High order recovery of geometric interfaces from cell-average data

Lions-Magenes days 2024

Agustín Somacal

Collaborators: Albert Cohen, Matthieu Dolbeault, Olga Mula

Inverse problem

$u \in V$ Banach

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Parametric PDEs

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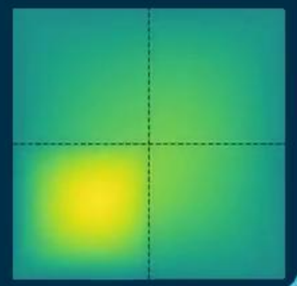

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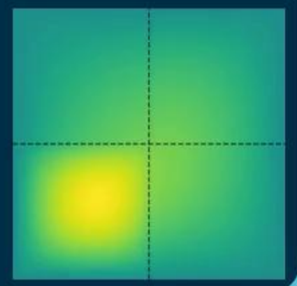

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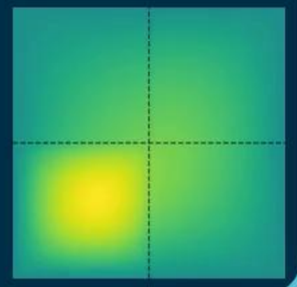

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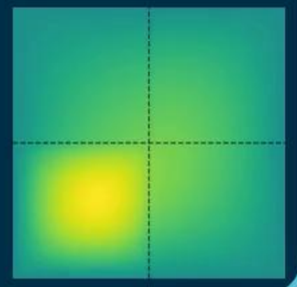

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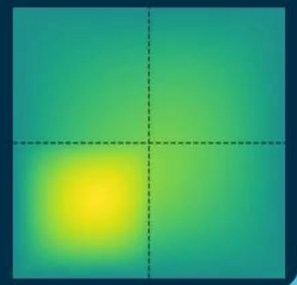

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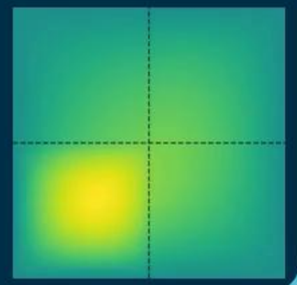

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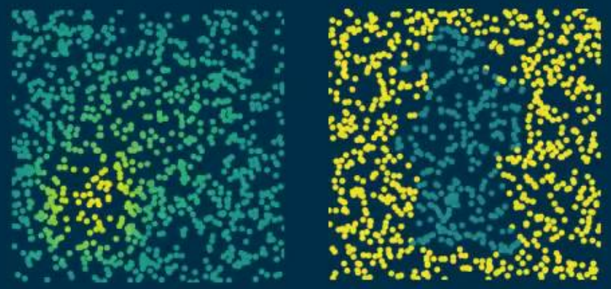

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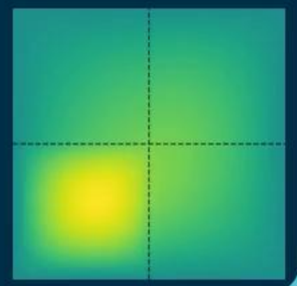

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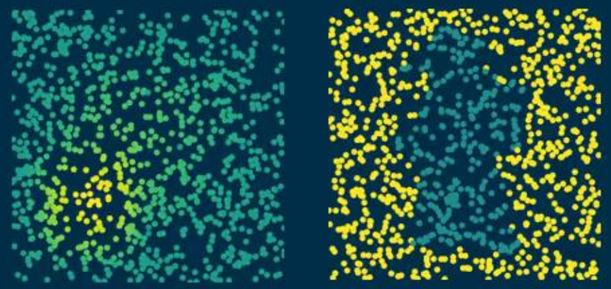

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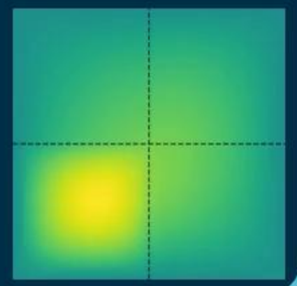

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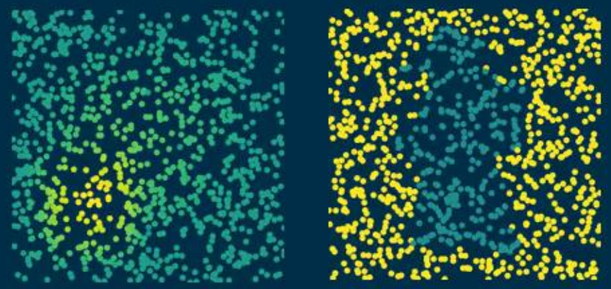

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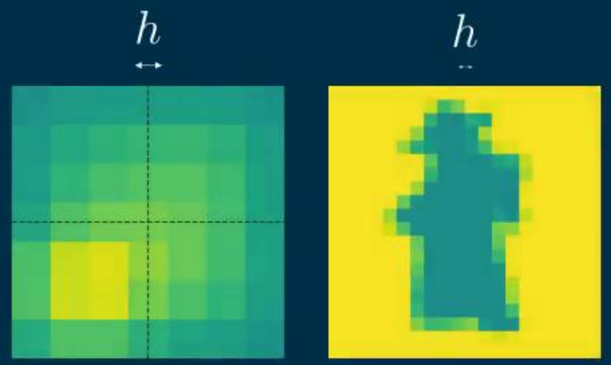

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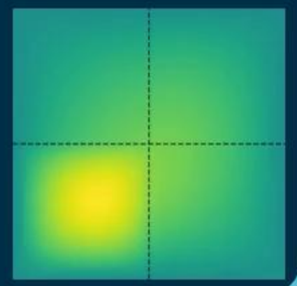

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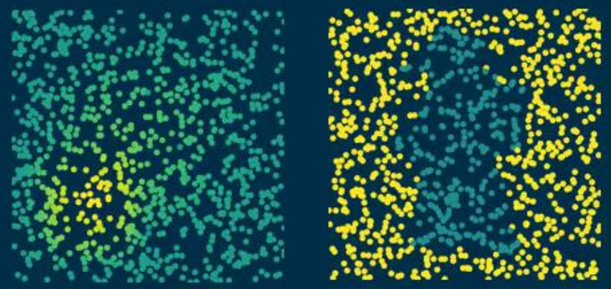

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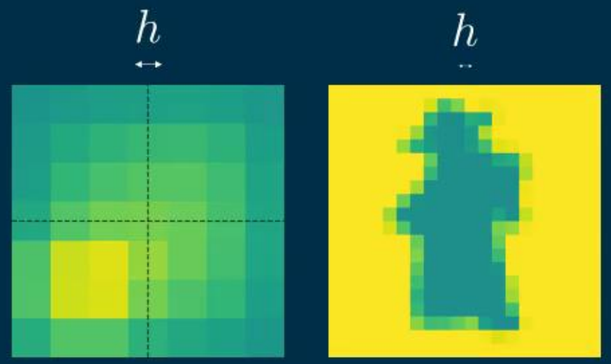

State estimation

$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$

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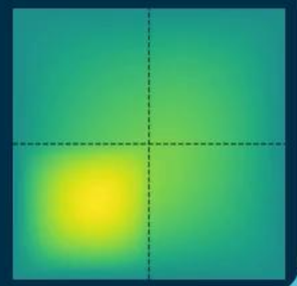

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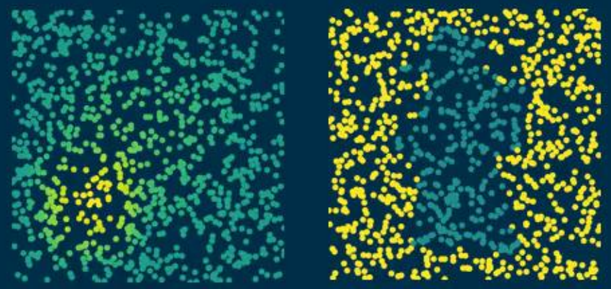

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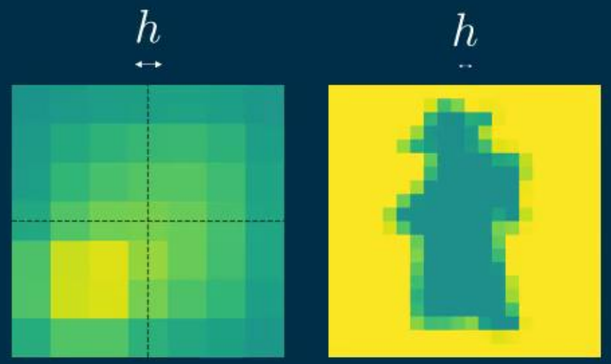

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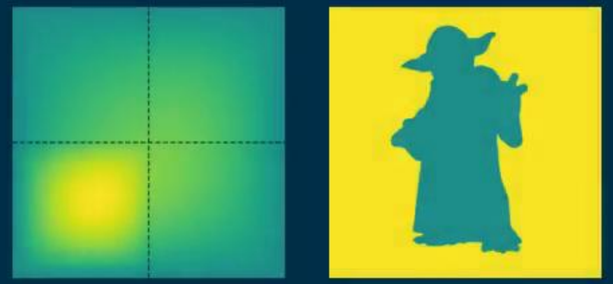

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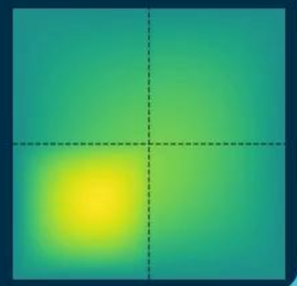

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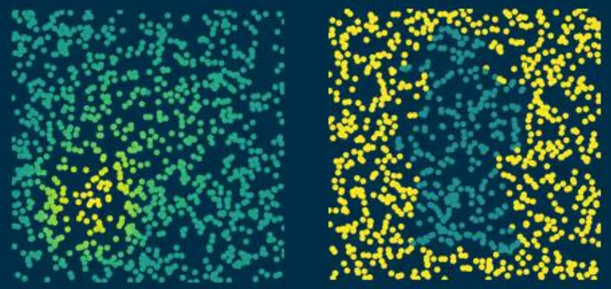

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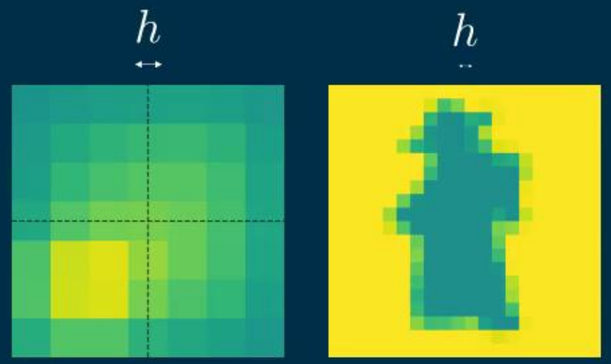

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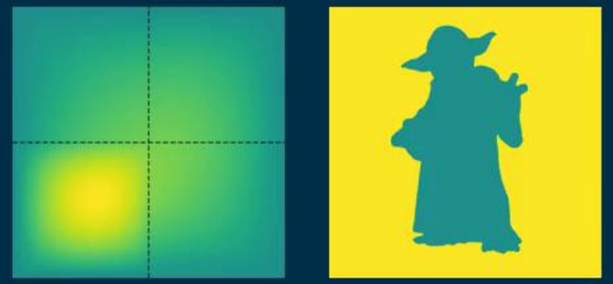

Parameter estimation

$\tilde{y} = R(z) \quad R : \mathbb{R}^m \to \mathbb{R}^d$

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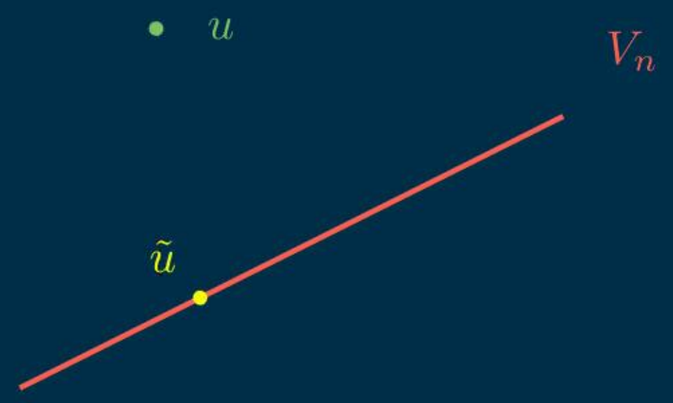

Near optimality property

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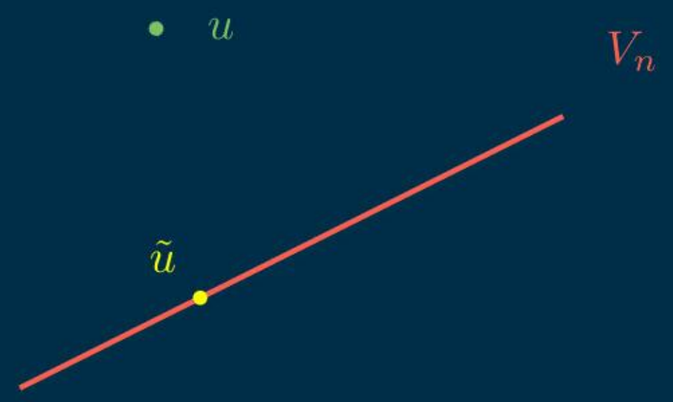

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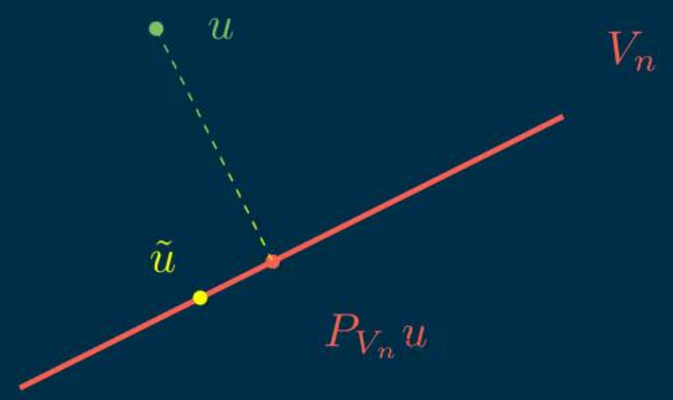

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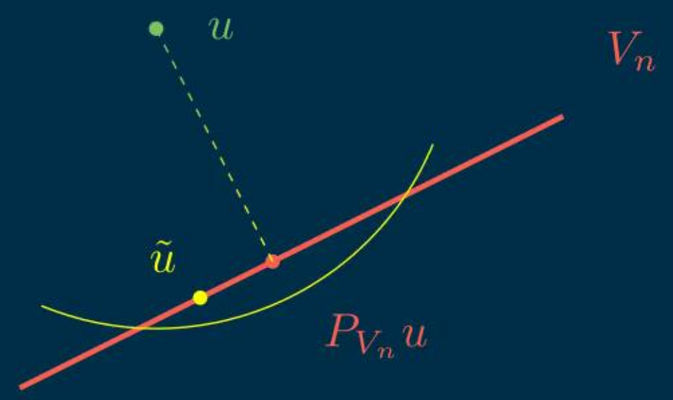

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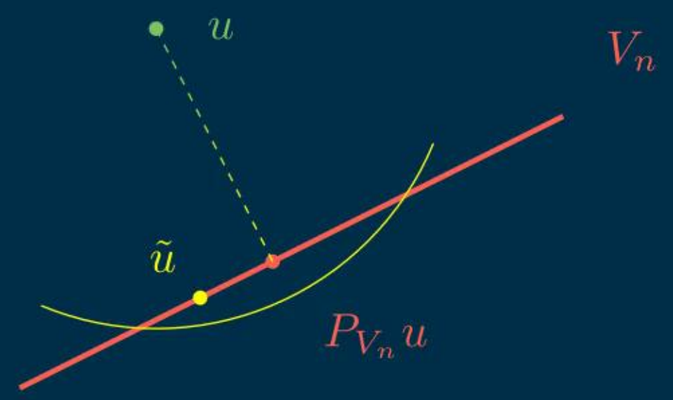

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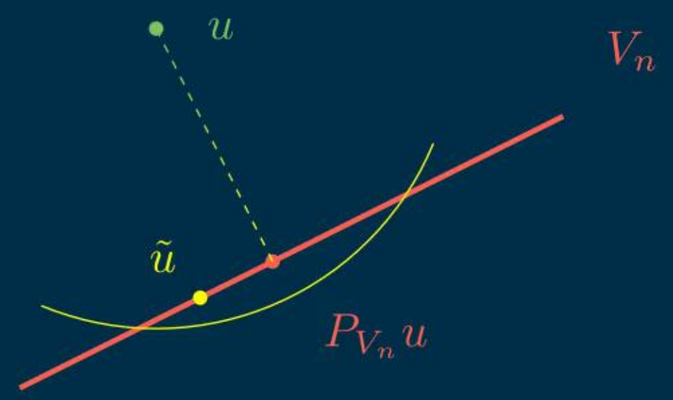

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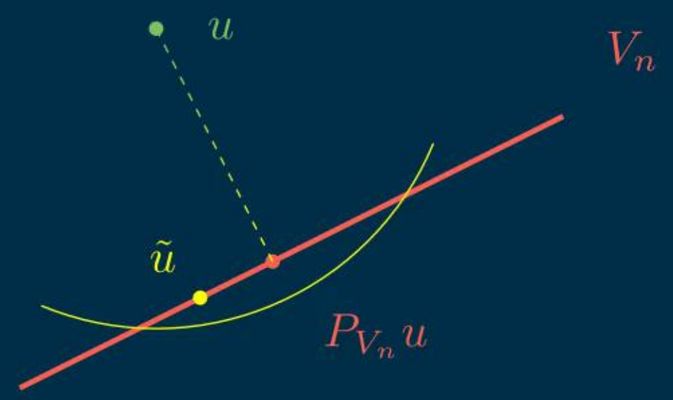

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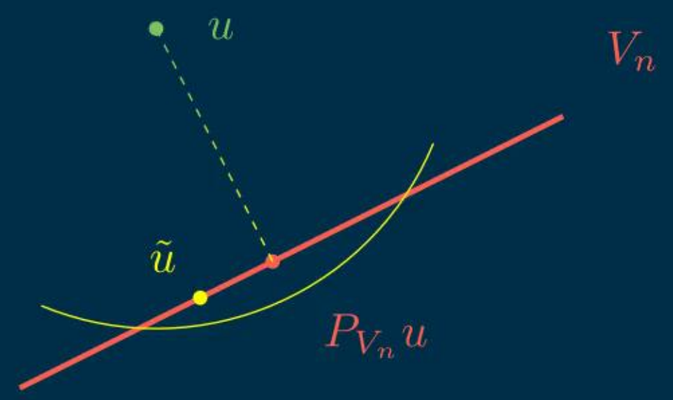

