Solutions to Assignment 7

Exercise 31: (Checking that the risk-neutral probability is a probability)

To see that $\tilde{p} = \frac{1+r-d}{u-d} \in (0,1)$ observe that by assumption $u > d > 0$ and hence $u - d > 0$. Furthermore by assumption $1 + r > d$ and hence $1 + r - d > 0$. Hence, $\tilde{p} > 0$. Furthermore,

$$\frac{1+r-d}{u-d} < 1 \Leftrightarrow 1+r-d < u-d \Leftrightarrow 1+r < u$$

which is satisfied by assumption of the no-arbitrage condition. Hence, indeed $\tilde{p} \in (0,1)$.

Furthermore,

$$1 - \tilde{p} = \frac{u-d-(1+r-d)}{u-d} = \frac{u-1-r}{u-d}.$$

Exercise 32: (Sufficient condition for no-arbitrage)

Let $d < 1 + r < u$. We show that there exists no trading strategy φ satisfying

$$X_0^\varphi = 0, \quad P[X_1^\varphi \geq 0] = 1, \quad P[X_1^\varphi > 0] > 0.$$

Observe that

$$\begin{aligned} d < 1+r < u &\Longleftrightarrow \frac{d}{1+r} < 1 < \frac{u}{1+r} \\ \Longleftrightarrow \frac{dS_0}{1+r} < S_0 < \frac{uS_0}{1+r} &\Longleftrightarrow \frac{S_1(T)}{1+r} < S_0 < \frac{S_1(H)}{1+r} \\ &\Longleftrightarrow \frac{S_1(T)}{1+r} - S_0 < 0 < \frac{S_1(H)}{1+r} - S_0 \end{aligned}$$

and the last inequalities implies that there is no arbitrage. To see this consider any trading strategy $\varphi = (\beta_0, \Delta_0)$ with $X_0^\varphi = \beta_0 B_0 + \Delta_0 S_0 = \beta_0 + \Delta_0 S_0 = 0$. Then, $\beta_0 = -\Delta_0 S_0$ and

$$X_1^\varphi = -\Delta_0 S_0 B_1 + \Delta_0 S_1 = \Delta_0 (1+r) \left(-S_0 + \frac{S_1}{1+r} \right).$$

We have just seen that $-S_0 + \frac{S_1}{1+r}$ can take both a positive and a negative value with positive probability and hence $P[X_1^\varphi \geq 0] = 1$ cannot hold and hence there is no arbitrage in the one-period binomial model.

Exercise 33: (Pricing a European put option)

With $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$ the time-0 price of the European put option with strike price $K = 5$ is given by

$$
\begin{aligned}
V_0 &= \frac{1}{1+r}\left(V_1(H)\frac{1+r-d}{u-d} + V_1(T)\frac{u-1-r}{u-d}\right) \\
&= \frac{1}{1+\frac{1}{4}}\left(V_1(H)\frac{1}{2} + V_1(T)\frac{1}{2}\right) \\
&= \frac{4}{5}\frac{1}{2}\left((K-uS_0)^+ + (K-dS_0)^+\right) \\
&= \frac{2}{5}\left((5-8)^+ + (5-2)^+\right) \\
&= \frac{2}{5}3 = \frac{6}{5}.
\end{aligned}
$$

We can implement this in Python as follows:

In [1]:

```
import numpy as np
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In [2]:

```
def putprice(S0, u, d, r, K):
    ptilde = (1+r-d) / (u-d)
    V1H = np.maximum(K - u * S0, 0)
    V1T = np.maximum(K - d * S0, 0)
    price = (1 / (1 + r)) * (ptilde * V1H + (1 - ptilde) * V1T)
    return(price)

print(putprice(S0=4, u=2, d=0.5, r=0.25, K=5))
```

```
1.2000000000000002
```

In this example it turns out that the time-0 call price is equal to the time-0 put price. This is not in general true. In general, the so-called put-call parity holds which in the binomial model is given by $C_0 - P_0 = S_0 - K/(1+r)^N$, where C_0, P_0 denote the price of the European call and put option, respectively. Since in this example $S_0 = 4$, $K = 5$, $(1+r)^1 = 5/4$ we find indeed that $S_0 - K/(1+r)^N = 0$ and hence $C_0 = P_0$.

In [ ]: