Solutions to Assignment 10

Exercise 40: (Central Limit Theorem)

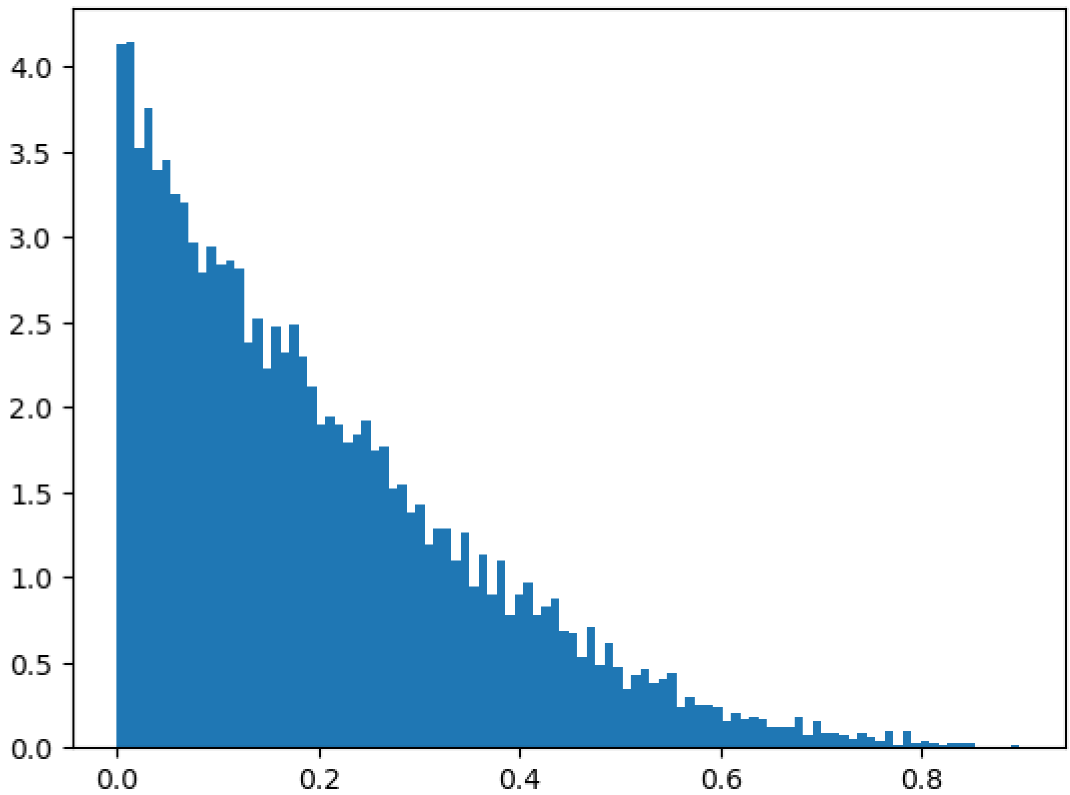
Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables having finite mean μ and finite variance σ^2. Then, for all $x \in \mathbb{R}$

$$\lim_{n\to\infty} \mathrm{P}\left[\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\frac{\sigma^2}{n}}} \leq x\right] = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Exercise 41: (Illustration of the Central Limit Theorem)

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
rng = np.random.default_rng(987654321)
```

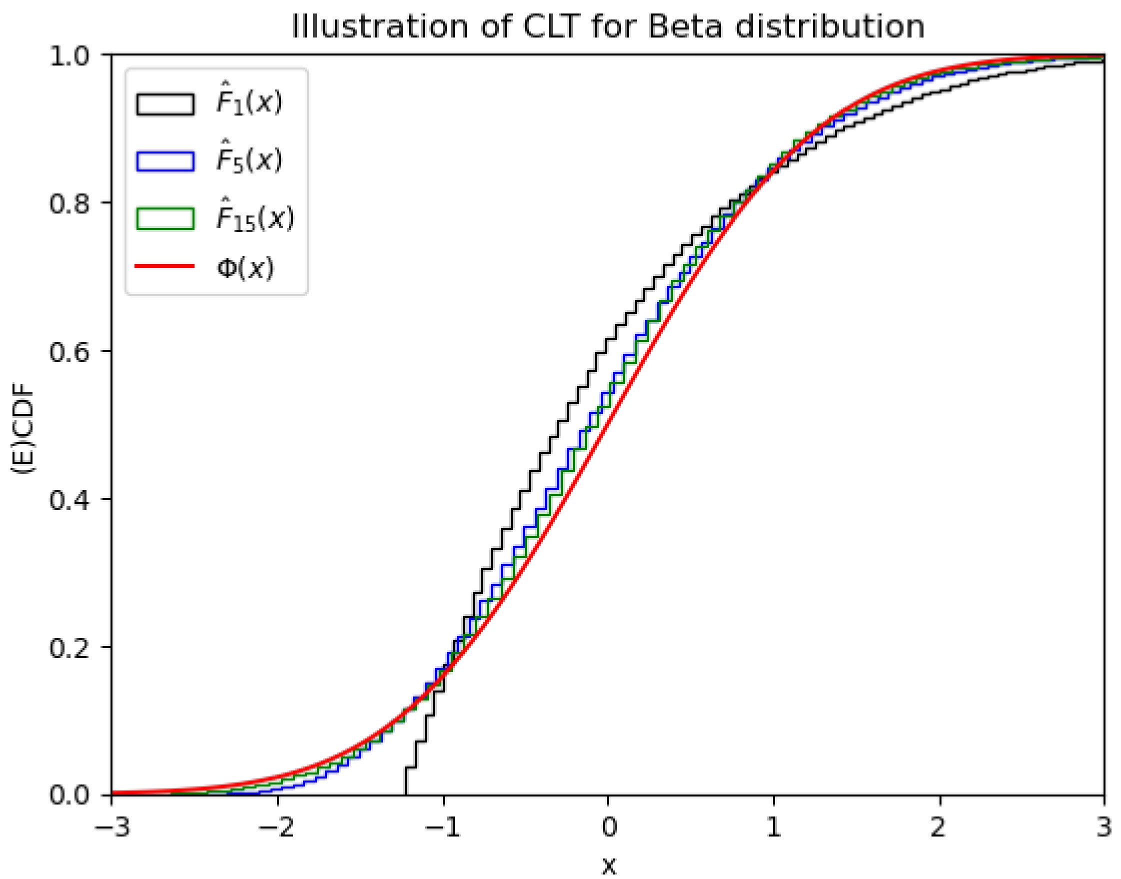
In [2]:
```
samplesize = 10000
sample = rng.beta(a=1, b=4, size=samplesize)

