11.1. Given

$$\phi_{ss}[i,k] = \sum_{n=-\infty}^{\infty} s[n-i]s[n-k],$$

let $m = n - i$. Then

$$\begin{aligned}\phi_{ss}[i,k] &= \sum_{m=-\infty}^{\infty} s[m]s[m+i-k] \\ &= r_{ss}[i-k].\end{aligned}$$

Now $r_{ss}[k-i]$ is given by

$$\begin{aligned}r_{ss}[k-i] &= \sum_{m=-\infty}^{\infty} s[m]s[m+k-i] \\ &= \sum_{m=-\infty}^{\infty} s[m+k-i]s[m].\end{aligned}$$

Substituting $n = m + k - i$ gives

$$\begin{aligned}r_{ss}[k-i] &= \sum_{n=-\infty}^{\infty} s[n]s[n+i-k] \\ &= r_{ss}[i-k].\end{aligned}$$

Thus we can write

$$r_{ss}\left[|i-k|\right] = \sum_{m=-\infty}^{\infty} s[m]s[m+i-k].$$

Therefore

$$\phi_{ss}[i,k] = r_{ss}\left[|i-k|\right],$$

which is a function of $|i-k|$.

11.2. Given

$$E = \left\langle \left(s[n] - \sum_{k=1}^{p} a_k s[n-k] \right)^2 \right\rangle.$$

A. Then

$$\begin{aligned}
E &= \left\langle s^2[n] - s[n]\sum_{i=1}^{p} a_i s[n-i] - s[n]\sum_{k=1}^{p} a_k s[n-k] + \sum_{i=1}^{p} a_i s[n-i] \sum_{i=k}^{p} a_k s[n-k] \right\rangle \\
&= \left\langle s^2[n] \right\rangle - 2\left\langle s[n]\sum_{k=1}^{p} a_k s[n-k] \right\rangle + \left\langle \sum_{i=1}^{p} a_i s[n-i] \sum_{k=1}^{p} a_k s[n-k] \right\rangle \\
&= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \left\langle s[n]s[n-k] \right\rangle + \sum_{i=1}^{p}\sum_{k=1}^{p} a_i a_k \left\langle s[n-i]s[n-k] \right\rangle \\
&= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \phi_{ss}[0,k] + \sum_{i=1}^{p} a_i \sum_{k=1}^{p} a_k \phi_{ss}[i,k],
\end{aligned}$$

as required.

B. Now suppose

$$\sum_{k=1}^{p} a_k \phi_{ss}[i,k] = \phi_{ss}[i,0], \quad i = 1, \ldots, p.$$

Substituting gives

$$\begin{aligned}
E &= \phi_{ss}[0,0] - 2\sum_{k=1}^{p} a_k \phi_{ss}[0,k] + \sum_{i=1}^{p} a_i \phi_{ss}[i,0] \\
&= \phi_{ss}[0,0] - \sum_{k=1}^{p} a_k \phi_{ss}[0,k],
\end{aligned}$$

since $\phi_{ss}[k,0] = \phi_{ss}[0,k]$.

11.3. Given

$$h[n] = \sum_{k=1}^{p} a_k h[n-k] + G\delta[n].$$

A. We have

$$\begin{aligned} r_{hh}[-m] &= \sum_{n=-\infty}^{\infty} h[n]h[n-m] \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=1}^{p} a_k h[n-k] + G\delta[n] \right) h[n-m] \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=1}^{p} a_k h[n-k]h[n-m] + \sum_{n=-\infty}^{\infty} G\delta[n]h[n-m] \\ &= \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m] + Gh[-m] \\ &= \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m], \quad m = 1, \ldots, p, \end{aligned}$$

since the all-pole system with impulse response $h[k]$ is causal. Now substitute $j = n-k$ to obtain

$$\begin{aligned} r_{hh}[-m] &= \sum_{k=1}^{p} a_k \sum_{j=-\infty}^{\infty} h[j]h[j+k-m] \\ &= \sum_{k=1}^{p} a_k r_{hh}[k-m] \\ &= \sum_{k=1}^{p} a_k r_{hh}[|m-k|], \quad m = 1, \ldots, p, \end{aligned}$$

where the last line reflects the fact that $r_{hh}[m]$ is an even function. Further, since $r_{hh}[-m] = r_{hh}[m]$ we have

$$r_{hh}[m] = \sum_{k=1}^{p} a_k r_{hh}[|m-k|], \quad m = 1, \ldots, p.$$

B. As above, we have

$$r_{hh}[-m] = \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n-m] + Gh[-m].$$

Now let $m = 0$. We have

$$\begin{aligned} r_{hh}[0] &= \sum_{k=1}^{p} a_k \sum_{n=-\infty}^{\infty} h[n-k]h[n] + Gh[0] \\ &= \sum_{k=1}^{p} a_k r_{hh}[k] + Gh[0]. \end{aligned}$$

But $h[0] = G\delta[0] = G$. Then

$$r_{hh}[0] = \sum_{k=1}^{p} a_k r_{hh}[k] + G^2,$$

as was to have been shown.

11.4. Appears in: Spring04 PS6, Fall03 PS6.

Problem

Consider a signal $x[n] = s[n] + w[n]$, where $s[n]$ is a first order autoregressive process that satisfies the difference equation

$$s[n] = 0.8s[n-1] + v[n]$$

where $v[n]$ is a white noise sequence with variance $\sigma_v^2 = 0.49$ and $w[n]$ is a white noise sequence with variance $\sigma_w^2 = 1$. The processes $v[n]$ and $w[n]$ are uncorrelated.

Determine the autocorrelation sequences $\phi_{ss}[m]$ and $\phi_{xx}[m]$.

Solution from Spring04 PS6

In addition to the information stated in the problem, students were also told that $v[n]$ and $w[n]$ were 0 mean sequences.

The signal $s[n]$ is generated by passing a 0 mean WSS sequence $v[n]$ through the LTI system

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

Assuming that $h[n]$ is causal, $h[n] = (0.8)^n u[n]$, and

$$\phi_{ss}[m] = \phi_{vv}[n] * h[n] * h[-n]|_{n=m}$$

We can evaluate $h[n] * h[-n]$ using the same method as in problem 6.2 part (c), and we obtain

$$h[n] * h[-n] = \frac{1}{0.36}(0.8)^{|m|}$$

Therefore,

$$\phi_{ss}[m] = \frac{0.49}{0.36}(0.8)^{|m|}$$

Since $w[n]$ is uncorrelated with $v[n]$, $w[n]$ is also uncorrelated with $s[n]$, and

$$\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ww}[m] = \phi_{ss}[m] + \delta[m]$$

Solution from Fall03 PS6

In addition to the information stated in the problem, students were also told that $v[n]$ and $w[n]$ were 0 mean sequences.

The signal $s[n]$ is generated by passing a 0 mean WSS sequence $v[n]$ through the LTI system

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

Assuming that $h[n]$ is causal, $h[n] = (0.8)^n u[n]$, and

$$\phi_{ss}[m] = \phi_{vv}[n] * h[n] * h[-n]|_{n=m}$$

We can evaluate $h[n] * h[-n]$ using the same method as in problem 1 part (c), and we obtain

$$h[n] * h[-n] = \frac{1}{0.36}(0.8)^{|m|}$$

Therefore,

$$\phi_{ss}[m] = \frac{0.49}{0.36}(0.8)^{|m|}$$

Since $w[n]$ is uncorrelated with $v[n]$, $w[n]$ is also uncorrelated with $s[n]$, and

$$\phi_{xx}[m] = \phi_{ss}[m] + \phi_{ww}[m] = \phi_{ss}[m] + \delta[m]$$

11.5. Problem 5 in Spring2005 Midterm exam.

Problem

Recall: In the inverse filter approach to all-pole modeling of a deterministic signal $s[n]$, we consider

$$s[n] \rightarrow \boxed{\frac{1}{A}\left[1 - \sum_{k=1}^{p} a_k z^{-k}\right]} \rightarrow g[n]$$

and choose the coefficients $a_1, a_2, \ldots, a_p$ to minimize

$$\mathcal{E} = \sum_{n=0}^{\infty} (g[n] - \delta[n])^2.$$

We then consider

$$\frac{A}{1 - \sum_{k=1}^{p} a_k z^{-k}}$$

to be the best available approximation to $S(z)$.

(a) Find the coefficients a_1 and a_2 of the best all-pole model for $s[n] = \delta[n] + \delta[n-2]$ with $p = 2$.

(b) Find the coefficients a_1, a_2 and a_3 of the best all-pole model for $s[n] = \delta[n] + \delta[n-2]$ with $p = 3$.

Solution from Spring05 midterm

Every term and notation in this problem matches the Parametric Signal Modeling handout. The relevant samples of the deterministic correlation sequence of $s[n]$ are:

$$\begin{aligned} \phi_{ss}[0] &= 2 \\ \phi_{ss}[1] &= 0 \\ \phi_{ss}[2] &= 1 \end{aligned}$$

In matrix form, the autocorrelation normal equations are

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From this we have (uniquely) that $a_1 = 0$ and $a_2 = \frac{1}{2}$.

We now need one more sample of the deterministic autocorrelation:

$$\phi_{ss}[3] = 0.$$

The autocorrelation normal equations are now

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which have the (unique) solution $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = 0$.

11.6. The all-pole spectrum estimate is given by

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{G}{1-\sum_{i=1}^{p} a_i e^{-j\omega i}}\right|.$$

For simplicity let us assume that $N > p$. If $\omega_k = 2\pi k/N, \quad k = 0,\ldots,N-1$, then

$$\left|H\left(e^{j\omega_k}\right)\right| = \left|\frac{G}{1-\sum_{i=1}^{p} a_i e^{-j2\pi ki/N}}\right|, \quad k = 0,\ldots,N-1.$$

Now let $a[0]=1$, $a[i]=-a_i, \quad i=1,\ldots,p$, and $a[i]=0, \quad i=p+1,\ldots,N-1$. Then

$$A[k] = 1-\sum_{i=1}^{p} a_i e^{-j2\pi ki/N}$$

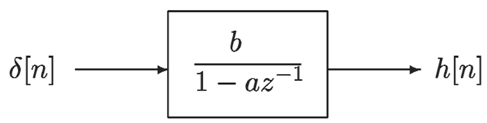
is the DFT of the finite sequence $a[i], \quad i=0,\ldots,N-1$. The algorithm is

1. Given the coefficients $a_i, \quad i=1,\ldots,p$, form $a[i], \quad i=0,\ldots,N-1$.
2. Use the FFT program to find $A[k]$, the DFT of $a[i]$.
3. The spectrum estimate is $\left|\dfrac{G}{A[k]}\right|, \quad k=0,\ldots,N-1$.

11.7. Appears in: Spring05 PS7, Fall04 PS6, Fall02 PS6, Fall01 PS9. Note: The Fall01 version has a few minor differences.

Problem

Consider a desired, causal impulse response $h_d[n]$ that we wish to approximate by the following system:



Our optimality criterion is to minimize the error function given by:

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - h[n])^2$$

(a) Supose a is given and we wish to determine the unknown parameter b which minimizes ε. Assume that $|a| < 1$. Is this a nonlinear problem? If so, show why. If not, determine b.

(b) Suppose b is given and we wish to determine the unknown parameter a which minimizes ε. Is this a nonlinear problem? If so, show why. If not, determine a.

Solution from Spring05 PS7

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b \frac{1}{1 - a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall04 PS6

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b\frac{1}{1-a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall02 PS6

$H(z)$ is causal, thus:

$$H(z) = \frac{b}{1 - az^{-1}} \longrightarrow h[n] = ba^n u[n]$$

$$\varepsilon = \sum_{n=0}^{\infty} (h_d[n] - ba^n)^2$$

(a)

$$\frac{\partial \varepsilon}{\partial b} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(-a^n) = 0$$

$$\sum_{n=0}^{\infty} a^n h_d[n] = b \sum_{n=0}^{\infty} (a^2)^n = b \frac{1}{1 - a^2}$$

$$\Rightarrow b = (1 - a^2) \sum_{n=0}^{\infty} a^n h_d[n]$$

(b)

$$\frac{\partial \varepsilon}{\partial a} = \sum_{n=0}^{\infty} 2(h_d[n] - ba^n)(bna^{n-1}) = 0 \longrightarrow \textbf{Nonlinear}$$

This problem illustrates that even for this simple first order all-pole system, it's intractable to obtain coefficients by directly matching $h_d[n]$ and $h[n]$, thus justifying the inverse approach, which yields the normal equations.

Solution from Fall01 PS9

N/A

11.8. A. The prediction error sequence $\tilde{e}[n] = s[n] - \sum_{k=1}^{p} \beta_k s[n+k]$ is the convolution of the sequence $s[n]$ with the impulse response $h_B[n] = \delta[n] - \sum_{k=1}^{p} \beta_k \delta[n+k]$ of the prediction-error filter. Now $s[n]$ takes non-zero values only in the interval $0 \le n \le M-1$ and $h_B[n]$ takes non-zero values only in the interval $-p \le n \le 0$. Thus the convolution is non-zero only in the interval $N_1 = -p \le n \le N_2 = M-1$.

B. The mean-squared backward prediction error is defined as

$$\tilde{E} = \sum_{m=-\infty}^{\infty} \left(s[m] - \sum_{k=1}^{p} \beta_k s[m+k] \right)^2.$$

Setting the derivatives of $\tilde{E}$ with respect to β_i, $i = 1, \ldots, p$, equal to zero gives

$$\frac{\partial \tilde{E}}{\partial \beta_i} = -\sum_{m=-\infty}^{\infty} \left(s[m] - \sum_{k=1}^{p} \beta_k s[m+k] \right) s[m+i] = 0, \quad i = 1, \ldots, p.$$

That is,

$$\sum_{m=-\infty}^{\infty} s[m] s[m+i] = \sum_{k=1}^{p} \beta_k \sum_{m=-\infty}^{\infty} s[m+k] s[m+i], \quad i = 1, \ldots, p.$$

Now define the autocorrelation function $r_{ss}[i-k]$ by

$$r_{ss}[i-k] = \sum_{m=-\infty}^{\infty} s[m+k] s[m+i], \quad i = 1, \ldots, p, \quad k = 1, \ldots, p.$$

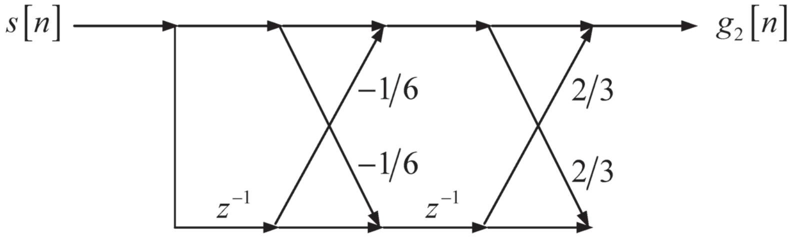
Then we have the required normal equations

$$r_{ss}[i] = \sum_{k=1}^{p} \beta_k r_{ss}[i-k], \quad i = 1, \ldots, p.$$

C. The normal equations derived in part B for backward prediction are the same as those derived in the text for forward prediction. Consequently, the backward predictor coefficients $\{\beta_k\}$ are identical to the forward predictor coefficients $\{\alpha_k\}$.

11.9. A. By tracing paths through the flow graph it is easy to see that the coefficient of z^{-1} is $h_{inv}^{(4)}[1] = -1/18$.

B.



C. First we convert the reflection coefficients into system function coefficients for the inverse filter. We have

$a_1^{(1)} = k_1 = 1/6$

$a_2^{(2)} = k_2 = -2/3$

$a_1^{(2)} = a_1^{(1)} - k_2 a_1^{(1)} = 5/18$.

Then $H_{inv}^{(2)}(z) = 1 - \frac{5}{18} z^{-1} + \frac{2}{3} z^{-2}$, and

$$H^{(2)}(z) = \frac{1}{1 - \frac{5}{18} z^{-1} + \frac{2}{3} z^{-2}}.$$

11.10. We have

$$A^{(i)}(z) = 1 - a_1^{(i)} z^{-1} - a_2^{(i)} z^{-2} - \cdots - a_i^{(i)} z^{-i}$$
$$= \left(1 - z_1^{(i)} z^{-1}\right)\left(1 - z_2^{(i)} z^{-1}\right)\cdots\left(1 - z_i^{(i)} z^{-1}\right).$$

Multiplying out, we see that the coefficient $a_i^{(i)}$ of z^{-i} is

$$k_i = a_i^{(i)} = -(-1)^i z_1^{(i)} z_2^{(i)} \cdots z_i^{(i)}.$$

Then

$$|k_i| = \left|z_1^{(i)} z_2^{(i)} \cdots z_i^{(i)}\right| = \left|z_1^{(i)}\right|\left|z_2^{(i)}\right|\cdots\left|z_i^{(i)}\right|.$$

Now if $\left|z_j^{(i)}\right| < 1, \quad j = 1, \ldots, i$, it follows that $|k_i| < 1$. Consequently, if $|k_i| \geq 1$, then we must have $\left|z_j^{(i)}\right| \geq 1$ for some j.

11.11. A. We have a white noise input to the system with zero mean, unit variance, and $r_{xx}[m] = \sigma_x^2\delta[m] = \delta[m]$. Taking the inverse z-transform of the given system function $H(z)$, we have the following impulse response:

$$h[n] = h_0\delta[n] + h_1\delta[n-1].$$

The autocorrelation of the output of the system can be determined as follows:

$$\begin{aligned}
r_{yy}[m] &= h[m] * h[-m] * r_{xx}[m] \\
&= h[m] * h[-m] * \delta[m] \\
&= h[m] * h[-m] \\
&= h_0h_1\delta[m+1] + \left(h_0^2 + h_1^2\right)\delta[m] + h_0h_1\delta[m-1].
\end{aligned}$$

B. The impulse response of the forward prediction error filter is

$$h_A[n] = \delta[n] - a_1\delta[n-1] - a_2\delta[n-2],$$

and we have the following relation for the autocorrelation function of the error:

$$r_{ee}[m] = h_A[m] * h_A[-m] * r_{yy}[m].$$

Now

$$\begin{aligned}
h_A[m] * h_A[-m] = &-a_2\delta[m+2] + (-a_1 + a_1a_2)\delta[m+1] + \left(1 + a_1^2 + a_2^2\right)\delta[m] \\
&+ (-a_1 + a_1a_2)\delta[m-1] - a_2\delta[m-2]
\end{aligned}$$

and we have $r_{yy}[m]$ from part A. Convolving the two will give $r_{ee}[m]$. However, we want to minimize the variance of $e[n]$, and so we just need to compute $r_{ee}[0]$. We obtain

$$\begin{aligned}
r_{ee}[0] &= (-a_1 + a_1b_1)(h_0h_1) + \left(1 + a_1^2 + a_2^2\right)\left(h_0^2 + h_1^2\right) + (-a_1 + a_1b_1)(h_0h_1) \\
&= \left(1 + a_1^2 + a_2^2\right)\left(h_0^2 + h_1^2\right) + 2(-a_1 + a_1b_1)(h_0h_1).
\end{aligned}$$

Now we take partial derivatives with respect to a_1 and a_2, set them equal to zero, and solve for a_1 and a_2 respectively.

$$\frac{\partial r_{ee}[0]}{\partial a_1} = 2a_1\left(h_0^2 + h_1^2\right) + 2h_0h_1(-1 + a_2) = 0$$

$$\frac{\partial r_{ee}[0]}{\partial a_2} = 2a_2\left(h_0^2 + h_1^2\right) + 2h_0h_1a_1 = 0.$$

We have two equations in two unknowns. Solving for a_1 and a_2 gives

$$a_1 = \frac{h_0h_1\left(h_0^2 + h_1^2\right)}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}$$

$$a_2 = \frac{-h_0^2h_1^2}{\left(h_0^2 + h_1^2\right)^2 - h_0^2h_1^2}.$$

C. The impulse response of the backward prediction error filter is

$$h_B[n] = \delta[n] - b_1\delta[n+1] - b_2\delta[n+2],$$

and we have the following relation for the autocorrelation function of the error:

$$r_{\tilde{e}\tilde{e}}[m] = h_B[m] * h_B[-m] * r_{yy}[m].$$

Now

$$h_B[m] * h_B[-m] = -b_2\delta[m+2] + (-b_1 + b_1b_2)\delta[m+1] + (1 + b_1^2 + b_2^2)\delta[m] + (-b_1 + b_1b_2)\delta[m-1] - b_2\delta[m-2]$$

and we have $r_{yy}[m]$ from part A. Convolving the two will give $r_{\tilde{e}\tilde{e}}[m]$. We note, however, that $r_{\tilde{e}\tilde{e}}[m]$ is identical to $r_{ee}[m]$ with b_1 and b_2 substituted for a_1 and a_2, respectively. The solution proceeds exactly as in part B, yielding

$$b_1 = \frac{h_0h_1(h_0^2 + h_1^2)}{(h_0^2 + h_1^2)^2 - h_0^2h_1^2}$$

$$b_2 = \frac{-h_0^2h_1^2}{(h_0^2 + h_1^2)^2 - h_0^2h_1^2}.$$

We see that $b_1 = a_1$ and $b_2 = a_2$.

11.12. A. For the given all-pole model, $a_1 = a, a_2 = 0, a_3 = b$. Then

$$ar_{ss}[i-1] + br_{ss}[i-3] = r_{ss}[i], \quad i = 1,2,3.$$

B. First observe that

$$v[n] = x[n-1] + \frac{1}{2}x[n-2] + z[n].$$

Now

$$\phi_{vv}[n] = \overline{v[n]v[n+m]}$$
$$= \overline{\left(x[n-1] + \frac{1}{2}x[n-2] + z[n]\right)\left(x[n-1+m] + \frac{1}{2}x[n-2+m] + z[n]\right)}.$$

Multiplying out and averaging term-by-term gives

$$\phi_{vv}[m] = \frac{1}{2}\delta[m+1] + \frac{9}{4}\delta[m] + \frac{1}{2}\delta[m-1],$$

where we have used the facts that $\phi_{xx}[m] = \delta[m]$, $\phi_{zz}[m] = \delta[m]$, and $x[n]$ and $z[n]$ are independent and each of zero mean.

C. Let the output of the system $H_1(z)$ with input $x[n]$ be designated $x_1[n]$. Then $y_1[n] = x_1[n] + z[n]$. Let $H_2(z) = 1 - az^{-1} - bz^{-3}$.

Now the response of an LTI system to a zero mean input also has zero mean. Thus $\overline{x_1[n]} = 0$. Then $\overline{y_1[n]} = \overline{x_1[n] + z[n]} = \overline{x_1[n]} + \overline{z[n]} = 0$, and consequently $\overline{w_1[n]} = 0$.

By linearity, $w_1[n] = w_x[n] + w_z[n]$, where $w_x[n]$ is the response of $H_2(z)$ to $x_1[n]$ and $w_z[n]$ is the response of $H_2(z)$ to $z[n]$. It can be shown by direct calculation that $w_x[n]$ and $w_z[n]$ are uncorrelated owing to the independence of $x[n]$ and $z[n]$, and consequently $\phi_{w_1w_1}[m] = \phi_{w_xw_x}[m] + \phi_{w_zw_z}[m]$. Now $w_x[n]$ results from passing $x[n]$ through $H_1(z)$ and then through $H_2(z)$. Since $H_2(z)$ is the inverse of $H_1(z)$, $w_x[n]$ will be white because $x[n]$ is white. On the other hand, $w_z[n]$ is not white; its spectrum is shaped by the frequency response of $H_2(z)$. We have $\phi_{w_1w_1}[m] = \phi_{w_xw_x}[m] + \phi_{w_zw_z}[m] = \delta[m] + \phi_{w_zw_z}[m] \neq c\delta[m]$ for any constant c.

D. Since $w_1[n]$ has zero mean, the variance of $w_1[n]$ is $\phi_{w_1w_1}[0]$. Now

$$\phi_{w_1w_1}[0] = \phi_{w_xw_x}[0] + \phi_{w_zw_z}[0]$$
$$= 1 + \phi_{w_zw_z}[0]$$
$$= 1 + \overline{w_z^2[n]}.$$

We have

$$\overline{w_z^2[n]} = \overline{\left(z[n] - az[n-1] - bz[n-3]\right)^2}$$
$$= 1 + a^2 + b^2.$$

Therefore $\phi_{w_1w_1}[0] = 2 + a^2 + b^2$.

11.13. A. Using $g[n]=s[n]-a_1s[n-1]-a_2s[n-2]$,

$$g[0]-G\delta[0]=4-G$$
$$g[1]-G\delta[1]=8-4a_1$$
$$g[2]-G\delta[2]=4-8a_1-4a_2$$
$$g[3]-G\delta[3]=2-4a_1-8a_2$$
$$g[4]-G\delta[4]=1-2a_1-4a_2$$
$$g[5]-G\delta[5]=\tfrac{1}{2}-a_1-2a_2$$

B. The linear equations are $4-G=0$ and

$$\begin{bmatrix} 4 & 8 & 4 & 2 & 1 \\ 0 & 4 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 8 & 4 \\ 4 & 8 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} 4 & 8 & 4 & 2 & 1 \\ 0 & 4 & 8 & 4 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 4 \\ 2 \\ 1 \\ \frac{1}{2} \end{bmatrix}.$$

C. $G=4$.

D. Evaluating $g[1]-G\delta[1]$ yields $a_1=2$.
Using this value and evaluating $g[2]-G\delta[2]$ yields $a_2=-3$.
Using these values, we get a nonzero value for $g[3]-G\delta[3]$. Therefore we do not expect $\mathcal{E}$ to be zero.

E. Using $g[n]=s[n]-as[n-1]$ and $r[n]=b_0\delta[n]+b_1\delta[n-1]$,

$$g[0]-r[0]=4-b_0$$
$$g[1]-r[1]=8-4a-b_1$$
$$g[2]-r[2]=4-8a$$
$$g[3]-r[3]=2-4a$$
$$g[4]-r[4]=1-2a$$
$$g[5]-r[5]=\tfrac{1}{2}-a$$

F. From the equations in part E involving only a, $a=\frac{1}{2}$.
From the equations involving b_0 and b_1, if the system needs to be minimum phase then setting $b_0=5$ and $b_1=5$ puts the system at the edge of being minimum phase while minimizing $\mathcal{E}_2$. If the system does not need to be minimum phase then $b_0=4$ and $b_1=6$.

G. If the system needs to be minimum phase, $\mathcal{E}_2=2$.
If the system does not need to be minimum phase, $\mathcal{E}_2=0$.

11.14. Given $s[n]=\alpha^n, \quad n=0,\ldots,M$. In the following it is assumed that $|\alpha|<1$.

A. Using the autocorrelation method,

$$r_{ss}[m]=r_{ss}[-m]=\begin{cases}\sum_{n=0}^{M-m} s[n]s[n+m], & m=0,\ldots,M\\ 0, & m>M.\end{cases}$$

Then for $m=-M,\ldots,M$,

$$r_{ss}[m]=\sum_{m=0}^{M-|m|}\alpha^n\alpha^{n+|m|}=\sum_{m=0}^{M-|m|}\alpha^{2n+|m|}=\alpha^{|m|}\sum_{m=0}^{M-|m|}\left(\alpha^2\right)^n$$
$$=\alpha^{|m|}\frac{1-\alpha^{2(M-|m|+1)}}{1-\alpha^2}.$$

B. Now $a_1 r_{ss}[0]=r_{ss}[1]$. Then

$$a_1=\frac{\alpha\dfrac{1-\alpha^{2M}}{1-\alpha^2}}{\dfrac{1-\alpha^{2(M+1)}}{1-\alpha^2}}=\alpha\frac{1-\alpha^{2M}}{1-\alpha^{2(M+1)}}.$$

C. Now as $M\to\infty$ we have $\alpha^{2M}\to 0$ and $\alpha^{2(M+1)}\to 0$. Then $a_1\to\alpha$.

D. We have $E=r_{ss}[0]-a_1 r_{ss}[1]$. Then

$$E=\frac{1-\alpha^{2(M+1)}}{1-\alpha^2}-\alpha\frac{1-\alpha^{2M}}{1-\alpha^{2(M+1)}}\alpha\frac{1-\alpha^{2M}}{1-\alpha^2}$$
$$=\frac{1-\alpha^{4M+2}}{1-\alpha^{2M+2}}.$$

As $M\to\infty$ we have $\alpha^{4M+2}\to 0$ and $\alpha^{2M+2}\to 0$, so $E\to 1$.

E. Using the covariance method,

$$\begin{aligned}
\phi_{ss}[i,k] &= \sum_{n=1}^{M} s[n-i]s[n-k] \\
&= \sum_{n=1}^{M} \alpha^{(n-i)}\alpha^{(n-k)} = \alpha^{-i-k}\sum_{n=1}^{M}\left(\alpha^2\right)^n \\
&= \alpha^{-i-k}\sum_{n=0}^{M}\left(\alpha^2\right)^n - \alpha^{-i-k} \\
&= \alpha^{-i-k}\frac{1-\alpha^{2(M+1)}}{1-\alpha^2} - \alpha^{-i-k}\frac{1-\alpha^2}{1-\alpha^2} \\
&= \alpha^{-i-k}\frac{\alpha^2-\alpha^{2(M+1)}}{1-\alpha^2} = \alpha^{2-i-k}\frac{1-\alpha^{2M}}{1-\alpha^2}.
\end{aligned}$$

F. Now $a_1\phi_{ss}[1,1] = \phi_{ss}[1,0]$. Then

$$a_1 = \frac{\alpha\dfrac{1-\alpha^{2M}}{1-\alpha^2}}{\dfrac{1-\alpha^{2M}}{1-\alpha^2}} = \alpha.$$

This is identical to the result given in the example in the chapter. The result found in part B approaches this value as M becomes large.

G. The minimum mean-squared prediction error is given by

$$E = \phi_{ss}[0,0] - a_1\phi_{ss}[0,1].$$

That is,

$$E = \alpha^2\frac{1-\alpha^{2M}}{1-\alpha^2} - \alpha\times\alpha\frac{1-\alpha^{2M}}{1-\alpha^2} = 0.$$

The covariance method produces zero prediction error in this case. This is smaller than the error produced by either the autocorrelation method above or by the example in the chapter.

11.15. Problem 5 in Spring 2002 Final exam.
Appears in: Fall04 PS6, Fall02 PS6.

Problem

(a) Consider the signal

$$p[n] = 3\left(\frac{1}{2}\right)^n u[n] + 4\left(-\frac{2}{3}\right)^n u[n] .$$

(i) We want to use a causal, *second*-order all-pole model, i.e., a model of the form

$$\widehat{P}(z) = \frac{A}{1 - a_1 z^{-1} - a_2 z^{-2}} ,$$

to optimally represent the signal $p[n]$, in the least-square error sense. Find a_1, a_2, and A.

(ii) Now suppose we want to use a causal, *third*-order all-pole model, i.e., a model of the form

$$\widetilde{P}(z) = \frac{B}{1 - b_1 z^{-1} - b_2 z^{-2} - b_3 z^{-3}} ,$$

to optimally represent the signal $p[n]$, in the least-square error sense. Find, b_1, b_2, b_3, and B.

(b) Consider the signal

$$q[n] = 3\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{2}{3}\right)^n u[n] .$$

We want to use a *second*-order linear-predictive model of the form

$$\widehat{q}[n] = c_1 q[n-1] + c_2 q[n-2]$$

to optimally represent the signal $q[n]$, in the least-square error sense. Find c_1 and c_2, and the range of n for which the model is exact, i.e., find c_1, c_2, and the range of n for which $q[n]$ is *exactly* linearly predictable from $q[n-1]$ and $q[n-2]$? **Hint:** Look at the difference equation that $q[n]$ satisfies.

Solution from Fall04 PS6

(a) (i)

$$P(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{4}{1+\frac{2}{3}z^{-1}} = \frac{3+2z^{-1}+4-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{2}{3}z^{-1})}$$

$$P(z) = \frac{7}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

We can perfectly match $\hat{P}(z)$ with $P(z)$ by choosing:

$$a_1 = -\frac{1}{6}, : a_2 = \frac{1}{3}, : and : A = 7$$

$p[n]$ has a second-order all-pole model.

(ii) Increasing the model order does not buy us anything in this case, because $p[n]$ itself has a second-order all-pole model. Hence:

$$b_1 = a_1 = -\frac{1}{6},\ b_2 = a_2 = \frac{1}{3},\ B = A = 7,\ and\, b_3 = 0$$

(b)

$$Q(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{2}{3}z^{-1}} = \frac{4+\frac{3}{2}z^{-1}}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

$q[n]$ has an ARMA model, not an exact all-pole model. However, $q[n]$ satisfies the difference equation:

$$q[n] = -\frac{1}{6}q[n-1] + \frac{1}{3}q[n-2] + 4\delta[n] + \frac{3}{2}\delta[n-1]$$

Clearly, for $n \geq 2$ the last two terms vanish and $q[n]$ becomes perfectly linearly predictable from $q[n-1]$ and $q[n-2]$ for $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{3}$.

Solution from Fall02 PS6

(a) (i)

$$P(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{4}{1+\frac{2}{3}z^{-1}} = \frac{3+2z^{-1}+4-2z^{-1}}{(1-\frac{1}{2}z^{-1})(1+\frac{2}{3}z^{-1})}$$

$$P(z) = \frac{7}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

We can perfectly match $\hat{P}(z)$ with $P(z)$ by choosing:

$$a_1 = -\frac{1}{6},\ a_2 = \frac{1}{3},\ and\ A = 7$$

$p[n]$ has a second-order all-pole model.

(ii) Increasing the model order does not buy us anything in this case, because $p[n]$ itself has a second-order all-pole model. Hence:

$$b_1 = a_1 = -\frac{1}{6},\ b_2 = a_2 = \frac{1}{3},\ B = A = 7,\ and\ b_3 = 0$$

(b)

$$Q(z) = \frac{3}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{2}{3}z^{-1}} = \frac{4+\frac{3}{2}z^{-1}}{1+\frac{1}{6}z^{-1}-\frac{1}{3}z^{-2}}$$

$q[n]$ has an ARMA model, not an exact all-pole model. However, $q[n]$ satisfies the difference equation:

$$q[n] = -\frac{1}{6}q[n-1] + \frac{1}{3}q[n-2] + 4\delta[n] + \frac{3}{2}\delta[n-1]$$

Clearly, for $n \geq 2$ the last two terms vanish and $q[n]$ becomes perfectly linearly predictable from $q[n-1]$ and $q[n-2]$ for $c_1 = -\frac{1}{6}$ and $c_2 = \frac{1}{3}$.

Solution from Spring02 Final

N/A

11.16. Appears in: Fall05 PS6, Fall04 PS6, Fall02 PS6. Note: There are versions of the problem that use different numbers, only solutions for this version are included here.

Problem

Consider the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

We wish to model this signal using a second-order ($p = 2$) all-pole model, or equivalently using second-order linear prediction.

For this problem, since we are given an analytical expression for $s[n]$ and $s[n]$ is the impulse response of an all-pole filter, we can obtain the linear prediction coefficients directly from the Z-transform of $s[n]$ (you're asked to do this in part (a)). In practical situations, we are typically given data, i.e. a set of signal values, and not an analytical expression. In this case, even when the signal to be modeled is the impulse response of an all-pole filter, we need to perform some computation on the data, such as the methods discussed in the lecture and in the notes, in order to determine the linear prediction coefficients.

There are also situations in which an analytical expression is available for the signal but the signal is not the impulse response of an all-pole filter and we would like to model it as such, in which case, we again need to carry out computations such as those discussed in lecture and the notes.

(a) Determine the linear prediction coefficients a_1, a_2 directly from the Z-transform of $s[n]$.

(b) Write the normal equations for $p = 2$ to obtain equations for a_1, a_2 in terms of $\phi_s[m]$.

(c) Find the values of $\phi_s[0]$, $\phi_s[1]$, and $\phi_s[2]$ for the signal $s[n]$ given above.

(d) Solve the system of equations from part (a) using the values you found in part (b) to get values for the a_k's. You may find MATLAB or another computer tool useful to solve these equations.

(e) Are the values of a_k from part (c) what you would expect for this signal? Justify your answer clearly.

(f) Suppose you wish to model the signal now with $p = 3$. Write the normal equations for this case.

(g) Find the value of $\phi_s[3]$.

(h) Solve for the values of a_k when $p = 3$. You may find MATLAB or another computer tool useful to solve these equations.

(i) Are the values of a_k found in part (h) what you would expect given $s[n]$? Justify your answer clearly.

(j) Would the values of a_1, a_2 you found in (h) change if we model the signal with $p = 4$?

Solution from Fall05 PS6

(a)
$$S(z) = \frac{2}{1-\frac{1}{3}z^{-1}} + \frac{3}{1+\frac{1}{2}z^{-1}} = \frac{5}{1+\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}.$$
Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:
$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$
or, in matrix form:
$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,
$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$
$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$
$$\phi_{s12}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$
$$\phi_{s21}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$
Thus,
$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$
So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:
$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

Solution from Fall04 PS6

(a)

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \quad i = 1, 2,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s12}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_{21}}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,

$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$

So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \quad i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

Solution from Fall02 PS6

(a)
$$S(z) = \frac{2}{1-\frac{1}{3}z^{-1}} + \frac{3}{1+\frac{1}{2}z^{-1}} = \frac{5}{1+\frac{1}{6}z^{-1}-\frac{1}{6}z^{-2}}.$$
Thus, $a_1 = -1/6, a_2 = 1/6$.

(b) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{2} a_k \phi_s[i-k], \;\; i = 1, 2,$$

or, in matrix form:
$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

(c) Denote $s_1[n] = 2(\frac{1}{3})^n u[n]$, $s_2[n] = 3(-\frac{1}{2})^n u[n]$. Then for $m > 0$,

$$\phi_{s_1}[m] = \frac{9}{2}\left(\frac{1}{3}\right)^m$$

$$\phi_{s_2}[m] = 12\left(-\frac{1}{2}\right)^m$$

$$\phi_{s12}[m] = \frac{36}{7}\left(\frac{1}{3}\right)^m$$

$$\phi_{s21}[m] = \frac{36}{7}\left(-\frac{1}{2}\right)^m.$$

Thus,
$$\phi_s[m] = \frac{135}{14}\left(\frac{1}{3}\right)^{|m|} + \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|}.$$
So, $\phi_s[0] = 26.78$, $\phi_s[1] = -5.36$ and $\phi_s[2] = 5.36$.

(d) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6$.

(e) These values are the same as those we found in part (a), as expected.

(f) The normal (Yule-Walker) equations are:

$$\phi_s[i] = \sum_{k=1}^{3} a_k \phi_s[i-k], \;\; i = 1, 2, 3,$$

or, in matrix form:

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] & \phi_s[2] \\ \phi_s[1] & \phi_s[0] & \phi_s[1] \\ \phi_s[2] & \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \\ \phi_s[3] \end{bmatrix}.$$

(g) $\phi_s[3] = -1.79$.

(h) Substituting the values of $\phi_s[i]$ in the normal equations and solving for the a_i's results in $a_1 = -1/6, a_2 = 1/6, a_3 = 0$.

(i) The signal $s[n]$ is the impulse response of an all-pole filter with two poles, i.e. second order. Therefore, $a_k = 0$ for $k > 2$.

(j) No, since the signal corresponds to the impulse response of a second order filter. The higher order coefficients will all be 0.

11.17. Problem 5 in Spring 2003 final exam Appears in: Fall03 PS9.

Problem

The following information is known for $x[n]$ and $y[n]$, which are wide sense stationary, zero mean signals:

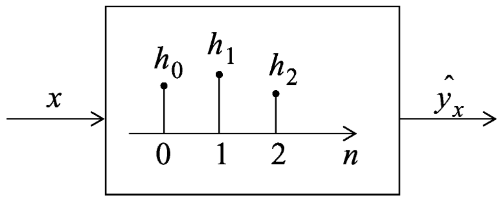
$$\phi_{xx}[m] = \begin{cases} 0 & m \text{ odd} \\ \frac{1}{2^{|m|}} & m \text{ even} \end{cases}$$

$\phi_{yx}[-1] = 2 \qquad \phi_{yx}[0] = 3 \qquad \phi_{yx}[1] = 8 \qquad \phi_{yx}[2] = -3 \qquad \phi_{yx}[3] = 2 \qquad \phi_{yx}[4] = -0.75$

(a) The linear estimate of y given x is $\hat{y}_x$. The estimator of y is designed to minimize

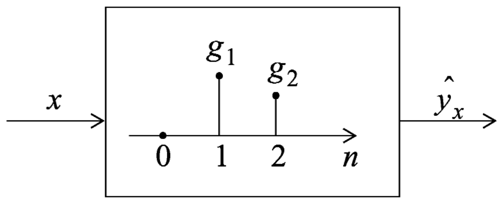
$$\mathcal{J} = E\left(\mid y[n] - \hat{y}_x[n] \mid^2 \right) \tag{1}$$

where the estimate is formed by passing $x[n]$ through $h[n]$, an FIR filter with 3 taps, as shown below.



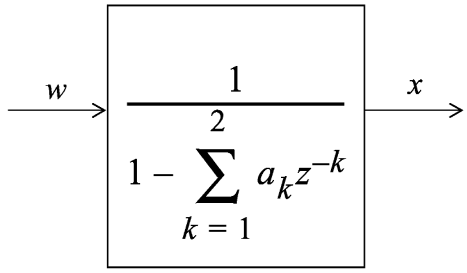
Find $h[n]$.

(b) In this part $\hat{y}_x$, the linear estimate of y given x, is also designed to minimize $\mathcal{J}$ in equation (??). Here the estimate is formed by passing $x[n]$ through $g[n]$, an FIR filter with 2 taps, as shown below.



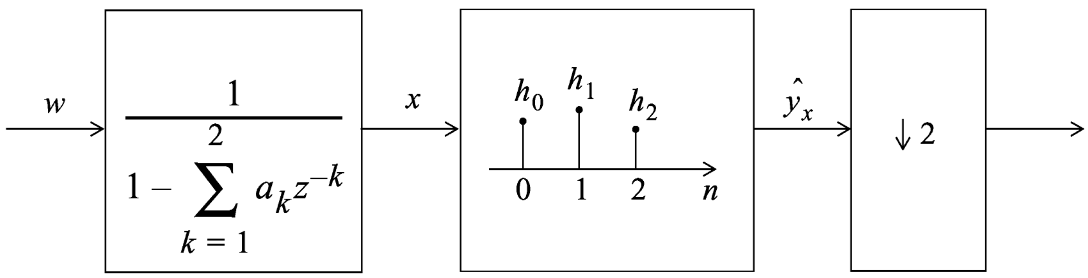
Find $g[n]$.

(c) As shown below, $x[n]$ can be modeled as the output from a two-pole filter whose input is $w[n]$, which is a wide sense stationary, zero mean, unit-variance white noise signal.



Find a_1 and a_2.

(d) We want to implement the following system:



where the coefficients a_i are from all-pole modeling in part (c) and the coefficients h_i are the taps of the linear estimator in part (a). Draw an implementation that minimizes the total cost of delays, where the cost of each individual delay is weighted linearly by its clock rate.

(e) Let $\mathcal{J}_a$ be the cost in part (a) and let $\mathcal{J}_b$ be the cost in part (b), where each $\mathcal{J}$ is defined as in equation (**??**). Is $\mathcal{J}_a$ larger than, equal to, or smaller than $\mathcal{J}_b$, or is there not enough information to compare them?

(f) Calculate $\mathcal{J}_a$ and $\mathcal{J}_b$ when $\phi_{yy}[0] = 88$. (Hint: the optimum FIR filters calculated in parts (a) and (b) are such that $E\left[\hat{y}_x[n](y[n] - \hat{y}_x[n])\right] = 0$.)

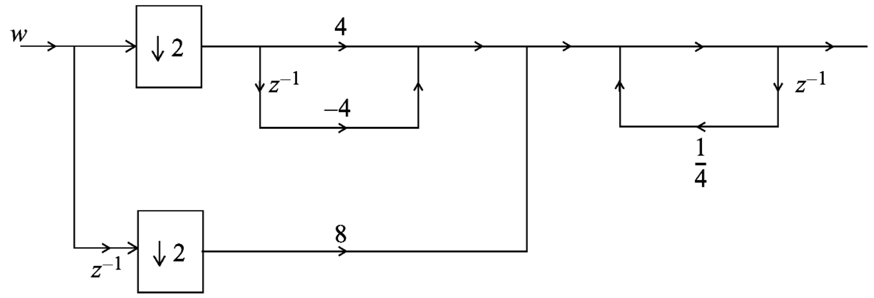
Solution from Fall03 PS9

(a) $h[n] = 4\delta[n] + 8\delta[n-1] - 4\delta[n-2]$

(b) $g[n] = 8\delta[n-1] - 3\delta[n-2]$

(c) $a_1 = 0, \quad a_2 = 1/4$

(d) Below is a realization with a weighted delay cost of 4. This is the same weighted delay cost for the system drawn in direct form II, with the downsampling at the end.



(e) Since $h[n]$ has one more degree of freedom than $g[n]$, $\mathcal{J}_a \leq \mathcal{J}_b$.

(f) $\mathcal{J}_a = 0, \quad \mathcal{J}_b = 15$

Solution from Spring03 Final

N/A

11.18. Problem 6 in Spring2005 final exam.

Problem

A discrete-time communication channel with impulse response $h[n]$ is to be compensated for with a system with impulse response $h_c[n]$ as indicated below:

$$s[n] \longrightarrow \underset{\text{channel}}{\boxed{h[n]}} \xrightarrow{r[n]} \underset{\text{compensator}}{\boxed{h_c[n]}} \longrightarrow \hat{s}[n]$$

The channel $h[n]$ is known to be a one sample delay, *i.e.*,

$$h[n] = \delta[n-1].$$

The compensator $h_c[n]$ is an N-point causal FIR filter, *i.e.*,

$$H_c(z) = \sum_{k=0}^{N-1} a_k z^{-k}.$$

The compensator $h_c[n]$ is designed to invert (or compensate for) the channel. Specifically, $h_c[n]$ is designed so that with $s[n] = \delta[n]$, $\hat{s}[n]$ is as "close" as possible to an impulse. Mathematically, define this as designing $h_c[n]$ so that the error

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} |\hat{s}[n] - \delta[n]|^2$$

is minimized.

Find the optimal compensator of length N, *i.e.*, determine $a_0, a_1, \ldots, a_{N-1}$ to minimize $\mathcal{E}$.

Solution from Spring05 Final

Since $h_c[n]$ is causal and $h[n]$ is a one-sample delay, we know that $\hat{s}[n]$ is zero for $n < 1$. $\mathcal{E}$ then simplifies to

$$\mathcal{E} = 1 + \sum_{n=1}^{\infty} |\hat{s}[n]|^2 = 1 + \sum_{n=1}^{\infty} |h_c[n-1]|^2.$$

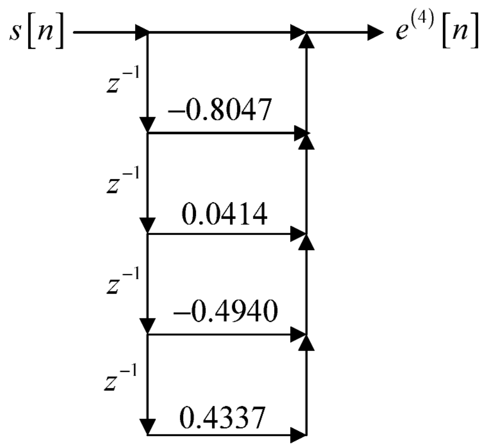
$\mathcal{E}$ is therefore minimized when $h_c[n] = 0$ for all n. (For *any* value of N, the a_ks are all 0.)

One could also obtain this result by writing out $\mathcal{E}$ in terms of the a_ks and taking partial derivatives with respect to the a_ks.

11.19. A. The coefficients of $A^{(4)}(z)$ can be taken directly from the table provided. We have

$$A^{(4)}(z) = 1 - 0.8047z^{-1} + 0.0414z^{-2} - 0.4940z^{-3} + 0.4337z^{-4}.$$

The direct form implementation is



B. In general, $k_i = a_i^{(i)}$. We have

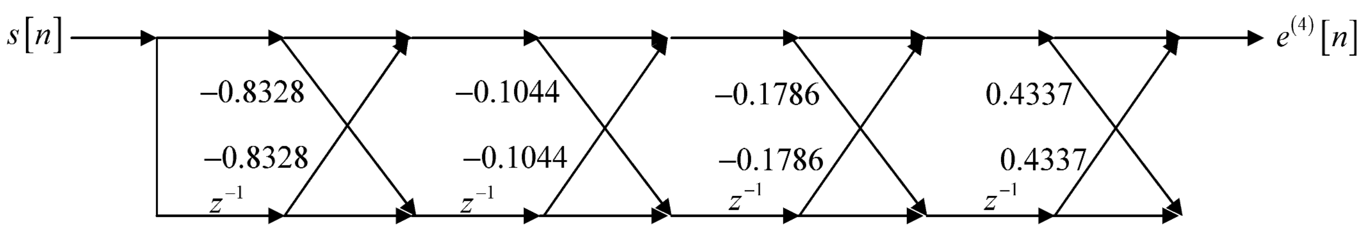
$k_1 = a_1^{(1)} = 0.8328$

$k_2 = a_2^{(2)} = 0.1044$

$k_3 = a_3^{(3)} = 0.1786$

$k_4 = a_4^{(4)} = -0.4337.$

The lattice implementation is



C. We are given that $E^{(2)} = 0.5803$. Since $E^{(i)} = \left(1 - k_i^2\right)E^{(i-1)}$ we have

$$E^{(3)} = \left(1 - (0.1786)^2\right)(0.5803) = 0.5681.$$

Working backwards we have

$$E^{(1)} = \frac{E^{(2)}}{\left(1-k_2^2\right)} = \frac{0.5803}{\left(1-(0.1044)^2\right)} = 0.5867$$

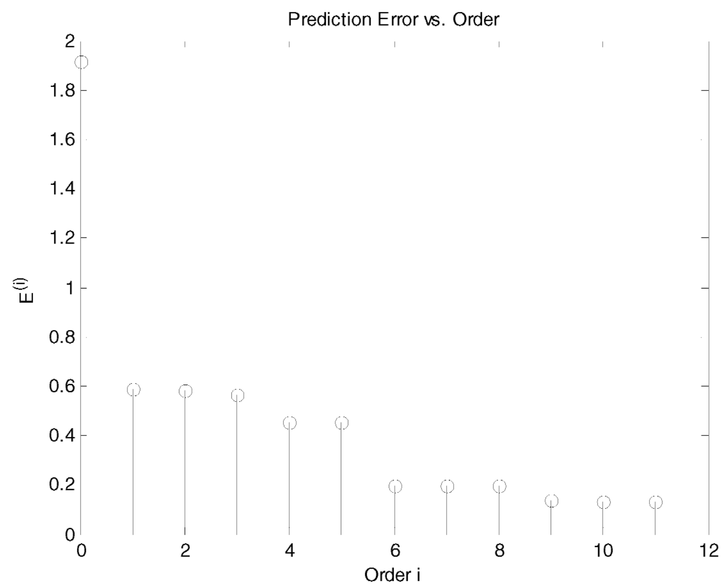
$$E^{(0)} = \frac{E^{(1)}}{\left(1-k_1^2\right)} = \frac{0.5867}{\left(1-(0.8328)^2\right)} = 1.915.$$

Since $E^{(0)} = r_{ss}(0)$, the total energy in the signal $s[n]$ is $E^{(0)} = 1.915$.

From the first step of the Levinson-Durbin algorithm we have $k_1 = r_{ss}[1]/r_{ss}[0]$. Then

$$r_{ss}[1] = k_1 r_{ss}[0] = (0.8328)(1.915) = 1.594.$$

D. Minimum mean square prediction error is plotted vs. filter order below.



It can be seen that the error drops abruptly in going from $i = 0$ to $i = 1$ and then makes another sharp decrease in going from $i = 5$ to $i = 6$. This is a consequence of the comparitively large magnitude of the reflection coefficient k_6, i.e. $|k_6| = 0.7505$.

E. The output $e^{(11)}[n]$ of the eleventh-order prediction-error filter should be very nearly white noise.

F. Five of the remaining zeros must be the complex conjugates of the zeros given in the table. The remaining zero must be real-valued. Now

$$\begin{aligned} a_{11}^{(11)} &= -(-1)^{11}\prod_{i=1}^{11} z_i = \prod_{i=1}^{11} z_i \\ &= z_6 (0.2567)^2 (0.9681)^2 (0.9850)^2 (0.8647)^2 (0.9590)^2 \\ &= 0.0371. \end{aligned}$$

Solving gives $z_6 = 0.900$.

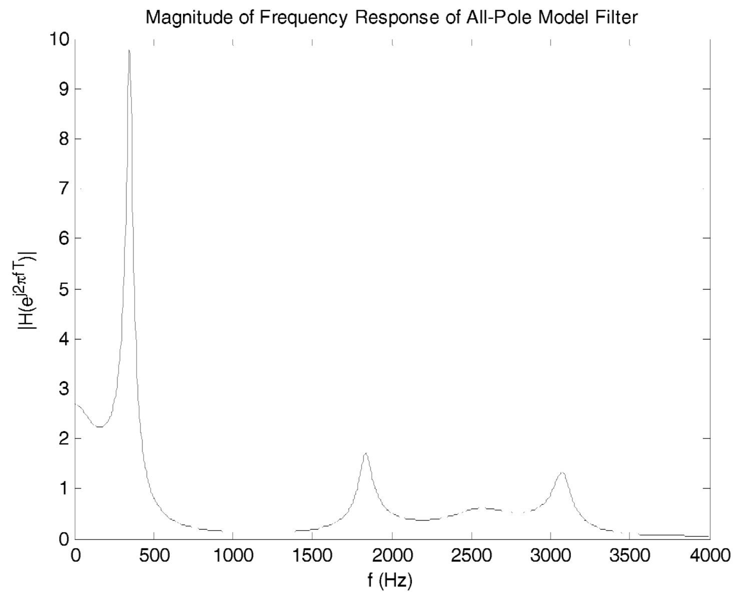
G. For random process modeling, the gain G is chosen so that $\langle \hat{s}^2[n] \rangle = \langle s^2[n] \rangle$. We know from part C that $\langle s^2[n] \rangle = r_{ss}[0] = E^{(0)} = 1.915$. Now if the all-pole model filter is driven by white noise of unit average power, then

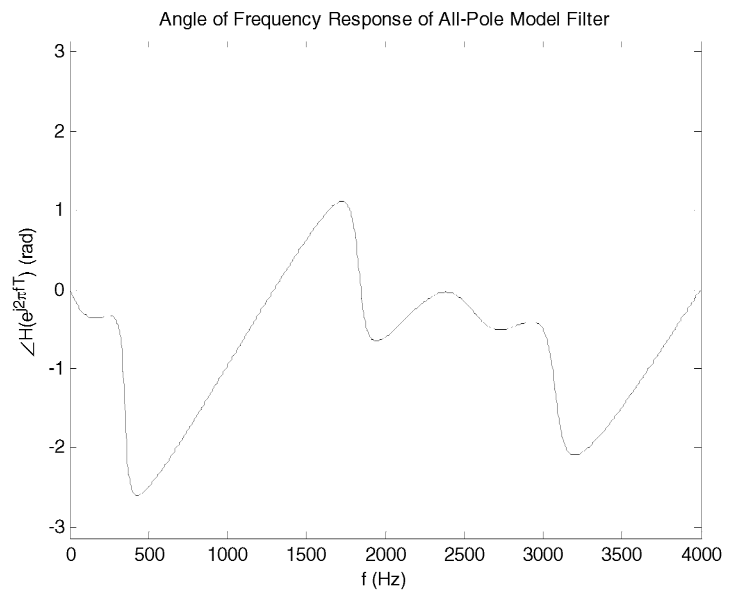
$$\begin{aligned} \langle \hat{s}^2[n] \rangle &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H(e^{j\omega}) \right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{G^2}{\left| A^{(11)}(e^{j\omega}) \right|^2} d\omega \\ &= 1.915. \end{aligned}$$

The frequency response $A^{(11)}(e^{j\omega})$ of the prediction-error filter can be written out using the values of $a_1^{(11)},\ldots,a_{11}^{(11)}$ given in the problem statement. The necessary integral can then be evaluated numerically to give $G = 0.363$.

As an alternative and numerically much simpler approach, suppose the input to the all-pole model filter is white noise $v[n]$ of unit variance. The output of this filter is $\hat{s}[n]$. Now suppose $\hat{s}[n]$ is the input to a filter with system function $A^{(11)}(z)/G$. This filter is the exact inverse of the all-pole model filter, so the output of this filter is $v[n]$. This means that if $\hat{s}[n]$ were the input to the prediction-error filter with system function $A^{(11)}(z)$, then the output would be $Gv[n]$, having mean-square value $G^2 \langle v^2[n] \rangle = G^2$. Now we do not know the numerical value of the output power of the prediction-error filter when the input is $\hat{s}[n]$, but we do know that when $s[n]$ is the input, the output has mean-square value $E^{(11)} = E^{(2)} \prod_{i=3}^{11} (1 - k_i^2) = 0.1321$. Since $\hat{s}[n] \cong s[n]$, let us take the mean square response to $\hat{s}[n]$ as 0.1321 as well. Then we can write $G^2 = 0.1321$, or $G = 0.363$.

H. The frequency response of the all-pole model filter is plotted below.





Note that the magnitude has a sharp peak at about 350 Hz, corresponding to the sinusoidal component visible in the data segment provided in the problem statement.

11.20. A. Let $s[n] = A\cos(\omega_0 n + \theta)$, where $p_\theta(\phi) = \frac{1}{2\pi}$, $0 \le \phi < 2\pi$.

Then

$$\begin{aligned}
r_{ss}[m] &= E\{s[n]s[n+m]\} \\
&= E\{A\cos(\omega_0 n + \theta) A\cos(\omega_0(n+m)+\theta)\} \\
&= \frac{A^2}{2} E\{\cos(\omega_0 m)\} + \frac{A^2}{2} E\{\cos(\omega_0(2n+m)+2\theta)\} \\
&= \frac{A^2}{2}\cos(\omega_0 m) + \frac{A^2}{2}\int_0^{2\pi} \cos(\omega_0(2n+m)+2\phi)\frac{1}{2\pi}d\phi \\
&= \frac{A^2}{2}\cos(\omega_0 m).
\end{aligned}$$

B. The autocorrelation normal equations are

$$\begin{aligned}
a_1\frac{A^2}{2}\cos(\omega_0 0) + a_2\frac{A^2}{2}\cos(\omega_0 1) &= \frac{A^2}{2}\cos(\omega_0 1) \\
a_1\frac{A^2}{2}\cos(\omega_0 1) + a_2\frac{A^2}{2}\cos(\omega_0 0) &= \frac{A^2}{2}\cos(\omega_0 2),
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
a_1 + a_2\cos(\omega_0) &= \cos(\omega_0) \\
a_1\cos(\omega_0) + a_2 &= \cos(\omega_0 2).
\end{aligned}$$

C. Solving these two equations (and applying a little trigonometry) gives $a_1 = 2\cos(\omega_0)$ and $a_2 = -1$.

D. We have

$$\begin{aligned}
A(z) &= 1 - 2\cos(\omega_0)z^{-1} + z^{-2} \\
&= (1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}).
\end{aligned}$$

Note that the zeros lie on the unit circle.

E. The prediction error is

$$\begin{aligned}
E &= r_{ss}[0] - a_1 r_{ss}[1] - a_2 r_{ss}[2] \\
&= \frac{A^2}{2} - 2\cos(\omega_0)\frac{A^2}{2}\cos(\omega_0) + \frac{A^2}{2}\cos(2\omega_0) \\
&= \frac{A^2}{2}\left(1 - 2\cos^2(\omega_0) + \cos(2\omega_0)\right) \\
&= 0.
\end{aligned}$$

11.21. The ith-order prediction-error filter has system function given by

$$A^{(i)}(z) = 1 - \sum_{j=1}^{p} a_j^{(i)} z^{-j}.$$

The filter coefficients update according to

$$a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \quad j = 1, 2, \ldots, i-1,$$
$$a_i^{(i)} = k_i.$$

Substituting gives,

$$\begin{aligned} A^{(i)}(z) &= 1 - \sum_{j=1}^{i-1} \left(a_j^{(i-1)} - k_i a_{i-j}^{(i-1)} \right) z^{-j} - k_i z^{-i} \\ &= 1 - \sum_{j=1}^{i-1} a_j^{(i-1)} z^{-j} - k_i \left(\sum_{j=1}^{i-1} a_{i-j}^{(i-1)} z^{-j} + z^{-i} \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{j=1}^{i-1} a_{i-j}^{(i-1)} z^{i-j} + 1 \right). \end{aligned}$$

Now let $p = i - j$ in the summation. We have

$$\begin{aligned} A^{(i)}(z) &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{p=i-1}^{1} a_p^{(i-1)} z^{p} + 1 \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} \left(\sum_{p=1}^{i-1} a_p^{(i-1)} \left(z^{-1}\right)^{-p} + 1 \right) \\ &= A^{(i-1)}(z) - k_i z^{-i} A^{(i-1)}\left(z^{-1}\right). \end{aligned}$$

11.22. Appears in: Spring05 PS7, Fall04 PS6, Fall02 PS6. Appears in another form(different numbers) in Spring2001 PS9.

The problem has been modified from previous versions in Fall05. The Fall04 is included right after it. **Problem**

This problem considers the construction of lattice filters to implement the inverse filter for the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

(a) Find the values of k_1 and k_2 for the second-order $p = 2$ case.

(b) Draw the signal flow graph of a lattice filter implementation of the inverse filter, i.e., the filter which outputs $y[n] = A\delta[n]$ (a scaled impulse) when the input $x[n] = s[n]$.

(c) Verify that the signal flow graph you drew in part (b) has the correct impulse response by showing that the z-transform of this inverse filter is indeed the inverse of $S(z)$.

(d) Draw the signal flow graph for a lattice filter which implements an all-pole system such that when the input is $x[n] = \delta[n]$, the output is the signal $s[n]$ given above.

(e) Derive the system function of the signal flow graph you drew in part (d) and demonstrate that its impulse response $h[n]$ satisfies $h[n] = s[n]$.

Fall04 Version of Problem

This problem considers the construction of lattice filters to implement the inverse filter for the signal

$$s[n] = 2\left(\frac{1}{3}\right)^n u[n] + 3\left(-\frac{1}{2}\right)^n u[n].$$

(a) Find the values of k_1 and k_2 for the second-order $p = 2$ case.

(b) Draw the signal flow graph of a lattice filter implementation of the inverse filter, i.e., the filter which outputs $y[n] = A\delta[n]$ (a scaled impulse) when the input $x[n] = s[n]$.

(c) Verify that the signal flow graph you drew in part (b) has the correct impulse response based on the values of the a_k's found in Problem 6.4.

(d) Draw the signal flow graph for a lattice filter which implements an all-pole system such that when the input is $x[n] = \delta[n]$, the output is the signal $s[n]$ given above.

(e) Derive the system function of the signal flow graph you drew in part (d) and demonstrate that its impulse response $h[n]$ satisfies $h[n] = s[n]$.

Solution from Spring05 PS7

(a) Using the same technique as in Problem 3 part (c), we first find

$$\phi_s[m] = \frac{120}{7}\left(-\frac{1}{2}\right)^{|m|} + \frac{135}{14}\left(\frac{1}{3}\right)^{|m|}.$$

Therefore, $\phi_s[0] \approx 26.79$, $\phi_s[1] \approx -5.357$, and $\phi_s[2] \approx 5.357$.

Substituting $p = 0$ in the Levinson-Durbin recursion we have:

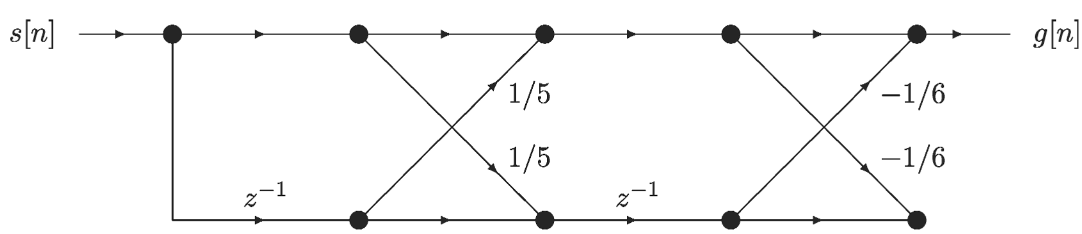
$$k_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}.$$

For $p = 1$,

$$a_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}$$

$$k_2 = \frac{\phi_s[2] - \phi_s[1]a_1}{\phi_s[0] - \phi_s[1]a_1} = \frac{1}{6}.$$

(b) Signal flow graph for inverse lattice filter:



(c) From the flow graph, $g[n] = s[n] + \frac{1}{6}s[n-1] - \frac{1}{6}s[n-2]$. Therefore,

$$H_{inv}(z) = \frac{G(z)}{S(z)} = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}.$$

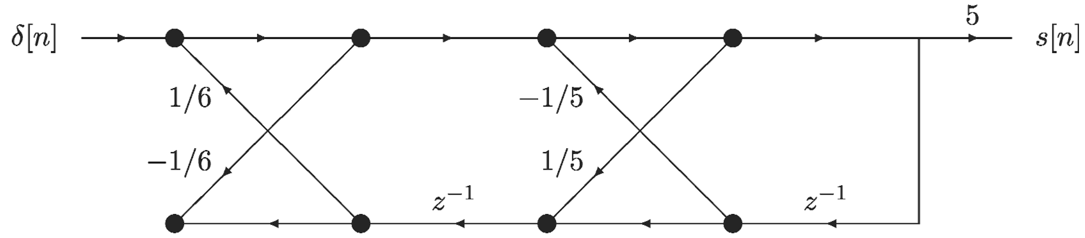
Taking the z-transform of $s[n]$,

$$S(z) = \frac{2}{1 - \frac{1}{3}z^{-1}} + \frac{3}{1 + \frac{1}{2}z^{-1}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}},$$

so

$$H_{inv}(z) = AS^{-1}(z).$$

(d) Signal flow graph for forward lattice filter:



(e) The system function is:

$$H(z) = \frac{5}{1 - k_1(1-k_2)z^{-1} - k_2 z^{-2}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Comparing with $S(z)$ as determined in part (c), we see that $h[n] = s[n]$.

Solution from Fall02 PS6

(a) Substituting $p = 0$ in the Levinson-Durbin recursion we have:

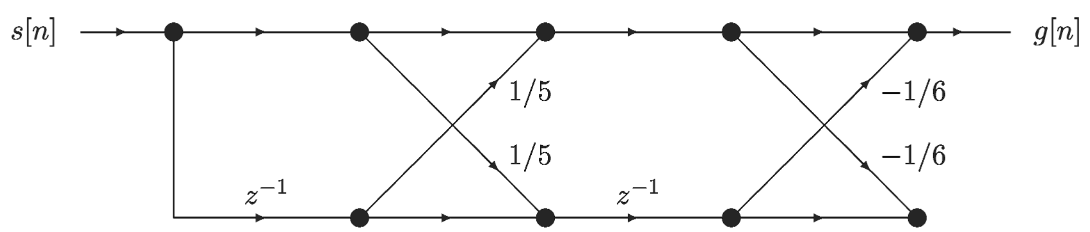
$$k_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}.$$

For $p = 1$,

$$a_1 = \frac{\phi_s[1]}{\phi_s[0]} = -\frac{1}{5}$$

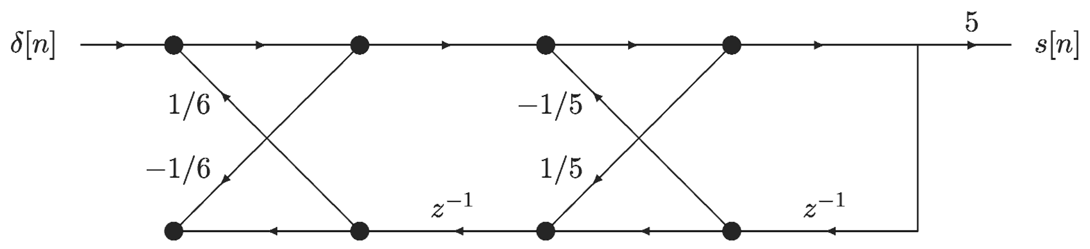
$$k_2 = \frac{\phi_s[2] - \phi_s[1]a_1}{\phi_s[0] - \phi_s[1]a_1} = \frac{1}{6}.$$

Figure for Part (b)



(c) From the flow graph, $g[n] = s[n] + \frac{1}{6}s[n-1] - \frac{1}{6}s[n-2]$.

Figure for Part (d):



(e) The system function is:

$$H(z) = \frac{5}{1 - k_1(1-k_2)z^{-1} - k_2 z^{-2}} = \frac{5}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}.$$

Comparing with the answer to part (a) of problem 6.3, we see that $h[n] = s[n]$.

11.23. A.

$$\begin{bmatrix} \phi_s[0] & \phi_s[1] \\ \phi_s[1] & \phi_s[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \phi_s[1] \\ \phi_s[2] \end{bmatrix}.$$

B.

$$S(z) = \alpha \frac{1}{1-\frac{2}{3}z^{-1}} + \beta \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{\alpha - \frac{\alpha}{4}z^{-1} + \beta - \frac{2\beta}{3}z^{-1}}{\left(1-\frac{2}{3}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{\alpha + \beta - \frac{\alpha}{4}z^{-1} - \frac{2\beta}{3}z^{-1}}{1-\frac{11}{12}z^{-1} + \frac{1}{6}z^{-2}}.$$

If $a_1 = 11/12$ and $a_2 = -1/6$ then we are modeling a second-order all-pole signal. Therefore the z^{-1} terms in the numerator must cancel. That is, $-\frac{\alpha}{4} = \frac{2\beta}{3}$, or $-3\alpha = 8\beta$.

One possibility is $\alpha = 8, \beta = -3$. The solution is not unique; and pair $c\times 8, c\times(-3)$ with $c \neq 0$ will work.

C. Since $s[n]$ is a second-order all-pole signal, if you were to solve the Levinson recursion for $p = 3$, then $k_3 = a_3^{(3)} = 0$. The k's do not change as the model order increases, therefore $k_3 = 0$ for any p.

11.24. A. Start with $\mathbf{\Gamma}_p \mathbf{b}_p = \mathbf{c}_p$. Expanded this is

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] \end{bmatrix} \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Adding an additional row and column gives us

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta^{(p)} \end{bmatrix},$$

where the first p rows are the original equations and the last row is valid provided

$$\delta^{(p)} = c[p+1] - \left(\gamma_p^b\right)^T \bar{b}_p.$$

From the proof of the Levinson-Durbin algorithm we have,

$$\begin{bmatrix} \phi[0] & \phi[1] & \phi[2] & \cdots & \phi[p] & \phi[p+1] \\ \phi[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ \phi[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} \gamma^{(p)} \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Let us disregard the first equation in this system of equations. Then we obtain

$$\begin{bmatrix} \phi[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ \phi[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ \phi[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Note that entry in the leftmost column of the autocorrelation matrix is multiplied by zero in every equation in this reduced system. The values of the entries in the leftmost column do not matter, then, and they can be replaced by other values. We can write

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

Let us now construct

$$\begin{bmatrix} c[1] & \phi[0] & \phi[1] & \cdots & \phi[p-1] & \phi[p] \\ c[2] & \phi[1] & \phi[0] & \cdots & \phi[p-2] & \phi[p-1] \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c[p] & \phi[p-1] & \phi[p-2] & \cdots & \phi[0] & \phi[1] \\ c[p+1] & \phi[p] & \phi[p-1] & \cdots & \phi[1] & \phi[0] \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ -b_1^p \\ \vdots \\ -b_p^p \\ 0 \end{bmatrix} - \hat{k}_{p+1} \begin{bmatrix} 0 \\ -a_p^p \\ \vdots \\ -a_1^p \\ 1 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \delta^{(p)} \end{bmatrix} - \hat{k}_{p+1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E^{(p)} \end{bmatrix}.$$

To make the right-hand side of the last equation equal to zero, we choose $\hat{k}_{p+1}$ so that $\delta^{(p)} - \hat{k}_{p+1} E^{(p)} = 0$. That is, $\hat{k}_{p+1} = \dfrac{\delta^{(p)}}{E^{(p)}} = \dfrac{c[p+1] - (\gamma_p^b)^T \bar{b}_p}{E^{(p)}} = \dfrac{c[p+1] - (\gamma_p^b)^T \bar{b}_p}{\phi[0] - (\gamma_p)^T \bar{a}_p}$.

Now let $b_{p+1}^{p+1} = \hat{k}_{p+1}$. Then we have $\mathbf{\Gamma}_{p+1}\mathbf{b}_{p+1} = \mathbf{c}_{p+1}$, where

$$b_m^{p+1} = b_m^p - b_{p+1}^{p+1} a_{p-m+1}^p, \quad m = 1, \ldots, p,$$

QED.

11.25. To solve for the fourth-order system we extend the second-order one using the Levinson recursion. The resulting lattice will have the same k_1 and k_2. However we need to know $\phi_s[m]$ to determine k_3 and k_4. To determine $H_4(z)$, i.e., $a_i^{(4)}$, we need to know all the k_i's. Therefore $k_1 = -2/7$, $k_2 = 1/8$, and the remaining parameters cannot be determined from the information provided.

11.26. A. The prediction-error filter has system function $A(z)=1-\sum_{m=1}^{p} a_m z^{-m}$ and impulse response $h_A[n]=\delta[n]-\sum_{m=1}^{p} a_m \delta[n-m]$. The function $Q(z)$ is defined as

$$Q(z)=\frac{A(z)A(z^{-1})}{G^2}.$$

Then

$$Q(z)=\frac{\left(1-a_1 z^{-1}-\ldots-a_p z^{-p}\right)\left(1-a_1 z-\ldots-a_p z^{p}\right)}{G^2}$$
$$=q_p z^p+q_{p-1}z^{p-1}+\ldots+q_1 z+q_0+q_1 z^{-1}+\ldots+q_{p-1}z^{-(p-1)}+q_p z^{-p}.$$

We make the following observations about the sequence $q[n]$:

i) $q[n]$ can be nonzero only for $-p\le n\le p$. That is, $q[n]$ contains at most $2p+1$ nonzero samples.

ii) $q[n]$ is an even function of n

iii) At $n=\pm p$, $q[\pm p]=q_p=-\frac{a_p}{G^2}$.

B. Given the numerical values

$$C[k]=\left|H\left(e^{j\pi k/p}\right)\right|^2=\left|\frac{G}{1-\sum_{i=1}^{p} a_i e^{-j\pi ki/p}}\right|^2, \quad k=0,1,2,\ldots p,$$

1. Take the reciprocal of the given data to form

$$Q[k]=\frac{1}{G^2}\left|1-\sum_{i=1}^{p} a_i e^{-j\pi ki/p}\right|^2, \quad k=0,1,2,\ldots p.$$

2. Except for the factor of G^2, $Q[k]$ consists of samples of the magnitude-squared of the system function of the prediction-error filter evaluated at equally spaced points around the unit circle from $\omega=0$ to $\omega=\pi$. Since a magnitude-squared function is an even function of frequency, we can extend $Q[k]$ to $k=0,1,\ldots,2p-1$. That is

$$Q[k]=Q[2p-k], \quad k=p+1,\ldots 2p-1.$$

3. We now wish to find the sequence $q[n], \quad n=-p,\ldots,p$ that is the inverse Fourier transform of $Q(e^{j\omega})$. Recall from part A that $q[n]$ contains $2p+1$ values. To begin, compute the $2p$-point inverse DFT $\hat{q}[n]$ of $Q[k]$. The sequence $\hat{q}[n], \quad n=0,1,\ldots,2p-1$ is a time-aliased version of the sequence $q[n]$ that we seek. Fortunately, the period of $\hat{q}[n]$ is $2p$, and the aliasing consists of only one point of overlap. Specifically, the sequence $\hat{q}[n]$ contains

$$\begin{aligned}\hat{q}[n]&=\left[\hat{q}[0],\hat{q}[1],\ldots,\hat{q}[2p-1]\right]\\&=\left[q[0],q[1],\ldots,q[p-1],2q[p],q[-(p-1)],\ldots,q[-1]\right].\end{aligned}$$

We can make the identification $q[-p]=q[p]=\frac{1}{2}\hat{q}[p]$ and rotate the sequence $\hat{q}[i]$ so that $q[n]$ contains

$$\begin{aligned}q[n]&=\left[q[-p],\ldots,q[-1],q[0],q[1],\ldots,q[p]\right]\\&=\left[\tfrac{1}{2}\hat{q}[p],\hat{q}[p+1]\ldots,\hat{q}[2p-1],\hat{q}[0],\hat{q}[1],\ldots,\hat{q}[p-1],\tfrac{1}{2}\hat{q}[p]\right].\end{aligned}$$

C.1. The function, $Q(z)$ will have zeros with conjugate-reciprocal symmetry, with half of its zeros inside the unit circle and half outside. Now we know that $H(z)$ is a stable all-pole filter, and therefore its inverse, the prediction-error filter, will have minimum phase. Consequently we can factor $Q(z)$, and group together the p factors corresponding to zeros that are inside the unit circle. The resulting polynomial is $A(z)$.

2. If $A(z)$ is written as a product of terms of the form $(1-z_i z^{-1})$, we will have

$$A(z)=\left[1-a_1 z^{-1}-\ldots-a_p z^{-p}\right],$$

giving us the coefficients $a_i, \quad i=1,\ldots,p$. Now recall that $q[-p]=-a_p/G^2$. Since we now know a_p, we can solve for G.

11.27. A. For the given lattice we have

$$E^{(i)}(z) = A^{(i)}(z)S(z)$$

for the prediction error at stage i. When $i = p$ this is

$$E^{(p)}(z) = A^{(p)}(z)S(z).$$

Similarly, the backward prediction error is given by

$$\begin{aligned}\tilde{E}^{(i)}(z) &= B^{(i)}(z)S(z)\\ &= z^{-i}A^{(i)}(z^{-1})S(z).\end{aligned}$$

The system function $H^{(i)}(z)$ relating $E^{(p)}(z)$ to $\tilde{E}^{(i)}(z)$ is

$$\begin{aligned}H^{(i)}(z) &= \frac{\tilde{E}^{(i)}(z)}{E^{(p)}(z)}\\ &= \frac{z^{-i}A^{(i)}(z^{-1})S(z)}{A^{(p)}(z)S(z)}\\ &= \frac{z^{-i}A^{(i)}(z^{-1})}{A^{(p)}(z)}.\end{aligned}$$

B. We have

$$\begin{aligned}H^{(p)}(z) &= \frac{z^{-p}A^{(p)}(z^{-1})}{A^{(p)}(z)}\\ &= \frac{z^{-p} - a_1^{(p)}z^{-(p-1)} - a_2^{(p)}z^{-(p-2)}\cdots - a_p^{(p)}}{1 - a_1^{(p)}z^{-1} - a_2^{(p)}z^{-2} - \cdots - a_p^{(p)}z^{-p}}\\ &= \frac{1 - a_1^{(p)}z - a_2^{(p)}z^2 - \cdots - a_p^{(p)}z^p}{z^p - a_1^{(p)}z^{p-1} - a_2^{(p)}z^{p-2} - \cdots - a_p^{(p)}}.\end{aligned}$$

Now suppose z_1 is a pole of $H^{(p)}(z)$. (Note that $H^{(p)}(z)$ has no poles at zero or infinity.) Then

$$z_1^p - a_1^{(p)}z_1^{p-1} - a_2^{(p)}z^{p-2} - \cdots - a_p^{(p)} = 0.$$

Multiplying through by z_1^{-p} gives

$$1 - a_1^{(p)}z_1^{-1} - a_2^{(p)}z_1^{-2} - \cdots - a_p^{(p)}z_1^{-p} = 0.$$

This shows that z_1^{-1} is a root of the numerator, and hence is a zero of $H^{(p)}(z)$. A very similar argument shows that if z_2 is a zero of $H^{(p)}(z)$, then z_2^{-1} is a pole. Since the coefficients $a_1^{(p)},\ldots,a_p^{(p)}$ are assumed to be real-valued, the reciprocal pairing of poles and zeros guarantees that $H^{(p)}(z)$ is an all-pass function.

C. From the block diagram we see that

$$
\begin{aligned}
H(z) &= \frac{Y(z)}{X(z)} \\
&= \frac{Y(z)}{E^{(p)}(z)} \\
&= \frac{\sum_{i=0}^{p} c_i \tilde{E}^{(i)}(z)}{E^{(p)}(z)} \\
&= \sum_{i=0}^{p} c_i \frac{\tilde{E}^{(i)}(z)}{E^{(p)}(z)} \\
&= \sum_{i=0}^{p} c_i \frac{z^{-i} A^{(i)}\left(z^{-1}\right)}{A^{(p)}(z)} \\
&= \frac{1}{A^{(p)}(z)} \sum_{i=0}^{p} c_i z^{-i} A^{(i)}\left(z^{-1}\right) \\
&= \frac{Q(z)}{A^{(p)}(z)},
\end{aligned}
$$

where $Q(z)=\sum_{i=0}^{p} c_i z^{-1} A^{(i)}\left(z^{-1}\right)$. Now $z^{-i}A^{(i)}\left(z^{-1}\right)$ is a polynomial of degree $-i, \quad i=0,\ldots,p$. Since $Q(z)$ is a linear combination of such polynomials, $Q(z)$ is a polynomial of degree $-p$. We can write $Q(z)=\sum_{m=0}^{p} q_m z^{-m}$.

Continuing, we have $z^{-0}A^{(0)}\left(z^{-1}\right)=1$ and $z^{-i}A^{(i)}\left(z^{-1}\right)=z^{-i}-\sum_{j=1}^{i} a_j^{(i)} z^{-(i-j)}, \quad i=1,\ldots,p$.

Then

$$
\begin{aligned}
Q(z) = c_0 &+ c_1\left(z^{-1}-a_1^{(1)}\right) + c_2\left(z^{-2}-a_1^{(2)}z^{-1}-a_2^{(2)}\right) \\
&+ c_3\left(z^{-3}-a_1^{(3)}z^{-2}-a_2^{(3)}z^{-1}-a_3^{(3)}\right)+\cdots \\
&+ c_p\left(z^{-p}-a_1^{(p)}z^{-(p-1)}-\cdots-a_p^{(p)}\right).
\end{aligned}
$$

Collecting terms in like powers of z^{-1} we see that

$$
\begin{aligned}
q_p &= c_p \\
q_{p-1} &= c_{p-1} - c_p a_1^{(p)} \\
q_{p-2} &= c_{p-2} - c_{p-1} a_1^{(p-1)} - c_p a_2^{(p)},
\end{aligned}
$$

etc. That is,

$$
q_m = c_m - \sum_{i=m+1}^{p} c_i a_{i-m}^{(i)}, \quad m = p, p-1, \ldots, 0.
$$

Rearranging gives

$$c_m = q_m + \sum_{i=m+1}^{p} c_i a_{i-m}^{(i)}, \quad m = p, p-1, \ldots, 0.$$

D. Given a system function $H(z)$, apply the Coefficients-to-k-Parameters algorithm to the coefficients of the denominator to obtain the reflection coefficients of the lattice. Then apply the recursion obtained in the answer to part C above to obtain the coefficients $c_0, \ldots, c_p$.

E. Given

$$H(z) = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{1-0.9z^{-1}+0.64z^{-2}-0.576z^{-3}}$$

we have $a_1^{(3)} = 0.9, a_2^{(3)} = -0.64, a_3^{(3)} = 0.576$. Then $k_3 = a_3^{(3)} = 0.576$, and

$$a_1^{(2)} = \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1-k_3^2} = \frac{0.9+(0.576)(-0.64)}{1-(0.576)^2} = 0.795$$

$$a_2^{(2)} = \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1-k_3^2} = \frac{-0.64+(0.576)(0.9)}{1-(0.576)^2} = -0.182.$$

Continuing, $k_2 = a_2^{(2)} = -0.182$. Then

$$a_1^{(1)} = \frac{a_1^{(2)} + k_2 a_1^{(1)}}{1-k_2^2} = \frac{0.795+(-0.182)(0.795)}{1-(-0.182)^2} = 0.673$$

and $k_1 = a_1^{(1)} = 0.673$.

We now proceed to find the numerator coefficients as follows: First, $q_0 = 1, q_1 = 3, q_2 = 3, q_3 = 1$. Then

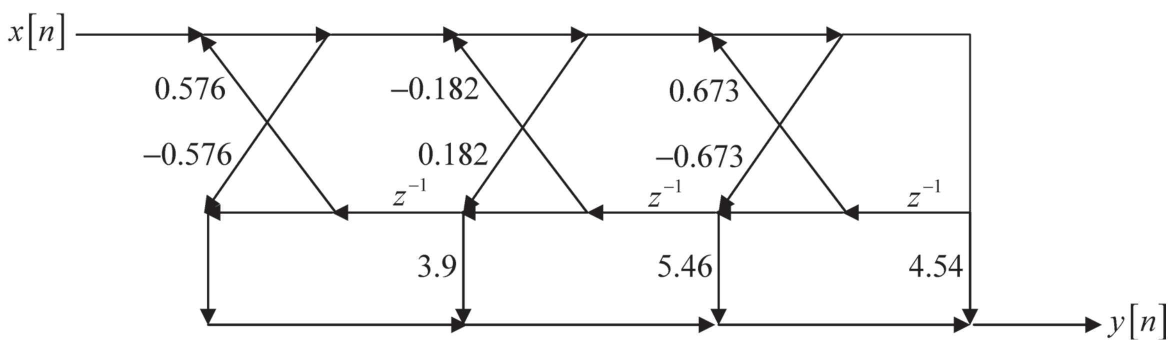
$$c_3 = q_3 = 1$$

$$c_2 = q_2 + c_3 a_1^{(3)} = 3 + (1)(0.9) = 3.9$$

$$c_1 = q_1 + c_2 a_1^{(2)} + c_3 a_2^{(3)} = 3 + (3.9)(0.795) + (1)(-0.64) = 5.46$$

$$c_0 = q_0 + c_1 a_1^{(1)} + c_2 a_2^{(2)} + c_3 a_3^{(3)} = 1 + (5.46)(0.673) + (3.9)(-0.182) + (1)(0.576) = 4.54.$$

The lattice is as follows:



11.28. Given the relations

$$e^{(i)}[n] = e^{(i-1)}[n] - k_i \tilde{e}^{(i-1)}[n-1]$$
$$\tilde{e}^{(i)}[n] = \tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n],$$

we can write

$$\begin{aligned}\mathcal{E}^{(i)} &= \sum_{n=-\infty}^{\infty}\left(e^{(i)}[n]\right)^2 \\ &= \sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]\right)^2 \\ &= \sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n]\right)^2 - 2k_i\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] \\ &\quad + k_i^2\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2.\end{aligned}$$

To maximize over k_i, differentiate and set the derivative equal to zero. That is,

$$\begin{aligned}\frac{d\left(\mathcal{E}^{(i)}\right)}{dk_i} &= -2\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] + 2k_i\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \\ &= 0.\end{aligned}$$

Solving gives

$$k_i^f = \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2}.$$

If, on the other hand, we start with

$$\tilde{\mathcal{E}}^{(i)} = \sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i)}[n]\right)^2,$$

we can follow an analogous sequence of steps and show that

$$k_i^b = \frac{\sum_{n=-\infty}^{\infty} \tilde{e}^{(i-1)}[n-1]e^{(i-1)}[n]}{\sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n]\right)^2}.$$

Taking the geometric mean of k_i^f and k_i^b we obtain

$$\begin{aligned}\sqrt{k_i^f k_i^b} &= \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n]\right)^2\right\}^{\frac{1}{2}}} \\ &= k_i^P.\end{aligned}$$

11.29. We begin with the expression for the PARCOR coefficient,

$$k_i^P = \frac{\sum_{n=-\infty}^{\infty} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=-\infty}^{\infty}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \sum_{n=-\infty}^{\infty}\left(e^{(i-1)}[n]\right)^2\right\}^{\frac{1}{2}}}.$$

Substitute

$$e^{(i)}[n] = s[n] - \sum_{k=1}^{i} a_k^{(i)} s[n-k]$$

and

$$\tilde{e}^{(i)}[n] = s[n-i] - \sum_{j=1}^{i} a_j^{(i)} s[n-i+j]$$

to give

$$k_i^P = \frac{\sum_{n=-\infty}^{\infty}\left(s[n]-\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\right)\left(s[n-i]-\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\right)}{\left\{\sum_{n=-\infty}^{\infty}\left(s[n-i]-\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\right)^2 \sum_{n=-\infty}^{\infty}\left(s[n]-\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\right)^2\right\}^{\frac{1}{2}}}.$$

To make the expressions easier to read we will expand the numerator and denominator separately. First, the numerator can be expanded to give

$$\begin{aligned} num = \sum_{n=-\infty}^{\infty}\Bigg(& s[n]s[n-i] - s[n-i]\sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k] \\ & - s[n]\sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j] + \sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k] \sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\Bigg) \\ = & \sum_{n=-\infty}^{\infty} s[n]s[n-i] - \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{n=-\infty}^{\infty} s[n-i]s[n-k] - \sum_{j=1}^{i-1} a_j^{(i-1)} \sum_{n=-\infty}^{\infty} s[n]s[n-i+j] \\ & + \sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} \sum_{n=-\infty}^{\infty} s[n-k]s[n-i+j]. \end{aligned}$$

Now we will use the fact that $r_{ss}[m] = r_{ss}[-m] = \sum_{n=-\infty}^{\infty} s[n]s[n+m]$. Substituting gives

$$\begin{aligned} num = & r_{ss}[i] - \sum_{k=1}^{i-1} a_k^{(i-1)} r_{ss}[i-k] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[i-j] \\ & + \sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} r_{ss}[j-i+k]. \end{aligned}$$

The double sum in the second line can be reduced if we recall the autocorrelation normal equations, which specify

$$\sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[m-j] = r_{ss}[m], \quad m = 1,\ldots,i-1.$$

Then we have

$$\begin{aligned}\sum_{k=1}^{i-1}\sum_{j=1}^{i-1} a_k^{(i-1)} a_j^{(i-1)} r_{ss}[j-i+k] &= \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[j-i+k] \\ &= \sum_{k=1}^{i-1} a_k^{(i-1)} \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[-j+i-k] \\ &= \sum_{k=1}^{i-1} a_k^{(i-1)} r_{ss}[i-k].\end{aligned}$$

The numerator is therefore

$$num = r_{ss}[i] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[i-j].$$

Each term in the denominator can be expanded using a sequence of steps analogous to those used to expand the numerator. We obtain

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left(s[n-i] - \sum_{j=1}^{i-1} a_j^{(i-1)} s[n-i+j]\right)^2 &= r_{ss}[0] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[j] \\ &= \mathcal{E}^{(i-1)}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\left(s[n] - \sum_{k=1}^{i-1} a_k^{(i-1)} s[n-k]\right)^2 &= r_{ss}[0] - \sum_{k=1}^{i-1} a_k^{(i-1)} r_{ss}[k] \\ &= \mathcal{E}^{(i-1)},\end{aligned}$$

which we can substitute into the denominator to obtain

$$den = \left\{\mathcal{E}^{(i-1)}\mathcal{E}^{(i-1)}\right\}^{\frac{1}{2}} = \mathcal{E}^{(i-1)}.$$

The PARCOR coefficient can now be written as

$$k_i^P = \frac{r_{ss}[i] - \sum_{j=1}^{i-1} a_j^{(i-1)} r_{ss}[i-j]}{\mathcal{E}^{(i-1)}} = k_i,$$

where the last equality recognizes that the expression obtained for k_i^P is precisely the expression for the reflection coefficient k_i obtained in the Levinson-Durbin algorithm.

11.30. Given

$$\mathcal{E}^{(i)} = \sum_{n=i}^{M}\left[\left(e^{(i)}[n]\right)^2 + \left(\tilde{e}^{(i)}[n]\right)^2\right].$$

A. Substitute $e^{(i)}[n] = e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]$ and $\tilde{e}^{(i)}[n] = \tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]$. This gives

$$\mathcal{E}^{(i)} = \sum_{n=i}^{M}\left[\left(e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]\right)^2 + \left(\tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]\right)^2\right].$$

Now set the derivative with respect to k_i equal to zero. We obtain

$$\begin{aligned}\frac{d\left(\mathcal{E}^{(i)}\right)}{dk_i} = \sum_{n=i}^{M}\Big[&-2\left(e^{(i-1)}[n] - k_i\tilde{e}^{(i-1)}[n-1]\right)\tilde{e}^{(i-1)}[n-1] \\ &-2\left(\tilde{e}^{(i-1)}[n-1] - k_i e^{(i-1)}[n]\right)e^{(i-1)}[n]\Big] \\ = 0.&\end{aligned}$$

That is,

$$\begin{aligned}&\sum_{i=1}^{M} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1] - k_i\sum_{i=1}^{M}\left(\tilde{e}^{(i-1)}[n-1]\right)^2 \\ &+\sum_{i=1}^{M}\tilde{e}^{(i-1)}[n-1]e^{(i-1)}[n] - k_i\sum_{i=1}^{M}\left(e^{(i-1)}[n]\right)^2 = 0.\end{aligned}$$

Solving for k_i gives the result,

$$k_i^B = \frac{2\sum_{n=i}^{M} e^{(i-1)}[n]\tilde{e}^{(i-1)}[n-1]}{\left\{\sum_{n=i}^{M}\left(e^{(i-1)}[n]\right)^2 + \sum_{n=i}^{M}\left(\tilde{e}^{(i-1)}[n-1]\right)^2\right\}}.$$

B. Start with the inequality

$$0 \le \sum_{n=i}^{M} \left(e^{(i-1)}[n] \pm \tilde{e}^{(i-1)}[n-1] \right)^2.$$

This inequality is true because the sum of squares cannot be negative. Now expand giving

$$\begin{aligned}0 &\le \sum_{n=i}^{M} \left[\left(e^{(i-1)}[n] \right)^2 \pm 2 e^{(i-1)}[n] \tilde{e}^{(i-1)}[n-1] + \left(\tilde{e}^{(i-1)}[n-1] \right)^2 \right] \\ &= \sum_{n=i}^{M} \left(e^{(i-1)}[n] \right)^2 \pm 2 \sum_{n=i}^{M} e^{(i-1)}[n] \tilde{e}^{(i-1)}[n-1] + \sum_{n=i}^{M} \left(\tilde{e}^{(i-1)}[n-1] \right)^2.\end{aligned}$$

If we divide through by $\sum_{n=i}^{M} \left(e^{(i-1)}[n] \right)^2 + \sum_{n=i}^{M} \left(\tilde{e}^{(i-1)}[n-1] \right)^2$ we obtain

$$0 \le 1 \pm k_i^B.$$

That is, $-1 \le k_i^B \le 1$, as was to have been shown.

C. To find $A^{(p)}(z)$, follow the steps of the Levinson-Durbin algorithm. That is, given

$k_i^B, \quad i = 1, \ldots, p,$

for $i = 1, 2, \ldots, p$

  $a_i^{(i)} = k_i^B$

  if $i > 1$ then for $j = 1, 2, \ldots, i-1$

    $a_j^{(i)} = a_j^{(i-1)} - k_i^B a_{i-j}^{(i-1)}$

  end

end

$a_j = a_j^{(p)}, \quad j = 1, 2, \ldots, p.$

Finally, $A^{(p)}(z) = 1 - \sum_{j=1}^{p} a_j z^{-j}$.