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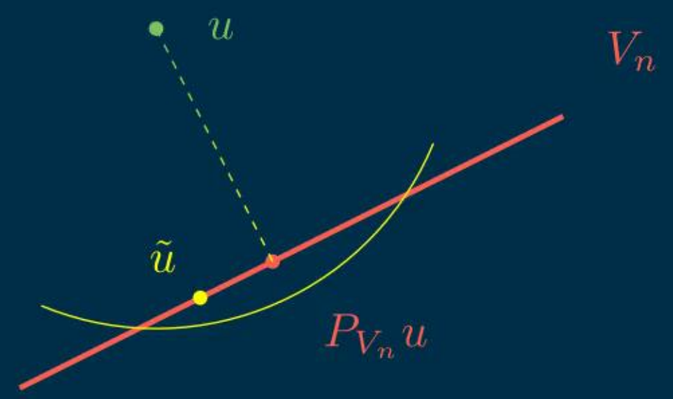
$$W := \operatorname{span}\{w_1, \ldots, w_m\}$$

$$w = P_W u$$



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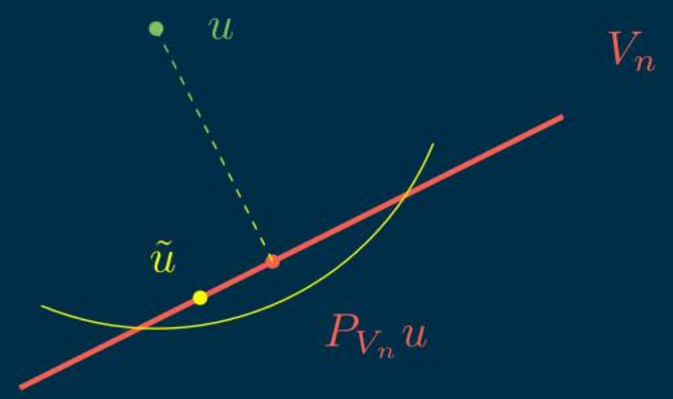

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$$\tilde{u} = R(z) \quad R : \mathbb{R}^m \to V$$

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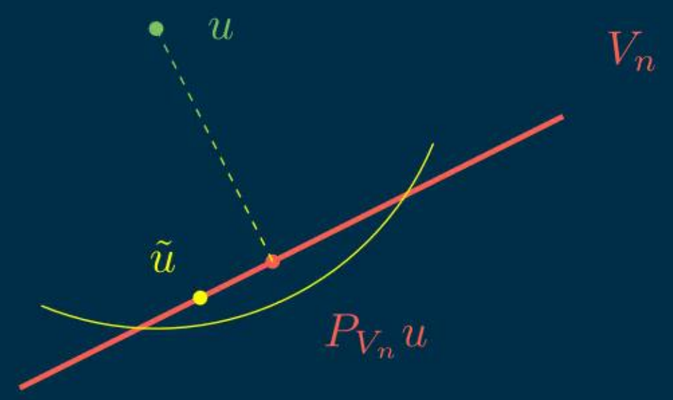

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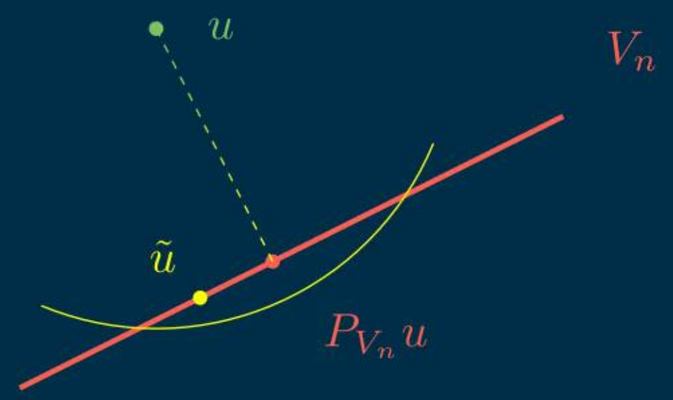
[Y. Maday, A. T. Patera, J. D. Penn, M. Yano, 2015]



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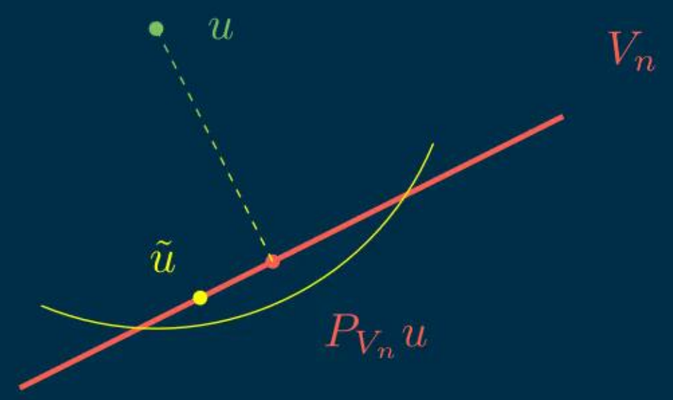

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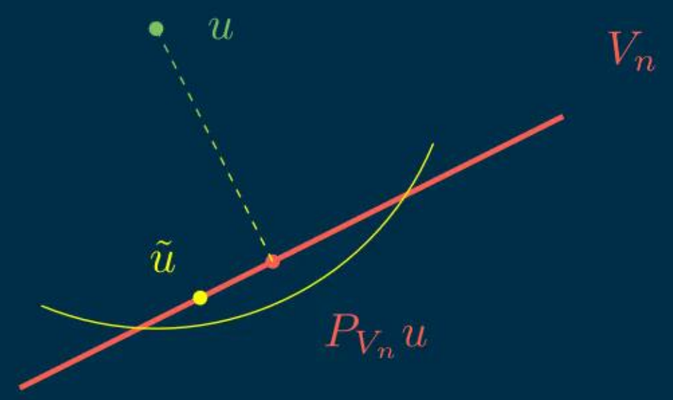

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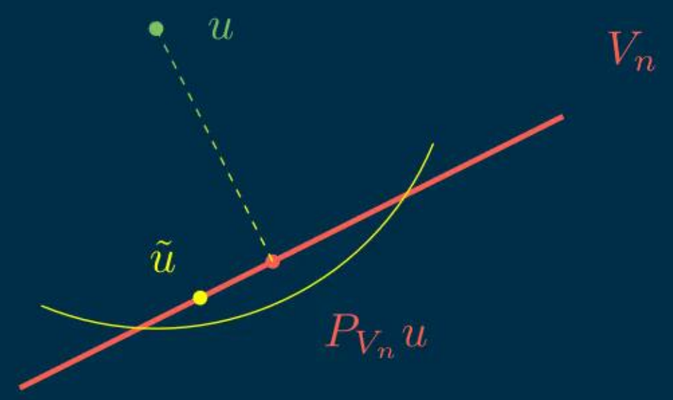

Towards non-linearity

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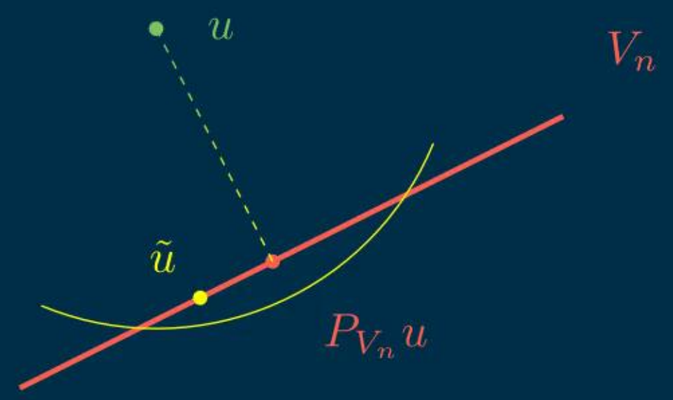

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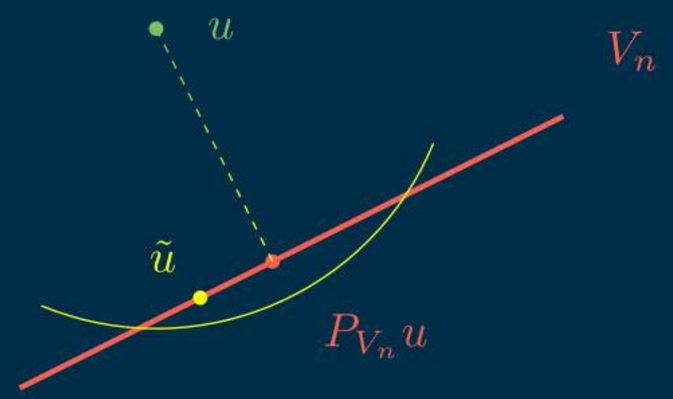

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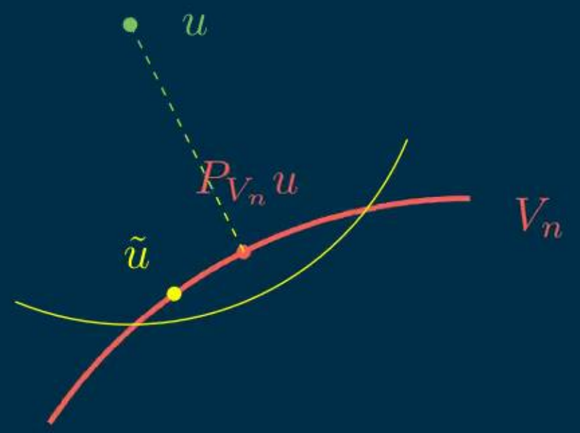

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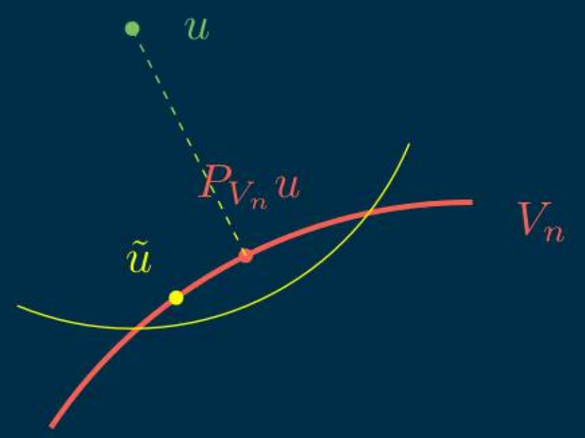

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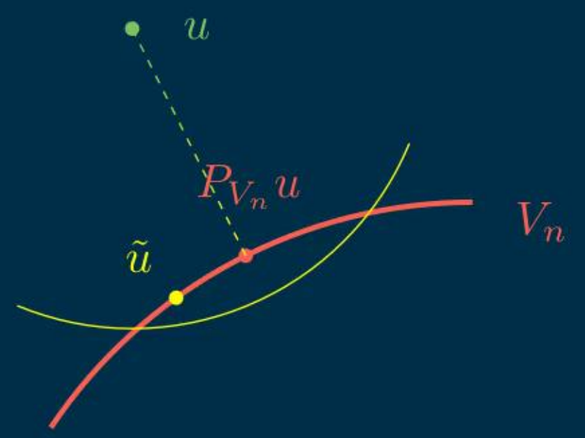

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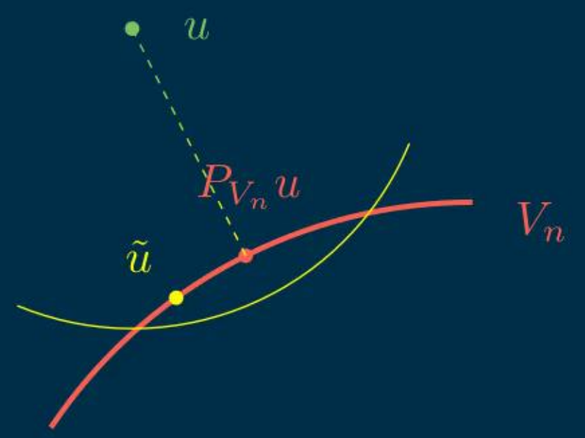

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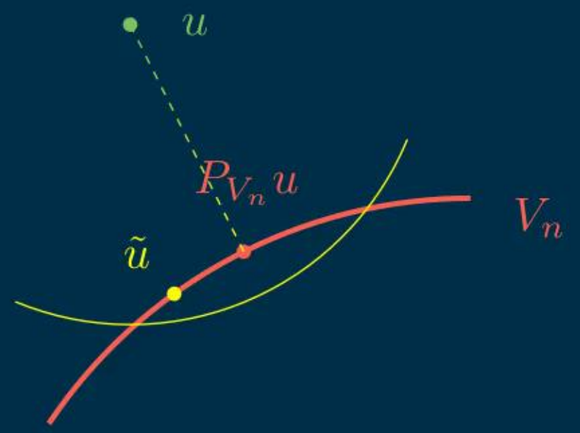

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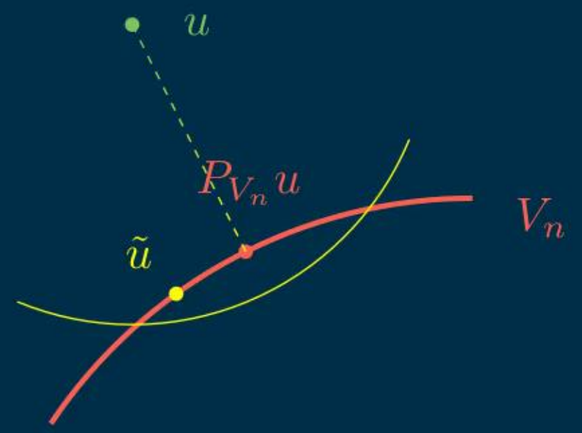

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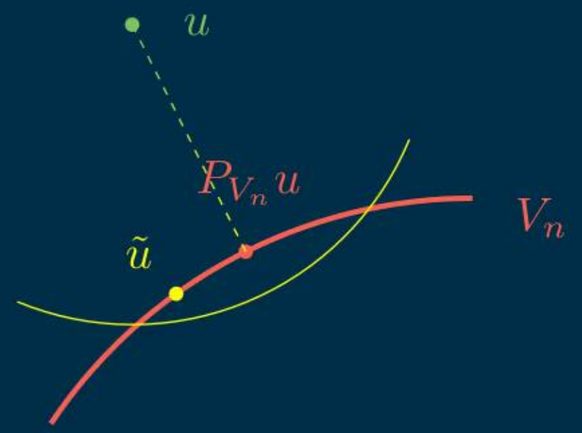

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[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

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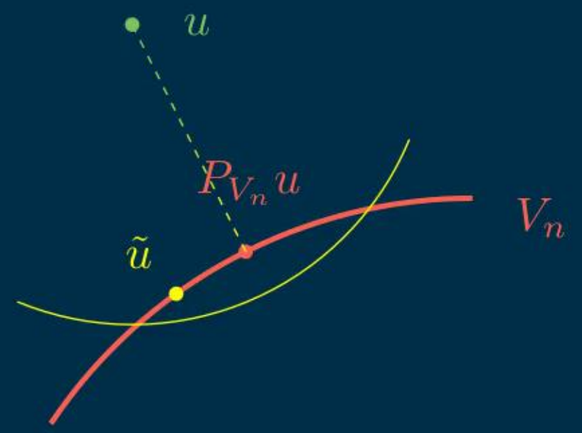
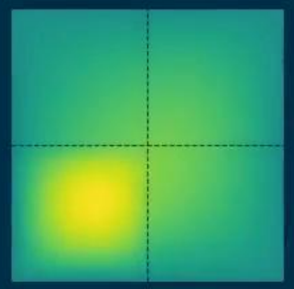
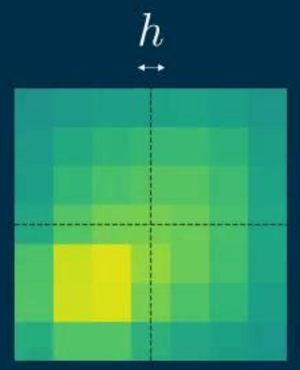


Interface reconstruction from cell averages



Interface reconstruction from cell averages

$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$



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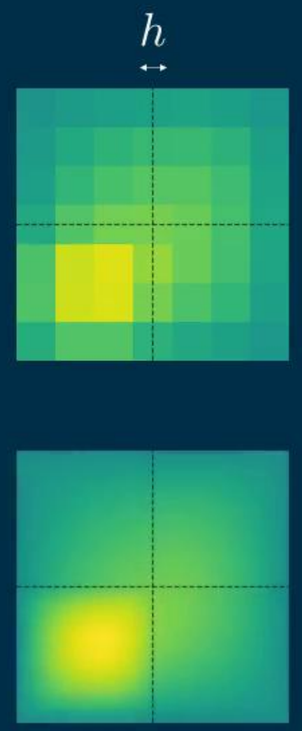

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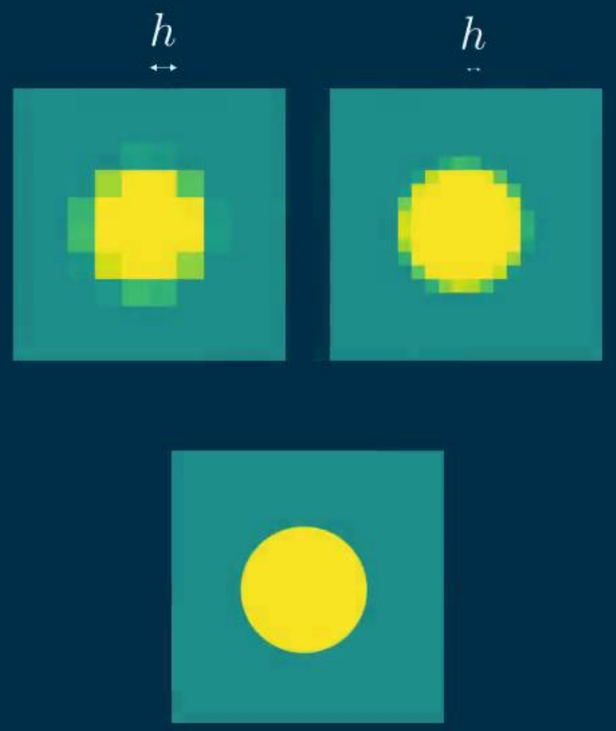

Application

1. Image superresolution.

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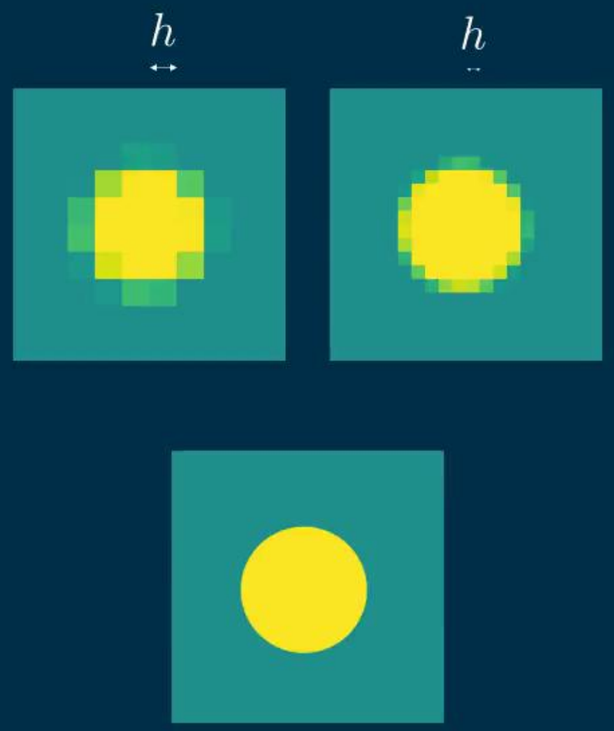

Application

1. Image superresolution.
2. Finite Volumes PDE Solvers

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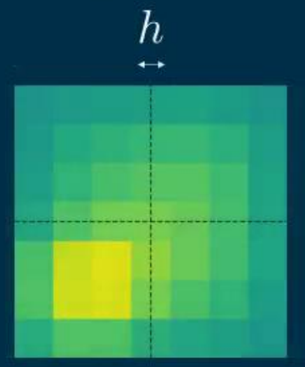

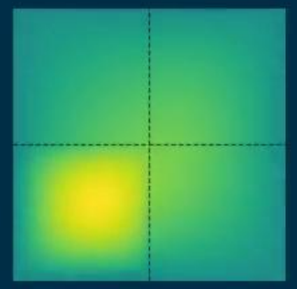

u smooth

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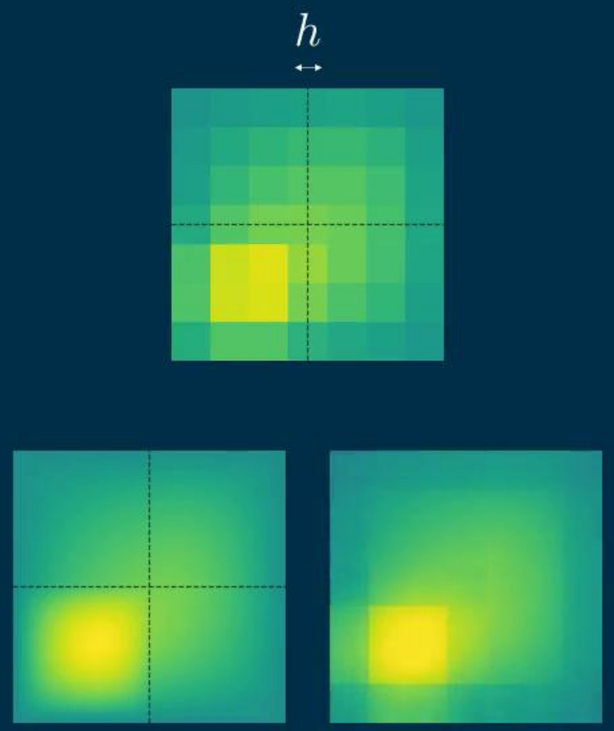
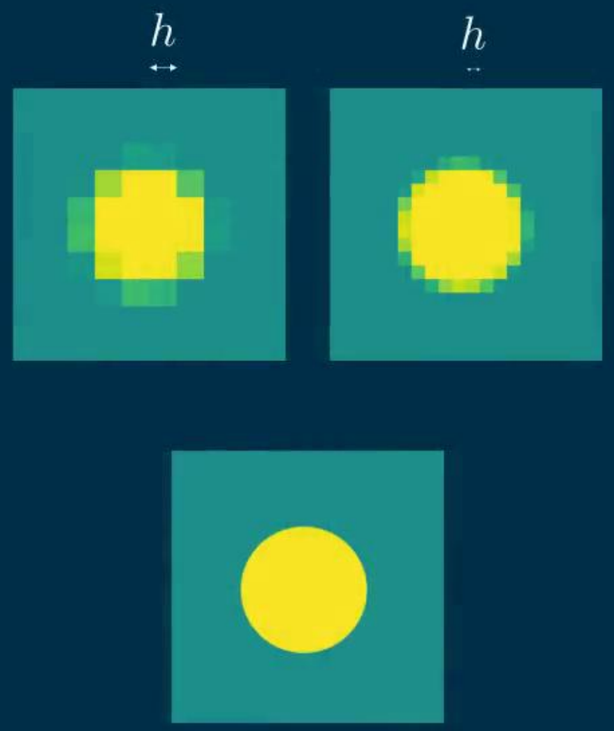


u smooth locally fit polynomials

Interface reconstruction

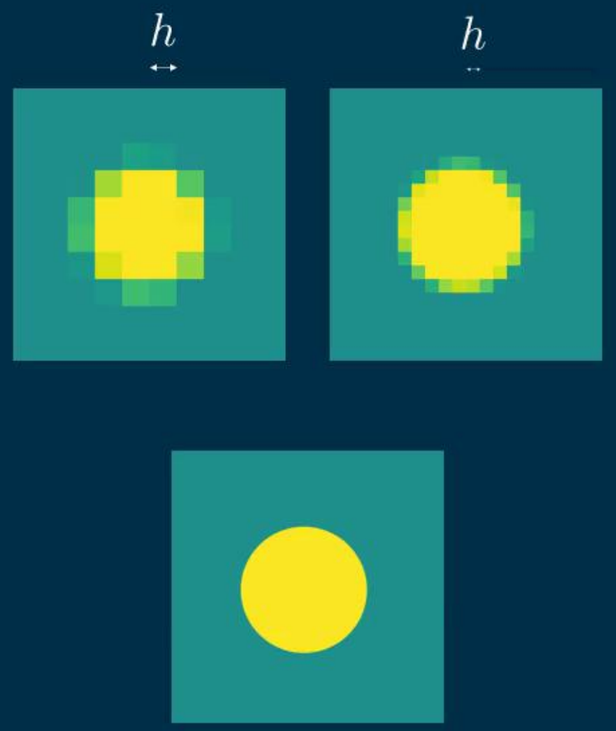
$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$

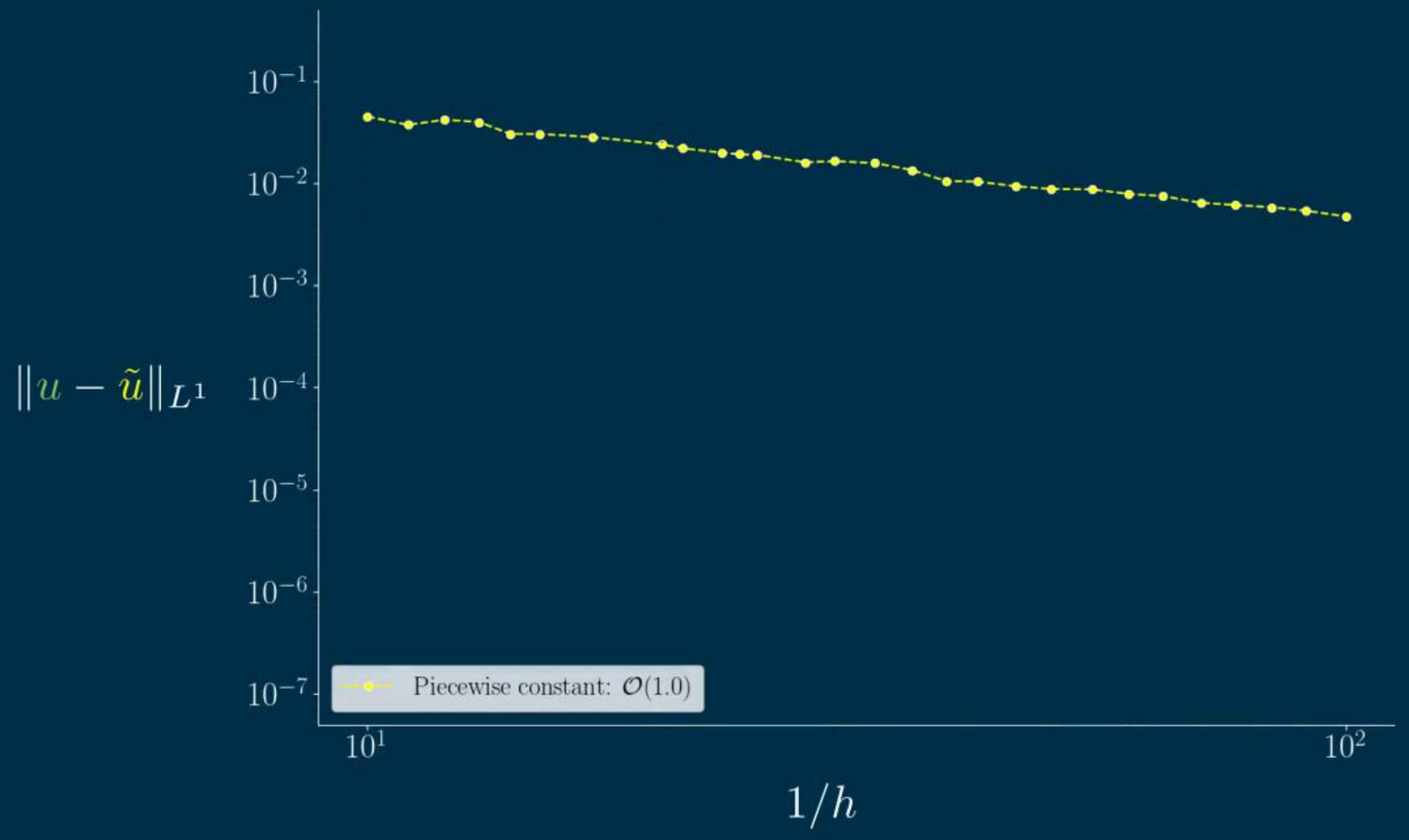


u presents jump discontinuities

Interface reconstruction: linear approach

$$\ell_i(u) = \int_{I_i} u(x)dx \quad I_i \in \mathcal{T}_h$$





[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2023]

Non-linear approximation

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Non-linear approximation: linear interface

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

Non-linear approximation: linear interface

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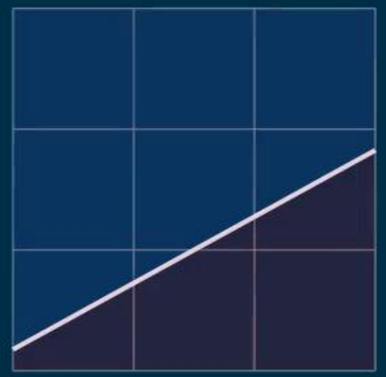
$v \in V_2$



0.0	0.0	0.0
0.0	0.1	0.6
0.4	0.9	1.0

$\ell(v) \in \mathbb{R}^m$

Non-linear approximation: linear interface

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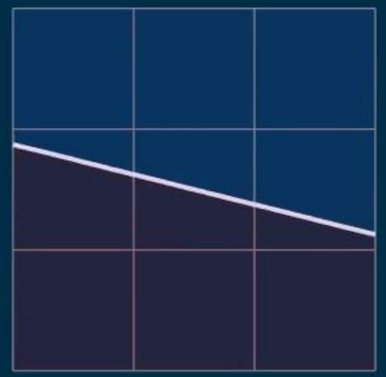
$v \in V_2$



0.0	0.0	0.0
0.7	0.5	0.3
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

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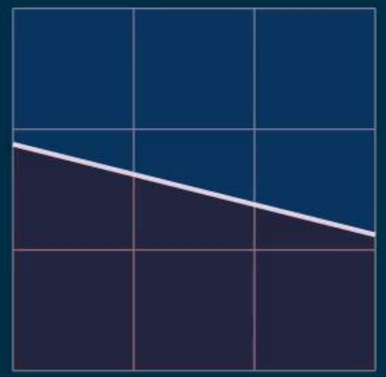
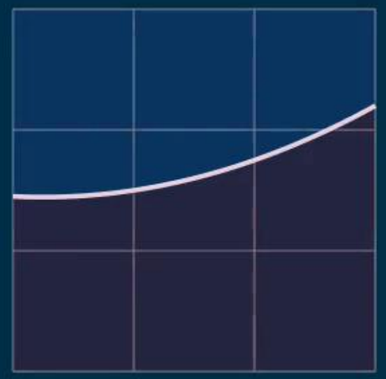


0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

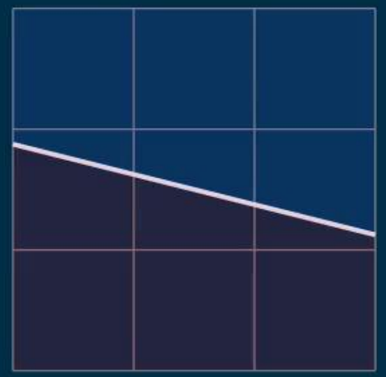
$z \in \mathbb{R}^m$
Observed averages

Best fit estimator: LVIRA

$$V_2 := \{\chi_{\vec{r}\cdot(x-\overline{x})\geq c} : \vec{r} \in \mathbb{S}^1, c \in \mathbb{R}\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \|z - \ell(v)\|_Z$$ LVIRA [E.G. Puckett, 1991]

$v \in V_2$

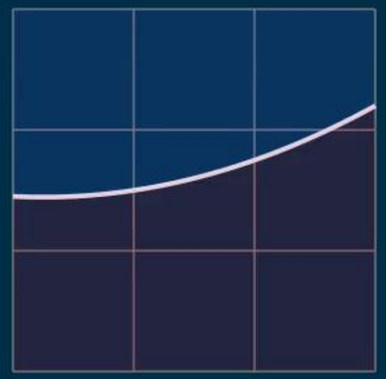


0.0	0.0	0.0
0.7	0.5	0.2
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
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$z \in \mathbb{R}^m$

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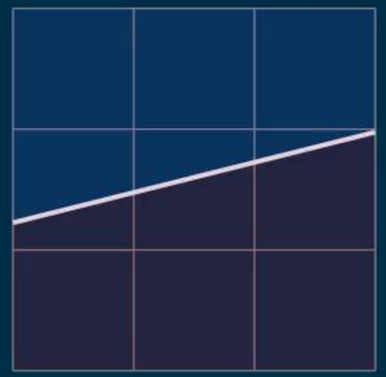
$v \in V_2$

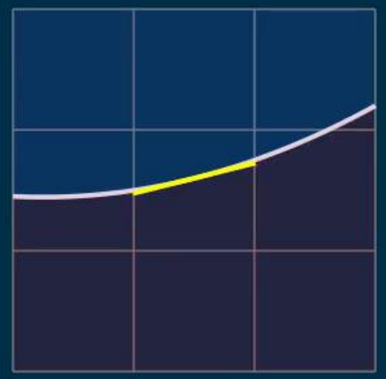


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
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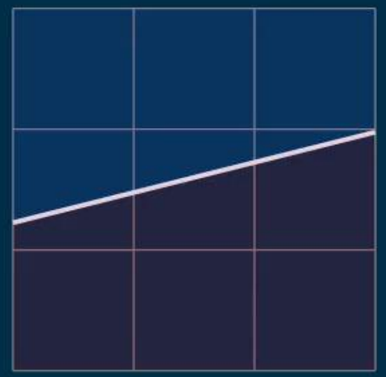
Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

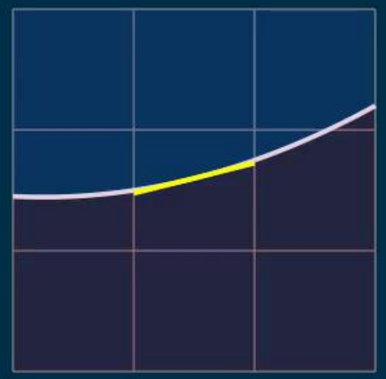
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
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$z \in \mathbb{R}^m$
Observed
averages

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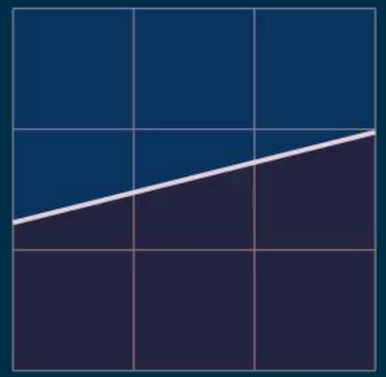
$v \in V_2$

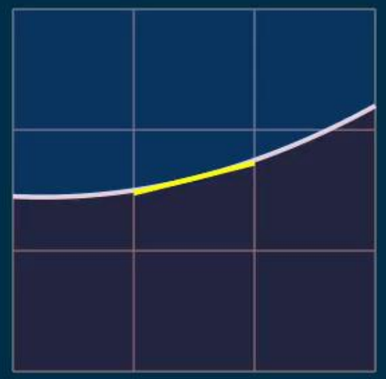


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_V \leq C\|u - P_{V_n} u\|_V$$

Lipschitz continuity

$$\|\ell(u) - \ell(v)\|_Z \leq \alpha_Z \|u - v\|_V, \quad u, v \in V$$

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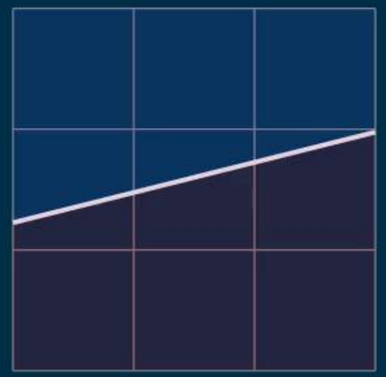
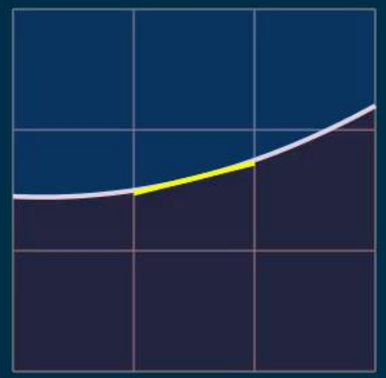


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



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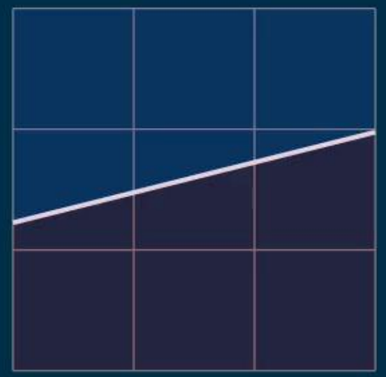
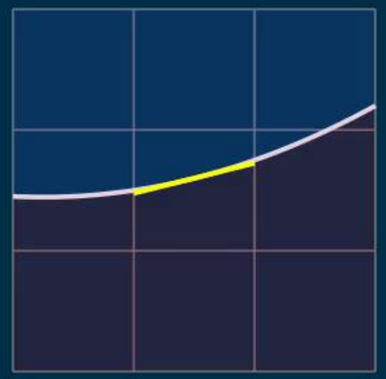
$v \in V_2$



0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



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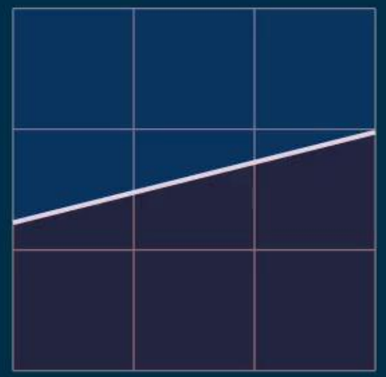
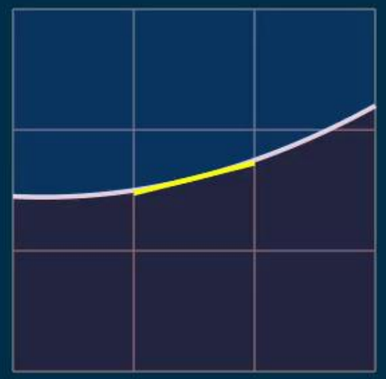


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
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[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

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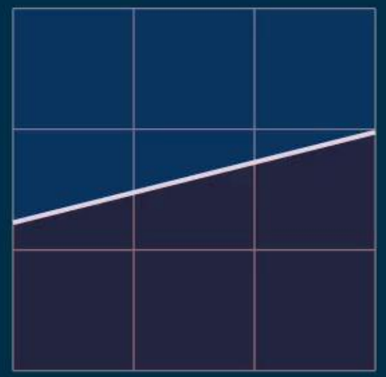
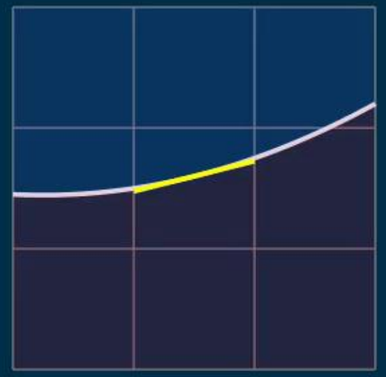


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
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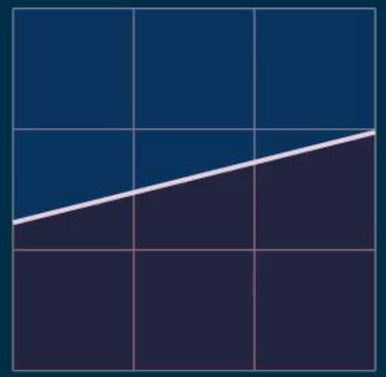
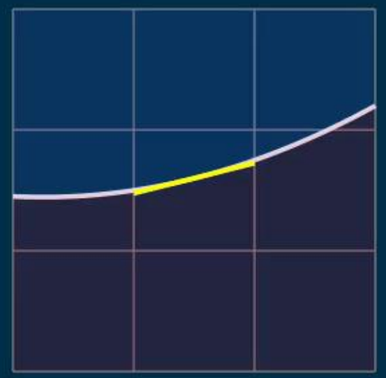


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown
u



0.0	0.0	0.0
0.5	0.6	0.9
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$z \in \mathbb{R}^m$
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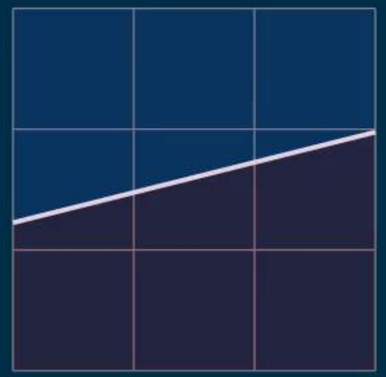
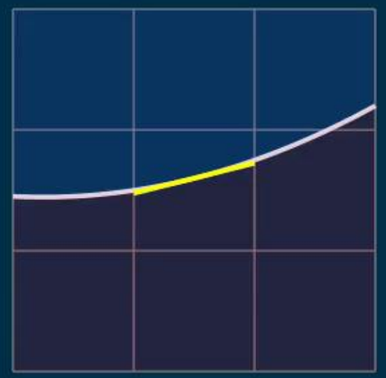


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

$z \in \mathbb{R}^m$
Observed averages

Near Optimality property

$$\|u - \tilde{u}\|_{L^1} \leq 10\|u - P_{V_n} u\|_{L^1}$$

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

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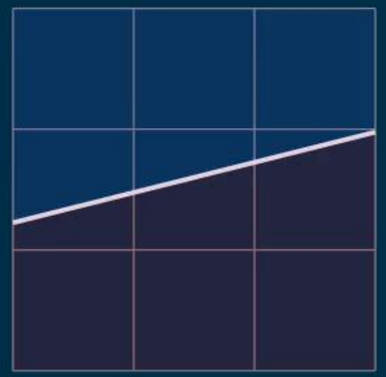
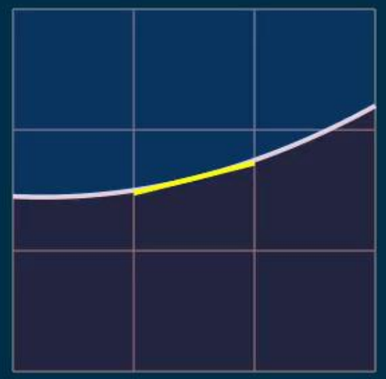


0.0	0.0	0.0
0.4	0.6	0.8
1.0	1.0	1.0

$\ell(v) \in \mathbb{R}^m$

Unknown

u



0.0	0.0	0.0
0.5	0.6	0.9
1.0	1.0	1.0

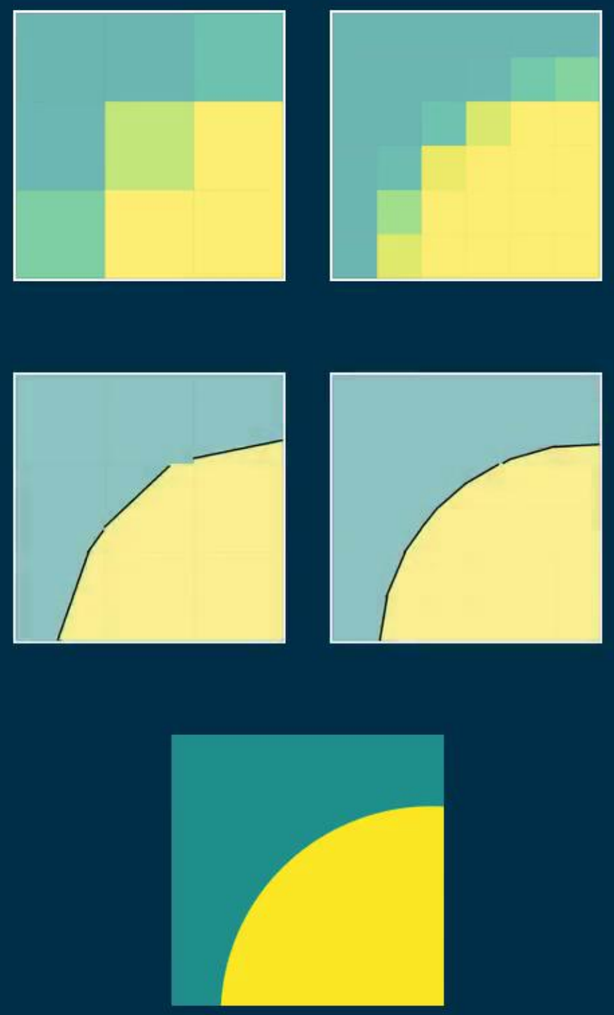
$z \in \mathbb{R}^m$
Observed averages

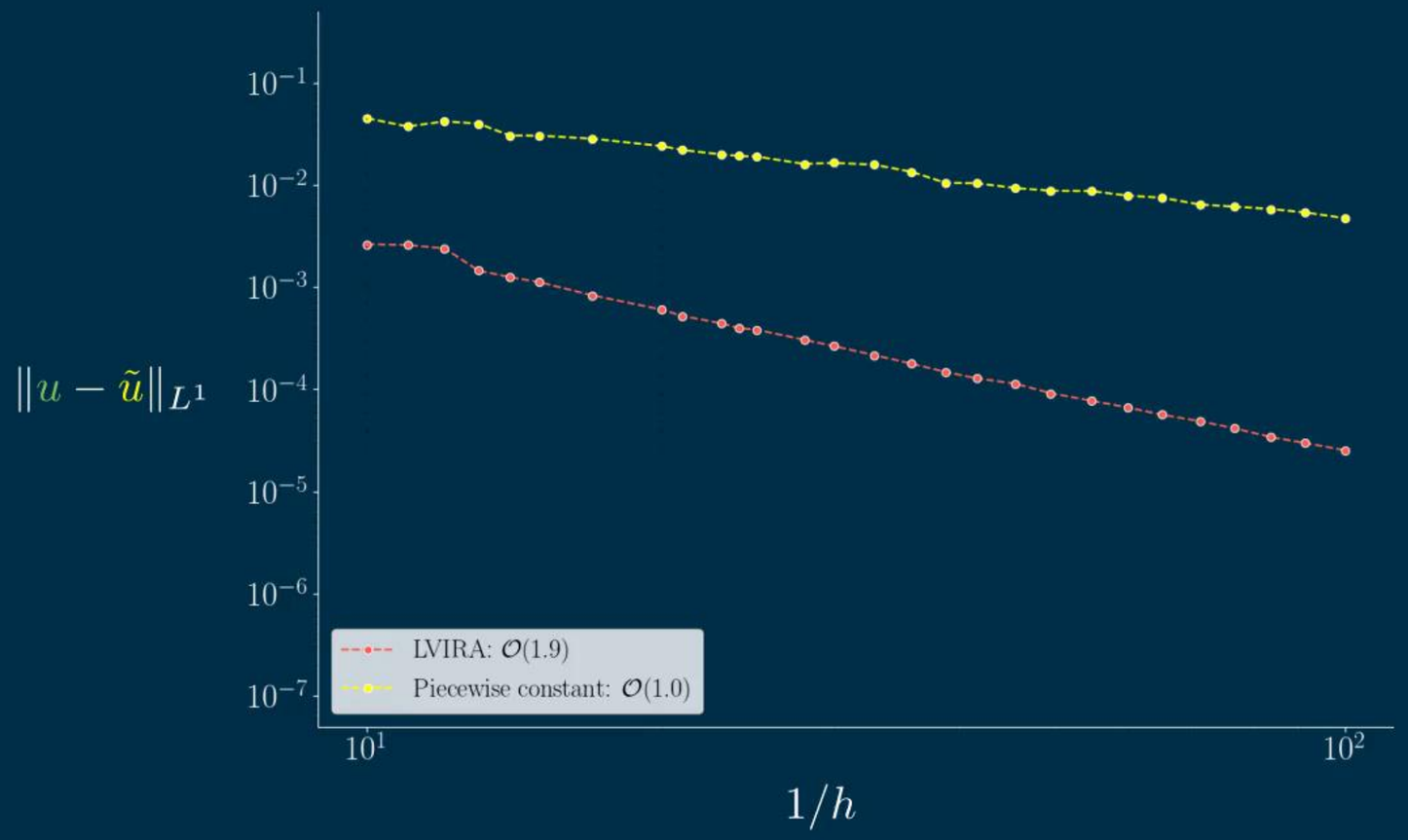
Global convergence

Corollary: If Ω is a $\mathcal{C}^s$ domain with $s \geq 2$, the LVIRA method converges in L^1 norm at rate h^2

[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Best fit estimator: LVIRA





[A. Cohen, M. Dolbeault, O. Mula, A. Somacal, 2024]

Higher order

Optimization Based Edge Reconstruction Algorithm

Higher order: OBERA

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

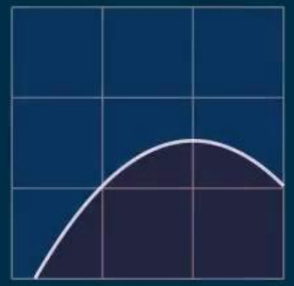
$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



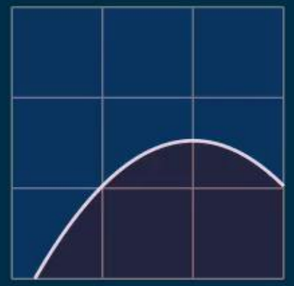
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

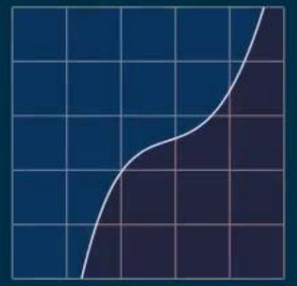
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



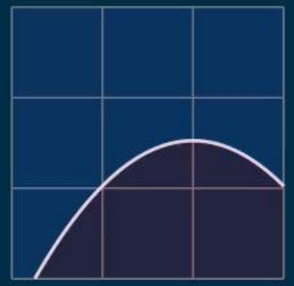
0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

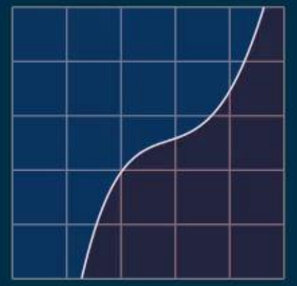
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



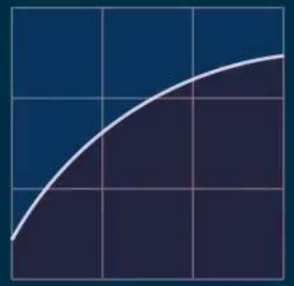
0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic



0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Circle



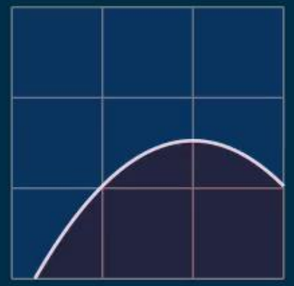
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

Higher order: OBERA

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

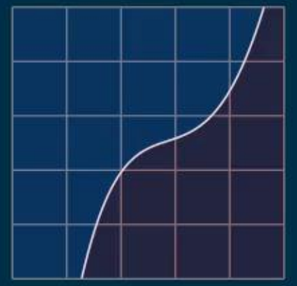
$$\tilde{u} = \underset{v \in V_n}{\arg\min} \, \|z - \ell(v)\|_Z$$

Quadratic



0.0	0.0	0.0
0.0	0.4	0.4
0.4	1.0	1.0

Quartic

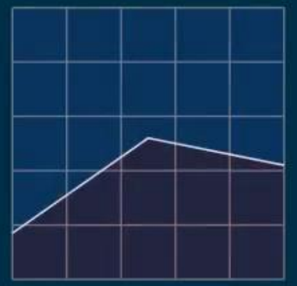


0.0	0.0	0.0	0.0	0.5
0.0	0.0	0.0	0.1	0.9
0.0	0.0	0.4	0.8	1.0
0.0	0.2	1.0	1.0	1.0
0.0	0.6	1.0	1.0	1.0

Circle

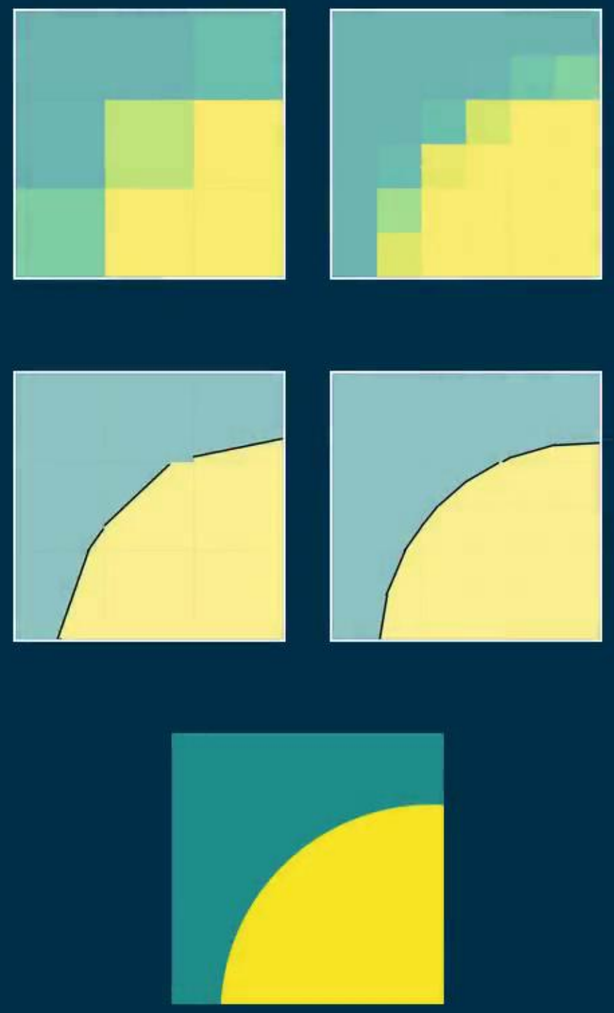
0.0	0.0	0.4
0.2	0.9	1.0
0.9	1.0	1.0

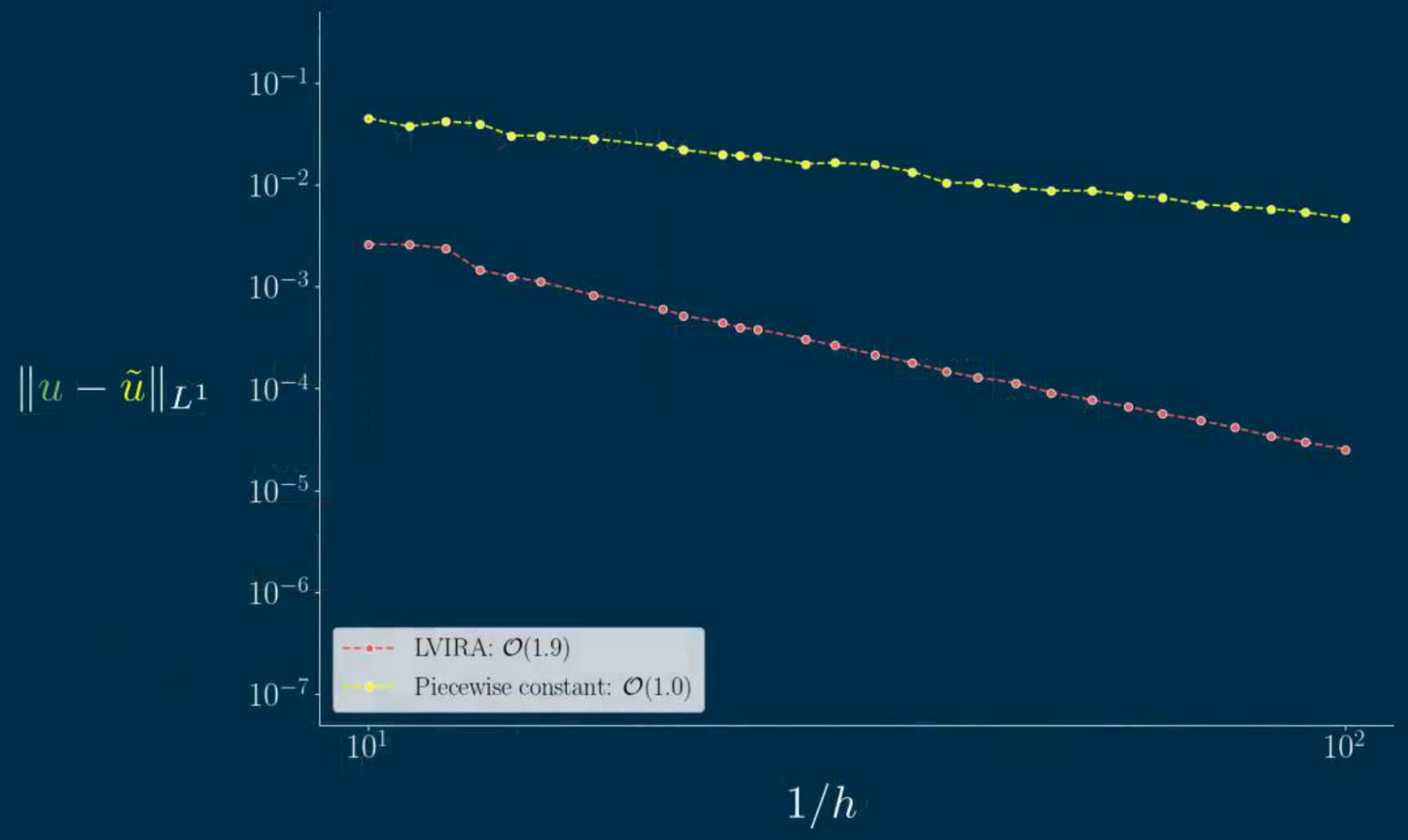
Vertex



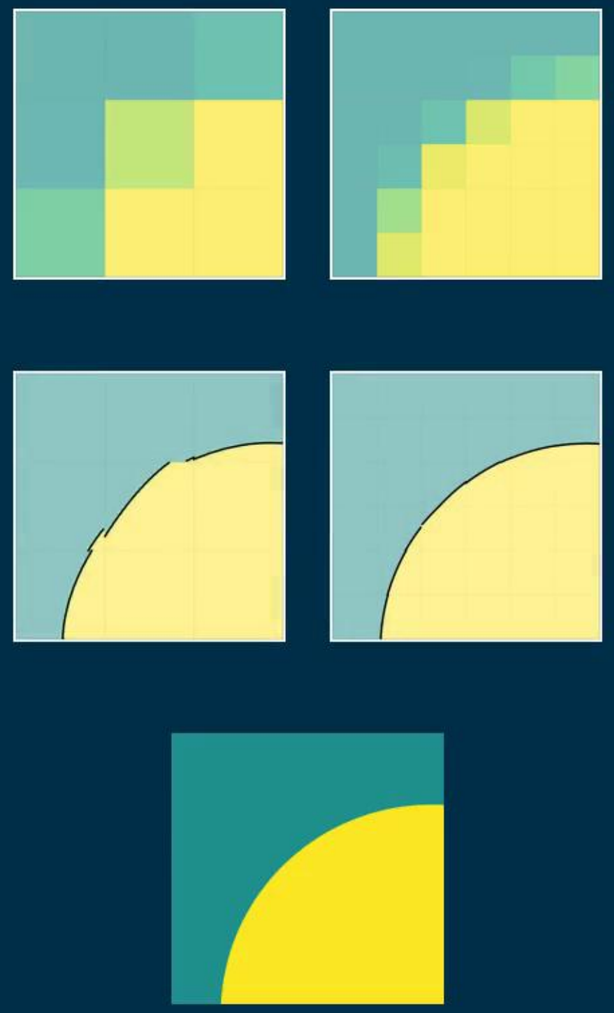
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.5	0.4	0.2
0.2	0.9	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

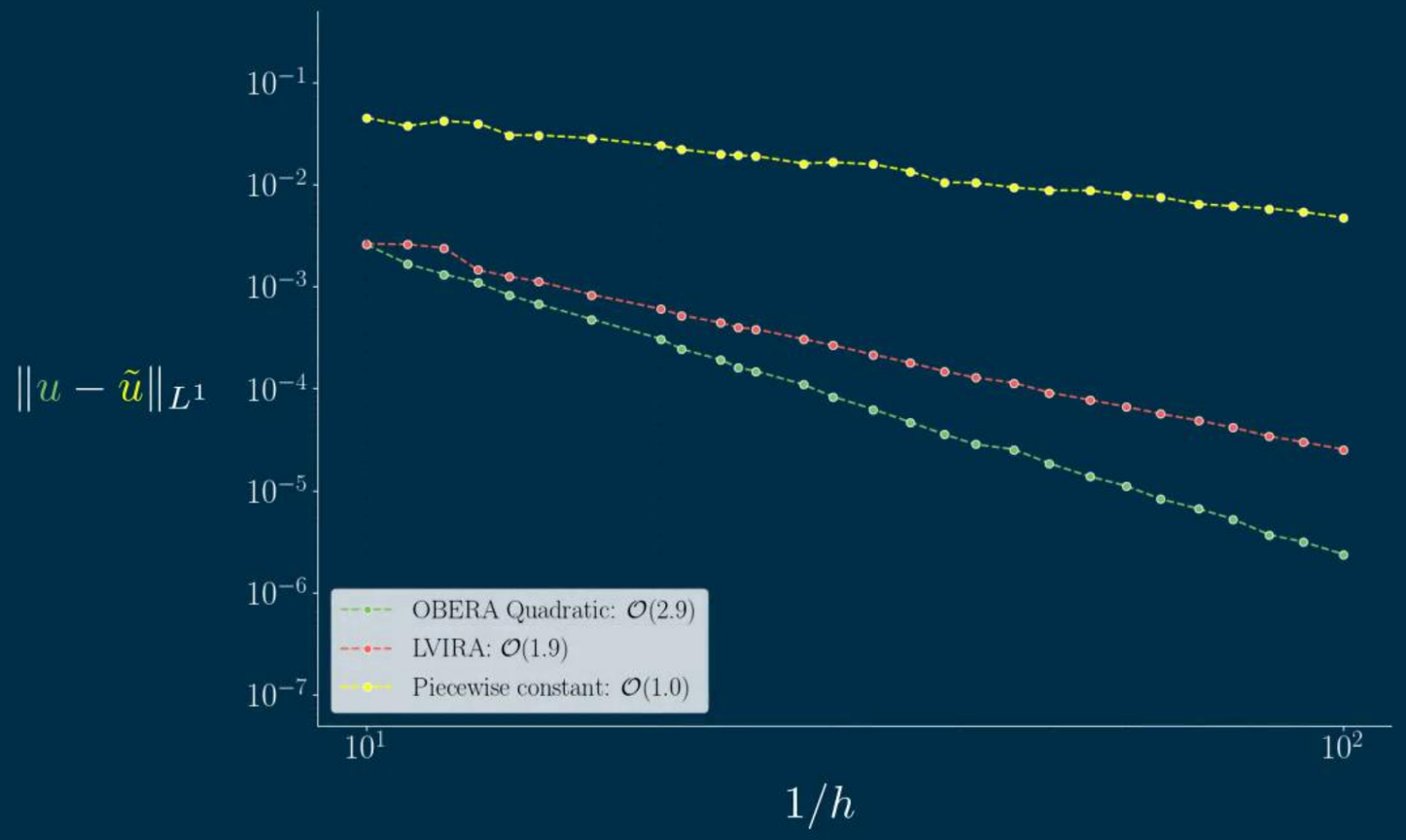
Higher order: OBERA





Higher order: OBERA





[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

Higher order: OBERA

Problem: long computation time

Higher order: OBERA

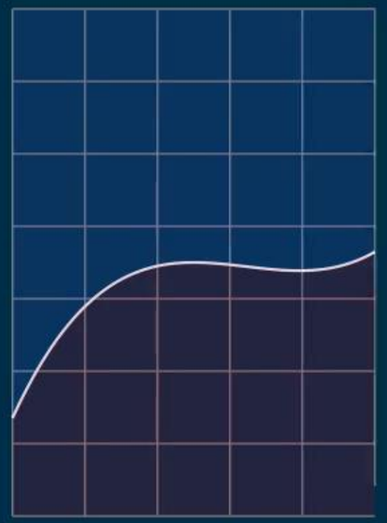
$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

Problem: long computation time

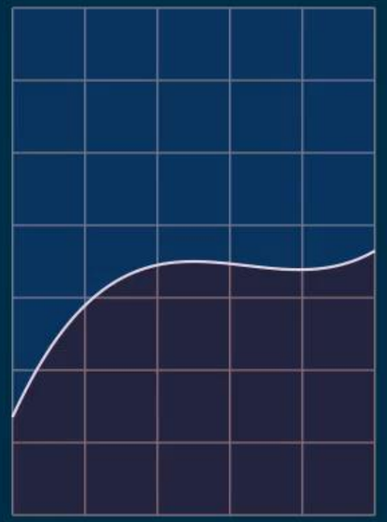
Fast higher order methods

Algorithm for Edge Reconstruction using Oriented Stencils

Higher order: AEROS

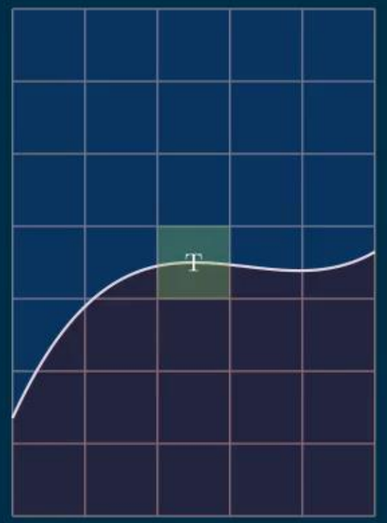


Higher order: AEROS



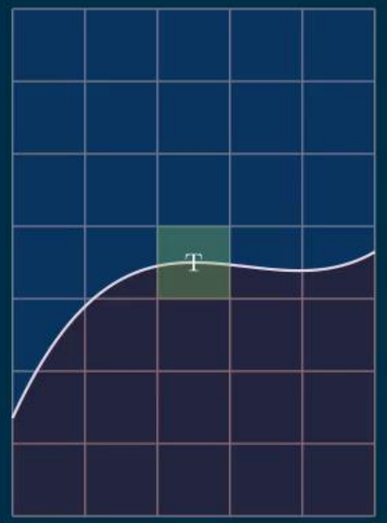
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

Higher order: AEROS

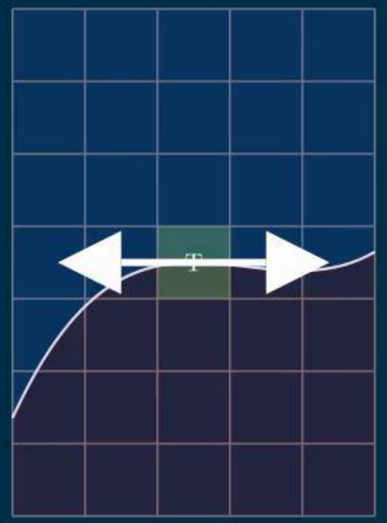


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Horizontal orientation

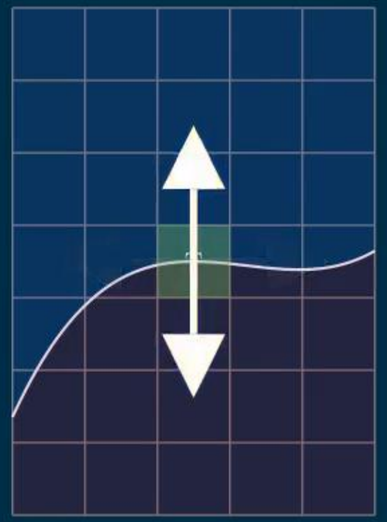


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation

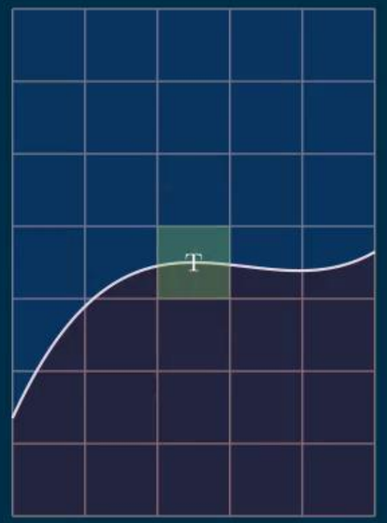


0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Higher order: AEROS

Vertical orientation



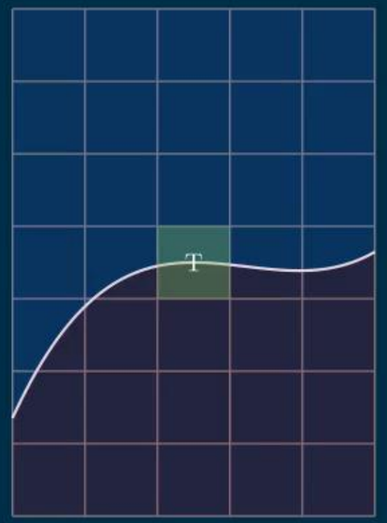
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

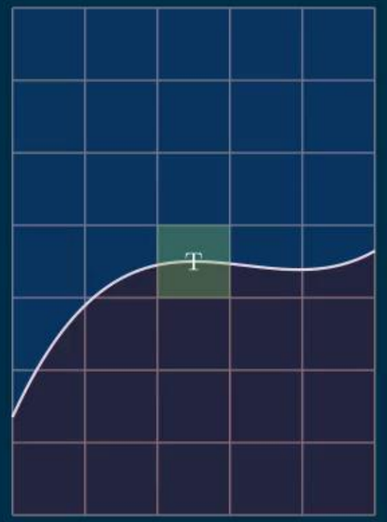
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

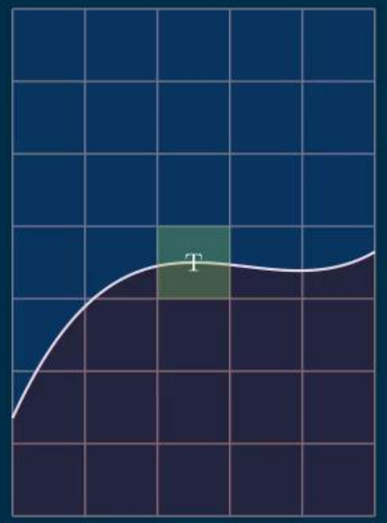
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \qquad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

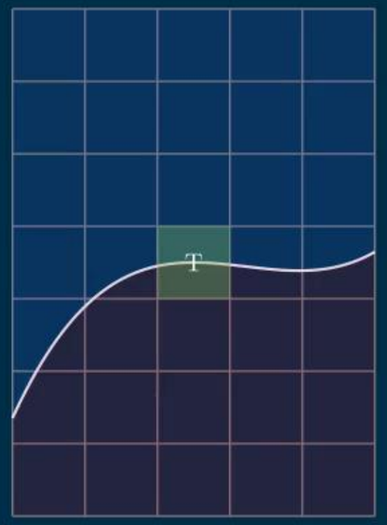
Orientation test based on Sobel filter

$$G_T = (H_T, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

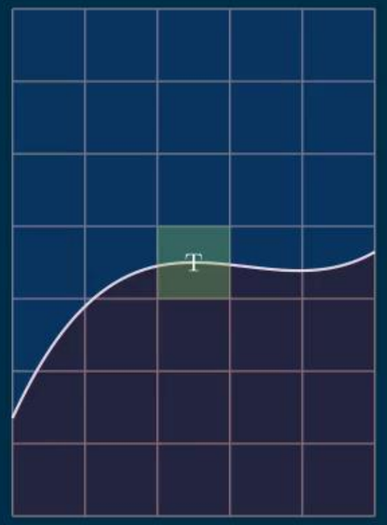
Orientation test based on Sobel filter

$$G_T = (-0.4, V_T)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$

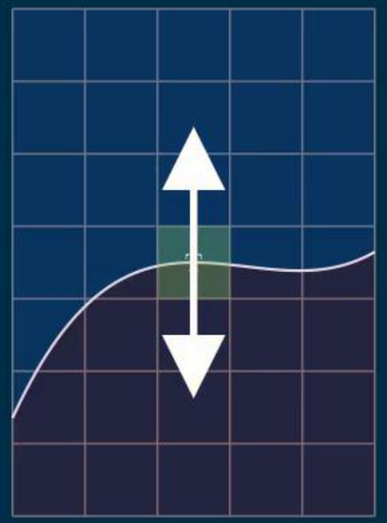
Orientation test based on Sobel filter

$$G_T = (-0.4, 4.0)$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Higher order: AEROS

Vertical orientation



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

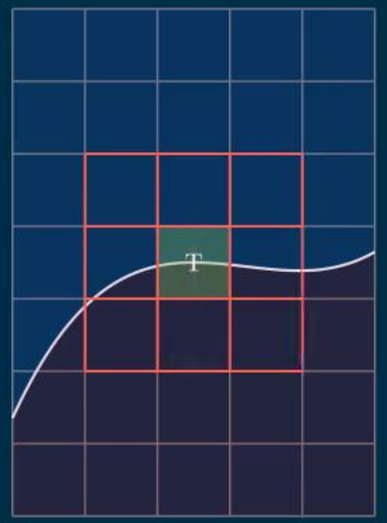
- Find orientation for cell $T = (i, j)$

Orientation test based on Sobel filter

$$0.4 = |H_T| < |V_T| = 4.0$$

$$\begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1.0 & 1.0 & 1.0 \\ 0.2 & 0.5 & 0.4 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

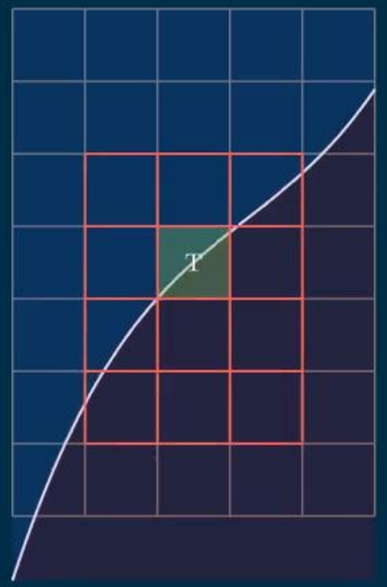
Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
0.3	1.0	1.0	1.0	1.0
0.9	1.0	1.0	1.0	1.0
1.0	1.0	1.0	1.0	1.0

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
- Define an appropriate stencil S_T

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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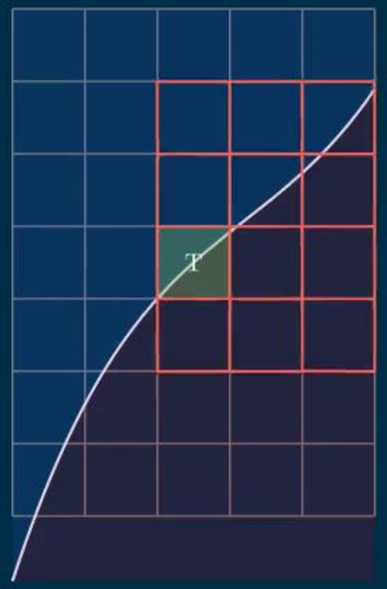

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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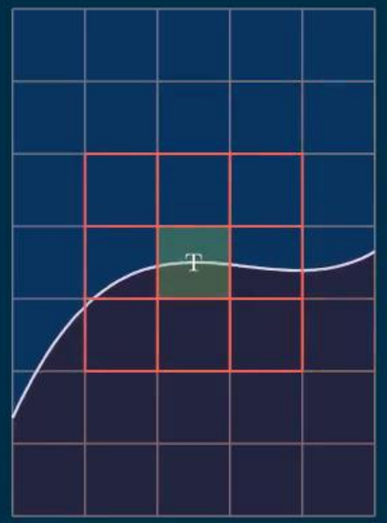

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.2	0.5	0.4	0.5
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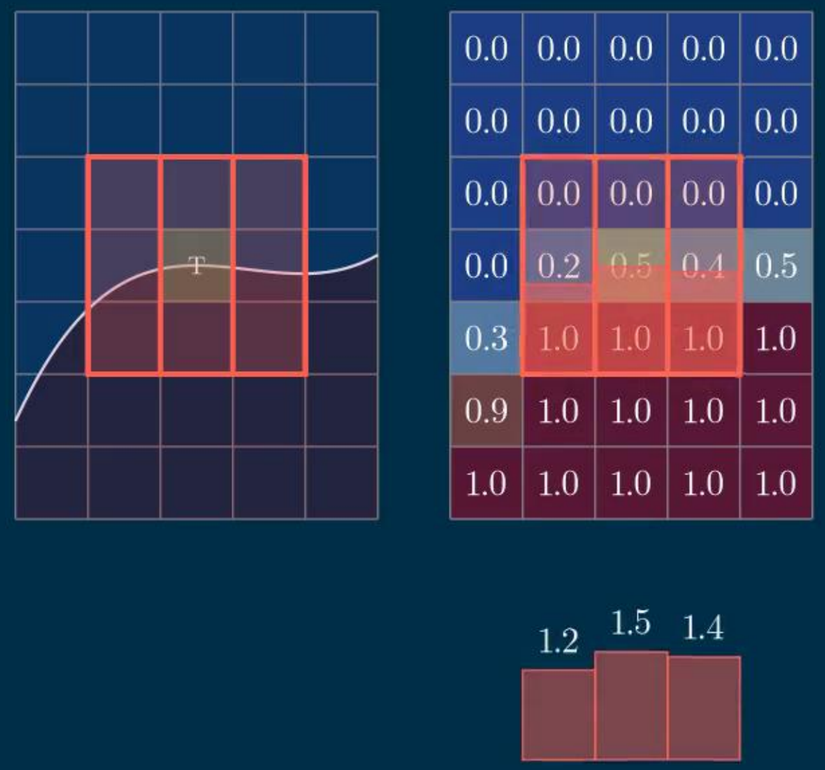

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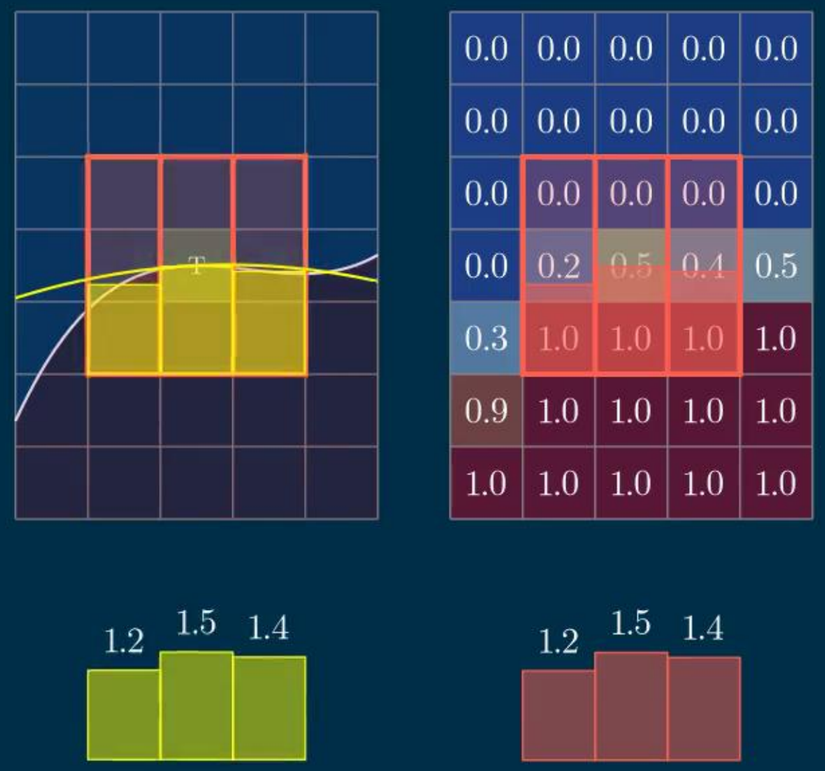
- Sum averages a_{S_T} column-wise: $C_i = \sum_{l=j-1}^{j+1} \ell_{il}(u)$

Higher order: AEROS



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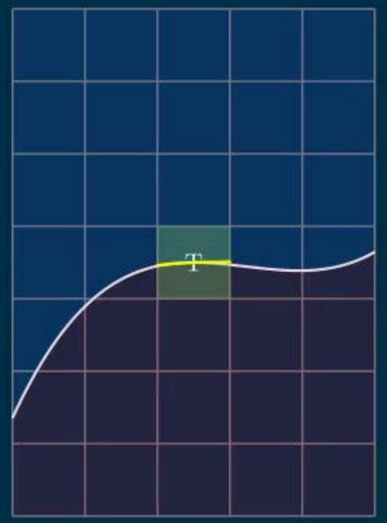
$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

Higher order: AEROS



0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
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AEROS theoretical results

Orientation test result

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AEROS global convergence theorem

$$C_i = (j-1)h + \int_{ih}^{(i+1)h} p(x)dx$$

AEROS theoretical results

Orientation test result

The orientation given by the Sobel filter ensures that if the mesh size h is below a critical value h^* then there is a tall or wide enough stencil so that the interface croses the sides of it.

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AEROS global convergence theorem

If Ω is a $\mathcal{C}^s$ domain with $s \geq n$, the AEROS method based on polynomials of degree n has a global convergence rate in L^1 norm of h^{n+1}

- Find orientation for cell $T = (i, j)$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix}$
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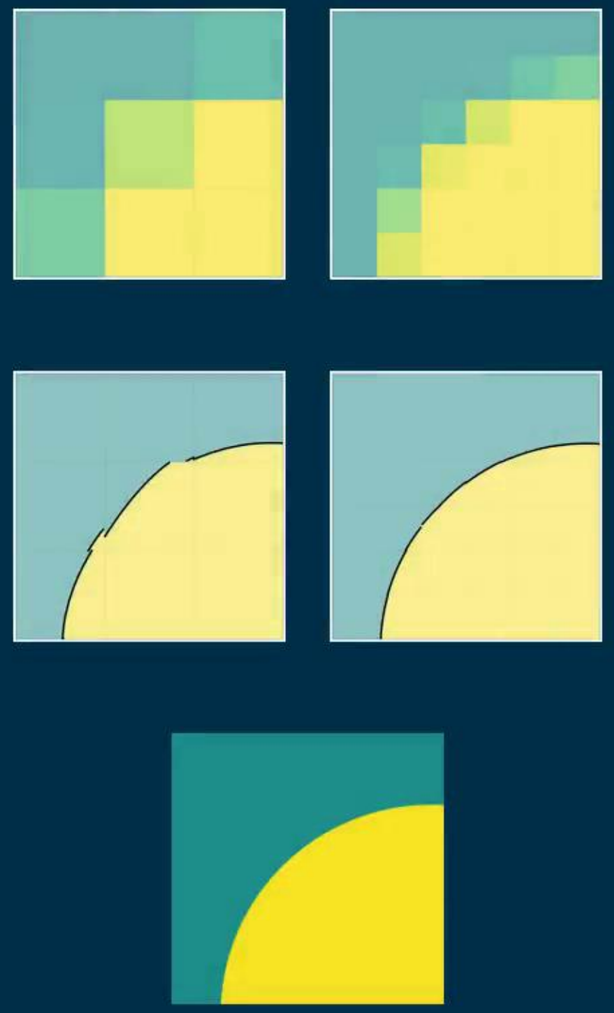
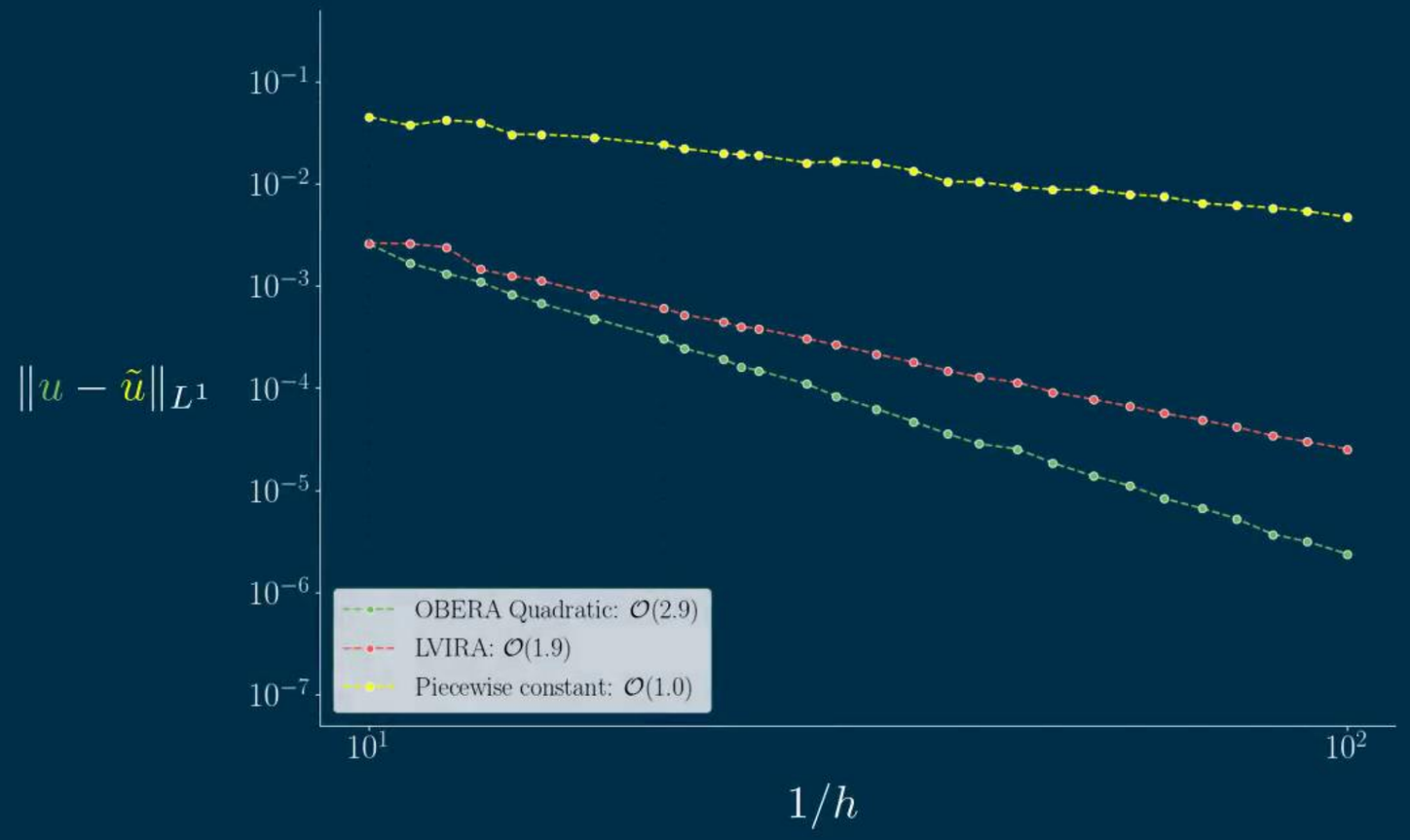
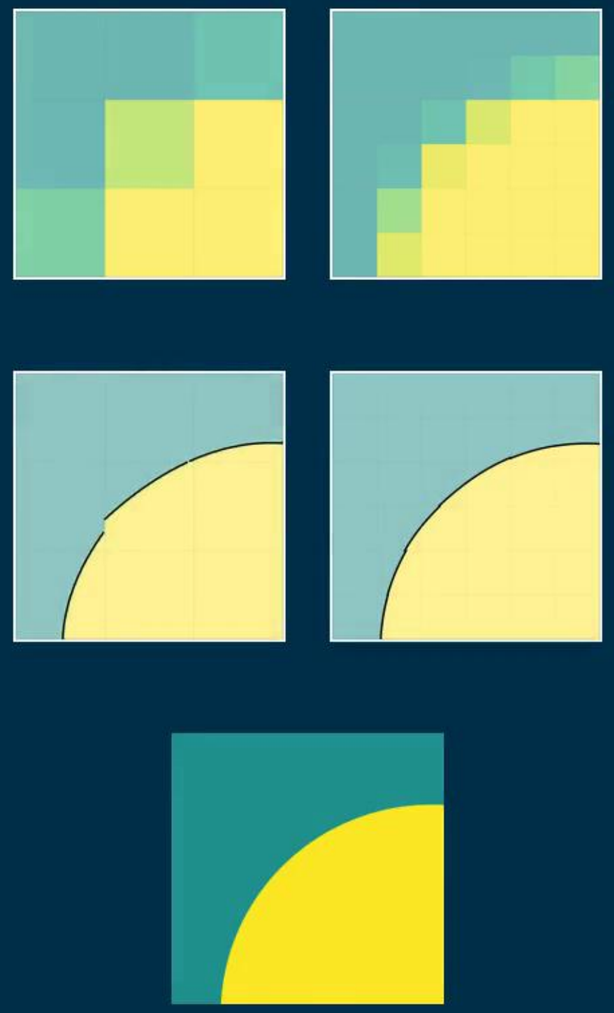
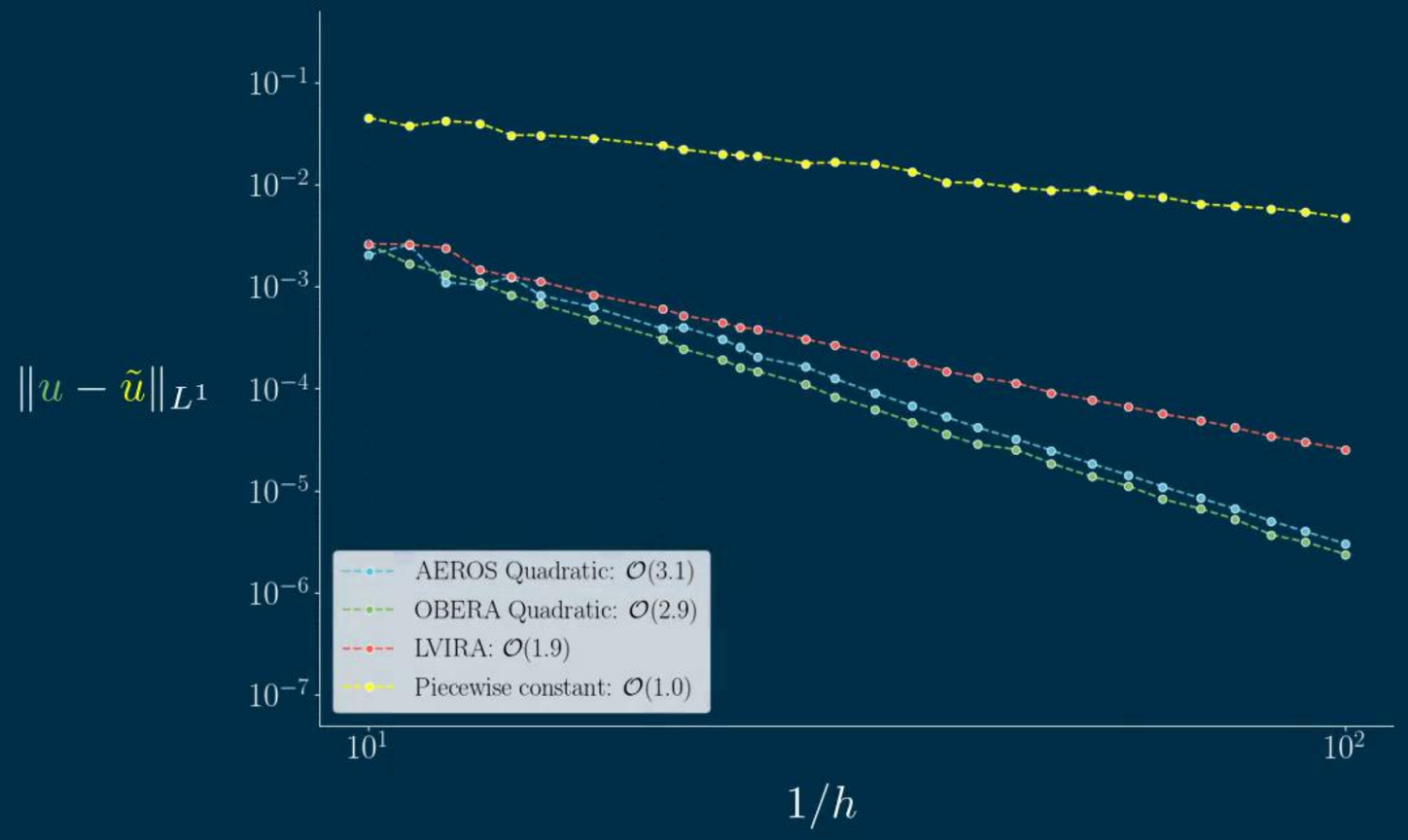
AEROS convergence





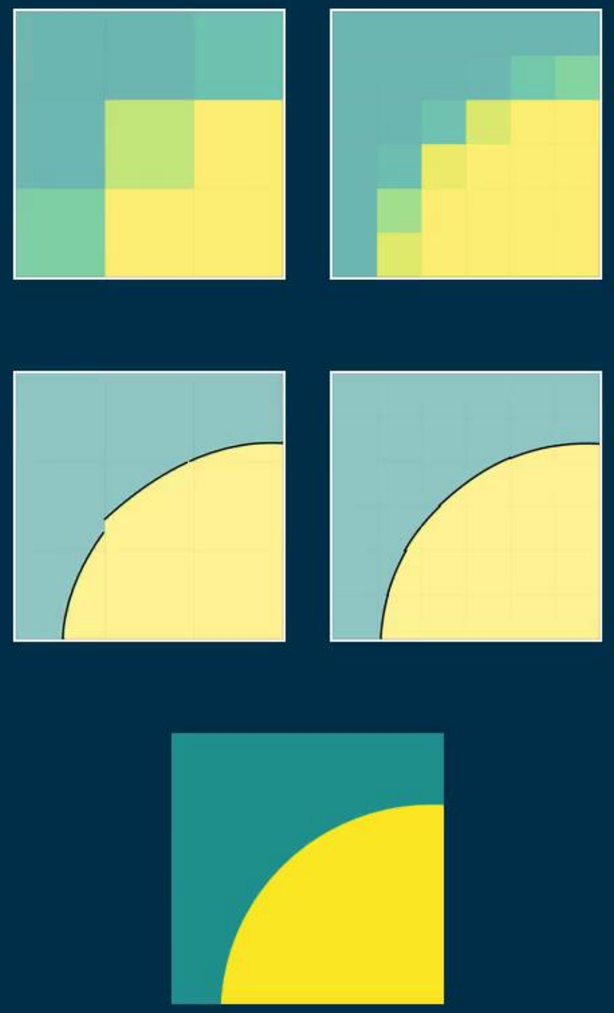
AEROS convergence: Quadratic

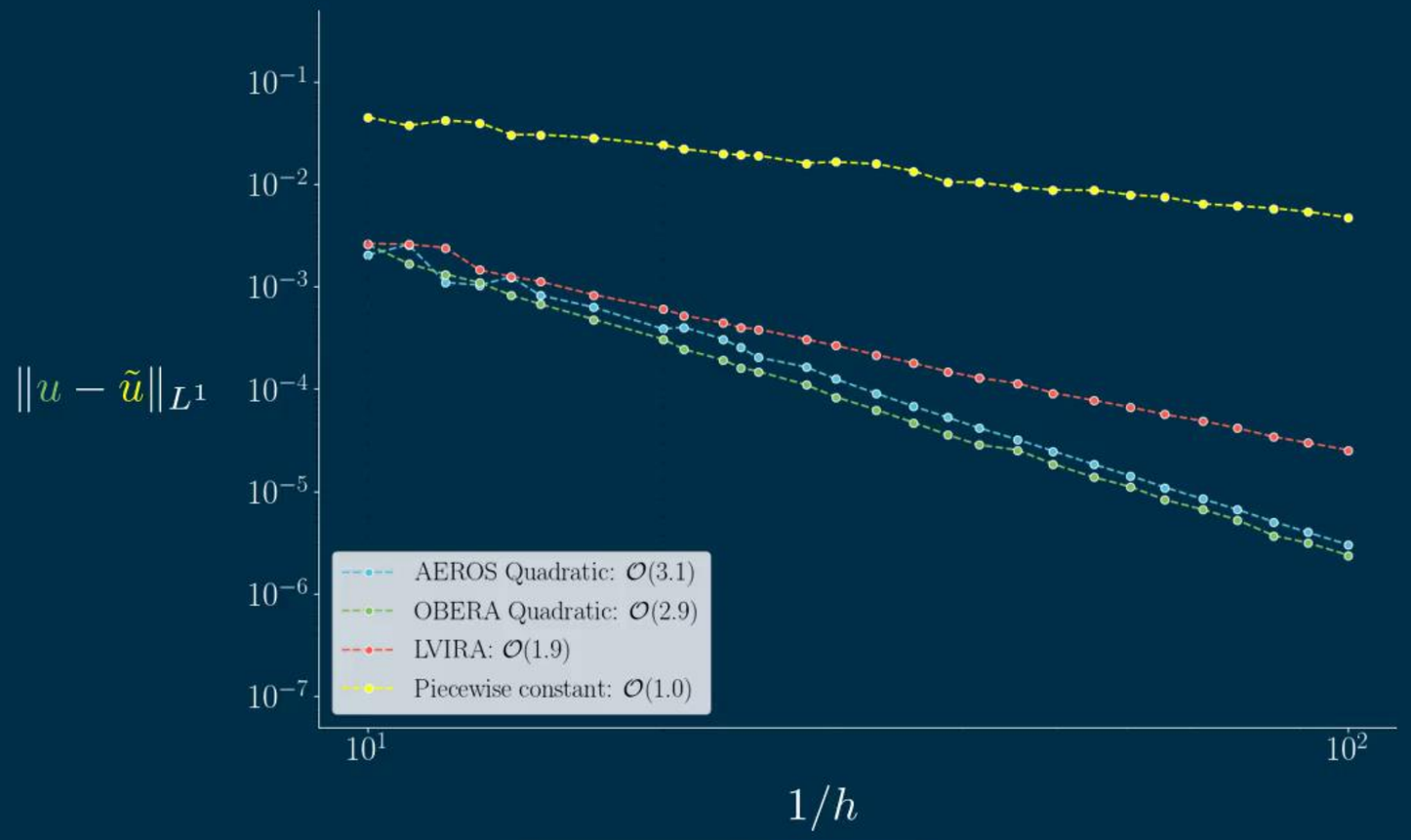




[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

AEROS convergence: Quadratic

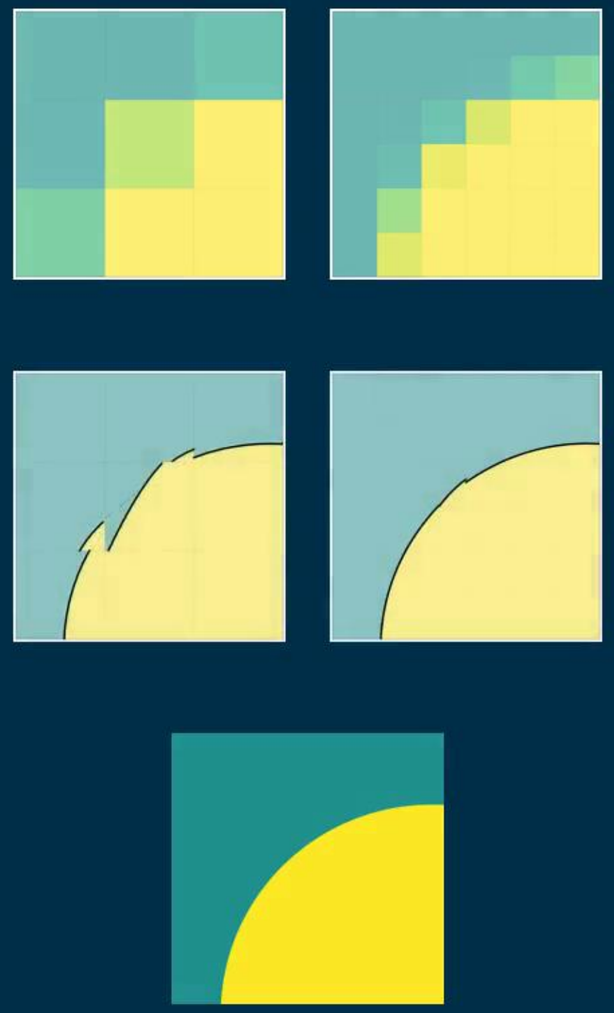


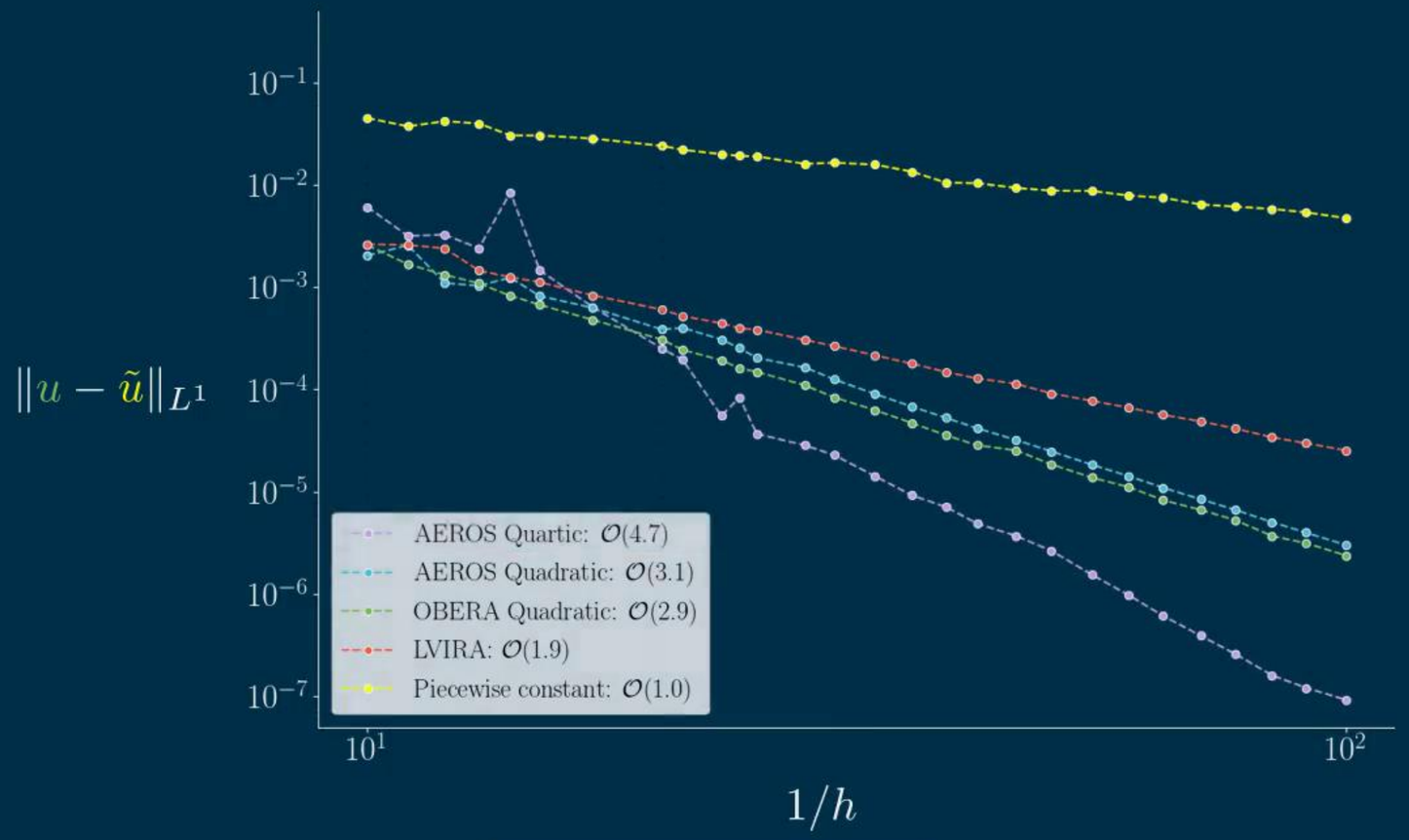


$$\tau_{AEROS} \lesssim \frac{\tau_{OBERA}}{100}$$

[A. Cohen, O. Mula, A. Somacal, ArXiv, 2024]

AEROS convergence: Quartic





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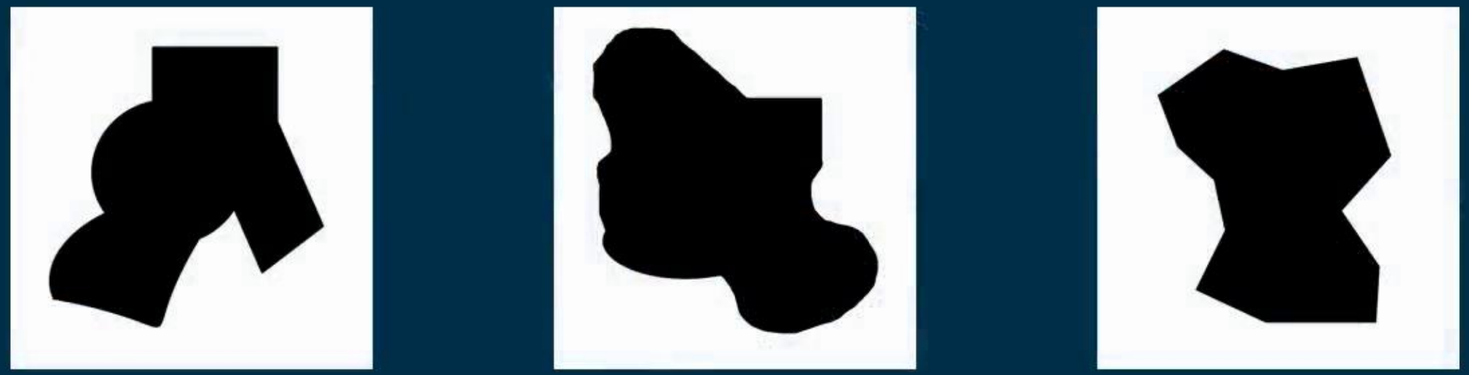
Corners

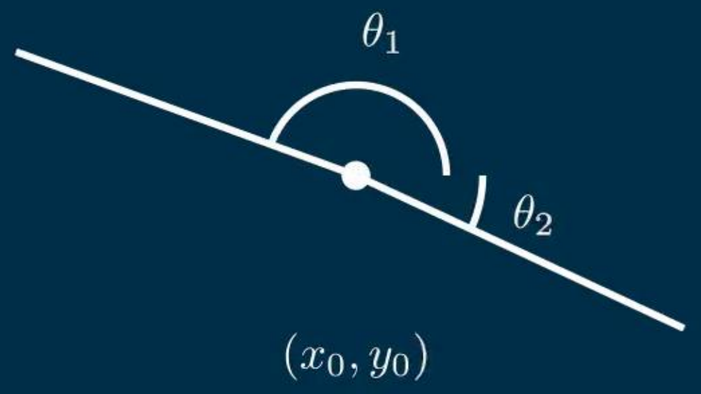


Corners

$$V_n := \{v_\mu : \mu \in \mathbb{R}^n\}$$

Corners

$$V_4 := \{v_\mu : \mu = (x_0, y_0, \theta_1, \theta_2)\}$$



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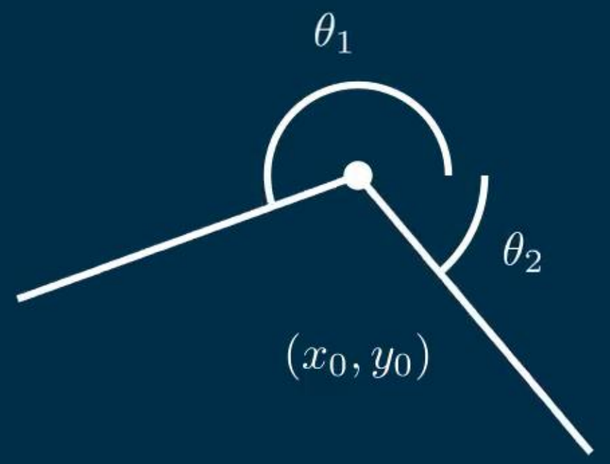

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AEROS Vertex

Tangent Extension Method (TEM)

Corners

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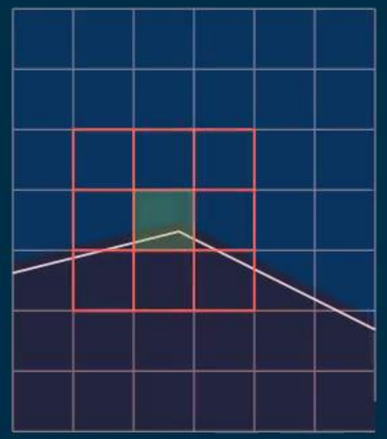
AEROS Vertex



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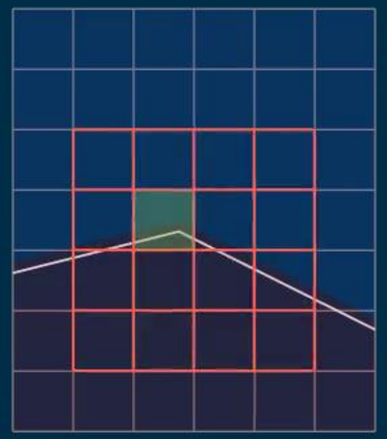

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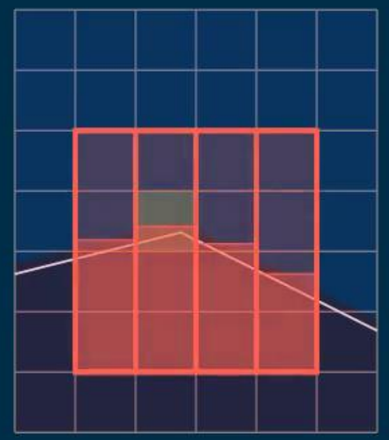

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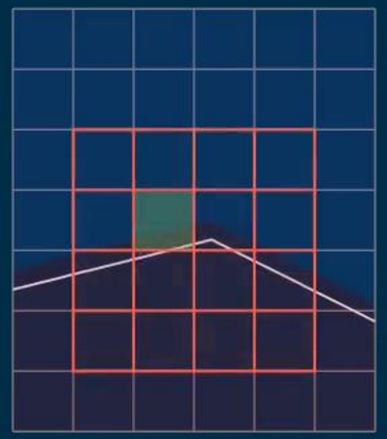

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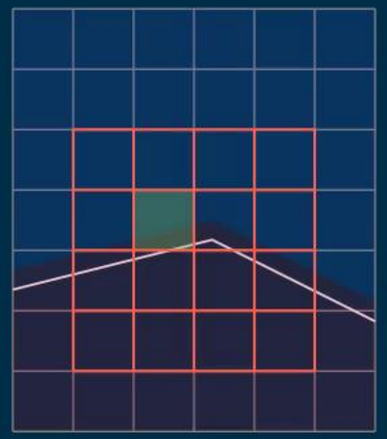

Tangent Extension Method (TEM)

- More than 1 posibility to test
- Can not deal with orientation change

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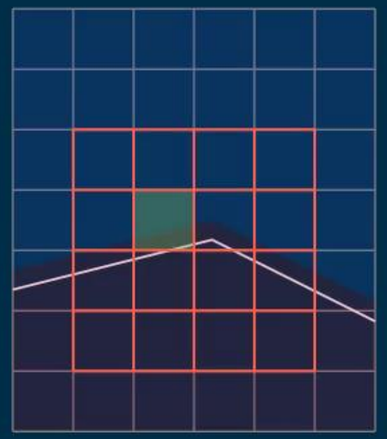

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[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

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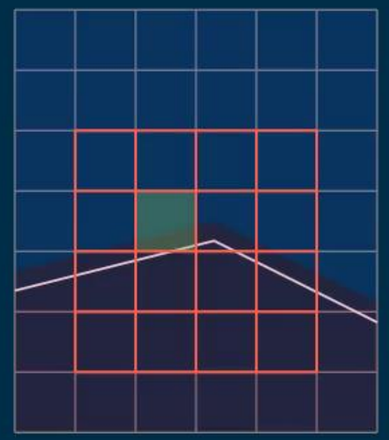

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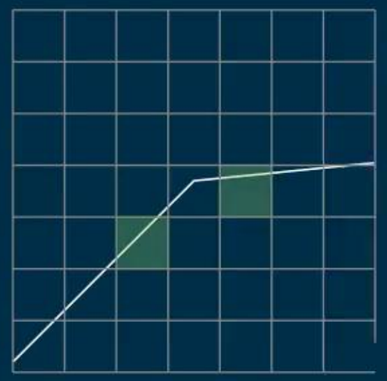

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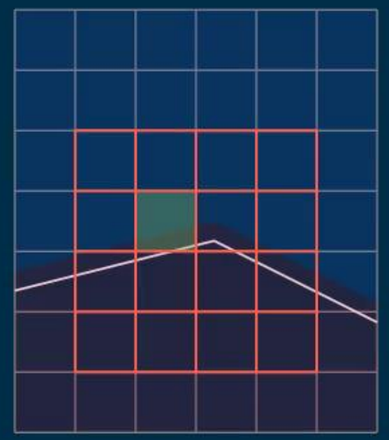

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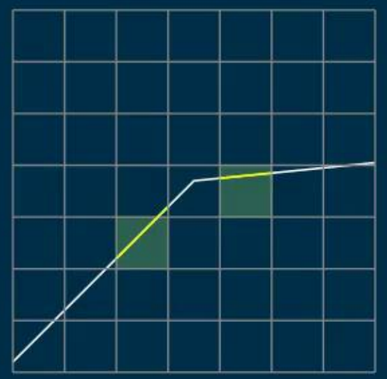

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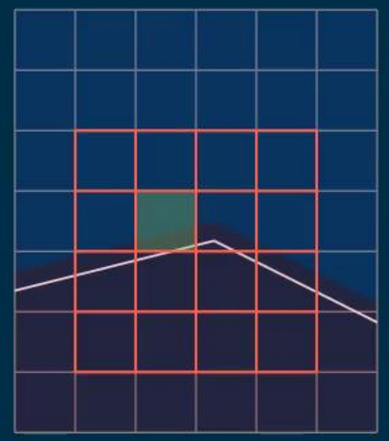

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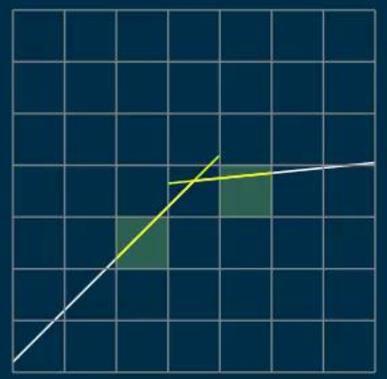

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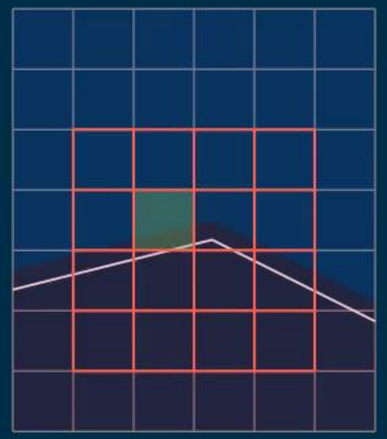

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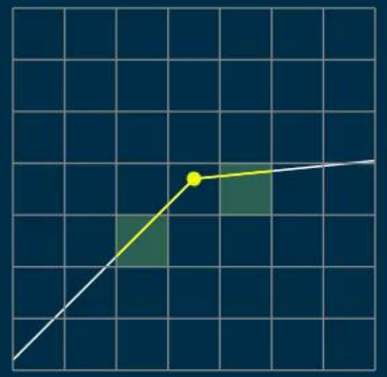

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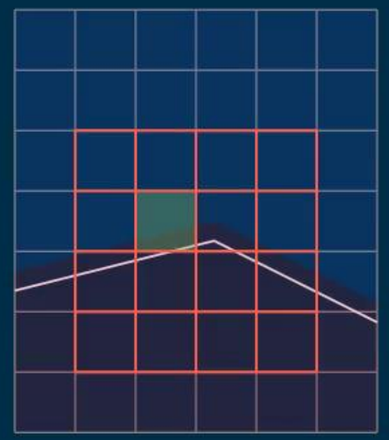
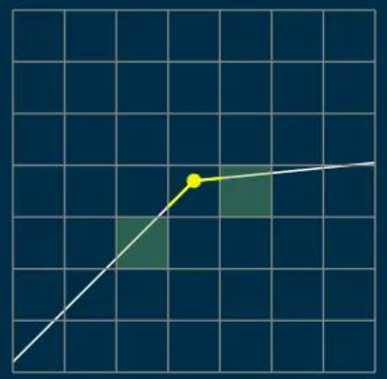


Tangent Extension Method (TEM)



- More than 1 posibility to test
- Can not deal with orientation change

[B. Després, H. Jourdren, Journal of Computational Physics, 2020]

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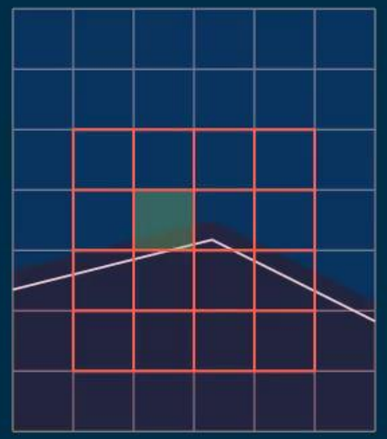

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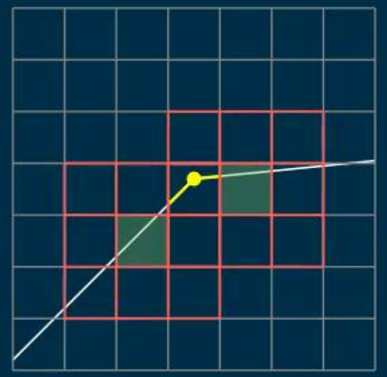

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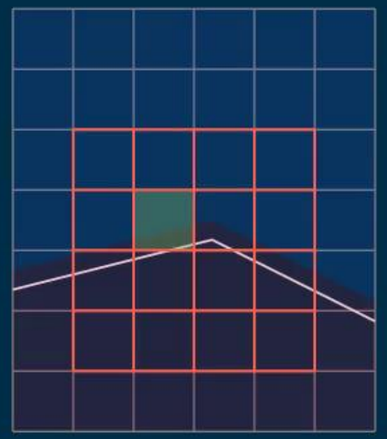

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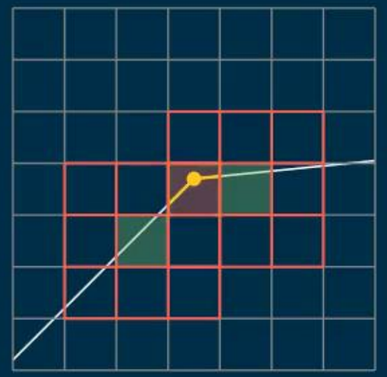

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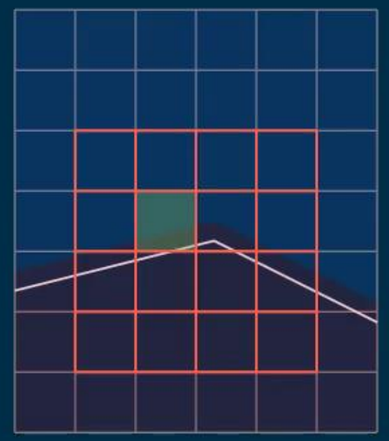

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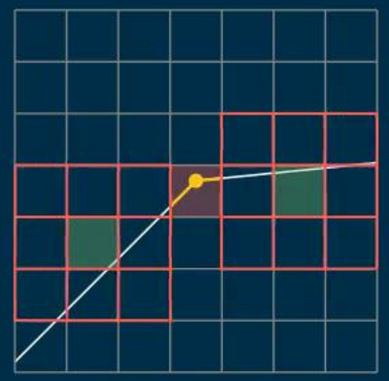

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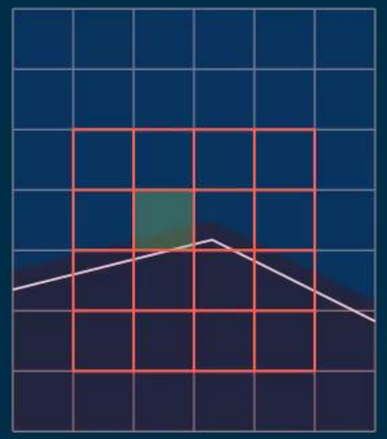

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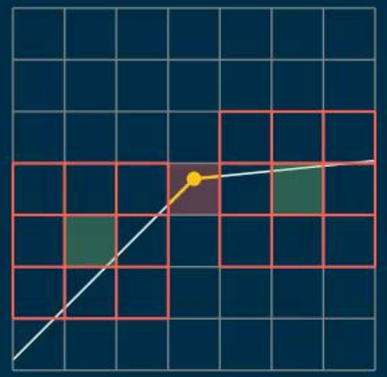

- Exact only if both sides of vertex are straight lines

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Reconstructing corners

$$\tilde{u} = \arg\min_{v \in V_n} \|z - \ell(v)\|_Z$$

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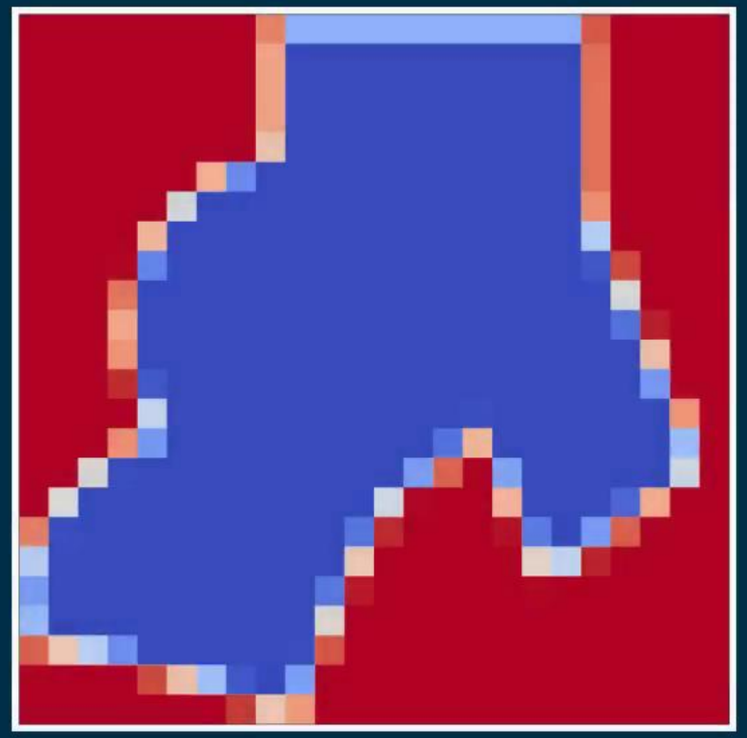

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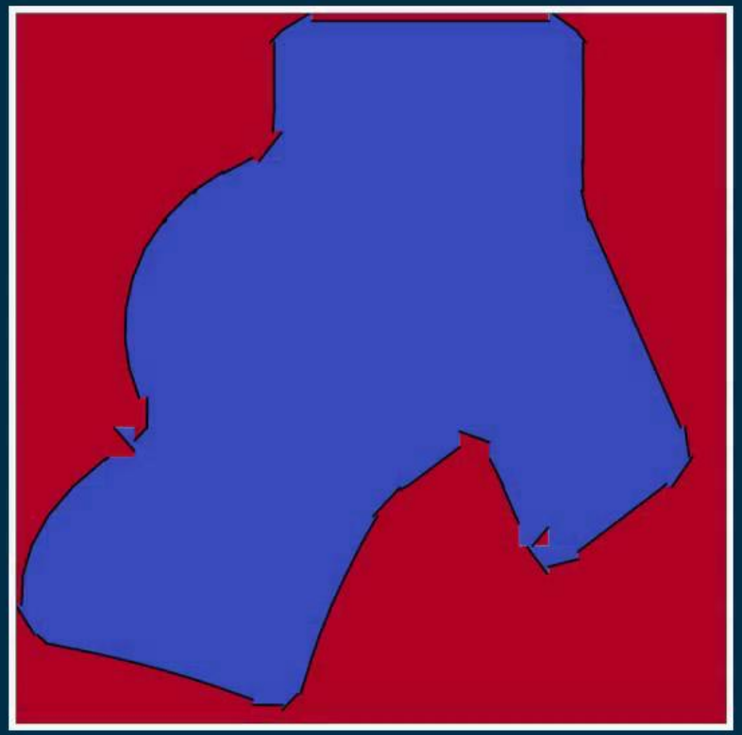

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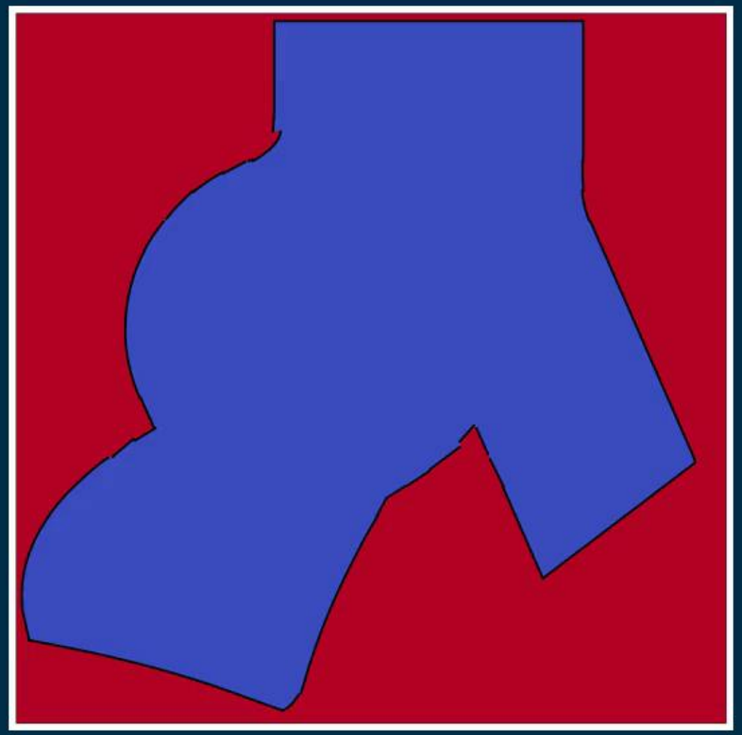

All code available open source at:
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ELVIRA [J. E. Pillod, E.G. Puckett, Second-order accurate volume-of-fluid algorithms for tracking material interfaces, Journal of Computational Physics, 2004]

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[A. Cohen, O. Mula, A. Somacal, High order recovery of geometric interfaces from cell-average data, ArXiv, 2024]