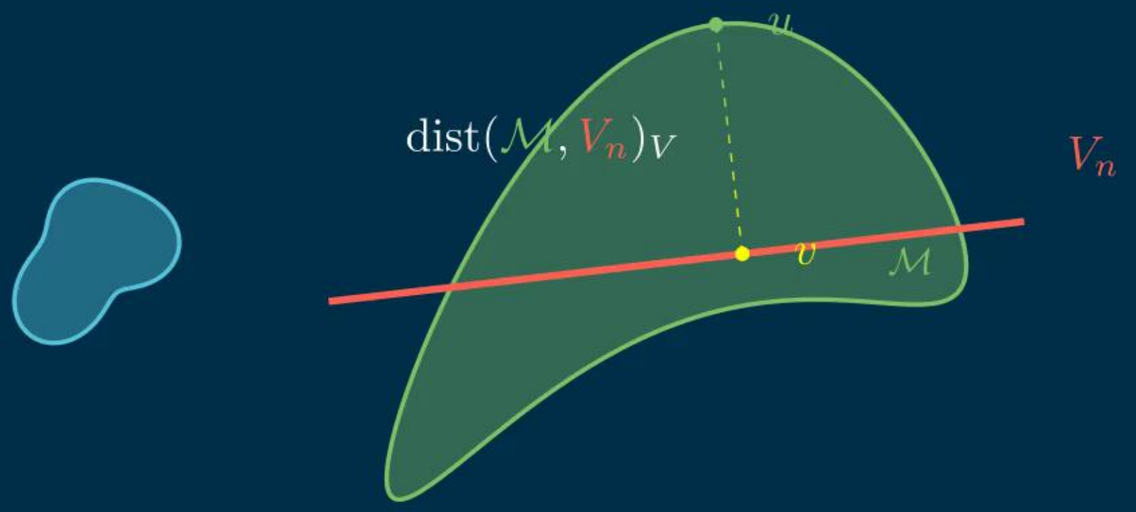

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$$d_n(\mathcal{M})_V = \inf_{V_n,\ \dim(V_n)=n} \mathrm{dist}(\mathcal{M}, V_n)_V$$

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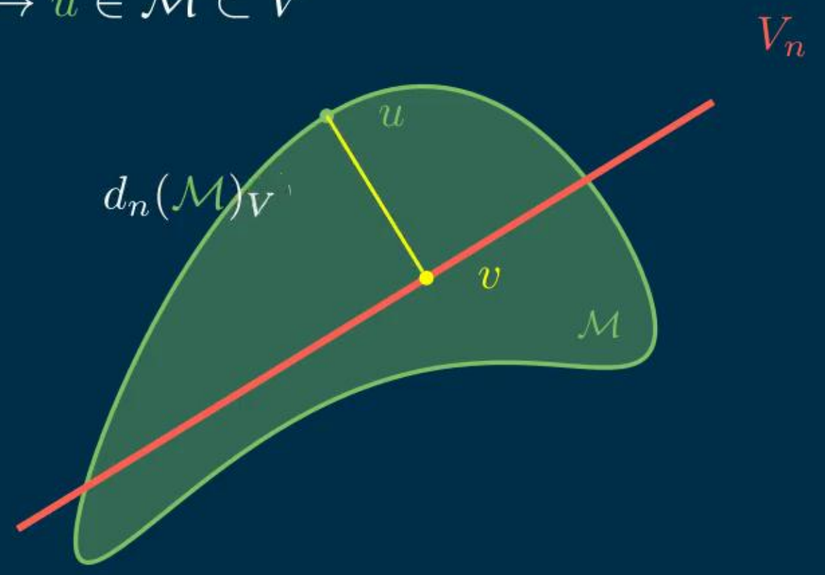

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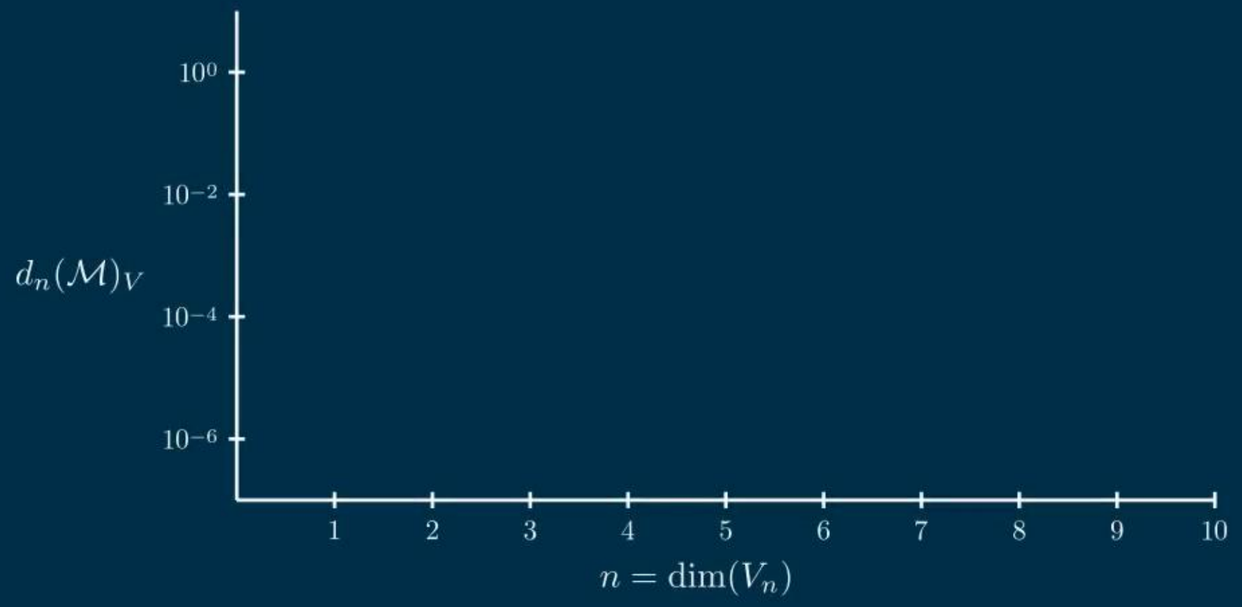

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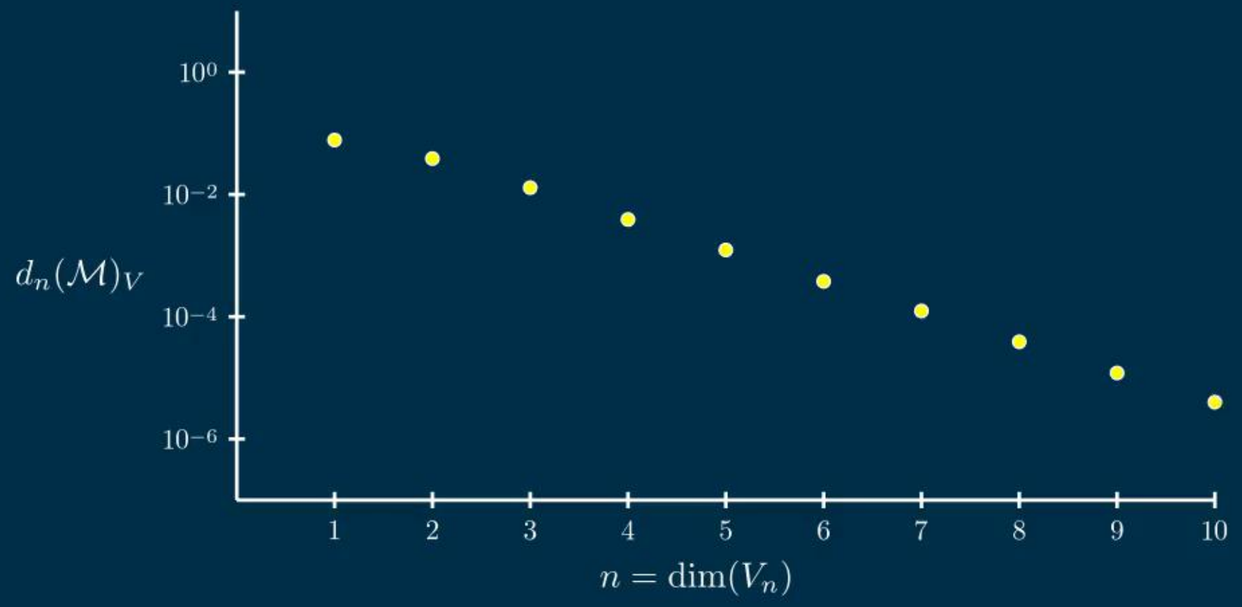

Elliptic PDE

$$-\operatorname{div}(a\nabla u) = f$$

$$a(x, y) = \overline{a}(x) + \sum_{j=1}^{d} y_j \psi_j(x)$$

Kolmogorov width decay

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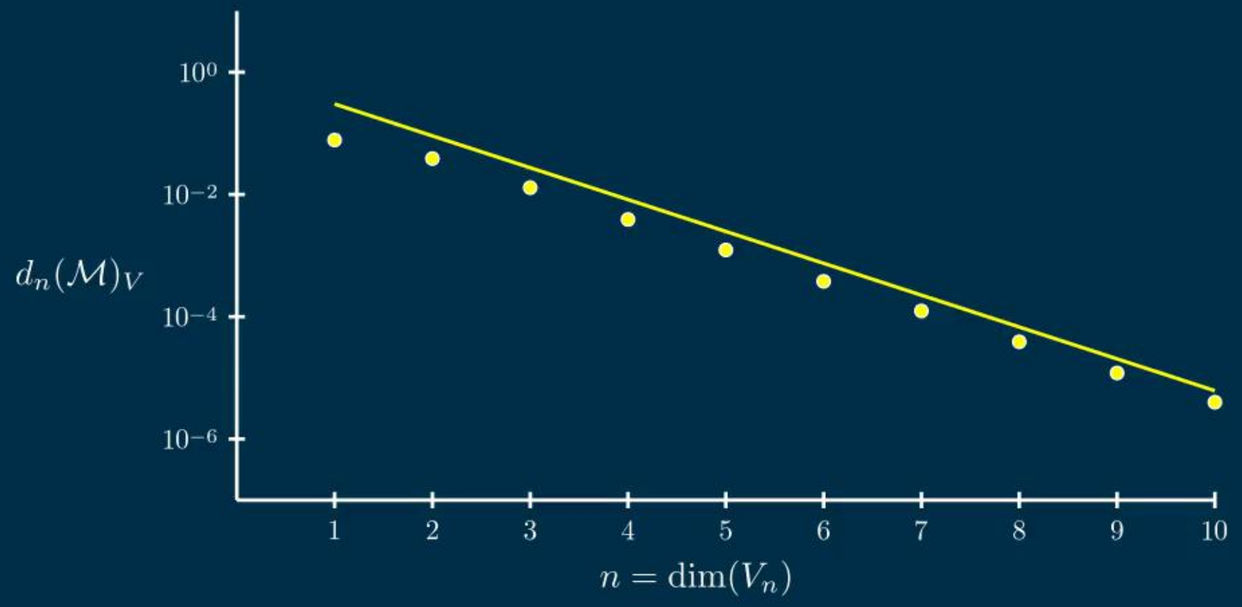

$$d_n(\mathcal{M})_V \leq Ce^{-cn^{1/d}}$$

[A. Cohen, R. DeVore. Acta Numerica, 2015]

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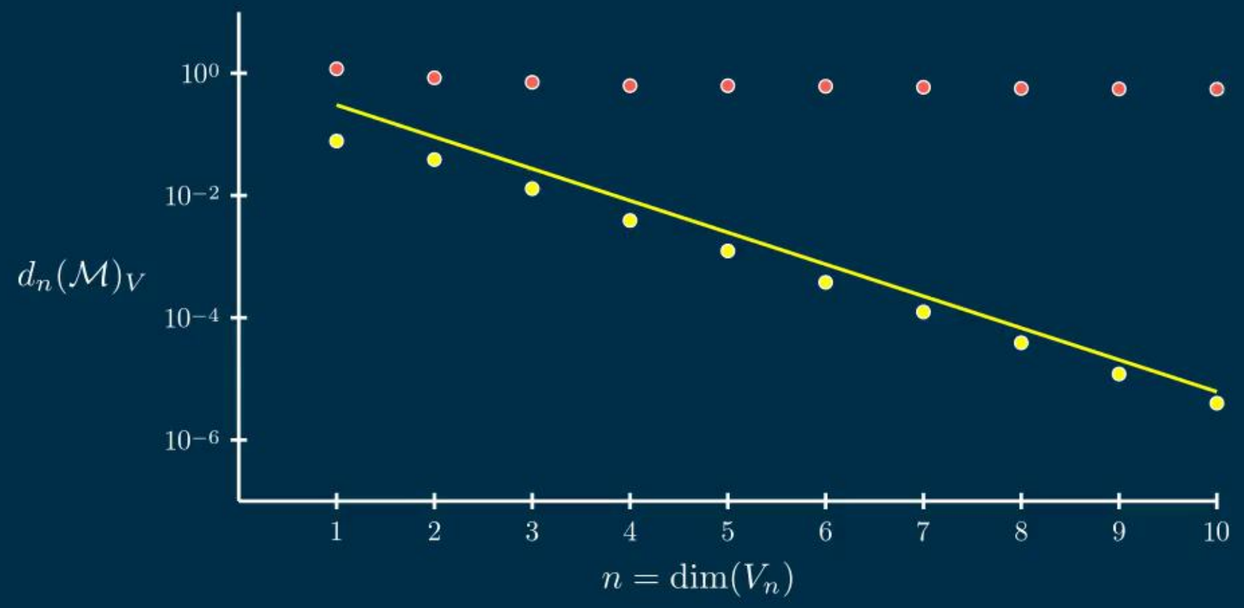

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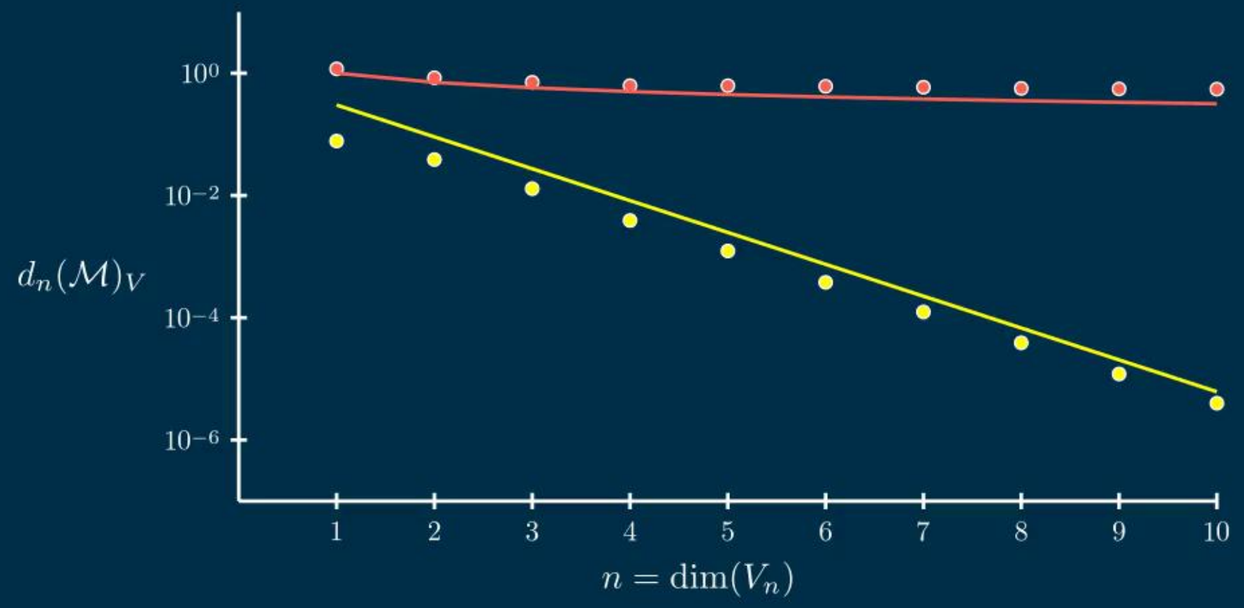

$d_n(\mathcal{M})_V \geq Cn^{-\frac{1}{2}}$

$d_n(\mathcal{M})_V \leq Ce^{-cn^{1/d}}$

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[M. Ohlberger, S. Rave. Proceedings of the Conference Algoritmy, 2016]
[C. Greif, K. Urban, Applied Mathematics Letters, 2019]

Contributions of the thesis

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Diffusion equation with
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[A. Cohen, C. Farhat, Y. Maday, A. Somacal, 2023]

Comparing traditional and
physics informed learning strategies
for transport-diffusion equations

[A. Beguinet, V. Ehrlacher, R. Flenghi, M. Fuente, O. Mula, A. Somacal, 2023]

Real case study:
estimation of pollution in a city

[M. Dolbeault, O. Mula, A. Somacal, 2024]

Contributions of the thesis

Linear approaches

Extend approximation guarantees
Diffusion equation with
unbounded parameter space Y

[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Non-linear approaches

Theoretical framework
to analyse
non-linear strategies

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Implementation of non-linear strategies
to reconstruct step-functions
out of cell average data

[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

Reconstruct step-functions
combining
linear and learning techniques

[A. Cohen, C. Farhat, Y. Maday, A. Somacal, 2023]

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High contrast diffusivity

Diffusion PDE

$$-\mathrm{div}(a\nabla u) = f$$

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$$a = \overline{a}(x) + \sum_{j=1}^{d} y_j \chi_{\Omega_j}(x)$$

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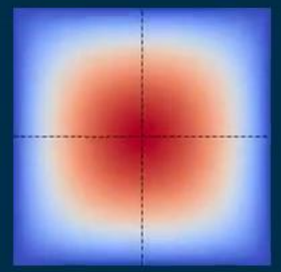
1.0	1.0
1.0	1.0



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Diffusion PDE

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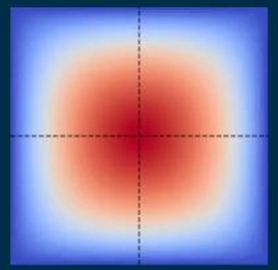

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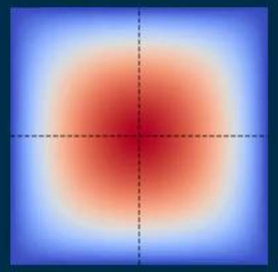

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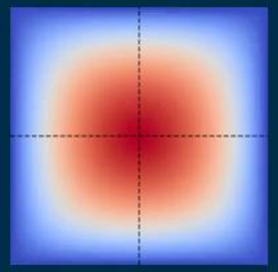

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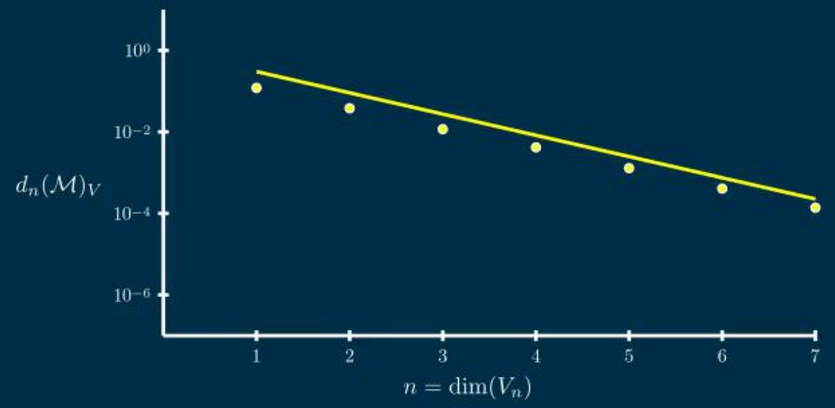
Uniform Ellipticity Assumption (UEA)

$$0 < r \leq a(x, y) \leq R < \infty$$



$$d_n(\mathcal{M})_V \leq Ce^{-cn^{\frac{1}{d}}}$$

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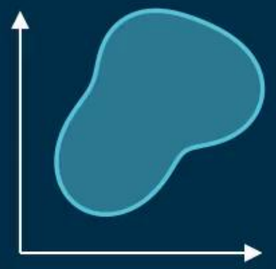
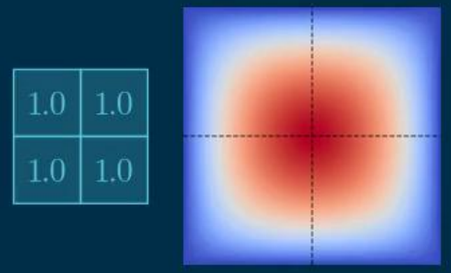
[H. Tran, C. G. Webster, G. Zhang., Numerische Mathematik, 2017]





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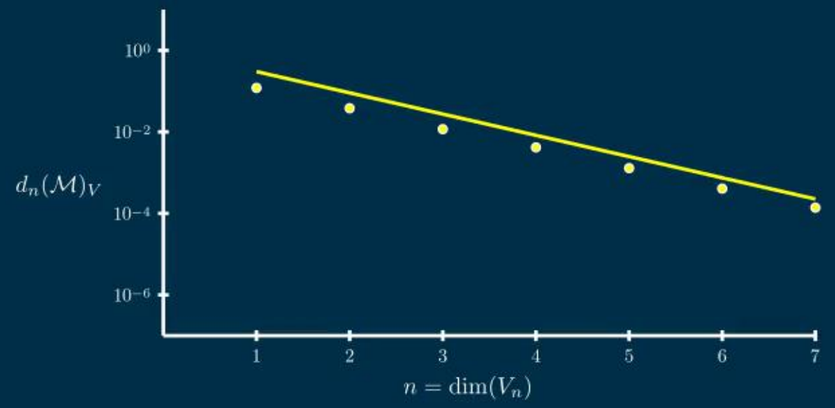

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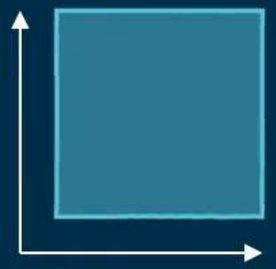

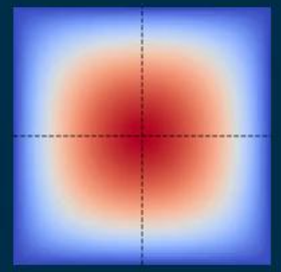

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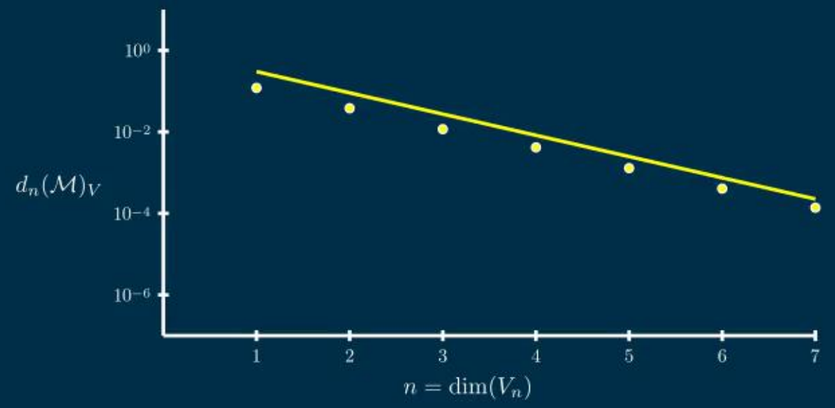

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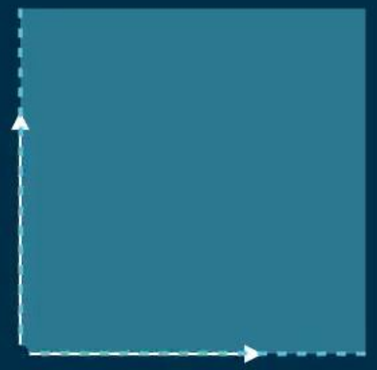

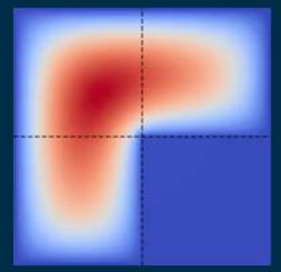

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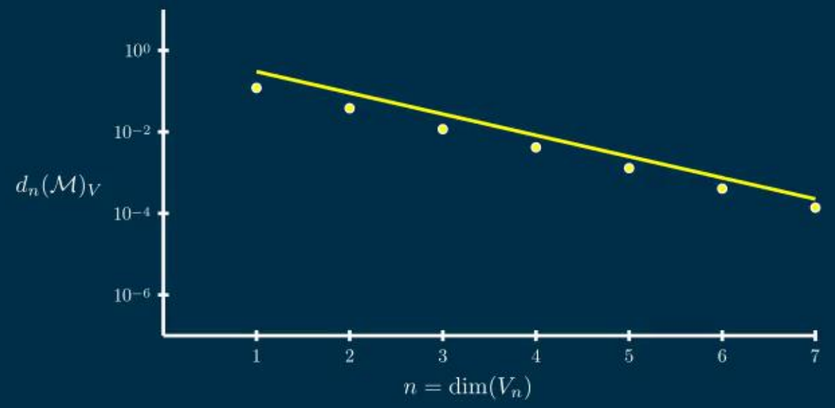

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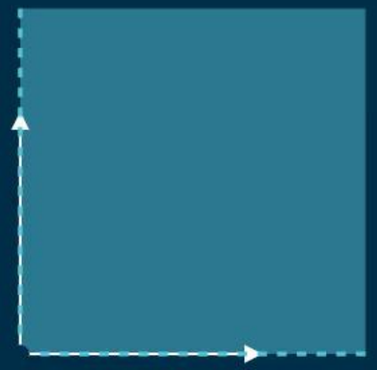

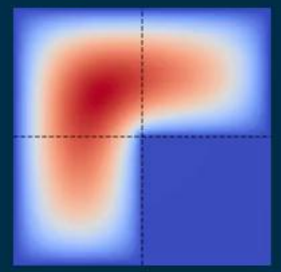

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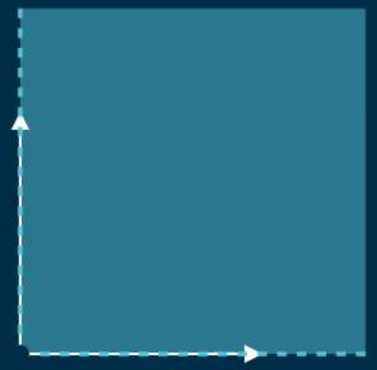

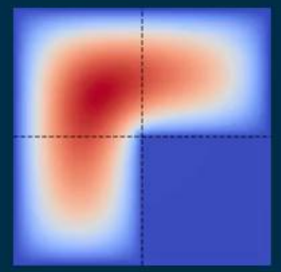

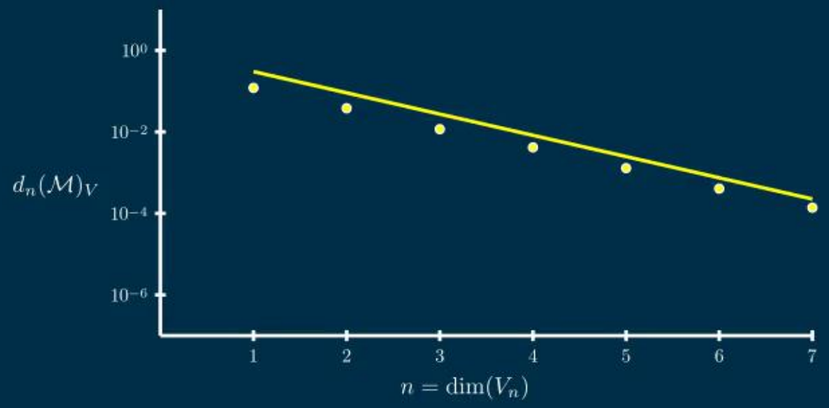

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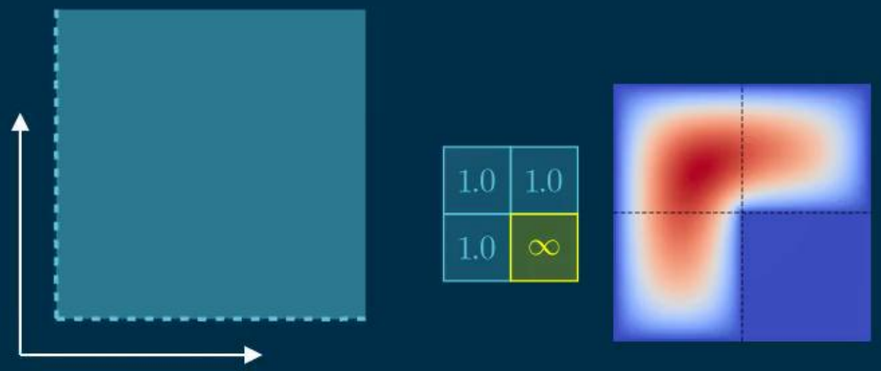

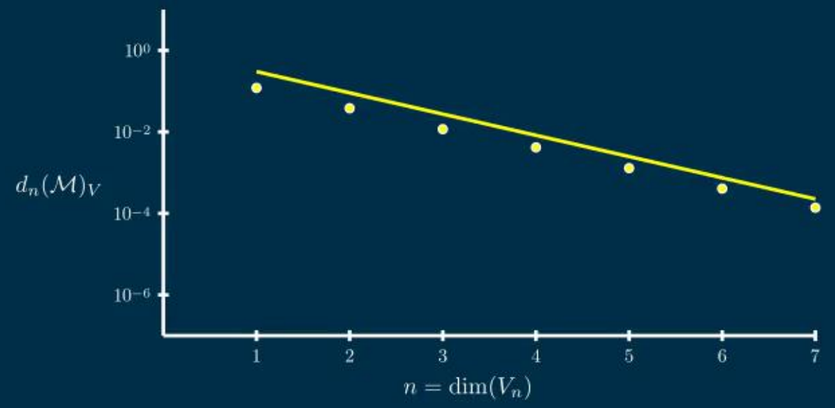

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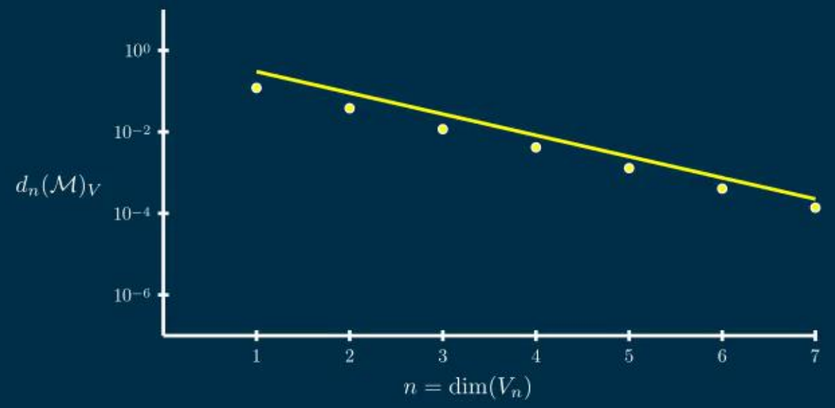

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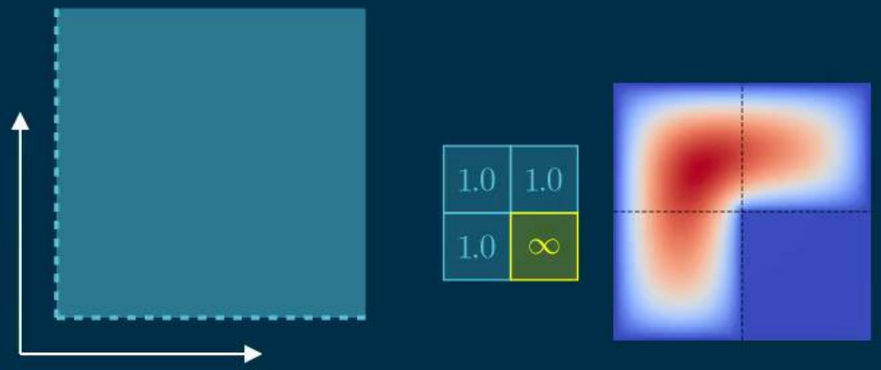
[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]



Three main results

Compactness

Convergence

Convergence rate

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$\mathcal{M}' := \{u(y) : y \in Y'\}$ is a compact set of $H_0^1(\Omega)$

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1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

$u(y_S^c, y_S)$

$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
1.0	1.0	1.0	1.0	1.0
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Lemma: $!\exists\ u_S(y_{S^c}) \in V_S$ the limit of $u(y_S^c, y_S)$ as $y_j \to \infty$ with $j \in S$

∞	1.0	1.0	1.0	1.0
1.0	1.0	∞	1.0	1.0
1.0	1.0	1.0	1.0	1.0
∞	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	∞

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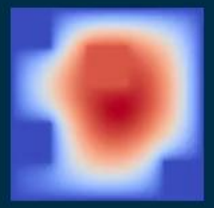

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Theorem: There exists a sequence of linear spaces $(V_n)_{n>1}$ and a zero converging sequence $(\varepsilon_n)_{n>1}$ for all $y \in Y'$ such that

$$\|u(y) - P_{V_n} u(y)\| \leq \varepsilon_n \|u(y)\|_{H_0^1}.$$

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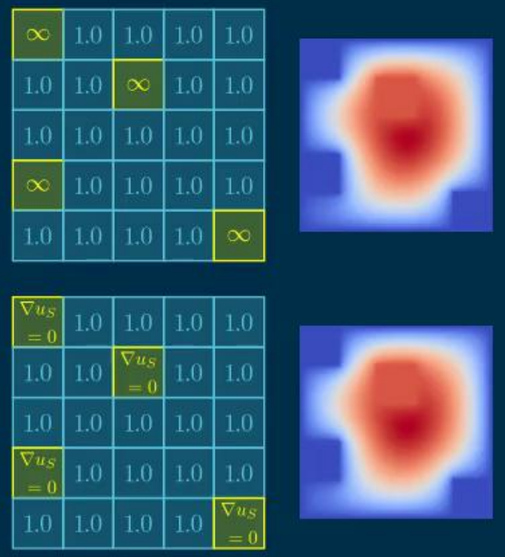

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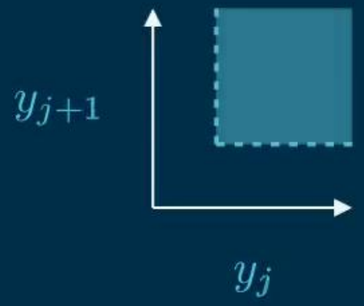

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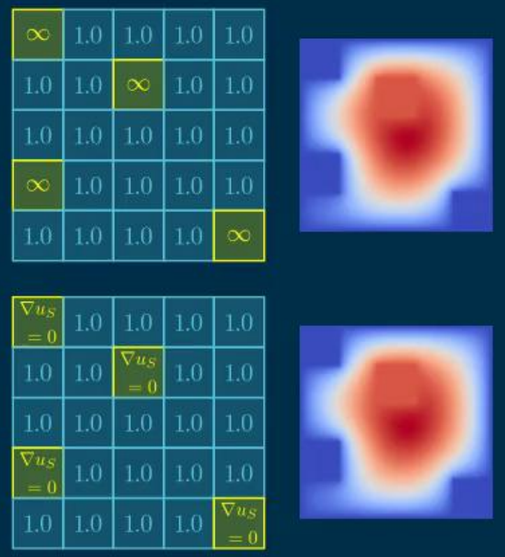

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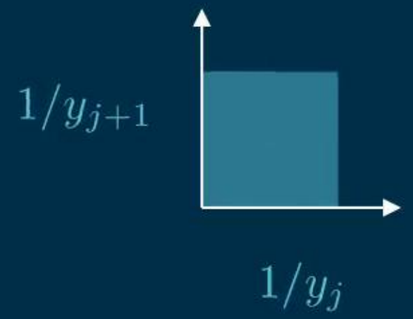

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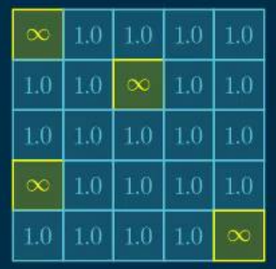

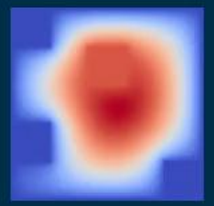

$u(y_S^c, y_S)$

u_S

Convergence

Theorem: There exists a sequence of linear spaces $(V_n)_{n>1}$ and a zero converging sequence $(\varepsilon_n)_{n>1}$ for all $y \in Y'$ such that

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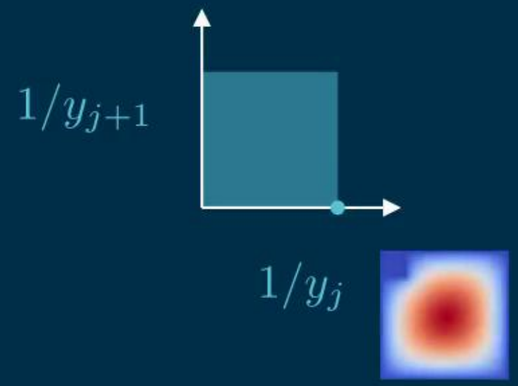

Convergence rate

Three main results

Compactness

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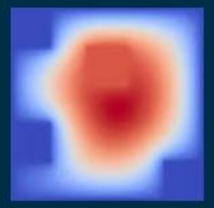
∞	1.0	1.0	1.0	1.0
1.0	1.0	∞	1.0	1.0
1.0	1.0	1.0	1.0	1.0
∞	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	∞

$u(y_S^c, y_S)$

$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
1.0	1.0	1.0	1.0	1.0
$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	$\nabla u_S = 0$



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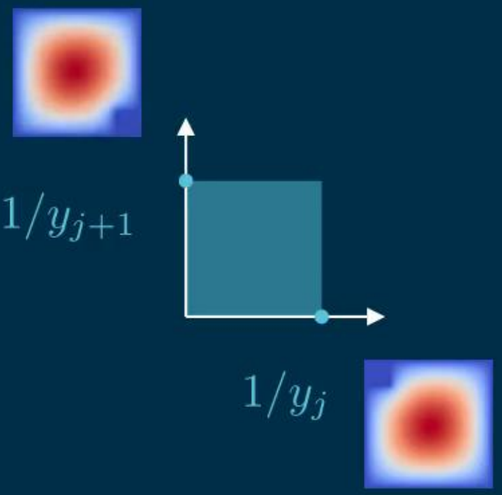

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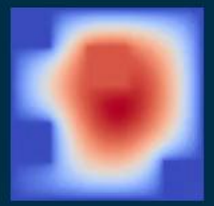

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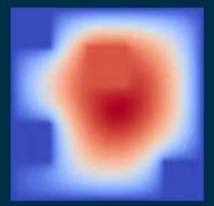

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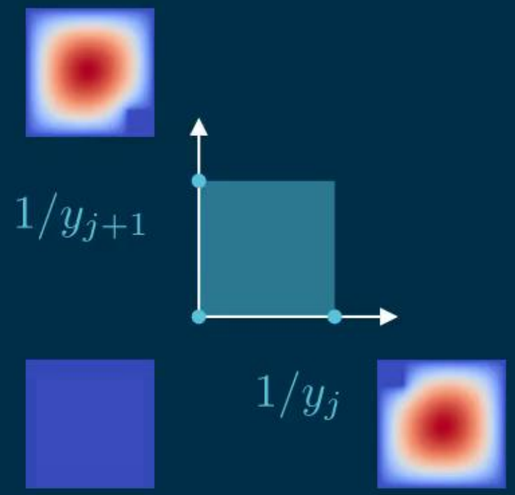

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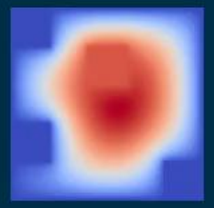
1.0	1.0	∞	1.0	1.0
1.0	1.0	1.0	1.0	1.0
∞	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	∞



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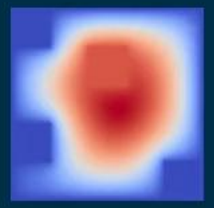
$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
1.0	1.0	1.0	1.0	1.0
$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	$\nabla u_S = 0$



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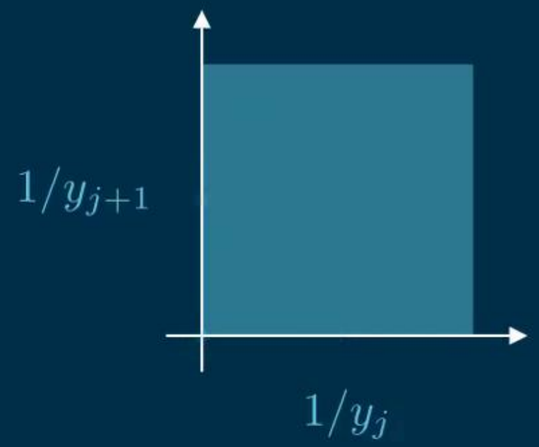

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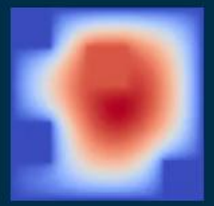
1.0	1.0	1.0	1.0	1.0
∞	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	∞



$u(y_S^c, y_S)$

$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
1.0	1.0	1.0	1.0	1.0
$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	$\nabla u_S = 0$

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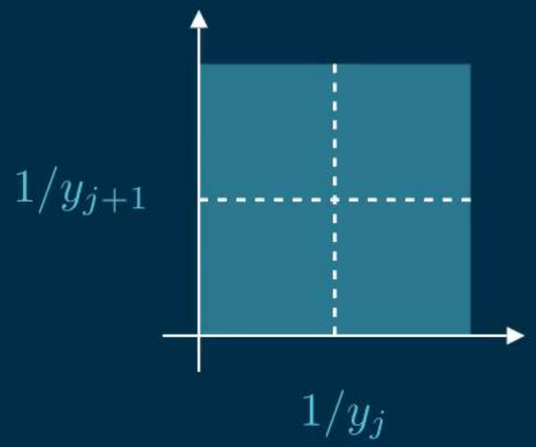
Dyadic partition



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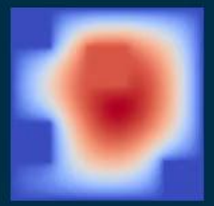

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$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
1.0	1.0	1.0	1.0	1.0
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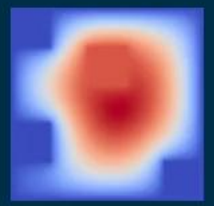

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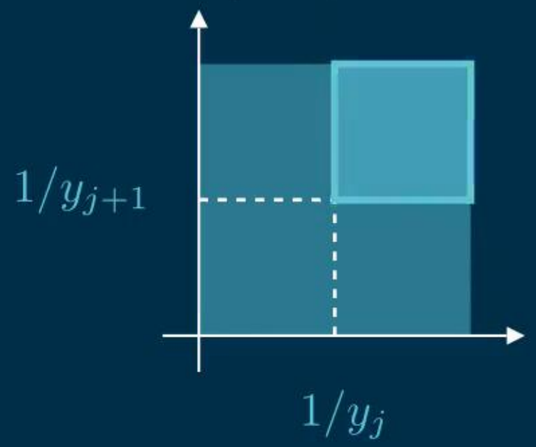
Dyadic partition



Convergence rate

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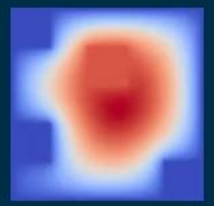

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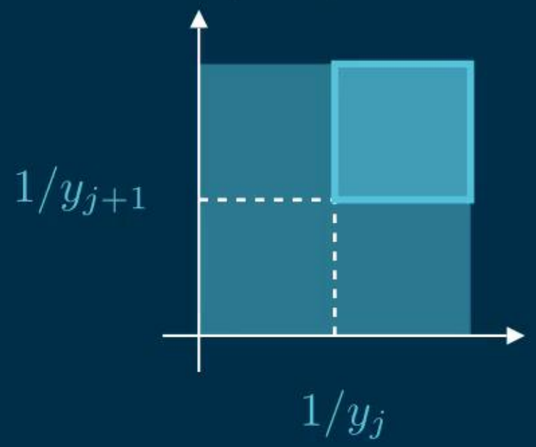

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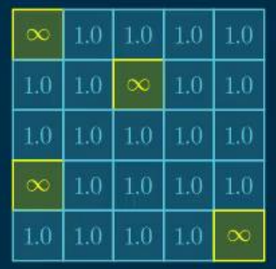

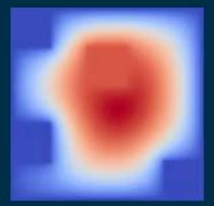

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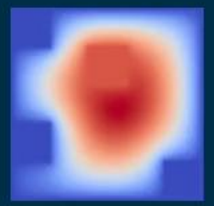

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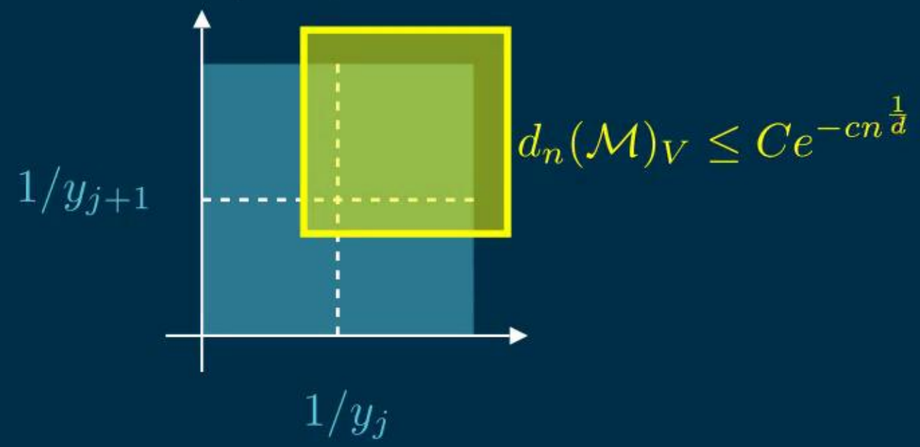

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[M. Bachmayr and A. Cohen., Mathematics of Computation, 2017]

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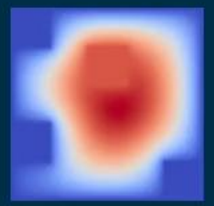
1.0	1.0	1.0	1.0	$\nabla u_S = 0$



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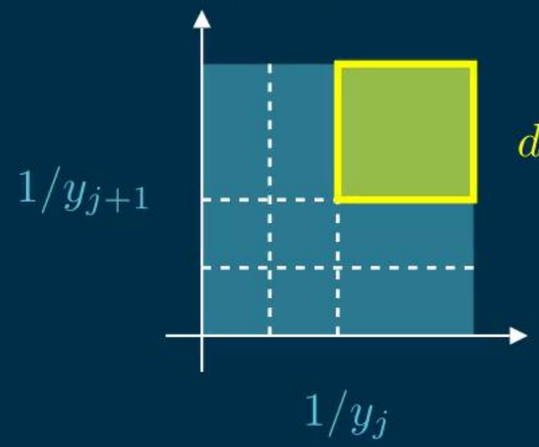

$$d_n(\mathcal{M})_V \leq Ce^{-cn^{\frac{1}{d}}}$$

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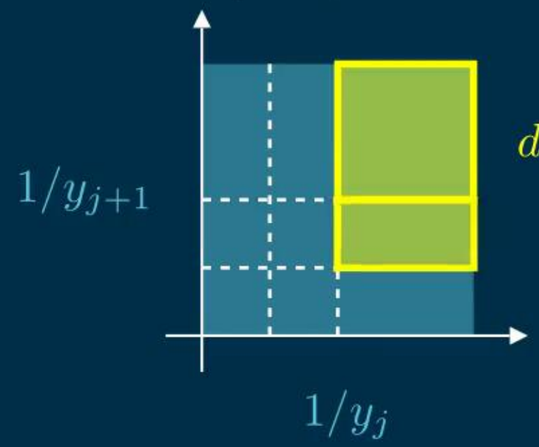

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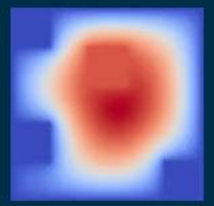

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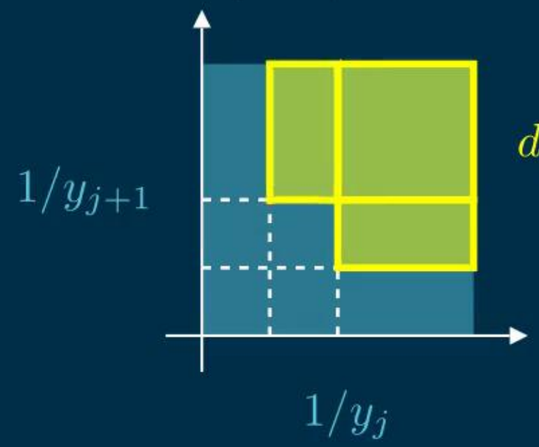

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[M. Bachmayr and A. Cohen., Mathematics of Computation, 2017]

Convergence rate

Three main results

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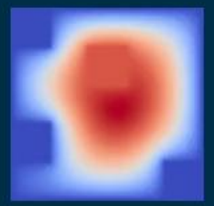

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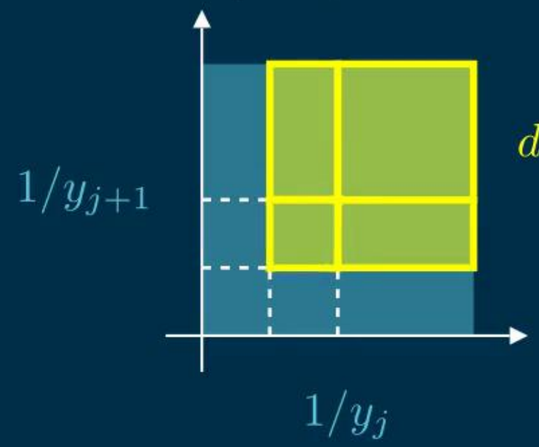

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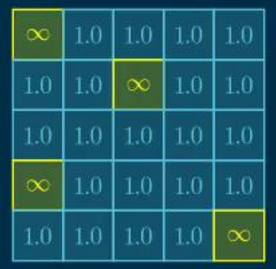

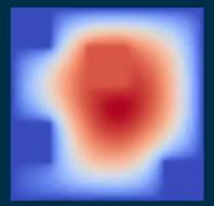

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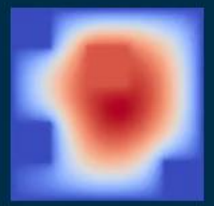

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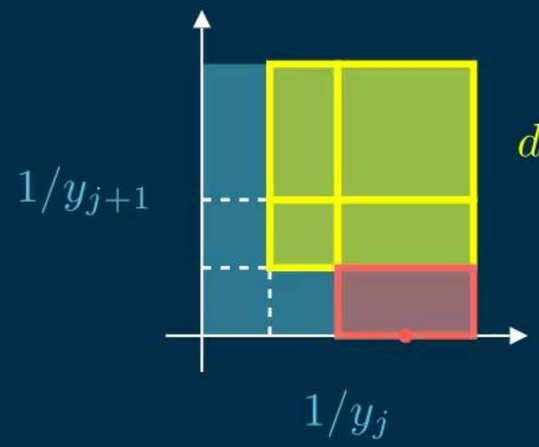

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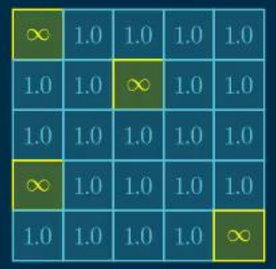

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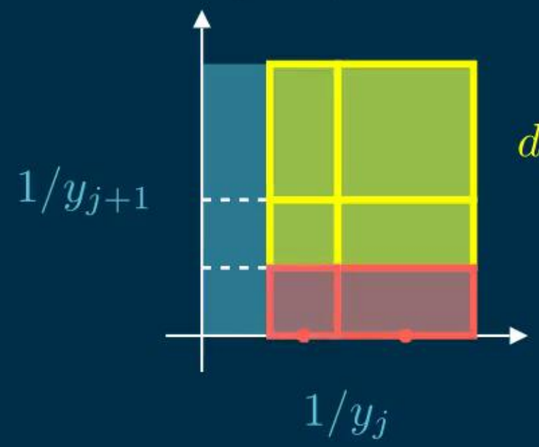

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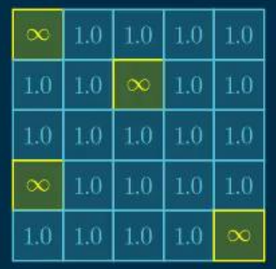

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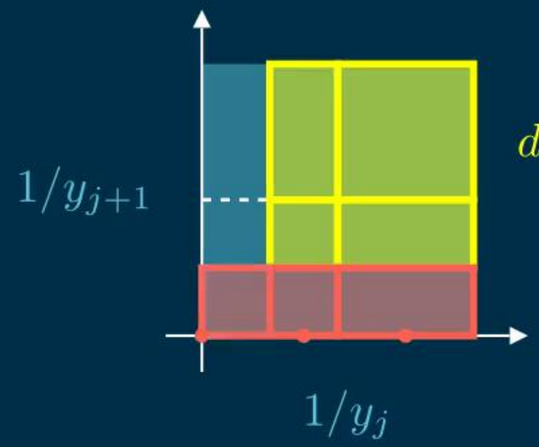

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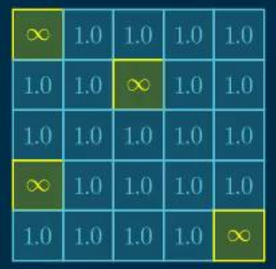

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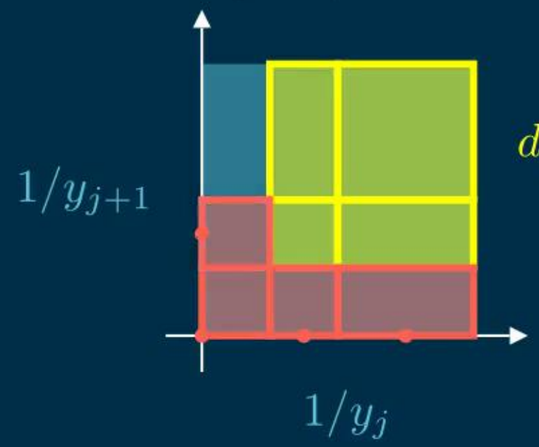

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1.0	1.0	1.0	1.0	1.0
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1.0	1.0	1.0	1.0	∞

$u(y_S^c, y_S)$

$\nabla u_S = 0$	1.0	1.0	1.0	1.0
1.0	1.0	$\nabla u_S = 0$	1.0	1.0
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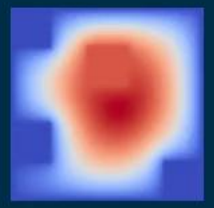

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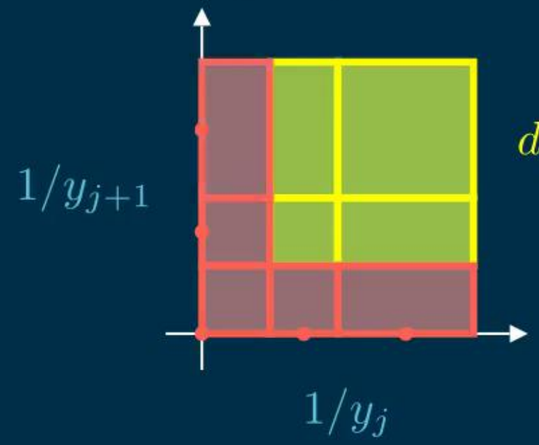

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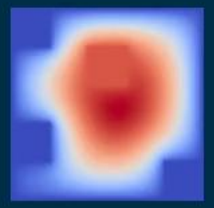

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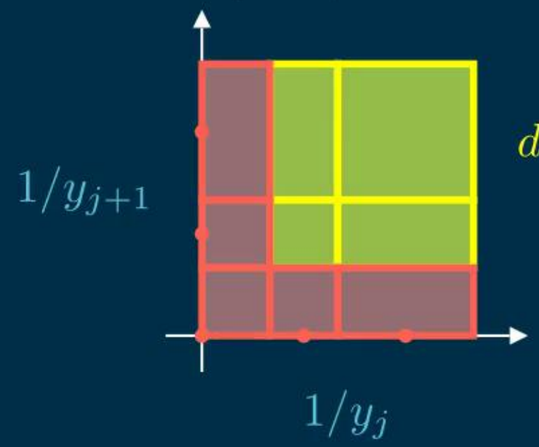

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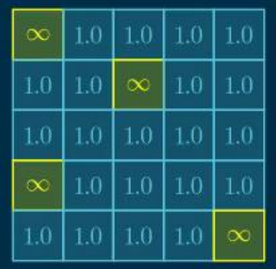

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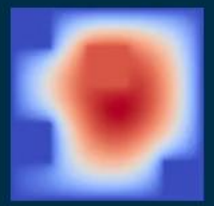

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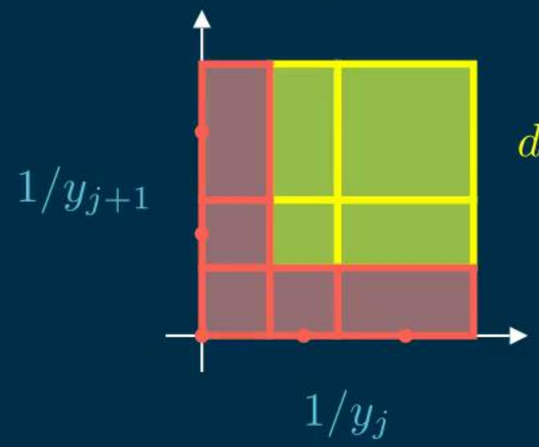

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[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Forward and inverse problems

Forward modelling

Inverse problem

Forward and inverse problems

Forward modelling

$$\|u(y) - P^y_{V_n} u(y)\| \leq C \exp(-cn^{\frac{1}{2d-2}})$$

Inverse problem

State estimation

$$\|u - \tilde{u}\| \leq C \exp(-cn^{\frac{1}{2d-2}}) \|u\|_{H^1_0}$$

Parameter estimation

$$y^*_j = \left(\sum_{i=1}^{n} \frac{c_i}{y^i_j} \right)^{-1}$$

$$\left\| \frac{1}{y^*} - \frac{1}{y} \right\|_\infty \leq C \exp(-cn^{\frac{1}{2d-2}}) \left\| \frac{1}{y} \right\|_\infty$$

Forward and inverse problems

Forward modelling

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$$\|u\|_y^2 = \sum_{j=1}^{d} y_j \int_{\Omega_j} |\nabla u|^2$$

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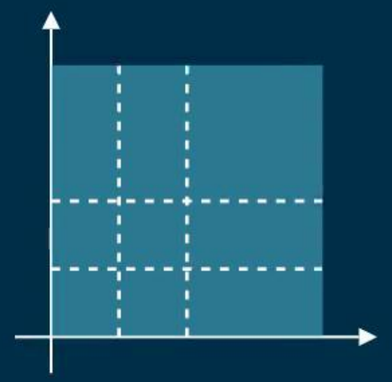

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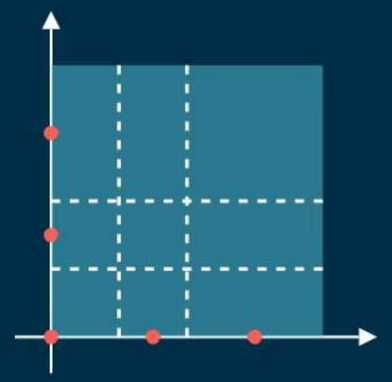

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Contributions of the thesis

Linear approaches

Extend approximation guarantees
Diffusion equation with
unbounded parameter space Y

[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Non-linear approaches

Theoretical framework
to analyse
non-linear strategies

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Implementation of non-linear strategies
to reconstruct step-functions
out of cell average data

[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

Reconstruct step-functions
combining
linear and learning techniques

[A. Cohen, C. Farhat, Y. Maday, A. Somacal, 2023]

Comparing traditional and
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[A. Beguinet, V. Ehrlacher, R. Flenghi, M. Fuente, O. Mula, A. Somacal, 2023]

Real case study:
estimation of pollution in a city

[M. Dolbeault, O. Mula, A. Somacal, 2024]

Contributions of the thesis

Linear approaches

Extend approximation guarantees
Diffusion equation with
unbounded parameter space Y

[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Non-linear approaches

Theoretical framework
to analyse
non-linear strategies

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Implementation of non-linear strategies
to reconstruct step-functions
out of cell average data

[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

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[A. Cohen, C. Farhat, Y. Maday, A. Somacal, 2023]

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Real case study:
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Inverse problem

$$\ell(u) + \eta = z \in \mathbb{R}^m$$

$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

Near optimality property

$$\ell(u) + \eta = z \in \mathbb{R}^m$$

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$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

Best fit estimator

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Parameterized Background Data Weak

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$w_j \in V$ the Riesz representers of ℓ_j

$$W := \operatorname{span}\{w_1, \ldots, w_m\}$$

$$w = P_W u$$

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[Y. Maday, A. T. Patera, J. D. Penn, M. Yano, 2015]

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Sufficient conditions

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Inverse stability

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$$\|\ell(u) - \ell(v)\|_Z \leq \alpha_Z \|u - v\|_V, \quad u, v \in V$$

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[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

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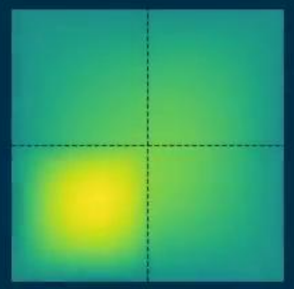
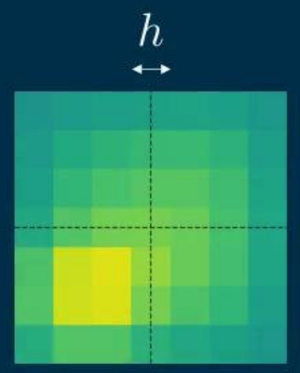
Interface reconstruction from cell averages



Interface reconstruction from cell averages

$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$



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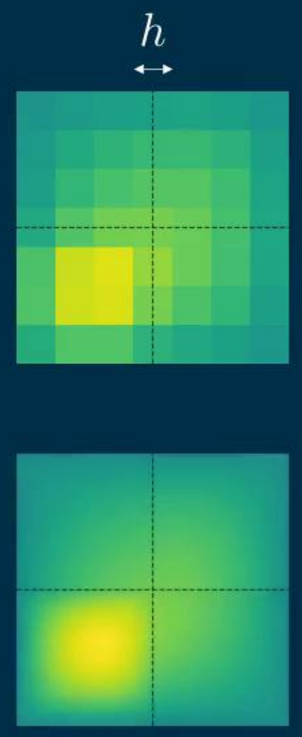

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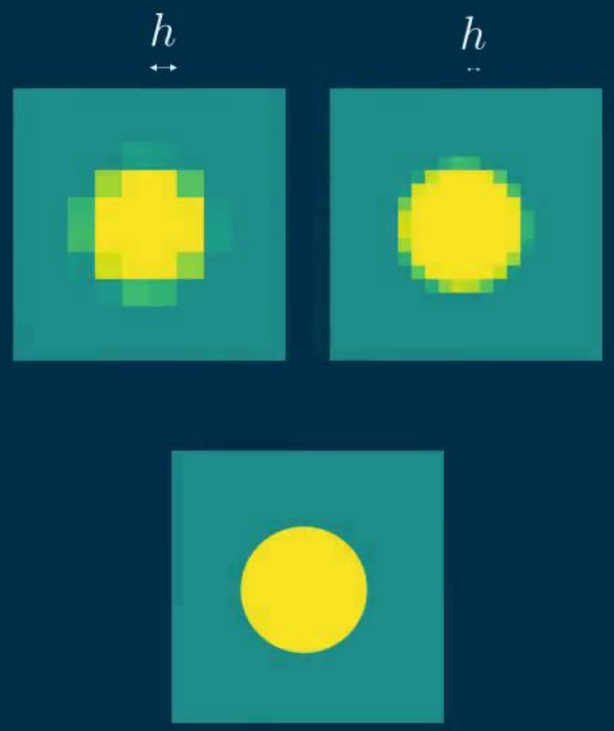

Application

1. Image superresolution.

Interface reconstruction from cell averages

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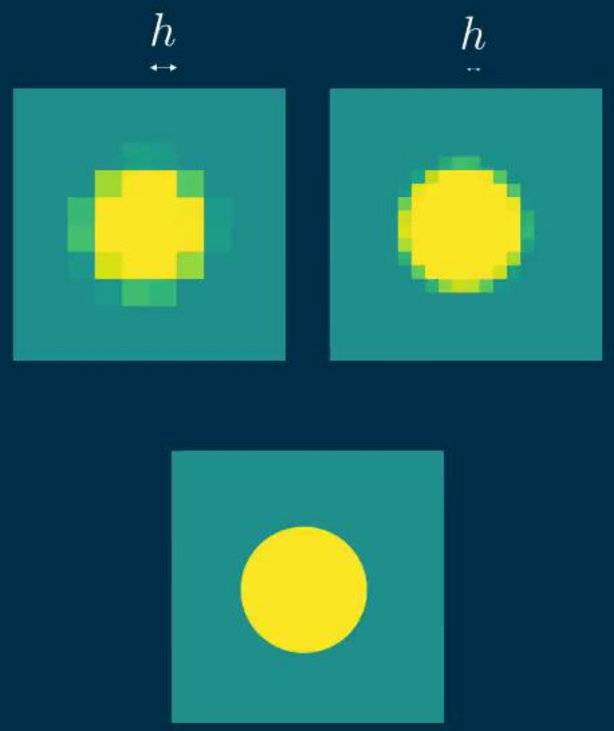

Application

1. Image superresolution.
2. Finite Volumes PDE Solvers

Regular case

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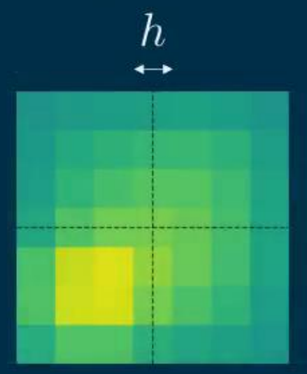

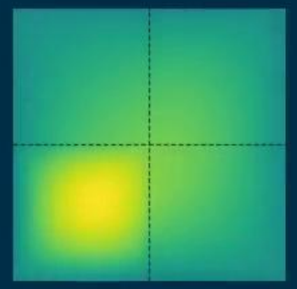

u smooth

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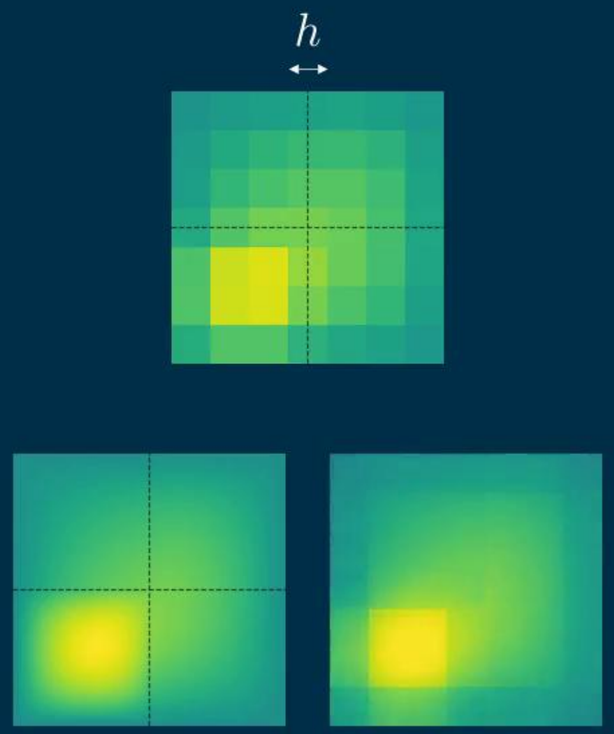

u smooth locally fit polynomials

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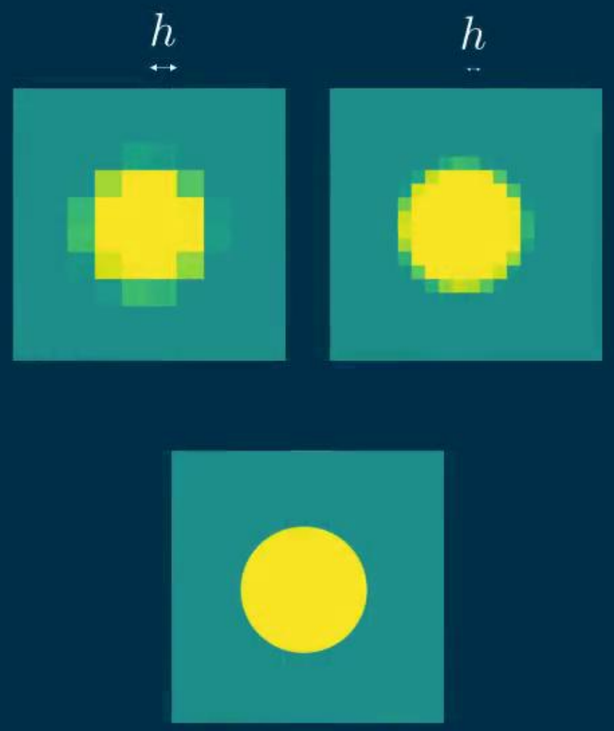
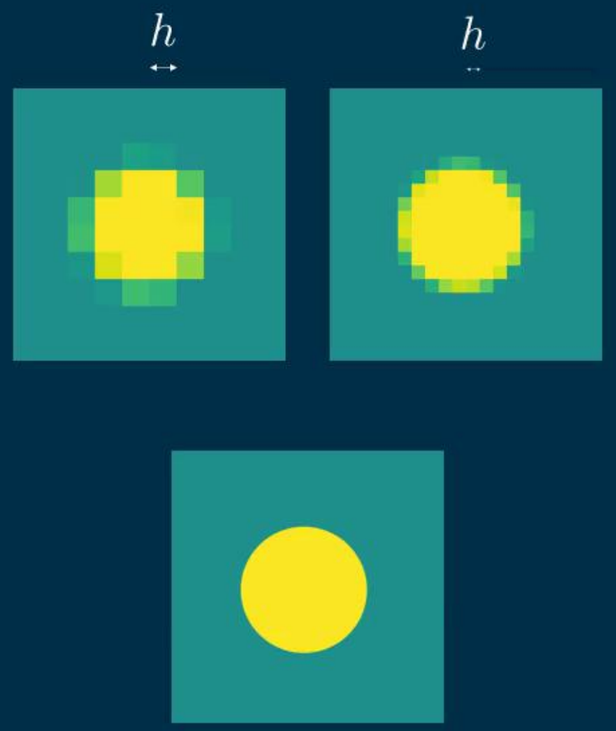
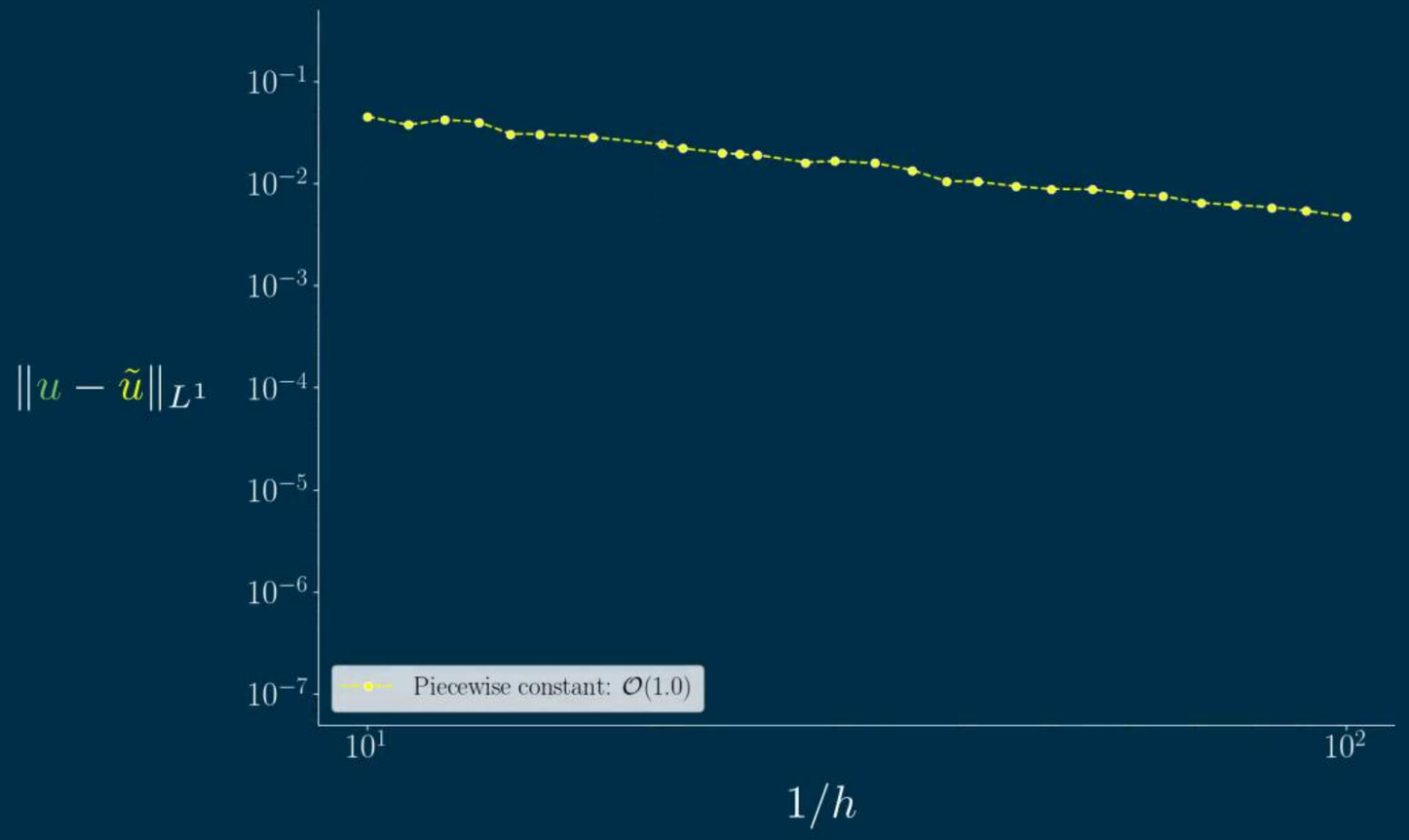


u presents jump discontinuities

Interface reconstruction: linear approach

$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$





[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Non-linear approximation

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Non-linear approximation: linear interface

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

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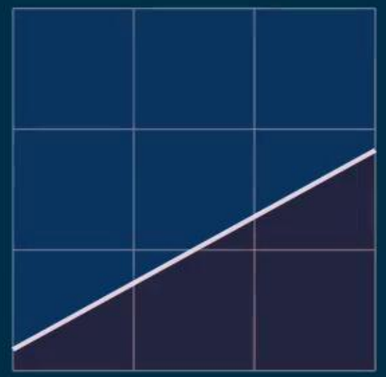
$v \in V_2$



0.0	0.0	0.0
0.0	0.1	0.6
0.4	0.9	1.0

$\ell(v) \in \mathbb{R}^m$

Non-linear approximation: linear interface

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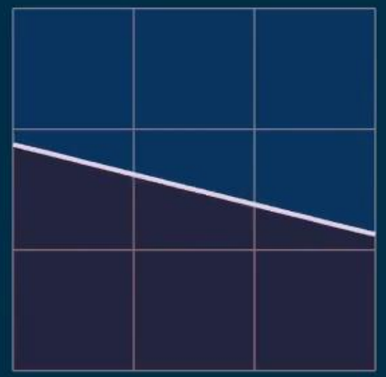
$v \in V_2$



0.0	0.0	0.0
0.7	0.5	0.3
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

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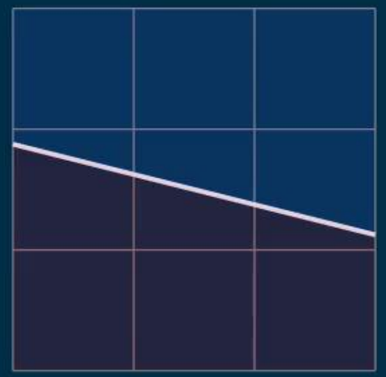
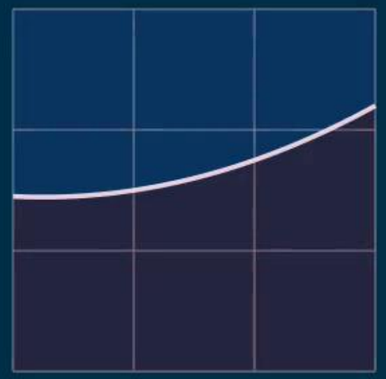


0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$

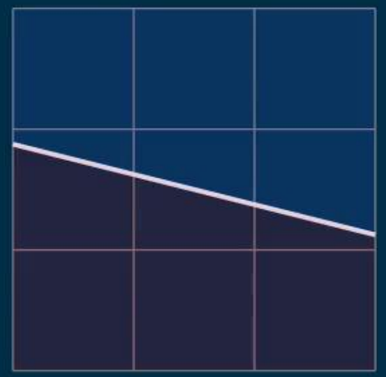
Observed averages

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$ LVIRA [E.G. Puckett, 1991]

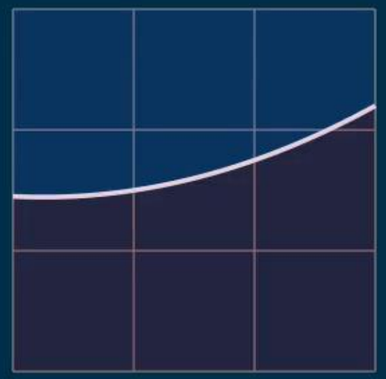
$v \in V_2$



0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

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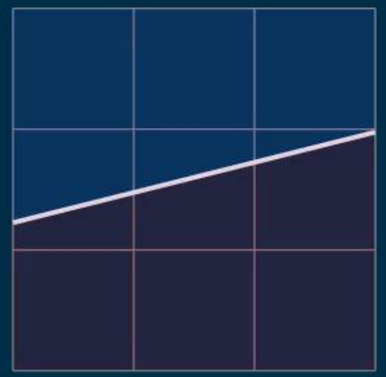
$v \in V_2$

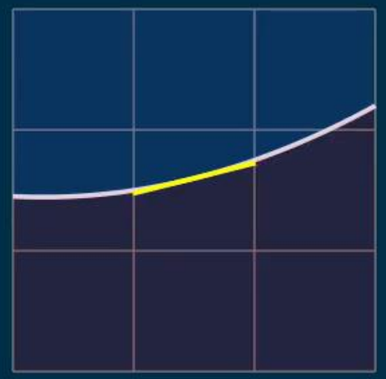


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
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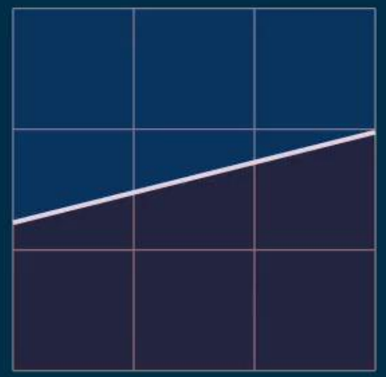
Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

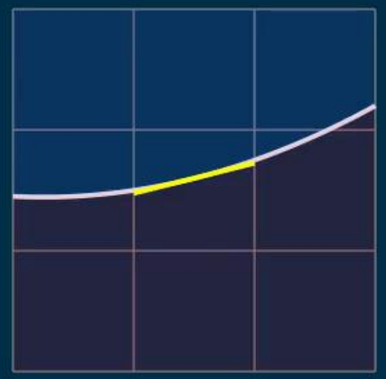
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed
averages

Best fit estimator: LVIRA

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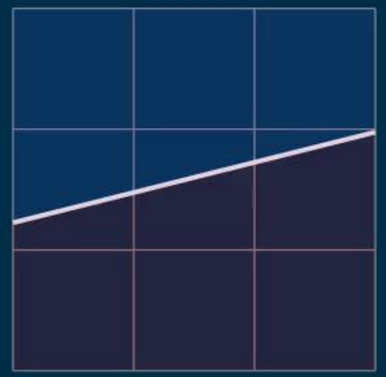
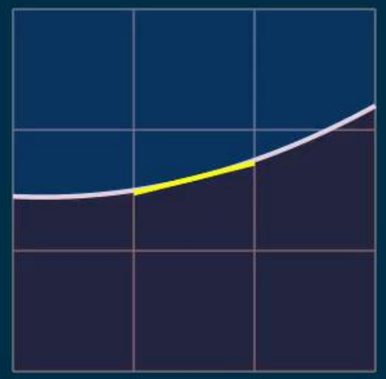
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed
averages

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

Lipschitz continuity

$$\|\ell(u) - \ell(v)\|_Z \leq \alpha_Z \|u - v\|_V, \quad u, v \in V$$

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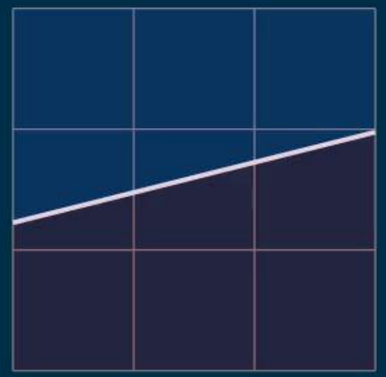
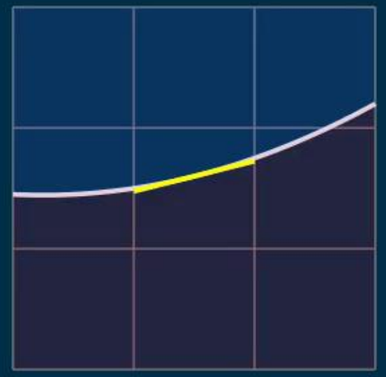
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

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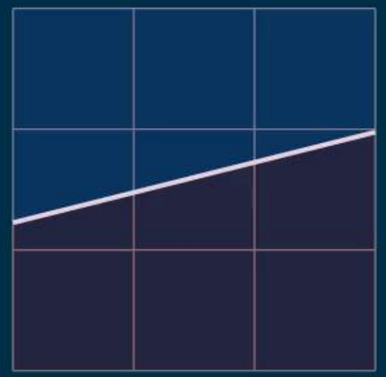

0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

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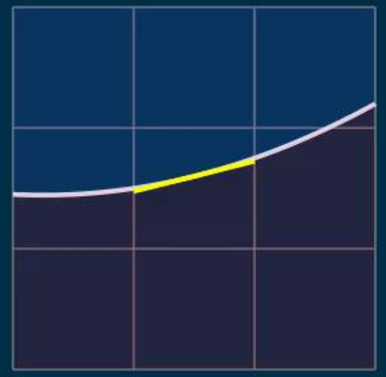
Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

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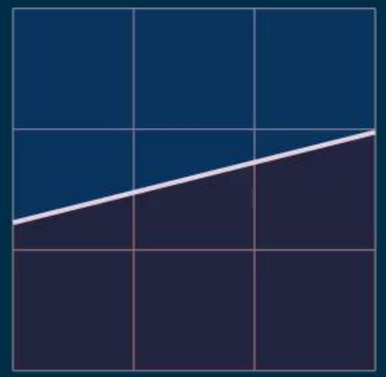
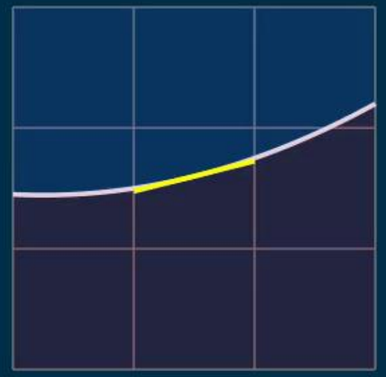


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
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$z \in \mathbb{R}^m$
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[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

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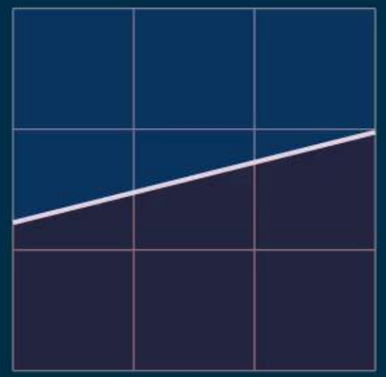
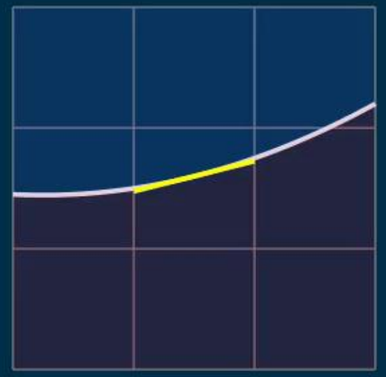


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
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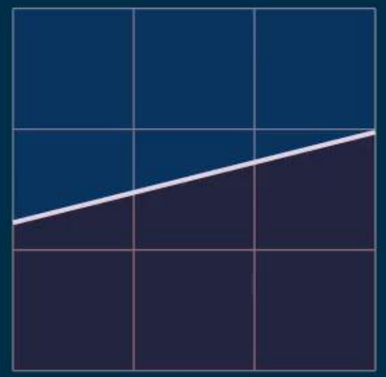
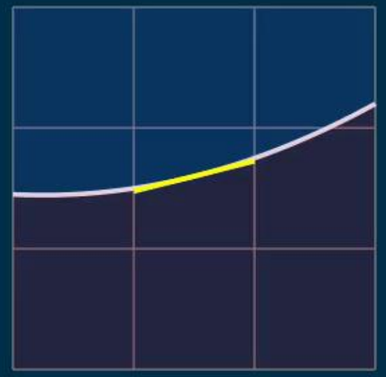


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

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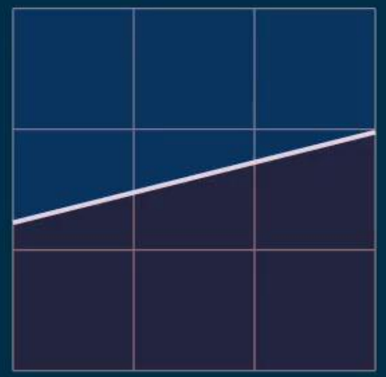
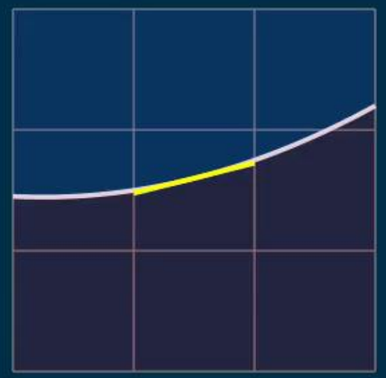


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_{L^1} \leq 10\|u - P_{V_n} u\|_{L^1}$$

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Best fit estimator: LVIRA

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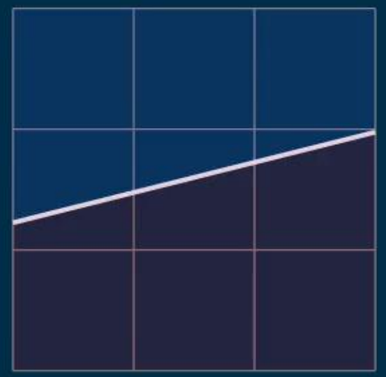
$v \in V_2$

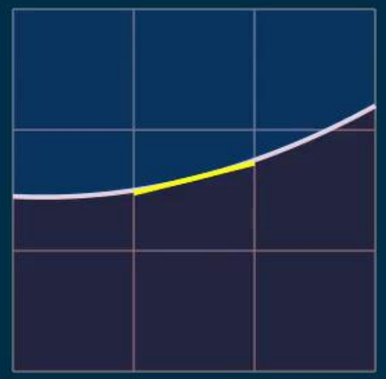


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

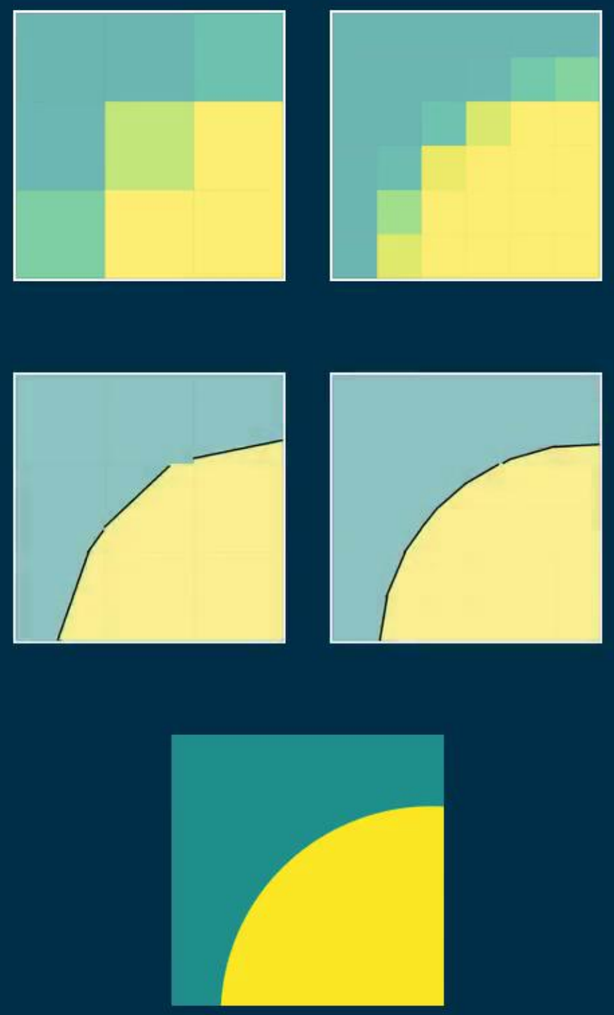
$z \in \mathbb{R}^m$
Observed averages

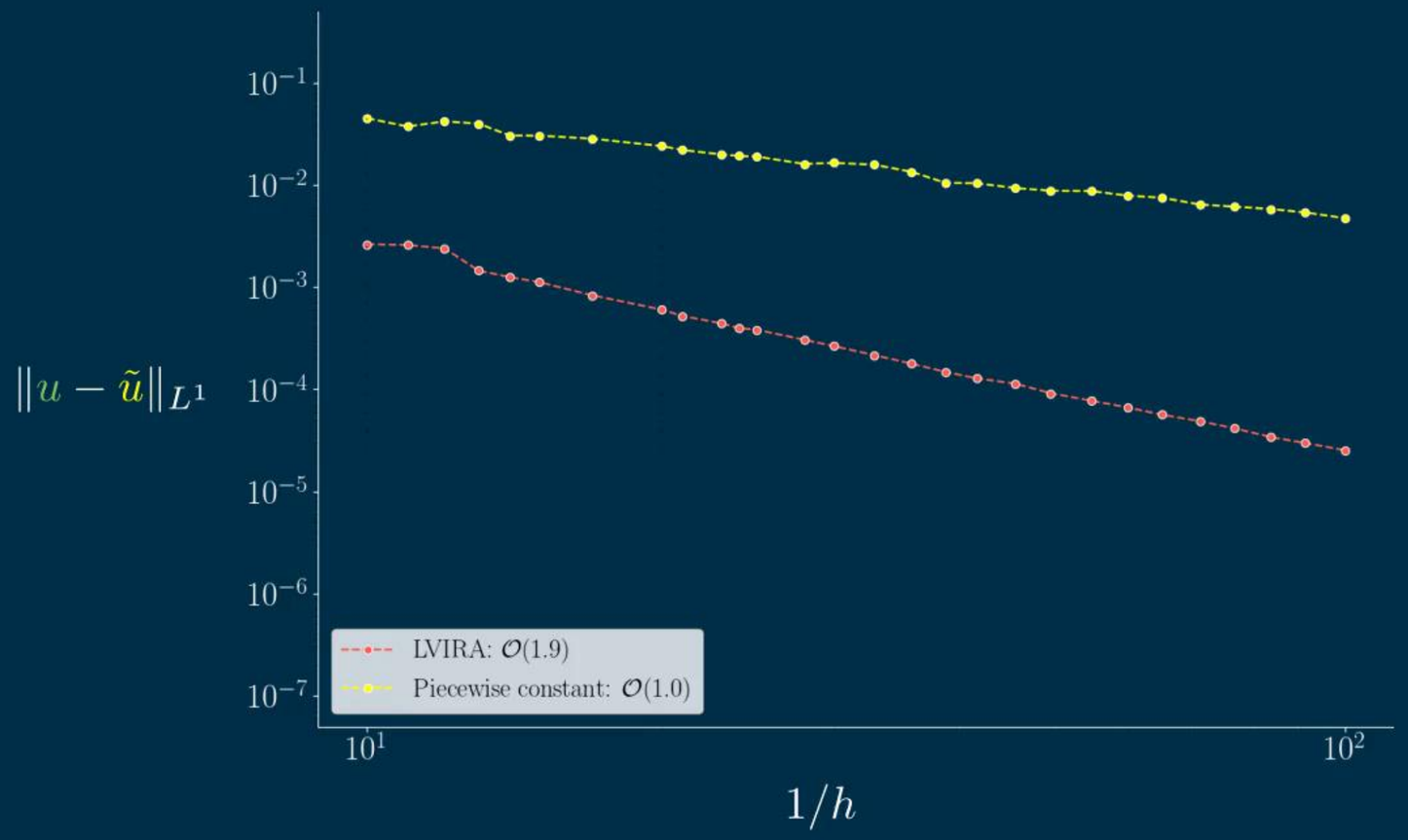
Global convergence

Corollary: If Ω is a $\mathcal{C}^s$ domain with $s \geq 2$, the LVIRA method converges in L^1 norm at rate h^2

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Best fit estimator: LVIRA





[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Contributions of the thesis

Linear approaches

Extend approximation guarantees
Diffusion equation with
unbounded parameter space Y

[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Non-linear approaches

Theoretical framework
to analyse
non-linear strategies

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Implementation of non-linear strategies
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Higher order

Optimization Based Edge Reconstruction Algorithm

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

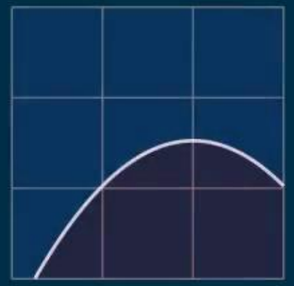
$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



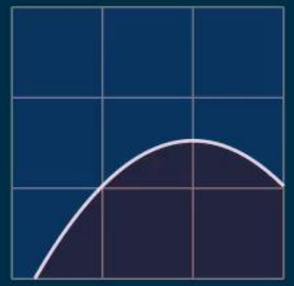
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

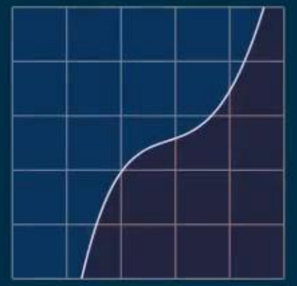
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



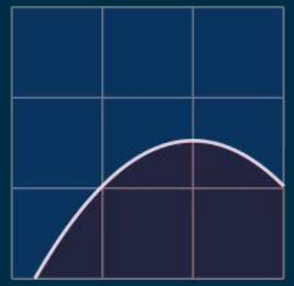
0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

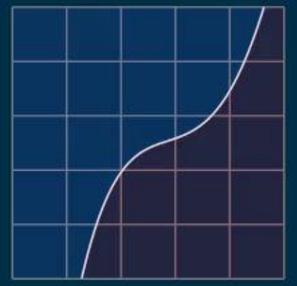
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



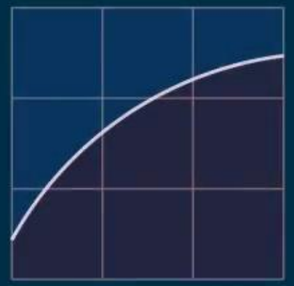
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Circle



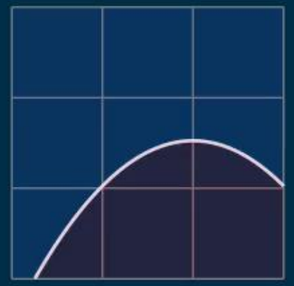
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

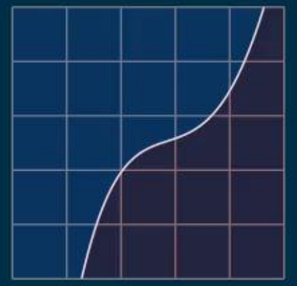
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic

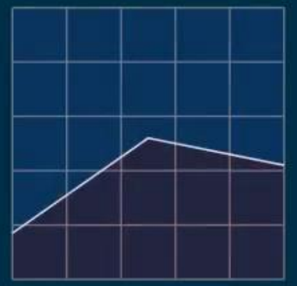


0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Circle

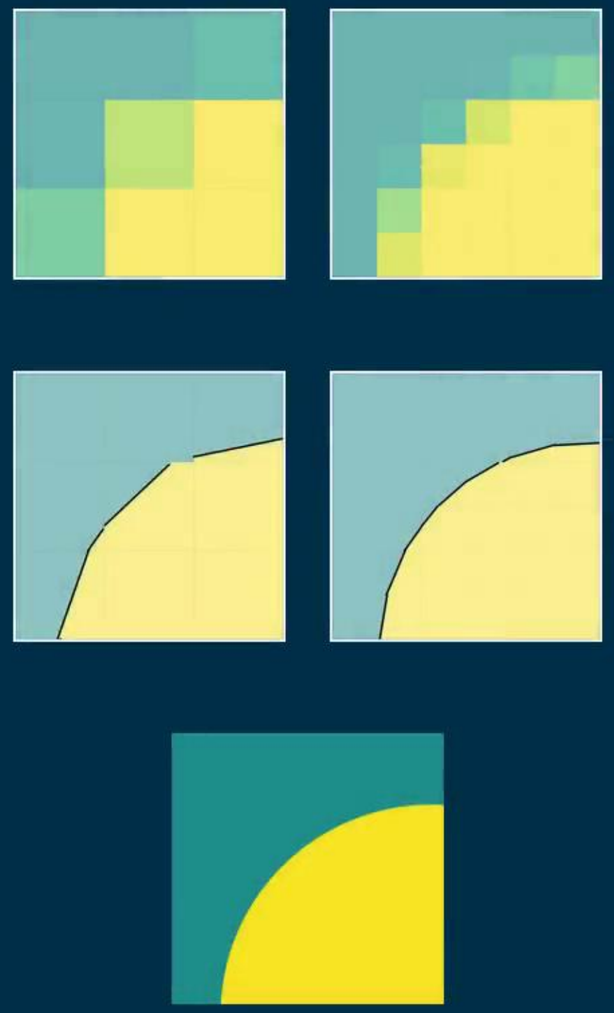
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

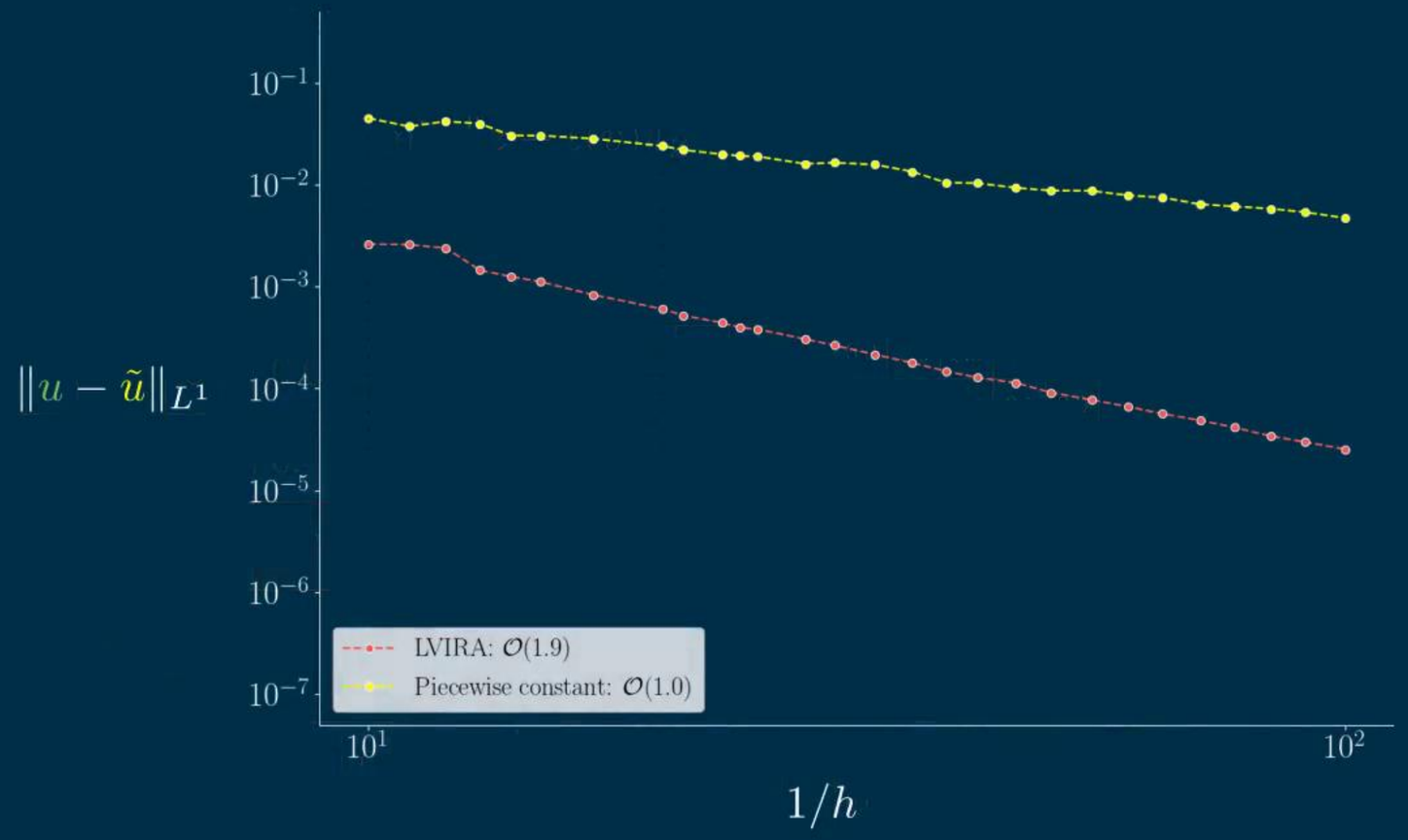
Vertex



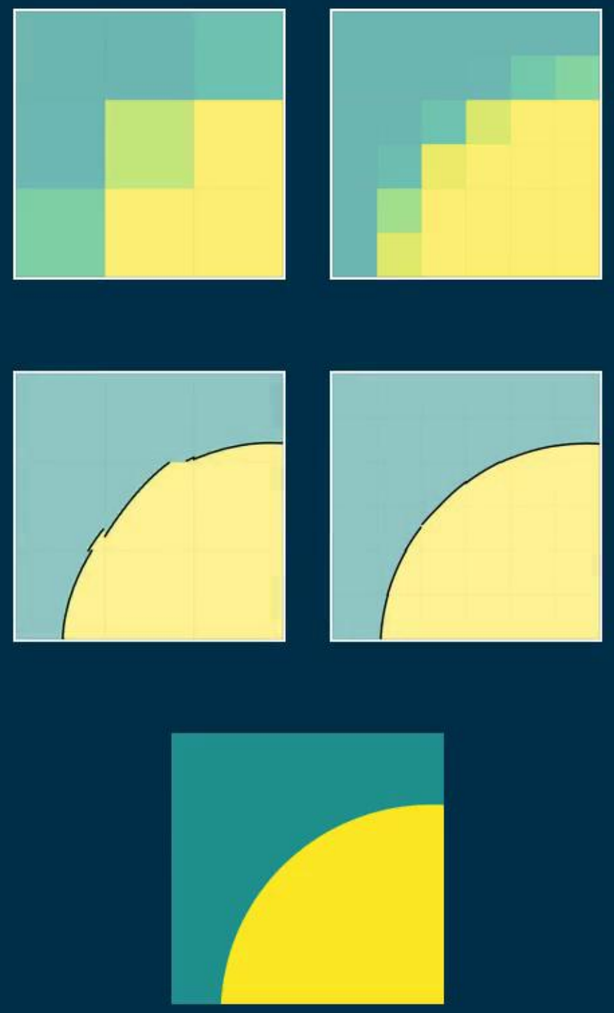
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.5	0.4	0.2
0.2	0.9	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

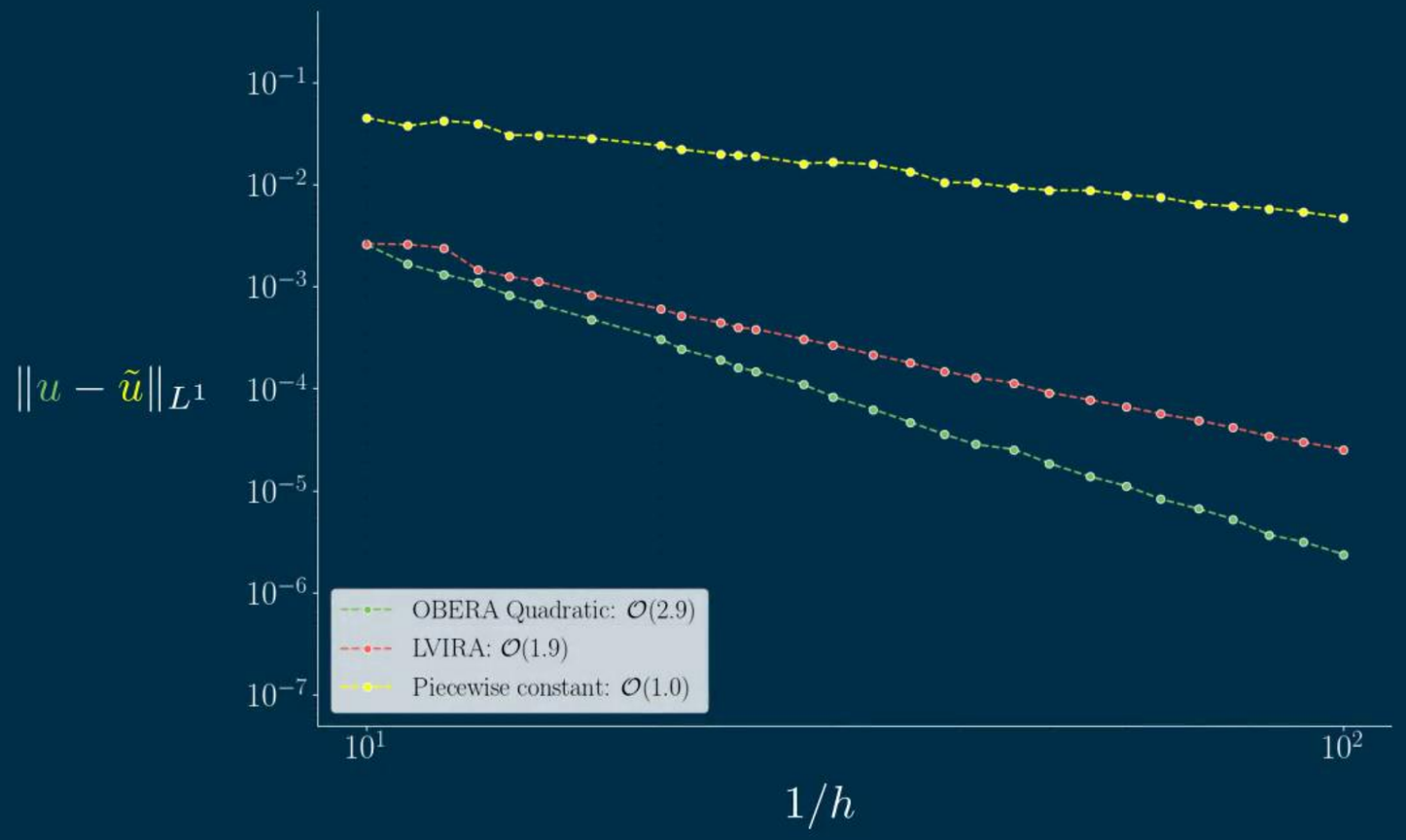
Higher order: OBERA





Higher order: OBERA





[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

Higher order: OBERA

Problem: long computation time

Higher order: OBERA

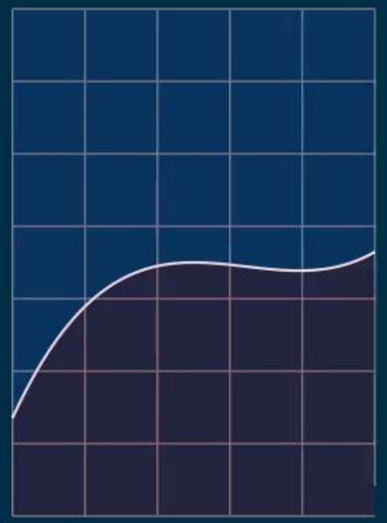
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Problem: long computation time

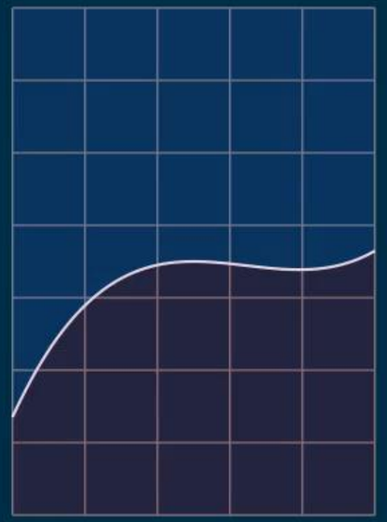
Fast higher order methods

Algorithm for Edge Reconstruction using Oriented Stencils

Higher order: AEROS

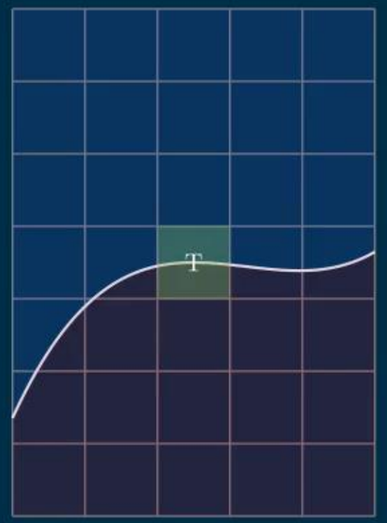


Higher order: AEROS



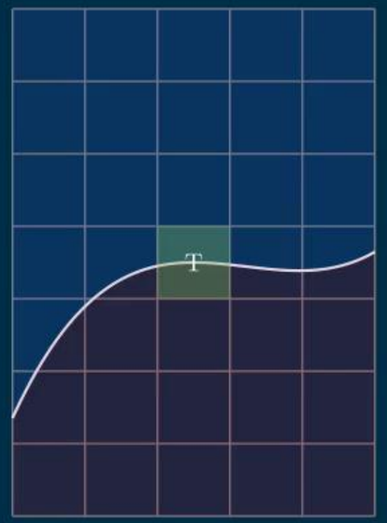
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

