AEROSP 740: Assignment 3

Note 1: Completed solutions should be uploaded into Gradescope in a single pdf file before 11:59 p.m. on the day due. **Put boxes around final answers (except for problems involving derivations/proofs, simulation figures and MATLAB codes). 1-2 Points could be deducted if not following the instruction**. Carefully explain and justify your solutions. Plots should be clearly labeled. Include print out of all the codes. Homework should be neat in appearance. You can develop your own code from scratch or follow coding hints given.

Note 2: When submitting the assignment, please follow the steps in the Gradescope and assign corresponding pages to each problem number. **5 points** deduction will be taken if not following this procedure.

Note 3: This assignment pdf together with other complementary documents are uploaded into Canvas: Files > Homeworks and Solutions > Homework 3.

Note 4: For other homework related policies, please consult the syllabus.

1. (20 points) Suppose an output constraint, defined as

$$y_{min} \leq y = Cx + Du \leq y_{max},$$

where y could be a vector and C and D are given matrices, is added to MPC problem formulation considered in Module 5. Derive a modified QP problem that needs to be solved to determine MPC action, i.e., what modifications to H, G, W, T, q will the introduction of such an output constraint induce? Assume the constraint is imposed in the MPC formulation as $y_{min} \leq y_k \leq y_{max}$, $k = 0, \cdots, N-1$.

2. The spacecraft attitude dynamics are represented by nonlinear ODEs

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \frac{1}{\cos(\theta)} \begin{bmatrix} \cos(\theta) & \sin(\phi)\sin(\theta) & \cos(\phi)\sin(\theta) \\ 0 & \cos(\phi)\cos(\theta) & -\sin(\phi)\cos(\theta) \\ 0 & \sin(\phi) & \cos(\phi) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},$$

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = \begin{bmatrix} ((J_2 - J_3)\omega_2\omega_3)/J_1 + M_1/J_1 \\ ((J_3 - J_1)\omega_3\omega_1)/J_2 + M_2/J_2 \\ ((J_1 - J_2)\omega_1\omega_2)/J_3 + M_3/J_3 \end{bmatrix},$$

where ϕ, θ, ψ are spacecraft roll, pitch and yaw angles, respectively; ω_1, ω_2, ω_3 are components of spacecraft angular velocity vector in the body fixed frame; $J_1 = 120$, $J_2 = 100$, $J_3 = 80$ are spacecraft principal moments of inertia; and $u = [M_1,\ M_2,\ M_3]^{\mathrm{T}}$ is the vector of control moments. Let $x = [\phi,\ \theta,\ \psi,\ \omega_1,\ \omega_2,\ \omega_3]^{\mathrm{T}}$ denote the state vector. The control moments are limited and control constraints are given by

$$-0.1 \leq M_i \leq 0.1, \quad i = 1, 2, 3.$$

The objective is to design and simulate LQ-MPC controller on the nonlinear model. The controller must detumble the spacecraft and achieve the desired orientation, specifically, the objective is to steer the spacecraft state to $x = 0$. The state and control weighting matrices are given by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.01 \end{bmatrix}, \quad R = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.01 \end{bmatrix},$$

and the terminal weighting matrix should be chosen as the solution to the corresponding DARE.

(a) (4 points) Implement a function of the form

begin code
```
function [xdot] = scdynamics(t,x,u)
```
end code

that can be passed to `ode45.m` to simulate the spacecraft attitude trajectory. Find the continuous-time linearized model at $x = 0$, $u = 0$ of the form,

$$\dot{x} = A_c x + B_c u,$$

and convert to discrete-time assuming the sampling period of $T_s = 2$ sec. Use `c2d.m`. As your answer give a print-out of the code of `scdynamics` and the matrices A_d, B_d in the discrete-time model, $x_{k+1} = A_d x_k + B_d u_k$. Hint: You can re-use the outcomes of the previous homework.

(b) (8 points) Use MPT3 toolbox to design MPC controller. Develop closed-loop simulations to the initial condition, $x_0 = [-0.4,\ -0.8,\ 1.2,\ -0.02,\ -0.02,\ 0.02]^{\mathrm{T}}$. Your simulation should compute and hold the control constant over each time window of length T_s while simulating the underlying continuous-time nonlinear spacecraft dynamics, i.e., `scdynamics.m`, using `ode45.m`. For two horizons, $N = 2$ and $N = 20$ provide the plots of time histories of orientation angles, angular velocity components, and control moments. Provide also plots of time to compute the control action versus time. Show these responses for 200 sec. For this, you can use `tic` and `toc` commands in Matlab. For instance, suppose your controller is defined somewhere in your code as

begin code
```
ctrl = MPCController(model,N);
```
end code

You can use the following code snippet to compute the control action and time to compute the control action based on the state, x

begin code
```
tic;
u  = ctrl.evaluate(x);
execTime = toc;
```
end code

Once you compute u, you can call `ode45.m` to perform the integration over the sampling period, $[t, t + T_s]$:

begin code
```
[T,X]   = ode45(@ (t,x) scdynamics(t,x,u), [t+h:h:t+Ts], x,options);
```
end code

where h can be chosen as $T_s/10$. You need to repeat this process of computing and applying the control action to obtain responses over 200 sec time interval.

(c) (8 points) Now repeat 2b but using Hybrid Toolbox for Matlab. Generate the same plots for the two horizons, $N = 2$, $N = 20$. Note that when you setup hybrid toolbox, you can specify prediction, control, and constraint horizons separately, but in this example they are the same. You could set them as follows

Begin code

```
moves = 20;  % or 2
horizon.Nu  = moves;        % input horizon    u(0),...,u(Nu-1)
horizon.N   = moves;            % output horizon    \sum_{k=0}^{Ny-1}
horizon.Ncu = moves-1;     % input constraints horizon    k=0,...,Ncu
horizon.Ncy = moves-1;          % output constraints horizon k=0,...,Ncy
```

end code

3. We consider the control of a car lateral motion with active front steering. The linearized vehicle dynamics are given in continuous time by

$$\dot{x} = A_c x + B_c u,$$

where the components of the state vector $x \in \mathbb{R}^2$ are the lateral velocity of vehicle CG (x_1) and yaw rate (x_2), while the control, u, is the steering angle. The matrices A_c and B_c are given by

$$A_c = \begin{bmatrix} 2(C_f + C_r)/(mv_x) & 2(C_f l_f - C_r l_r)/(mv_x) - v_x \\ 2(l_f C_f - l_r C_r)/(I_z v_x) & 2(C_f l_f^2 + C_r l_r^2)/(I_z v_x) \end{bmatrix}, \quad B_c = \begin{bmatrix} -2C_f/m \\ -2l_f C_f/I_z \end{bmatrix}$$

where vehicle mass is $m = 1891$, forward velocity is $v_x = 20$, front tire cornering stiffness is $C_f = -18 \times 10^3$, rear tire cornering stiffness is $C_r = -28 \times 10^3$, $l_f = 1.5$, $l_r = 1.55$, and vehicle moment of inertia is $I_z = 3200$. The front slip angle is given by

$$\alpha_f = \frac{x_1}{v_x} + \frac{x_2 l_f}{v_x} - u,$$

and the rear slip angle is given by

$$\alpha_r = \frac{x_1}{v_x} - \frac{x_2 l_r}{v_x}.$$

(a) (4 points) Assuming the sampling period of $T_s = 20 \times 10^{-3}$ sec, convert the linearized model to discrete-time. Use `c2d.m` command. Give discrete-time A_d and B_d matrices. Is A_d a Schur matrix? Is (A_d, B_d) controllable?

(b) (8 points) Consider the control objective of achieving offset-free tracking of the yaw rate command, r, i.e., we wish to achieve, $x_2 \to r$. Define the augmented state, $x_a = [x,\ u,\ r]^{\mathrm{T}}$ and the augmented model,

$$x_{a,k+1} = \begin{bmatrix} A_d & B_d & 0_{n_x \times 1} \\ 0_{1 \times n_x} & 1 & 0 \\ 0_{1 \times n_x} & 0 & 1 \end{bmatrix} x_{a,k} + \begin{bmatrix} B_d \\ 1 \\ 0 \end{bmatrix} \Delta u_k,$$

where $\Delta u_k = u_k - u_{k-1}$, $n_x = 2$. Let $E_x = [0,\ 1,\ 0,\ -1]$ so that the tracking error can be expressed as $e = x_2 - r = E_x x_a$. Consider the constraints,

$$-3 \leq x_1 \leq 3, \quad -1 \leq x_2 \leq 1, \quad -0.15 \leq u \leq 0.15,$$

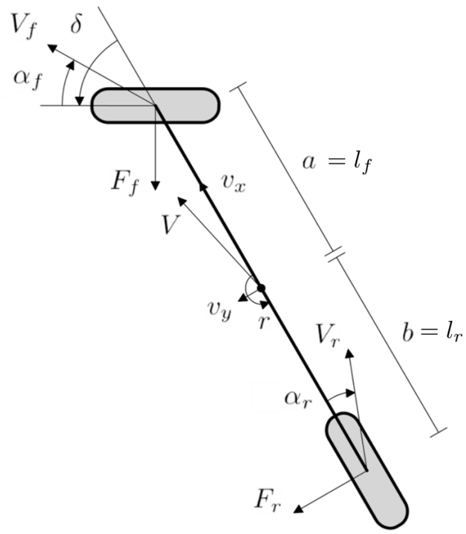

Figure 1: Vehicle schematics.

$$-0.1 \le \alpha_f \le 0.1, \ -0.07 \le \alpha_r \le 0.07,$$

and

$$-0.06 \le \Delta_u \le 0.06.$$

Use MPT3 toolbox for the augmented system, to generate MPC Controller for the horizon $N = 10$ steps. For the weights, use

$$Q = E_x^{\mathrm{T}} Q_y E_x, \ Q_y = 3, \ R = 0.1,$$

and assume zero terminal penalty weight. Simulate the response to the zero initial condition for the horizon $N = 10$ steps. Assume that the yaw rate command, r, changes in steps between 0.2 and -0.2 staying at each level for 1 sec. Give plots of the time histories of yaw rate, side slip angle ($= x_1/v_x$ for small angles), steering angle, front slip angle and rear slip angle. Are constraints satisfied?

(c) (8 points) Repeat the exercise in 3b using Hybrid Toolbox for Matlab.

4. The spacecraft relative motion dynamics (in orbital plane relative to a nominal orbital position on a circular orbit) and without thrust/other external forces acting are described by the following differential equations

$$\begin{aligned} \ddot{x} &= 3n^2 x + 2n\dot{y} \\ \ddot{y} &= -2n\dot{x}, \\ \ddot{z} &= -n^2 z \end{aligned}$$

where x, y, z are spacecraft relative position coordinates in km, and $n = 0.00114$ is the mean motion in rad/sec.

(a) (4 points) Let the state vector be $X = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^{\mathrm{T}}$. Write out equations of motion in the state space form, $\dot{X} = A_c X$. Give A_c as your answer. Describe stability properties of the open-loop spacecraft relative motion dynamics. You can answer this question for the model with the numerical parameter values as given.

(b) (4 points) Assume that the output measurements are

$$Y = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and you wish to write the output equation in the standard form, $Y = CX$. What is C matrix? Is the matrix pair (C, A) observable?

(c) (4 points) Assume that the output is sampled with the sampling period of $T_s = 30$ sec. Using c2d command, obtain discrete-time equations of motion in the form $X_{k+1} = A_d X_k$, $Y_k = C_d X_k$. Give A_d and C_d as your answers.

(d) (8 points) Implement a Moving Horizon Observer (MHO) assuming the following MHO parameters:

```
        P_0   = diag([1,1,1,0.001,0.001,0.001])
        R     = diag([0.01,0.01,0.01])
        Q     = diag([0.001,0.001,0.001,0.1,0.1,0.1])
```

Suppose the true initial state is $X_0 = [1,\ 1,\ -1,\ 0.002,\ -0.002,\ 0.004]^{\mathsf{T}}$, and the initial state estimate is $\hat{X}_0 = [0,\ 0,\ 0,\ 0,\ 0,\ 0]^{\mathsf{T}}$. Suppose that the output is measured with a measurement noise,

```
            Y = C_d*X + 0.01*randn(3,1)
```

Simulate for $k = 0, 1, 2, \cdots, 100$ and plot the true position states x_k, y_k, z_k and the estimated position states, $\hat{x}_k, \hat{y}_k, \hat{z}_k$ on one plot and the true velocity states $\dot{x}_k, \dot{y}_k, \dot{z}_k$ and the estimated velocity states, $\hat{\dot{x}}_k, \hat{\dot{y}}_k, \hat{\dot{z}}_k$ on the other plot.

5. The spacecraft relative motion dynamics (in orbital plane relative to a nominal orbital position on a circular orbit) are described by the following differential equations

$$\ddot{x} = 3n^2 x + 2n\dot{y} + \frac{u_x}{m}$$
$$\ddot{y} = -2n\dot{x} + \frac{u_y}{m},$$

where x and y are spacecraft position coordinates in km, u_x and u_y are components of the thrust force vector in kN, $n = 1.107 \times 10^{-3}$ is the mean motion in rad/sec, and $m = 100$ is the spacecraft mass in kg.

(a) (2 points) Let the state vector be $X = [x, y, \dot{x}, \dot{y}]^{\mathrm{T}}$ and control vector is $u = [u_x, u_y]^{\mathrm{T}}$. Write out equations of motion in the state space form, $\dot{X} = A_c X + B_c u$. Give A_c and B_c matrices as your answer. Describe stability properties of the open-loop spacecraft dynamics. Are these dynamics controllable? You can answer these questions for the model with the numerical parameter values as given.

(b) (3 points) Assume the sampling period is $T_s = 20$ sec and convert the model to discrete-time form using `c2d.m` command. For the resulting discrete-time model, $X_{k+1} = A_d X_k + B_d u_k$, suppose the cost function that we wish to minimize has the form,

$$J_N = q X_N^T X_N + \sum_{k=0}^{N-1} \|u_k\|_1. \tag{1}$$

This 1-norm cost function reflects the actual fuel consumption of the spacecraft with independent thrusters along x and y axes. The first term with the scalar weight $q > 0$ reflects our objective of driving the spacecraft to the origin (e.g., for a rendezvous or a docking mission). Let

$$U = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}$$

Using the state transition formula, you can express X_N in the form $X_N = P X_0 + R U$. Give expressions for P and R.

(c) (3 points) To handle the 1-norm cost, we introduce auxiliary variables, $\zeta_i \in \mathbb{R}^2$, $i = 0, \cdots, N-1$, and inequality constraints,

$$\begin{aligned} \zeta_0 \geq u_0, &\quad \zeta_0 \geq -u_0, \\ \zeta_1 \geq u_1, &\quad \zeta_1 \geq -u_1, \\ &\vdots \\ \zeta_{N-1} \geq u_{N-1}, &\quad \zeta_{N-1} \geq -u_{N-1}. \end{aligned}$$

We change the cost function to

$$\tilde{J}_N = q X_N^T X_N + \sum_{k=0}^{N-1} \mathbf{1}^{\mathrm{T}} \zeta_i,$$

where $\mathbf{1}^{\mathrm{T}} = \begin{bmatrix} 1 & 1 \end{bmatrix}$.
Let

$$Z = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ \zeta_0 \\ \zeta_1 \\ \vdots \\ \zeta_{N-1} \end{bmatrix},$$

be the "decision" vector. Give the expression for the cost $\tilde{J}_N$ as a quadratic function of Z, i.e., $\tilde{J}_N = \frac{1}{2} Z^{\mathrm{T}} \Gamma_1 Z + \gamma_2 Z +$ constant, where you determine expressions for Γ_1 and Γ_2. You can omit additive constant terms that do not depend on Z since they will not affect the result of the optimization.

(d) (3 points) Suppose that minimizing $\tilde{J}_N$ yields a control sequence, $\{u_0^*, \cdots, u_{N-1}^*\}$. Justify mathematically that this sequence minimizes J_N.

(e) (3 points) Let $N = 15$, $q = 10^4$ and $X_0 = [0,\ 10,\ 0,\ 0]^{\mathrm{T}}$. Using Matlab's quadratic programming solver, `quadprog.m`, solve the problem of minimizing $\tilde{J}_N$ subject to the above constraints. Extract from Z the corresponding control trajectory U and simulate the spacecraft motion based on your discrete-time model. Do not recompute the solution at subsequent time instants as in MPC, but just apply your computed input trajectory. Include a plot the spacecraft trajectory on y-x plane (note axes are switched). Include also a plot of the time histories of the control inputs versus time. Use Matlab's `stairs.m` command.

(f) (2 points) What is the "fuel cost" of the optimal maneuver, i.e., the value of $\sum_{k=0}^{N-1} \|u_k\|_1$?

(g) (4 points) Suppose now we impose additional constraints,

$$\|u_k\|_\infty \leq 0.05,\ k = 0, \cdots, N-1.$$

Add these constraints to your quadratic program and repeat problems (5e) and (5f). Include the same type of plots and compute the "fuel cost." Explain the reasons the fuel cost has increased.