fig, ax = plt.subplots(nrows=1, ncols=1)
ax.hist(sample, bins=100, density = True)
plt.show()
#ax.set_ylabel("density")
```



Next, we illustrate the CLT for the Beta distribution. To do so we consider samplesize=10000 realisations of $\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\frac{\sigma^2}{n}}}$, where $X_1, \ldots, X_n$ are i.i.d. random variables from the $Beta(a, b)$ distribution with mean

$\mu = \frac{a}{a+b}$ and variance $\sigma^2 = \frac{ab}{(a+b)^2(a+b+1)}$ for different choices of n. In particular, we consider $n \in \{1, 5, 15\}$ (stored in the array nsCLT below). We plot the empirical cumulative distribution functions (ECDFs) of $\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\frac{\sigma^2}{n}}}$ for the three different values of n and compare them to the CDF of the standard Normal distribution.

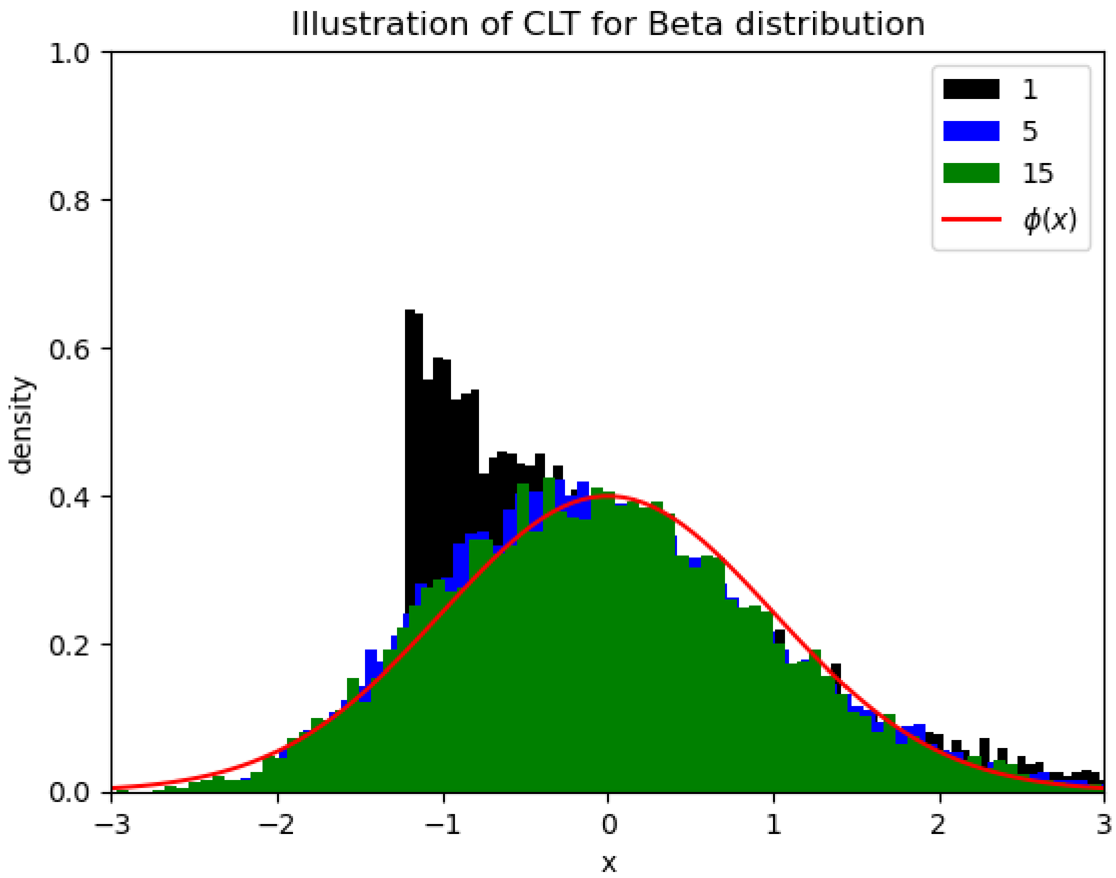
In [3]:
```
a = 1
b = 4
betamean = a / (a + b)
betavariance = a * b / (((a + b)**2) * (a + b + 1))

samplesize = 10000
nsCLT = np.array([1, 5, 15])
mycolours = ["black", "blue", "green"]
mylabels = ["$\hat{F}_1(x)$", "$\hat{F}_5(x)$", "$\hat{F}_{15}(x)$"]

xs = np.linspace(start=-3, stop=3, num=100)
fig, ax = plt.subplots(nrows=1, ncols=1)
for i in range(nsCLT.size):
    n = nsCLT[i]
    sample = rng.beta(a=1, b=4, size=samplesize * n)
    sample = sample.reshape(samplesize, n)
    xbar = (sample.mean(axis=1) - betamean) / np.sqrt(betavariance/n) #compute the row means
    ax.hist(xbar, bins=100, density = True, cumulative=True, histtype="step", color=mycolours
ax.set_xlim([-3, 3])
ax.set_ylim([0, 1])
ax.set_xlabel("x")
ax.set_ylabel("(E)CDF")
ax.plot(xs, norm.cdf(xs, loc=0, scale=1), color="red", label="$\Phi(x)$")
ax.set_title("Illustration of CLT for Beta distribution")
ax.legend(loc = "best")
plt.show()
```



Next, we repeat the analysis above, but plot the histogram/density rather than the (E)CDF.

In [4]:
```
fig, ax = plt.subplots(nrows=1, ncols=1)
for i in range(nsCLT.size):
```