Higher order: AEROS

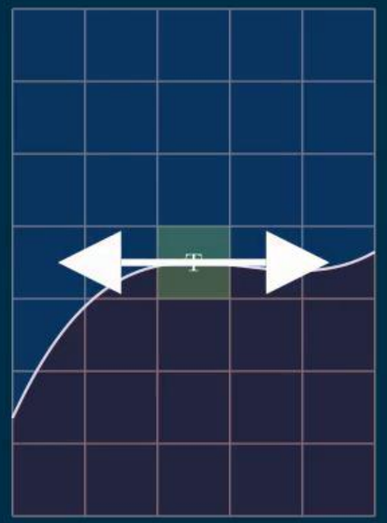


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Horizontal orientation

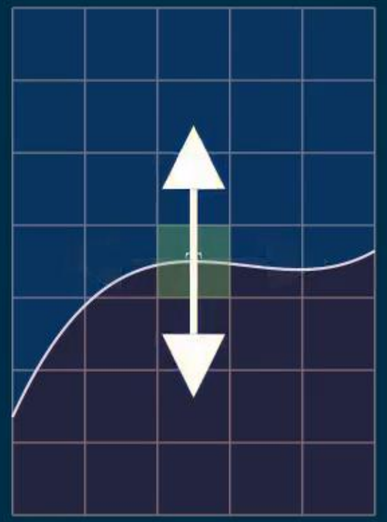


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation

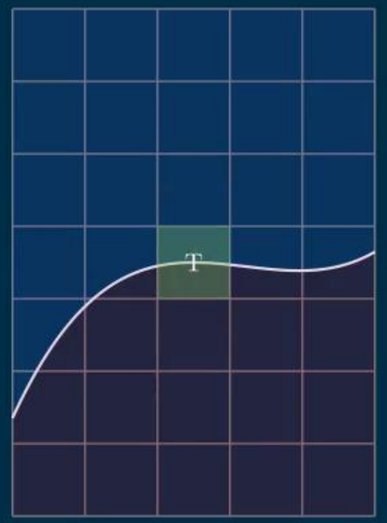


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation



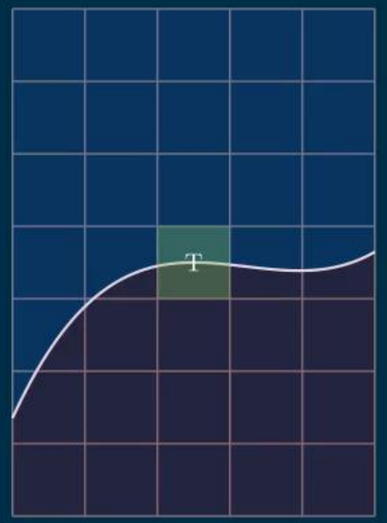
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

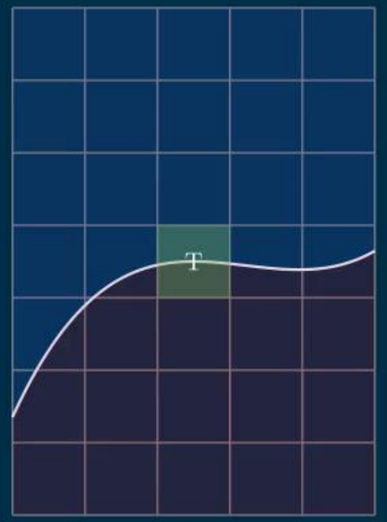
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

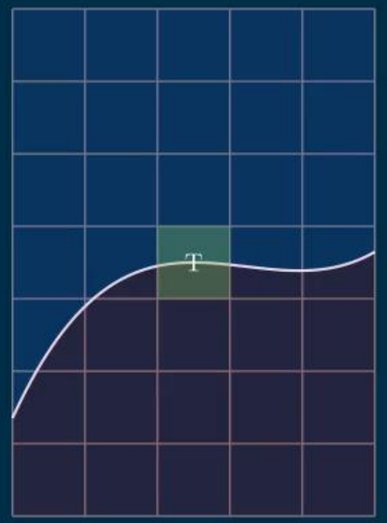
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

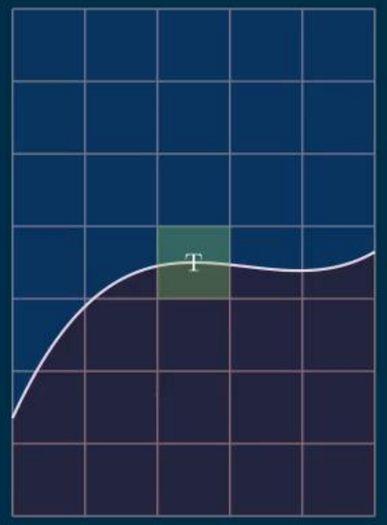
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

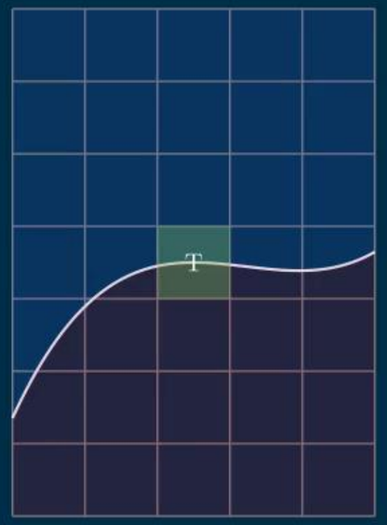
Orientation test based on Sobel filter

$$G_T = (-0.4, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

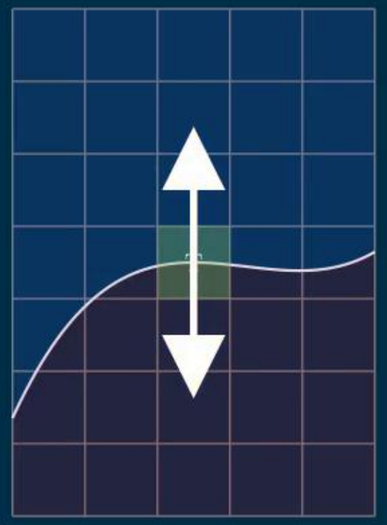
Orientation test based on Sobel filter

$$G_T = (-0.4, 4.0)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

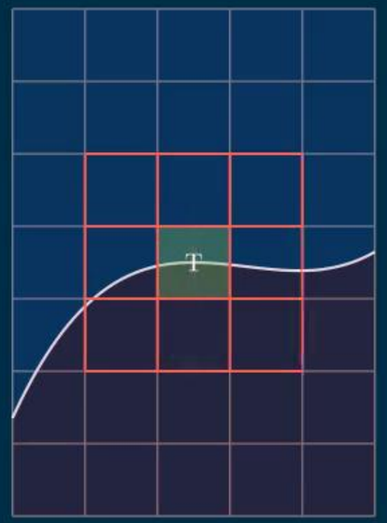
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$0.4 = |H_T| < |V_T| = 4.0$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

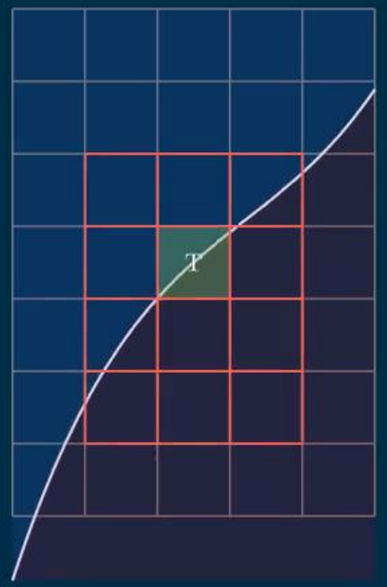
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
- Define an appropriate stencil S_T

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
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1.0	1.0	1.0	1.0	1.0

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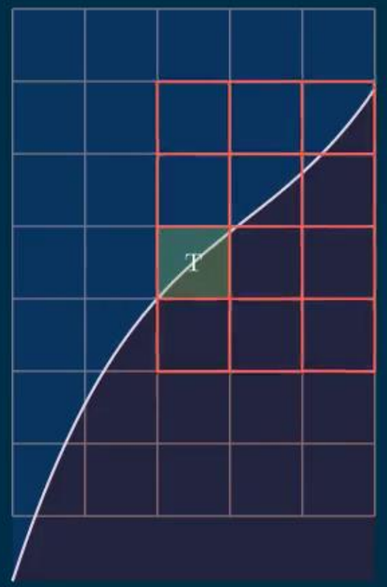
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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1.0	1.0	1.0	1.0	1.0

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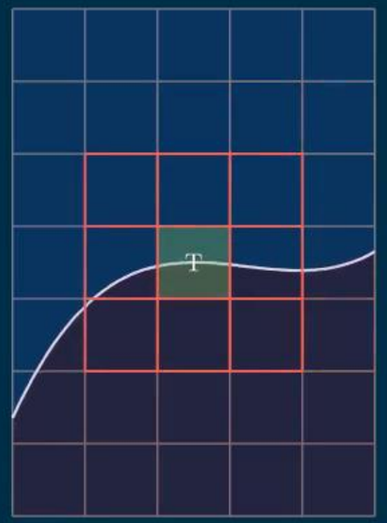
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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1.0	1.0	1.0	1.0	1.0

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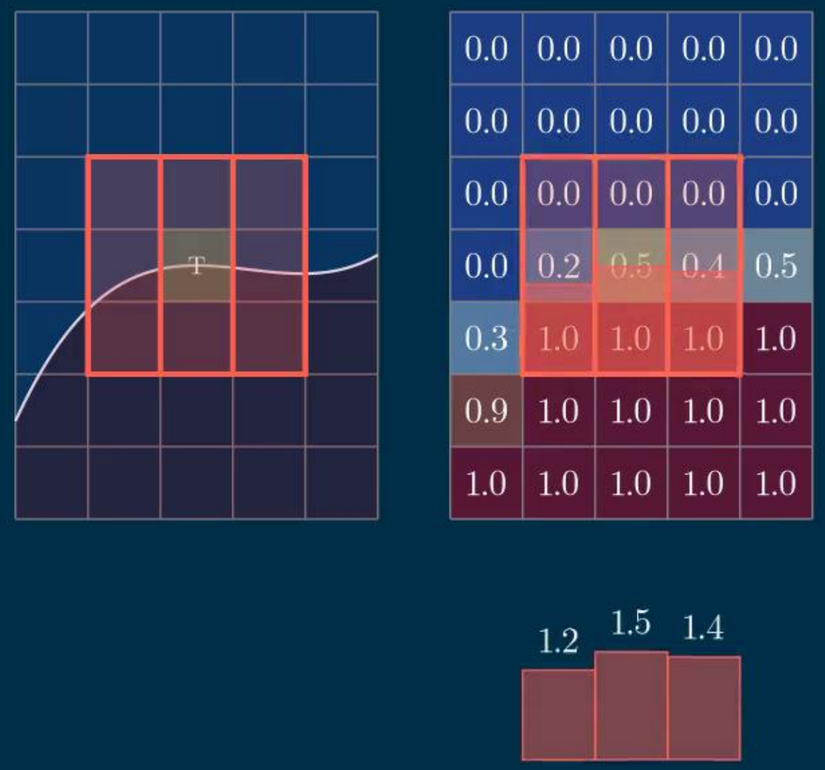

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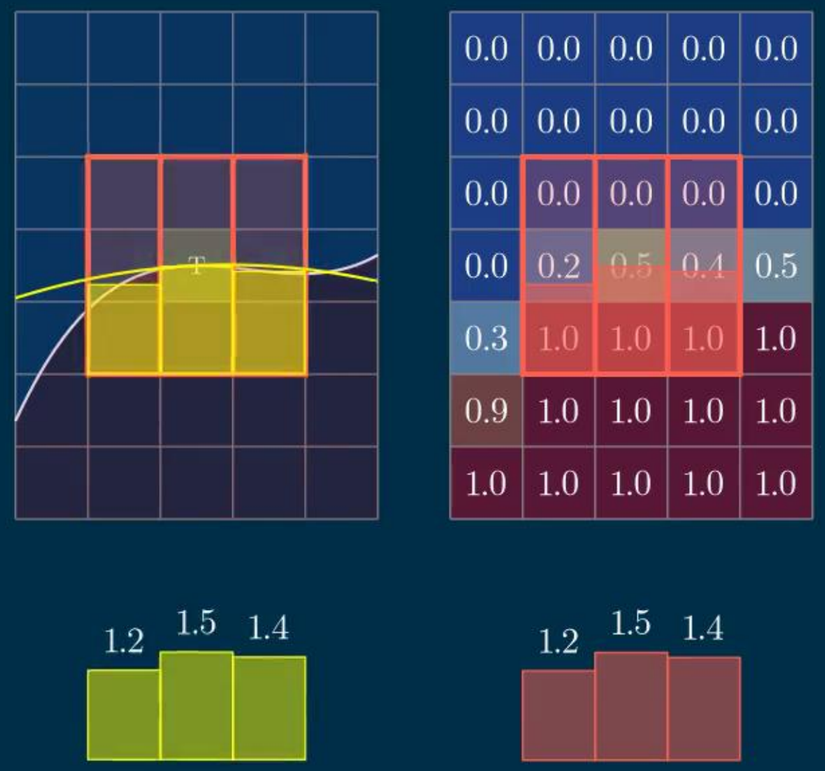
- Define an appropriate stencil S_T
- Sum averages a_{S_T} column-wise: $C_i = \sum_{l=j-1}^{j+1} \ell_{il}(u)$

Higher order: AEROS



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- Define an appropriate stencil S_T
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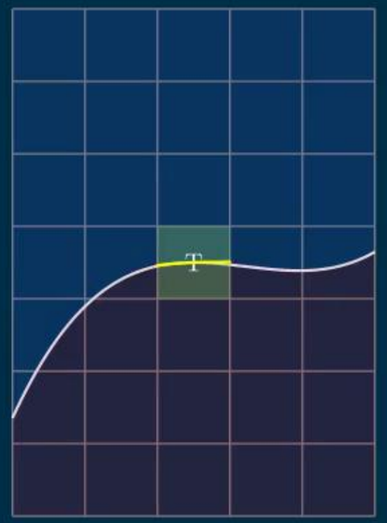
- Fit a polynomial to the associated 1D-problem

$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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AEROS theoretical results

Orientation test result

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
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AEROS global convergence theorem

$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

AEROS theoretical results

Orientation test result

The orientation given by the Sobel filter ensures that if the mesh size h is below a critical value h^* then there is a tall or wide enough stencil so that the interface croses the sides of it.

AEROS global convergence theorem

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AEROS theoretical results

Orientation test result

The orientation given by the Sobel filter ensures that if the mesh size h is below a critical value h^* then there is a tall or wide enough stencil so that the interface croses the sides of it.

AEROS global convergence theorem

If Ω is a $\mathcal{C}^s$ domain with $s \geq n$, the AEROS method based on polynomials of degree n has a global convergence rate in L^1 norm of h^{n+1}

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
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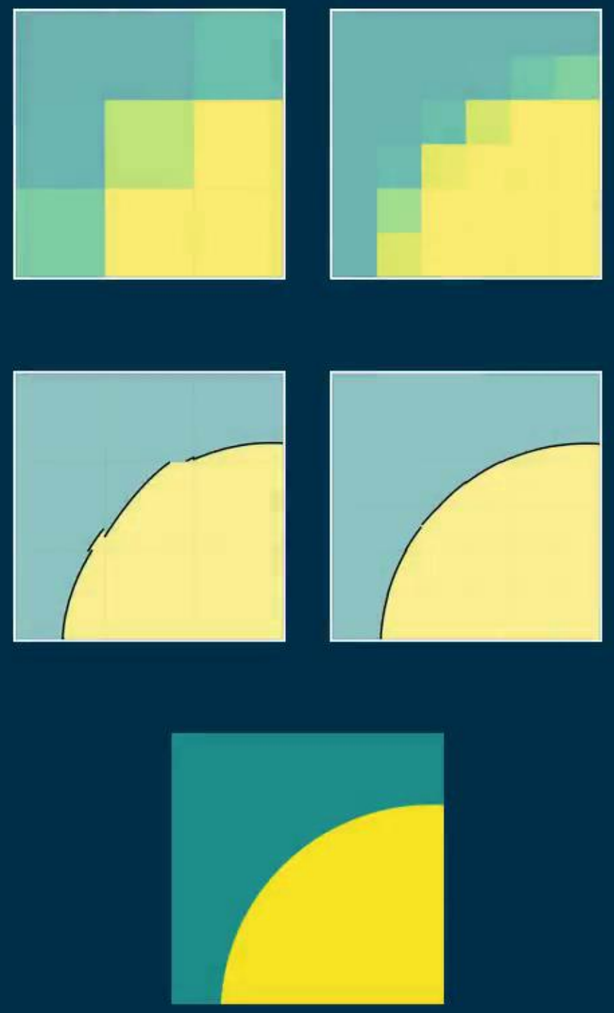
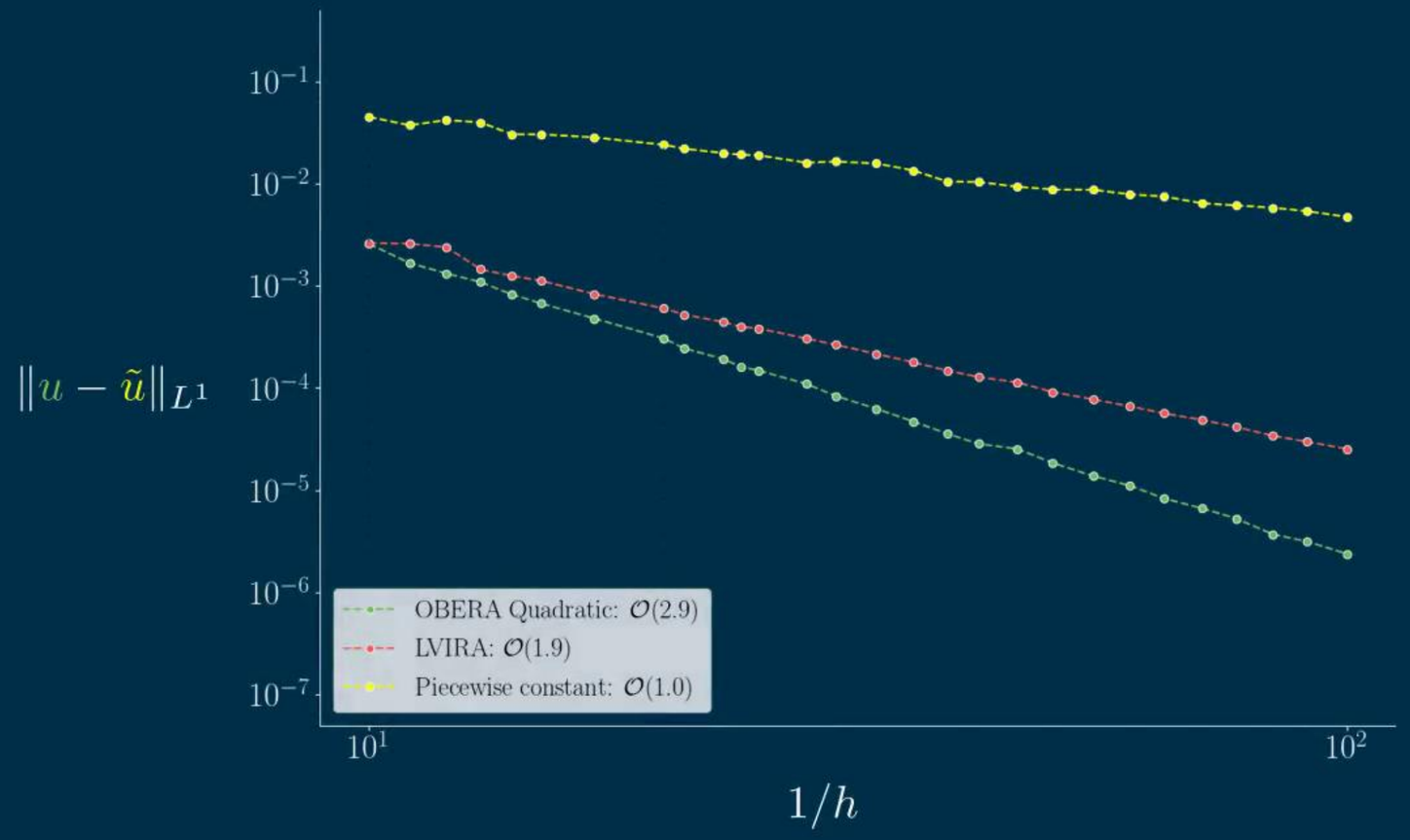
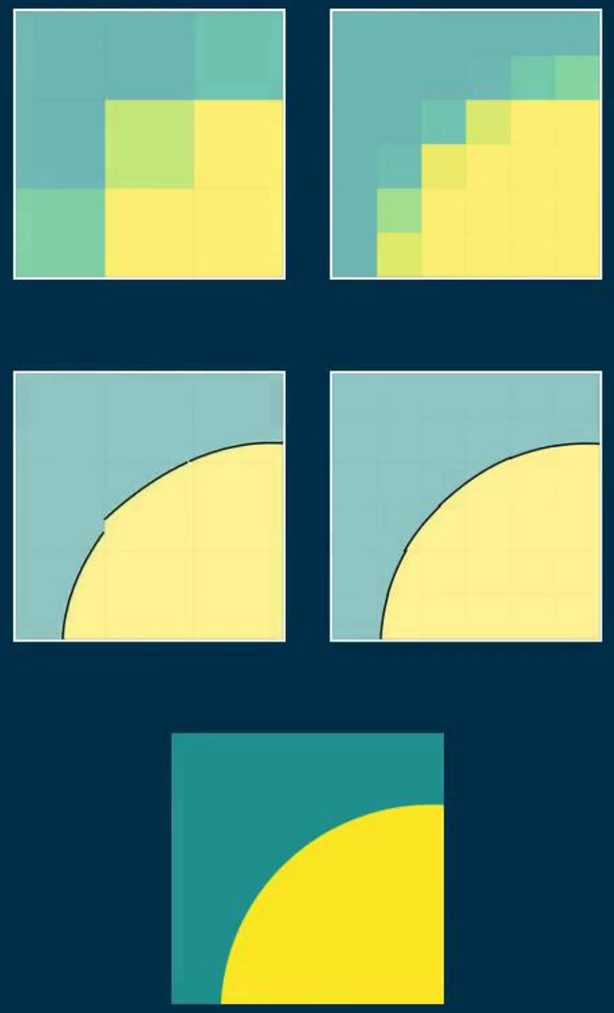
$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

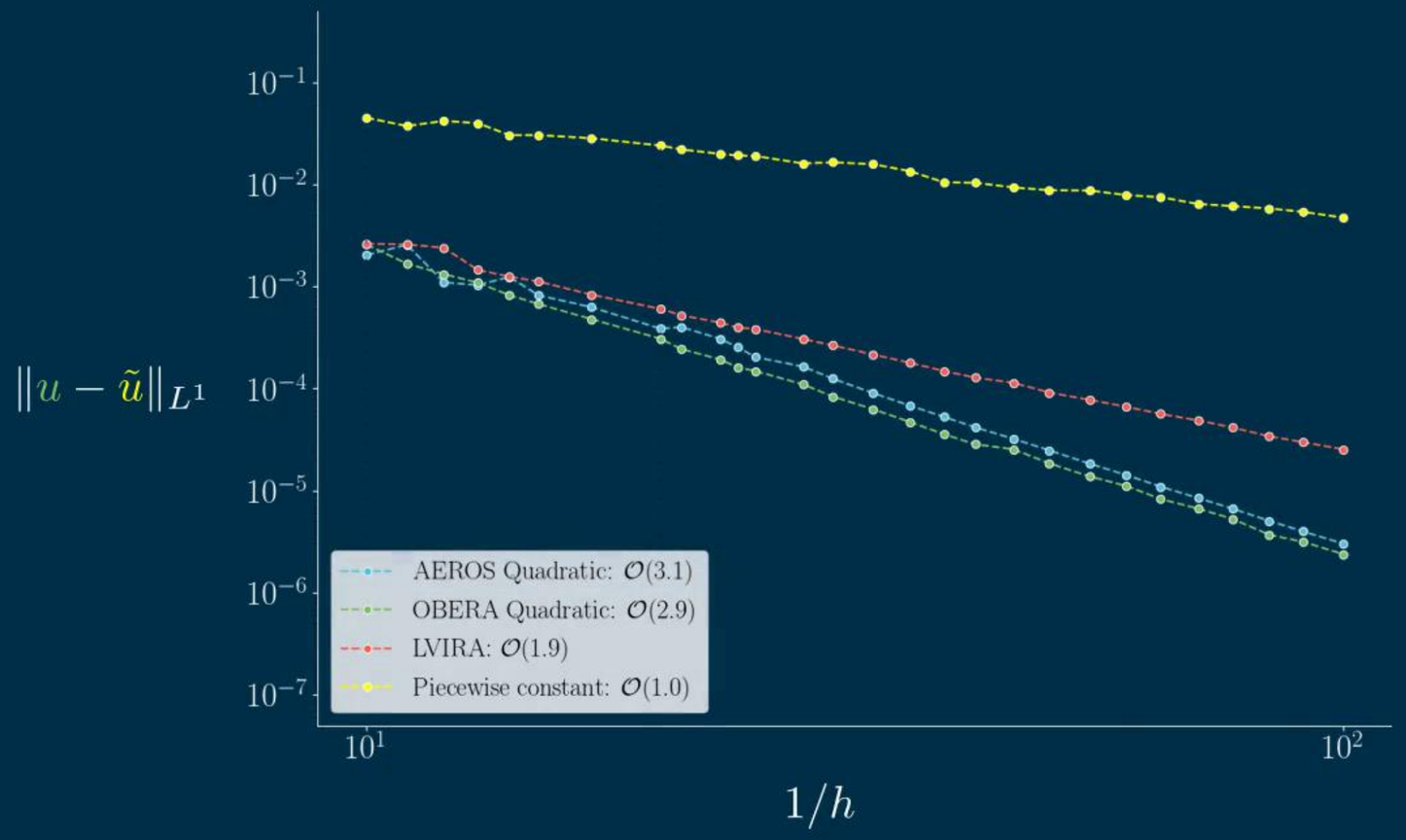
AEROS convergence





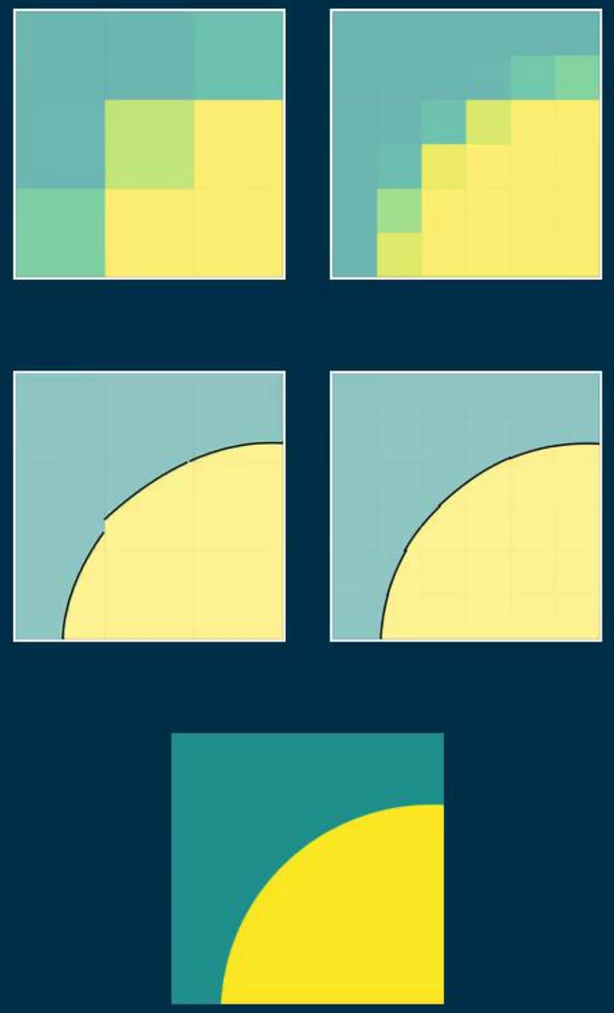
AEROS convergence: Quadratic

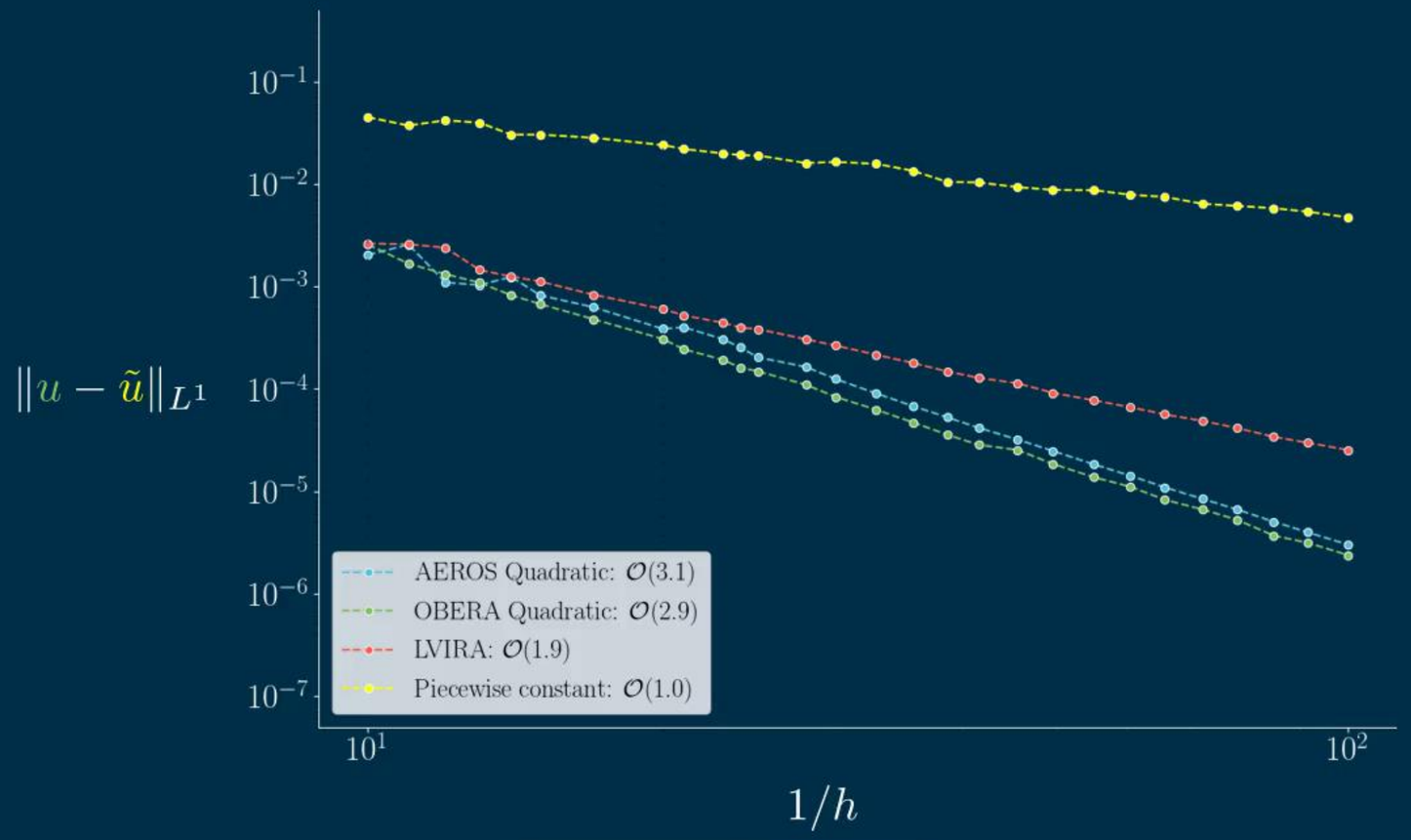




[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

AEROS convergence: Quadratic

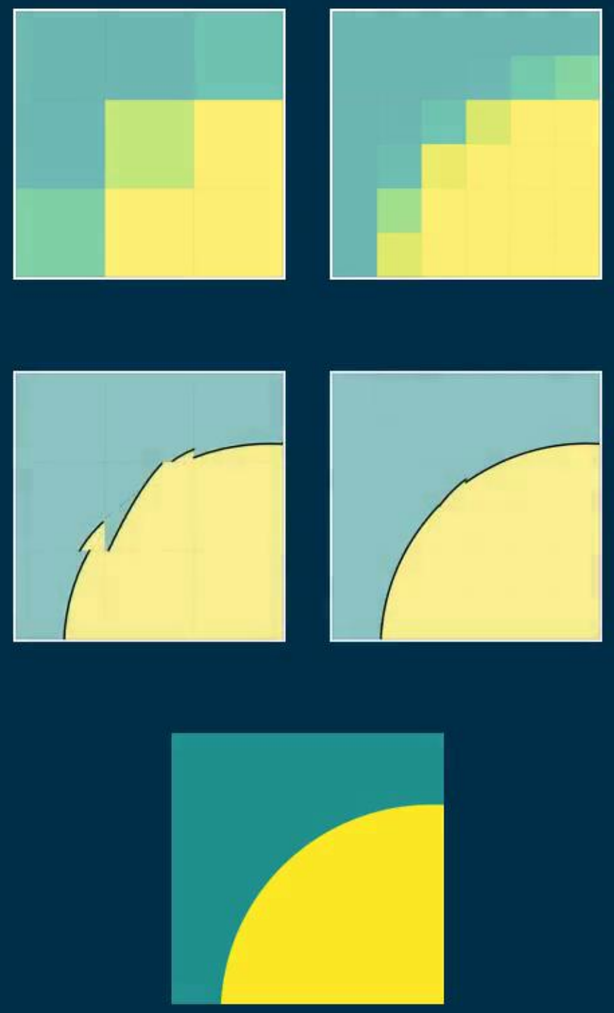


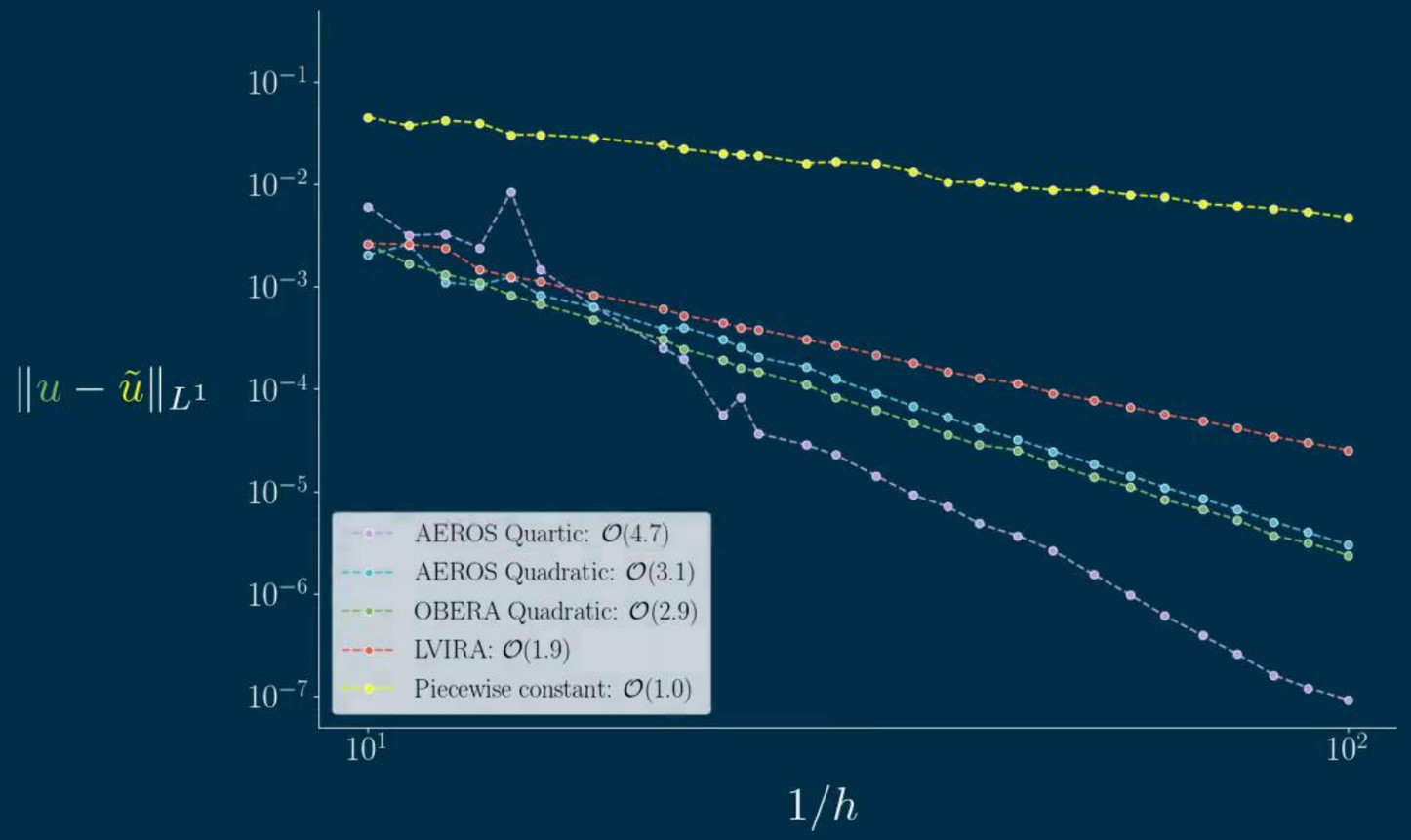


$$\tau_{AEROS} \lesssim \frac{\tau_{OBERA}}{100}$$

[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

AEROS convergence: Quartic





$$\tau_{AEROS} \lesssim \frac{\tau_{OBERA}}{100}$$

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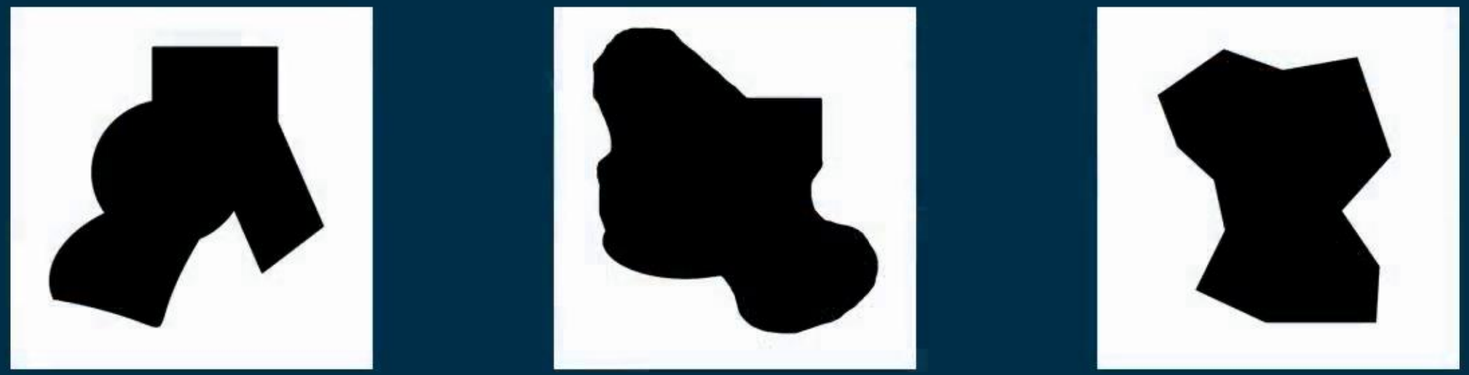
Corners

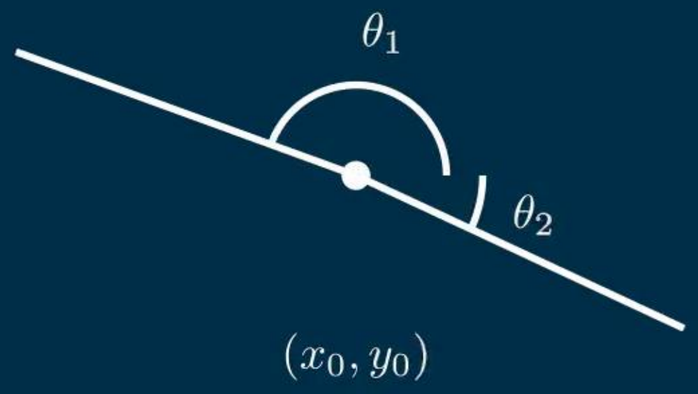


Corners

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Corners

$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$



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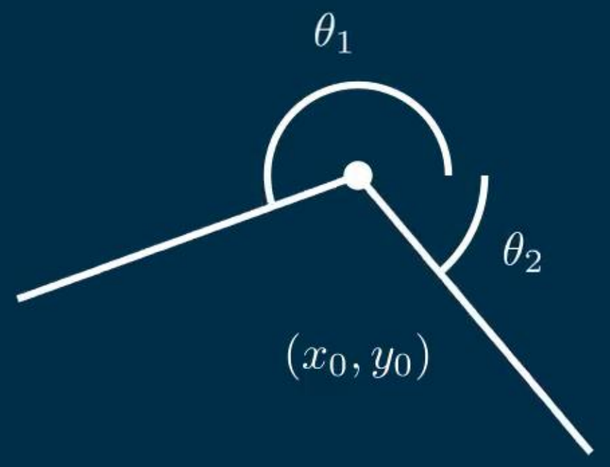
$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$



Corners

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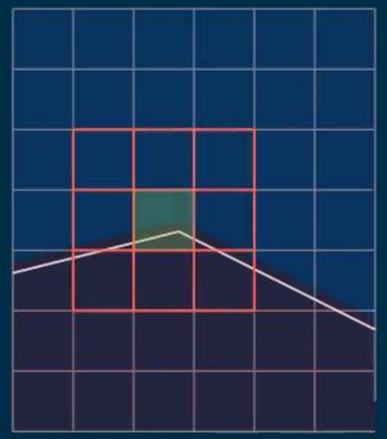
AEROS Vertex

Tangent Extension Method (TEM)

Corners

$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$

AEROS Vertex



Tangent Extension Method (TEM)

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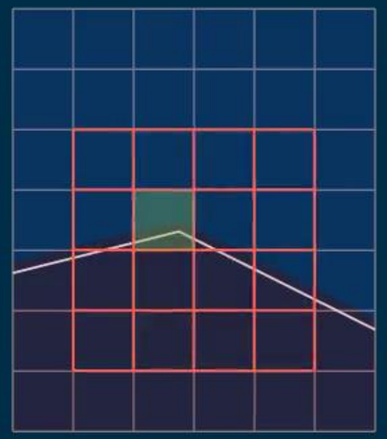
$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$

AEROS Vertex



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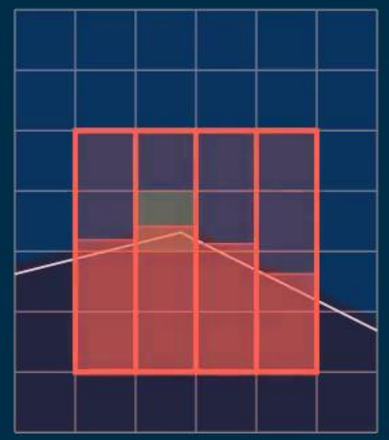
AEROS Vertex



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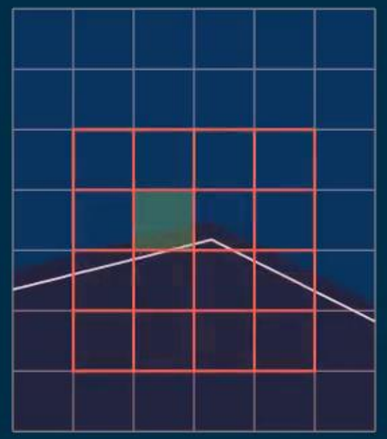
AEROS Vertex



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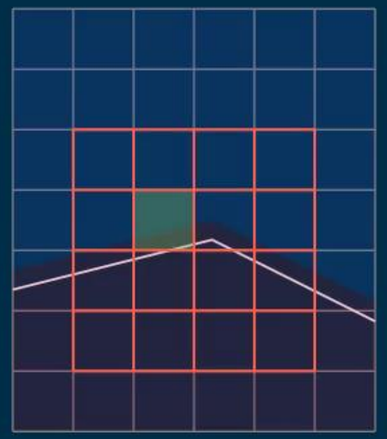

Tangent Extension Method (TEM)

- More than 1 posibility to test
- Can not deal with orientation change

Corners

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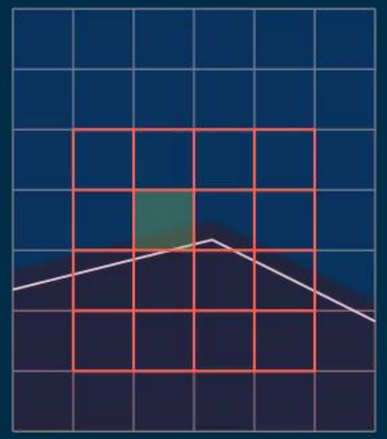
AEROS Vertex



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[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

Corners

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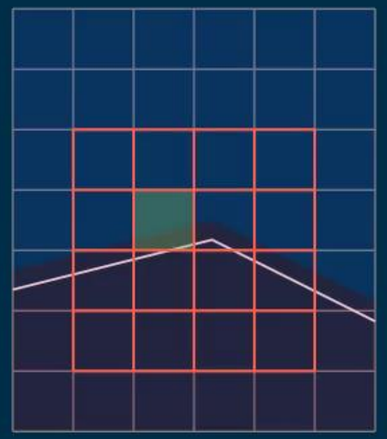
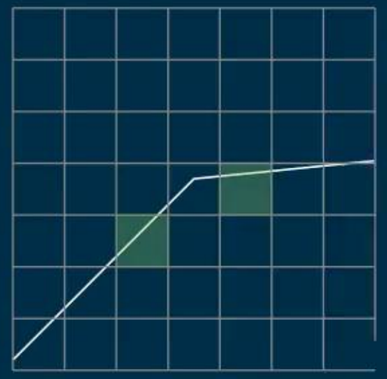


Tangent Extension Method (TEM)



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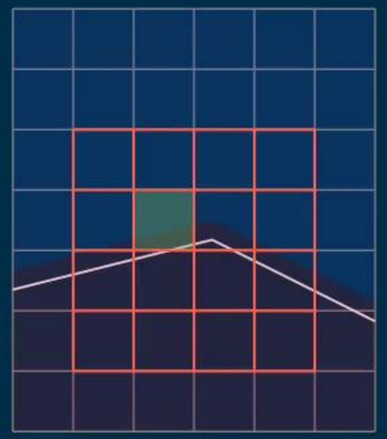

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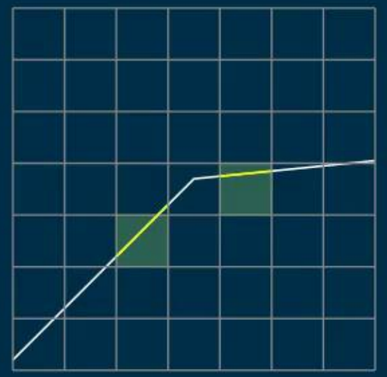

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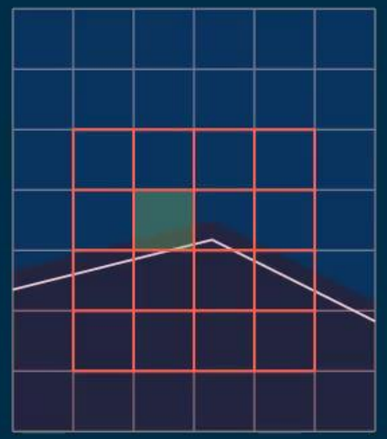

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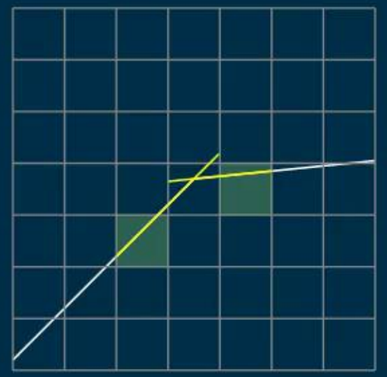

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[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

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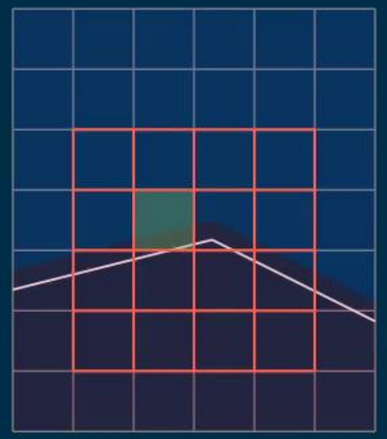

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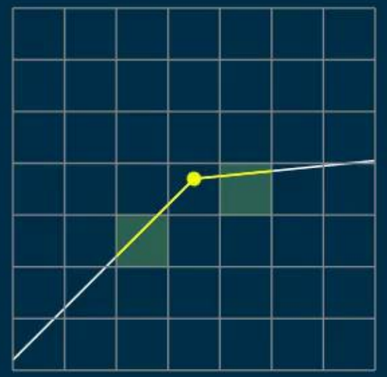

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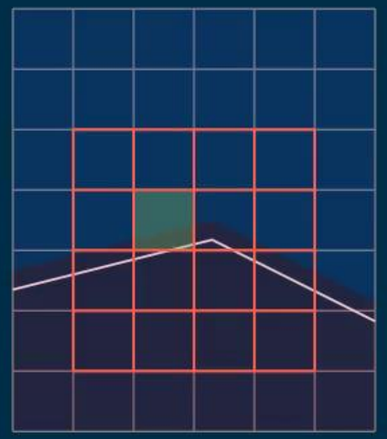

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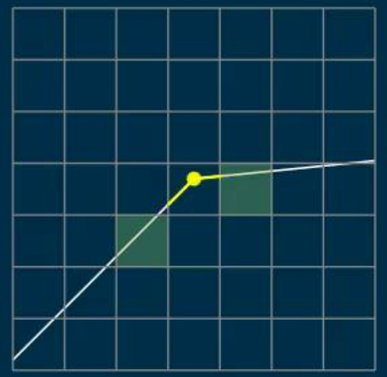

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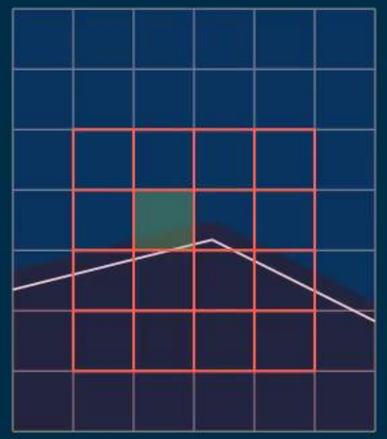

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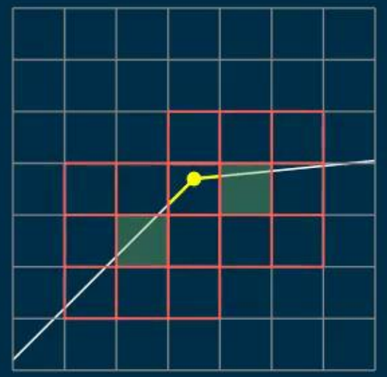

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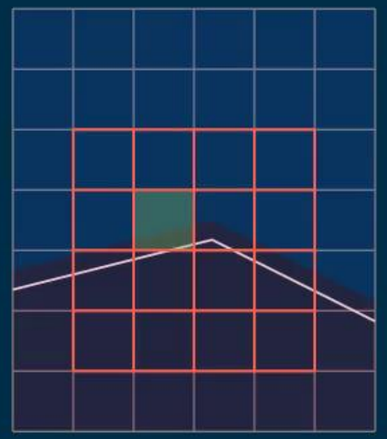

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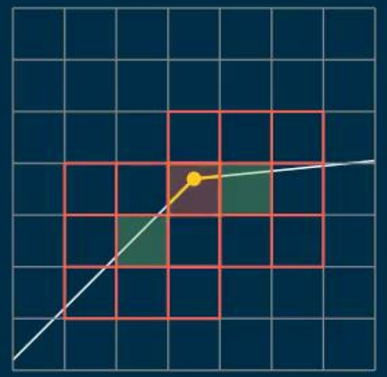

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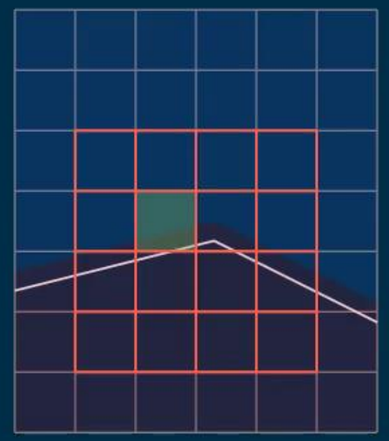

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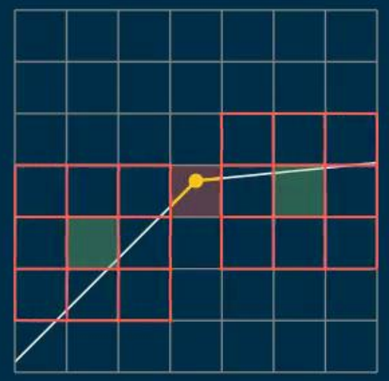

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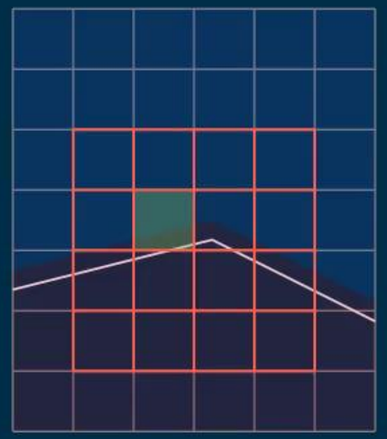

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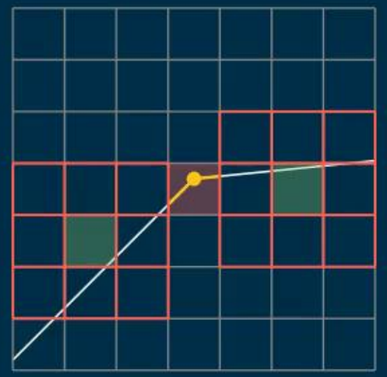

- Exact only if both sides of vertex are straight lines

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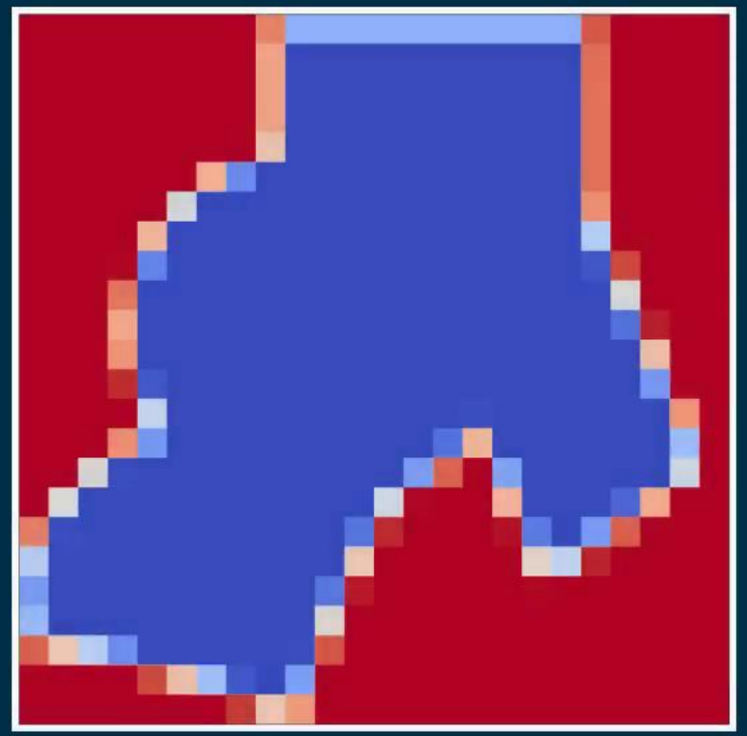

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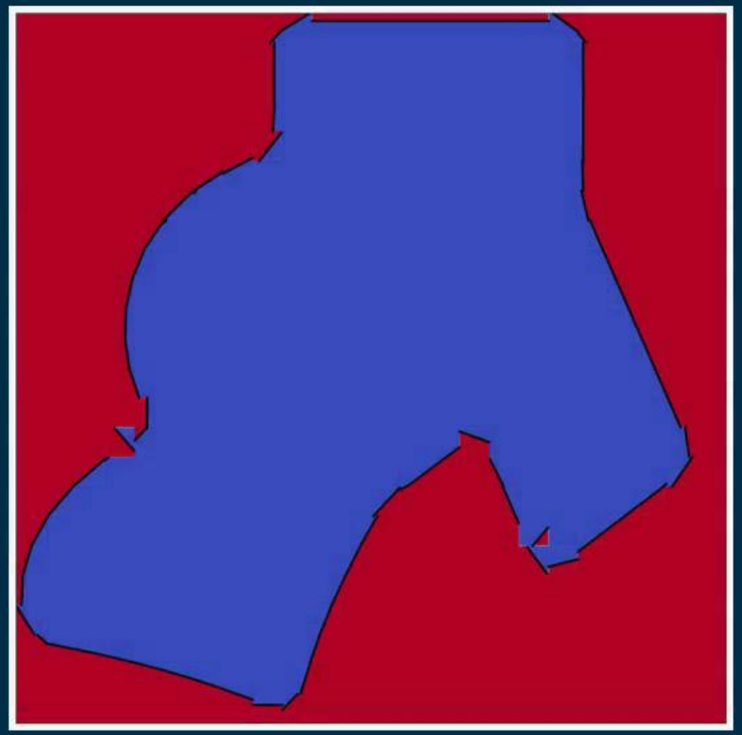

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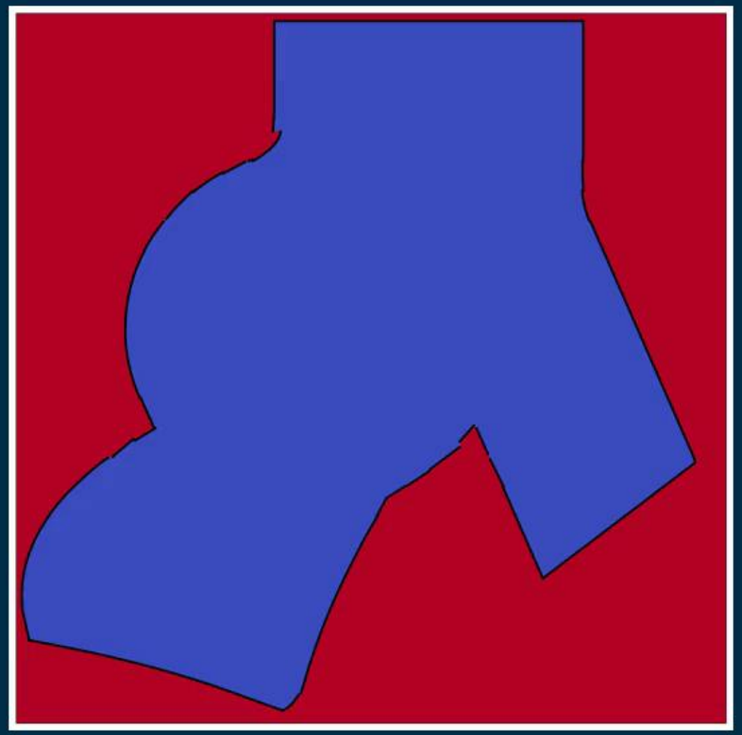

Contributions of the thesis

Linear approaches

Extend approximation guarantees
Diffusion equation with
unbounded parameter space Y

[A. Cohen, W. Dahmen, M. Dolbeault, A. Somacal, 2023]

Non-linear approaches

Theoretical framework
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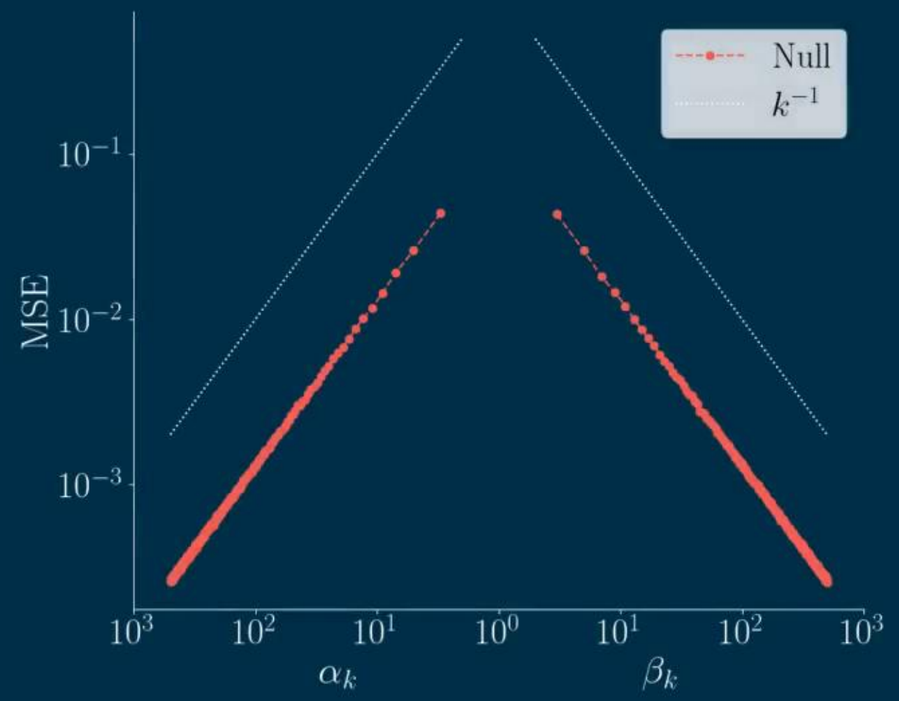

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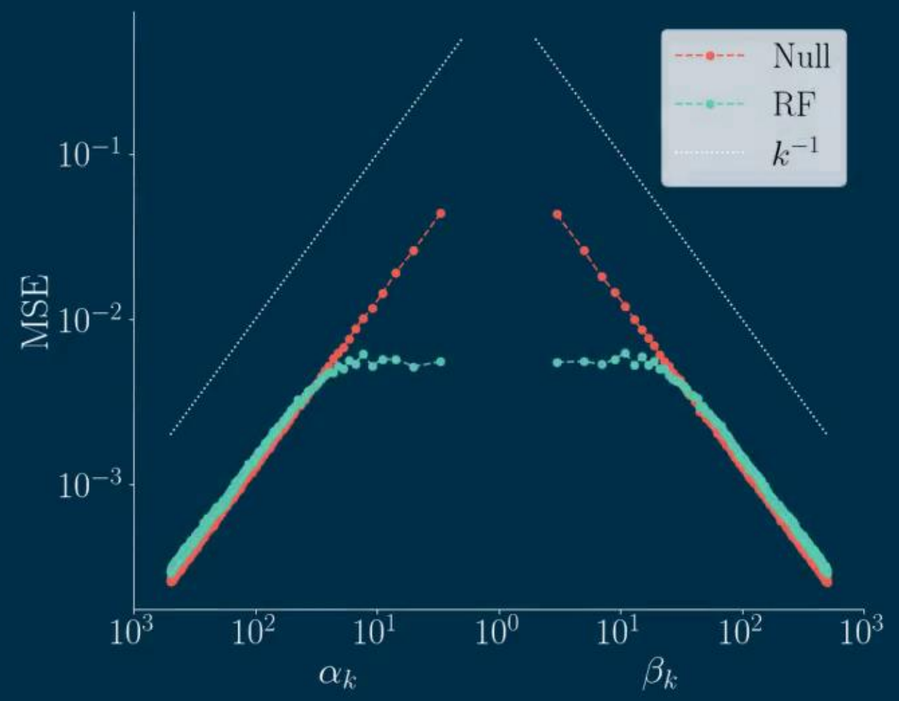

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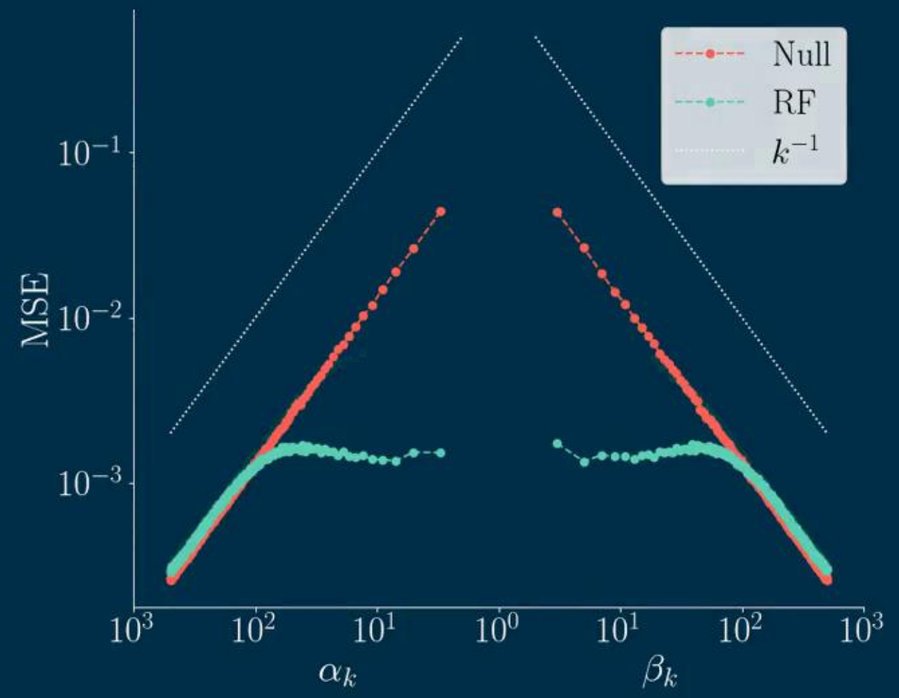
Non-linear reduced basis

Kolmogorov n-width

$$d_n(\mathcal{M})_V = \inf_{V_n,\ \dim(V_n)=n} \operatorname{dist}(\mathcal{M}, V_n)_V$$

$$u_\mu(x) := \begin{cases} b & \text{for } x \in (a, a+\ell) \pmod 1 \\ 0 & \text{for } x \in (a+\ell, a) \pmod 1 \end{cases}, \qquad \mu = (a, \ell, b)$$



Sensing numbers

$$s_n(\mathcal{M})_V := \inf_{R,\ell_1,\dots,\ell_n} \max_{u\in\mathcal{M}} \|u - R(\ell_1(u),\dots,\ell_n(u))\|_V,$$

$$u_\mu = \sum_{k\in\mathbb{N}} \alpha_k \cos(2\pi k x) + \sum_{k\in\mathbb{N}^*} \beta_k \sin(2\pi k x)$$

$$\begin{cases} a + \frac{\ell}{2} & = -\frac{1}{\pi}\arctan(\frac{\beta_1}{\alpha_1}) \\ \pi b \sin(\pi\ell) & = (\alpha_1^2 + \beta_1^2)^{1/2} \\ b\ell & = \alpha_0 \end{cases}$$

$$\begin{aligned} R(u) &= P_{V_n} u + P_{V_{N-n}} u \\ &= \sum_{i=1}^{n} \ell_i(u) v_i + \sum_{j=1}^{N-n} \tilde{\ell}_{n+j}(u) v_j \end{aligned}$$

$$\tilde{\ell}_{n+j}(u) = \psi_\theta(\ell_1(u),\dots,\ell_n(u))$$

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Codes are open sourced in GitHub

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