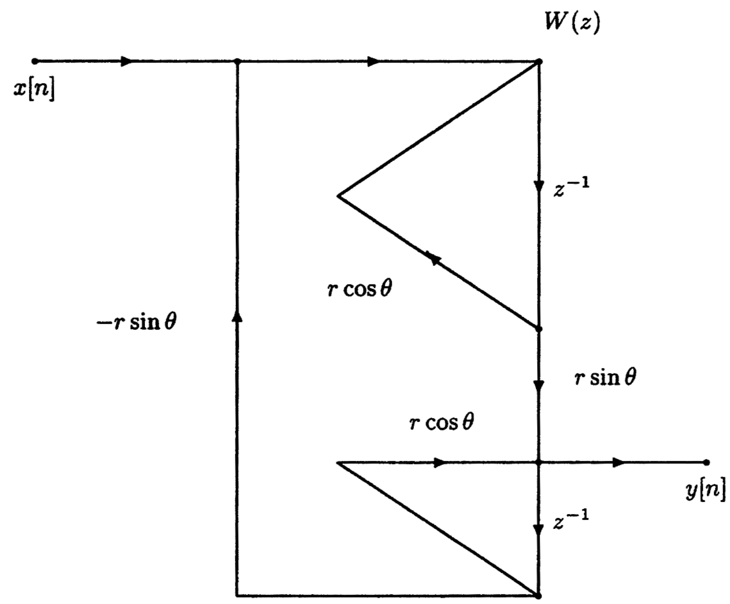
6.1. We proceed by obtaining the transfer functions for each of the networks. For network 1,

$$Y(z) = 2r\cos\theta z^{-1}Y(z) - r^2z^{-2}Y(z) + X(z)$$

or

$$H_1(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2z^{-2}}$$

For network 2, define $W(z)$ as in the figure below:



then

$$W(z) = X(z) - r\sin\theta z^{-1}Y(z) + r\cos\theta z^{-1}W(z)$$

and

$$Y(z) = r\sin\theta z^{-1}W(z) + r\cos\theta z^{-1}Y(z)$$

Eliminate $W(z)$ to get

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{r\sin\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2z^{-2}}$$

Hence the two networks have the same poles.

6.2. The only input to the $y[n]$ node is a unity branch connection from the $x[n]$ node. The rest of the network does not affect the input-output relationship. The difference equation is $y[n] = x[n]$.

6.3.

$$H(z) = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}$$

System (d) is recognizable as a transposed direct form II implementation of $H(z)$.

6.4. (a) From the flow graph, we have:

$$Y(z) = 2X(z) + (\frac{1}{4}X(z) - \frac{1}{4}Y(z) + \frac{3}{8}Y(z)z^{-1})z^{-1}$$

That is:

$$Y(z)(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}) = X(z)(2 + \frac{1}{4}z^{-1}).$$

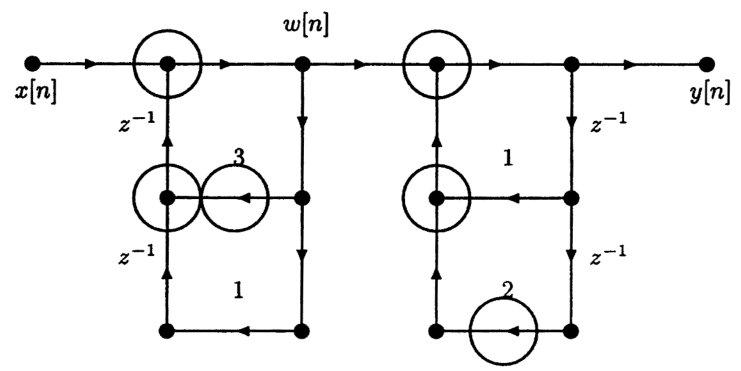
The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{4}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

(b) To get the difference equation, we just inverse $Z-$transform the equation in a. We get:

$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = 2x[n] + \frac{1}{4}x[n-1].$$

6.5. The flow graph for this system is drawn below.



(a)

$$w[n] = x[n] + 3w[n-1] + w[n-2]$$

$$y[n] = w[n] + y[n-1] + 2y[n-2]$$

(b)

$$W(z) = X(z) + 3z^{-1}W(z) + z^{-2}W(z)$$

$$Y(z) = W(z) + z^{-1}Y(z) + 2z^{-2}Y(z)$$

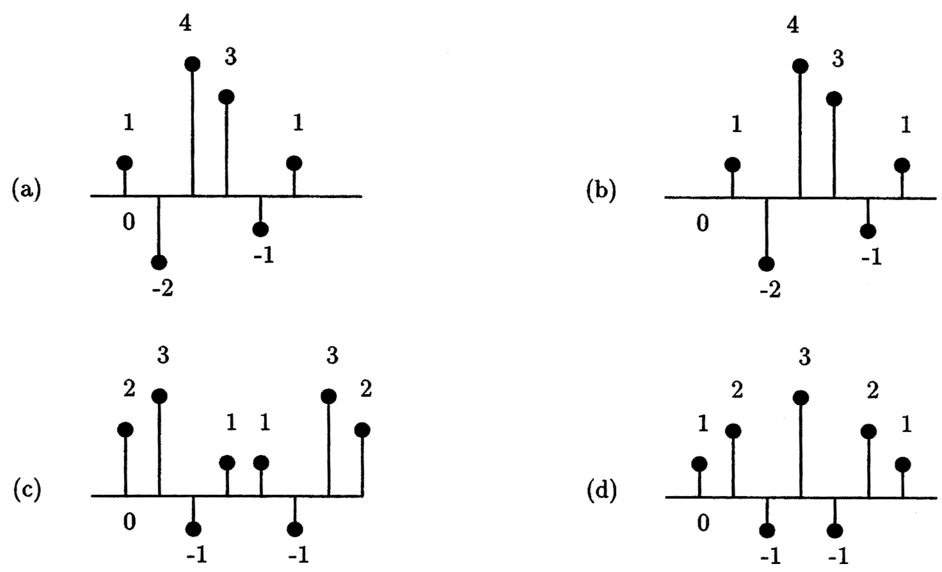
So

$$\begin{aligned}\frac{Y(z)}{X(z)} &= H(z) \\ &= \frac{1}{(1 - z^{-1} - 2z^{-2})(1 - 3z^{-1} - z^{-2})} \\ &= \frac{1}{1 - 4z^{-1} + 7z^{-3} + 2z^{-4}}.\end{aligned}$$

(c) Adds and multiplies are circled above: 4 real adds and 2 real multiplies per output point.

(d) It is not possible to reduce the number of storage registers. Note that implementing $H(z)$ above in the canonical direct form II (minimum storage registers) also requires 4 registers.

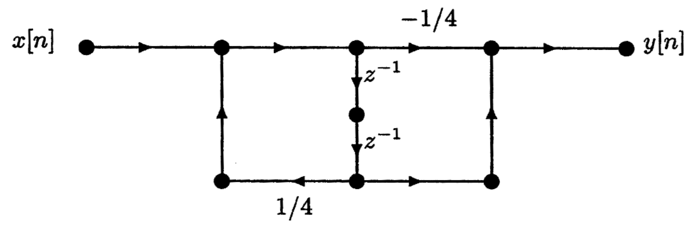
6.6. The impulse responses of each system are shown below.



6.7. We have

$$H(z) = \frac{-\frac{1}{4} + z^{-2}}{1 - \frac{1}{4}z^{-2}}.$$

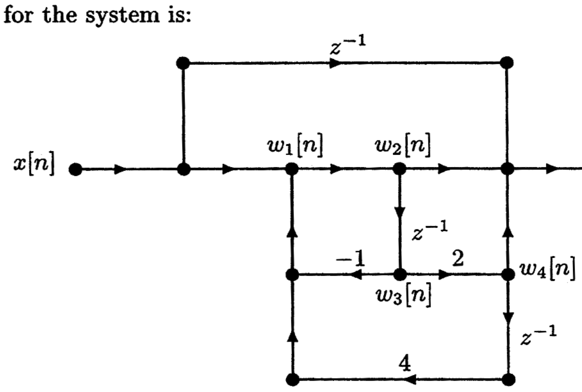
Therefore the direct form II is given by:



6.8. By looking at the graph, we get:

$$y[n] = 2y[n-2] + 3x[n-1] + x[n-2].$$

6.9. The signal flow graph for the system is:



(a) First we need to determine the transfer function. We have

$$\begin{aligned} w_1[n] &= x[n] - w_3[n] + 4w_4[n-1] \\ w_2[n] &= w_1[n] \\ w_3[n] &= w_2[n-1] \\ w_4[n] &= 2w_3[n] \\ y[n] &= w_2[n] + x[n-1] + w_4[n]. \end{aligned}$$

Taking the $Z-$transform of the above equations, rearranging and substituting terms, we get:

$$H(z) = \frac{1 + 3z^{-1} + z^{-2} - 8z^{-3}}{1 + z^{-1} - 8z^{-2}}.$$

The difference equation is thus given by:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

The impulse response is the response to an impulse, therefore:

$$h[n] + h[n-1] - 8h[n-2] = \delta[n] + 3\delta[n-1] + \delta[n-2] - 8\delta[n-3].$$

From the above equation, we have:

$$\begin{aligned} &h[0] = 1 \\ &h[1] = 3 - h[0] = 2. \end{aligned}$$

(b) From part (a) we have:

$$y[n] + y[n-1] - 8y[n-2] = x[n] + 3x[n-1] + x[n-2] - 8x[n-3].$$

6.10. (a)

$$\begin{aligned} w[n] &= \frac{1}{2}y[n] + x[n] \\ v[n] &= \frac{1}{2}y[n] + 2x[n] + w[n-1] \\ y[n] &= v[n-1] + x[n]. \end{aligned}$$

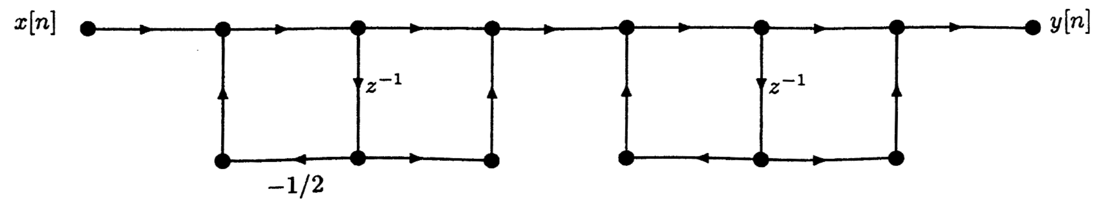
(b) Using the $Z-$transform of the difference equations in part (a), we get the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

We can rewrite it as :

$$H(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 + \frac{1}{2}z^{-1})(1 - z^{-1})}.$$

We thus get the following cascade form:

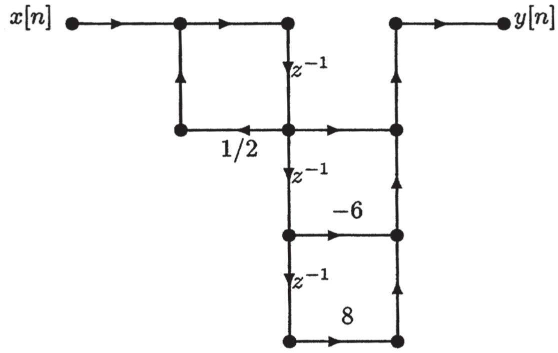


(c) The system function has poles at $z = -\frac{1}{2}$ and $z = 1$. Since the second pole is on the unit circle, the system is not stable.

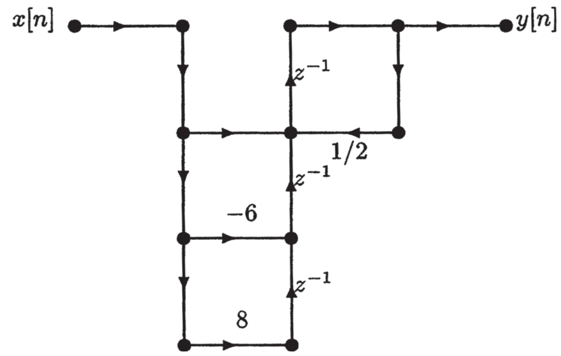
6.11. (a) $H(z)$ can be rewritten as:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}.$$

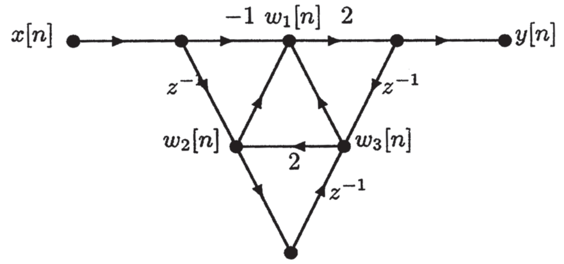
We thus get the following direct from II flow graph :



(b) To get the transposed form, we just reverse the arrows and exchange the input and the ouput. The graph can then be redrawn as:



6.12. We define the intermediate variables $w_1[n]$, $w_2[n]$ and $w_3[n]$ as follows:



We thus have the following relationships:

$$\begin{aligned} w_1[n] &= -x[n] + w_2[n] + w_3[n] \\ w_2[n] &= x[n-1] + 2w_3[n] \\ w_3[n] &= w_2[n-1] + y[n-1] \\ y[n] &= 2w_1[n]. \end{aligned}$$

$Z-$transforming the above equations and rearranging and grouping terms, we get:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-2 + 6z^{-1} + 2z^{-2}}{1 - 8z^{-1}}.$$

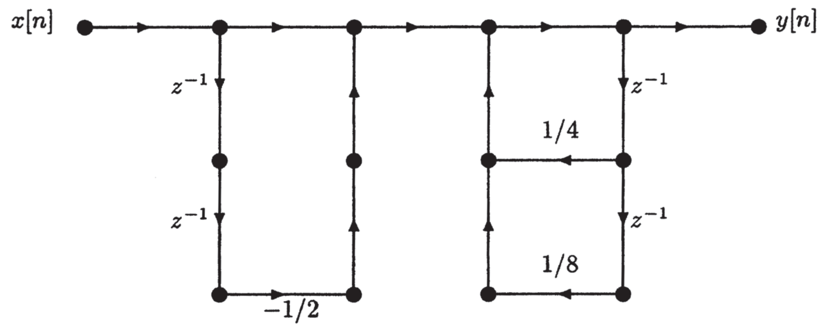
Taking the inverse $Z-$transform, we get the following difference equation:

$$y[n] - 8y[n-1] = -2x[n] + 6x[n-1] + 2x[n-2].$$

6.13.

$$H(z) = \frac{1 - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}.$$

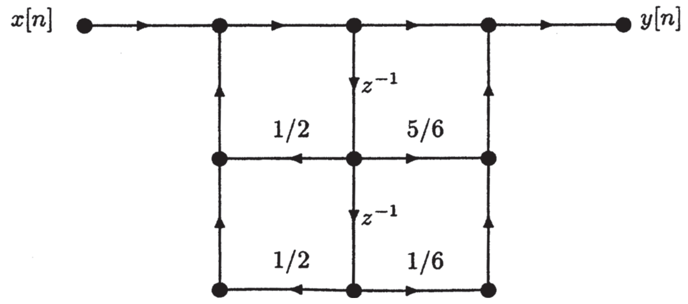
The direct form I implementation is:



6.14.

$$H(z) = \frac{1 + \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}.$$

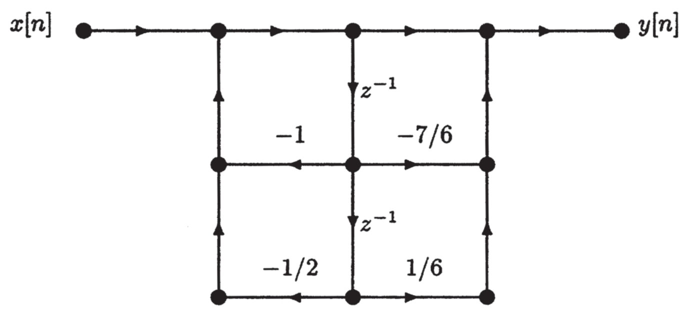
The direct form II implementation is:



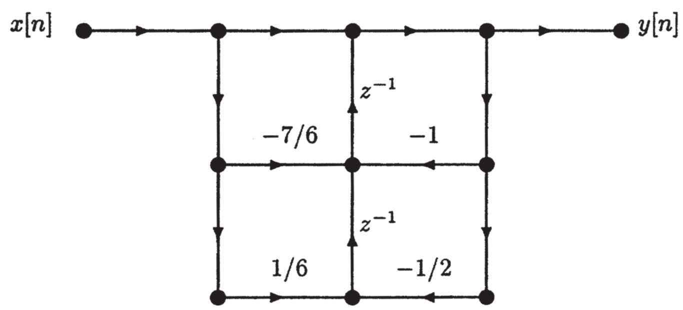
6.15.

$$H(z) = \frac{1 - \frac{7}{6}z^{-1} + \frac{1}{6}z^{-2}}{1 + z^{-1} + \frac{1}{2}z^{-2}}.$$

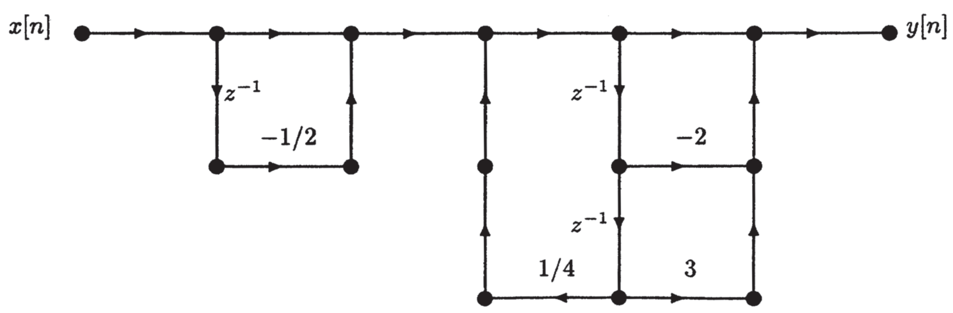
To get the transposed direct form II implementation, we first get the direct form II:



Now, we reverse the arrows and exchange the role of the input and the ouput to get the transposed direct form II:



6.16. (a) We just reverse the arrows and reverse the role of the input and the output, we get:



(b) The original system is the cascade of two transposed direct form II structures, therefore the system function is given by:

$$H(z) = \left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right)\left(1 - \frac{1}{2}z^{-1}\right).$$

The transposed graph, on the other hand, is the cascade of two direct form II structures, therefore the system function is given by:

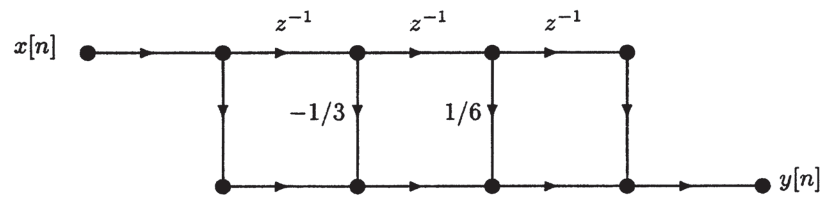
$$H(z) = \left(1 - \frac{1}{2}z^{-1}\right)\left(\frac{1 - 2z^{-1} + 3z^{-2}}{1 - \frac{1}{4}z^{-2}}\right).$$

This confirms that both graphs have the same system function $H(z)$.

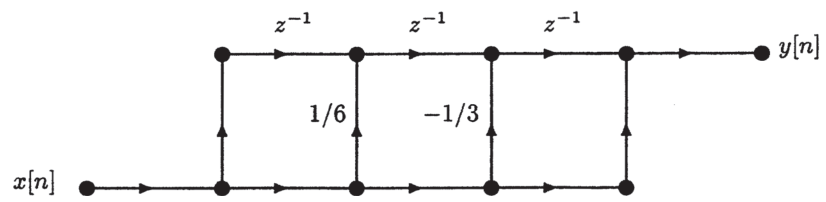
6.17.

$$H(z) = 1 - \frac{1}{3}z^{-1} + \frac{1}{6}z^{-2} + z^{-3}.$$

(a) Direct form implementation of this system:



(b) Transposed direct form implementation of the system:



6.18. The flow graph is just a cascade of two transposed direct form II structures, the system function is thus given by:

$$H(z) = \left(\frac{1 + \frac{4}{3}z^{-1} - \frac{4}{3}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}\right)\left(\frac{1}{1 - az^{-1}}\right).$$

Which can be rewritten as:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{(1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2})(1 - az^{-1})}.$$

In order to implement this system function with a second-order direct form II signal flow graph, a pole-zero cancellation has to occur, this happens if $a = \frac{2}{3}$, $a = -2$ or $a = 0$. If $a = \frac{2}{3}$, the overall system function is:

$$H(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

If $a = -2$, the overall system function is:

$$H(z) = \frac{1 - \frac{2}{3}z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

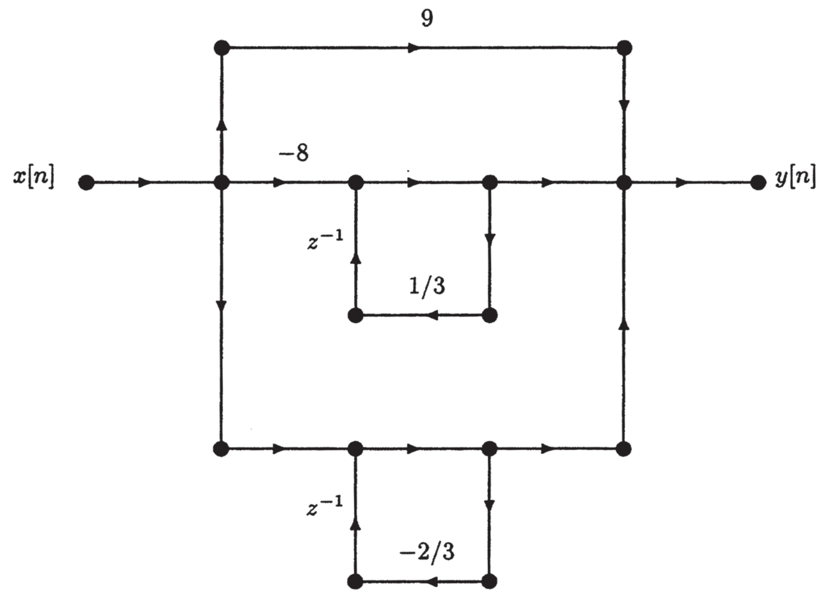
And finally if $a = 0$, the overall system function is:

$$H(z) = \frac{(1 + 2z^{-1})(1 - \frac{2}{3}z^{-1})}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}}.$$

6.19. Using partial fraction expansion, the system function can be rewritten as:

$$H(z) = \frac{-8}{1 - \frac{1}{3}z^{-1}} + \frac{1}{1 + \frac{2}{3}z^{-1}} + 9.$$

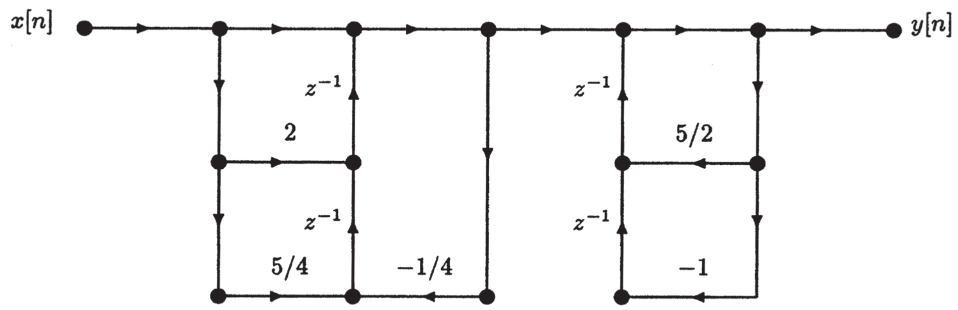
Now we can draw the flow graph that implements this system as a parallel combination of first-order transposed direct form II sections:



6.20. The transfer function can be rewritten as:

$$H(z) = \frac{(1 + 2z^{-1} + \frac{5}{4}z^{-2})}{(1 + \frac{1}{4}z^{-2})(1 - \frac{5}{2}z^{-1} + z^{-2})}$$

which can be implemented as the following cascade of second-order transposed direct form II sections:

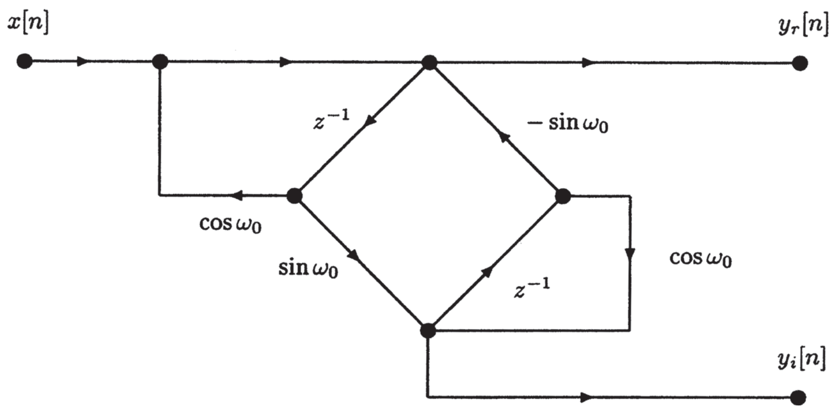


6.21.

$$h[n] = e^{j\omega_0 n}u[n] \longleftrightarrow H(z) = \frac{1}{1 - e^{j\omega_0}z^{-1}} = \frac{Y(z)}{X(z)}.$$

So $y[n] = e^{j\omega_0}y[n-1] + x[n]$. Let $y[n] = y_r[n] + jy_i[n]$. Then $y_r[n] + jy_i[n] = (\cos\omega_0 + j\sin\omega_0)(y_r[n-1] + jy_i[n-1]) + x[n]$. Separate the real and imaginary parts:

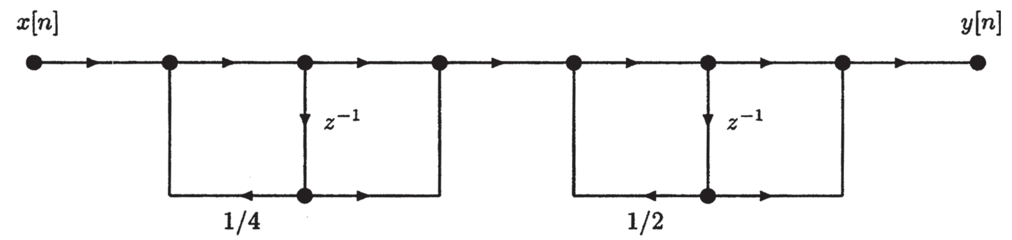
$$\begin{aligned} y_r[n] &= x[n] + \cos\omega_0 y_r[n-1] - \sin\omega_0 y_i[n-1] \\ y_i[n] &= \sin\omega_0 y_r[n-1] + \cos\omega_0 y_i[n-1]. \end{aligned}$$



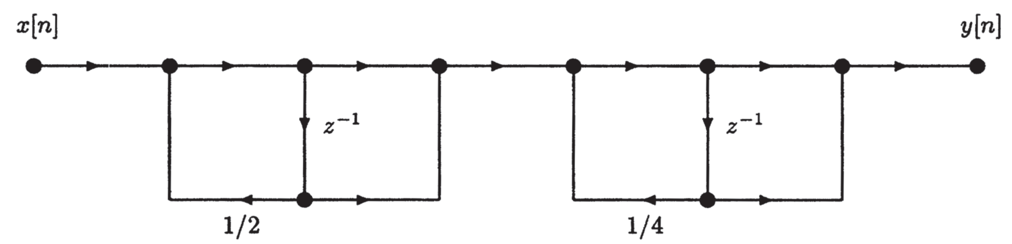
6.22.

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})}.$$

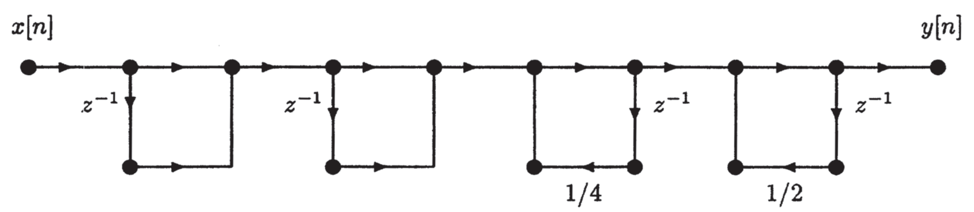
$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right).$$



$$H(z) = \left(\frac{1+z^{-1}}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}\right).$$



Plus 12 systems of this form:



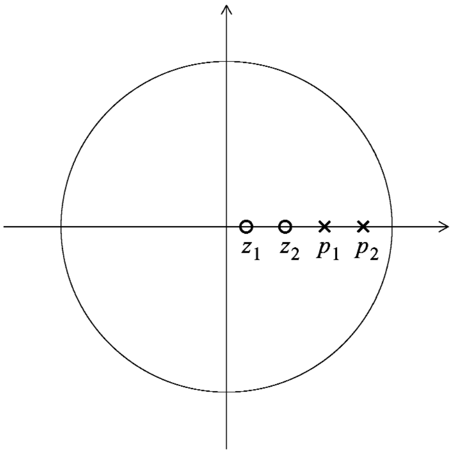
with the three types of 1st-order systems taken in various orders.

6.23. Problem 1 in Fall 2003 Midterm exam Appears in: Spring04 PS3.
Note: The Fall2003 Midterm version additionally includes a part (d):

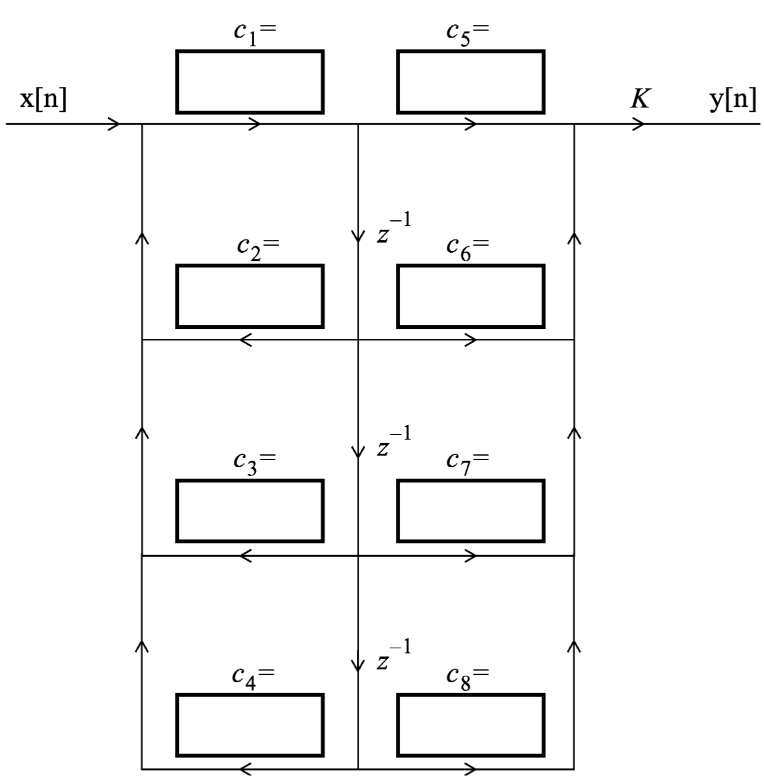
(d) (5%) For the most accurate placement of the **zeros**, which form(s) would you chose: direct, cascade, or parallel? Explain briefly.

Problem

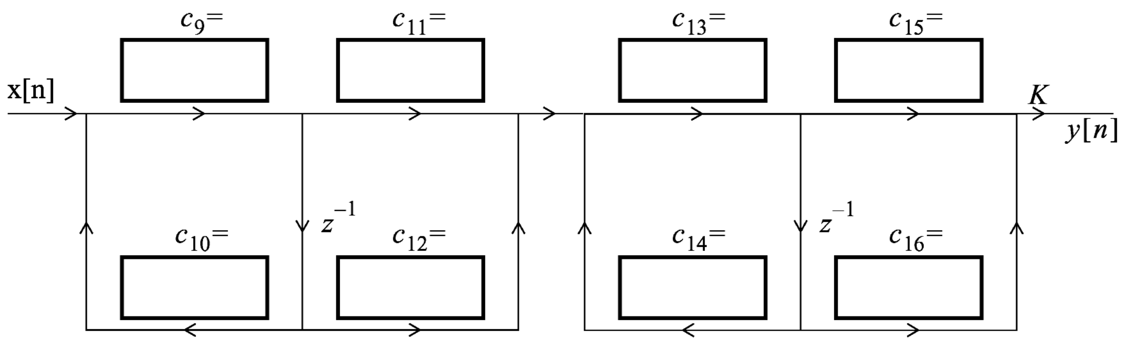
We want to implement a causal system $H(z)$ with the pole-zero diagram shown below. For all parts of this problem, z_1, z_2, p_1, and p_2 are real, and a gain constant that is independent of frequency can be absorbed into the K term in each flowgraph.



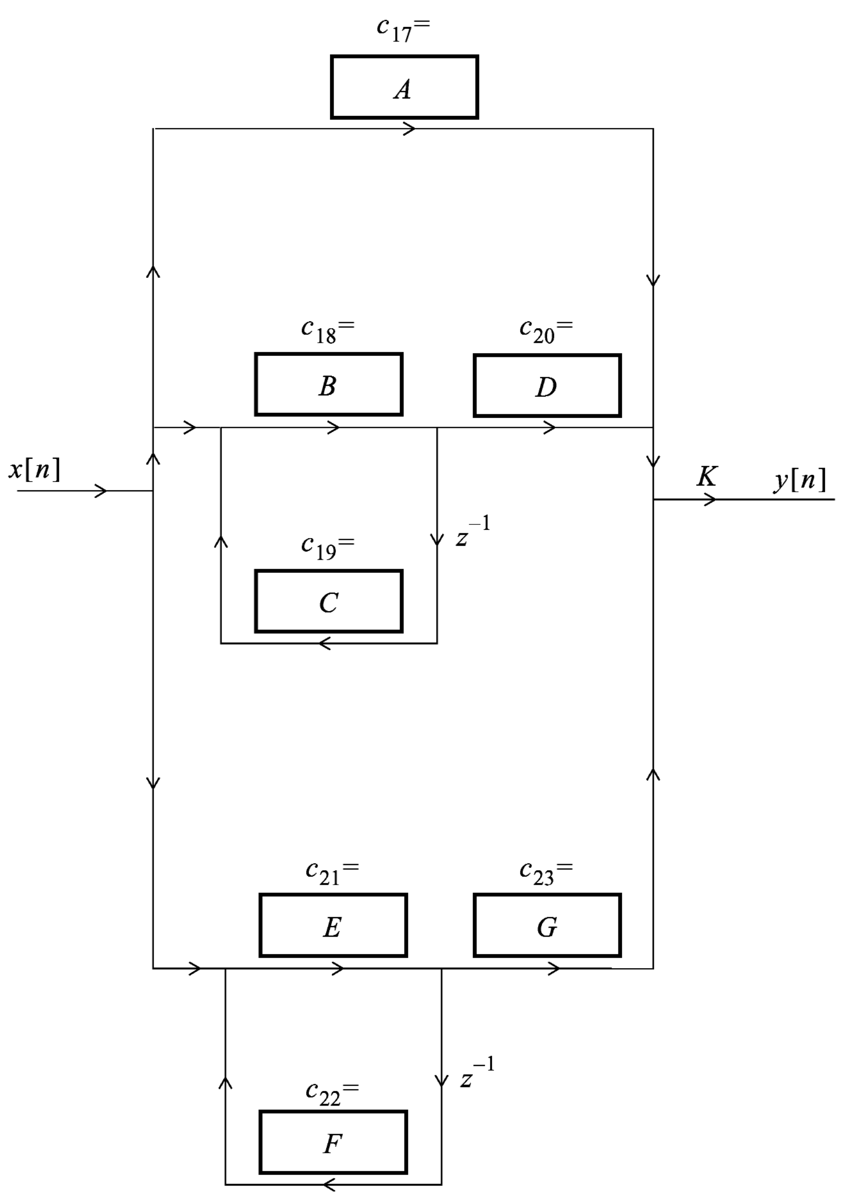
(a) Fill in the following flowgraph, in terms of the variables z_1, z_2, p_1, and p_2.



(b) Fill in the following flowgraph, in terms of the variables z_1, z_2, p_1, and p_2.



(c) Write down the system of linear equations for the variables $A, B, \ldots, G$ in terms of the variables z_1, z_2, p_1, and p_2.



Solution from Spring04 PS3

The system function is proportional to:

$$\begin{aligned} &\frac{\left(1 - z_1 z^{-1}\right)\left(1 - z_2 z^{-1}\right)}{\left(1 - p_1 z^{-1}\right)\left(1 - p_2 z^{-1}\right)} \\ = &\frac{1 - (z_1 + z_2) z^{-1} + z_1 z_2 z^{-2}}{1 - (p_1 + p_2) z^{-1} + p_1 p_2 z^{-2}} \end{aligned}$$

(a) $c_1 = 1, \quad c_2 = p_1 + p_2, \quad c_3 = -p_1 p_2, \quad c_4 = 0$

$c_5 = 1, \quad c_6 = -(z_1 + z_2), \quad c_7 = z_1 z_2, \quad c_8 = 0$

(b) $c_9 = 1, \quad c_{10} = p_1, \quad c_{11} = 1, \quad c_{12} = -z_1$

$c_{13} = 1, \quad c_{14} = p_2, \quad c_{15} = 1, \quad c_{16} = -z_2$

(c) $B = 1, \quad C = p_1, \quad E = 1, \quad F = p_2$

From a partial fraction expansion,

$$1 - (z_1 + z_2) z^{-1} + z_1 z_2 z^{-2} = A(1 - p_1 z^{-1})(1 - p_2 z^{-1}) + D(1 - p_2 z^{-1}) + G(1 - p_1 z^{-1})$$

Therefore A, D, and G can be found by solving the following system of equations:

$$\begin{aligned} 1 &= A + D + G \\ -(z_1 + z_2) &= -A(p_1 + p_2) - Dp_2 - Gp_1 \\ z_1 z_2 &= A p_1 p_2 \end{aligned}$$

$$A = \frac{z_1 z_2}{p_1 p_2}, \quad D = \frac{(z_1 - p_1)(z_2 - p_1)}{p_1(p_1 - p_2)}, \quad G = \frac{(z_1 - p_2)(z_2 - p_2)}{p_2(p_2 - p_1)}$$

Solution from Fall03 Midterm

Problem

The system function is proportional to:

$$\begin{aligned} & \frac{\left(1-z_1z^{-1}\right)\left(1-z_2z^{-1}\right)}{\left(1-p_1z^{-1}\right)\left(1-p_2z^{-1}\right)} \\ = & \frac{1-(z_1+z_2)z^{-1}+z_1z_2z^{-2}}{1-(p_1+p_2)z^{-1}+p_1p_2z^{-2}} \end{aligned}$$

(a) $c_1 = 1, \quad c_2 = p_1 + p_2, \quad c_3 = -p_1p_2, \quad c_4 = 0$

$c_5 = 1, \quad c_6 = -(z_1 + z_2), \quad c_7 = z_1z_2, \quad c_8 = 0$

(b) $c_9 = 1, \quad c_{10} = p_1, \quad c_{11} = 1, \quad c_{12} = -z_1$

$c_{13} = 1, \quad c_{14} = p_2, \quad c_{15} = 1, \quad c_{16} = -z_2$

(c) $B = 1, \quad C = p_1$

$E = 1, \quad F = p_2$

From a partial fraction expansion,

$$1-(z_1+z_2)z^{-1}+z_1z_2z^{-2} = A(1-p_1z^{-1})(1-p_2z^{-1}) + D(1-p_2z^{-1}) + G(1-p_1z^{-1})$$

Therefore A, D, and G can be found by solving the following system of equations:

$$\begin{aligned} 1 &= A + D + G \\ -(z_1+z_2) &= -A(p_1+p_2) - Dp_2 - Gp_1 \\ z_1z_2 &= Ap_1p_2 \end{aligned}$$

(d) For direct form, the zeros of the actual system are the roots of the second order equation

$$z^2 + c_6z + c_7 = 0$$

For cascade form, the zeros of the actual system are the roots of the first order equations:

$$z + c_{12} = 0, \quad z + c_{16} = 0$$

For parallel form, the zeros of the actual system are the roots of the second order equation:

$$c_{17}(z - c_{19})(z - c_{22}) + c_{20}z(z - c_{22}) + c_{23}z(z - c_{19}) = 0$$

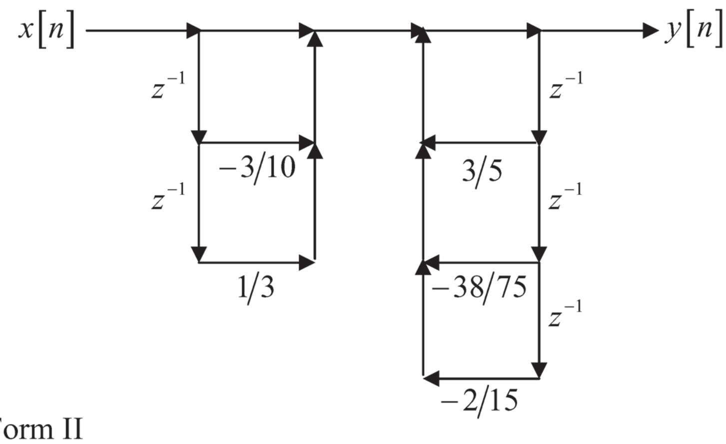
Cascade form is prefered because it is the only form with first order equations.

6.24.

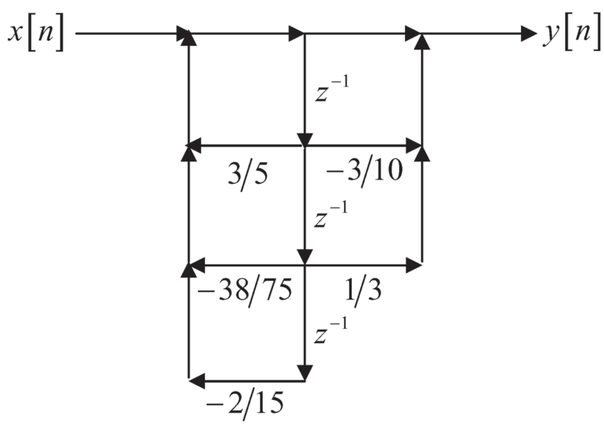
$$H(z)=\frac{1-\frac{3}{10}z^{-1}+\frac{1}{3}z^{-2}}{\left(1-\frac{4}{5}z^{-1}+\frac{2}{3}z^{-2}\right)\left(1+\frac{1}{5}z^{-1}\right)}$$

$$=\frac{1-\frac{3}{10}z^{-1}+\frac{1}{3}z^{-2}}{1-\frac{3}{5}z^{-1}+\frac{38}{75}z^{-2}+\frac{2}{15}z^{-3}}.$$

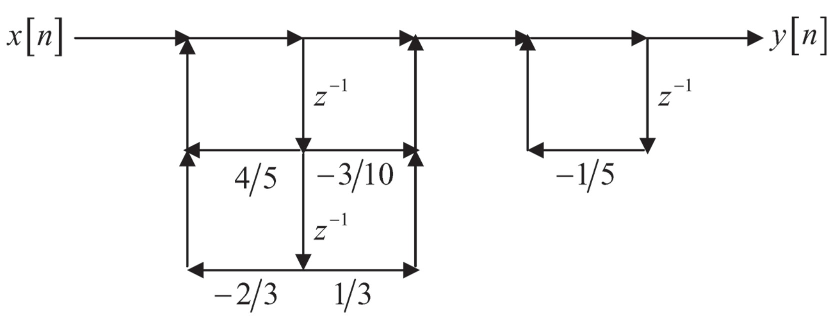
1. i) Direct Form I



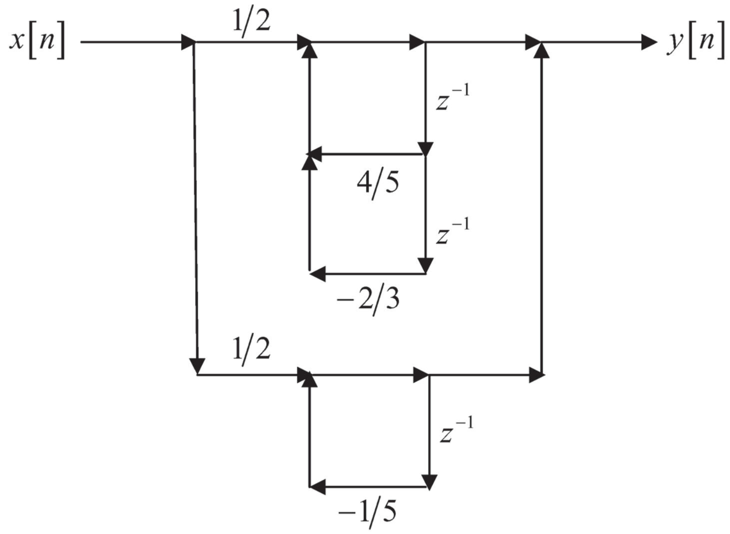
ii) Direct Form II



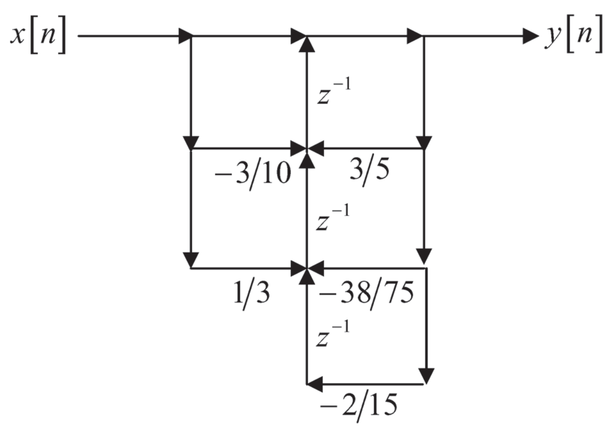
iii) Cascade Form



iv) Parallel Form



v) Transposed Direct Form II



2. Label the interior nodes of the transposed direct form II structure $v_0[n], v_1[n], v_2[n], v_3[n]$ counting from the top down. Then we have

$$y[n] = v_0[n]$$
$$v_0[n] = x[n] + v_1[n-1]$$
$$v_1[n] = \tfrac{3}{5}y[n] - \tfrac{3}{10}x[n] + v_2[n-1]$$
$$v_2[n] = -\tfrac{38}{75}y[n] + \tfrac{1}{3}x[n] + v_3[n-1]$$
$$v_3[n] = -\tfrac{2}{15}y[n].$$

Taking the z-transform of these equations gives

$$Y(z) = V_0(z)$$
$$V_0(z) = X(z) + z^{-1}V_1(z)$$
$$V_1(z) = \tfrac{3}{5}Y(z) - \tfrac{3}{10}X(z) + z^{-1}V_2(z)$$
$$V_2(z) = -\tfrac{38}{75}Y(z) + \tfrac{1}{3}X(z) + z^{-1}V_3(z)$$
$$V_3(z) = -\tfrac{2}{15}Y(z).$$

Substituting Eq. (5) into Eq. (4) gives

$$\begin{aligned} V_2(z) &= -\tfrac{38}{75}Y(z) + \tfrac{1}{3}X(z) - \tfrac{2}{15}z^{-1}Y(z) \\ &= -\left(\tfrac{38}{75} + \tfrac{2}{15}z^{-1}\right)Y(z) + \tfrac{1}{3}X(z). \end{aligned}$$

Substituting into Eq. (3) gives

$$\begin{aligned} V_1(z) &= \tfrac{3}{5}Y(z) - \tfrac{3}{10}X(z) + z^{-1}\left(-\left(\tfrac{38}{75} + \tfrac{2}{15}z^{-1}\right)Y(z) + \tfrac{1}{3}X(z)\right) \\ &= \left(\tfrac{3}{5} - \tfrac{38}{75}z^{-1} - \tfrac{2}{15}z^{-2}\right)Y(z) + \left(-\tfrac{3}{10} + \tfrac{1}{3}z^{-1}\right)X(z). \end{aligned}$$

Now substitute into Eq. (2):

$$\begin{aligned} V_0(z) &= X(z) + z^{-1}\left\{\left(\tfrac{3}{5} - \tfrac{38}{75}z^{-1} - \tfrac{2}{15}z^{-2}\right)Y(z) + \left(-\tfrac{3}{10} + \tfrac{1}{3}z^{-1}\right)X(z)\right\} \\ &= \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z) + \left(\tfrac{3}{5}z^{-1} - \tfrac{38}{75}z^{-2} - \tfrac{2}{15}z^{-3}\right)Y(z). \end{aligned}$$

Finally, substitute into Eq. (1):

$$Y(z) = \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z) + \left(\tfrac{3}{5}z^{-1} - \tfrac{38}{75}z^{-2} - \tfrac{2}{15}z^{-3}\right)Y(z)$$

$$\left(1 - \tfrac{3}{5}z^{-1} + \tfrac{38}{75}z^{-2} + \tfrac{2}{15}z^{-3}\right)Y(z) = \left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(1 - \tfrac{3}{10}z^{-1} + \tfrac{1}{3}z^{-2}\right)}{\left(1 - \tfrac{3}{5}z^{-1} + \tfrac{38}{75}z^{-2} + \tfrac{2}{15}z^{-3}\right)}.$$

This final expression is the correct system function.

6.25. A.

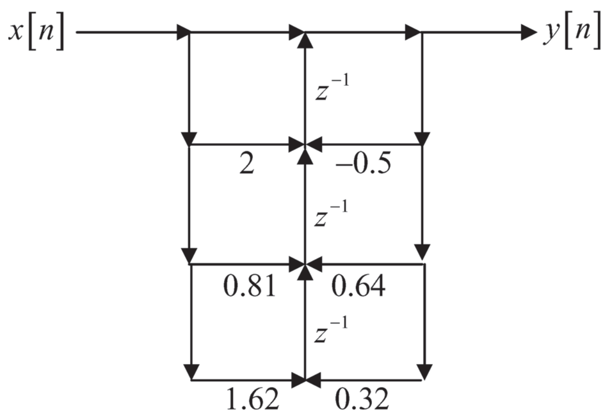
$$H(z) = \frac{1+0.81z^{-2}}{1-0.3z^{-1}-0.4z^{-2}} \times \frac{1+2z^{-1}}{1+0.8z^{-1}}$$
$$= \frac{\left(1+j0.9z^{-1}\right)\left(1-j0.9z^{-1}\right)\left(1+2z^{-1}\right)}{\left(1-0.8z^{-1}\right)\left(1+0.5z^{-1}\right)\left(1+0.8z^{-1}\right)}.$$

B. Yes, the overall system is stable. All of the poles are inside the unit circle, which guarantees stability for a causal system.

C. No, the system is not minimum-phase. There is a zero outside the unit circle at $z = -2$.

D.

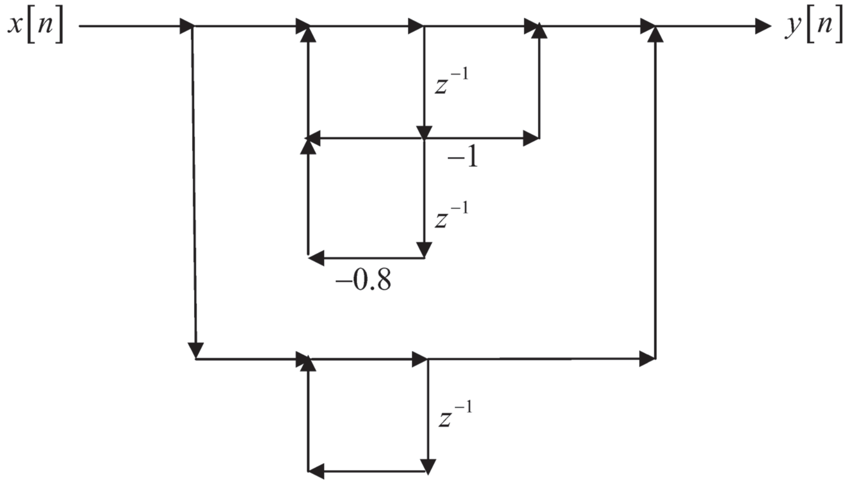
$$H(z) = \frac{1+2z^{-1}+0.81z^{-2}+1.62z^{-3}}{1+0.5z^{-1}-0.64z^{-2}-0.32z^{-3}}.$$



6.26. $$H(z)=\frac{1}{1-z^{-1}}+\frac{1-z^{-1}}{1-z^{-1}+0.8z^{-2}}.$$

A. This system is not stable since there is a pole on the unit circle at $z=1$.

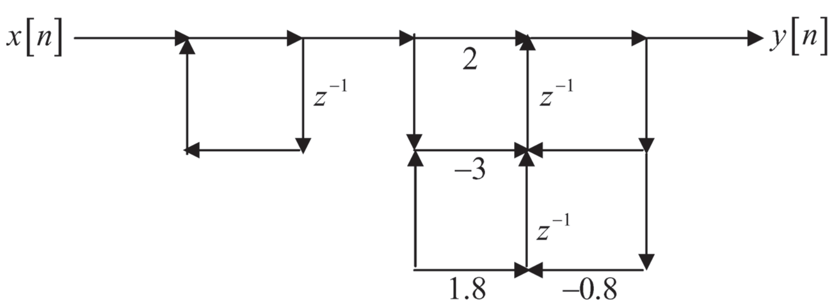
B. Parallel Form. (A direct form II implementation has been chosen for the second-order subsystem. Other possibilities would be acceptable.)



C. We can combine the terms in $H(z)$ to obtain

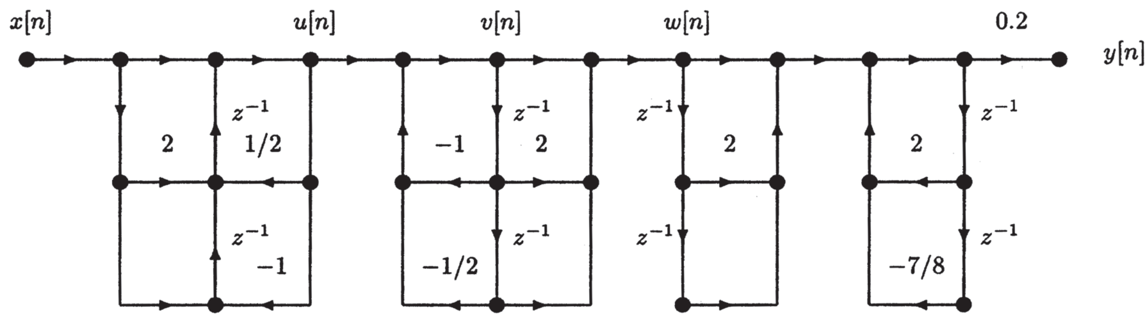
$$H(z)=\frac{2-3z^{-1}+1.8z^{-2}}{\left(1-z^{-1}\right)\left(1-z^{-1}+0.8z^{-2}\right)}.$$

Cascade Form. (Transposed direct form II has been used for the second-order section, as required.)



6.27. (a) We can rearrange $H(z)$ this way:

$$H(z) = \frac{(1+z^{-1})^2}{1-\frac{1}{2}z^{-1}+z^{-2}} \cdot \frac{(1+z^{-1})^2}{1+z^{-1}+\frac{1}{2}z^{-2}} \cdot (1+z^{-1})^2 \cdot \frac{1}{1-2z^{-1}+\frac{7}{8}z^{-2}} \cdot 0.2$$

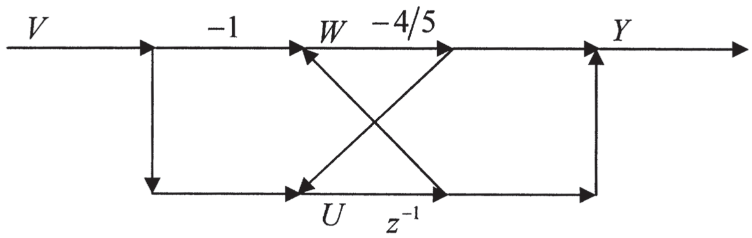


The solution is not unique; the order of the denominator 2nd-order sections may be rearranged.

(b)

$$\begin{aligned}
u[n] &= x[n] + 2x[n-1] + x[n-2] + \frac{1}{2}u[n-1] - u[n-2] \\
v[n] &= u[n] - v[n-1] - \frac{1}{2}v[n-2] \\
w[n] &= v[n] + 2v[n-1] + v[n-2] \\
y[n] &= w[n] + 2w[n-1] + w[n-2] + 2y[n-1] - \frac{7}{8}y[n-2].
\end{aligned}$$

6.28. (a) Working first with the inner part of the flowgraph:



Writing z-transform equations,

$$W = -V + z^{-1}U$$

$$U = V - \frac{4}{5}W$$

$$Y = -\frac{4}{5}W + z^{-1}U$$

Developing the equation for W,

$$W = -V + z^{-1}V - \frac{4}{5}z^{-1}W$$

$$W\left(1+\frac{4}{5}z^{-1}\right) = V\left(-1+z^{-1}\right)$$

$$W = V\left(\frac{-1+z^{-1}}{1+\frac{4}{5}z^{-1}}\right).$$

Developing the equation for U,

$$U = V + \frac{4}{5}V - \frac{4}{5}z^{-1}U$$

$$U\left(1+\frac{4}{5}z^{-1}\right) = \frac{9}{5}V$$

$$U = V\left(\frac{\frac{9}{5}}{1+\frac{4}{5}z^{-1}}\right).$$

Substituting these into the equation for Y,

$$\frac{Y}{V} = -\frac{4}{5}\left(\frac{-1+z^{-1}}{1+\frac{4}{5}z^{-1}}\right) + \frac{\frac{9}{5}}{1+\frac{4}{5}z^{-1}}$$

$$\frac{Y}{V} = \frac{\frac{4}{5}+z^{-1}}{1+\frac{4}{5}z^{-1}}.$$

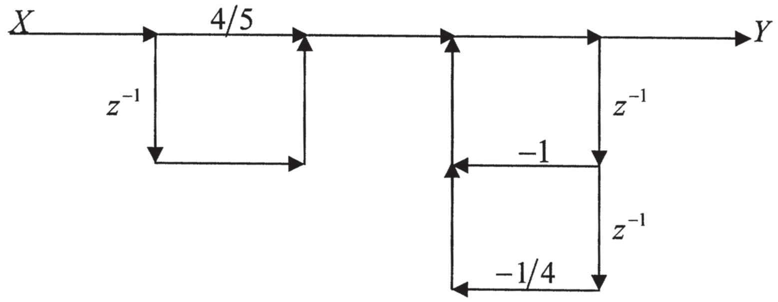
Let us call this $H_{inner}(z)$. The other parts of the flowgraph are connected to $H_{inner}(z)$ in a feedback configuration.

$$H(z)=\frac{Y(z)}{X(z)}=\frac{H_{inner}(z)}{1-H_{inner}(z)\left(-\frac{1}{4}z^{-1}\right)}$$

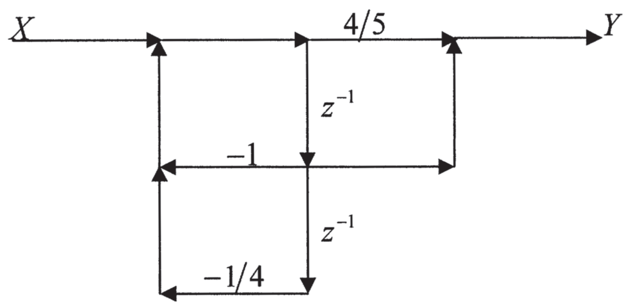
$$=\frac{\frac{4}{5}+z^{-1}}{1+\frac{4}{5}z^{-1}+\left(\frac{1}{4}z^{-1}\right)\left(\frac{4}{5}+z-1\right)}$$

$$=\frac{\frac{4}{5}+z^{-1}}{1+z^{-1}+\frac{1}{4}z^{-2}}.$$

(b) The direct form I flowgraph is



The direct form II flowgraph is



(c) To see where $H(z)$ has poles and zeros, write it as

$$H(z) = \frac{z\left(\frac{4}{5}z+1\right)}{\left(z+\frac{1}{2}\right)^2}.$$

The zero at $z=0$ and the double pole at $z=-\frac{1}{2}$ are both inside the unit circle. However, the zero at $z=-\frac{5}{4}$ is not. Cancel that zero by designing $H_1(z)$ as a unity gain all-pass filter with a pole at $z=-\frac{5}{4}$ and a zero at $z=-\frac{4}{5}$. That is,

$$H_1(z) = \frac{z^{-1}-a^*}{1-az^{-1}} = \frac{1-a^*z}{z-a}.$$

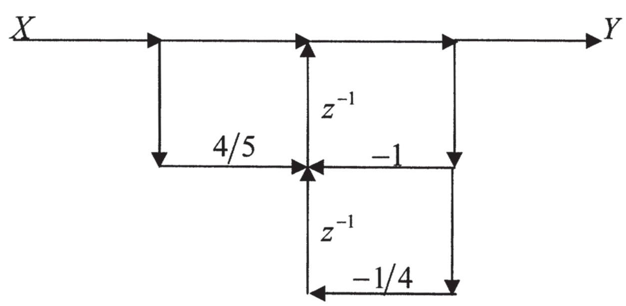
If we want a pole at $z=-\frac{5}{4}$, we can set $a=-\frac{5}{4}$.

(d) First,

$$H_2(z) = \frac{\left(\frac{4}{5}+z^{-1}\right)}{\left(1+z^{-1}+\frac{1}{4}z^{-2}\right)} \frac{\left(z^{-1}+\frac{5}{4}\right)}{\left(1+\frac{5}{4}z^{-1}\right)}$$

$$= \frac{1+\frac{4}{5}z^{-1}}{1+z^{-1}+\frac{1}{4}z^{-2}}.$$

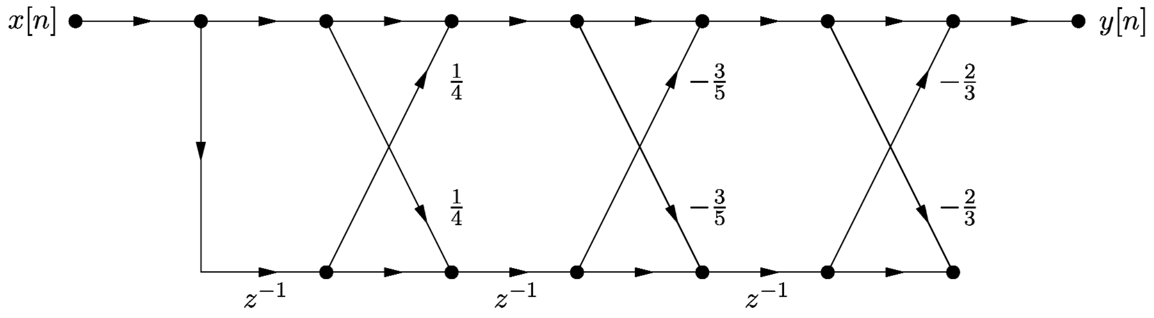
Then the transposed direct form II flowgraph is



6.29. Appears in: Fall05 PS4.

Problem

(a) Determine the system function $H(z)$ relating the input $x[n]$ to the output $y[n]$ for the FIR lattice filter depicted below:



(b) Draw the lattice filter structure for the all-pole filter $1/H(z)$.

Solution from Fall05 PS4

(a) The FIR lattice filter has three stages and is therefore third-order. The three k-parameters or reflection coefficients are:

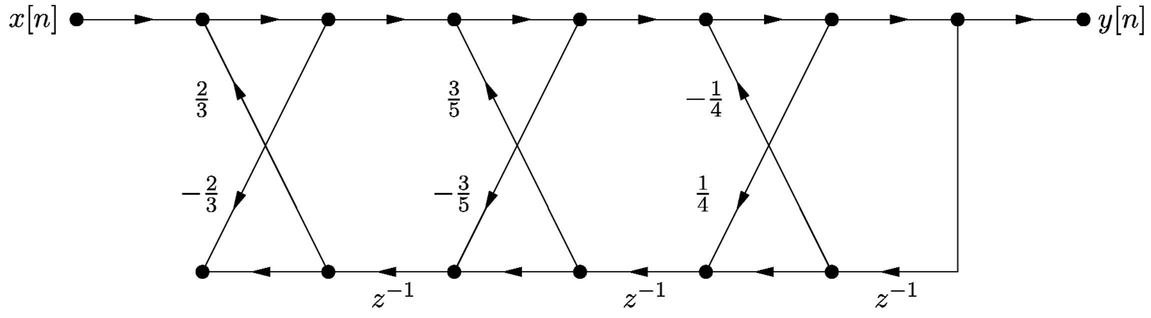
$$k_1 = -\frac{1}{4}, \quad k_2 = \frac{3}{5}, \quad k_3 = \frac{2}{3}.$$

We start the recursion given by equations (8) in the lattice filter notes at $p+1 = 1$ and proceed until we have found the coefficients $a_k^{(3)}, \quad k = 1, 2, 3.$

$$\begin{aligned}
a_1^{(1)} &= k_1 = -\frac{1}{4} \\
a_1^{(2)} &= a_1^{(1)} - k_2 a_1^{(1)} = -\frac{1}{10} \\
a_2^{(2)} &= k_2 = \frac{3}{5} \\
a_1^{(3)} &= a_1^{(2)} - k_3 a_2^{(2)} = -\frac{1}{2} \\
a_2^{(3)} &= a_2^{(2)} - k_3 a_1^{(2)} = \frac{2}{3} \\
a_3^{(3)} &= k_3 = \frac{2}{3}
\end{aligned}$$

$$H(z) = 1 + \frac{1}{2}z^{-1} - \frac{2}{3}z^{-2} - \frac{2}{3}z^{-3}.$$

(b) Armed with the k-parameters from part (a), we refer to Figure 5 in the lattice filter notes in drawing the lattice structure for the all-pole filter $1/H(z)$:



(Note: $1/H(z)$ is stable because all of the reflection coefficients have magnitudes less than unity.)

6.30. Appears in: Fall05 PS4.

Problem

Determine and draw the lattice filter implementation of the following causal all-pole system function:

$$H(z) = \frac{1}{1 + \frac{3}{2}z^{-1} - z^{-2} + \frac{3}{4}z^{-3} + 2z^{-4}}$$

Is the system stable?

Solution from Fall05 PS4

The all-pole filter is fourth-order with coefficients:

$$a_1^{(4)} = -\frac{3}{2}, \quad a_2^{(4)} = 1, \quad a_3^{(4)} = -\frac{3}{4}, \quad a_4^{(4)} = -2.$$

We know immediately that $k_4 = a_4^{(4)} = -2$. To find the remaining reflection coefficients, we need to run the recursion in reverse and find the coefficients for successively lower order filters. Letting $M = 4$ in equation (11) of the lattice filter notes,

$$\begin{aligned}
a_1^{(3)} &= \frac{a_1^{(4)} + k_4 a_3^{(4)}}{1 - k_4^2} = 0 \\
a_2^{(3)} &= \frac{a_2^{(4)} + k_4 a_2^{(4)}}{1 - k_4^2} = \frac{1}{3} \\
a_3^{(3)} &= \frac{a_3^{(4)} + k_4 a_1^{(4)}}{1 - k_4^2} = -\frac{3}{4}
\end{aligned}$$

We identify $k_3 = a_3^{(3)} = -\frac{3}{4}$ and proceed to $M = 3$:

$$\begin{aligned}
a_1^{(2)} &= \frac{a_1^{(3)} + k_3 a_2^{(3)}}{1 - k_3^2} = -\frac{4}{7} \\
a_2^{(2)} &= \frac{a_2^{(3)} + k_3 a_1^{(3)}}{1 - k_3^2} = \frac{16}{21}
\end{aligned}$$

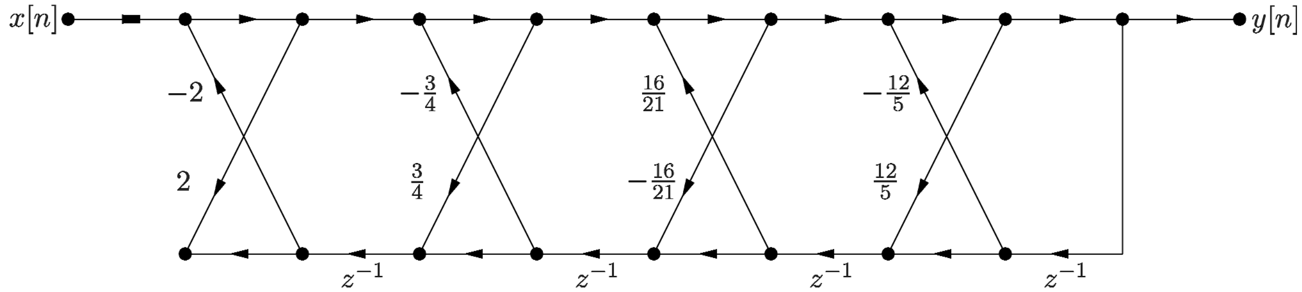
Thus $k_2 = a_2^{(2)} = \frac{16}{21}$. Finally,

$$a_1^{(1)} = \frac{(1 + k_2)a_1^{(2)}}{1 - k_2^2} = \frac{a_1^{(2)}}{1 - k_2} = -\frac{12}{5},$$

and $k_1 = -\frac{12}{5}$.

The lattice structure for $H(z)$ is shown below:

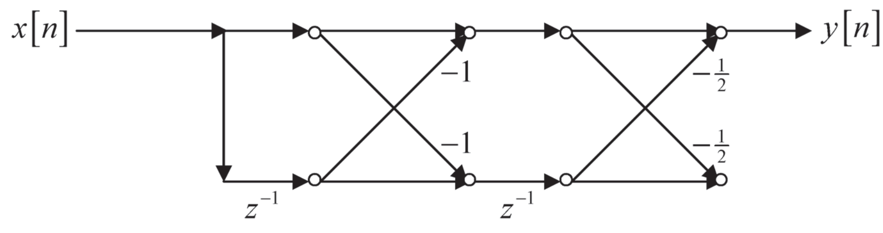
Since $|k_1| > 1$ and $|k_4| > 1$, the all-pole filter cannot be stable.



6.31. (a) To determine $y[1]$, sum the gains of all paths with a single delay to the output. This gives

$$y[1] = 1 + (-1)\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$

(b) The flow graph for the inverse filter will be a cascade of FIR stages with the *k*-coefficients in the reverse order.



(c) When the FIR lattice of part (b) is driven by an impulse, the response is seen to be

$$\begin{aligned} h_{FIR}[n] &= \delta[n] + \left(-1 + (-1)\left(-\tfrac{1}{2}\right)\right)\delta[n-1] - \tfrac{1}{2}\delta[n-2] \\ &= \delta[n] - \tfrac{1}{2}\delta[n-1] - \tfrac{1}{2}\delta[n-2]. \end{aligned}$$

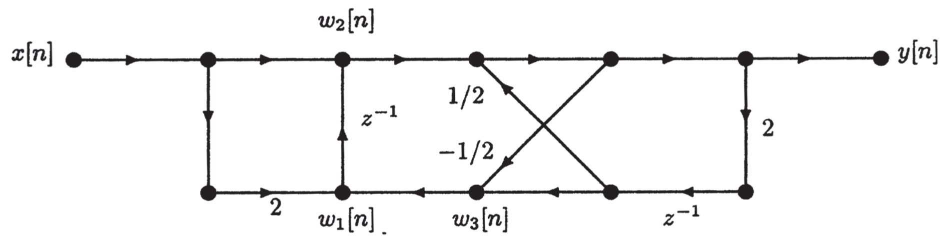
The transfer function is

$$H_{FIR}(z) = 1 - \tfrac{1}{2}z^{-1} - \tfrac{1}{2}z^{-2}.$$

This is the transfer function for the inverse filter. The transfer function for the given lattice is then

$$H(z) = \frac{1}{1 - \tfrac{1}{2}z^{-1} - \tfrac{1}{2}z^{-2}}.$$

6.32. (a) Transpose = reverse arrows direction and reverse the input/output, we get:



(b) From part (a), we have:

$$\begin{aligned}
(1)\ w_1[n] &= 2x[n] + w_3[n] \\
(2)\ w_2[n] &= x[n] + w_1[n-1] \\
(3)\ w_3[n] &= -\frac{1}{2}y[n] + 2y[n-1] \\
(4)\ y[n] &= w_2[n] + y[n-1]
\end{aligned}$$

Taking the $Z-$transform of the above equations, substituting and rearranging terms, we get:

$$(1 - \frac{1}{2}z^{-1} - 2z^{-2})Y(z) = (2z^{-1} + 1)X(z).$$

Finally, inverse $Z-$transforming, we get the following difference equation:

$$y[n] - \frac{1}{2}y[n-1] - 2y[n-2] = x[n] + 2x[n-1].$$

(c) From part (b), the system function is given by:

$$H(z) = \frac{1 + 2z^{-1}}{1 - \frac{1}{2}z^{-1} - 2z^{-2}}.$$

It has poles at

$$z = -\frac{8}{1-\sqrt{33}} \text{ and } z = -\frac{8}{1+\sqrt{33}}$$

which are outside the unit circle, therefore the system is NOT BIBO stable.

(d)

$$\begin{aligned}
&y[2] = x[2] + 2x[1] + \tfrac{1}{2}y[1] + 2y[0] \\
&y[0] = x[0] = 1 \\
&y[1] = x[1] + 2x[0] + \tfrac{1}{2}y[0] = \tfrac{1}{2} + 2 + \tfrac{1}{2} = 3
\end{aligned}$$

Therefore,

$$y[2] = \frac{1}{4} + 1 + \frac{3}{2} + 2 = \frac{19}{4}.$$

6.33. Problem 3 in midterm exam.

Problem

For this problem you may find the information on page ?? useful.

Consider the LTI system represented by the FIR lattice structure in Figure 1.

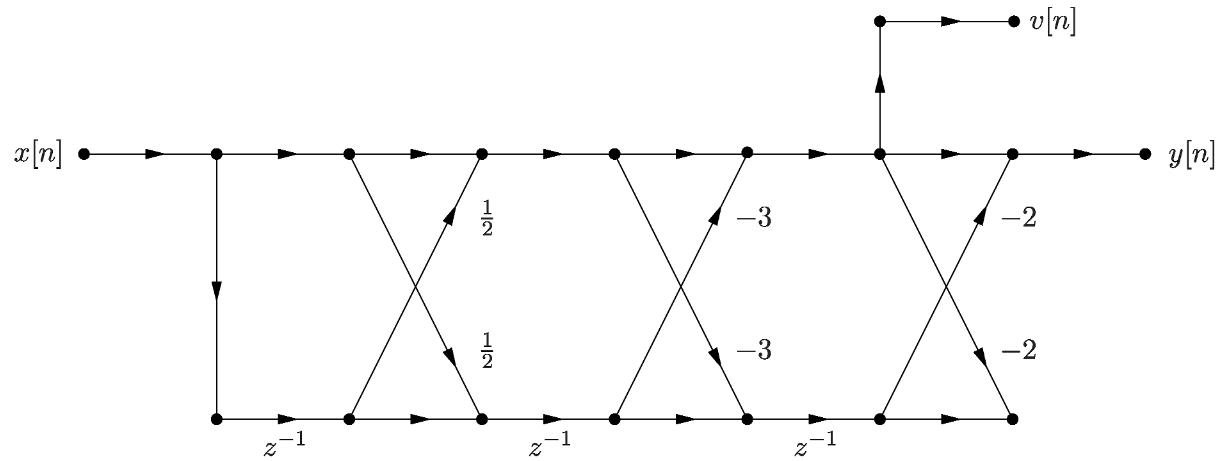
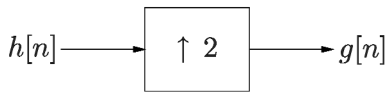


Figure 1:

(a) Determine the system function from the input $x[n]$ to the output $v[n]$ (NOT $y[n]$).

(b) Let $H(z)$ be the system function from the input $x[n]$ to the output $y[n]$, and let $g[n]$ be the result of expanding the associated impulse response $h[n]$ by 2:



The impulse response $g[n]$ defines a new system with system function $G(z)$.

We would like to implement $G(z)$ using an FIR lattice structure as defined by the figure on page **??**. Determine the k-parameters necessary for an FIR lattice implementation of $G(z)$.

Note: You should think carefully before diving into a long calculation.

Solution from Fall05 Midterm

The output $v[n]$ is taken after two stages, so we perform the lattice recursion up to order $p = 2$.

$$k_1 = -\frac{1}{2}, \quad k_2 = 3, \quad k_3 = 2$$

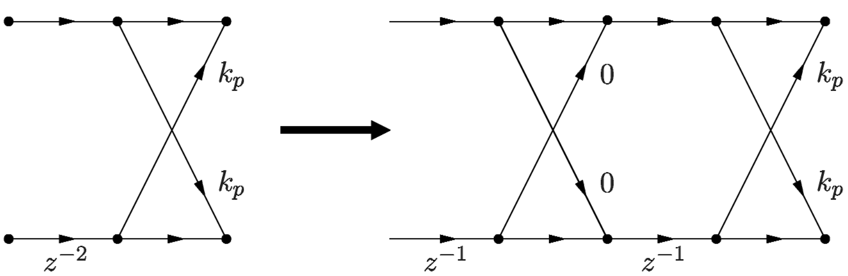
$$a_1^{(1)} = k_1 = -\frac{1}{2}$$

$$\begin{bmatrix} a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} \\ 0 \end{bmatrix} - k_2 \begin{bmatrix} a_1^{(1)} \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\frac{V(z)}{X(z)} = 1 - z^{-1} - 3z^{-2}$$

Note the change of signs in going from $a_k^{(2)}$ to the system function.

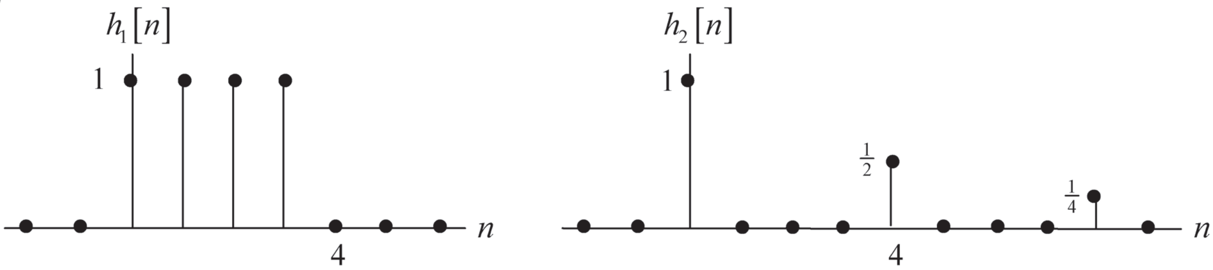
Since $g[n]$ is $h[n]$ expanded by 2, $G(z) = H(z^2)$. We replace z by z^2 in Figure 1. We then expand each of the three sections as shown below:



The resulting flowgraph for $G(z)$ is in the form of a 6th-order FIR lattice. We read off the six k-parameters as:

$$\begin{aligned} k_2 &= -\frac{1}{2} \\ k_4 &= 3 \\ k_6 &= 2 \\ k_p &= 0, \quad p = 1, 3, 5 \end{aligned}$$

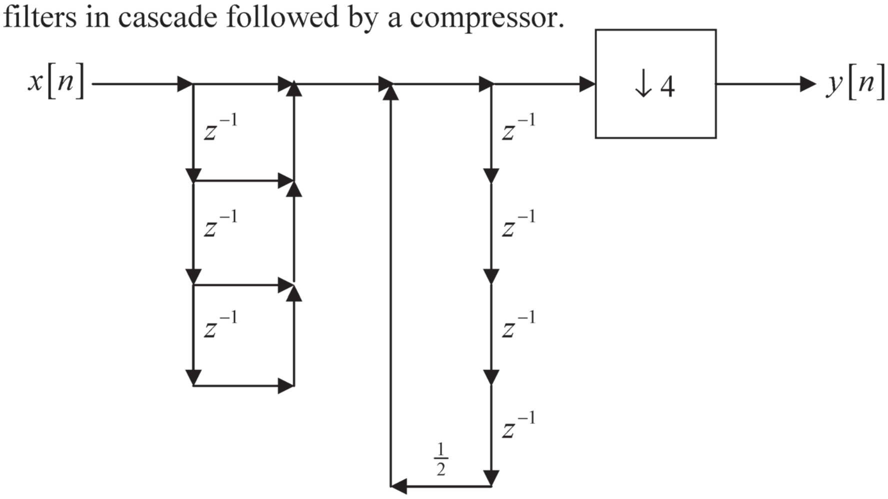
6.34. (a)



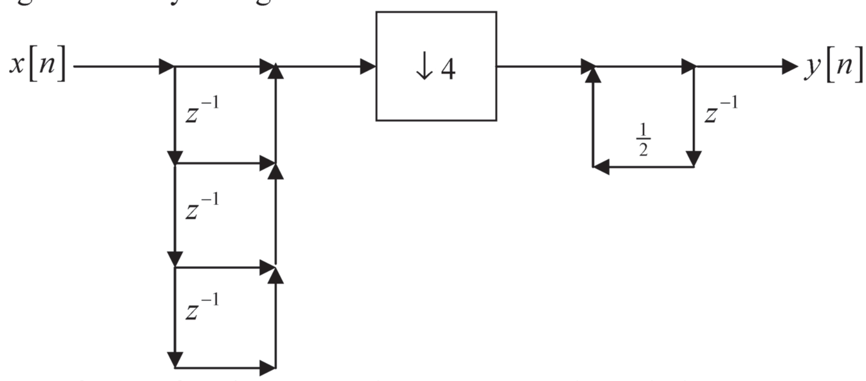
$$h_2[n] = \begin{cases} \left(\frac{1}{2}\right)^{n/4} u[n], & \text{for } n \text{ an integer multiple of 4} \\ 0, & \text{otherwise.} \end{cases}$$

The filter with impulse response $h_2[n]$ is an IIR filter.

(b) The two filters from part (a) have system functions $H_1(z) = 1 + z^{-1} + z^{-2} + z^{-3}$ and $H_2(z) = \dfrac{1}{1 - \frac{1}{2}z^{-4}}$ respectively. The flow graph below shows a direct form implementation of the two filters in cascade followed by a compressor.



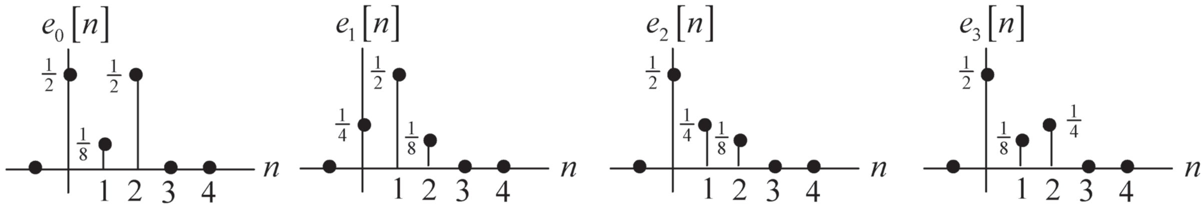
We can simplify the design (although without changing the number of coefficient multipliers) by applying the identity of Figure 4.31. The result is shown below.



(c) Only one out of every four input samples propagates through the compressor and gets multiplied by the coefficient $\frac{1}{2}$. Thus there is $\frac{1}{4}$ multiplication per input sample.

Every output sample is derived from a multiplication of the previous sample by $\frac{1}{2}$. Thus there is one multiplication per output sample.

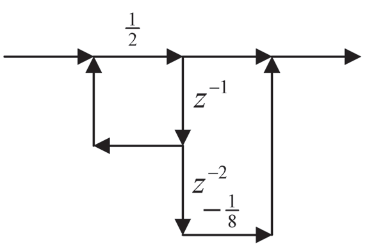
6.35. (a)



(b) First, $e_0[n]$ is a three-point system, but symmetry can be used to reduce the number of multiplies to two.

Next, $e_1[n]$ requires three multiplies per output sample.

Next $e_2[n]$ requires three multiplies per output sample. However, $e_2[n]$ can be implemented using pole-zero cancellation as shown. This requires only two multiplies per output sample.

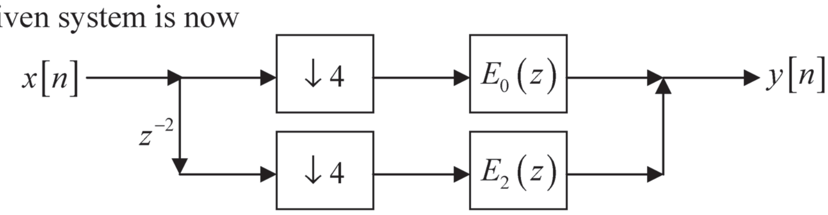


Finally, $e_3[n]$ requires three multiplies per output sample.

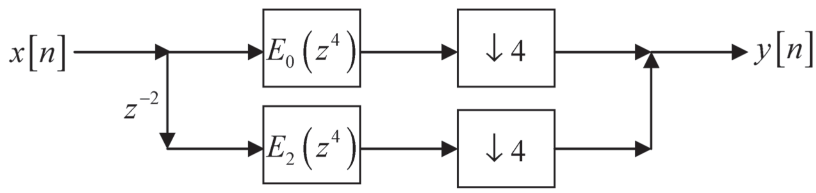
Altogether this is 10 multiplies per output sample.

The compressor reduces the required rate of multiplies relative to the input samples. We need $10/4 = 2.5$ multiplies per input sample.

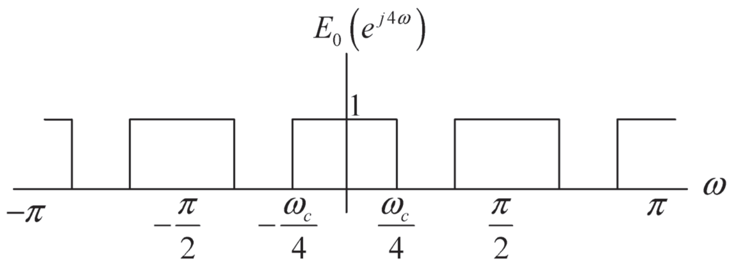
(c) The given system is now

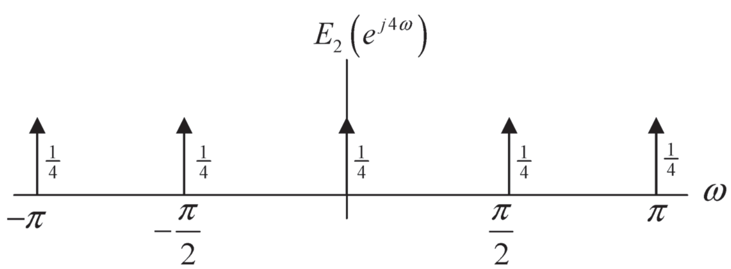


This is equivalent to



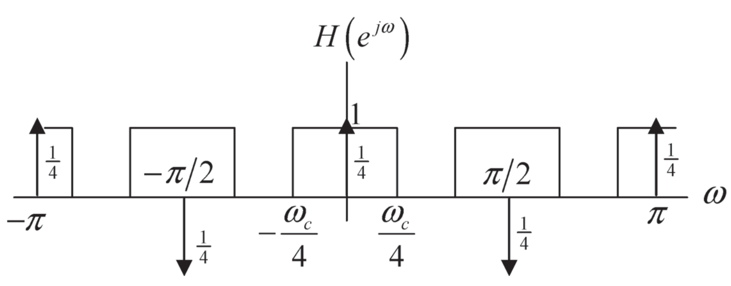
The system function is $H(z) = E_0(z^4) + z^{-2}E_2(z^4)$. This gives a frequency response of $H(e^{j\omega}) = E_0(e^{j4\omega}) + e^{-j2\omega}E_2(e^{j4\omega})$. The components of the frequency response are shown below.



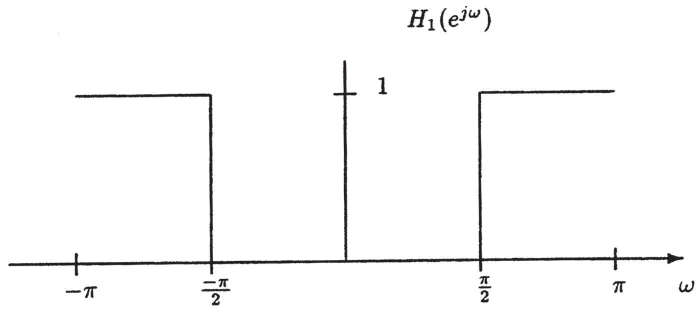


Note that the areas of the impulses scale when the frequency axis is scaled.

Including the factor of $e^{-j2\omega}$, we obtain the frequency response $H(e^{j\omega})$ shown below.

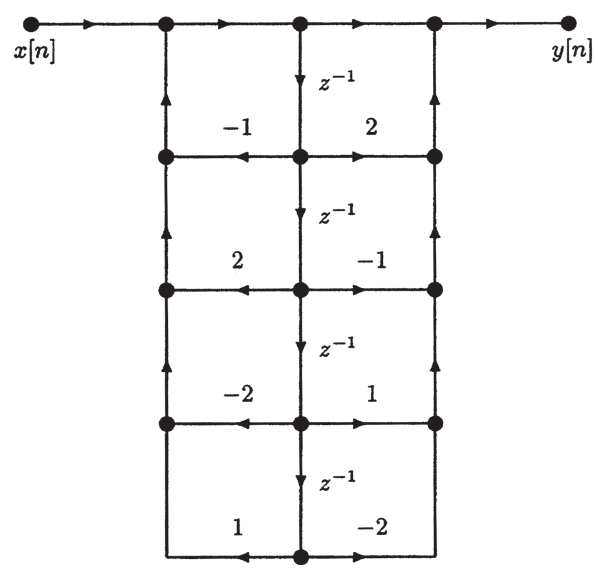


6.36. (a) $H_1(e^{j\omega}) = H(e^{j(\omega+\pi)})$.

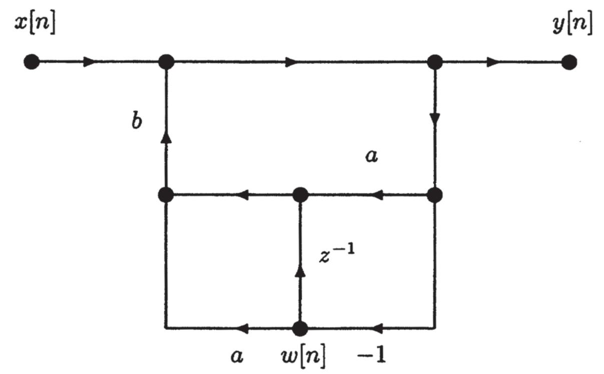


(b) For $H_1(z) = H(-z)$, replace each z^{-1} by $-z^{-1}$. Alternatively, replace each coefficient of an odd-delayed variable by its negative.

(c)



6.37.



(a)

$$
\begin{aligned}
y[n] &= x[n] + abw[n] + bw[n-1] + aby[n] \\
w[n] &= -y[n].
\end{aligned}
$$

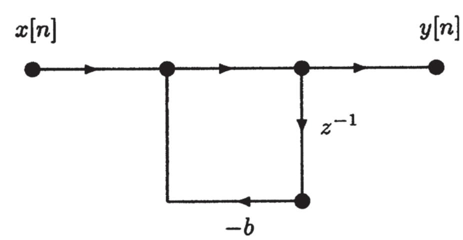
Eliminate w[n]:

$$
\begin{aligned}
y[n] &= x[n] - aby[n] - by[n-1] + aby[n] \\
y[n] &= x[n] - by[n-1]
\end{aligned}
$$

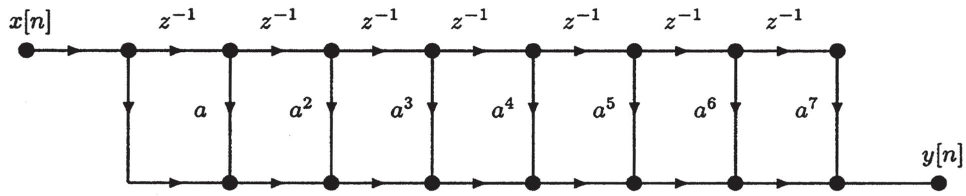
So:

$$H(z) = \frac{1}{1 + bz^{-1}}.$$

(b)



6.38. (a)



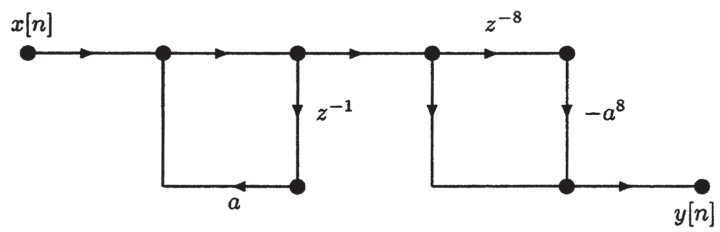
(b) From

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}$$

it follows that

$$\sum_{n=0}^{7} a^n z^{-n} = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}.$$

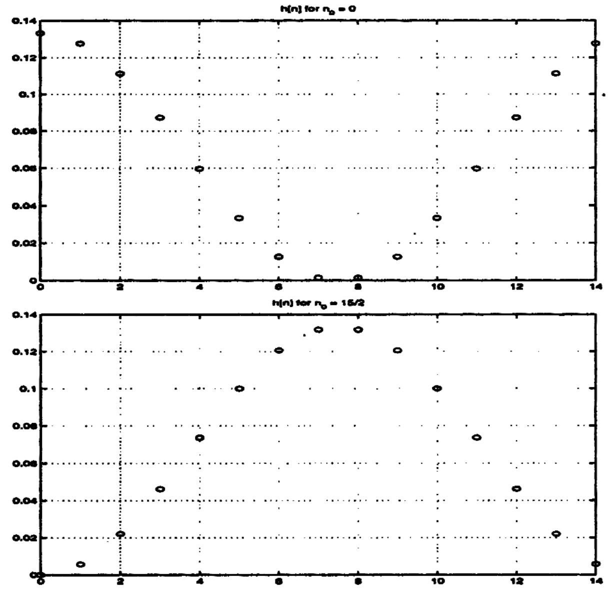
(c)



(d) (i) (c) has the most storage: 9 vs. 7.
(ii) (a) has the most arithmetic: 7 adds + 7 multiplies per sample, vs. 2 multiplies + 2 adds per sample.

6.39.

(a)



(b)

$$\begin{aligned}
H(z) &= \frac{1}{15}\sum_{n=0}^{14}\left[1+\cos\left(\frac{2\pi}{15}(n-n_0)\right)\right]z^{-n} \\
&= \frac{1}{15}\sum_{n=0}^{14}z^{-n}+\frac{1}{15}\sum_{n=0}^{14}\frac{1}{2}\left[e^{j\frac{2\pi}{15}(n-n_0)}+e^{-j\frac{2\pi}{15}(n-n_0)}\right]z^{-n} \\
&= \frac{1}{15}\frac{1-z^{-15}}{1-z^{-1}}+\frac{1}{15}\frac{1}{2}\frac{e^{-j\frac{2\pi}{15}n_0}[1-(e^{j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{j\frac{2\pi}{15}}z^{-1}} \\
&\quad +\frac{1}{15}\frac{1}{2}\frac{e^{j\frac{2\pi}{15}n_0}[1-(e^{-j\frac{2\pi}{15}}z^{-1})^{15}]}{1-e^{-j\frac{2\pi}{15}}z^{-1}} \\
&= \frac{1}{15}(1-z^{-15})\left[\frac{1}{1-z^{-1}}+\frac{\frac{1}{2}e^{-j\frac{2\pi}{15}n_0}}{1-e^{j\frac{2\pi}{15}}z^{-1}}+\right. \\
&\quad \left.\frac{\frac{1}{2}e^{j\frac{2\pi}{15}n_0}}{1-e^{-j\frac{2\pi}{15}}z^{-1}}\right].
\end{aligned}$$

(c)

$$H(e^{j\omega})=\frac{1}{15}e^{-j7\omega}\left[\frac{\sin((15\omega)/2)}{\sin(\omega/2)}-\frac{1}{2}\frac{e^{-j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega-\frac{2\pi}{15})/2)}-\frac{1}{2}\frac{e^{j\frac{\pi}{15}}\sin((15\omega)/2)}{\sin((\omega+(2\pi)/15)/2)}\right]$$

$$H(e^{j\omega})=\frac{1-e^{-j15\omega}}{15}\left[\frac{1}{1-e^{-j\omega}}+\frac{\frac{1}{2}e^{-j\frac{2\pi n_0}{15}}}{1-e^{j\frac{2\pi}{15}}e^{-j\omega}}+\frac{\frac{1}{2}e^{j\frac{2\pi n_0}{15}}}{1-e^{-j\frac{2\pi}{15}}e^{-j\omega}}\right]$$

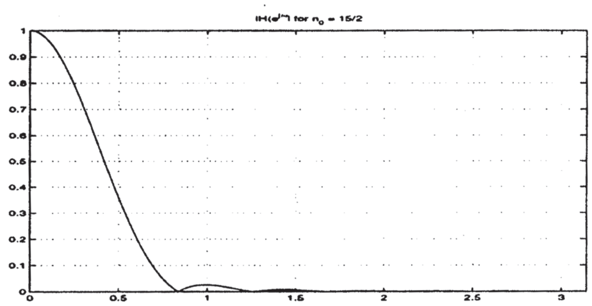
When $n_0 = 15/2$, .

$$H(e^{j\omega}) = \frac{1}{15}\left[\frac{e^{j\frac{\omega}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}}-\frac{\frac{1}{2}e^{j\frac{\omega-(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega-(2\pi/15)}{2}}-e^{-j\frac{\omega-(2\pi/15)}{2}}}-\right.$$

$$\left.\frac{\frac{1}{2}e^{j\frac{\omega+(2\pi/15)}{2}}(1-e^{-j15\omega})}{e^{j\frac{\omega+(2\pi/15)}{2}}-e^{-j\frac{\omega+(2\pi/15)}{2}}}\right]$$

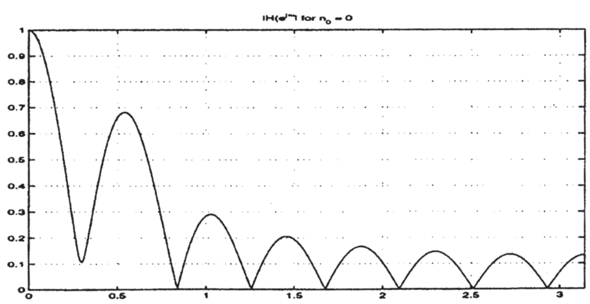
$$= \frac{1}{15}\left[\frac{e^{-j\omega 7}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\frac{\omega}{2}} - \frac{\frac{1}{2}e^{-j\omega 7}e^{-j\frac{\pi}{15}}(e^{j\omega\frac{15}{2}}-e^{-j\omega\frac{15}{2}})}{2j\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \frac{\frac{1}{2}e^{-j\omega 7}e^{j\frac{\pi}{15}}(e^{j\frac{15}{2}\omega}-e^{-j\frac{15}{2}\omega})}{2j\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$

$$= \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} - \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} - \frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



When $n_0 = 0$,

$$H(e^{j\omega}) = \frac{e^{-j\omega 7}}{15}\left[\frac{\sin(15\omega/2)}{\sin(\omega/2)} + \frac{\frac{1}{2}e^{-j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega-(2\pi/15)}{2}\right)} + \frac{\frac{1}{2}e^{j\frac{\pi}{15}}\sin(15\omega/2)}{\sin\left(\frac{\omega+(2\pi/15)}{2}\right)}\right]$$



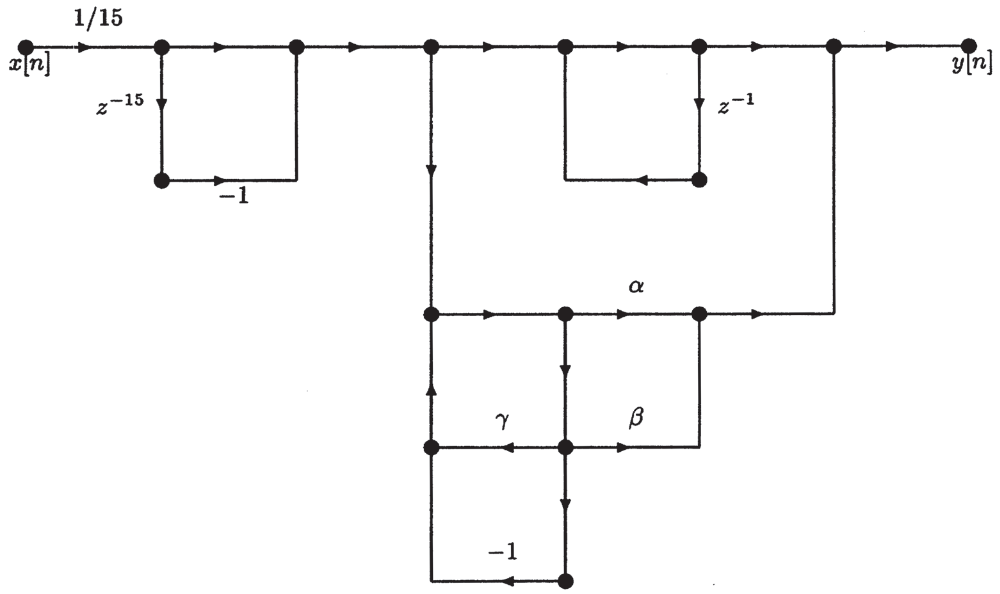
The system will have generalized linear phase if the impulse response has even symmetry (note it cannot have odd symmetry), or alternatively, if the frequency response can be expressed as:

$$H(e^{j\omega}) = e^{-jw7}A_e(e^{j\omega})$$

where $A_e(e^{j\omega})$ is a real, even, periodic function in ω. We thus conclude that the system will have generalized linear phase for $n_0 = \frac{15}{2}k$, where k is an odd integer.

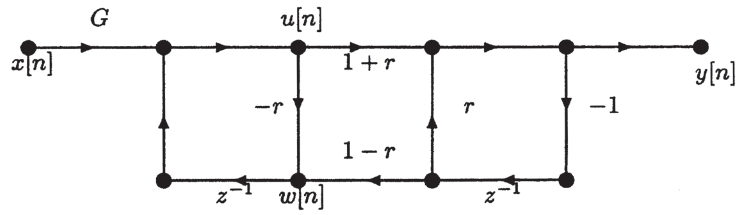
(d) Rewrite $H(z)$ as

$$H(z) = \frac{1 - z^{-15}}{15}\left[\frac{1}{1 - z^{-1}} + \frac{\cos\frac{2\pi n_0}{15} - \cos\left(\frac{2\pi}{15} + \frac{2\pi n_0}{15}\right)z^{-1}}{1 - 2\cos\frac{2\pi}{15}z^{-1} + z^{-2}}\right]$$



where $\alpha = \cos(2\pi n_0/15)$, $\beta = -\cos(2\pi(n_0 + 1)/15)$, and $\gamma = 2\cos(2\pi/15)$.

6.40. (a)



$$\begin{aligned} u[n] &= Gx[n] + w[n-1] \\ w[n] &= -ru[n] - (1-r)y[n-1] \\ y[n] &= (1+r)u[n] - ry[n-1]. \end{aligned}$$

(b)

$$\begin{aligned} U(z) &= GX(z) + z^{-1}W(z) \\ W(z) &= -ru(z) - (1-r)z^{-1}Y(z) \\ Y(z) &= (1+r)U(z) - rz^{-1}Y(z). \end{aligned}$$

Solve for $U(z)$ in terms of $X(z)$ and $Y(z)$:

$$U(z) = \frac{GX(z) - (1-r)z^{-2}Y(z)}{1 + rz^{-1}}$$

Then

$$Y(z) = (1+r)\left\{\frac{GX(z) - (1-r)z^{-2}Y(z)}{1+rz^{-1}}\right\} - rz^{-1}Y(z)$$

$$Y(z)(1 + rz^{-1}) = G(1+r)X(z) - (1-r^2)z^{-2}Y(z) - rz^{-1}Y(z) - r^2z^{-2}Y(z)$$

$$Y(z)(1 + 2rz^{-1} + z^{-2}) = G(1+r)X(z)$$

$$H_1(z) = \frac{G(1+r)}{1 + 2rz^{-1} + z^{-2}}.$$

From the quadratic formula, the poles are at $(-r + j\sqrt{1-r^2})^{-1}$ and $(-r - j\sqrt{1-r^2})^{-1}$. The magnitude of each pole is 1. The angles are

$$-\tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right) \quad \text{and} \quad \tan^{-1}\left(\frac{\sqrt{1-r^2}}{r}\right),$$

respectively.

(c) $U(z) = z^{-1}(GX(z) + W(z))$, $W(z) = -rU(z) - (1-r)Y(z)$, and $Y(z) = z^{-1}((1+r)U(z) - rY(z))$ lead to

$$H_2(z) = \frac{G(1+r)z^{-2}}{1 + 2rz^{-1} + z^{-2}} = z^{-2}H_1(z).$$

6.41. (a)

$$\begin{aligned} y_1[n] &= (1+r)x_1[n] + rx_2[n] \\ y_2[n] &= -rx_1[n] + (1-r)x_2[n]. \end{aligned}$$

(b)

$$\begin{aligned} y_1[n] &= (1+a)x_1[n] + dx_2[n] \quad (a = r = d) \\ y_2[n] &= (1+cd)x_2[n] + abx_1[n] \quad (c = d = -1). \end{aligned}$$

(c)

$$\begin{aligned} y_1[n] &= (1+e)x_1[n] + ex_2[n] \quad (e = r) \\ y_2[n] &= efx_1[n] + (1+ef)x_2[n] \quad (f = -1). \end{aligned}$$

(d) B and C preferred over A:

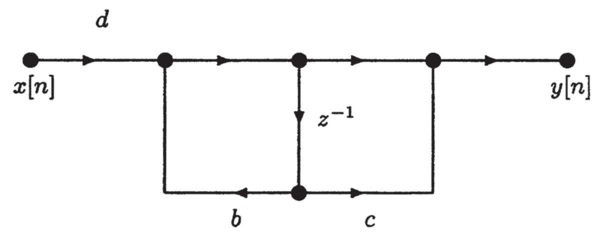
(i) coefficient quantization. If r is small, $1+r$ may not be precisely representable even in floating point. Also, network A has 4 multipliers that must be quantized, while B and C have only 1.

(ii) computational complexity. Networks B and C require fewer multiplications per output sample.

6.42.

$$H(z) = \frac{z^{-1} - 0.54}{1 - 0.54z^{-1}}.$$

(a)

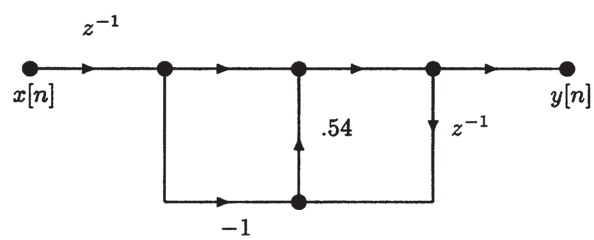


$$H(z) = \frac{cdz^{-1} + d}{1 - bz^{-1}}$$

so set $b = 0.54$, $c = -1.852$, and $d = -0.54$.

(b) With quantized coefficients $\hat{b}$, $\hat{c}$, and $\hat{d}$, $\hat{c}\hat{d} \neq 1$ and $\hat{d} \neq -\hat{b}$ in general, so the resulting system would not be allpass.

(c)

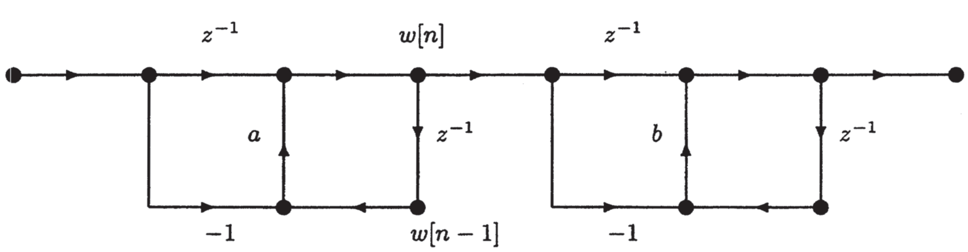


(d) Yes, since there is only one "0.54" to quantize.

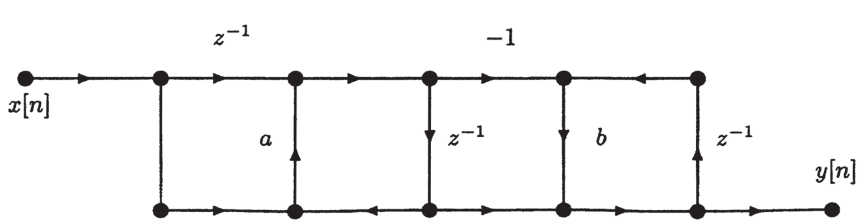
(e)

$$H(z) = \left(\frac{z^{-1} - a}{1 - az^{-1}}\right)\left(\frac{z^{-1} - b}{1 - bz^{-1}}\right)$$

Cascading two sections like (c) gives

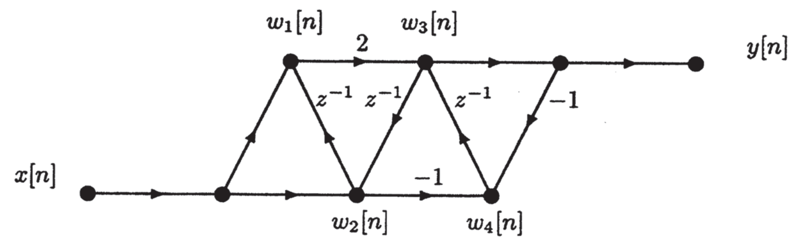


The first delay in the second section has output $w[n-1]$ so we can combine with the second delay of the first section.



(f) Yes, same reason as part (d).

6.43. (a) We have:



First, we find the system function, we have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] + w_4[n-1] \\
(4)\ \ y[n] &= w_3[n] \\
(5)\ w_4[n] &= -y[n] - w_2[n]
\end{aligned}$$

Taking the Z-transform of the above equations and combining terms, we get:

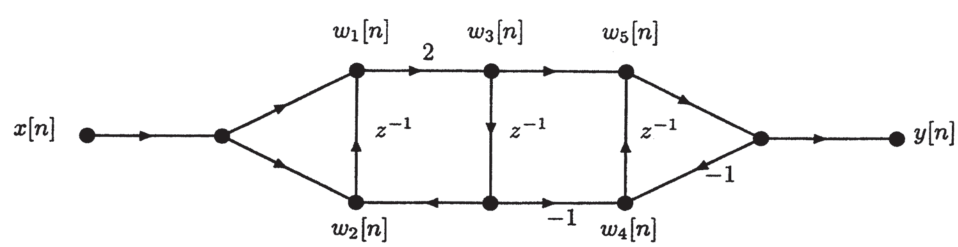
$$(1 - z^{-1})Y(z) + z^{-1}Y(z) = (2 + z^{-1})X(z).$$

The system function is thus given by:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + z^{-1}}{1 + z^{-1} - z^{-2}}$$

Since the system function is second order (highest order term is z^{-2}), we should be able to implement this system using only 2 delays, this can be done with a direct form II implementation. Therefore, the minimum number of delays required to implement an equivalent system is 2.

(b) Now we have:



Let's find the transfer function, we have:

$$\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= x[n] + w_3[n-1] \\
(3)\ w_3[n] &= 2w_1[n] \\
(4)\ w_4[n] &= -w_3[n-1] - y[n] \\
(5)\ w_5[n] &= w_3[n] + w_4[n-1] \\
(6)\ \ y[n] &= w_5[n]
\end{aligned}$$

Taking the Z-transform of the above equations and combining terms, we get:

$$(1 + z^{-1})Y(z) = \frac{(1 - z^{-2})(2 + 2z^{-1})}{1 - 2z^{-2}}X(z).$$

The system function is thus given by:

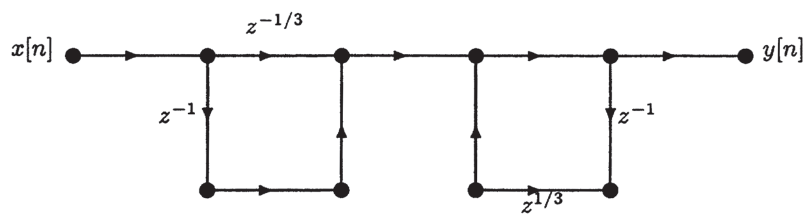
$$H(z) = \frac{Y(z)}{X(z)} = \frac{2(1 + z^{-1})(1 - z^{-1})}{1 - 2z^{-2}}.$$

Since the transfer function is not the same as the one in part a, we conclude that system B does not represent the same input-output relationship as system A. This should not be surprising since in system B we added two unidirectional wires and therefore changed the input-output relationship.

6.44.

$$H(z) = \frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}}.$$

(a) Direct form I:



From the graph above, it is clear that 2 delays and 2 multipliers are needed.

(b)

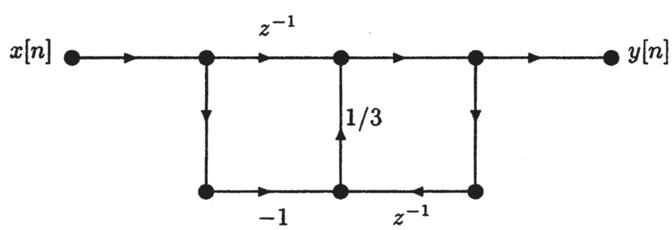
$$(1 - \frac{1}{3}z^{-1})Y(z) = (-\frac{1}{3} + z^{-1})X(z)$$

Inverse Z−transforming, we get:

$$y[n] - \frac{1}{3}y[n-1] = -\frac{1}{3}x[n] + x[n-1]$$

$$y[n] = \frac{1}{3}(y[n-1] - x[n]) + x[n-1]$$

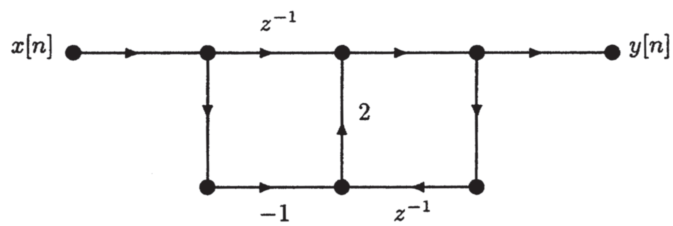
Which can be implemented with the following flow diagram:



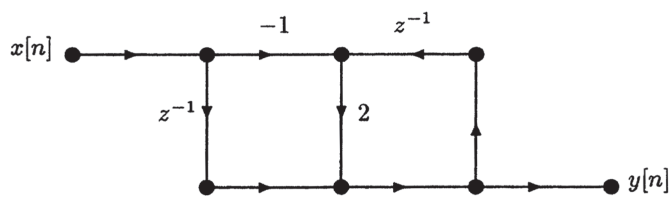
(c)

$$H(z) = (\frac{z^{-1} - \frac{1}{3}}{1 - \frac{1}{3}z^{-1}})(\frac{z^{-1} - 2}{1 - 2z^{-1}}).$$

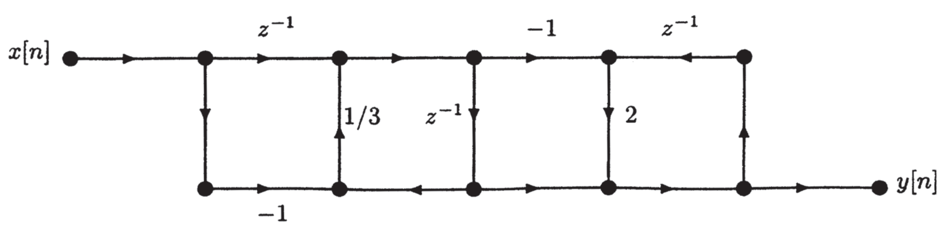
This can be implemented as the cascade of the flow graph in part (b) with the following flow graph:



However the above flow graph can be redrawn as:



Now cascading the above flow graph with the one from part (b) and grouping the delay element we get the following system with two multipliers and three delays:



6.45. (a) From Eq. (6.120),

$$x_{\max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h[m]|}.$$

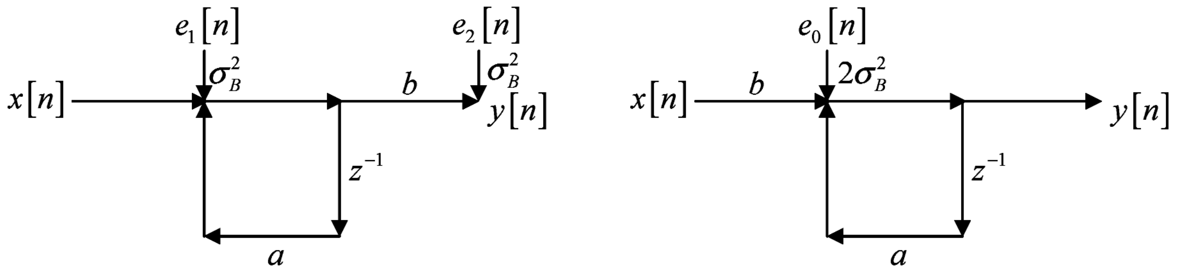
For the given systems, $h[n] = ba^n u[n]$. Then we have

$$x_{\max} < \frac{1}{\sum_{m=0}^{\infty} |ba^m|}$$

$$= \frac{1}{|b| \sum_{m=0}^{\infty} |a|^m}$$

$$= \frac{1-|a|}{|b|}.$$

(b)



(c) For System 1 we have $H_1(z) = \dfrac{b}{1-az^{-1}}$, so $H_1(e^{j\omega}) = \dfrac{b}{1-ae^{-j\omega}}$. Then

$$\Phi_{f_1 f_1}(e^{j\omega}) = \frac{\sigma_B^2 |b|^2}{|1-ae^{-j\omega}|^2} + \sigma_B^2$$

$$= \frac{\sigma_B^2 b^2}{1+a^2-2a\cos\omega} + \sigma_B^2.$$

For System 2 we have $H_0(z) = \dfrac{1}{1-az^{-1}}$, so $H_0(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$. Then

$$\Phi_{f_2 f_2}(e^{j\omega}) = \frac{2\sigma_B^2}{|1-ae^{-j\omega}|^2}$$

$$= \frac{2\sigma_B^2}{1+a^2-2a\cos\omega}.$$

(d) For System 1,

$$\begin{aligned}\sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_1 f_1}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_B^2 |b|^2}{\left|1-ae^{-j\omega}\right|^2} d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_B^2 d\omega \\ &= \frac{\sigma_B^2 b^2}{1-a^2} + \sigma_B^2,\end{aligned}$$

where the method of Example 6.11 was used to evaluate the integral.

For System 2,

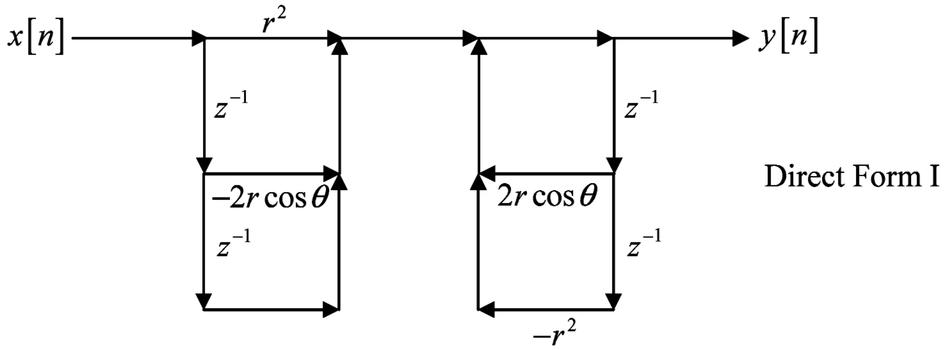
$$\begin{aligned}\sigma_{f_2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_2 f_2}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2\sigma_B^2}{\left|1-ae^{-j\omega}\right|^2} d\omega \\ &= \frac{2\sigma_B^2}{1-a^2}.\end{aligned}$$

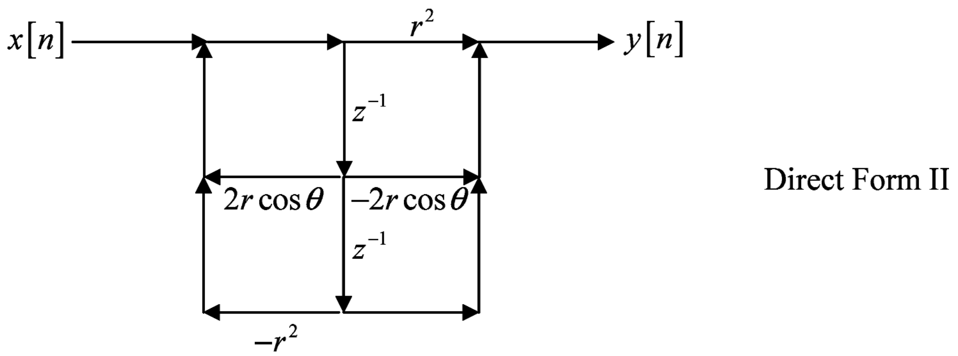
6.46. (a) For the allpass system

$$H(z) = \frac{(z^{-1}-a^*)(z^{-1}-a)}{(1-az^{-1})(1-a^*z^{-1})}$$

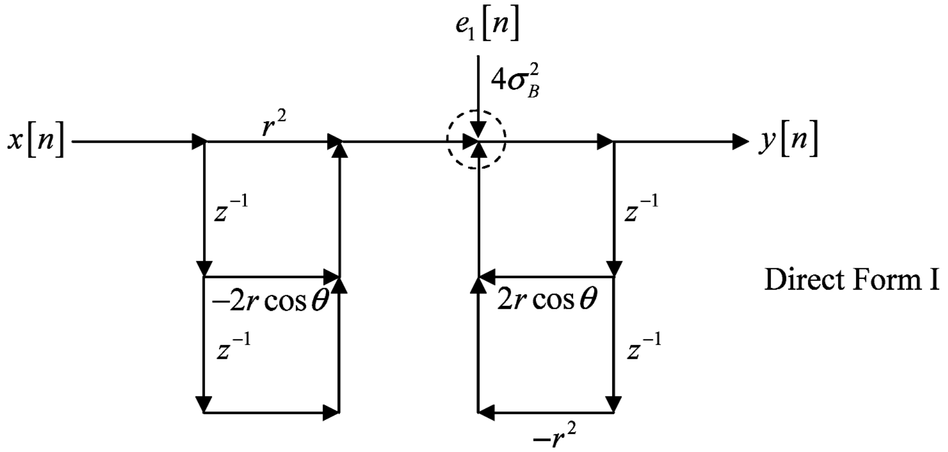
$$= \frac{z^{-2}-(a+a^*)z^{-1}+|a|^2}{1-(a+a^*)z^{-1}+|a|^2 z^{-2}}$$

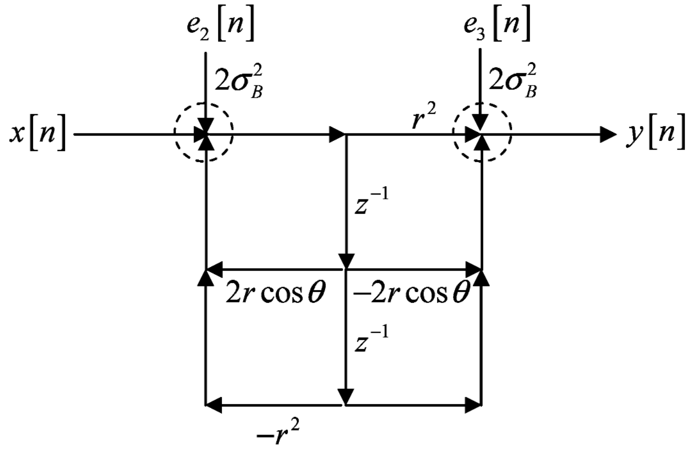
$$= \frac{z^{-2}-2r\cos\theta z^{-1}+r^2}{1-2r\cos\theta z^{-1}+r^2 z^{-2}}.$$





(b), (c)





Direct Form II

(d) In the direct form II realization, the noise power spectrum at the output is given by

$$\Phi_{f_2 f_2}\left(e^{j\omega}\right) = 2\sigma_B^2 \left|H\left(e^{j\omega}\right)\right|^2 + 2\sigma_B^2.$$

Since the system is allpass, $\left|H\left(e^{j\omega}\right)\right| = 1$. Consequently, the parameters of the system have no effect on the output noise power.

In the direct form I realization, the noise power spectrum at the output is given by

$$\Phi_{f_1 f_1}\left(e^{j\omega}\right) = \frac{4\sigma_B^2}{\left|\left(1-ae^{-j\omega}\right)\left(1-a^* e^{-j\omega}\right)\right|^2}.$$

This spectrum will show a peak near $\omega = \theta$ caused by the system poles. This spectral peak will become more prominent as $r \to 1$, enhancing the output noise power.

(e) For the direct form II realization,

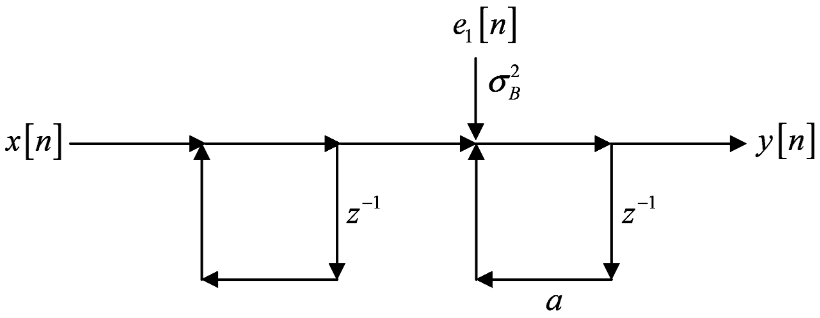
$$\begin{aligned}\sigma_{f_2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_2 f_2}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sigma_B^2 d\omega \\ &= 4\sigma_B^2.\end{aligned}$$

For the direct form I realization,

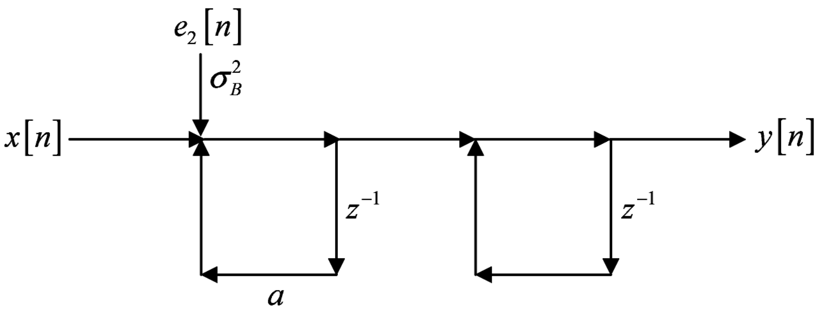
$$\begin{aligned}\sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_1 f_1}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{4\sigma_B^2}{\left|\left(1-ae^{-j\omega}\right)\left(1-a^* e^{-j\omega}\right)\right|^2} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{4\sigma_B^2}{\left|1-2r\cos\theta e^{-j\omega}+r^2 e^{-j2\omega}\right|^2} d\omega \\ &= 4\sigma_B^2\left(\frac{1+r^2}{1-r^2}\right)\frac{1}{1-2r\cos(2\theta)+r^4},\end{aligned}$$

where the technique of Example 6.12 was used to evaluate the integral.

6.47. (a) Flow graph #1:



Flow graph #2:



(b) The power density spectrum of the output noise for flow graph #1 is

$$\Phi_{f_1 f_1}\left(e^{j\omega}\right) = \sigma_B^2 \left|\frac{1}{1-ae^{j\omega}}\right|^2 = \frac{\sigma_B^2}{\left|1-ae^{j\omega}\right|^2}.$$

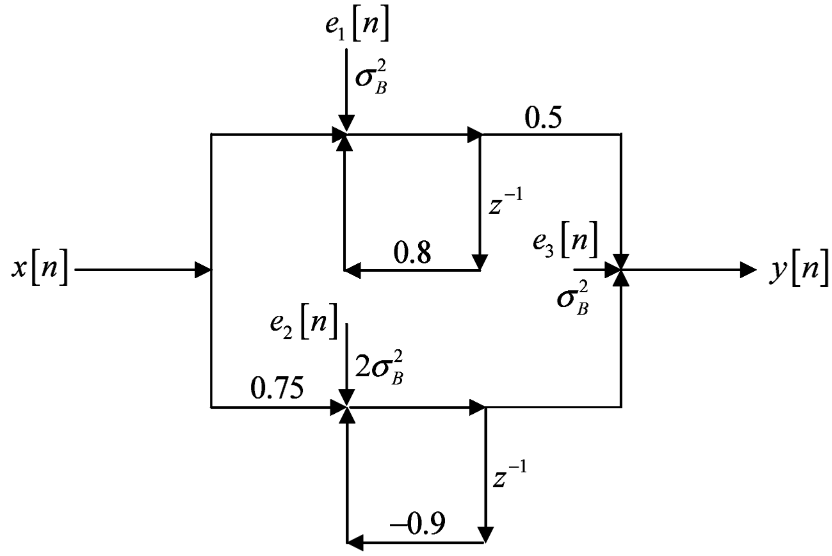
The total output noise power is

$$\begin{aligned} \sigma_{f_1}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{f_1 f_1}\left(e^{j\omega}\right) d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\sigma_B^2}{\left|1-ae^{j\omega}\right|^2} d\omega \\ &= \frac{\sigma_B^2}{1-a^2}, \end{aligned}$$

where the method of Example 6.11 was used to evaluate the integral.

(c) The noise in flow graph #2 is filtered through a cascade of two stages. The second stage has a pole on the unit circle at $\omega = 0$. This pole will cause a peak in the noise power density spectrum. Consequently, flow graph #2 would be expected to produce the largest total noise power at the output. (In fact, the noise power at the output of flow graph #2 will be infinite.)

6.48. (a)



(b) Let $H_1(z)$ be the system function of the upper branch, i.e. $H_1(z)=\dfrac{0.5}{1-0.8z^{-1}}$. Let $H_2(z)$ be the system function of the lower branch to the right of the noise source, i.e. $H_2(z)=\dfrac{1}{1+0.9z^{-1}}$. Then the power density spectrum of the output noise is given by

$$\begin{aligned}\Phi_{ff}(e^{j\omega}) &= \sigma_B^2\left|H_1(e^{j\omega})\right|^2 + 2\sigma_B^2\left|H_2(e^{j\omega})\right|^2 + \sigma_B^2 \\ &= \sigma_B^2\frac{(0.5)^2}{\left|1-0.8e^{-j\omega}\right|^2} + 2\sigma_B^2\frac{1}{\left|1+0.9e^{-j\omega}\right|^2} + \sigma_B^2 \\ &= \frac{0.25\sigma_B^2}{1.64-1.6\cos\omega} + \frac{2\sigma_B^2}{1.81+1.8\cos\omega} + \sigma_B^2.\end{aligned}$$

(c) The total noise component of the output is given by

$$\begin{aligned}\sigma_f^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{ff}(e^{j\omega})d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{0.25\sigma_B^2}{\left|1-0.8e^{-j\omega}\right|^2}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2\sigma_B^2}{\left|1+0.9e^{-j\omega}\right|^2}d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_B^2 d\omega \\ &= \frac{0.25\sigma_B^2}{1-(0.8)^2} + \frac{2\sigma_B^2}{1-(-0.9)^2} + \sigma_B^2 \\ &= 12.2\sigma_B^2,\end{aligned}$$

where the method of Example 6.11 was used to evaluate the integrals.

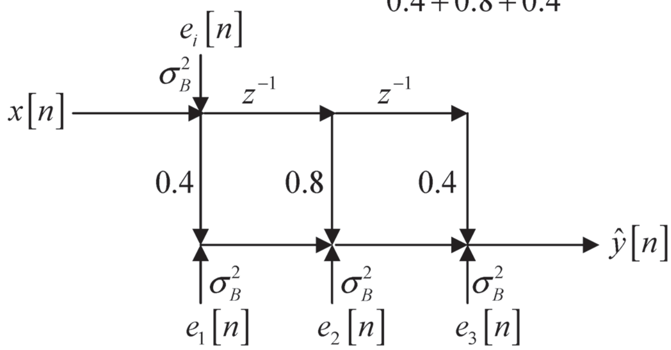
6.49. (a) From Eq. (6.120),

$$x_{\max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h[m]|}.$$

Substituting the coefficients of the given impulse response gives

$$x_{\max} < \frac{1}{0.4+0.8+0.4} = 0.625.$$

(b)



Since quantization is to $(B+1)=16$ bits, we have $\sigma_B^2 = \dfrac{2^{-2\times15}}{12} = 77.6\times10^{-12}$.

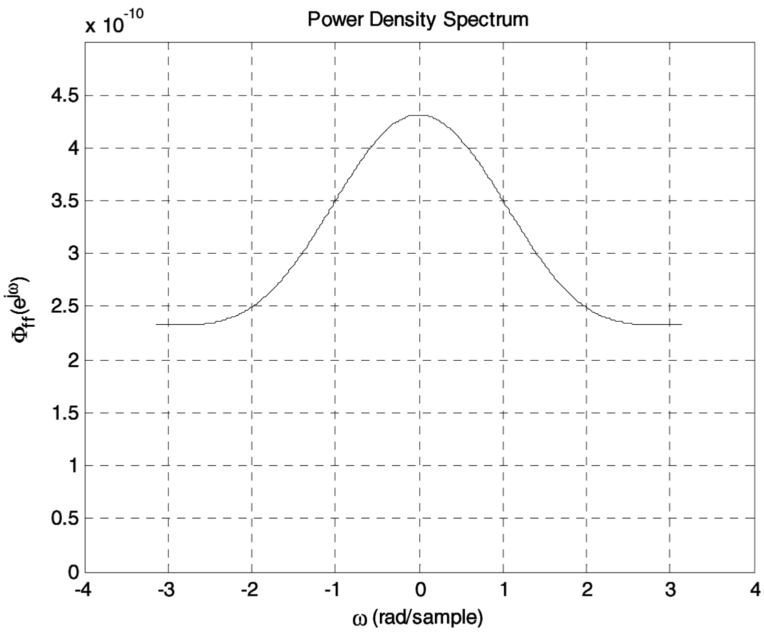
(c) Let $H(e^{j\omega})$ be the frequency response of the digital filter. The total noise power at the output is given by

$$\begin{aligned}
\sigma_f^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi_{ff}\left(e^{j\omega}\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma_B^2\left|H\left(e^{-j\omega}\right)\right|^2 d\omega + \frac{1}{2\pi}\int_{-\pi}^{\pi}3\sigma_B^2 d\omega \\
&= \sigma_B^2\sum_{n=0}^{2}h^2[n] + 3\sigma_B^2 \\
&= \left[(0.4)^2+(0.8)^2+(0.4)^2+3\right]\sigma_B^2 \\
&= 307\times10^{-12}.
\end{aligned}$$

(d) From the given impulse response, $H(z) = 0.4+0.8z^{-1}+0.4z^{-2}$. Then

$$\begin{aligned}
\Phi_{ff}\left(e^{j\omega}\right) &= \sigma_B^2\left|H\left(e^{j\omega}\right)\right|^2 + 3\sigma_B^2 \\
&= \sigma_B^2\left|0.4+0.8e^{-j\omega}+0.4e^{-j2\omega}\right|^2 + 3\sigma_B^2 \\
&= 0.64(1+\cos\omega)^2\sigma_B^2 + 3\sigma_B^2 \\
&= (3.96+1.28\cos\omega+0.32\cos2\omega)\sigma_B^2 \\
&= 307\times10^{-12} + 99.3\times10^{-12}\cos\omega + 24.8\times10^{-12}\cos2\omega.
\end{aligned}$$

This power density spectrum is plotted below.



6.50. Problem from Exam 1 Spring2001 Appears in: Fall02 PS4.

Problem

This is a problem from Exam 1 from Spring 2001. In this problem, we consider the implementation of a causal filter with system function

$$H(z) = \frac{1}{(1 - .63z^{-1})(1 - .83z^{-1})} = \frac{1}{1 - 1.46z^{-1} + 0.5229z^{-2}}$$

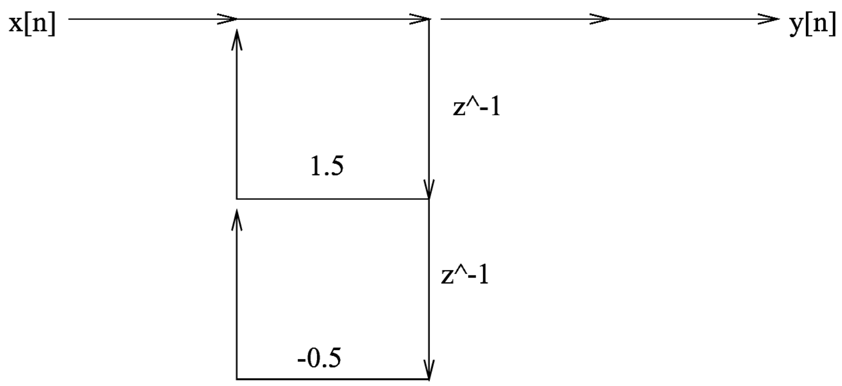
This system is to be implemented with $(B+1)$-bit 2's-complement rounding arithmetic with products rounded before additions are performed. The input to the system is a zero-mean, white, wide-sense stationary random process, with values uniformly distributed between $-S$ and S, where S is a parameter.

1. Draw the direct form flow graph implementation for the filter, with all coefficient multipliers rounded to the nearest tenth.
2. Draw a flow graph implementation of this system as a cascade of two first-order systems, with all coefficient multipliers rounded to the nearest tenth.
3. Only one of the implementations from parts (1) and (2) above is usable. Which one? Explain.
4. To prevent overflow at the output node, we must carefully choose the parameter S. For the implementation you selected in part (3), determine a value for S which guarantees the output will stay between -1 and 1. (Ignore any potential overflow at nodes other than the output).
5. Redraw the flow graph you selected in part (3), this time including linearized noise models representing quantization roundoff error.
6. Whether you chose the direct form or cascade implementation for part (3), there is still at least one more design alternative:
   (a) If you chose the direct form, you could also use a transposed direct form.
   (b) If you chose the cascade form, you could implement the smaller pole first or the larger pole first.

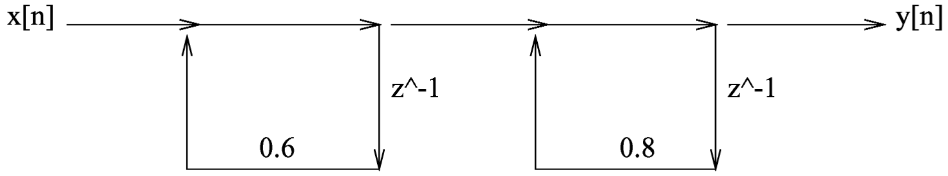
   For the system you chose in part (3), which alternative (if any) has lower output quantization noise power? Note you do not need to explicitly calculate the total output quantization noise power, but you must justify your answer with some analysis.

Solution from Fall02 PS4

(1) Direct Form Implementation:



(2) Cascade Implementation:



(3) Only the cascade implementation is usable. The quantized direct form implementation is not stable:

$$H(z) = \frac{1}{1 - 1.5z^{-1} + .5z^{-2}} = \frac{1}{(1 - z^{-1})(1 - .5z^{-1})}$$

Pole at 1 on the unit circle, system is not stable b/c ROC does not include pole.

(4)

$$S < \frac{1}{\sum_{n=0}^{\infty} |h[n]|}$$

Solving by partial fractions we get:

$$h[n] = 1.923((.6)^n u[n] + (.8)^n u[n])$$

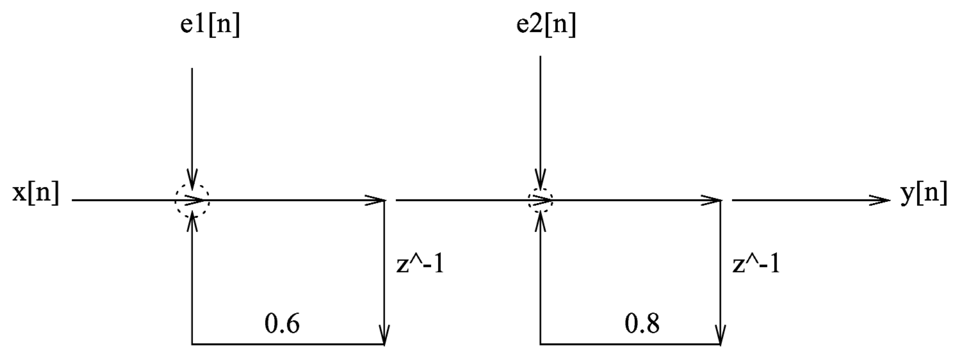
$$\sum_{n=0}^{\infty} |h[n]| = 1.923(\sum_{n=0}^{\infty}(.6)^n + \sum_{n=0}^{\infty}(.8)^n u[n])$$

Summation of an infinite geometric series:

$$1.923\left(\frac{1}{1-.6}+\frac{1}{1-.8}\right)=14.4225$$

$$S<\frac{1}{14.4225}=0.69$$

(5) Quantize after each multiply, add linear error sources there:

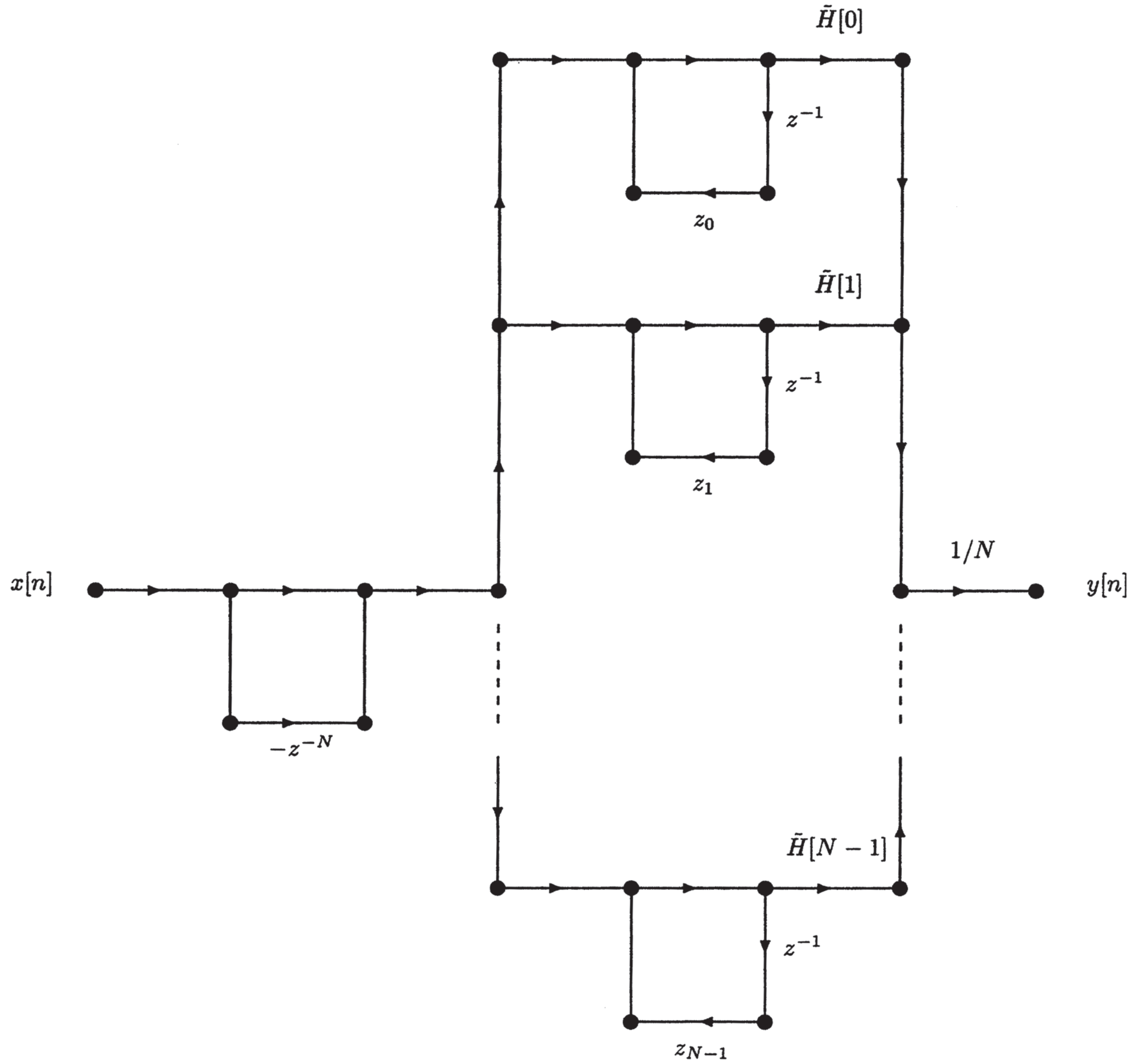


(6) We should put the large pole first in the cascade. The larger pole, at z = .8 is closer to the unit circle and thus amplifies the noise variance more at the output. Thus by putting it second the second noise source, gets amplified by a lesser amount by only going through the second pole. The first noise source will always go through the entire system, so we do not have control over that.

Solution from Spring01 exam

N/A

6.51. (a)



(b) Note that the z_k's are the zeros of $(1 - z^{-N})$. Then write $H(z)$ over a common denominator:

$$\begin{aligned} H(z) &= \frac{\prod_{\ell=0}^{N-1}(1 - z_\ell z^{-1}) \sum_{k=0}^{N-1} \frac{\tilde{H}[k]}{N} \prod_{\substack{i=0 \\ i\neq k}}^{N-1}(1 - z_i z^{-1})}{\prod_{k=0}^{N-1}(1 - z_k z^{-1})} \\ &= \sum_{k=0}^{N-1} \frac{\tilde{H}[k]}{N} \prod_{\substack{i=0 \\ i\neq k}}^{N-1} (1 - z_i z^{-1}). \end{aligned}$$

Therefore, $H(z)$ is the sum of polynomials in z^{-1} with degree $\leq N-1$. Hence, the system impulse response has length $\leq N$.

(c)

$$\mathcal{Z}^{-1}\left[(1 - z^{-N})\frac{\tilde{H}[k]/N}{1 - z_k z^{-1}}\right] = \frac{\tilde{H}[k]}{N}\mathcal{Z}^{-1}[1 - z^{-N}] * \mathcal{Z}^{-1}\left[\frac{1}{1 - z_k z^{-1}}\right]$$

$$\begin{aligned}
&= \frac{\tilde{H}[k]}{N}(\delta[n]-\delta[n-N]) * (z_k^n u[n]) \\
&= \frac{\tilde{H}[k]}{N}[z_k^n u[n] - z_k^{n-N}u[n-N]] \\
&= \frac{\tilde{H}[k]}{N} z_k^n \{u[n]-u[n-N]\}.
\end{aligned}$$

So

$$h[n] = \left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{H}[k]e^{j\frac{2\pi}{N}kn}\right)(u[n]-u[n-N]).$$

(d) Note that, since $(1-z_m^{-N})=0$,

$$\begin{aligned}
H(z_m) &= \left.\frac{(1-z^{-N})\tilde{H}[m]/N}{1-z_m z^{-1}}\right|_{z=z_m} \\
&= \frac{\frac{d}{dz}\{(1-z^{-N})\tilde{H}[m]/N\}|_{z=z_m}}{\frac{d}{dz}\{1-z_m z^{-1}\}|_{z=z_m}} \\
&= \frac{Nz_m^{-N-1}\tilde{H}[m]/N}{z_m z_m^{-2}} \\
&= \tilde{H}[m]z_m^{-N} \\
&= \tilde{H}[m].
\end{aligned}$$

(e) If $h[n]$ is real, $|H(e^{j\omega})| = |H(e^{j(2\pi-\omega)})|$, and $\angle H(e^{j\omega}) = -\angle H(e^{j(2\pi-\omega)})$. $H(e^{j2\pi k/N}) = \tilde{H}[k] = |\tilde{H}[k]|e^{j\tilde{\theta}[k]}$, so $|\tilde{H}[k]| = |\tilde{H}[N-k]|$ and $\tilde{\theta}[k] = -\tilde{\theta}[N-k]$, $k=0,1,\ldots,N-1$.

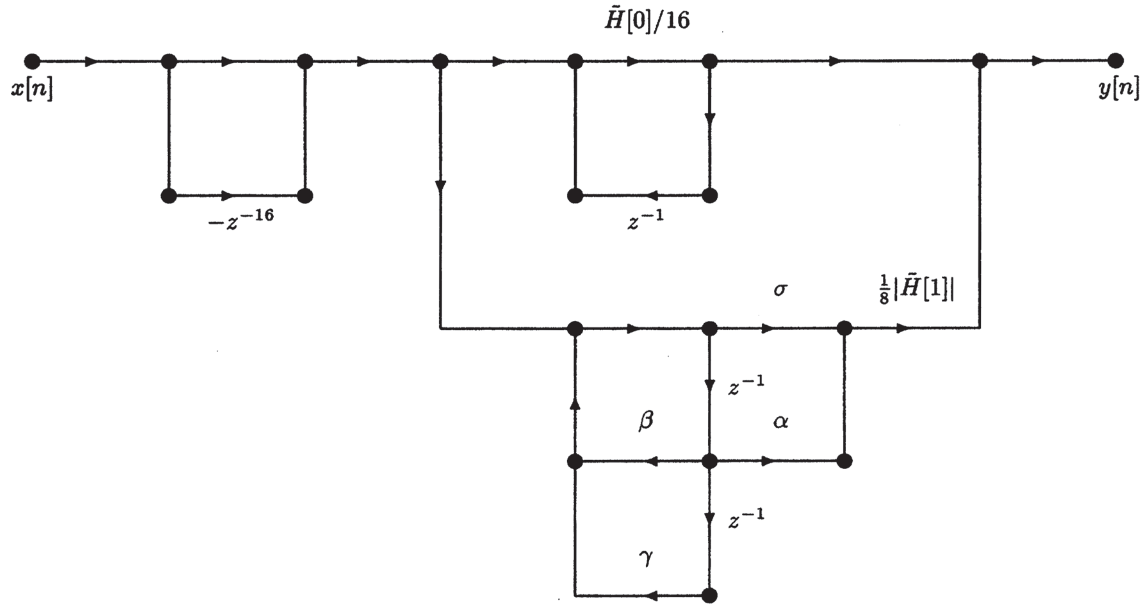
$$\begin{aligned}
H(z) &= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \frac{\tilde{H}[N/2]/N}{1-z_{N/2}z^{-1}} + \sum_{\ell=\frac{N}{2}+1}^{N-1}\frac{\tilde{H}[\ell]/N}{1-z_\ell z^{-1}}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \sum_{p=1}^{\frac{N}{2}-1}\frac{\tilde{H}[N-p]/N}{1-z_{N-p}z^{-1}}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \sum_{k=1}^{\frac{N}{2}-1}\left(\frac{\tilde{H}[k]/N}{1-z_k z^{-1}} + \frac{\tilde{H}[N-k]/N}{1-z_{-k}z^{-1}}\right)\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \right. \\
&\qquad \left. \frac{1}{N}\sum_{k=1}^{\frac{N}{2}-1}\frac{\tilde{H}[k](1-z_{-k}z^{-1}) + \tilde{H}[N-k](1-z_k z^{-1})}{(1-z_k z^{-1})(1-z_{-k}z^{-1})}\right] \\
&= (1-z^{-N})\left[\frac{\tilde{H}[0]/N}{1-z^{-1}} + \frac{\tilde{H}[N/2]/N}{1+z^{-1}} + \right. \\
&\qquad \left. \sum_{k=1}^{\frac{N}{2}-1}\frac{2|\tilde{H}[k]}{N}\cdot\frac{\cos(\tilde{\theta}[k]) - z^{-1}\cos(\tilde{\theta}[k]-2\pi k/N)}{1-2\cos\frac{2\pi k}{N}z^{-1}+z^{-2}}\right]
\end{aligned}$$

And since $\tilde{H}[0] = H(1)$, $\tilde{H}[N/2] = H(-1)$,

$$H(z) = (1-z^{-N})\left[\frac{H(1)/N}{1-z^{-1}} + \frac{H(-1)/N}{1+z^{-1}} + \right.$$

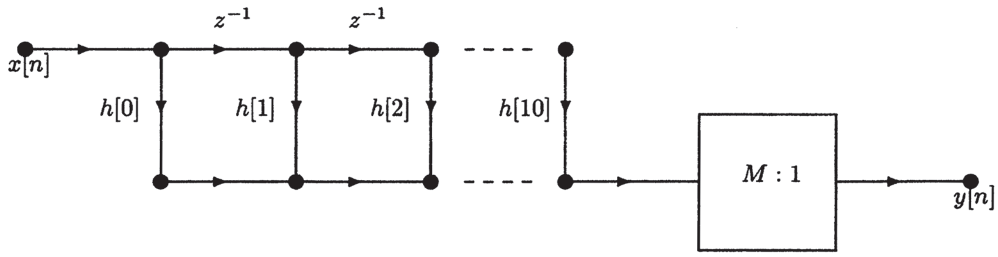
$$\left.\sum_{k=1}^{\frac{N}{2}-1} \frac{2|H(e^{j\frac{2\pi}{N}k}|}{N} \cdot \frac{\cos[\theta(2\pi k/N)] - z^{-1}\cos[\theta(2\pi k/N) - 2\pi k/N]}{1 - 2\cos(2\pi k/N)z^{-1} + z^{-2}}\right]$$

If $\tilde{H}[14] = 0$,then $\tilde{H}[16-14] = \tilde{H}[2] = 0$ also.



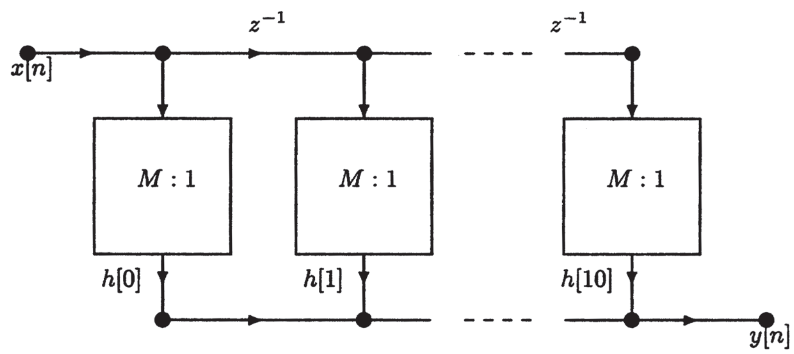
where $\sigma = \cos(\tilde{\theta}[1])$, $\alpha = -\cos(\tilde{\theta}[1] - (2\pi/16))$, $\beta = 2\cos(2\pi/16)$, and $\gamma = -1$.

6.52. (a)



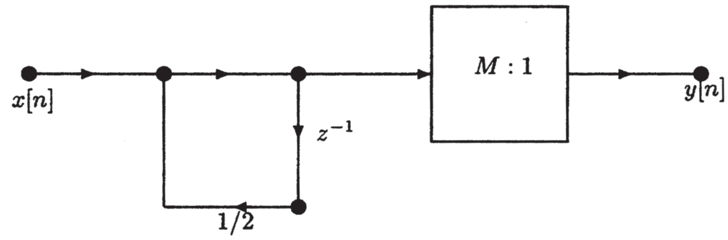
$M(N+1)$ multiplies per output sample; MN adds per output sample.

(b)



$N+1$ multiplies per output sample; N adds per output sample. The number of computations has been reduced by a factor of M in both adds and multiplies.

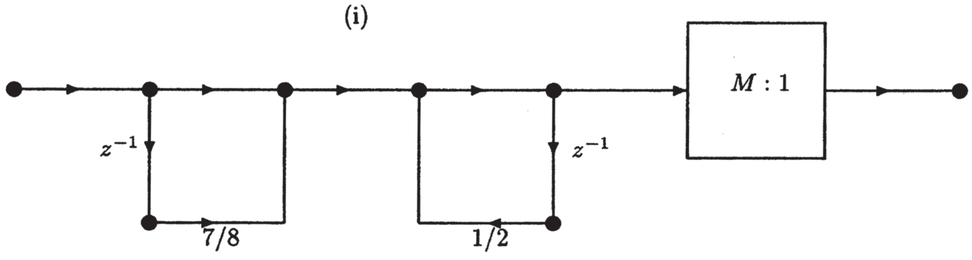
(c)



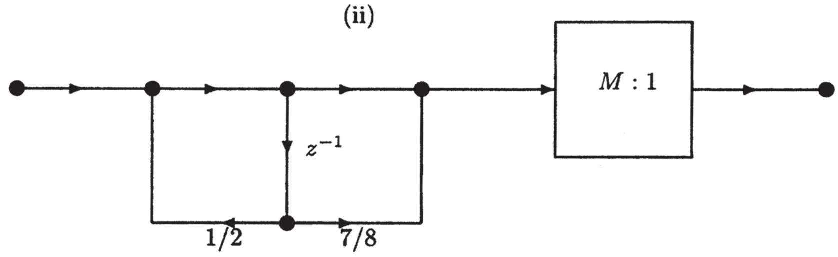
The total computation can not be reduced because to compute the value of any given output sample, the previous output value must be known.

(d)

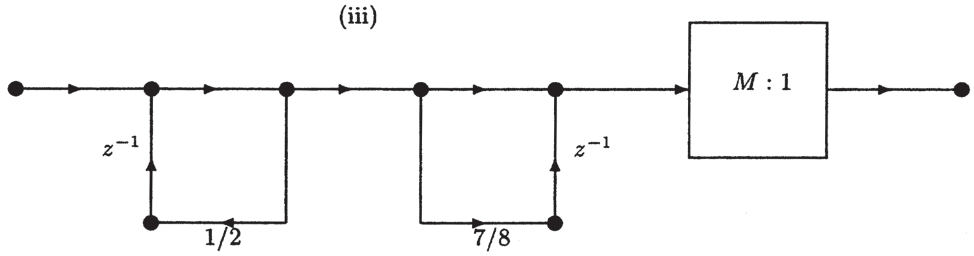
(i)



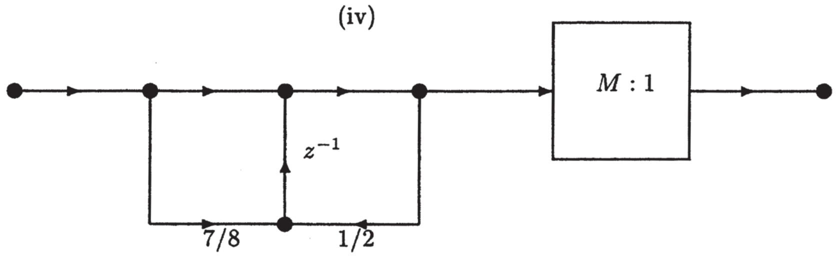
(ii)



(iii)



(iv)



Only direct form II (ii) can be implemented more efficiently by commuting operations with the downsamplers.

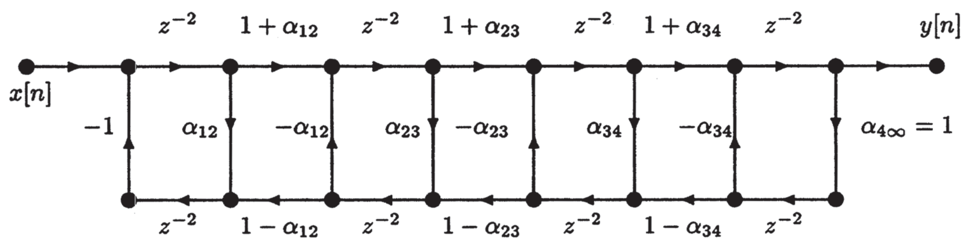
6.53. Since each section is 3.4cm long, it takes

$$\frac{3.4\text{cm}}{3.4\frac{\text{cm}}{\text{sec}} \cdot 10^4} = 10^{-4}\text{sec}$$

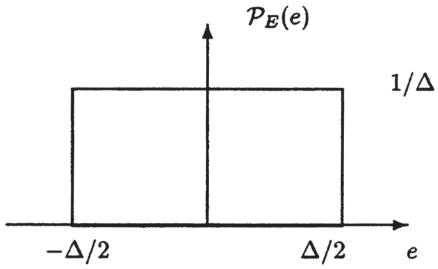
to traverse one section. Since the sampling rate is 20kHz ($T_s = 0.5 \cdot 10^{-4}$sec), it takes two sampling intervals to traverse a section. The entire system is linear and so the forward going and backward going waves add at a boundary. Let

$$\alpha_{kn} = \frac{A_n - A_k}{A_n + A_k}$$

(from A_k into A_n); then $\alpha_{kn} = -\alpha_{nk}$ and we get:



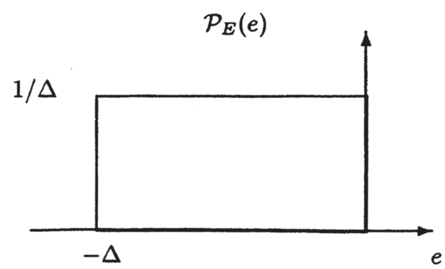
6.54. (a) For rounding:



$$m_e = \frac{1}{\Delta}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} e\, de = \frac{1}{\Delta}\frac{e^2}{2}\Big]_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} = 0$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} e^2\, de = \frac{e^3}{3\Delta}\Big]_{\frac{-\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12}.$$

(b) For truncation:



$$m_e = \frac{1}{\Delta}\int_{-\Delta}^{0} e\, de = \frac{1}{\Delta}\frac{e^2}{2}\Big]_{-\Delta}^{0} = \frac{-\Delta}{2}$$

$$\sigma_e^2 = \frac{1}{\Delta}\int_{-\Delta}^{0} e^2\, de - \frac{\Delta^2}{4} = \frac{e^3}{3\Delta}\Big]_{-\Delta}^{0} - \frac{\Delta^2}{4} = \frac{\Delta^2}{12}.$$

6.55. Since the system is linear, $y[n]$ is the sum of the outputs due to $x_1[n]$ and $x_2[n]$. Therefore

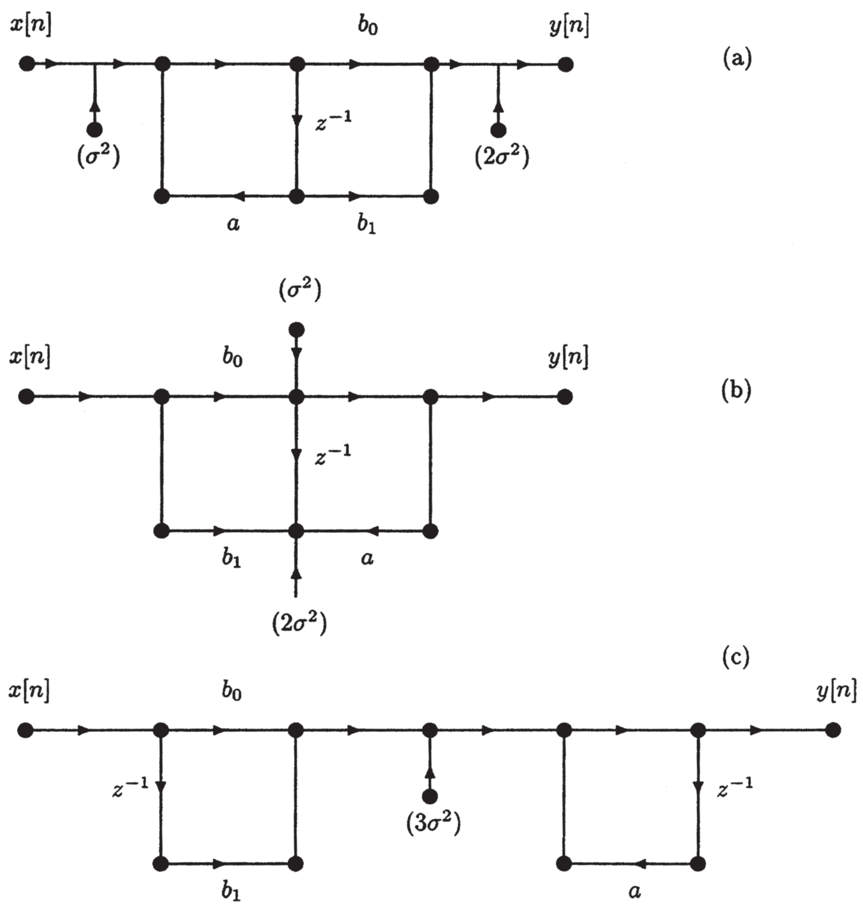
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h_1[k]x_1[n-k] + \sum_{k=-\infty}^{\infty} h_2[k]x_2[n-k] \\ &= y_1[n] + y_2[n]. \end{aligned}$$

The correlation between $y_1[n]$ and $y_2[n]$ is

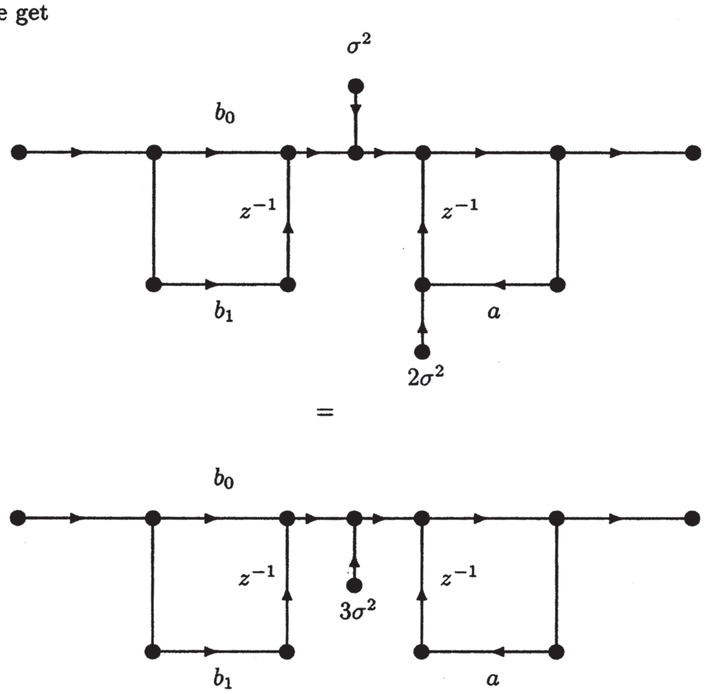
$$\begin{aligned} E\{y_1[m]y_2[n]\} &= E\left\{\sum_{\ell=-\infty}^{\infty} h_1[\ell]x_1[m-\ell] \cdot \sum_{k=-\infty}^{\infty} h_2[k]x_2[n-k]\right\} \\ &= \sum_{\ell=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_1[\ell]h_2[k]E\{x_1[m-\ell]x_2[n-k]\} \end{aligned}$$

If $x_1[n]$ and $x_2[n]$ are uncorrelated, $E\{x_1[m-\ell]x_2[n-k]\} = 0$; hence, $E\{y_1[m]y_2[n]\} = 0$. Therefore, $y_1[n]$ and $y_2[n]$ are uncorrelated.

6.56. (a) The linear noise model for each system is drawn below.



(b) Clearly (a) and (c) are different. Thus the answer is either (a) and (b) or (b) and (c). If we take (b) apart, we get



We see that the noise all goes through the poles. Note that the $1\sigma^2$ source sees a system function $(1 - az^{-1})^{-1}$ while the $2\sigma^2$ source sees $z^{-1}/(1 - az^{-1})$. However, the delay (z^{-1}) does not affect the average power. Hence, the answer is (b) and (c).

(c) For network (c),

$$\sigma_f^2 = 3\sigma^2 \sum_{n=0}^{\infty} (a^n)^2 = \frac{3\sigma^2}{1-a^2},$$

or using the frequency domain formula,

$$\begin{aligned}
\sigma_f^2 &= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{1}{1-az^{-1}} \cdot \frac{1}{1-az} \frac{dz}{z} \\
&= 3\sigma^2 \frac{1}{2\pi j} \oint \frac{dz}{(z-a)(1-az)} \\
&= \frac{3\sigma^2}{1-a^2}.
\end{aligned}$$

For network (a),

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1-az^{-1}}$$

$$h[n] = b_0\delta[n] + (b_0 + \frac{b_1}{a})a^n u[n]$$

- Time domain calculation:

$$\begin{aligned}
\sigma_f^2 &= 2\sigma^2 + \sigma^2 \sum_n h^2[n] \\
&= 2\sigma^2 + \sigma^2 \left(b_0^2 + \left(b_0 + \frac{b_1}{a}\right)^2 \underbrace{\sum_{n=1}^{\infty} a^{2n}}_{\frac{a^2}{1-a^2}} \right) \\
&= 2\sigma^2 + \sigma^2 \left(b_0^2 + \frac{(ab_0 + b_1)^2}{1-a^2} \right).
\end{aligned}$$

- Frequency domain calculation:

$$\begin{aligned}
\sum_n h^2[n] &= \frac{1}{2\pi j} \oint H(z)H(z^{-1}) \frac{dz}{z} \\
&= \sum \left(\text{residues of } \frac{H(z)H(z^{-1})}{z} \text{inside unit circle} \right).
\end{aligned}$$

$$\begin{aligned}
\frac{H(z)H(z^{-1})}{z} &= \frac{(b_0 + b_1 z^{-1})(b_0 + b_1 z)}{(z-a)(1-az)} \frac{z}{z} \\
&= \frac{(b_0 z + b_1)(b_0 + b_1 z)}{z(z-a)(1-az)}.
\end{aligned}$$

$$\text{residue } (z=0) = \frac{-b_1 b_0}{a}$$

$$\text{residue } (z=a) = \frac{(b_0 a + b_1)(b_0 + b_1 a)}{a(1-a^2)} = \frac{b_0^2 a + b_1^2 a + b_1 b_0 + b_1 b_0 a^2}{a(1-a^2)}.$$

$$\begin{aligned}
\oint H(z)H(z^{-1}) \frac{dz}{z} &= \frac{b_0^2 a + b_1^2 a + b_1 b_0 + b_1 b_0 a^2 - b_1 b_0 + b_1 b_0 a^2}{a(1-a^2)} \\
&= \frac{b_0^2 + b_1^2 + 2b_0 b_1 a}{1-a^2} \\
&= b_0^2 + \frac{(ab_0 + b_1)^2}{1-a^2}
\end{aligned}$$

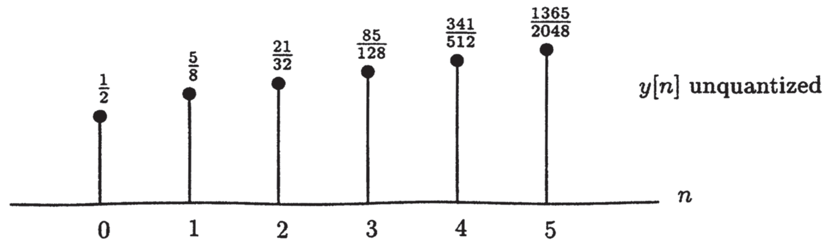
6.57. (a)

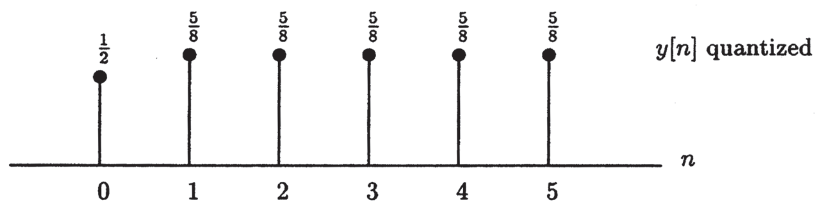
$$y[n] = \frac{1}{4}y[n-1] + \frac{1}{2}, \quad n \geq 0$$

$$y[n] = \frac{1}{2}\sum_{i=0}^{n}\left(\frac{1}{4}\right)^i = \frac{1}{2}\frac{1-\left(\frac{1}{4}\right)^{n+1}}{\frac{3}{4}}$$

For large n, $y[n] = (1/2)/(3/4) = 2/3$.

(b) Working from the difference equation and quantizating after multiplication, it is easy to see that, in the quantized case, $y[0] = 1/2$ and $y[n] = 5/8$ for $n \geq 1$. In the unquantized case, the output monotonically approaches 2/3.





(c) The system diagram is direct form II:

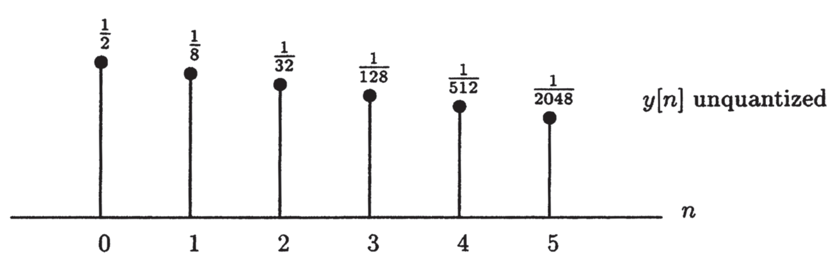
$$H(e^{j\omega}) = \frac{1+e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

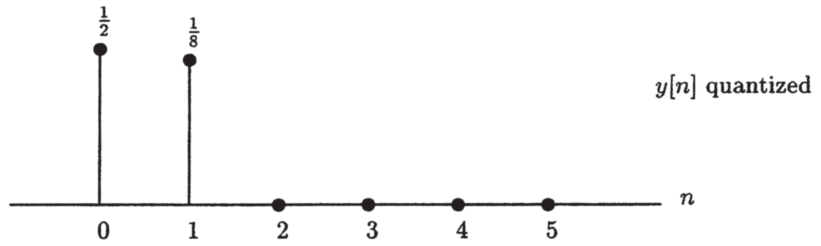
$$X(e^{j\omega}) = \frac{\frac{1}{2}}{1+e^{-j\omega}}$$

So

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{\frac{1}{2}}{1-\frac{1}{4}e^{-j\omega}}$$

which implies that $y[n] = (1/2)(1/4)^n$, which approaches 0 as n grows large.
To find the quantized output (working from the difference equation): $y[0] = 1/2$, $y[1] = 1/8$, and $y[n] = 0$ for $n \geq 2$.





6.58. (a) To check for stability, we look at the poles location. The poles are located at

$$z \approx 0.52 + 0.84j \text{ and } z \approx 0.52 - 0.84j.$$

Note that

$$|z|^2 \approx 0.976 < 1.$$

The poles are inside the unit circle, therefore the system function is stable.

(b) If the coefficients are rounded to the nearest tenth, we have

$$1.04 \to 1.0 \text{ and } 0.98 \to 1.0.$$

Now the poles are at

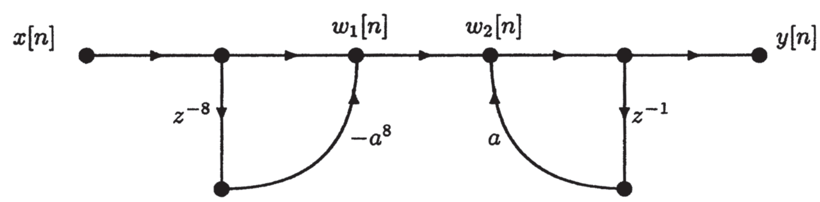
$$z = \frac{1 - j\sqrt{3}}{2} \text{ and } z = \frac{1 - j\sqrt{3}}{2}.$$

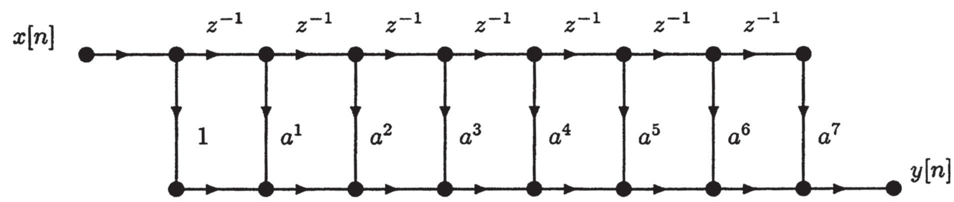
Note that now,

$$|z|^2 = 1.$$

The poles are on the unit circle, therefore the system is not stable.

6.59. The flow graphs for networks 1 and 2 respectively are:





(a) For Network 1, we have:

$$\begin{aligned} w_1[n] &= x[n] - a^8x[n-8] \\ w_2[n] &= ay[n-1] + w_1[n] \\ y[n] &= w_2[n] \end{aligned}$$

Taking the $Z-$transform of the above equations and combining terms, we get:

$$Y(z)(1 - az^{-1}) = (1 - a^8z^{-8})X(z)$$

That is:

$$H(z) = \frac{1 - a^8z^{-8}}{1 - az^{-1}}.$$

For Network 2, we have:

$$y[n] = x[n] + ax[n-1] + a^2x[n-2] + ... + a^7x[n-7].$$

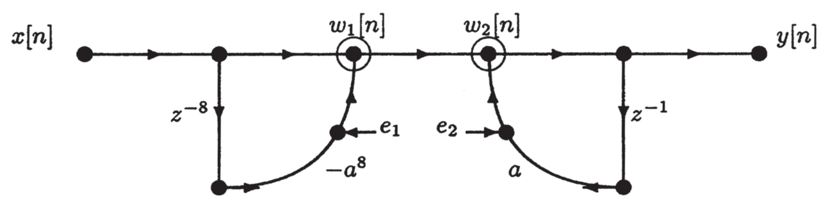
Taking the $Z-$transform, we get:

$$Y(z) = (1 + az^{-1} + a^2z^{-2} + ... + a^7z^{-7})X(z).$$

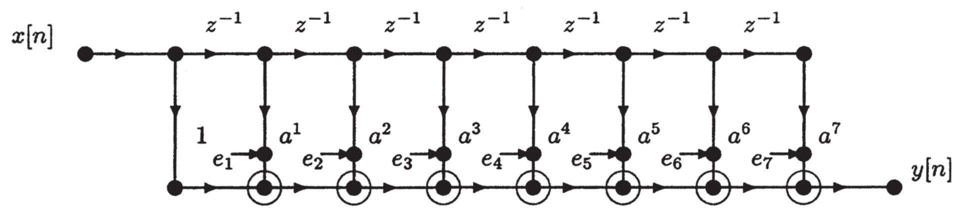
So:

$$H(z) = 1 + az^{-1} + a^2z^{-2} + ... + a^7z^{-7} = \frac{1 - a^8z^{-8}}{1 - az^{-1}}.$$

(b) Network 1:



Network 2:



(c) The nodes are circled on the figures in part (b).

(d) In order to avoid overflow in the system, each node in the network must be constrained to have a magnitude less than 1. That is if $w_k[n]$ denotes the value of the kth node variable and $h_k[n]$ denotes the impulse response from the input $x[n]$ to the node variable $w_k[n]$, a sufficient condition for $|w_k[n]| < 1$ is

$$x_{max} < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k[m]|}.$$

In this problem, we need to make sure overflow does not occur in each node, i.e. we need to take the tighter bound on x_{max}. For network 1, the impulse response from $w_2[n]$ to $y[n]$ is $a^n u[n]$, therefore the condition to avoid overflow from that node to the output is

$$w_{max} < 1 - |a|.$$

Where we assumed that $|a| < 1$. The transfer function from $x[n]$ to $w_1[n]$ is $1 - a^8 z^{-8}$, therefore to avoid overflow at that node we need:

$$w_1[n] < x_{max}(1 - a^8) < 1 - |a|.$$

We thus conclude that to avoid overflow in network 1, we need:

$$x_{max} < \frac{1 - |a|}{1 - a^8}.$$

Now, for network 2, the transfer function from input to output is given by $\delta[n] + a\delta[n-1] + a^2\delta[n-2] + ... + a^7\delta[n-7]$, therefore to avoid overflow, we need:

$$x_{max} < \frac{1}{1 + |a| + a^2 + ... + |a|^7}.$$

(e) For network 1, the total noise power is $\frac{2\sigma_e^2}{1-|a|}$. For network 2, the total noise power is $7\sigma_e^2$. For network 1 to have less noise power than network 2, we need

$$\frac{2\sigma_e^2}{1 - |a|} < 7\sigma_e^2.$$

That is:

$$|a| < \frac{5}{7}.$$

The largest value of $|a|$ such that the noise in network 1 is less than network 2 is therefore $\frac{5}{7}$.