```
    n = nsCLT[i]
    sample = rng.beta(a=1, b=4, size=samplesize * n)
    sample = sample.reshape(samplesize, n)
    xbar = (sample.mean(axis=1) - betamean) / np.sqrt(betavariance/n) #compute the row means
    ax.hist(xbar, bins=100, density = True, cumulative=False, histtype="barstacked", color=my
ax.set_xlim([-3, 3])
ax.set_ylim([0, 1])
ax.set_xlabel("x")
ax.set_ylabel("density")
ax.plot(xs, norm.pdf(xs, loc=0, scale=1), color="red", label="$\phi(x)$")
ax.set_title("Illustration of CLT for Beta distribution")
ax.legend(loc = "best")
plt.show()
```



Exercise 42: (Black-Scholes option pricing formula - European call option)

Let $X \sim \mathrm{N}(0,1)$. Then,

$$\begin{aligned}\mathrm{E}\left[e^{-rT}(S_T-K)^+\right] &= \mathrm{E}\left[e^{-rT}(S_0\exp\left((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}X\right)-K)^+\right]\\ &= \int_{-\infty}^{+\infty} e^{-rT}(S_0\exp\left((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right)-K)^+\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx=(*).\end{aligned}$$

Now observe that

$$S_0\exp\left((r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x\right)\geq K \Longleftrightarrow$$

$$x\geq -\frac{\log\left(\frac{S_0}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}=-(D_1-\sigma\sqrt{T})=-D_1+\sigma\sqrt{T}.$$

Hence,

$$
\begin{aligned}
(*) &= \int_{-D_1+\sigma\sqrt{T}}^{+\infty} e^{-rT}(S_0 \exp\Big((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\Big) - K)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\,dx \\
&= e^{-rT}S_0 \int_{-D_1+\sigma\sqrt{T}}^{+\infty} \exp\Big((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\Big)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\,dx \\
&\quad - e^{-rT}K \int_{-D_1+\sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\,dx.
\end{aligned}
$$

We now look at the two integrals separately. We start with the first one:

$$
\begin{aligned}
&\int_{-D_1+\sigma\sqrt{T}}^{+\infty} \exp\Big((r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x\Big)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\,dx \\
&= \exp\Big((r-\frac{\sigma^2}{2})T\Big)\int_{-D_1+\sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\left(x^2 - 2\sigma\sqrt{T}x + \sigma^2T - \sigma^2T\right)\Big)\,dx \\
&= \exp\Big((r-\frac{\sigma^2}{2})T + \frac{\sigma^2T}{2}\Big)\int_{-D_1+\sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{1}{2}\left(x - \sigma\sqrt{T}\right)^2\Big)\,dx \\
&= e^{rT}(1 - \Phi(-D_1 + \sigma\sqrt{T} - \sigma\sqrt{T})) \\
&= e^{rT}\Phi(D_1).
\end{aligned}
$$

Next we evaluate the second integral:

$$
\int_{-D_1+\sigma\sqrt{T}}^{+\infty} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{x^2}{2}\Big)\,dx = 1 - \Phi(-D_1 + \sigma\sqrt{T}) = \Phi(D_1 - \sigma\sqrt{T}).
$$

Combining these results, gives

$$
\mathrm{E}\left[e^{-rT}(S_T - K)^+\right] = S_0\Phi(D_1) - Ke^{-rT}\Phi(D_1 - \sigma\sqrt{T}).
$$

In [ ]: