5.1.

$$y[n] = \begin{cases} 1, & 0 \le n \le 10, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

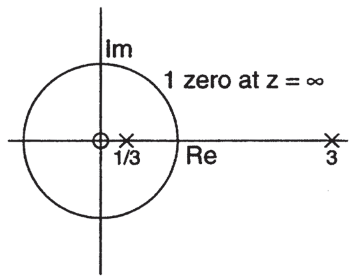
$$Y(e^{j\omega}) = e^{-j5\omega} \frac{\sin \frac{11}{2}\omega}{\sin \frac{\omega}{2}}$$

This $Y(e^{j\omega})$ is full band. Therefore, since $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$, the only possible $x[n]$ and ω_c that could produce $y[n]$ is $x[n] = y[n]$ and $\omega_c = \pi$.

5.2. We have $y[n-1] - \frac{10}{3}y[n] + y[n+1] = x[n]$ or $z^{-1}Y(z) - \frac{10}{3}Y(z) + zY(z) = X(z)$. So,

$$\begin{aligned} H(z) &= \frac{1}{z^{-1} - \frac{10}{3} + z} \\ &= \frac{z}{(z-\frac{1}{3})(z-3)} \\ &= \frac{-\frac{1}{8}}{z-\frac{1}{3}} + \frac{\frac{9}{8}}{z-3} \end{aligned}$$

(a)



(b)

$$H(z) = \frac{-\frac{1}{8}z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{\frac{9}{8}z^{-1}}{1-3z^{-1}}$$

Stable $\Rightarrow$ ROC is $\frac{1}{3} \leq |z| \leq 3$. Therefore,

$$h[n] = -\frac{1}{8}\left(\frac{1}{3}\right)^{n-1} u[n-1] - \frac{9}{8}(3)^{n-1}u[-n]$$

5.3.

$$y[n-1] + \frac{1}{3}y[n-2] = x[n]$$

$$z^{-1}Y(z) + \frac{1}{3}z^{-2}Y(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1}{z^{-1} + \frac{1}{3}z^{-2}} \\ H(z) &= \frac{z}{1 + \frac{1}{3}z^{-1}} \end{aligned}$$

i) $\frac{1}{3} < |z|$, $h[n] = (-\frac{1}{3})^{n+1}u[n+1] \Rightarrow$ answer (a)
ii) $\frac{1}{3} > |z|$,

$$\begin{aligned} h[n] &= -\left(-\frac{1}{3}\right)^{n+1} u[-n-2] \\ &= -\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)^{n} u[-n-2] \\ &= \frac{1}{3}\left(-\frac{1}{3}\right)^{n} u[-n-2] \Rightarrow \text{answer (d)} \end{aligned}$$

5.4. (a)

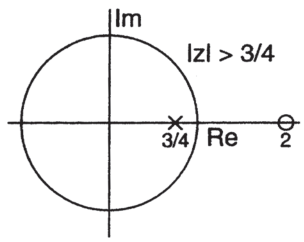
$$x[n] = \left(\frac{1}{2}\right)^n u[n] + (2)^n u[-n-1]$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{z}{z-2}, \qquad \frac{1}{2} < |z| < 2$$

$$y[n] = 6\left(\frac{1}{2}\right)^n u[n] - 6\left(\frac{3}{4}\right)^n u[n]$$

$$Y(z) = \frac{6}{1-\frac{1}{2}z^{-1}} - \frac{6}{1-\frac{3}{4}z^{-1}}, \qquad \frac{3}{4} < |z|$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{3}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}{-\frac{3}{2}z^{-1}} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}, \quad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}} - \frac{2z^{-1}}{1-\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$h[n] = \left(\frac{3}{4}\right)^n u[n] - 2\left(\frac{3}{4}\right)^{n-1} u[n-1]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-2z^{-1}}{1-\frac{3}{4}z^{-1}}$$

$$Y(z) - \frac{3}{4}z^{-1}Y(z) = X(z) - 2z^{-1}X(z)$$

$$y[n] - \frac{3}{4}y[n-1] = x[n] - 2x[n-1]$$

(d) The ROC is outside $|z| = \frac{3}{4}$, which includes the unit circle. Therefore the system is stable. The $h[n]$ we found in part (b) tells us the system is also causal.

5.5.

$$y[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{4}\right)^n u[n] + u[n]$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$x[n] = u[n]$$

$$X(z) = \frac{1}{1-z^{-1}}, \qquad |z| > 1$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3 - \frac{19}{6}z^{-1} + \frac{2}{3}z^{-2}}{1 - \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}, \qquad |z| > \frac{1}{3}$$

(a) Cross multiplying and equating z^{-1} with a delay in time:

$$y[n] - \frac{7}{12}y[n-1] + \frac{1}{12}y[n-2] = 3x[n] - \frac{19}{6}x[n-1] + \frac{2}{3}x[n-2]$$

(b) Using partial fractions on $H(z)$ we get:

$$H(z) = \frac{1}{1-\frac{1}{3}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{4}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{4}z^{-1}} + 1, \qquad |z| > \frac{1}{3}$$

So,

$$h[n] = \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^{n-1} u[n-1] + \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n-1] + \delta[n]$$

(c) Since the ROC of $H(z)$ includes $|z| = 1$ the system is stable.

5.6. (a)

$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$X(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{4}{3}}{1-2z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}, \qquad \frac{1}{2} < |z| < 2$$

(b)

$$Y(z) = \frac{1-z^{-2}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

This has the same poles as the input, therefore the ROC is still $\frac{1}{2} < |z| < 2$.

(c)

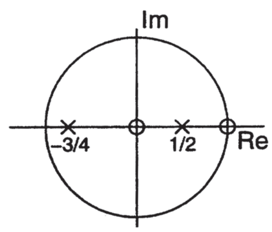
$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \Leftrightarrow h[n] = \delta[n] - \delta[n-2]$$

5.7. (a)

$$x[n] = 5u[n] \Leftrightarrow X(z) = \frac{5}{1-z^{-1}}, \qquad |z| > 1$$

$$y[n] = \left(2\left(\frac{1}{2}\right)^n + 3\left(-\frac{3}{4}\right)^n\right)u[n] \Leftrightarrow Y(z) = \frac{2}{1-\frac{1}{2}z^{-1}} + \frac{3}{1+\frac{3}{4}z^{-1}}, \qquad |z| > \frac{3}{4}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}, \qquad |z| > \frac{3}{4}$$



(b)

$$H(z) = \frac{1-z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} = \frac{-\frac{2}{5}}{(1-\frac{1}{2}z^{-1})} + \frac{\frac{7}{5}}{(1+\frac{3}{4}z^{-1})}, \qquad |z| > \frac{3}{4}$$

$$h[n] = -\frac{2}{5}\left(\frac{1}{2}\right)^n u[n] + \frac{7}{5}\left(-\frac{3}{4}\right)^n u[n]$$

(c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}$$

$$Y(z) + \frac{1}{4}z^{-1}Y(z) - \frac{3}{8}z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

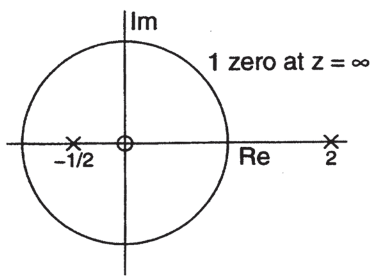
$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - x[n-1]$$

5.8. (a)

$$\begin{aligned} y[n] &= \frac{3}{2}y[n-1] + y[n-2] + x[n-1] \\ Y(z) &= \frac{3}{2}z^{-1}Y(z) + z^{-2}Y(z) + z^{-1}X(z) \end{aligned}$$

Therefore,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{3}{2}z^{-1} - z^{-2}} = \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})}, \quad |z| > 2$$



(b)

$$\begin{aligned} H(z) &= \frac{z^{-1}}{(1-2z^{-1})(1+\frac{1}{2}z^{-1})} = \frac{A}{(1-2z^{-1})} + \frac{B}{(1+\frac{1}{2}z^{-1})}, \quad |z| > 2 \\ A &= \left.\frac{z^{-1}}{(1+\frac{1}{2}z^{-1})}\right|_{z^{-1}=\frac{1}{2}} = \frac{2}{5} \\ B &= \left.\frac{z^{-1}}{(1-2z^{-1})}\right|_{z^{-1}=-2} = -\frac{2}{5} \end{aligned}$$

$$h[n] = \frac{2}{5}\left[(2)^n - \left(-\frac{1}{2}\right)^n\right]u[n]$$

(c) Use ROC of $\frac{1}{2} < |z| < 2$ since the ROC must include $|z| = 1$ for a stable system.

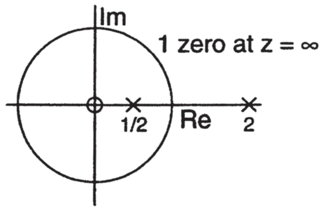
$$h[n] = -\frac{2}{5}(2)^n u[-n-1] - \frac{2}{5}\left(-\frac{1}{2}\right)^n u[n]$$

5.9.

$$y[n-1] - \frac{5}{2}y[n] + y[n+1] = x[n]$$

$$z^{-1}Y(z) - \frac{5}{2}Y(z) + zY(z) = X(z)$$

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}} \\ &= \frac{z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{\frac{2}{3}}{1-2z^{-1}} - \frac{\frac{2}{3}}{1-\frac{1}{2}z^{-1}} \end{aligned}$$



Three regions of convergence:

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] + \frac{2}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}(2)^n u[-n-1] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

Includes $|z| = 1$, so this is stable.

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}(2)^n u[n] - \frac{2}{3}\left(\frac{1}{2}\right)^n u[n]$$

ROC outside of largest pole, so this is causal.

5.10. Figure P5.16 shows two zeros and three poles inside the unit circle. Since the number of poles must equal the number of zeros, there must be an additional zero at $z = \infty$.

$H(z)$ is causal, so the ROC lies outside the largest pole and includes the unit circle. Therefore, the system is also stable.

The inverse system switches poles and zeros. The inverse system could have a ROC that includes $|z| = 1$, making it stable. However, the zero at $z = \infty$ of $H(z)$ is a pole for $H_i(z)$, so the system $H_i(z)$ cannot be causal.

5.11. (a) *It cannot be determined.* The ROC might or might not include the unit circle.

(b) *It cannot be determined.* The ROC might or might not include $z = \infty$.

(c) *False.* Given that the system is causal, we know that the ROC must be outside the outermost pole. Since the outermost pole is outside the unit circle, the ROC will not include the unit circle, and thus the system is not stable.

(d) *True.* If the system is stable, the ROC must include the unit circle. Because there are poles both inside and outside the unit circle, any ROC including the unit circle must be a ring. A ring-shaped ROC means that we have a two-sided system.

5.12. (a) *Yes.* The poles $z = \pm j(0.9)$ are inside the unit circle so the system is stable.

(b) First, factor $H(z)$ into two parts. The first should be minimum phase and therefore have all its poles and zeros inside the unit circle. The second part should contain the remaining poles and zeros.

$$H(z) = \underbrace{\frac{1+0.2z^{-1}}{1+0.81z^{-2}}}_{\text{minimum phase}} \qquad \underbrace{\frac{1-9z^{-2}}{1}}_{\substack{\text{poles \& zeros}\\ \text{outside unit circle}}}$$

Allpass systems have poles and zeros that occur in conjugate reciprocal pairs. If we include the factor $(1-\frac{1}{9}z^{-2})$ in both parts of the equation above the first part will remain minimum phase and the second will become allpass.

$$\begin{aligned} H(z) &= \frac{(1+0.2z^{-1})(1-\frac{1}{9}z^{-2})}{1+0.81z^{-2}} \cdot \frac{1-9z^{-2}}{1-\frac{1}{9}z^{-2}} \\ &= H_1(z)H_{ap}(z) \end{aligned}$$

5.13. *An aside*: Technically, this problem is not well defined, since a pole/zero plot does not uniquely determine a system. That is, many system functions can have the same pole/zero plot. For example, consider the systems

$$\begin{aligned} H_1(z) &= z^{-1} \\ H_2(z) &= 2z^{-1} \end{aligned}$$

Both of these systems have the same pole/zero plot, namely a pole at zero and a zero at infinity. Clearly, the system $H_1(z)$ is allpass, as it passes all frequencies with unity gain (it is simply a unit delay). However, one could ask whether $H_2(z)$ is allpass. Looking at the standard definition of an allpass system provided in this chapter, the answer would be no, since the system does not pass all frequencies with *unity* gain.

A broader definition of an allpass system would be a system for which the system magnitude response $|H(e^{j\omega})| = a$, where a is a real constant. Such a system would pass all frequencies, and scale the output by a constant factor a. In a practical setting, this definition of an allpass system is satisfactory. Under this definition, both systems $H_1(z)$ and $H_2(z)$ would be considered allpass.

For this problem, it is assumed that none of the poles or zeros shown in the pole/zero plots are scaled, so this issue of using the proper definition of an allpass system does not apply. The standard definition of an allpass system is used.

Solution:

(a) Yes, the system is allpass, since it is of the appropriate form.

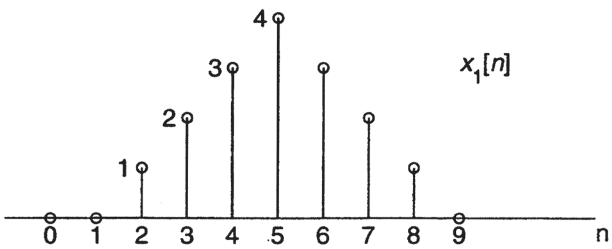
(b) No, the system is not allpass, since the zero does not occur at the conjugate reciprocal location of the pole.

(c) Yes, the system is allpass, since it is of the appropriate form.

(d) Yes, the system is allpass. This system consists of an allpass system in cascade with a pole at zero. The pole at zero is simply a delay, and does not change the magnitude spectrum.

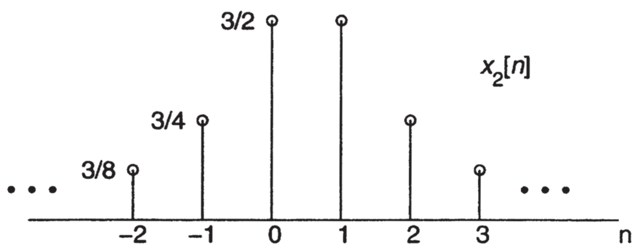
5.14. (a) By the symmetry of $x_1[n]$ we know it has linear phase. The symmetry is around $n = 5$ so the continuous phase of $X_1(e^{jw})$ is $\arg[X_1(e^{j\omega})] = -5\omega$. Thus,

$$\text{grd}[X_1(e^{j\omega})] = -\frac{d}{d\omega}\left\{\arg[X_1(e^{j\omega})]\right\} = -\frac{d}{d\omega}\left\{-5\omega\right\} = 5$$



(b) By the symmetry of $x_2[n]$ we know it has linear phase. The symmetry is around $n = 1/2$ so we know the phase of $X_2(e^{j\omega})$ is $\arg[X_2(e^{j\omega}] = -\omega/2$. Thus,

$$\text{grd}[X_2(e^{j\omega})] = -\frac{d}{d\omega}\left\{\arg[X_2(e^{j\omega})]\right\} = -\frac{d}{d\omega}\left\{-\frac{\omega}{2}\right\} = \frac{1}{2}$$



5.15. (a) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 2 + e^{-j\omega} + 2e^{-2j\omega} \\ &= e^{-j\omega}(2e^{j\omega} + 1 + 2e^{-j\omega}) \\ &= (1 + 4\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 1 + 4\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase but not Linear Phase since $A(\omega)$ is not always positive.

(b) This sequence has no even or odd symmetry, so it does not possess generalized linear phase.

(c) $h[n]$ is symmetric about $n = 1$.

$$\begin{aligned} H(e^{j\omega}) &= 1 + 3e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 3 + e^{-j\omega}) \\ &= (3 + 2\cos\omega)e^{-j\omega} \end{aligned}$$

$$A(\omega) = 3 + 2\cos\omega,\ \alpha = 1,\ \beta = 0$$

Generalized Linear phase & Linear Phase.

(d) $h[n]$ has even symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 + e^{-j\omega} \\ &= e^{-j(1/2)\omega}(e^{j(1/2)\omega} + e^{-j(1/2)\omega}) \\ &= 2\cos(\omega/2)e^{-j(1/2)\omega} \end{aligned}$$

$$A(\omega) = 2\cos(\omega/2),\ \alpha = \frac{1}{2},\ \beta = 0$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

(e) $h[n]$ has odd symmetry.

$$\begin{aligned} H(e^{j\omega}) &= 1 - e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} - e^{-j\omega}) \\ &= e^{-j\omega}2j\sin\omega \\ &= (2\sin\omega)e^{-j\omega + j\frac{\pi}{2}} \end{aligned}$$

$$A(\omega) = 2\sin\omega,\ \alpha = 1,\ \beta = \frac{\pi}{2}$$

Generalized Linear Phase but not Linear Phase since $A(\omega)$ is not always positive.

5.16. The causality of the system cannot be determined from the figure. A causal system $h_1[n]$ that has a linear phase response $\angle H(e^{jw}) = -\alpha w$, is:

$$\begin{aligned}
h_1[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \\
H_1(e^{j\omega}) &= 1 + 2e^{-j\omega} + e^{-j2\omega} \\
&= e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega}) \\
&= e^{-j\omega}(2 + 2\cos(\omega)) \\
\left|H_1(e^{j\omega})\right| &= 2 + 2\cos(\omega) \\
\angle H_1(e^{j\omega}) &= -\omega
\end{aligned}$$

An example of a non-causal system with the same phase response is:

$$\begin{aligned}
h_2[n] &= \delta[n+1] + \delta[n] + 4\delta[n-1] + \delta[n-2] + \delta[n-3] \\
H_2(e^{j\omega}) &= e^{j\omega} + 1 + 4e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} \\
&= e^{-jw}(e^{j2\omega} + e^{j\omega} + 4 + e^{-j\omega} + e^{-j2\omega}) \\
&= e^{-j\omega}(4 + 2\cos(\omega) + 2\cos(2\omega)) \\
\left|H_2(e^{j\omega})\right| &= 4 + 2\cos(\omega) + 2\cos(2\omega) \\
\angle H_2(e^{j\omega}) &= -\omega
\end{aligned}$$

Thus, both the causal sequence $h_1[n]$ and the non-causal sequence $h_2[n]$ have a linear phase response $\angle H(e^{j\omega}) = -\alpha\omega$, where $\alpha = 1$.

5.17. A minimum phase system is one which has all its poles and zeros inside the unit circle. It is causal, stable, and has a causal and stable inverse.

(a) $H_1(z)$ has a zero outside the unit circle at $z = 2$ so it is not minimum phase.

(b) $H_2(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(c) $H_3(z)$ is minimum phase since its poles and zeros are inside the unit circle.

(d) $H_4(z)$ has a zero outside the unit circle at $z = \infty$ so it is not minimum phase. Moreover, the inverse system would not be causal due to the pole at infinity.

5.18. A minimum phase system with an equivalent magnitude spectrum can be found by analyzing the system function, and reflecting all poles are zeros that are outside the unit circle to their conjugate reciprocal locations. This will move them inside the unit circle. Then, all poles and zeros for $H_{min}(z)$ will be inside the unit circle. Note that a scale factor may be introduced when the pole or zero is reflected inside the unit circle.

(a) Simply reflect the zero at $z = 2$ to its conjugate reciprocal location at $z = \frac{1}{2}$. Then, determine the scale factor.

$$H_{min}(z) = 2\left(\frac{1 - \frac{1}{2}z^{-1}}{1 + \frac{1}{3}z^{-1}}\right)$$

(b) First, simply reflect the zero at $z = -3$ to its conjugate reciprocal location at $z = -\frac{1}{3}$. Then, determine the scale factor. This results in

$$H_{min}(z) = 3\frac{\left(1 + \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}\left(1 + \frac{1}{3}z^{-1}\right)}$$

The $\left(1 + \frac{1}{3}z^{-1}\right)$ terms cancel, leaving

$$H_{min}(z) = 3\frac{\left(1 - \frac{1}{2}z^{-1}\right)}{z^{-1}}$$

Note that the term $\frac{1}{z^{-1}}$ does not affect the frequency response magnitude of the system. Consequently, it can be removed. Thus, the remaining term has a zero inside the unit circle, and is therefore minimum phase. As a result, we are left with the system

$$H_{min}(z) = 3\left(1 - \frac{1}{2}z^{-1}\right)$$

(c) Simply reflect the zero at 3 to its conjugate reciprocal location at $\frac{1}{3}$ and reflect the pole at $\frac{4}{3}$ to its conjugate reciprocal location at $\frac{3}{4}$. Then, determine the scale factor.

$$H_{min}(z) = \frac{9}{4}\frac{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}{\left(1 - \frac{3}{4}z^{-1}\right)^2}$$

5.19. Due to the symmetry of the impulse responses, all the systems have generalized linear phase of $\arg[H(e^{j\omega})] = \beta - n_o\omega$ where n_o is the point of symmetry in the impulse response graphs. The group delay is

$$\text{grd}\left[H_i(e^{j\omega})\right] = -\frac{d}{d\omega}\left\{\arg\left[H_i(e^{j\omega})\right]\right\} = -\frac{d}{d\omega}\{\beta - n_o\omega\} = n_o$$

To find each system's group delay we need only find the point of symmetry n_o in each system's impulse response.

$$\begin{aligned} \text{grd}\left[H_1(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_4(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_2(e^{j\omega})\right] &= 1.5 & \qquad \text{grd}\left[H_5(e^{j\omega})\right] &= 3 \\ \text{grd}\left[H_3(e^{j\omega})\right] &= 2 & \qquad \text{grd}\left[H_6(e^{j\omega})\right] &= 3.5 \end{aligned}$$

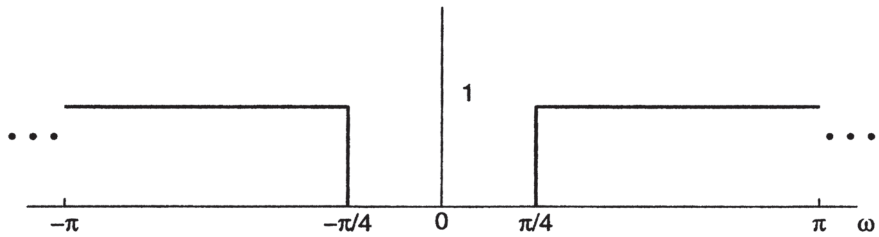
5.20. (a) *Yes.* The system function could be a generalized linear phase system implemented by a linear constant-coefficient differential equation (LCCDE) with real coefficients. The zeros come in a set of four: a zero, its conjugate, and the two conjugate reciprocals. The pole-zero plot could correspond to a Type I FIR linear phase system.

(b) *No.* This system function could not be a generalized linear phase system implemented by a LCCDE with real coefficients. Since the LCCDE has real coefficients, its poles and zeros must come in conjugate pairs. However, the zeros in this pole-zero plot do not have corresponding conjugate zeros.

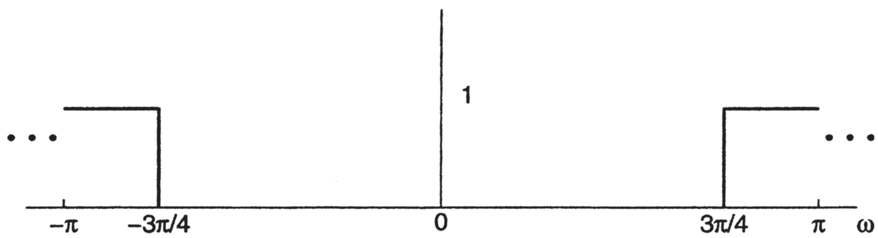
(c) *Yes.* The system function could be a generalized linear phase system implemented by a LCCDE with real coefficients. The pole-zero plot could correspond to a Type II FIR linear phase system.

5.21. $h_{lp}[n]$ is an ideal lowpass filter with $\omega_c = \frac{\pi}{4}$

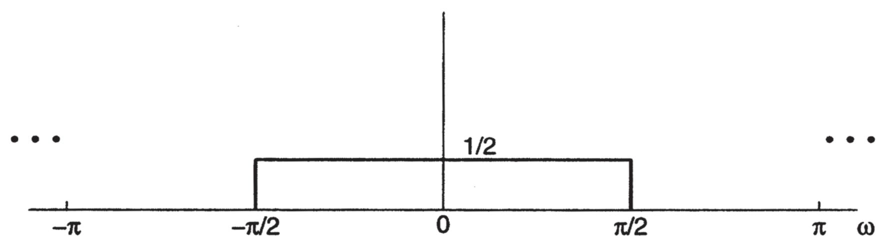
(a) $y[n] = x[n] - x[n] * h_{lp}[n] \Rightarrow H(e^{j\omega}) = 1 - H_{lp}(e^{j\omega})$
This is a highpass filter.



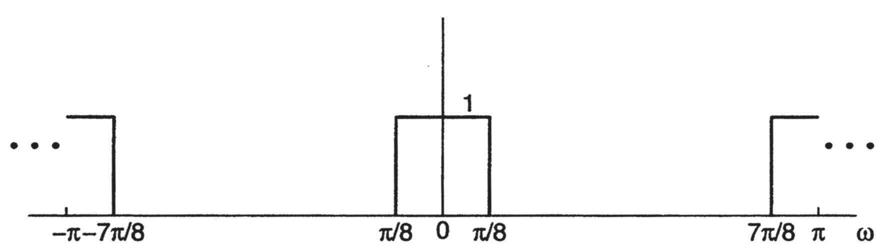
(b) $x[n]$ is first modulated by π, lowpass filtered, and demodulated by π. Therefore, $H_{lp}(e^{j\omega})$ filters the high frequency components of $X(e^{j\omega})$.
This is a highpass filter.



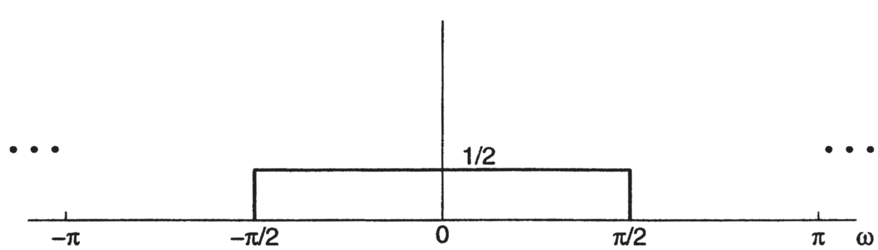
(c) $h_{lp}[2n]$ is a downsampled version of the filter. Therefore, the frequency response will be "spread out" by a factor of two, with a gain of $\frac{1}{2}$.
This is a lowpass filter.



(d) This system upsamples $h_{lp}[n]$ by a factor of two. Therefore, the frequency axis will be compressed by a factor of two.
This is a bandstop filter.



(e) This system upsamples the input before passing it through $h_{lp}[n]$. This effectively doubles the frequency bandwidth of $H_{lp}(e^{j\omega})$.
This is a lowpass filter.



5.22. Problem 2 from sp 2005 final exam
Appears in: Fall05 PS2.

Problem

Many properties of a discrete-time sequence $h[n]$ or an LTI system with impulse response $h[n]$ can be discerned from a pole-zero plot of $H(z)$. **In this problem we are concerned only with causal** $h[n]$**.** Clearly describe the z-plane characteristic that corresponds to each of the following properties:

(i) Real-valued impulse response:

(ii) Finite impulse response:

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

(iv) Minimum phase:

(v) All-pass:

Solution from Fall05 PS2

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin. The ROC is the entire z-plane, except possibly $z = 0$.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iv) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(v) All-pass:

Each pole is paired with a zero at the conjugate reciprocal location.

Solution from Spring05 Final

(i) Real-valued impulse response:

Poles that aren't real must be in complex conjugate pairs. Zeros that aren't real must be in complex conjugate pairs.

(ii) Finite impulse response:

All poles are at the origin.

(iii) $h[n] = h[2\alpha - n]$ where 2α is an integer:

Causality combined with the given symmetry property implies a finite-length $h[n]$ that can only be nonzero between time zero and time 2α. Thus we must have all poles at the origin and at most 2α zeros. The z transform of $h[2\alpha - n]$ is $z^{-2\alpha}H(1/z)$, so any zero of $H(z)$ at $c \neq 0$ must be paired with a zero at $1/c$.

(iv) Minimum phase:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

(v) All-pass:

All poles and zeros are inside the unit circle (so that the inverse can be stable and causal).

5.23. Problem 1 from sp 2005 midterm exam
Appears in: Fall05 PS2.

Problem

For all parts of this problem $H(e^{j\omega})$ is the frequency response of a DT filter and can be expressed in polar coordinates as

$$H(e^{j\omega}) = A(\omega)e^{j\theta(\omega)}$$

where $A(\omega)$ is even and real-valued and $\theta(\omega)$ is a continuous, odd function of ω for $-\pi < \omega < \pi$, *i.e.*, $\theta(\omega)$is what we have referred to as the *unwrapped phase*. Recall:

- The *group delay* $\tau(\omega)$ associated with the filter is defined as

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega} \qquad \text{for} |\omega| < \pi.$$

- An LTI filter is called *minimum phase* if it is stable and causal and has a stable and causal inverse.

For each of the following statements, state whether it is **TRUE** or **FALSE**. If you state that it is **TRUE**, give a clear, brief justification. If you state that it is **FALSE**, give a simple counterexample with a clear, brief explanation of why it is a counterexample. **No credit will be given for true/false answers without explanations.**

(a) "If the filter is causal, its group delay must be nonnegative at all frequencies in the range $|\omega| < \pi$."

(b) "If the group delay of the filter is a positive constant integer for $|\omega| < \pi$ the filter must be a simple integer delay."

(c) "If the filter is minimum phase and all the poles and zeros are on the real axis then $\int_0^{\pi} \tau(\omega)d\omega = 0$."

Solution from Fall05 PS2

(a) False. As a counterexample, consider a filter described by

$$H(e^{j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$

We know that $H(z)$ can describe a causal, stable filter since its single pole is at $z = 0$. In the vicinity of $\omega = 0$, $\theta(\omega)$ is increasing, so the group delay $\tau(\omega)$ is negative.
The group delay evaluated at $\omega = 0$ for this example can be determined using equation (5.67) in OSB:

$$grd[1 - re^{j\theta}e^{-j\omega}] = \frac{r^2 - r\cos(\omega - \theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}$$

and substituting $r = \frac{1}{2}$ and $\theta = 0$.

(b) False. Any zero-phase FIR filter may be delayed to become causal, and the resulting filter will have the same phase as the delay block which was applied to it. One counterexample is the causal, stable filter described by $H(z) = 1 + 2z^{-1} + z^{-2}$.

(c) True. Note that

$$\int_0^{\pi} \tau(\omega)d\omega = -\left.\theta(\omega)\right|_{\omega=0}^{\pi} = \theta(0) - \theta(\pi).$$

Setting this to 0, we realize that the assertion is true if having minimum phase and all poles and zeros on the real axis implies

$$\theta(0) = \theta(\pi).$$

A minimum phase filter with poles and zeros on the real axis will have all poles and zeros between -1 and 1. Factoring $H(e^{j\omega})$ to find poles and zeros, we notice that for a given pole term $\frac{1}{1-ae^{-j\omega}}$, the restriction $-1 < a < 1$ yields a phase contribution of 0 when $\omega = 0$ or $\omega = \pi$. Likewise, a given zero term $1 - be^{-j\omega}$ gives a phase contribution of 0 when $\omega = 0$ or $\omega = \pi$ as long as $-1 < b < 1$. Since all poles and zeros of such a minimum phase filter meet these criteria, $\theta(0) = \theta(\pi) = 0$ Therefore, $\int_0^{\pi} \tau(\omega)d\omega = 0$. (Note that $\int_0^{\pi} \tau(\omega)d\omega = 0$ for all real minimum phase filters, even if there are complex poles and zeros.)

Solution from Spring05 Midterm

(7%) (a) "If the filter is causal, its group delay must be nonnegative at all frequencies in the range $|\omega| < \pi$."

False. As a counterexample, consider a filter described by

$$H(e^{-j\omega}) = 1 - \frac{1}{2}e^{-j\omega}.$$

We know that $H(z)$ can describe a causal, stable filter since its single pole is at $z = 0$. In the vicinity of $\omega = 0$, $\theta(\omega)$ is increasing, so the group delay $\tau(\omega)$ is negative.

In general, any non-trivial minimum phase filter must have regions of both positive and negative group delay; see the solution to 1(c).

(7%) (b) "If the group delay of the filter is a positive constant integer for $|\omega| < \pi$ the filter must be a simple integer delay."

False. Any zero-phase FIR filter may be delayed to become causal, and the resulting filter will have the same phase as the delay block which was applied to it. One counterexample is the causal, stable filter described by $H(z) = 1 + 2z^{-1} + z^{-2}$.

(8%) (c) "If the filter is minimum phase and all the poles and zeros are on the real axis then $\int_0^\pi \tau(\omega)\, d\omega = 0$."

True. Note that

$$\int_0^\pi \tau(\omega)d\omega = -\left.\theta(\omega)\right|_{\omega=0}^{\pi} = \theta(0) - \theta(\pi).$$

Setting this to 0, we realize that the assertion is true if having minimum phase and all poles and zeros on the real axis implies

$$\theta(0) = \theta(\pi).$$

A minimum phase filter with poles and zeros on the real axis will have all poles and zeros between -1 and 1. Factoring $H(e^{j\omega})$ to find poles and zeros, we notice that for a given pole term $\frac{1}{a-e^{j\omega}}$, the restriction $-1 < a < 1$ yields a phase contribution of 0 when $\omega = 0$ and a phase contribution of $-\pi$ when $\omega = \pi$. Likewise, a given zero term $b - e^{j\omega}$ gives a phase contribution of 0 when $\omega = 0$ and π when $\omega = \pi$ as long as $-1 < b < 1$. Since all poles and zeros of such a minimum phase filter meet these criteria, $\theta(0) = \theta(\pi) = 0$ (the phase contributions from the poles cancel with those from zeros). Therefore, $\int_0^\pi \tau(\omega)d\omega = 0$. (Note that $\int_0^\pi \tau(\omega)d\omega = 0$ for all real minimum phase filters, even if there are complex poles and zeros.)

5.24. Problem 1 in Fall2005 Midterm exam.

Problem

A stable system with system function $H(z)$ has the pole-zero diagram shown in Figure 1. It can be represented as the cascade of a stable minimum-phase system $H_{min}(z)$ and a stable all-pass system $H_{ap}(z)$.

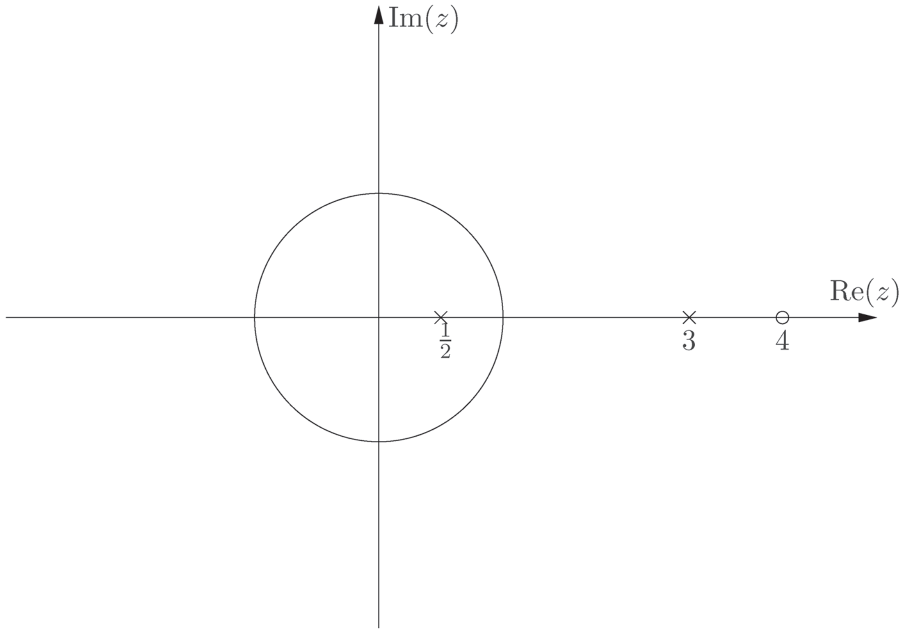


Figure 1: Pole-zero diagram for $H(z)$.

Determine a choice for $H_{min}(z)$ and $H_{ap}(z)$ (up to a scale factor) and draw their corresponding pole-zero plots. Indicate whether your decomposition is unique up to a scale factor.

Solution from Fall05 Midterm

We find a minimum-phase system $H_{min}(z)$ that has the same frequency response magnitude as $H(z)$ up to a scale factor. Poles and zeros that were outside the unit circle are moved to their conjugate reciprocal locations (3 to $\frac{1}{3}$, 4 to $\frac{1}{4}$, ∞ to 0).

$$H_{min}(z) = K_1 \frac{(1 - \frac{1}{4}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

There is no need to include an explicit z term to account for the zero at the origin.

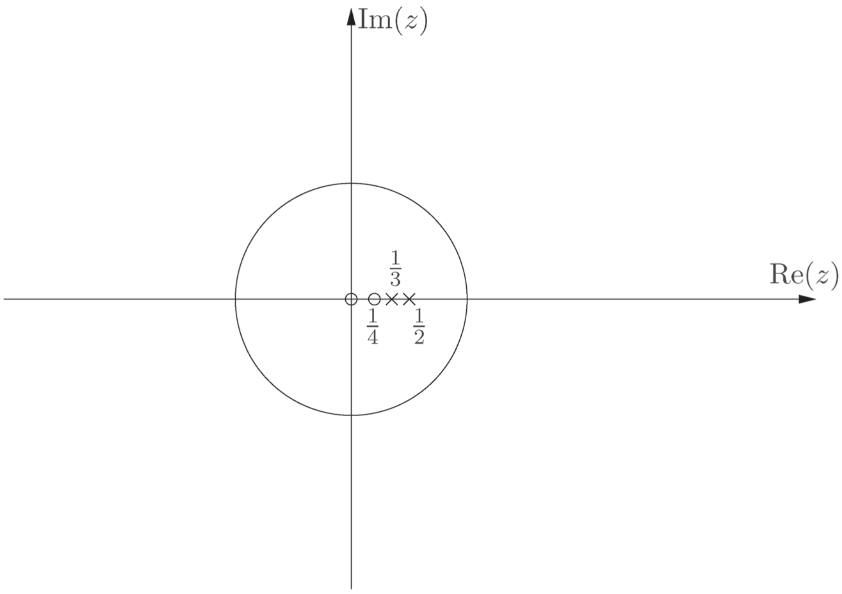


Figure 2: Pole-zero diagram for $H_{min}(z)$.

We now include all-pass terms in $H_{ap}(z)$ to move poles and zeros back to their original locations in $H(z)$. The term $\frac{z^{-1}-3}{1-3z^{-1}}$ moves the pole at $\frac{1}{3}$ back to 3, the term z^{-1} moves the zero from 0 to ∞, and so on:

$$H_{ap}(z) = z^{-1}\left(\frac{z^{-1} - 3}{1 - 3z^{-1}}\right)\left(\frac{z^{-1} - \frac{1}{4}}{1 - \frac{1}{4}z^{-1}}\right)$$

The decomposition is unique up to a scale factor. We cannot include additional all-pass terms in $H_{ap}(z)$, since it is not possible for $H_{min}(z)$ to cancel the resulting poles and zeros outside the unit circle.

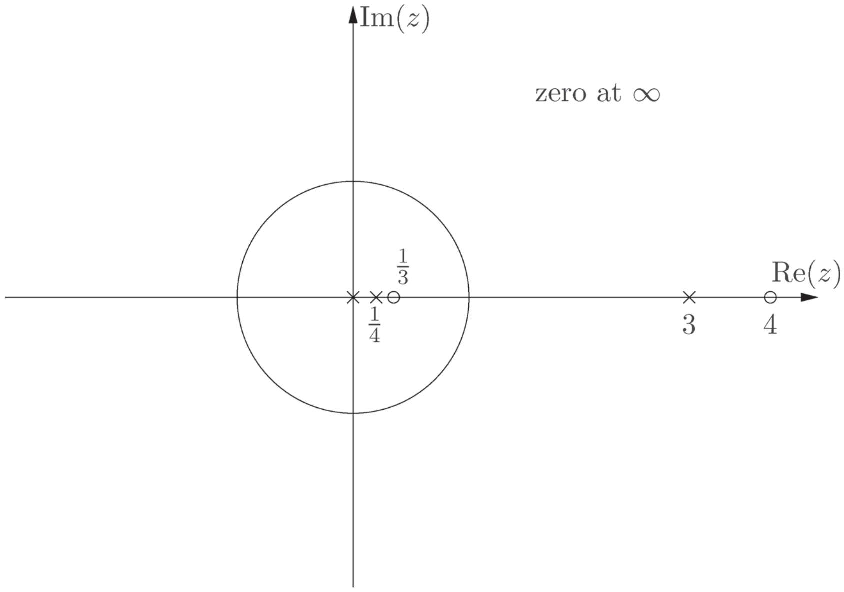
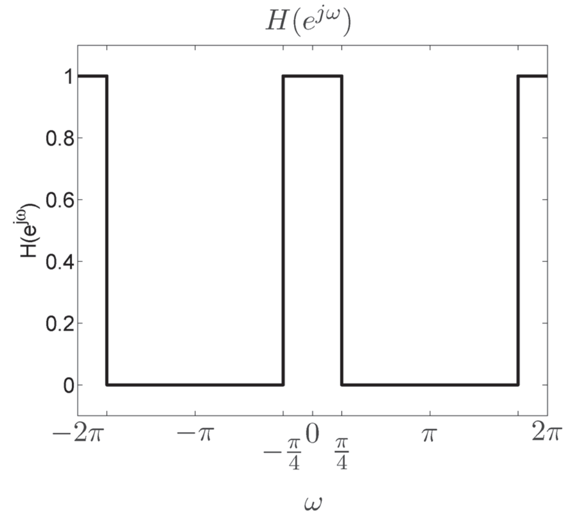


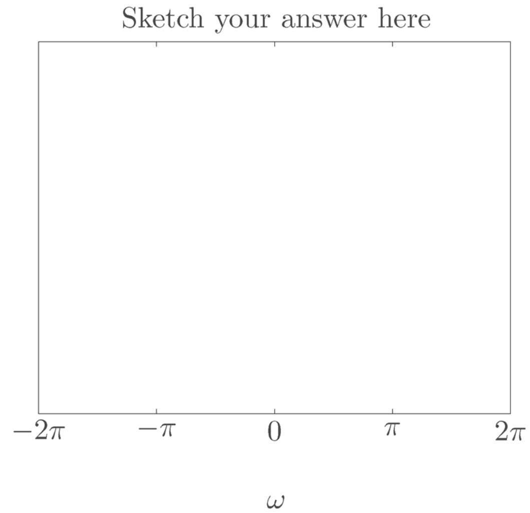
Figure 3: Pole-zero diagram for $H_{ap}(z)$.

5.25. Problem 1 in Spring2005 Final exam.

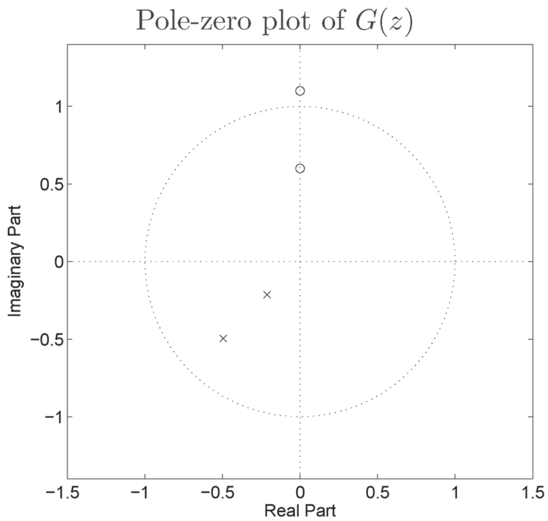
Problem

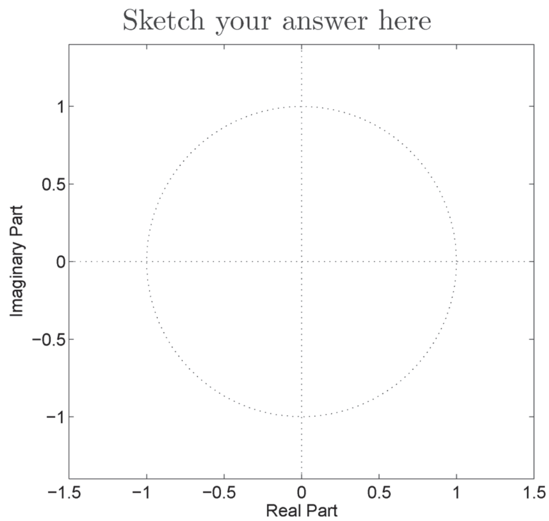
(a) An ideal lowpass filter $h[n]$ is designed with zero phase, a cutoff frequency of $\omega_c = \pi/4$, a passband gain of 1, and a stopband gain of 0. ($H(e^{j\omega})$ is shown below on the left.) Sketch the discrete-time Fourier transform of $(-1)^n h[n]$.





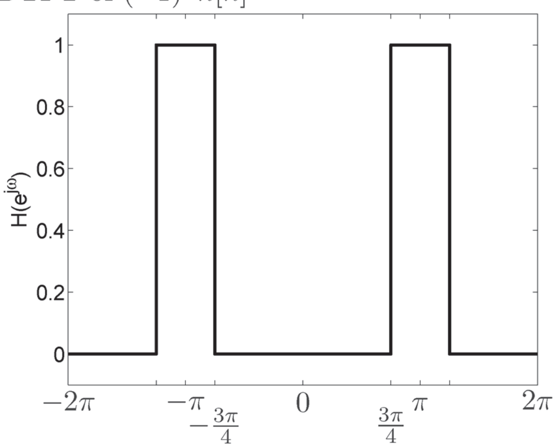
(b) A (complex) filter $g[n]$ has the pole-zero diagram shown below. Sketch the pole-zero diagram for $(-1)^n g[n]$. If there is not sufficient information provided, explain why.



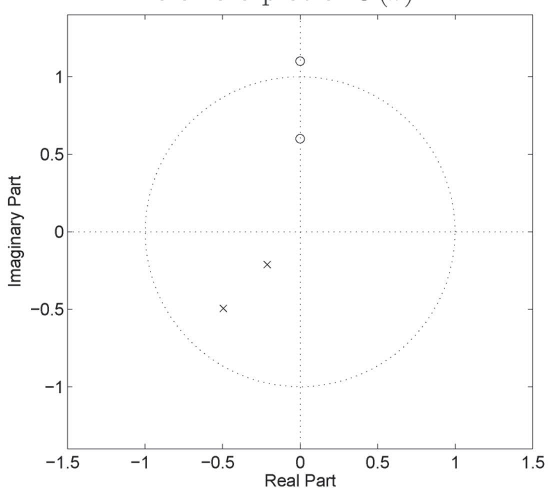


Solution from Spring2005 Final

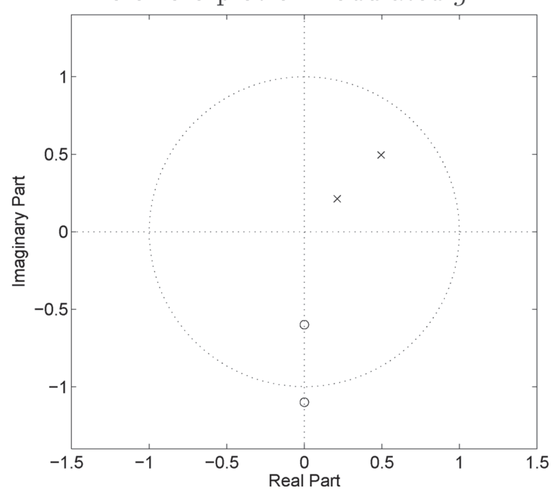
DTFT of $(-1)^n h[n]$



Pole-zero plot of $G(z)$



Pole-zero plot of modulated g



5.26. Problem 9 from Fall 2005 Background exam

Problem

Consider a discrete-time LTI system for which the frequency response $H(e^{j\omega})$ is described by:

$$H(e^{j\omega}) = -j, \qquad 0 < \omega < \pi$$
$$H(e^{j\omega}) = j, \qquad -\pi < \omega < 0$$

(a) Is the impulse response of the system $h[n]$ real-valued? (i.e. is $h[n] = h^*[n]$ for all n)

(b) Calculate the following:

$$\sum_{n=-\infty}^{\infty} |h[n]|^2$$

(c) Determine the response of the system to the input $x[n] = s[n]\cos(\omega_c n)$, where $0 < \omega_c < \frac{\pi}{2}$ and $S(e^{j\omega}) = 0$ for $\frac{\omega_c}{3} \leq |\omega| \leq \pi$.

Solution from Fall05 background exam

(a) $h[n]$ real-valued? $\boxed{\textbf{YES}}$ **NO**

(b) $\sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega = 1$

(c) Response of the system: $y[n] = s[n]\cos(\omega_c n - \frac{\pi}{2})$

5.27. Problem 12 from Fall 2005 Background exam

Problem

We process the signal $x[n] = \cos(\frac{\pi}{3}n)$ with a unity-gain all-pass LTI system, with frequency response $H(e^{j\omega})$ and a group delay of 4 samples at frequency $\frac{\pi}{3}$, to get the output $y[n]$. We also know that $\angle H(e^{j\frac{\pi}{3}}) = \theta$ and $\angle H(e^{-j\frac{\pi}{3}}) = -\theta$. Choose the most accurate statement:

A. $y[n] = \cos(\frac{\pi}{3}n + \theta)$

B. $y[n] = \cos(\frac{\pi}{3}(n-4) + \theta)$

C. $y[n] = \cos(\frac{\pi}{3}(n-4-\theta))$

D. $y[n] = \cos(\frac{\pi}{3}(n-4))$

E. $y[n] = \cos(\frac{\pi}{3}(n-4+\theta))$

Solution from Fall05 background exam

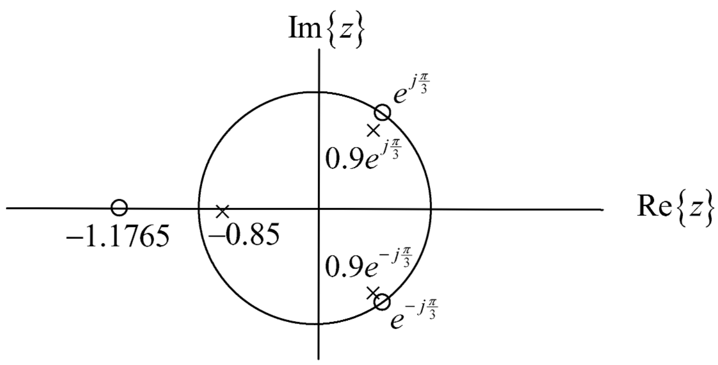
(Circle one) $\boxed{A}$ B C D E

5.28. A.

$$
\begin{aligned}
H(z) &= \frac{\left(1-e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+1.1765z^{-1}\right)}{\left(1-0.9e^{j\frac{\pi}{3}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{3}}z^{-1}\right)\left(1+0.85z^{-1}\right)} \\
&= \frac{1+0.1765z^{-1}-0.1765z^{-2}+1.1765z^{-3}}{1-0.05z^{-1}+0.045z^{-2}+0.6885z^{-3}} \\
&= \frac{Y(z)}{X(z)}.
\end{aligned}
$$

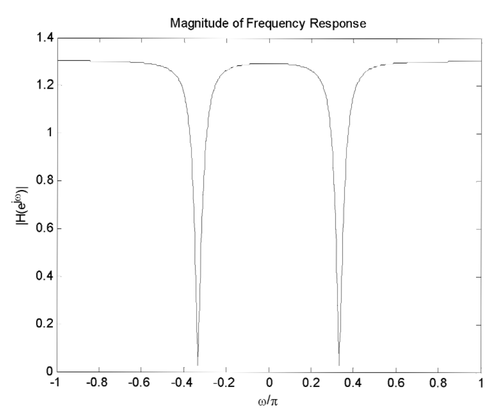
$$
\begin{aligned}
y[n] = {} & 0.05y[n-1]-0.45y[n-2]-0.6885y[n-3] \\
& + x[n]+0.1765x[n-1]-0.1765x[n-2]+1.1765x[n-3].
\end{aligned}
$$

B.



Since the system is causal, the ROC is the region outside the outermost pole. That is, $|z| > 0.9$.

C.



The zeros on the unit circle null the frequency response at $\omega = \pm\pi/3$. The sharpness of the nulls depend on how close the nearby poles are to the zeros. The factor $\dfrac{1+1.1765z^{-1}}{1+0.85z^{-1}} = 1.1765\dfrac{z^{-1}+0.85}{1+0.85z^{-1}}$ is allpass and does not affect the magnitude response.

D. 1. True. The system is stable because the ROC contains the unit circle.

2. False. The impulse response must approach zero for large n because the system is stable.

3. False. The system function has a zero on the unit circle at $\omega = \pi/3$. This negates the effect of the pole, and since the pole is not on the unit circle, the pole does not cancel the zero. Instead, the sharpness of the notch depends on how close the pole is to the zero.

4. False. There is a zero outside the unit circle.

5. False. The system is not a minimum-phase system so it does not have a causal and stable inverse.

5.29. A. For the inverse system $H(e^{j\omega})H_i(e^{j\omega}) = 1$. This means

$$H_i(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$
$$= \frac{1}{1+2e^{-j\omega}}.$$

The ROC of $H_i(e^{j\omega})$ must include the unit circle if the inverse system is to be stable. That is, $|z| < 2$. Taking the inverse Fourier transform,

$$h_i[n] = -(-2)^n u[-n-1].$$

The inverse system is not causal.

B. For this part

$$H_i(e^{j\omega}) = \frac{1}{1+\alpha e^{-j\omega}}.$$

For the inverse system to be causal and stable we require $|\alpha| < 1$. Then the ROC of a stable $H_i(e^{j\omega})$ will be $|z| > |\alpha|$, and the impulse response will be

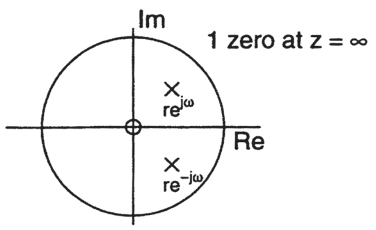
$$h_i[n] = (-\alpha)^n u[n]$$

corresponding to a causal system.

5.30. A. True. The poles and zeros of $H(z)=H_1(z)H_2(z)$ will be the combined set of poles and zeros of $H_1(z)$ and $H_2(z)$. If all of the poles and zeros of $H_1(z)$ are inside the unit circle and all of the poles and zeros of $H_2(z)$ are inside the unit circle, then all of the poles and zeros of $H(z)$ will be inside the unit circle.

B. False. For systems in parallel, $H(z)=H_1(z)+H_2(z)$. The poles of $H(z)$ will be inside the unit circle, but the zeros depend in a complicated way on both the poles and zeros of $H_1(z)$ and $H_2(z)$. Consequently, minimum phase cannot be guaranteed.

5.31. (a) A labeled pole-zero diagram appears below.



The table of common z-transform pairs gives us

$$(r^n \sin\omega_0 n)u[n] \longleftrightarrow \frac{(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, \quad |z|>r$$

which enables us to derive $h[n]$.

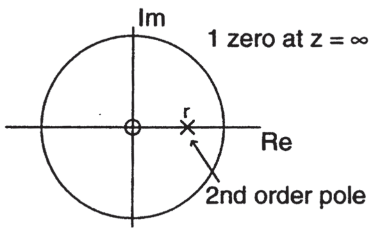
$$h[n] = \left(\frac{1}{\sin\omega_0}\right)(r^n \sin\omega_0 n)u[n]$$

(b) When $\omega_0 = 0$

$$H(z) = \frac{rz^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}} = \frac{rz^{-1}}{(1-rz^{-1})^2}, \quad |z|>r$$

Again, using a table lookup gives us

$$h[n] = nr^n u[n]$$



5.32. The given causal LTI system has system function

$$H(z) = 0.5 + 0.2z^{-1} - 0.3z^{-2} + cz^{-3} + 0.75z^{-4} - z^{-5}.$$

This system will be minimum phase if all of its zeros lie inside the unit circle. Note that it is equivalent to require the zeros of the function $2H(z)$ to lie inside the unit circle. Let us write

$$\begin{aligned} A^{(5)}(z) = 2H(z) &= 1 + 0.4z^{-1} - 0.6z^{-2} + 2cz^{-3} + 1.5z^{-4} - 2z^{-5} \\ &= 1 - \left(-0.5z^{-1} + 0.6z^{-2} - 2cz^{-3} - 1.5z^{-4} + 2z^{-5}\right). \end{aligned}$$

If we were to implement the system $2H(z)$ as a lattice, we would use the expression for $A^{(5)}(z)$ to find the *k*-parameters $k_1, k_2, \ldots, k_5$. A necessary and sufficient condition for the zeros of $A^{(5)}(z)$ to lie inside the unit circle is $|k_i| < 1, \quad i = 1, 2, \ldots, 5$. Thus we can use the lattice synthesis as a way of testing the given system for the minimum phase property.

The lattice synthesis begins with $k_5 = \alpha_5^{(5)} = 2$. We see immediately that $|k_5| \geq 1$. This implies that the given system cannot be minimum phase, regardless of the value of the parameter c.

5.33. Appears in: Fall05 PS2, Spring05 PS2, Spring04 PS2, Fall03 PS2, Spring03 PS2.

Problem

$H(z)$ is the system function for a stable LTI system and is given by:

$$H(z) = \frac{(1 - 2z^{-1})(1 - 0.75z^{-1})}{z^{-1}(1 - 0.5z^{-1})}.$$

(a) $H(z)$ can be represented as a cascade of a minimum phase system $H_1(z)$ and a unity-gain all-pass system $H_A(z)$, *i.e.*

$$H(z) = H_1(z)H_A(z).$$

Determine a choice for $H_1(z)$ and $H_A(z)$ and specify whether or not they are unique up to a scale factor.

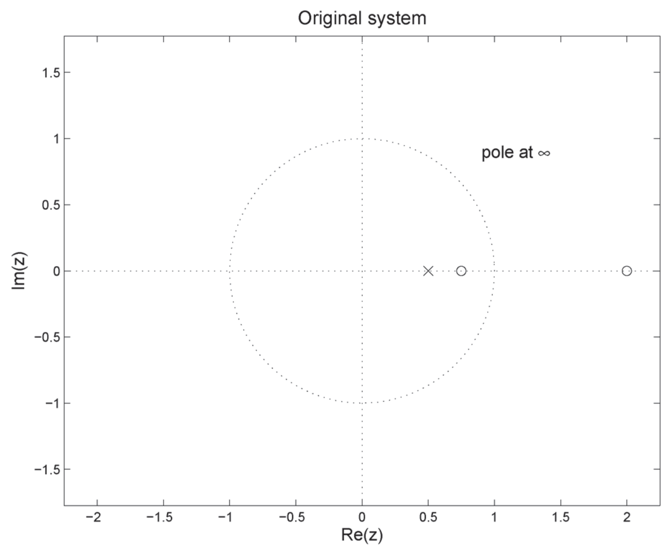
(b) $H(z)$ can be expressed as a cascade of a minimum-phase system $H_2(z)$ and a generalized linear phase FIR system $H_L(z)$:

$$H(z) = H_2(z)H_L(z).$$

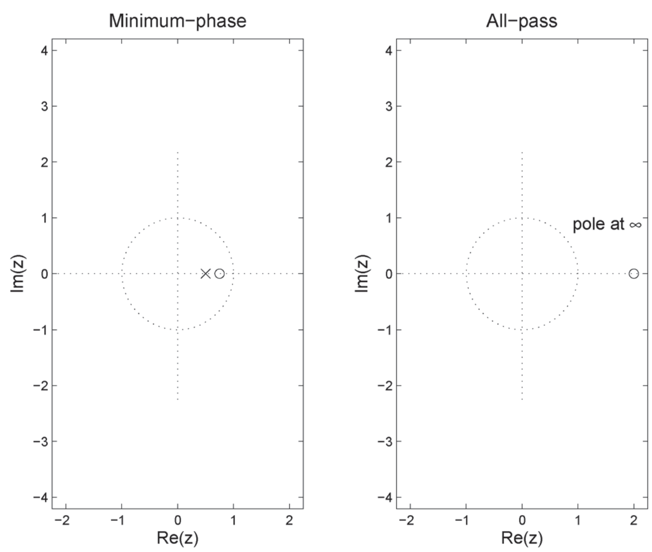
Determine a choice for $H_2(z)$ and $H_L(z)$ and specify whether or not these are unique up to a scale factor.

Solution from Fall05 PS2

The pole-zero diagram for the original system is as follows:



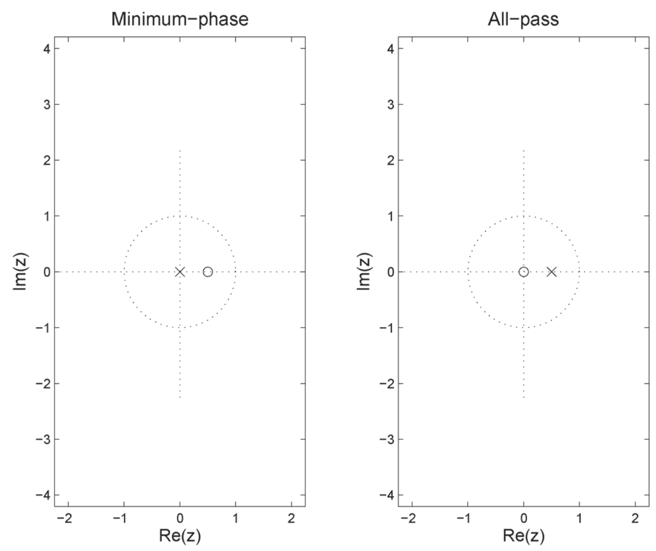
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



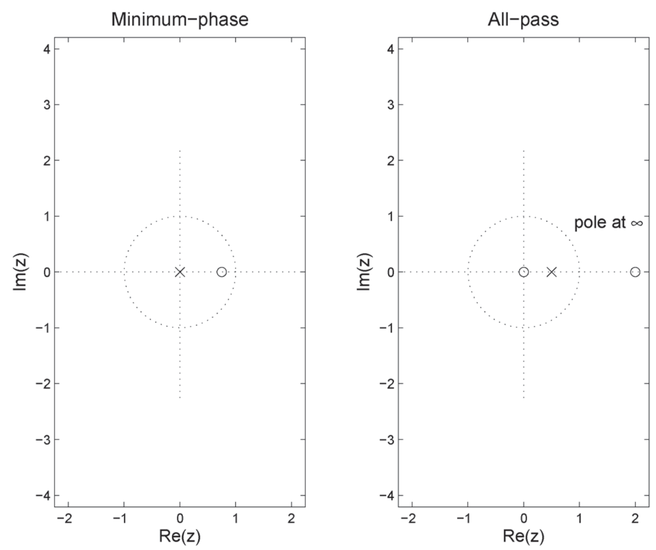
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

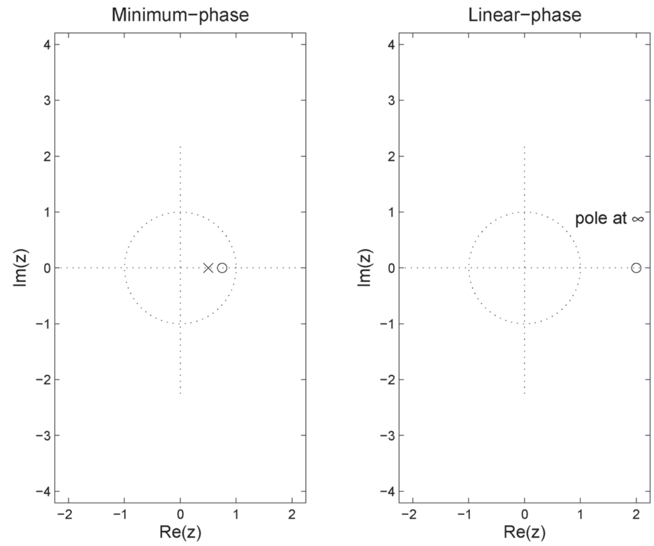
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

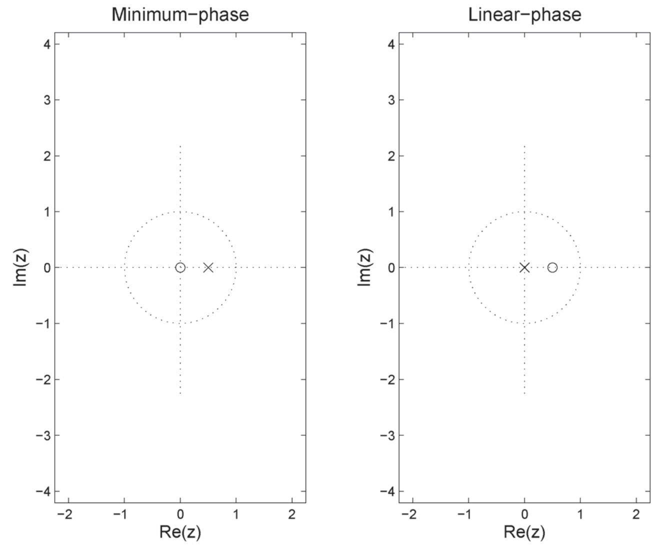
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique up to a scale factor.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

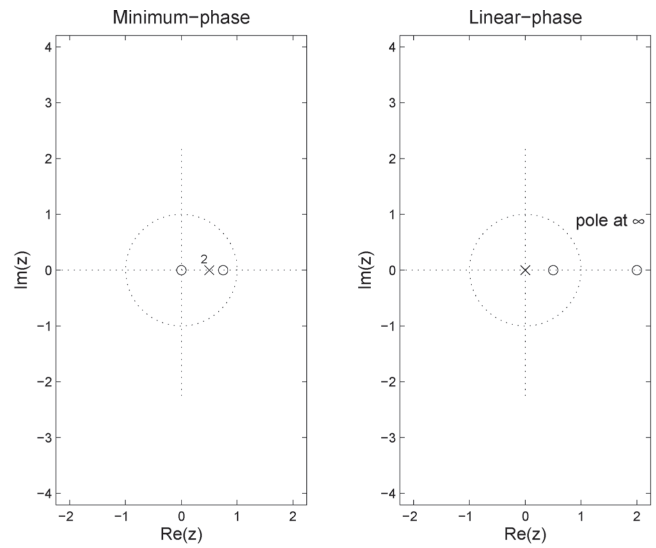


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$\begin{aligned} H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\ &= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

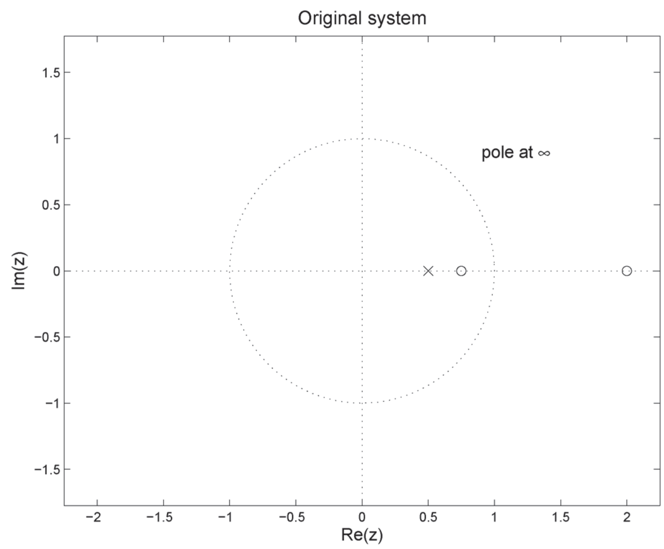
The FIR generalized linear-phase system function is:

$$\begin{aligned} H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\ &= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\ &= z-2.5+z^{-1} \end{aligned}$$

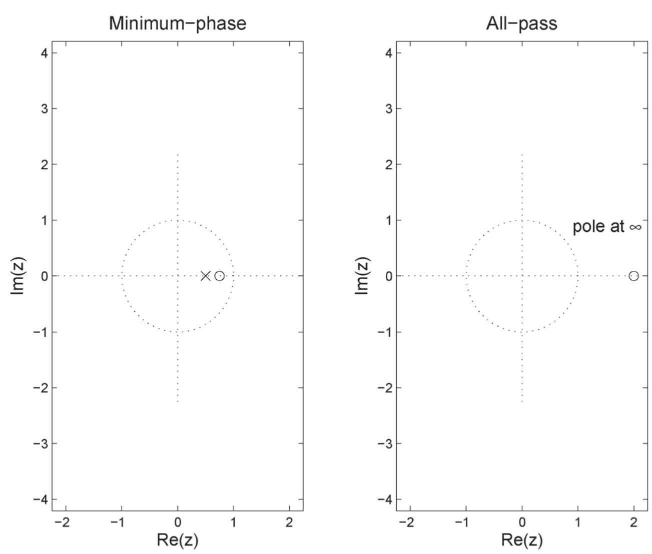
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=1$ or at $z=-1$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring05 PS2

The pole-zero diagram for the original system is as follows:



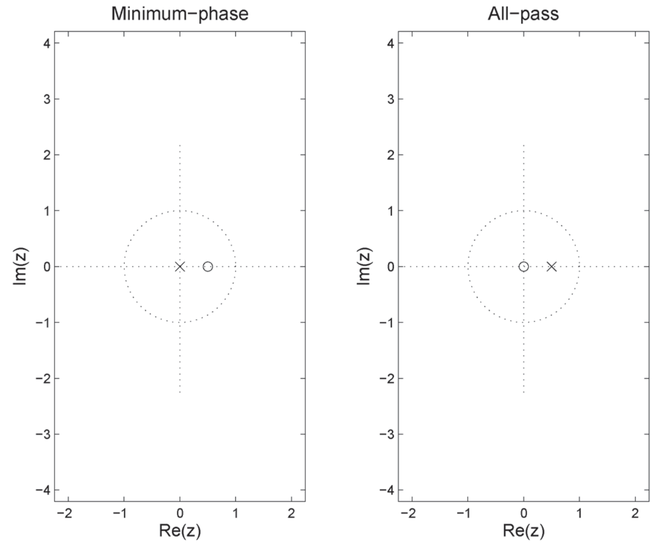
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



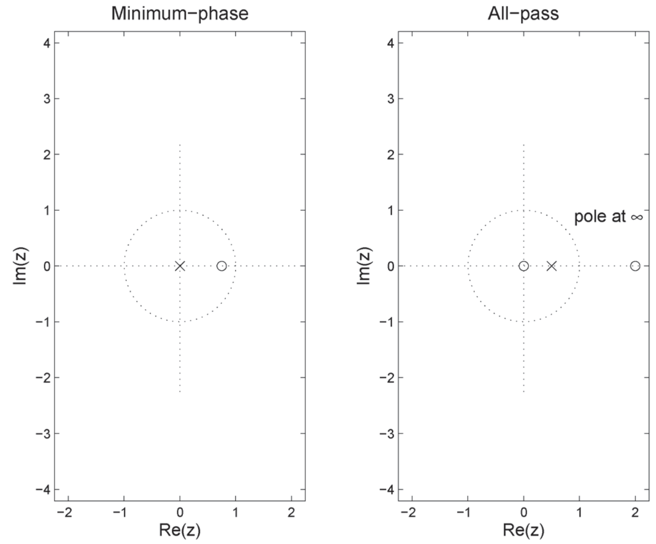
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

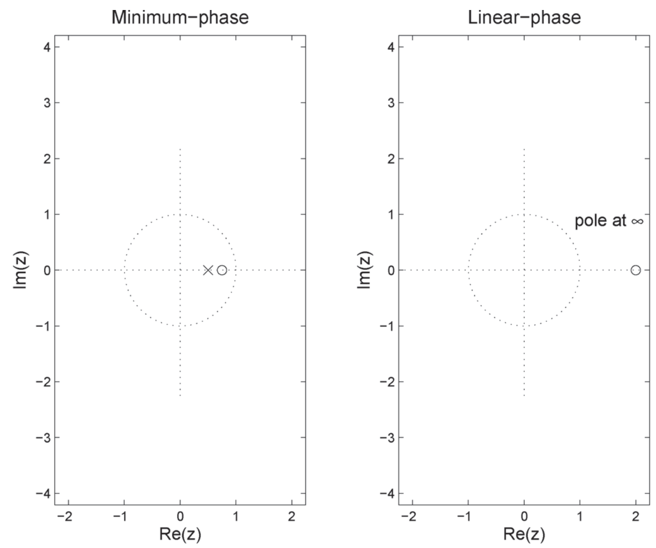
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

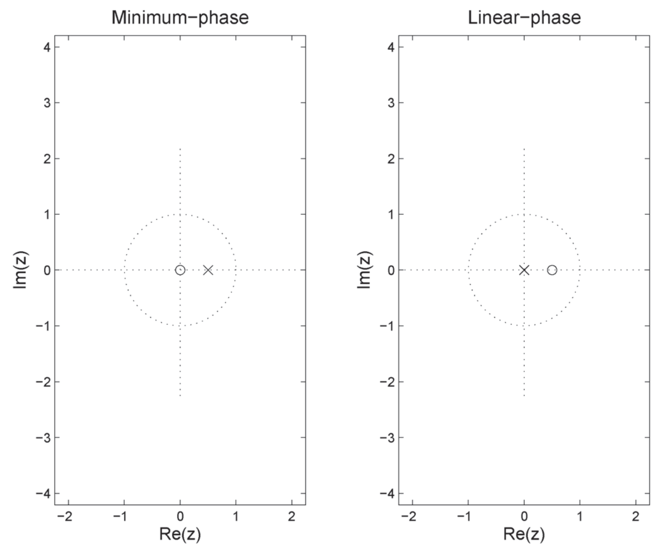
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

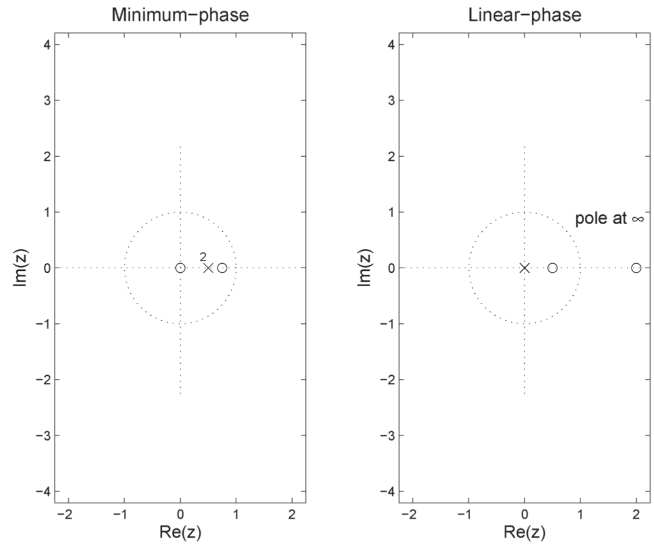


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$\begin{aligned} H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\ &= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

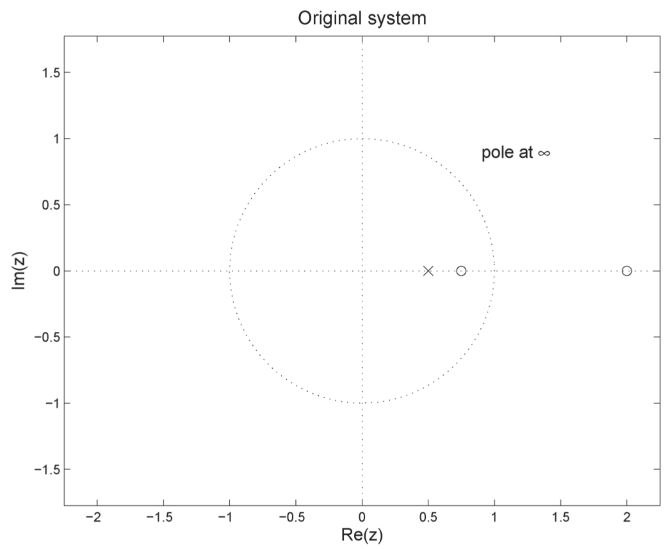
The FIR generalized linear-phase system function is:

$$\begin{aligned} H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\ &= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\ &= z-2.5+z^{-1} \end{aligned}$$

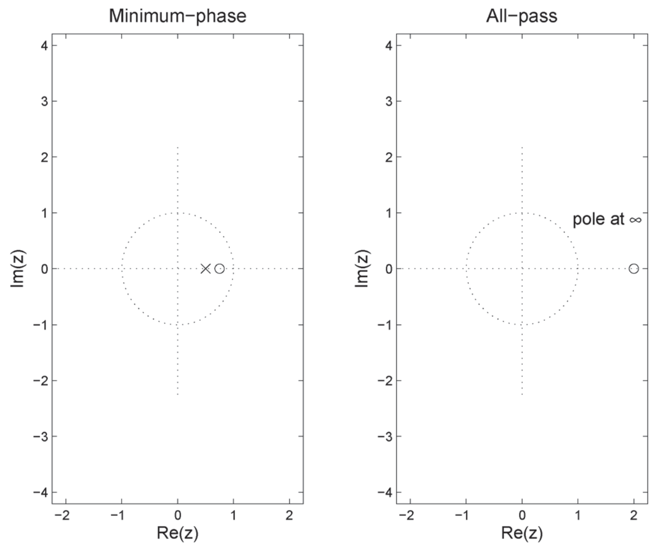
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring04 PS2

The pole-zero diagram for the original system is as follows:



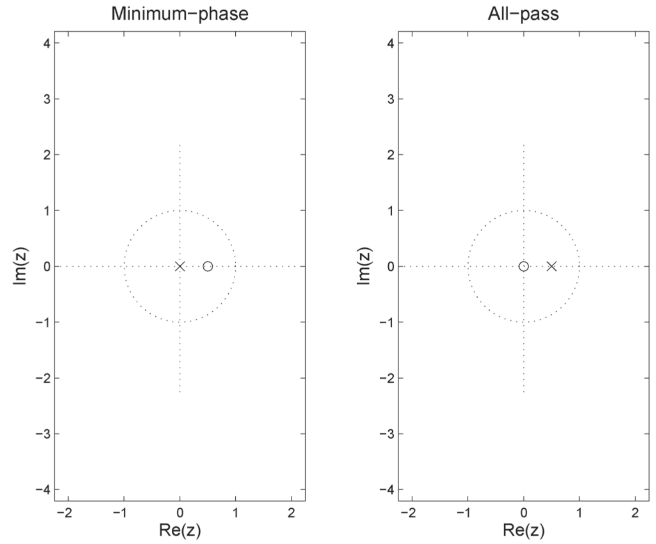
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



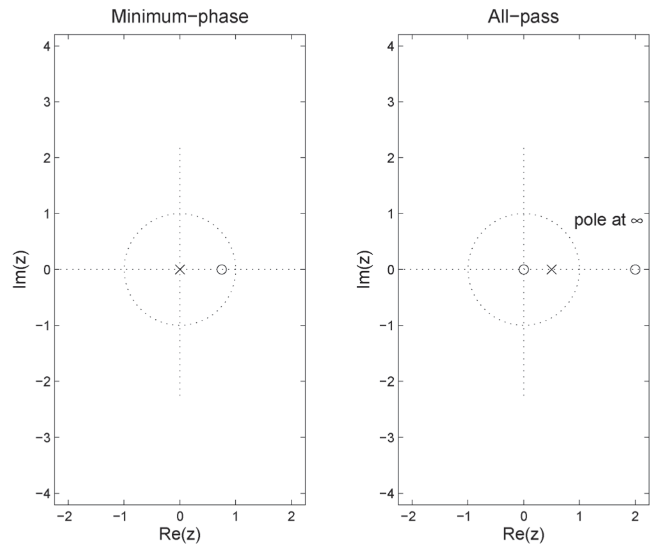
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$
\begin{aligned}
H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\
&= 1 - \frac{3}{4}z^{-1}
\end{aligned}
$$

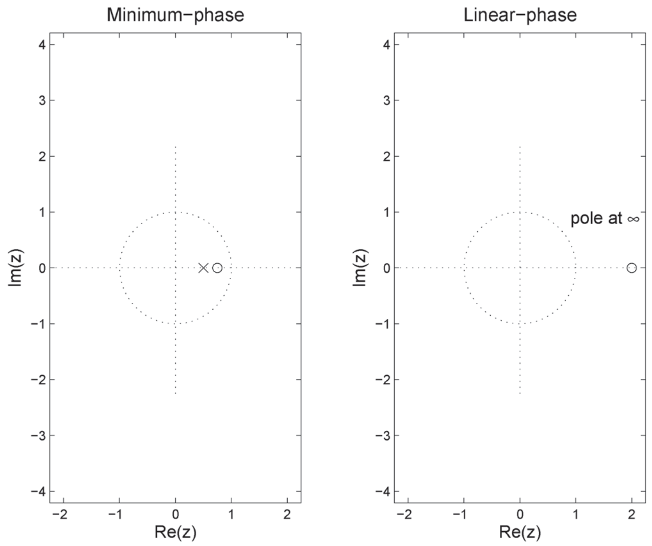
The all-pass system function is:

$$
\begin{aligned}
H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\
&= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}}
\end{aligned}
$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

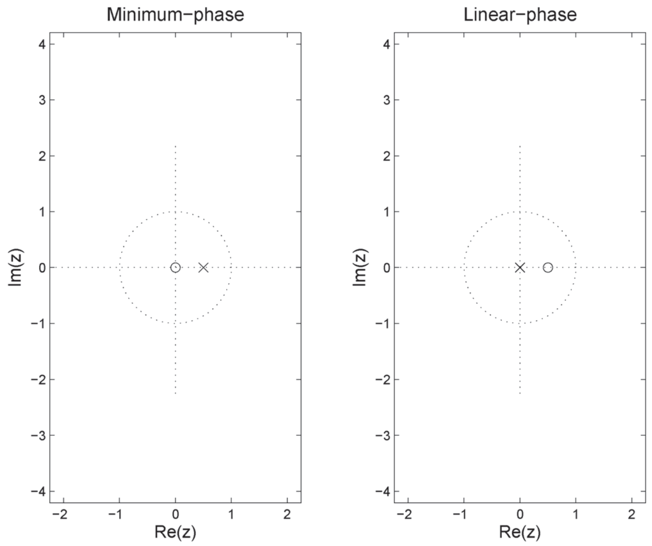
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

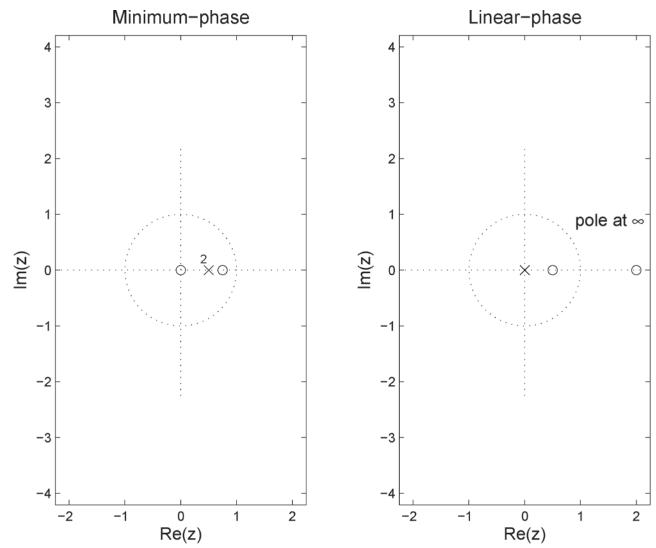


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$
\begin{aligned}
H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\
&= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}
\end{aligned}
$$

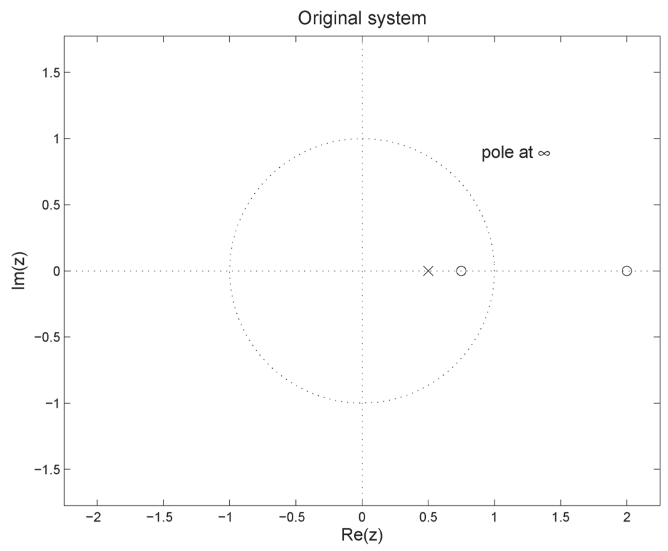
The FIR generalized linear-phase system function is:

$$
\begin{aligned}
H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\
&= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\
&= z-2.5+z^{-1}
\end{aligned}
$$

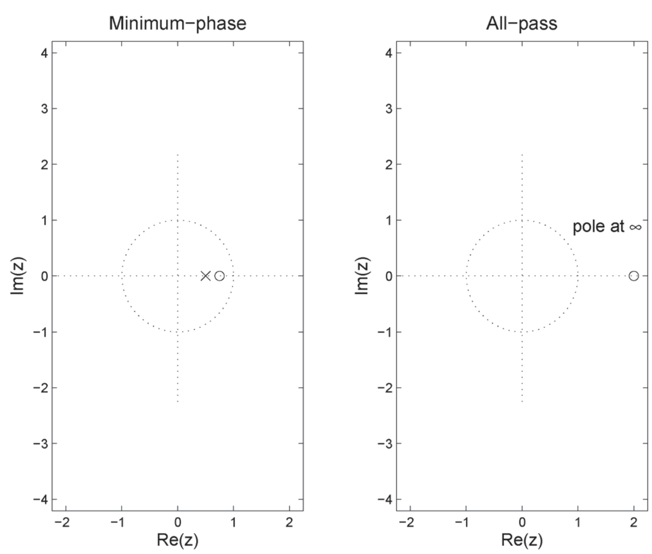
Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Fall03 PS2

The pole-zero diagram for the original system is as follows:



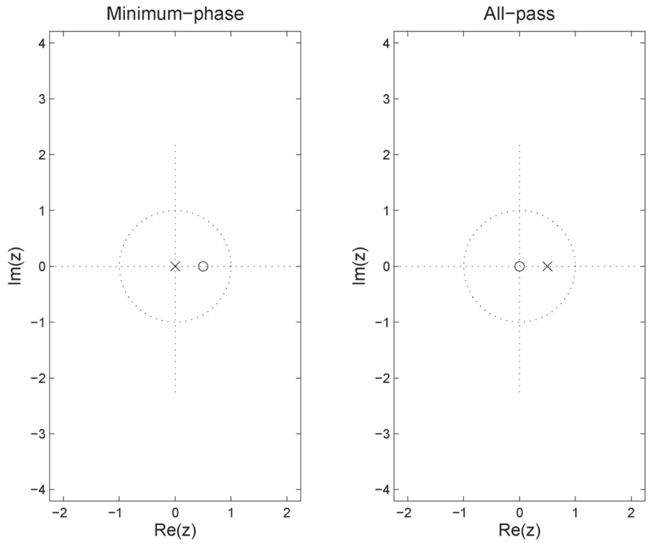
(a) One way to carry out the minimum-phase and all-pass decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the all-pass system.



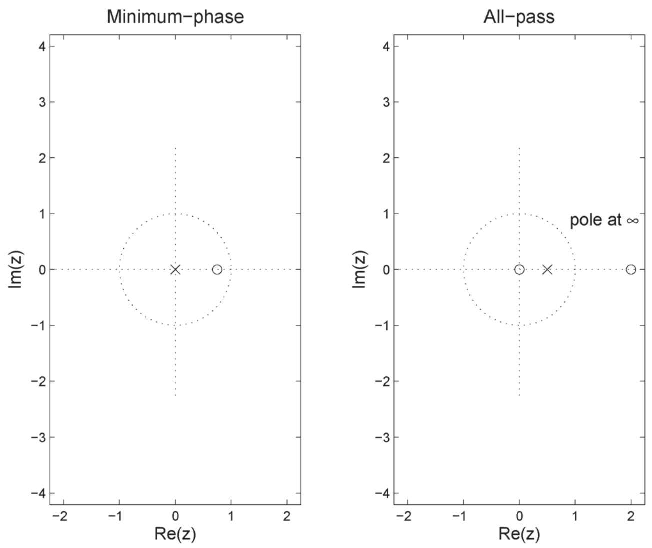
Next, we need to modify the first stage because we need to make sure that the all-pass

system really is all-pass, so add a pole at $z = 1/2$ and a zero at $z = 0$.

To preserve the original system, we can cancel these newcomers by placing a zero at $z = 1/2$ and a pole at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and all-pass systems are as shown below.



In the minimum-phase system, the pole at $z = 1/2$ from the first stage has been cancelled by the zero added in the second stage. Another way to look at that is that for this

particular system, we started with an all-pass pair (a pole at $z = 1/2$ and a zero at $z = 2$, so we could have put these into the all-pass system initially.

The minimum-phase system function is:

$$\begin{aligned} H_{M1}(z) &= \frac{z - \frac{3}{4}}{z} \\ &= 1 - \frac{3}{4}z^{-1} \end{aligned}$$

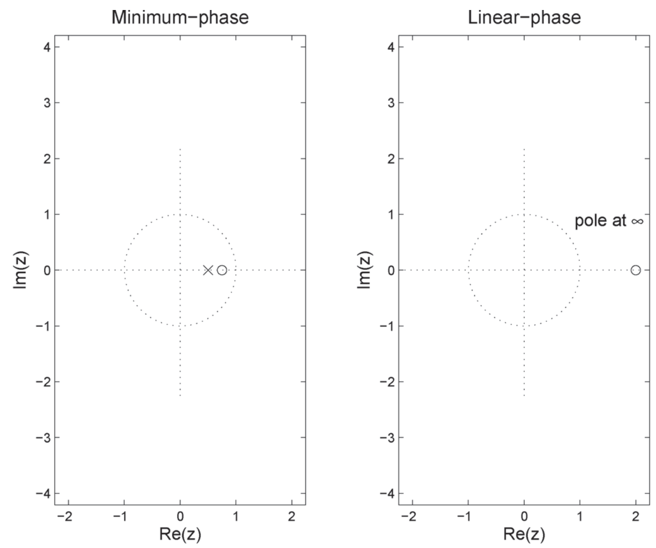
The all-pass system function is:

$$\begin{aligned} H_{ap}(z) &= \frac{z(z-2)}{z - \frac{1}{2}} \\ &= \frac{1 - 2z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right) z^{-1}} \end{aligned}$$

In constructing these systems, we didn't come across any decision where we could have chosen different routes. If we wanted to change one of the systems, we would have to add the same number of poles and zeros to it, and these would have to be cancelled by zeros and poles in the other system to preserve the original system.

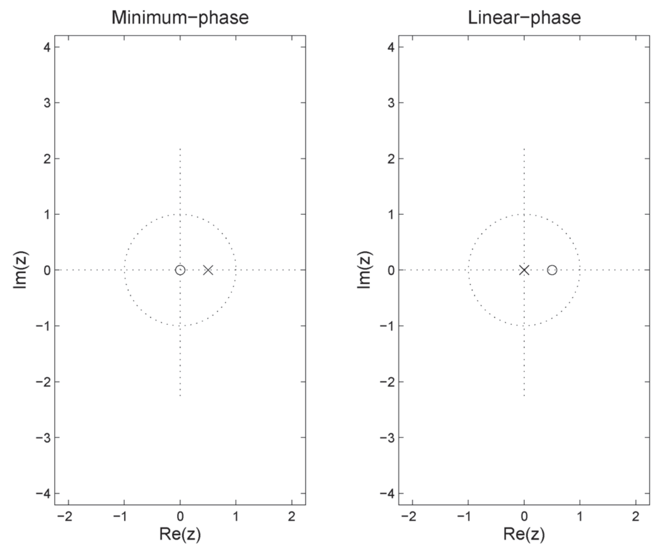
We can't add poles or zeros to the minimum phase system, because if we did, then when we added the cancelling zeros or poles to the all-pass system, they would have to be reflected outside the unit circle to keep the latter system all-pass. These items outside the unit circle could not be cancelled in the minimum phase system. Finally, we cannot change the all-pass system because if we added a zero and a pole, then to keep the system all-pass, we would have to reflect a pole or zero to the other side of the unit circle, and the items outside the unit circle could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

(b) One way to carry out the minimum-phase and FIR linear-phase decomposition is as follows. In the first stage, collect all zeros and poles that are inside the unit circle (zero at $z = 3/4$, pole at $z = 1/2$) into the minimum-phase system. The other zeros and poles (zero at $z = 2$, pole at $z = \infty$) go into the linear-phase system.

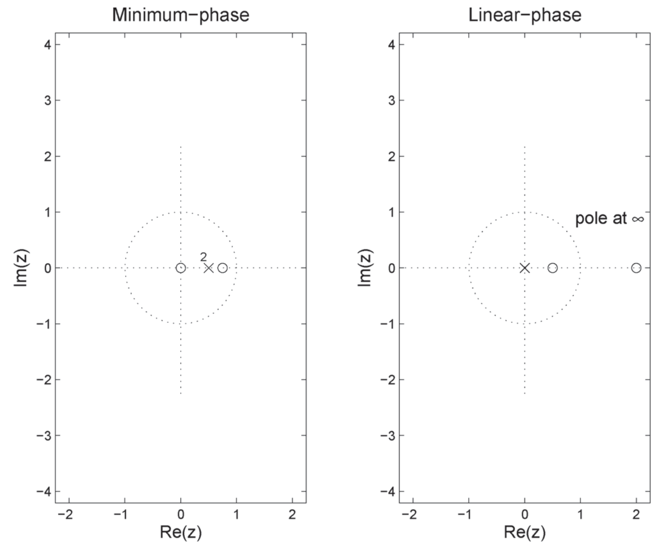


Next, we need to modify the first stage because we need to make sure that the linear-phase FIR system really is linear-phase FIR, so add a zero at $z = 1/2$. Since the system has to have the same number of zeros and poles, we also need to add a pole. For an FIR system, the pole must be at $z = 0$ or at $z = \infty$. We choose to add the pole at $z = 0$ because we will have to cancel the pole by a zero in the minimum-phase system.

To preserve the original system, we can cancel these newcomers by placing a pole at $z = 1/2$ and a zero at $z = 0$ in the minimum-phase system.



Combining these, the minimum-phase system and FIR linear-phase systems are as shown below.



The minimum phase system function is:

$$\begin{aligned} H_{M2}(z) &= \frac{z\left(z-\frac{3}{4}\right)}{\left(z-\frac{1}{2}\right)^2} \\ &= \frac{1-\frac{3}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \end{aligned}$$

The FIR generalized linear-phase system function is:

$$\begin{aligned} H_L(z) &= \frac{\left(z-\frac{1}{2}\right)(z-2)}{z} \\ &= z\left[\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right)\right] \\ &= z-2.5+z^{-1} \end{aligned}$$

Since this expression for $H_L(z)$ has even symmetry and an odd number of taps, we would not necessarily expect a zero at $z=0$ or at $z=\pi$, and this is consistent with the pole-zero diagram above. In constructing these systems, we didn't come across any decisions where we could have chosen different routes. Furthermore, we cannot change the minimum phase system. If we tried adding a pole and zero to it, these would have to be cancelled in the FIR linear phase system. But the zero in the linear-phase system would have to be reflected outside the unit circle to maintain linear-phase, and this could not be compensated for in the minimum-phase system. Similarly, we cannot add a pole and zero to the FIR linear-phase system because if we did, then to keep it linear-phase, we would have to reflect the zero outside the unit circle, and this could not be cancelled in the minimum-phase system. Therefore, the decomposition is unique.

Solution from Spring03 PS2

(a) All-pass: Poles at $z = 1/2$ and ∞, and zeros at $z = 0$ and 2.
Min-Phase: Pole at $z = 0$, and zero at $z = 3/4$.

$$H_A(z) = \frac{A(1-2z^{-1})}{z^{-1}(1-0.5z^{-1})} \qquad H_1(z) = B(1-0.75z^{-1})$$

$H_A(z)$ and $H_1(z)$ are unique up to a scale factor, since we can't add poles and zeros to the all-pass sytem, because one of them would be outside the unit circle; which the min-phase system can't cancel.

(b) FIR: Poles at $z = 0$ and ∞, and zeros at $z = 1/2$ and 2.
Min-Phase: Second order pole at $z = 1/2$, and zeros at $z = 0$ and $3/4$.

$$H_L(z) = Cz(1-2z^{-1})(1-0.5z^{-1}) \qquad H_2(z) = \frac{D(1-0.75z^{-1})}{(1-0.5z^{-1})^2}$$

Due to similar reasoning with part (a), $H_L(z)$ and $H_2(z)$ are unique up to a scale factor.

5.34. Appears in: Fall05 PS1, Spring05 PS1, Fall04 PS1, Fall02 PS1, Spring01 PS2. Note: Spring01 PS2 uses different plots than Fall04 and Fall02. The problem statement in Spring01 has also been modified for Fall02 and Fall04. The Spring01 version of the problem is included after the Fall04 and Fall02 version.

Problem

Filter A is a discrete-time LTI system with input $x[n]$ and output $y[n]$.

The frequency response magnitude and group delay functions for Filter A are shown in Figure 1. The signal $x[n]$, also shown in Figure 1, is the sum of three narrowband pulses. In particular, Figure 1 contains the following plots:

- $x[n]$.
- $|X(e^{j\omega})|$, the Fourier transform magnitude of $x[n]$.
- Frequency response magnitude plot for filter A.
- Group delay plot for filter A.

In Figure 2 you are given 4 possible output signals, $y_i[n]$ $i = 1, 2, \ldots, 4$. Determine which one of the possible output signals is the output of filter A when the input is $x[n]$. Provide a justification for your choice.

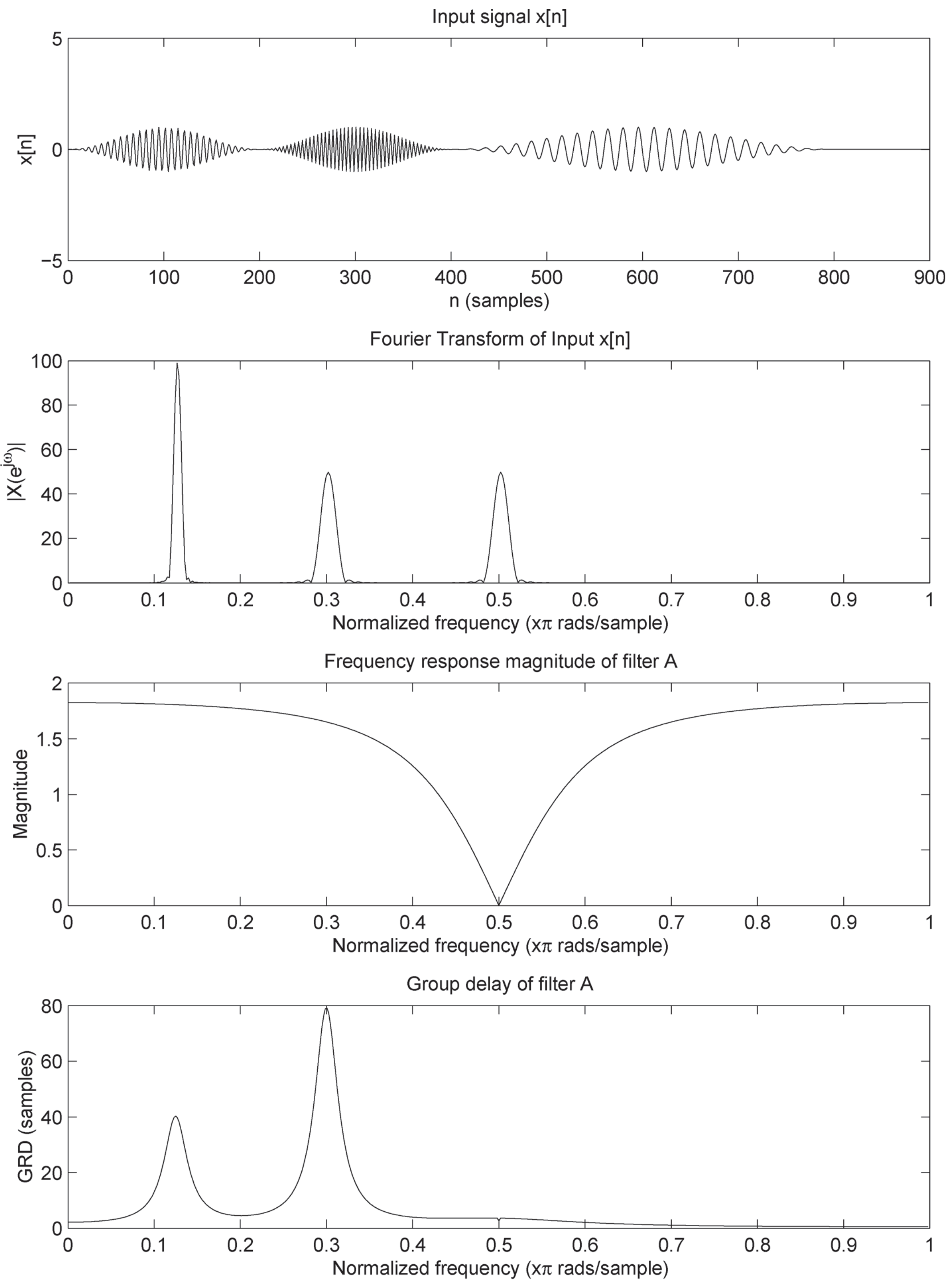


Figure 1: The signal and the filter for Problem

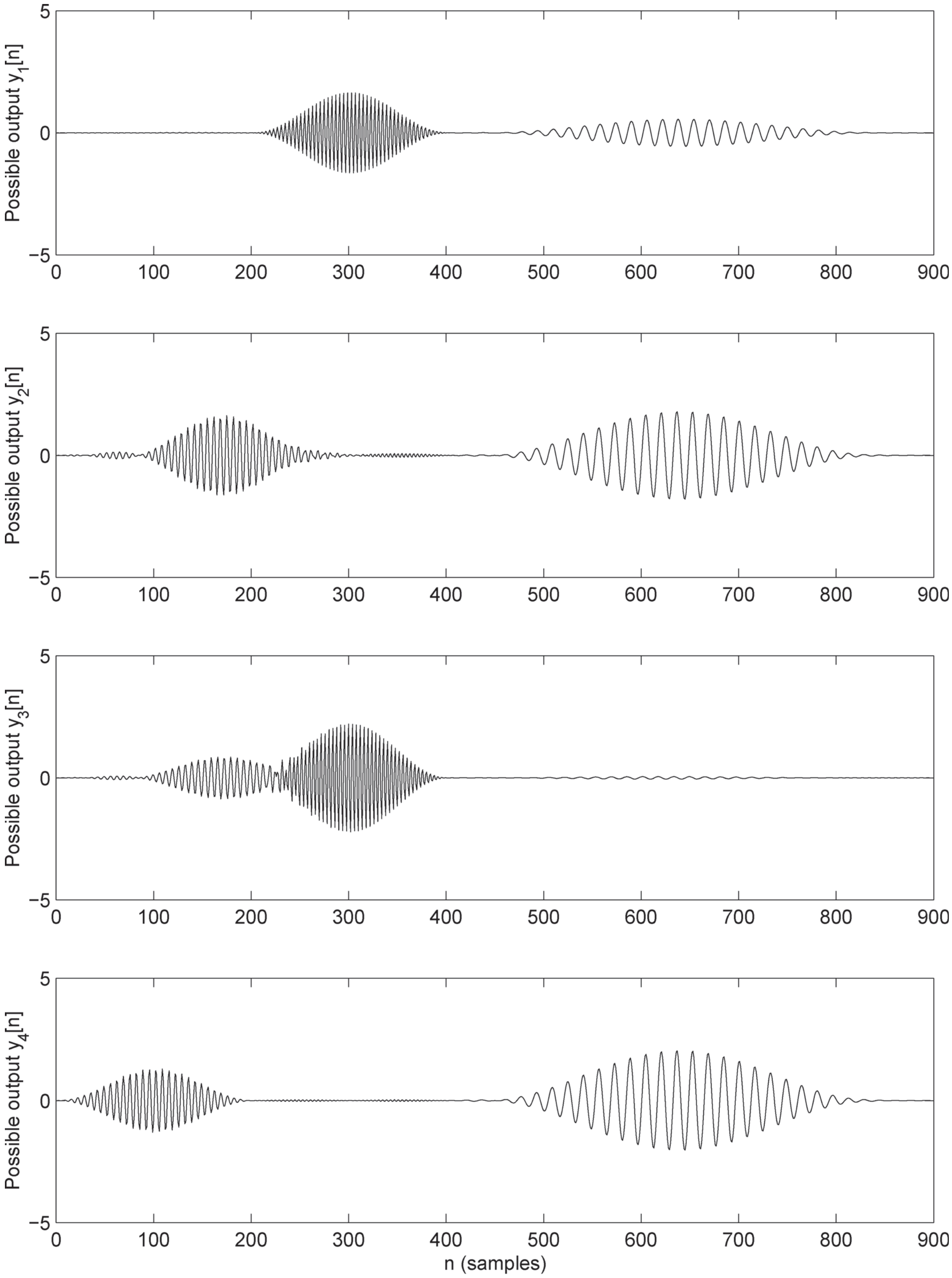


Figure 2: Possible output signals for Problem.

Spring01 Version of Problem Filter A is a discrete-time LTI system. Its frequency response magnitude and group delay functions are shown in Figure 2.6a. A signal, $x[n]$, also shown in Figure 2.6a, is the sum of three narrowband pulses which do not overlap in time. In Figures 2.6b and 2.6c you are given 8 possible output signals, $y_i[n]$ $i = 1, 2, \ldots, 8$. Determine which of the possible output signals is the output of filter A when the input is $x[n]$. **Clearly state your reasoning.**

Figure 2.6a contains the following plots:

- $x[n]$.
- $|X(e^{j\omega})|$, the Fourier transform magnitude of $x[n]$.
- Group delay plot for filter A.
- Frequency response magnitude plot for filter A.

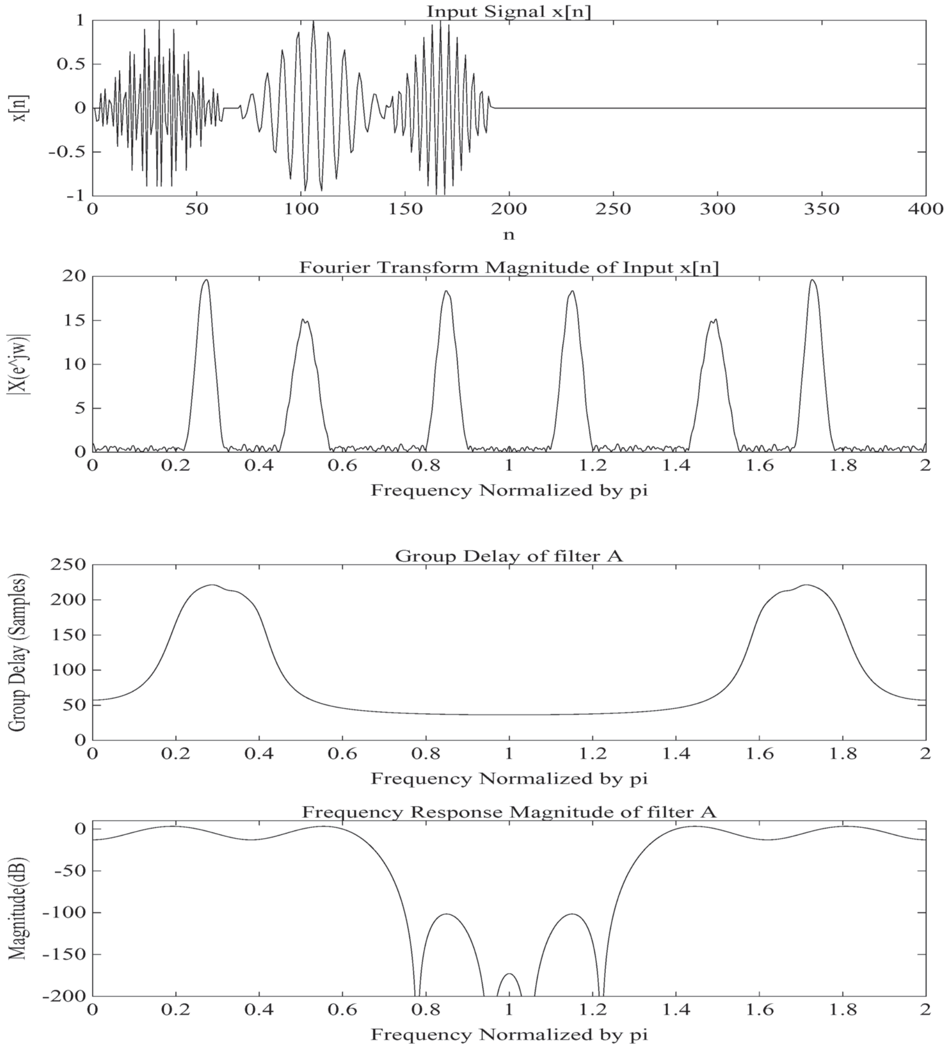


Figure 2.6a

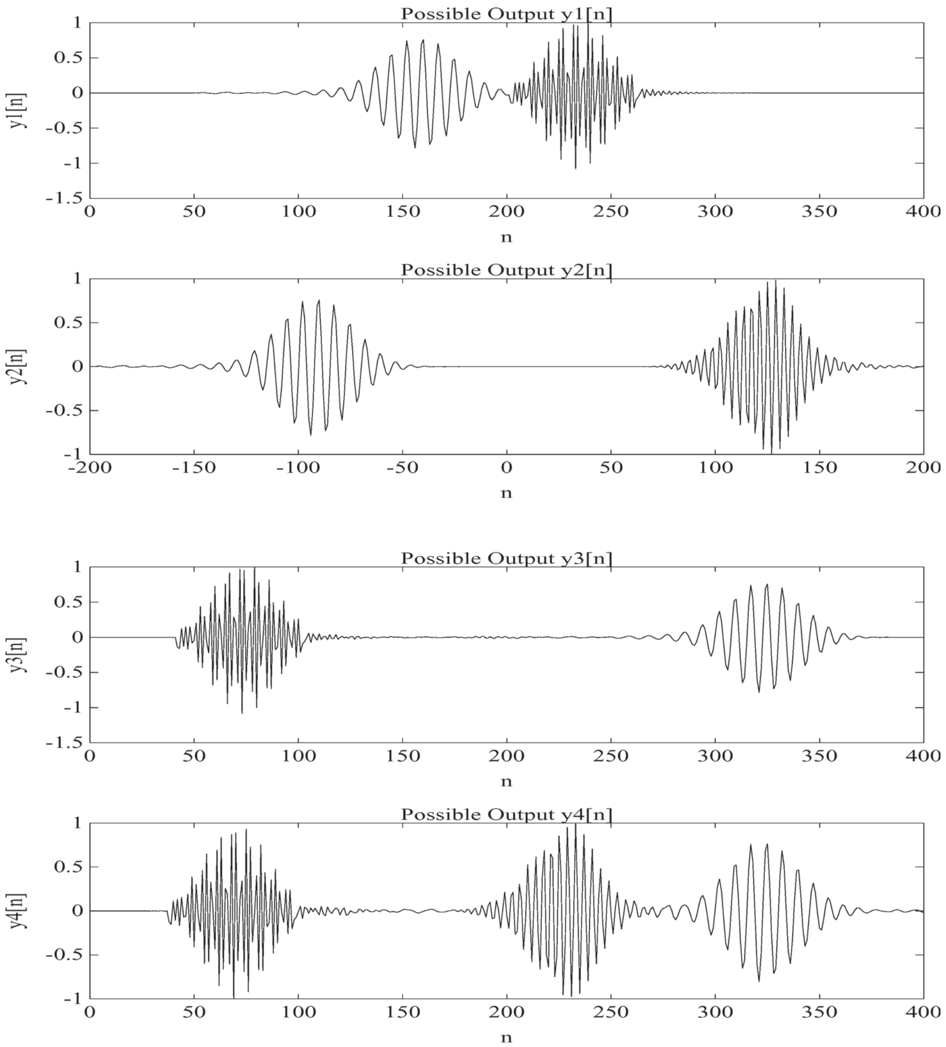

Possible Output y1[n]
y1[n]
n
Possible Output y2[n]
y2[n]
n
Possible Output y3[n]
y3[n]
n
Possible Output y4[n]
y4[n]
n

Figure 2.6b

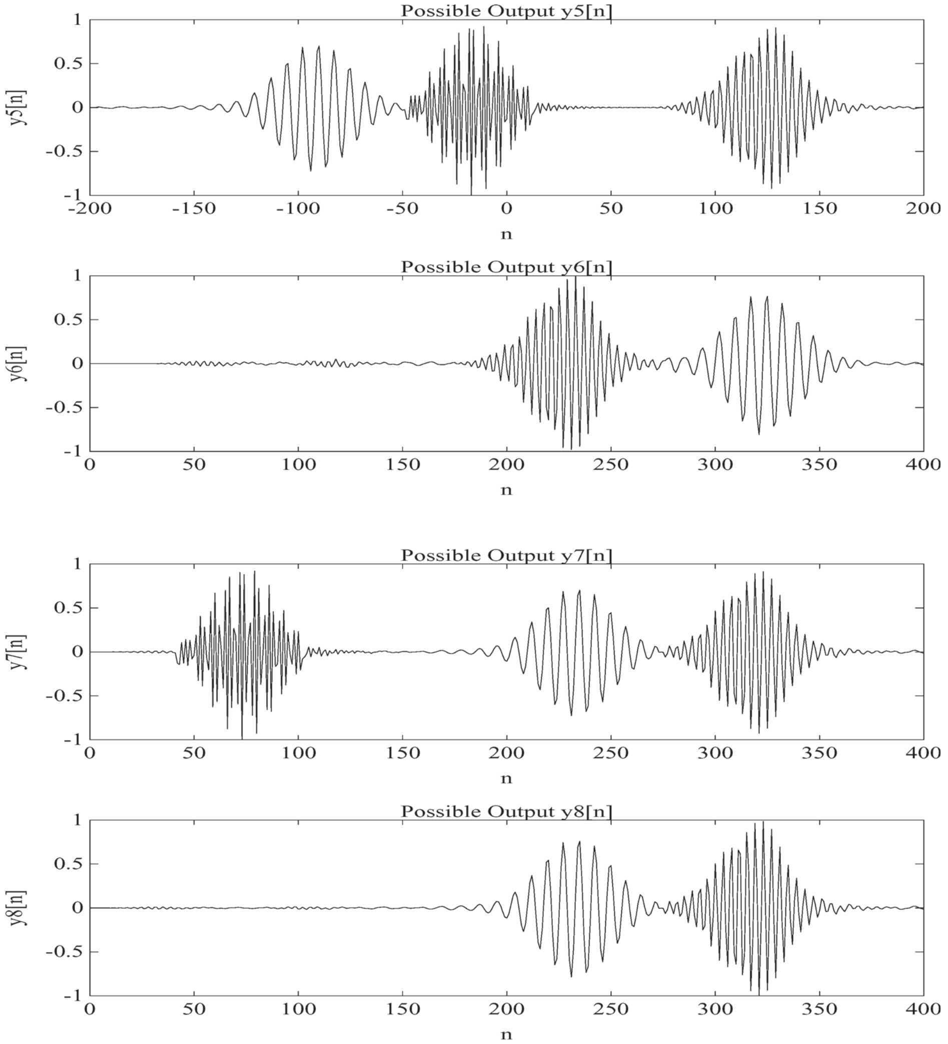

Possible Output y5[n]
y5[n]
n
Possible Output y6[n]
y6[n]
n
Possible Output y7[n]
y7[n]
n
Possible Output y8[n]
y8[n]
n

Figure 2.6c

Solution from Fall05 PS1

$y[n] = y_2[n]$

Justification:

The input signal $x[n]$ is made up of three narrow-band pulses: pulse-1 is a low-frequency pulse (whose peak is around 0.12π radians), pulse-2 is a higher-frequency pulse (0.3π radians), and pulse-3 is the highest-frequency pulse (0.5π radians).

Let $H(e^{j\omega})$ be the frequency response of Filter A. We read off the following values from the frequency response magnitude and group delay plots:

$$\begin{aligned}
|H(e^{j(0.12\pi)})| &\approx 1.8 \\
|H(e^{j(0.3\pi)})| &\approx 1.7 \\
|H(e^{j(0.5\pi)})| &\approx 0 \\
\tau_g(0.12\pi) &\approx 40 \text{ samples} \\
\tau_g(0.3\pi) &\approx 80 \text{ samples}
\end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. Pulse-1 will be scaled up by a factor of 1.8 and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.7 and its envelope delayed by about 80 samples. The correct output is thus $y_2[n]$.

Solution from Spring05 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse (whose peak is at $.12\pi$ radians) pulse-2 is a higher-frequency pulse (whose peak is at $.3\pi$ radians), and pulse-3 is the highest-frequency pulse (whose peak is at $.5\pi$ radians).

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned}
|H(e^{j(.12\pi)})| &\approx 1.8 \\
|H(e^{j(.3\pi)})| &\approx 1.75 \\
|H(e^{j(.5\pi)})| &\approx 0 \\
\tau_g(.12\pi) &\approx 40 \text{ samples} \\
\tau_g(.3\pi) &\approx 80 \text{ samples}
\end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Fall04 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse of frequency $.12\pi$ radians, pulse-2 is a higher-frequency pulse of frequency $.3\pi$ radians, and pulse-3 is the highest-frequency pulse of frequency $.5\pi$ radians.

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned} |H(e^{j(.12\pi)})| &\approx 1.8 \\ |H(e^{j(.3\pi)})| &\approx 1.75 \\ |H(e^{j(.5\pi)})| &\approx 0 \\ \tau_g(.12\pi) &\approx 40 \text{ samples} \\ \tau_g(.3\pi) &\approx 80 \text{ samples} \end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Fall02 PS1

$y[n] = y_2[n]$

Justification:

We see that the input signal $x[n]$ is made up of three narrow-band pulses; pulse-1 is a low-frequency pulse of frequency $.12\pi$ radians, pulse-2 is a higher-frequency pulse of frequency $.3\pi$ radians, and pulse-3 is the highest-frequency pulse of frequency $.5\pi$ radians.

From the given figure, we can also read off the following values of the filters frequency response magnitude and group delay. Call $H(e^{j\omega})$ the frequency response magnitude of Filter A. Then

$$\begin{aligned} |H(e^{j(.12\pi)})| &\approx 1.8 \\ |H(e^{j(.3\pi)})| &\approx 1.75 \\ |H(e^{j(.5\pi)})| &\approx 0 \\ \tau_g(.12\pi) &\approx 40 \text{ samples} \\ \tau_g(.3\pi) &\approx 80 \text{ samples} \end{aligned}$$

From these values, we would expect pulse-3 to be totally absent from the output signal $y[n]$. We would expect pulse-1 to be scaled up by a factor of 1.8, and its envelope delayed by about 40 samples. Pulse-2 will be scaled up by a factor of 1.75, with its envelope delayed by about 80 samples. The output which corresponds to this is $y_2[n]$.

Solution from Spring01 PS2

N/A

5.35. Problem 5 from Spring 2005 final exam
Appears in: Fall05 PS3.

Problem

Suppose a discrete-time filter has group delay $\tau(\omega)$. Does $\tau(\omega) > 0$ for all $\omega \in (-\pi, \pi)$ imply that the filter is necessarily causal? Clearly explain your reasoning.

Solution from Fall05 PS3

No, it is not necessarily causal. As a counterexample consider the case of a non-integer delay such as $H(e^{j\omega}) = e^{-j\frac{\omega}{2}}$, where $\tau(\omega) = \frac{1}{2}$ for all ω but the corresponding impulse response $h[n]$ is certainly noncausal.

Solution from Spring05 Final

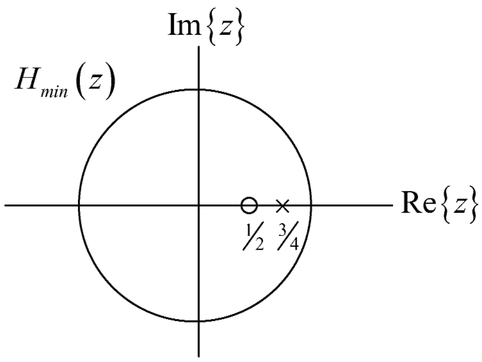
Suppose a discrete-time filter has group delay $\tau(\omega)$. Does $\tau(\omega) > 0$ for all $\omega \in (-\pi, \pi)$ imply that the filter is necessarily causal? Clearly explain your reasoning.

No. Consider the noncausal filter $h_a[n] = \delta[n-2] + \delta[n+2]$. Because it is real and even-symmetric, its discrete-time Fourier transform is real. It therefore has an unwrapped phase $\theta_a(\omega) = 0$ for all $\omega \in (-\pi, \pi)$. Delaying $h_a[n]$ by one sample, we have the noncausal filter $h_b[n] = \delta[n-3] + \delta[n+1]$ with unwrapped phase $\theta_b(\omega) = -\omega$ and group delay $\tau_b(\omega) = 1 > 0$ for all $\omega \in (-\pi, \pi)$).

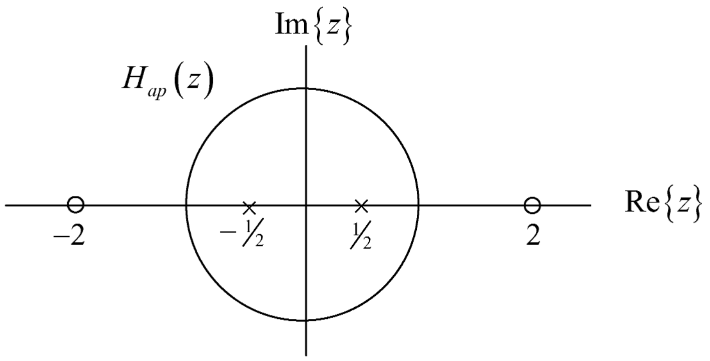
In general, any real, noncausal filter $h[n]$ which has even symmetry about some $n > 0$ will satisfy $\tau(\omega) > 0$ for all $\omega \in (-\pi, \pi)$.

5.36.

$$
\begin{aligned}
H(z) &= \frac{\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} \\
&= \frac{\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{3}{4}z^{-1}\right)} \times \frac{\left(1-2z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} \\
&= H_{min}(z)H_{ap}(z).
\end{aligned}
$$



ROC: $|z| > 3/4$



ROC: $|z| > 1/2$

The regions of convergence reflect the requirement that both $H_{min}(z)$ and $H_{ap}(z)$ must be stable.

5.37. Problem 1 in Spring2004 midterm exam.

Problem

System S is causal and linear phase with z-transform $H(z) = h_1 + h_2 z^{-1} + h_3 z^{-2}$. The impulse response has unit energy, $h_1 \geq 0$ and $H(e^{j\pi}) = H(e^{j0}) = 0$.

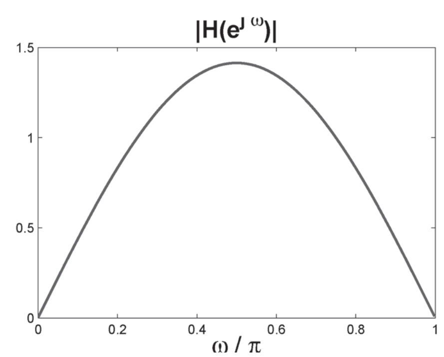
(a) Find the impulse response $h[n]$.

(b) Draw $|H(e^{j\omega})|$.

Solution from Spring04 midterm

Part (a): $h[n] = \frac{1}{\sqrt{2}}\delta[n] - \frac{1}{\sqrt{2}}\delta[n-2]$.

Part (b): $|H(e^{j\omega})| = \sqrt{1 - cos(2\omega)} = \sqrt{2}\,|\sin(\omega)|$.



5.38. Problem 1 in spring2003 midterm exam.

Problem

$H(z)$ is the system function for a stable LTI system and is given by:

$$H(z) = \frac{(1 - 9z^{-2})(1 + \frac{1}{3}z^{-1})}{1 - \frac{1}{3}z^{-1}}$$

(a) $H(z)$ can be represented as a cascade of a min-phase system $H_{min}(z)$ and a unity-gain all-pass system $H_A(z)$. Determine a choice for $H_{min}(z)$ and $H_A(z)$ and specify whether or not they are unique up to a scale factor.

(b) Is the min-phase system, $H_{min}(z)$, an FIR system? Explain.

(c) Is the min-phase system, $H_{min}(z)$, a generalized linear phase system? If not, can $H(z)$ be represented as a cascade of a generalized linear-phase system $H_{lp}(z)$ and an all-pass system $H_{A2}(z)$? If your answer is yes, determine $H_{lp}(z)$ and $H_{A2}(z)$. If your answer is no, explain why such representation does not exist.

Solution from Spring03 midterm

(a) To find the poles and zeros of $H(z)$, rewrite it as

$$H(z) = \frac{\left(z^2 - 9\right)\left(z + \frac{1}{3}\right)}{z^2\left(z - \frac{1}{3}\right)}$$

Zeros: $z = 3,\ -3,\ -1/3$

Poles: $z = 0,\ 0,\ 1/3$

The zeros at 3 and -3 can't be in the minimum phase system, so they must go in the all-pass system. In order to make the latter all-pass, it must also have poles at $1/3$ and $-1/3$. Since these were not part of $H(z)$, they must be cancelled by zeros in the minimum phase system. Inserting the zero at $1/3$ in the minimum phase system cancels the pole that was there.

$$\begin{aligned} H_{min}(z) &= \frac{1}{K}\left(1 + \frac{1}{3}z^{-1}\right)^2 \\ H_A(z) &= K\frac{\left(1 - 3z^{-1}\right)\left(1 + 3z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right)} \end{aligned}$$

The product of these two functions is the original $H(z)$ given in the problem. Since we want the all-pass system to have unity gain, $|H_A(z)| = 1$ for any z on the unit circle, e.g. $z = 1$. This yields $|K| = 1/9$.

Decompositions into minimum phase and all-pass systems are unique up to a scale factor.

(b) Yes, $H_{min}(z)$ is FIR. All its poles are at the origin.

(c) The phase of $H_{min}\left(e^{j\omega}\right)$ is

$$-\arctan\left(\frac{-\frac{2}{3}\sin(\omega) - \frac{1}{9}\sin(2\omega)}{1 + \frac{2}{3}\cos(\omega) + \frac{1}{9}\cos(2\omega)}\right)$$

This is not a linear or affine function of ω. However, we can rewrite $H(z)$ as the product of the following two systems:

$$\begin{aligned} H_{lp}(z) &= \left(1 + 3z^{-1}\right)\left(1 + \frac{1}{3}z^{-1}\right) \\ H_{A2}(z) &= \frac{1 - 3z^{-1}}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

This is equivalent to $H_{lp}(z) = 1 + (10/3)z^{-1} + z^{-2}$. The impulse response has even symmetry and the system is linear phase.

5.39. A. We are given

$$H(z) = \frac{z-2}{z\left(z-\frac{1}{3}\right)},$$

and the system is stable.

This system has poles at the origin and at $z = \frac{1}{3}$. Since the poles are inside the unit circle, stability requires the ROC to be $\frac{1}{3} < |z|$. This ROC corresponds to a causal system.

B. We wish to write $H(z)$ as $H(z) = H_M(z)H_L(z)$, where $H_M(z)$ is minimum phase and $H_L(z)$ has generalized linear phase. Let us begin by factoring $H(z)$ into

$$H(z) = \frac{1}{z-\frac{1}{3}}\frac{z-2}{z}.$$

The first factor has a pole inside the unit circle at $z = \frac{1}{3}$, but it has a zero outside the unit circle at $z = \infty$. We can move the zero if we multiply and divide by z, producing

$$H(z) = \frac{z}{z-\frac{1}{3}}\frac{z-2}{z^2}.$$

Now the first factor is minimum phase, with a pole at $z = \frac{1}{3}$ and a zero at the origin. The second factor is not linear phase, however. This factor has a zero at $z = 2$, and it needs an additional zero at $z = \frac{1}{2}$ to be a generalized linear phase function. We can add this additional zero if we add a corresponding pole. That is,

$$H(z) = \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}\frac{\left(z-\frac{1}{2}\right)(z-2)}{z^3}.$$

We now have $H(z) = H_M(z)H_L(z)$ as required, with

$$H_M(z) = \frac{z^2}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} = \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)},$$

and

$$H_L(z) = \frac{\left(z-\frac{1}{2}\right)(z-2)}{z^3} = z^{-1}\left(1-\frac{1}{2}z^{-1}\right)\left(1-2z^{-1}\right) = z^{-1}\left(1-\frac{5}{2}z^{-1}+z^{-2}\right).$$

5.40. Problem 5 from Spring 2004 Background exam

Note: Part (c) is problematic because there are non-rational causal IIR filters with linear phase (I can't recall the authors of a paper published about this)

System S_1 has a real impulse response $h_1[n]$ and a **real** frequency response $H_1(e^{j\omega})$.

(a) Does the impulse response have any symmetry properties? Explain.

(b) System S_2 is a linear phase system with the same magnitude response as system S_1. What is the relationship betwee $h_2[n]$, the impulse response of system S_2, and $h_1[n]$?

(c) Can a causal IIR filter have a linear phase? Explain. If your answer is yes, provide an example.

Solution from Spring 2004 background exam

Part(a): $h_1[n]$ is even.

Part(b): When the group delay is an integer, $\tau = n_0$, we have the following relation:

$$h_2[n] = h_1[n - n_0]$$

Part(c): There is no rational IIR filter that is causal with linear phase

5.41. The inverse Fourier transform gives

$$h[n]=\frac{1}{2\pi}\int_{(2\pi)} H(e^{j\omega})e^{j\omega n}d\omega.$$

Then

$$h[0]=\frac{1}{2\pi}\int_{(2\pi)} H(e^{j\omega})d\omega$$

$$=\frac{1}{2\pi}4\pi=2.$$

Next, the fact that $H(e^{j\omega})=A(e^{j\omega})e^{-j2\omega}$ with $A(e^{j\omega})$ real tells us that the given filter is a generalized linear-phase filter. Further, the fact that $M=4$ (five consecutive nonzero values) makes this a filter of Type I or Type III. Since $A(e^{j0})$ and $A(e^{j\pi})$ are non-zero, the system is of Type I. Therefore $H(e^{j\omega})$ has the form

$$H(e^{j\omega})=2+h_1e^{-j\omega}+h_2e^{-j2\omega}+h_1e^{-j3\omega}+2e^{-j4\omega}$$

$$=\left(h_2+2h_1\cos(\omega)+4\cos(2\omega)\right)e^{-j2\omega},$$

which gives $A(e^{j\omega})$ as

$$A(e^{j\omega})=h_2+2h_1\cos(\omega)+4\cos(2\omega).$$

Now

$$A(e^{j0})=h_2+2h_1+4=8$$

and

$$A(e^{j\pi})=h_2-2h_1+4=12.$$

Solving gives $h_1=-1$ and $h_2=6$. The required impulse response is

$$h[0]=2,\quad h[1]=-1,\quad h[2]=6,\quad h[3]=-1,\quad h[4]=2.$$

5.42. A. From the pole-zero diagram,

$$H(z) = K\frac{(z-2)}{z(z-\frac{1}{2})(z+3)}$$
$$= K\frac{z^{-2}(1-2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}.$$

We are given that the system is BIBO stable, and this implies that the ROC must include the unit circle. We therefore have the ROC $\frac{1}{2} < |z| < 3$.

Since the ROC includes the unit circle, the system has a frequency response given by

$$H(e^{j\omega}) = K\frac{e^{-j2\omega}(1-2e^{-j\omega})}{(1-\frac{1}{2}e^{-j\omega})(1+3e^{-j\omega})}.$$

The frequency reponse is the Fourier transform of the impulse response. That is,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}.$$

At $\omega = \pi$ this is

$$H(e^{j\pi}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\pi n} = \sum_{n=-\infty}^{\infty} (-1)^n h[n].$$

Now

$$H(e^{j\pi}) = K\frac{e^{-j2\pi}(1-2e^{-j\pi})}{(1-\frac{1}{2}e^{-j\pi})(1+3e^{-j\pi})}$$
$$= K\frac{(1+2)}{(1+\frac{1}{2})(1-3)}$$
$$= -K$$
$$= -1.$$

Therefore $K = 1$ and $H(z)$ is given by

$$H(z) = \frac{z^{-2}(1-2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}.$$

B. Now $g[n] = h[n+n_0]$. That is,

$$G(z) = z^{n_0}H(z)$$
$$= \frac{z^{n_0-2}(1-2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+3z^{-1})}$$
$$= \frac{z^{n_0}(z-2)}{z(z-\frac{1}{2})(z-3)}.$$

We will have $G(z)\big|_{z=0} = 0$ if $n_0 \geq 2$. In addition, we will have $\lim_{z\to\infty} G(z) < \infty$ only if $n_0 \leq 2$. Therefore $n_0 = 2$.

Applying the inverse Fourier transform,

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})e^{j\omega n}d\omega.$$

Then

$$\begin{aligned} g[0] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-2e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1+3e^{-j\omega}\right)}d\omega \\ &= -\frac{3}{7}. \end{aligned}$$

C. Let $f[n] = h[n] * h[-n]$. Then $F(z) = H(z)H(1/z)$. That is,

$$\begin{aligned} F(z) &= \frac{z^{-2}\left(1-2z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)} \frac{z^{2}\left(1-2z\right)}{\left(1-\frac{1}{2}z\right)\left(1+3z\right)} \\ &= \frac{4}{3}\frac{z}{(z+3)\left(z+\frac{1}{3}\right)} \\ &= \frac{4}{3}\frac{z^{-1}}{\left(1+3z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}. \end{aligned}$$

Since $H(z)$ has ROC $\frac{1}{2} < |z| < 3$, then $H(1/z)$ has ROC $\frac{1}{3} < |z| < 2$. The ROC of $F(z)$ must include the intersection of these two regions. Therefore $F(z)$ has ROC $\frac{1}{3} < |z| < 3$.

D. We want $e[n] * h[n] = u[n]$. Then $E(z)H(z) = \frac{1}{1-z^{-1}}$. That is,

$$\begin{aligned} E(z) &= \frac{1}{H(z)}\frac{1}{1-z^{-1}} \\ &= \frac{\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}{z^{-2}\left(1-2z^{-1}\right)\left(1-z^{-1}\right)} \\ &= \frac{z^{2}\left(1-\frac{1}{2}z^{-1}\right)\left(1+3z^{-1}\right)}{\left(1-2z^{-1}\right)\left(1-z^{-1}\right)}. \end{aligned}$$

The ROC of $\frac{1}{H(z)}$ can be $|z| < 2$ or $2 < |z|$. We will choose the latter to obtain a right-sided sequence. Then the ROC of $E(z)$ is also $2 < |z|$. Thus there is a right-sided sequence with the desired property.

The factor of z^2 in the numerator of $E(z)$ determines that $e[n]$ will take non-zero values at $n = -1$ and $n = -2$. Thus $e[n]$ is a right-sided sequence, but not a causal sequence.

5.43. A. Given

$$H(z) = \frac{z^{-2}\left(1-2z^{-1}\right)}{2\left(1-\frac{1}{2}z^{-1}\right)}, \quad |z| > \tfrac{1}{2},$$

we have

$$H\left(e^{j\omega}\right) = \frac{e^{-j2\omega}\left(1-2e^{-j\omega}\right)}{2\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$= -e^{-j3\omega}\frac{\left(1-\frac{1}{2}e^{j\omega}\right)}{\left(1-\frac{1}{2}e^{-j\omega}\right)}.$$

Now $1-\frac{1}{2}e^{j\omega}$ and $1-\frac{1}{2}e^{-j\omega}$ are complex conjugates, and therefore have the same magnitude. Further, $\left|-e^{-j3\omega}\right| = 1$. We conclude that $H(z)$ is an all-pass system.

B. We can write

$$H(z) = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}z^{-1}\right)}\left(1-2z^{-1}\right)z^{-2}, \quad |z| > \tfrac{1}{2}.$$

Then $H_{\min}(z) = \dfrac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}}$, $|z| > \frac{1}{2}$; $H_{\max}(z) = 1-2z^{-1}$; and $H_d(z) = z^{-2}$.

Inverse transforming gives $h_{\min}[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$, $h_{\max}[n] = \delta(n) - 2\delta[n-1]$, and $h_d[n] = \delta[n-2]$.

(Note that the factor of $\frac{1}{2}$ can be alternatively placed in the maximum-phase term.)

5.44. Assume that each impulse response corresponds to at most one frequency response.

A. $h_1[n]$ is a Type I FIR filter. Frequency reponse C corresponds to a Type I filter, as $\left|H_C\left(e^{j\omega}\right)\right| \neq 0$ for $\omega = 0$ or $\omega = \pi$.

B. $h_2[n]$ is a Type II FIR filter. Frequency response B corresponds to a Type II filter, as $\left|H_B\left(e^{j\omega}\right)\right| = 0$ for $\omega = \pi$.

C. $h_3[n]$ is a Type III filter. The frequency response must be D, as D is the only frequency response for which $\left|H_D\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$ and $\omega = \pi$.

D. $h_4[n]$ is a Type IV filter. Frequency response A corresponds to a Type IV filter, as $\left|H_A\left(e^{j\omega}\right)\right| = 0$ for $\omega = 0$.

5.45. A. Systems B, C, D, and E are IIR systems. All of these have poles at places other than the origin and infinity.

B. Systems A and F are FIR systems. These have poles only at the origin.

C. A causal LTI system is stable if and only if all of its poles lie inside the unit circle. Systems A, B, C, E, and F (i.e., all but D) are stable.

D. A stable causal system is minimum phase if its inverse system is also stable and causal. This means that all of the zeros as well as all of the poles must lie inside the unit circle. System E is the only minimum-phase system.

E. A system that is causal with a rational frequency response must be an FIR system to have linear phase. Both systems A and F are linear phase systems, as for both of these systems the zeros occur in reciprocal pairs or at $z = \pm 1$.

F. System C is allpass. It is the only system for which poles and zeros occur in conjugate reciprocal pairs.

G. Only System E has a stable and causal inverse. This is the only system having all of its zeros inside the unit circle.

H. System F has the shortest impulse response, with seven nonzero samples. System A has 12 nonzero samples, and the remaining systems are IIR.

I. Systems A and F are lowpass systems. Systems B and D are eliminated as they each have a zero at $\omega = 0$. System C is an allpass system. System E will have a frequency response whose magnitude tends to peak at frequencies near the system poles. None of these poles are near $\omega = 0$ and one of them is near $\omega = \pi$.

J. The "minimum group delay" property is an attribute of a minimum phase system. System E is the only minimum phase system in the given set. (Note that System E has the minimum group delay among systems with the same magnitude response. System E may not have the minimum group delay among the systems shown.)

5.46. A.

$$\begin{aligned} H\left(e^{j\omega}\right) &= H_1\left(e^{j\omega}\right)H_2\left(e^{j\omega}\right) \\ &= jA_1\left(e^{j\omega}\right)A_2\left(e^{j\omega}\right)e^{-j\omega(M_1+M_2)/2}. \end{aligned}$$

B. The overall impulse response has length $M_1 + M_2 + 1$.

C. The delay of the overall system is $(M_1 + M_2)/2$ samples.

D. The overall system is a Type-IV generalized linear-phase system. Note that $M_1 + M_2$ is an odd integer.

5.47. A. The system function $H(z)$ is a fifth-degree polynomial in z^{-1}. Consequently, the impulse has a length of six samples.

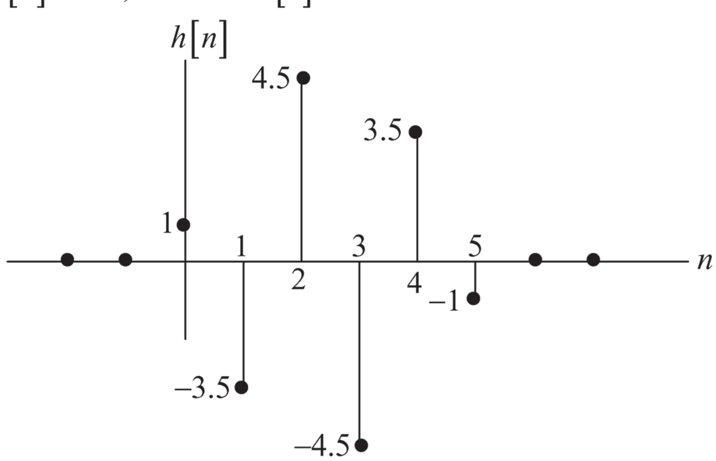
B. This is a Type IV system, since M is odd and $H(e^{j0})=0$.

C. The delay of the system is $M/2=5/2$ samples.

D. Since $H(e^{j0})=0$, there is a zero at $z=1$. Set $a=1$.

There is a zero on the unit circle at $z=e^{j\frac{\pi}{2}}$. Since the impulse response is zero there must be zero at the conjugate location. Set $b=e^{-j\frac{\pi}{2}}$.
Finally, there is a zero at $z=0.5$. For linear phase, there must a zero at the reciprocal location $z=2$. Therefore set $c=2$.

E. For a Type IV system, $h[n]=-h[M-n]$. We have $h[0]=1$ from the given expression for $H(z)$, and therefore $h[5]=-1$. Given $h[1]=-3.5$ we have $h[4]=3.5$. Finally, given $h[2]=4.5$, we have $h[3]=-4.3$.



You can also find $h[n]$ by multiplying out the factors in $H(z)$. This gives

$$H(z)=1-3.5z^{-1}+4.5z^{-2}-4.5z^{-3}+3.5z^{-4}-z^{-5}.$$

5.48. (a)

$$H(z) = \frac{A}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \quad h[n] \text{ causal}$$

$$H(1) = 6 \Rightarrow A = 4$$

(b)

$$\begin{aligned} H(z) &= \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > \frac{1}{2} \\ &= \frac{(\frac{12}{5})}{1-\frac{1}{2}z^{-1}} + \frac{(\frac{8}{5})}{1+\frac{1}{3}z^{-1}} \\ h[n] &= \frac{12}{15}\left(\frac{1}{2}\right)^n u[n] + \frac{8}{5}\left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(c) (i)

$$x[n] = u[n] - \frac{1}{2}u[n-1] \Leftrightarrow X(z) = \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}}, \qquad |z| > 1$$

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}} \cdot \frac{4}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{3}z^{-1})}, \qquad |z| > 1 \\ &= \frac{4}{(1-z^{-1})(1+\frac{1}{3}z^{-1})} \\ &= \frac{3}{1-z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}} \\ y[n] &= 3u[n] + \left(-\frac{1}{3}\right)^n u[n] \end{aligned}$$

(ii)

$$x(t) = 50 + 10\cos(20\pi t) + 30\cos(40\pi t)$$

$$T = \frac{1}{40} \qquad t = nT$$

$$\begin{aligned} x[n] &= 50 + 10\cos\frac{\pi}{2}n + 30\cos\pi n \\ &= 50 + 5e^{j(n\pi/2)} + 5e^{-j(n\pi/2)} + 15e^{jn\pi} + 15e^{-jn\pi} \end{aligned}$$

Using the eigenfunction property:

$$y[n] = 50H(e^{j0}) + 5e^{j(n\pi/2)}H(e^{j(\pi/2)}) + 5e^{-j(n\pi/2)}H(e^{-j(\pi/2)}) + 15e^{jn\pi}H(e^{j\pi}) + 15e^{-jn\pi}H(e^{-j\pi})$$

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{6}e^{-j\omega} - \frac{1}{6}e^{-j2\omega}}$$

$H(e^{j0}) = 6$, $H(e^{j(\pi/2)}) = 7\left(\frac{12}{25}\right) - j\frac{12}{25}$, $H(e^{-j(\pi/2)}) = 7\left(\frac{12}{25}\right) + j\frac{12}{25}$,
$H(e^{j\pi}) = 4$, $H(e^{-j\pi}) = 4$

$$y[n] = 300 + 24\sqrt{2}\cos\left(\frac{\pi}{2}n - \tan^{-1}\left(\frac{1}{7}\right)\right) + 120\cos\pi n$$

5.49.

$$\begin{aligned} H(z) &= \frac{21}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})(1-4z^{-1})} \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}} + \frac{48}{1-4z^{-1}} \end{aligned}$$

Since we know the sequence is not stable, the ROC must not include $|z| = 1$, and since it is two-sided, the ROC must be a ring. This leaves only one possible choice: the ROC is $2 < |z| < 4$.

(a)

$$h[n] = \left(\frac{1}{2}\right)^n u[n] - 28(2)^n u[n] - 48(4)^n u[-n-1]$$

(b)

$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{28}{1-2z^{-1}}$$

$$H_2(z) = \frac{48}{1-4z^{-1}}$$

5.50. Since $H(e^{jw})$ has a zero on the unit circle, its inverse system will have a pole on the unit circle and thus is not stable.

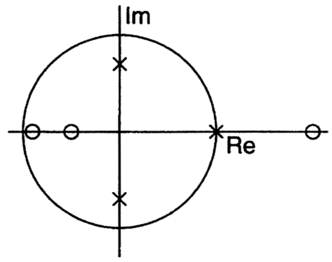
5.51. (a)

$$\begin{aligned} H(z) &= \frac{(1-2z^{-1})(1+\frac{1}{2}z^{-1})(1+0.9z^{-1})}{(1-z^{-1})(1+0.7jz^{-1})(1-0.7jz^{-1})} \\ &= \frac{1-0.6z^{-1}-2.35z^{-2}-0.9z^{-3}}{1-z^{-1}+0.49z^{-2}-0.49z^{-3}} \\ &= \frac{Y(z)}{X(z)} \end{aligned}$$

Cross multiplying and taking the inverse z-transform gives

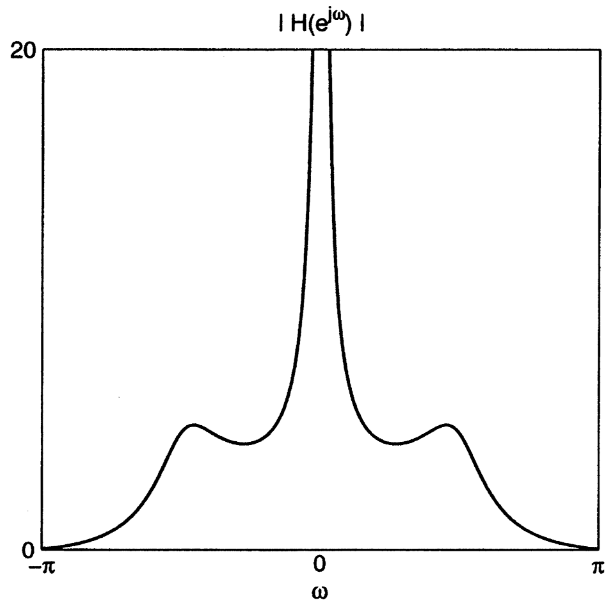
$$y[n]-y[n-1]+0.49y[n-2]-0.49y[n-3] = x[n]-0.6x[n-1]-2.35x[n-2]-0.9x[n-3]$$

(b)



Note that since $h[n]$ is causal, ROC is $|z| > 1$.

(c)

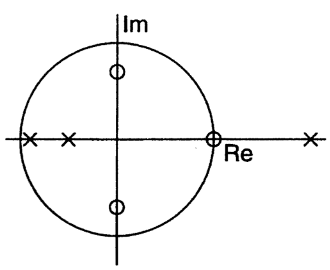


(d) (i) The system is not stable since the ROC does not include $|z| = 1$.

(ii) Because $h[n]$ is not stable, $h[n]$ does not approach a constant as $n \to \infty$.

(iii) We can see peaks at $\omega = \pm\frac{\pi}{2}$ in the graph of $|H(e^{j\omega})|$ shown in part (c), so this is false.

(iv) Swapping poles and zeros gives:



There is a ROC that includes the unit circle ($0.9 < |z| < 2$). However, this stable system would be two sided, so we must conclude the statement is false.

5.52.

$$X(z) = \frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{5}z)}{(1-\frac{1}{6}z)} = \frac{6}{5}\frac{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})(1-5z^{-1})}{(1-6z^{-1})}$$

$$\alpha^n x[n] \Leftrightarrow X(\alpha^{-1}z) = \frac{6}{5}\frac{(1-\frac{1}{2}\alpha z^{-1})(1-\frac{1}{4}\alpha z^{-1})(1-5\alpha z^{-1})}{(1-6\alpha z^{-1})}$$

A minimum phase sequence has all poles and zeros inside the unit circle.

$$\begin{aligned}
|\alpha/2| < 1 \quad &\Rightarrow \quad |\alpha| < 2 \\
|\alpha/4| < 1 \quad &\Rightarrow \quad |\alpha| < 4 \\
|5\alpha| < 1 \quad &\Rightarrow \quad |\alpha| < \frac{1}{5} \\
|6\alpha| < 1 \quad &\Rightarrow \quad |\alpha| < \frac{1}{6}
\end{aligned}$$

Therefore, $\alpha^n x[n]$ is real and minimum phase iff α is real and $|\alpha| < \frac{1}{6}$.

5.53. (a) The causal systems have conjugate zero pairs inside or outside the unit circle. Therefore

$$\begin{aligned}
H(z) &= (1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1})(1-1.25e^{-j0.8\pi}z^{-1}) \\
H_1(z) &= (0.9)^2(1.25)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1}) \\
&\quad (1-0.8e^{j0.8\pi}z^{-1})(1-0.8e^{-j0.8\pi}z^{-1}) \\
H_2(z) &= (0.9)^2(1-(10/9)e^{j0.6\pi}z^{-1})(1-(10/9)e^{-j0.6\pi}z^{-1})(1-1.25e^{j0.8\pi}z^{-1}) \\
&\quad (1-1.25e^{-j0.8\pi}z^{-1}) \\
H_3(z) &= (1.25)^2(1-0.9e^{j0.6\pi}z^{-1})(1-0.9e^{-j0.6\pi}z^{-1})(1-0.8e^{j0.8\pi}z^{-1}) \\
&\quad (1-0.8e^{-j0.8\pi}z^{-1})
\end{aligned}$$

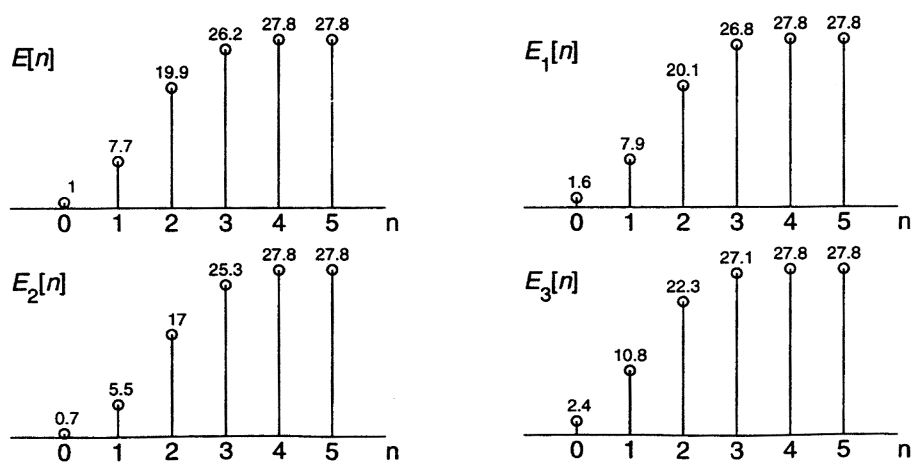
$H_2(z)$ has all its zeros outside the unit circle, and is a maximum phase sequence. $H_3(z)$ has all its zeros inside the unit circle, and thus is a minimum phase sequence.

(b)

$$\begin{aligned}
H(z) &= 1+2.5788z^{-1}+3.4975z^{-2}+2.5074z^{-3}+1.2656z^{-4} \\
h[n] &= \delta[n]+2.5788\delta[n-1]+3.4975\delta[n-2]+2.5074\delta[n-3]+1.2656\delta[n-4] \\
H_1(z) &= 1.2656+2.5074z^{-1}+3.4975z^{-2}+2.5788z^{-3}+z^{-4} \\
h_1[n] &= 1.2656\delta[n]+2.5074\delta[n-1]+3.4975\delta[n-2]+2.5788\delta[n-3]+\delta[n-4] \\
H_2(z) &= 0.81+2.1945z^{-1}+3.3906z^{-2}+2.8917z^{-3}+1.5625z^{-4} \\
h_2[n] &= 0.81\delta[n]+2.1945\delta[n-1]+3.3906\delta[n-2]+2.8917\delta[n-3]+1.5625\delta[n-4] \\
H_3(z) &= 1.5625+2.8917z^{-1}+3.3906z^{-2}+2.1945z^{-3}+0.81z^{-4} \\
h_3[n] &= 1.5625\delta[n]+2.8917\delta[n-1]+3.3906\delta[n-2]+2.1945\delta[n-3]+0.81\delta[n-4]
\end{aligned}$$

(c)

n	$E(n)$	$E_1(n)$	$E_2(n)$	$E_3(n)$
0	1.0	1.6	0.7	2.4
1	7.7	7.9	5.5	10.8
2	19.9	20.1	17.0	22.3
3	26.2	26.8	25.3	27.1
4	27.8	27.8	27.8	27.8
5	27.8	27.8	27.8	27.8



The plot of $E_3[n]$ corresponds to the minimum phase sequence.

5.54. All zeros inside the unit circle means the sequence is minimum phase. Since

$$\sum_{n=0}^{M} |h_{min}[n]|^2 \geq \sum_{n=0}^{M} |h[n]|^2$$

is true for all M, we can use $M = 0$ and just compute $h^2[0]$. The largest result will be the minimum phase sequence.

A	B	C	D	E	F	G	H
44.5	28.4	1.8	2.8	1.8	177.7	113.8	7.1

The answer is F.

5.55.

(i) A zero phase sequence has all its poles and zeros in conjugate reciprocal pairs. Generalized linear phase systems are zero phase systems with additional poles or zeros at $z = 0, \infty, 1$ or -1.

(ii) A stable system's ROC includes the unit circle.

(a) The poles are not in conjugate reciprocal pairs, so this does not have zero or generalized linear phase. $H_i(z)$ has a pole at $z = 0$ and perhaps $z = \infty$. Therefore, the ROC is $0 < |z| < \infty$, which means the inverse is stable. If the ROC includes $z = \infty$, the inverse will also be causal.

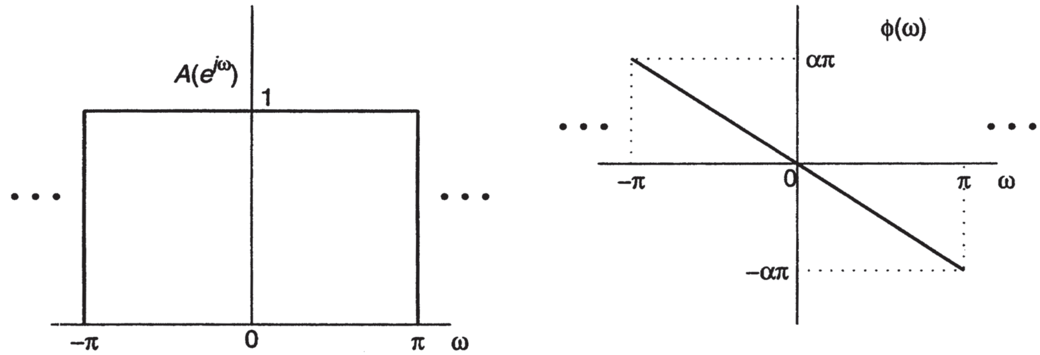
(b) Since the poles are not conjugate reciprocal pairs, this does not have zero or generalized linear phase either. $H_i(z)$ has poles inside the unit circle, so ROC is $|z| > \frac{2}{3}$ to match the ROC of $H(z)$. Therefore, the inverse is both stable and causal.

(c) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. The inverse has poles both inside and outside the unit circle. Therefore, a stable non-causal inverse exists.

(d) The zeros occur in conjugate reciprocal pairs, so this is a zero phase system. Since the poles of the inverse system are on the unit circle a stable inverse does not exist.

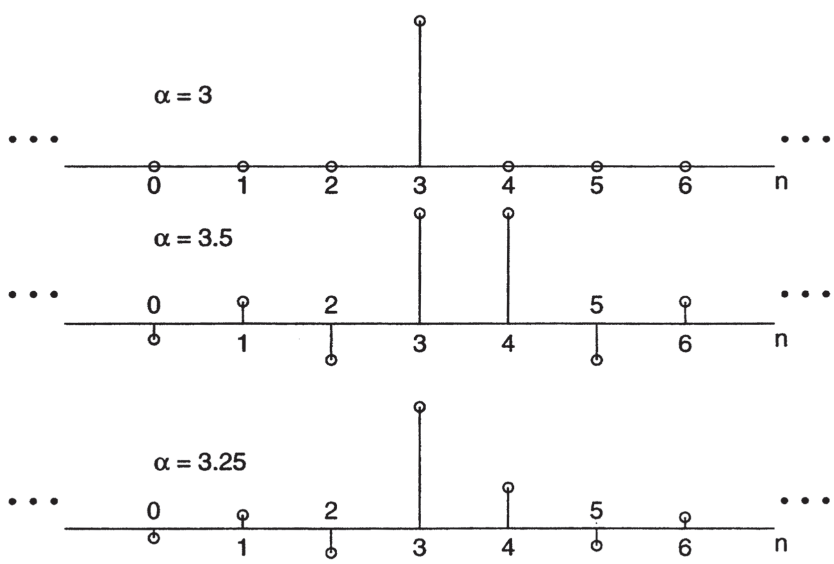
5.56. (a)

$$A(e^{j\omega}) = 1, \quad |\omega| < \pi$$
$$\phi(\omega) = -\alpha\omega, \quad |\omega| < \pi$$



(b)

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\alpha\omega} e^{j\omega n} d\omega = \frac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}$$



(c) If α is an integer, then $h[n]$ is symmetric about about the point $n = \alpha$. If $\alpha = \frac{M}{2}$, where M is odd, then $h[n]$ is symmetric about $\frac{M}{2}$, which is not a point of the sequence. For α in general, $h[n]$ will not be symmetric.

5.57. Type I: Symmetric, M Even, Odd Length

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=(M+2)/2}^{M} h[n]e^{-j\omega n} + h[M/2]e^{-j\omega(M/2)} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} + h[M/2]e^{-j\omega(M/2)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)} + h[M/2]\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} 2h[m]\cos\omega((M/2)-m) + h[M/2]\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\cos\omega n + h[M/2]\right)
\end{aligned}
$$

Let

$$
a[n] = \begin{cases} h[M/2], & n = 0 \\ 2h[(M/2)-n], & n = 1, \ldots, M/2 \end{cases}
$$

Then

$$
H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=0}^{M/2} a[n]\cos\omega n
$$

and we have

$$
A(\omega) = \sum_{n=0}^{M/2} a[n]\cos(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = 0
$$

Type II: Symmetric, M Odd, Even Length

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} + \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} 2h[m]\cos\omega((M/2)-m)\right) \\
&= e^{-j\omega(M/2)}\left(\sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\cos\omega(n-(1/2))\right)
\end{aligned}
$$

Let

$$
b[n] = 2h[(M+1)/2-n], \quad n = 1, \ldots, (M+1)/2
$$

Then

$$
H(e^{j\omega}) = e^{-j\omega(M/2)} \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2))
$$

and we have

$$A(\omega) = \sum_{n=1}^{(M+1)/2} b[n]\cos\omega(n-(1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = 0$$

Type III: Antisymmetric, M Even, Odd Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + 0 + \sum_{(M+2)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-2)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-2)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-2)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\
&= e^{-j\omega(M/2)}\left(j\sum_{m=0}^{(M-2)/2} 2h[m]\sin\omega((M/2)-m)\right) \\
&= e^{-j\omega(M/2)}e^{j(\pi/2)}\left(\sum_{n=1}^{M/2} 2h[(M/2)-n]\sin\omega n\right)
\end{aligned}$$

Let

$$c[n] = h[(M/2)-n], \quad n = 1, \ldots, M/2$$

Then

$$H(e^{j\omega}) = e^{-j\omega(M/2)}e^{j(\pi/2)}\sum_{n=1}^{M/2} c[n]\sin\omega n$$

and we have

$$A(\omega) = \sum_{n=1}^{M/2} c[n]\sin(\omega n), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

Type IV: Antisymmetric, M Odd, Even Length

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=(M+1)/2}^{M} h[n]e^{-j\omega n} \\
&= \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{m=0}^{(M-1)/2} h[M-m]e^{-j\omega(M-m)} \\
&= e^{-j\omega(M/2)}\left(\sum_{m=0}^{(M-1)/2} h[m]e^{j\omega((M/2)-m)} - \sum_{m=0}^{(M-1)/2} h[m]e^{-j\omega((M/2)-m)}\right) \\
&= e^{-j\omega(M/2)}\left(j\sum_{m=0}^{(M-1)/2} 2h[m]\sin\omega((M/2)-m)\right) \\
&= e^{-j\omega(M/2)}e^{j(\pi/2)}\sum_{n=1}^{(M+1)/2} 2h[(M+1)/2-n]\sin\omega(n-(1/2))
\end{aligned}$$

Let

$$d[n] = 2h[(M+1)/2 - n], \quad n = 1, \ldots, (M+1)/2$$

Then

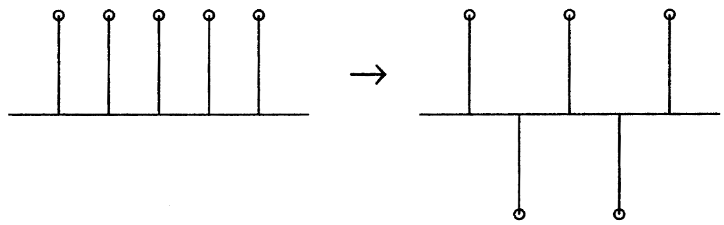
$$H(e^{j\omega}) = e^{-j\omega(M/2)} e^{j(\pi/2)} \sum_{n=1}^{(M+1)/2} d[n] \sin\omega(n - (1/2))$$

and we have

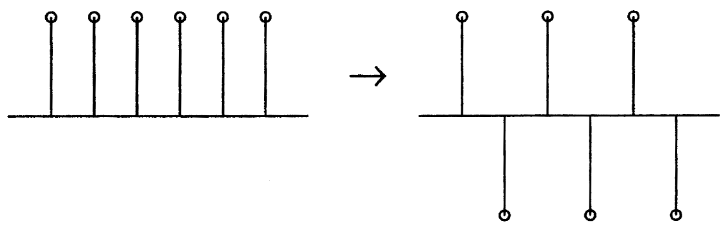
$$A(\omega) = \sum_{n=1}^{(M+1)/2} d[n] \sin\omega(n - (1/2)), \quad \alpha = \frac{M}{2}, \quad \beta = \frac{\pi}{2}$$

5.58. Filter Types II and III cannot be highpass filters since they both must have a zero at $z = 1$.

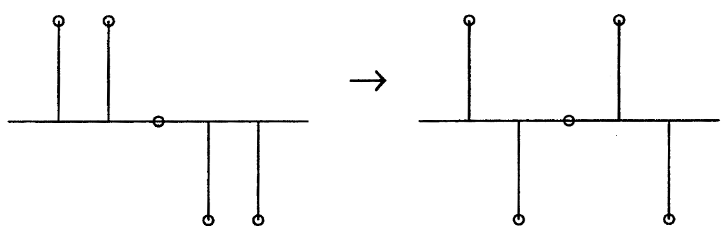
Type I $\rightarrow$ Type I could be highpass:



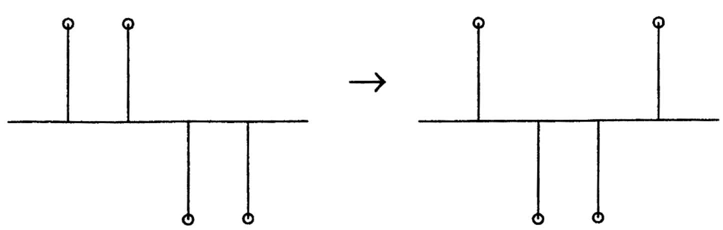
Type II $\rightarrow$ Type IV can be highpass:



Type III $\rightarrow$ Type III cannot be highpass:



Type IV $\rightarrow$ Type II cannot be highpass:



5.59. (a) Minimum phase systems have all poles and zeros inside $|z| = 1$. Allpass systems have pole-zero pairs at conjugate reciprocal locations. Generalized linear phase systems have pole pairs and zero pairs in conjugate reciprocal locations and at $z = 0, 1, -1$ and ∞. This implies that all the poles and zeros of $H_{min}(z)$ are second-order. When the allpass filter flips a pole or zero outside the unit circle, one is left in the conjugate reciprocal location, giving us linear phase.

(b) We know that $h[n]$ is length 8 and therefore has 7 zeros. Since it is an even length generalized linear phase filter with real coefficients and odd symmetry it must be a Type IV filter. It therefore has the property that its zeros come in conjugate reciprocal pairs stated mathematically as $H(z) = H(1/z^*)$. The zero at $z = -2$ implies a zero at $z = -\frac{1}{2}$, while the zero at $z = 0.8e^{j(\pi/4)}$ implies zeros at $z = 0.8e^{-j(\pi/4)}$, $z = 1.25e^{j(\pi/4)}$ and $z = 1.25e^{-j(\pi/4)}$. Because it is a IV filter, it also must have a zero at $z = 1$. Putting all this together gives us

$$\begin{aligned} H(z) &= (1 + 2z^{-1})(1 + 0.5z^{-1})(1 - 0.8e^{j(\pi/4)}z^{-1})(1 - 0.8e^{-j(\pi/4)}z^{-1}) \\ &\quad (1 - 1.25e^{j(\pi/4)}z^{-1})(1 - 1.25e^{-j(\pi/4)}z^{-1})(1 - z^{-1}) \end{aligned}$$

5.60. *False.* Let $h[n]$ equal

$$h[n] = \frac{\sin \omega_c(n - 4.3)}{\pi(n - 4.3)} \longleftrightarrow H(e^{jw}) = \begin{cases} e^{-4.3\omega}, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}$$

Proof: Although the group delay is constant (grd $\left[H(e^{jw})\right] = 4.3$) the resulting M is not an integer.

$$\begin{aligned} h[n] &= \pm h[M - n] \\ H(e^{j\omega}) &= \pm e^{jM\omega} H(e^{-j\omega}) \\ e^{-j4.3\omega} &= \pm e^{j(M+4.3)\omega}, \qquad |\omega| < \omega_c \\ M &= -8.6 \end{aligned}$$

5.61. The type II FIR system $H_{II}(z)$ has generalized linear phase. Therefore, it can be written in the form

$$H_{II}(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2}$$

where M is an odd integer and $A_e(e^{j\omega})$ is a real, even, periodic function of ω. Note that the system $G(z) = (1 - z^{-1})$ is a type IV generalized linear phase system.

$$\begin{aligned}
G(e^{j\omega}) &= 1 - e^{-j\omega} \\
&= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) \\
&= e^{-j\omega/2}(2j\sin(\omega/2)) \\
&= 2\sin(\omega/2)e^{-j\omega/2+j\pi/2} \\
&= A_o(e^{j\omega})e^{-j\omega/2+j\pi/2} \\
A_o(e^{j\omega}) &= 2\sin(\omega/2) \\
\angle G(e^{j\omega}) &= -\frac{\omega}{2} + \frac{\pi}{2}
\end{aligned}$$

The cascade of $H_{II}(z)$ with $G(z)$ results in a generalized linear phase system $H(z)$.

$$\begin{aligned}
H(e^{j\omega}) &= A_e(e^{j\omega})A_o(e^{j\omega})e^{-j\omega M/2}e^{-j\omega/2+j\pi/2} \\
&= A'_o(e^{j\omega})e^{j\omega M'/2+j\pi/2}
\end{aligned}$$

where $A'_o(e^{j\omega})$ is a real, odd, periodic function of ω and M' is an even integer.

Thus, the resulting system $H(e^{j\omega})$ has the form of a type III FIR generalized linear phase system. It is antisymmetric, has odd length (M is even), and has generalized linear phase.

5.62. (a) The LTI system S_2 is characterized as a lowpass filter.
The z-transform of $h_1[n]$ is found below.

$$\begin{aligned} y[n] - y[n-1] + \frac{1}{4}y[n-2] &= x[n] \\ Y(z) - Y(z)z^{-1} + \frac{1}{4}Y(z)z^{-2} &= X(z) \\ Y(z)\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right) &= X(z) \end{aligned}$$

$$H_1(z) = \frac{1}{(1 - z^{-1} + \frac{1}{4}z^{-2})} = \frac{1}{(1 - \frac{1}{2}z^{-1})^2}$$

This system function has a second order pole at $z = \frac{1}{2}$. (There is also a second order zero at $z = 0$). Evaluating this pole-zero plot on the unit circle yields a low pass filter, as the second order pole boosts the low frequencies.

Since

$$\begin{aligned} H_2(e^{j\omega}) &= H_1(-e^{j\omega}) \\ H_2(z) &= H_1(-z) \end{aligned}$$

If we replace all references to z in $H_1(z)$ with $-z$, we will get $H_2(z)$.

$$H_2(z) = \frac{1}{(1 + \frac{1}{2}z^{-1})^2}$$

Consequently, $H_2(z)$ has two poles at $z = -\frac{1}{2}$. (There is also a second order zero at $z = 0$). Evaluating this pole-zero plot on the unit circle yields a high pass filter, as the second order pole now boosts the high frequencies.

5.63. (a)

$$x[n] = s[n]\cos\omega_0 n = \frac{1}{2}s[n]e^{j\omega_0 n} + \frac{1}{2}s[n]e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{1}{2}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}S(e^{j(\omega+\omega_0)})$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)} \\ &= s[n]\cos(\omega_0 n - \phi_0) \end{aligned}$$

(b) This time,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{2}e^{-j\phi_0}e^{-j\omega n_d}S(e^{j(\omega-\omega_0)}) + \frac{1}{2}e^{j\phi_0}e^{-j\omega n_d}S(e^{j(\omega+\omega_0)})$$

$$\begin{aligned} y[n] &= \delta[n-n_d] * \left(\frac{1}{2}s[n]e^{j(\omega_0 n-\phi_0)} + \frac{1}{2}s[n]e^{-j(\omega_0 n-\phi_0)}\right) \\ &= \delta[n-n_d] * s[n]\cos(\omega_0 n - \phi_0) \\ &= s[n-n_d]\cos(\omega_0 n - \omega_0 n_d - \phi_0) \end{aligned}$$

Therefore, if $\phi_1 = \phi_0 + \omega_0 n_d$ then

$$y[n] = s[n-n_d]\cos(\omega_0 n - \phi_1)$$

for narrowband $s[n]$.

(c)

$$\begin{aligned} \tau_{gr} &= -\frac{d}{d\omega}arg[H(e^{j\omega})] = -\frac{d}{d\omega}[-\phi_0 - \omega n_d] = n_d \\ \tau_{ph} &= -\frac{1}{\omega}arg[H(e^{j\omega})] = -\frac{1}{\omega}[-\phi_0 - \omega n_d] = \frac{\phi_0}{\omega} - n_d \end{aligned}$$

$$y[n] = s[n - \tau_{gr}(\omega_0)]\cos[\omega_0(n - \tau_{ph}(\omega_0))]$$

(d) The effect would be the same as the following:

(i) Bandlimit interpolate the composite signal to a C–T signal with some rate T.
(ii) Delay the envelope by $T \cdot \tau_{gr}$, and delay the carrier by $T \cdot \tau_{ph}$.
(iii) Sample to a D–T signal at rate T

5.64. (a)

$$m_x = 0 \Rightarrow \phi_{yy}[m] = \Gamma_{yy}[m] \Leftrightarrow \Gamma_{yy}(z) = \Phi_{yy}(z)$$

$$\phi_{yy}[m] = y[n] * y[-n] = x[n] * x[-n] * h[n] * h[-n]$$

$$\Phi_{yy}(z) = X(z)X(z^{-1})H(z)H(z^{-1}) = \Phi_{xx}(z)H(z)H(z^{-1})$$

$$\phi_{xx}[m] = \sigma_x^2\delta[m] \Leftrightarrow \Phi_{xx}(z) = \sigma_x^2$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k], \quad b_0 = 1$$

$$Y(z) = \sum_{k=1}^{N} a_k Y(z)z^{-k} + X(z) + \sum_{k=1}^{M} b_k X(z)z^{-k}$$

$$H(z) = \frac{1 + \sum_{k=1}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = A\frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

So,

$$\Gamma_{yy}(z) = \Phi_{xx}(z)H(z)H(z^{-1}) = \sigma_x^2 \frac{\left(1 + \sum_{k=1}^{M} b_k z^{-k}\right)\left(1 + \sum_{k=1}^{M} b_k z^{k}\right)}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)\left(1 - \sum_{k=1}^{N} a_k z^{k}\right)}$$

Or equivalently,

$$\Gamma_{yy}(z) = A^2\sigma_x^2 \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k z)}{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k z)}$$

(b) To "whiten" the signal $y[n]$ we need a system:

$$H_w(z)H_w(z^{-1}) = \frac{1}{H(z)H(z^{-1})}$$

Therefore,

$$H_w(z)H_w(z^{-1}) = \frac{\prod_{k=1}^{N}(1 - d_k z^{-1})(1 - d_k z)}{\prod_{k=1}^{M}(1 - c_k z^{-1})(1 - c_k z)}$$

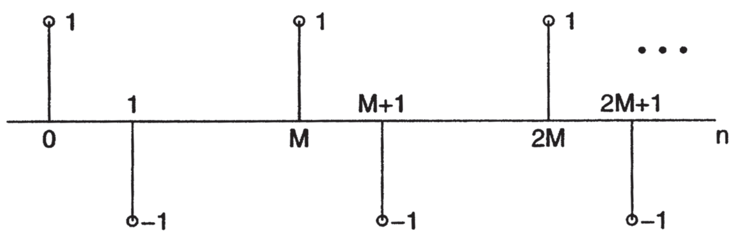
The poles of $H_w(z)$ are the zeros of $H(z)$ and the zeros of $H_w(z)$ are the poles of $H(z)$. We must now decide which N of the $2N$ zeros of $H_w(z)H_w(z^{-1})$ to associate with $H_w(z)$. The remaining N zeros and M poles will be reciprocals and will be associated with $H_w(z^{-1})$. In order for $H_w(z)$ to be stable, we must chose all its poles inside the unit circle. Thus for a pair c_k, c_k^{-1} we chose the one which is inside the unit circle.

(c) There is no real constraint on the zeros of $H_w(z)$, so we can select either d_k or d_k^{-1}. Thus, it is not unique.

5.65. (a)

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$H_i(e^{j\omega}) = \frac{1 - e^{-j\omega}}{1 - e^{-j\omega M}} \Leftrightarrow h_i[n] = \sum_{k=0}^{\infty} \delta[n - kM] - \delta[n - kM - 1]$$



$h_i[n]$ has infinite length, so we can never get a result without infinite sums. Therefore, it is not a real time filter. We can use the transform approach but we must have all the input data available to do this.

(b) The proposed system is a windowed version of $h_i[n]$:

$$h_1[n] * h_2[n] = h_i[n]p[n]$$

Where

$$p[n] = \begin{cases} 1, & 0 \le n \le qM \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] * h[n] * h_i[n]p[n] = w[n]$$

Therefore, if $x[n]$ is shorter than qM points, we can recover it by looking at $w[n]$ in the range $0 \le n \le qM - 1$.

(c)

$$H_i(z) = \frac{1}{H(z)} = H_1(z)H_2(z)$$

$$h_1[n] = \sum_{k=0}^{q} \delta[n - kM] \Leftrightarrow H_1(z) = \frac{1 - z^{-qM}}{1 - z^{-M}}$$

Thus,

$$H_2(z) = \frac{1}{H(z)} \frac{1 - z^{-M}}{1 - z^{-qM}}$$

Note that

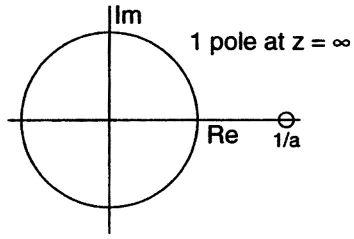
$$\frac{1 - z^{-M}}{1 - z^{-qM}}$$

has M zeros and qM poles. Since $H_2(z)$ is causal, there are no poles at $z = \infty$. If $H(z)$ has P poles and Z zeros:

$$Z + M \le P + qM$$

5.66. (a)

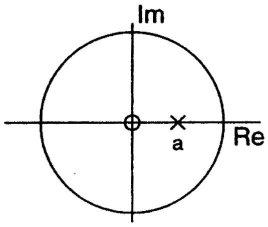
$$H(z) = z - \frac{1}{a} = \frac{az-1}{a} = \frac{a - z^{-1}}{az^{-1}}$$



$$H(e^{j\omega}) = e^{j\omega} - \frac{1}{a} = \cos\omega + j\sin\omega - \frac{1}{a}$$

$$\arg[H(e^{j\omega})] = \tan^{-1}\left(\frac{\sin\omega}{\cos\omega - \frac{1}{a}}\right)$$

(b)



$$G(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

$$G(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = \frac{1}{1 - a\cos\omega + ja\sin\omega}$$

$$\begin{aligned}\arg[G(e^{j\omega})] &= -\tan^{-1}\left(\frac{a\sin\omega}{1-a\cos\omega}\right)\\ &= \tan^{-1}\left(\frac{a\sin\omega}{a\cos\omega - 1}\right)\\ &= \arg[H(e^{j\omega})]\end{aligned}$$

5.67. (a) Because $h_1[n], h_2[n]$ are minimum phase sequences, all poles and zeros of their transforms must be inside the unit circle.

$$h_1[n] * h_2[n] \leftrightarrow H_1(z)H_2(z)$$

Since $H_1(z)$ and $H_2(z)$ have all their poles and zeros inside the unit circle, their product will also.

(b)

$$h_1[n] + h_2[n] \leftrightarrow H_1(z) + H_2(z)$$

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1 - \frac{1}{2}z^{-1}} = X_1(z)$$

$$x_2[n] = 2\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{2}{1 - \frac{1}{2}z^{-1}} = X_2(z)$$

Both of these are minimum phase, with a zero at $z = 0$ and a pole at $z = \frac{1}{2}$.

$$X_1(z) + X_2(z) = \frac{3}{1 - \frac{1}{2}z^{-1}}$$

This is minimum phase, with the same pole and zero as $X_1(z)$ and $X_2(z)$.

$$x_1[n] = 6\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{6}{1 - \frac{1}{2}z^{-1}} = X_1(z)$$

$$x_2[n] = -6\left(\frac{1}{3}\right)^n u[n] \longleftrightarrow \frac{-6}{1 - \frac{1}{3}z^{-1}} = X_2(z)$$

$X_1(z)$ has a pole at $z = \frac{1}{2}$ and a zero at $z = 0$. $X_2(z)$ has a pole at $z = \frac{1}{3}$ and a zero at $z = 0$.

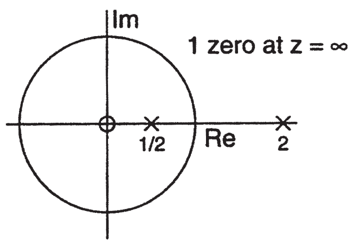
$$X_1(z) + X_2(z) = \frac{z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

This has zeros at $z = 0, \infty$ and poles at $z = \frac{1}{2}, \frac{1}{3}$. Therefore, it is not minimum phase.

5.68. (a)

$$r[n] = \frac{4}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{4}{3}(2)^n u[-n-1]$$

$$\begin{aligned} R(z) &= \frac{\frac{4}{3}}{1-\frac{1}{2}z^{-1}} - \frac{\frac{4}{3}}{1-2z^{-1}} \\ &= \frac{-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})} \\ &= \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}, \quad \text{ROC: } \frac{1}{2} < |z| < 2 \end{aligned}$$



(b)

$$r[n] = h[n] * h[-n] \Leftrightarrow R(z) = H(z)H(z^{-1})$$

$$R(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{2}z)}$$

We have two choices from $H(z)$. Since $h[n]$ is minimum phase we need the one which has the pole at $z = \frac{1}{2}$, which is inside the unit circle.

$$H(z) = \frac{\pm 1}{(1-\frac{1}{2}z^{-1})}, \quad \text{ROC: } |z| > \frac{1}{2}$$

$$h[n] = \pm\left(\frac{1}{2}\right)^n u[n]$$

5.69. (a) Maximum phase systems are of the form

$$H(z) = \frac{\prod_{k=1}^{M}(z - c_k)}{\prod_{k=1}^{M}(z - d_k)}, \qquad |c_k|, |d_k| > 1$$

Since the poles are outside the unit circle, the only stable system will have a ROC of $|z| < \min|d_k|$. This implies the poles will all contribute to the $h[n]$ with terms of the form $-(d_k)^n u[-n-1]$, which are anticausal. The zeros are all positive powers of z, which means they are shifting to left, and $h[n]$ is still anticausal.

(b)

$$H_{min}(z) = h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1})$$

$$H_{max}(z) = h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1}) \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right)$$

$$H_{ap}(z) = \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right)$$

(c)

$$\begin{aligned}
H_{max}(z) &= h_{min}[0] \prod_{k=1}^{M}(1 - c_k z^{-1}) \prod_{k=1}^{M}\left(\frac{z^{-1} - c_k^*}{1 - c_k z^{-1}}\right) \\
&= h_{min}[0] \prod_{k=1}^{M}(z^{-1} - c_k^*) \\
&= z^{-M} h_{min}[0] \prod_{k=1}^{M}(1 - c_k^* z) \\
&= z^{-M} H_{min}(z^{-1})
\end{aligned}$$

(d)

$$H_{max}(z) = z^{-M} H_{min}(z^{-1})$$

$$h_{max}[n] = \delta[n - M] * h_{min}[-n] = h_{min}[-n + M]$$

5.70. (a) We desire $|H(z)H_c(z)| = 1$, where $H_c(z)$ is stable and causal and $H(z)$ is not minimum phase. So,

$$|H_{ap}(z)H_{min}(z)H_c(z)| = 1$$

Since $|H_{ap}(z)| = 1$, we want

$$|H_{min}(z)H_c(z)| = 1$$

This means we have

$$H_c(z) = \frac{1}{H_{min}(z)}$$

which will be stable and causal since all the zeros of $H_{min}(z)$, which become the poles of $H_c(z)$, are inside the unit circle.

(b) Since

$$H_c(z) = \frac{1}{H_{min}(z)}$$

We have

$$G(z) = H_{ap}(z)$$

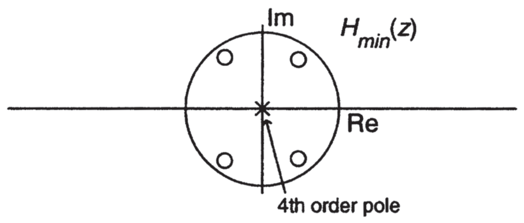
(c)

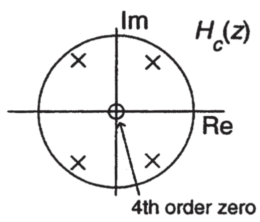
$$H(z) = (1-0.8e^{j0.3\pi}z^{-1})(1-0.8e^{-j0.3\pi}z^{-1})(1-1.2e^{j0.7\pi}z^{-1})(1-1.2e^{-j0.7\pi}z^{-1})$$

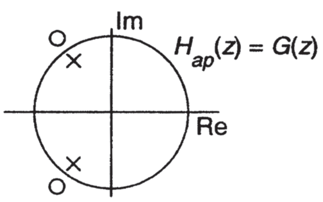
$$H_{min}(z) = (1.44)(1-0.8e^{j0.3\pi}z^{-1})(1-0.8e^{-j0.3\pi}z^{-1})(1-(5/6)e^{j0.7\pi}z^{-1})(1-(5/6)e^{-j0.7\pi}z^{-1})$$

$$H_c(z) = \frac{1}{(1.44)(1-0.8e^{j0.3\pi}z^{-1})(1-0.8e^{-j0.3\pi}z^{-1})(1-(5/6)e^{j0.7\pi}z^{-1})(1-(5/6)e^{-j0.7\pi}z^{-1})}$$

$$G(z) = H_{ap}(z) = \frac{(z^{-1}-(5/6)e^{-j0.7\pi})(z^{-1}-(5/6)e^{j0.7\pi})}{(1-(5/6)e^{j0.7\pi}z^{-1})(1-(5/6)e^{-j0.7\pi}z^{-1})}$$







5.71.

$$H(z) = H_{min}(z)\frac{z^{-1} - a}{1 - az^{-1}}, \quad |a| < 1$$

Thus,

$$\lim_{z\to\infty} H_{min}(z) = \lim_{z\to\infty} \frac{1 - az^{-1}}{z^{-1} - a} H(z)$$

$$h_{min}[0] = -\frac{1}{a}h[0]$$

Therefore, $|h_{min}[0]| > |h[0]|$ since $|a| < 1$. This process can be repeated if more than one allpass system is cascaded. In each case, the factor for each will be larger than unity in the limit.

5.72. (a) We use the allpass principle and place a pole at $z = z_k$ and a zero at $z = \frac{1}{z_k^*}$.

$$\begin{aligned} H(z) &= H_{min}(z)\frac{z^{-1} - z_k^*}{1 - z_k z^{-1}} \\ &= Q(z)(z^{-1} - z_k^*) \end{aligned}$$

(b)

$$H(z) = Q(z)z^{-1} - z_k^* Q(z)$$

$$h[n] = q[n-1] - z_k^* q[n]$$

$$H_{min}(z) = Q(z) - z_k Q(z) z^{-1}$$

$$h_{min}[n] = q[n] - z_k q[n-1]$$

(c)

$$\begin{aligned} \varepsilon &= \sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \\ &= \sum_{m=0}^{n} (|q[m]|^2 - z_k q[m-1]q^*[m] - z_k^* q^*[m-1]q[m] + |z_k|^2 |q[m-1]|^2) \\ &\quad - \sum_{m=0}^{n} (|q[m-1]|^2 - z_k^* q^*[m-1]q[m] - z_k q[m-1]q^*[m] + |z_k|^2 |q[m]|^2) \\ &= (1 - |z_k|^2) \sum_{m=0}^{n} (|q[m]|^2 - |q[m-1]|^2) \\ &= (1 - |z_k|^2)|q[n]|^2 \end{aligned}$$

(d)

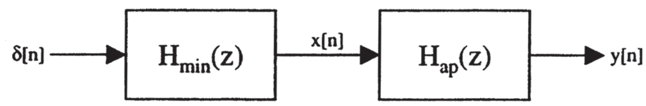
$$\varepsilon = (1 - |z_k|^2)|q[n]|^2 \geq 0 \quad \forall n \text{ since } |z_k| < 1$$

Then

$$\sum_{m=0}^{n} |h_{min}[m]|^2 - \sum_{m=0}^{n} |h[m]|^2 \geq 0$$

$$\sum_{m=0}^{n} |h[m]|^2 \leq \sum_{m=0}^{n} |h_{min}[m]|^2 \quad \forall n$$

5.73. (a) $x[n]$ is real, minimum phase and $x[n] = 0$ for $n < 0$. Consider the system:



$x[n]$ is the impulse response of a minimum phase system. $y[n]$ is the impulse response of a system which has the same frequency response magnitude as that of $x[n]$ but it is not minimum phase. Therefore, the equation applies.

$$\sum_{k=0}^{n} |x[k]|^2 \geq \sum_{k=0}^{n} |y[k]|^2$$

Since $h_{ap}[n]$ is causal and $x[n]$ is causal, $y[n]$ is also causal, and these sums are meaningful.

(b) As discussed in the book, the group delay for a rational allpass system is always positive. That is,

$$\text{grd}[H_{ap}(e^{j\omega})] \geq 0$$

Therefore, filtering a signal $x[n]$ by such a system will delay the energy in the output $y[n]$. If we require that $x[n]$ is causal, then $y[n]$ will be causal as well, and the equation

$$\sum_{k=0}^{n} |x[k]|^2 \geq \sum_{k=0}^{n} |y[k]|^2$$

applies to the system.

5.74. (a)

$$\begin{aligned} g[n] &= x[n] * h[n] \\ r[n] &= g[-n] * h[n] \end{aligned}$$

$$s[n] = r[-n] = g[n] * h[-n] = x[n] * (h[n] * h[-n])$$

$$\begin{aligned} h_1[n] &= h[n] * h[-n] \\ H_1(e^{j\omega}) &= H(e^{j\omega})H^*(e^{j\omega}) = |H(e^{j\omega})|^2 \end{aligned}$$

Since $H_1(e^{j\omega})$ is real, it is zero phase.

(b)

$$\begin{aligned} g[n] &= x[n] * h[n] \\ r[n] &= x[-n] * h[n] \end{aligned}$$

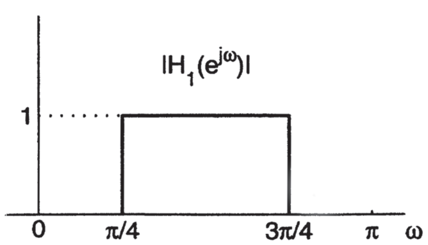
$$y[n] = g[n] + r[-n] = x[n] * h[n] + x[n] * h[-n] = x[n] * (h[n] + h[-n])$$

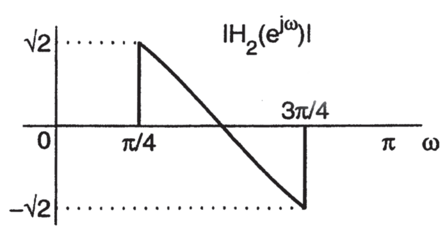
$$\begin{aligned} h_2[n] &= h[n] + h[-n] \\ H_2(e^{j\omega}) &= H(e^{j\omega}) + H^*(e^{j\omega}) \\ &= 2\mathcal{R}e\{H(e^{j\omega})\} \end{aligned}$$

$H_2(e^{j\omega})$ is real, so it is also zero phase.

$$|H_2(e^{j\omega})| = 2|H(e^{j\omega})| \cos\left(\angle H(e^{j\omega})\right)$$

(c)





In general, method A is preferable since method B causes a magnitude distortion which is a function of the (possibly non-linear) phase of $h[n]$.

5.75. *False.* Consider

$$H(z) = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

This system function has poles at $z = 1/2$ and $z = 2$. However, as the following shows it is a generalized linear phase filter.

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{1 - \frac{5}{2}e^{-j\omega} + e^{-j2\omega}} \\ &= \frac{e^{j\omega}}{e^{j\omega} - \frac{5}{2} + e^{-j\omega}} \\ &= \left(\frac{1}{2\cos w - \frac{5}{2}}\right) e^{j\omega} \end{aligned}$$

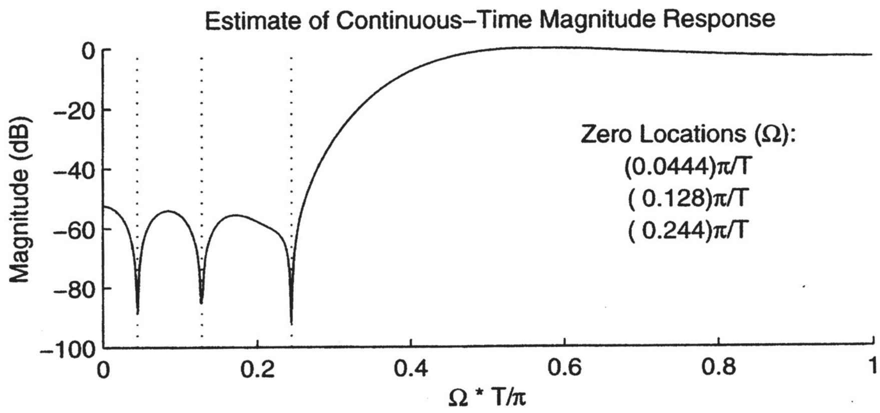
5.76. (a) Since $h[n]$ is a real causal linear phase filter the zeros must occur in sets of 4. Thus, if z_1 is a zero of $H(z)$ then $z_1^*, 1/z_1$ and $1/z_1^*$ must also be zeros. We can use this to find 4 zeros of $H(z)$ from the given information.

$$\begin{array}{lll} z_1, & \text{magnitude} = 0.5, & \text{angle} = 153 \text{ degrees} \\ z_1^*, & \text{magnitude} = 0.5, & \text{angle} = 207 \text{ degrees} \\ 1/z_1, & \text{magnitude} = 2, & \text{angle} = 207 \text{ degrees} \\ 1/z_1^*, & \text{magnitude} = 2, & \text{angle} = 153 \text{ degrees} \end{array}$$

(b) There are 24 zeros so the length of $h[n]$ is 25. Since it is a linear phase filter it has a delay of $(L-1)/2 = (25-1)/2 = 12$ samples. That corresponds to a time delay of

$$\left(0.5 \frac{\text{ms}}{\text{sample}}\right)(12 \text{ samples}) = 6 \text{ ms}$$

(c) The zero locations used to create the following plot were estimated from the figure using a ruler and a protractor.



5.77. (a) There are many possible solutions to this problem. The idea behind any solution is to have $h[n]$ be an upsampled (by a factor of 2) version of $g[n]$. That is,

$$h[n] = \begin{cases} g[n/2], & n = 0, \pm 2, \pm 4, ... \\ 0, & otherwise \end{cases}$$

Thus, $h[n]$ will process only the even-indexed samples. One such system would be described by

$$\begin{aligned} h[n] &= 1 + \delta[n-2] \\ g[n] &= 1 + \delta[n-1] \\ H(z) &= 1 + z^{-2} \\ G(z) &= 1 + z^{-1} \end{aligned}$$

(b) As in part a, there are many possible solutions to this problem. The idea behind any solution is to choose an $h[n]$ that cannot be an upsampled (by a factor of 2) version of $g[n]$. Clearly, choosing $h[n]$ to filter odd-indexed samples satisfies this criterion. One such $h[n]$ would be

$$\begin{aligned} h[n] &= 1 + \delta[n-1] + \delta[n-2] \\ H(z) &= 1 + z^{-1} + z^{-2} \end{aligned}$$

(c) In general, the odd-indexed samples of $h[n]$ must be zero, in order for a $g[n]$ to be found for which $r[n] = y[n]$. Thus, there must not be any odd powers of z^{-1} in $H(z)$.

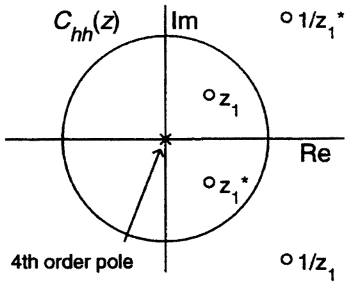
(d) For the conditions determined in part c, $g[n]$ is a downsampled (by a factor of 2) version of $h[n]$. That is,

$$g[n] = h[2n]$$

5.78. (a) *No.* You cannot uniquely recover $h[n]$ from $c_{hh}[l]$.

$$\begin{aligned} c_{hh}[l] &= h[l] * h[-l] \\ C_{hh}(e^{j\omega}) &= H(e^{j\omega})H(e^{-j\omega}) = \left|H(e^{j\omega})\right|^2 \\ C_{hh}(z) &= H(z)H^*(1/z^*) \end{aligned}$$

Causality and stability put restrictions on the poles of $H(z)$ (they must be inside the unit circle) but not its zeros. We know the zeros of $C_{hh}(z)$ in general occur in sets of 4. Here is why. A complex conjugate pair of zeros occur in $H(z)$ due to the fact that $h[n]$ is real. These 2 zeros and their conjugate reciprocals occur in $C_{hh}(z)$ due to the formula above for a total of 4. Thus, $H(z)$ is not uniquely determined since we do not know which 2 out of these 4 zeros to factor into $H(z)$. This is illustrated with a simple example below.



Let the above be the pole-zero diagram for $C_{hh}(z)$ and

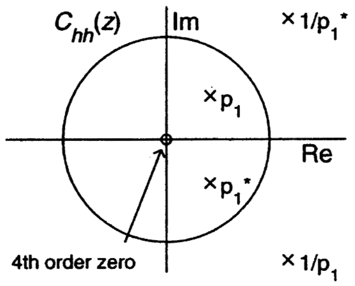
$$\begin{aligned} H_1(z) &= \left(1 - z_1 z^{-1}\right)\left(1 - z_1^* z^{-1}\right) \\ H_2(z) &= \left(1 - \frac{1}{z_1} z^{-1}\right)\left(1 - \frac{1}{z_1^*} z^{-1}\right) \end{aligned}$$

Since

$$C_{hh}(z) = H_1(z)H_1^*(1/z^*) = H_2(z)H_2^*(1/z^*)$$

we cannot determine whether $h_1[n]$ or $h_2[n]$ generated $c_{hh}[l]$.

(b) *Yes.* The poles of $C_{hh}(z)$ must occur in sets of 4 for the same reasons outlined above for the zeros. However, since the poles of $h[n]$ must be inside the unit circle to be causal and stable we do not have any ambiguity in determining which poles to group into $h[n]$. We always choose the complex conjugate poles inside the unit circle. Here is an example



Let the above be the zero/pole diagram for $C_{hh}(z)$. Then, if $h[n]$ is to be real, causal, and stable $H(z)$ must equal

$$H(z) = \frac{1}{(1 - p_1 z^{-1})(1 - p_1^* z^{-1})}$$

5.79. As shown in the book, the most general form of the system function of an allpass system with a real-valued impulse response is

$$H(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}, \quad |z| \in R_x$$

where R_x is the ROC which includes the unit circle. Correspondingly, the associated inverse system is

$$\begin{aligned}
H_i(z) &= \frac{1}{H(z)} \\
&= \prod_{k=1}^{M_r} \frac{1 - d_k z^{-1}}{z^{-1} - d_k} \prod_{k=1}^{M_c} \frac{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}{(z^{-1} - e_k^*)(z^{-1} - e_k)} \\
&= \prod_{k=1}^{M_r} \frac{z^{-1}(z - d_k)}{z^{-1} - d_k} \prod_{k=1}^{M_c} \frac{z^{-2}(z - e_k)(z - e_k^*)}{(z^{-1} - e_k^*)(z^{-1} - e_k)} \\
&= \prod_{k=1}^{M_r} \frac{z - d_k}{1 - d_k z} \prod_{k=1}^{M_c} \frac{(z - e_k)(z - e_k^*)}{(1 - e_k^* z)(1 - e_k z)} \\
&= H(1/z), \quad |z| \in \frac{1}{R_x}
\end{aligned}$$

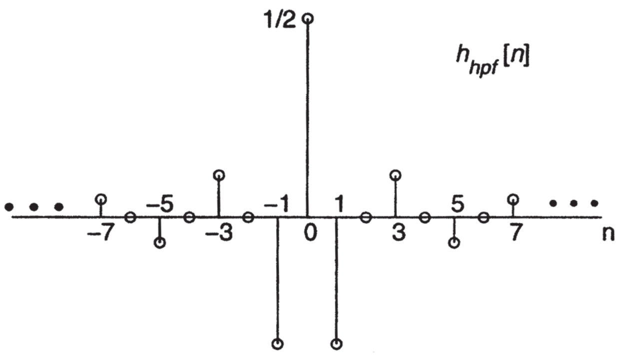
which in the time domain is

$$h_i[n] = h[-n]$$

5.80. We can model $g[n]$ as

$$g[n] = x[n] + \alpha\delta[n - n_0]$$

Now send the corrupted signal $g[n]$ through a highpass filter $h_{hpf}[n]$ with a cutoff of $w_c = \pi/2$.



$$h_{hpf}[n] = (-1)^n \frac{\sin \frac{\pi}{2} n}{\pi n}$$

The highpass filter completely filters out the lowpass signal $x[n]$. The output $y[n]$ is

$$\begin{aligned} y[n] &= (x[n] + \alpha\delta[n - n_0]) * h_{hpf}[n] \\ &= \alpha h_{hpf}[n - n_0] \\ &= \alpha(-1)^{(n-n_0)} \frac{\sin \frac{\pi}{2}(n - n_0)}{\pi(n - n_0)} \end{aligned}$$

$y[n]$ looks similar to the picture of $h_{hpf}[n]$ above except that it is scaled by α and shifted to n_0. Thus,

$$\alpha = 2y[n_0]$$

$$x[n] = g[n] - 2y[n_0]\delta[n - n_0]$$

(a) When n_0 is odd, $y[n] = 0$ at all odd values of n except $n = n_0$. This leads to a procedure to find $x[n]$ from $g[n]$:
- Filter $g[n]$ with the highpass filter described above.
- Find the only nonzero value at an odd index in the output $y[n]$. This value is $y[n_0]$.
- $x[n] = g[n] - 2y[n_0]\delta[n - n_0]$

(b) The only time three consecutive nonzero samples occur in $y[n]$ is at $n = n_0$. The procedure to find $x[n]$ is
- Filter $g[n]$ with the highpass filter described above.
- Look for three consecutive nonzero output samples. The middle value is $y[n_0]$.
- $x[n] = g[n] - 2y[n_0]\delta[n - n_0]$

5.81. Looking at the z-transform of the FIR filter,

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty} h[n]z^{-n} \\ &= \sum_{n=0}^{N-1} h[N-1-n]z^{-n} \end{aligned}$$

Substituting $m = N - 1 - n$ into the summation gives

$$\begin{aligned} H(z) &= \sum_{m=N-1}^{0} h[m]z^{m-N+1} \\ &= \sum_{m=0}^{N-1} h[m]z^{m}z^{-N+1} \\ &= z^{-N+1}\sum_{m=0}^{N-1} h[m]z^{m} \\ &= z^{-N+1}H(z^{-1}) \end{aligned}$$

Thus, for such a filter,

$$H(1/z) = z^{N-1}H(z)$$

If z_0 is a zero of $H(z)$, then $H(z_0) = 0$, and

$$H(1/z_0) = z_0^{N-1}H(z_0) = 0$$

Consequently, even-symmetric linear phase FIR filters have zeros that are reciprocal images.