(1) Let $f(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2$. Let $g(x,y,z)$ be the linear approximation of f at (x_0, y_0, z_0). Show that $f(x,y,z) - g(x,y,z) \geq 0$ for all values of x, y, z if all of $\alpha, \beta, \gamma \geq 0$ and vice-versa. [5 Marks]

Solution:

The linear approximation of $f(x,y,z)$ at (x_0,y_0,z_0) is $g(x,y,z) = f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) = \alpha x_0^2 + \beta y_0^2 + \gamma z_0^2 + 2\alpha x_0(x-x_0) + 2\beta y_0(y-y_0) + 2\gamma z_0(z-z_0)$. Then $f(x,y,z) - g(x,y,z)$ simplifies to $\alpha(x-x_0)^2 + \beta(y-y_0)^2 + \gamma(z-z_0)^2$. From this last expression we note that if α, β, γ are all greater than or equal to zero, $f(x,y,z) - g(x,y,z) \geq 0$. For the reverse direction, assume that $f(x,y,z) - g(x,y,z) \geq 0$ for all x, y, z. Then put $y = y_0$ and $z = z_0$ so that $f(x,y_0,z_0) - g(x,y_0,z_0) = \alpha(x-x_0)^2$ which must be greater than or equal to zero. This can be true only when $\alpha \geq 0$. Similarly by setting $x = x_0$ and $z = z_0$, we can show that $\beta \geq 0$. By setting $x = x_0$ and $y = y_0$, we can show that $\gamma \geq 0$.

Suggested Marking Scheme: 3 Marks $\rightarrow$ computing $f(x,y,z) - g(x,y,z) = \alpha(x-x_0)^2 + \beta(y-y_0)^2 + \gamma(z-z_0)^2$. 2 Marks $\rightarrow$ remaining argument.

(2) Consider the set of all $n \times n$ diagonalizable real matrices that have the same eigenvectors and real eigenvalues. Does this set constitute a vector space under standard operations of matrix addition and scalar multiplication over the field of real numbers? If so, find the dimension of the vector space, and a basis for it. Otherwise, show why it is not a vector space. [5 Marks]

Solution: Each of the matrices in the set can be factorized in the form of $S\Lambda S^{-1}$, where S is the common matrix of eigenvectors. Note that such a decomposition exists since it is given that the matrices are all diagonalizable. The given set is a subset of the set of all $n \times n$ matrices which is a vector space since matrix addition and scalar mutliplication satisfy all the vector space axioms. Therefore we can show that the given set has all the properties of a vector space by showing that any linear combination of two members of the set belong to the set: if $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$ are in the set, then $c_1 A + c_2 B$ is also in the set since $c_1 S\Lambda_1 S^{-1} + c_2 S\Lambda_2 S^{-1} = S(c_1\Lambda_1)S^{-1} + S(c_2\Lambda_2)S^{-1} = S(c_1\Lambda_1 + c_2\Lambda_2)S^{-1}$.

The dimension of this vector space is of dimension n, and a possible basis is SF_1S^{-1}, $SF_2S^{-1}, \ldots SF_nS^{-1}$, where F_i is a matrix where the $(i,i)th$ element is 1 and all other elements are 0. It can be easily seen that these matrices are linearly independent and they span the entire space as any element in the set can be expressed as a linear combination of these basis vectors.

Suggested Marking Scheme: 3 Marks $\rightarrow$ showing that the given set constitutes a vector space, 2 Marks $\rightarrow$ calculating the dimension of the vector space and producing a basis.

(3) Let $\boldsymbol{x} = [x_1, x_2]^T, \boldsymbol{y} = [y_1, y_2]^T \in \mathbb{R}^2$. Define $op(\boldsymbol{x}, \boldsymbol{y})$ to be $x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2$. Is op an inner product? Give adequate justification for

your answer. [5 Marks]

Solution:

The given expression for $op(\boldsymbol{x}, \boldsymbol{y})$ can be written in the form $\boldsymbol{x}^T \boldsymbol{A}\boldsymbol{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \sum_{i=1}^{i=2} \sum_{j=1}^{j=2} a_{ij} x_i y_j$. By comparing coefficients we have $\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Using the theorem proved in class regarding inner products, we need to show that $\boldsymbol{A}$ is positive-definite. To do this we calculate the eigenvalues of the matrix as the solutions to the equation $\lambda^2 - 3\lambda + 2 = 0$ and obtain the eigenvalues as 2 and 1. Since the eigenvalues are positive the matrix is positive-definite, and *op* is an inner product.

Another way of solving this problem is to explicitly show how the properties of symmetry, bilinearity and positive-definiteness are satisfied by the given expression for *op* using the definition of these properties. Marks will be awarded as appropriate for such steps.

Suggested Marking Scheme: 2 Marks $\rightarrow$ obtaining the entries of matrix $\boldsymbol{A}$, 3 Marks $\rightarrow$ showing the matrix is positive-definite, for the other approach: 1 Mark $\rightarrow$ symmetry, 3 Marks $\rightarrow$ bilinearity, 1 Mark $\rightarrow$ positive-definiteness.

(4) Consider matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -c_0 & -c_1 & -c_2 & -c_3 & \cdots & -c_{n-1} \end{bmatrix}$.

(i) If l is a root of equation $l^n + c_{n-1}l^{n-1} + \cdots c_0 = 0$, then prove or disprove that $[1, l, \cdots, l^{n-1}]^T$ is an eigenvector of $\boldsymbol{A}$ [1.5 Marks]

(ii) Find matrices $\boldsymbol{P}, \boldsymbol{D}$ such that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} = \boldsymbol{PDP}^{-1}$$

[2 Marks]

(iii) Find eigenvalues of $\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -4 \end{bmatrix}$ without actually evaluating the eigenvalues using the standard method. [1.5 Marks]

Solution (i) Now, $l^n + c_{n-1}l^{n-1} + \cdots c_0 = 0 \Rightarrow l^n = -\sum_{i=0}^{n-1} c_i l^i$ [0.5 Marks]

Let $\boldsymbol{v} = [1, l, \cdots, l^{n-1}]^T$, then $\boldsymbol{v} \neq \boldsymbol{0}$. Consider

$$\begin{aligned}
\boldsymbol{Av} &= \begin{bmatrix} l \\ l^2 \\ \vdots \\ l^{n-1} \\ -\sum_{i=0}^{n-1} c_i l^i \end{bmatrix} \\
&= \begin{bmatrix} l \\ l^2 \\ \vdots \\ l^{n-1} \\ l^n \end{bmatrix} \\
&= l \begin{bmatrix} 1 \\ l \\ \vdots \\ l^{n-1} \end{bmatrix} \\
&= l\boldsymbol{v}
\end{aligned}$$

Thus proved. [1 Mark]

(ii) Consider

$$\begin{aligned}
& l^3 - 6l^2 + 11l - 6 = 0 \\
\Rightarrow \quad & (l-1)(l-2)(l-3) = 0 \\
\Rightarrow \quad & l = 1, 2, 3
\end{aligned}$$

Therefore, from(i), we get the eigenvalues are $1, 2, 3$ and corresponding eigenvectors are $[1, 1, 1]^T$, $[1, 2, 4]^T$ and $[1, 3, 9]^T$ respectively. [1.5 Marks]

Hence, $\boldsymbol{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, $\boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ such that $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix} = \boldsymbol{PDP}^{-1}$. [0.5 Marks]

(Kindly award full marks for any other correct method and alternate correct $\boldsymbol{P}$ and $\boldsymbol{D}$.)

(iii) Clearly $\boldsymbol{B}^T$ is of the form of $\boldsymbol{A}$ in (i) and therefore eigenvalues of $\boldsymbol{B}^T$ are roots of equation $x^4 + 4x^3 + 6x^2 + 4x + 1 = 0$ [0.5 Marks]
Clearly, the roots of the equation $x^4 + 4x^3 + 6x^2 + 4x + 1 = (x+1)^4 = 0$ are $-1, -1, -1, -1$. [0.5 Marks]
Now the eigenvalues of $\boldsymbol{B}$ and $\boldsymbol{B}^T$ are the same and hence the eigenvalues of $\boldsymbol{B}$ are $-1, -1, -1, -1$. [0.5 Marks]

(5) Given that $z = xy + x^2$ and $x = e^t \sin(t), y = e^{-t}$, find $\frac{dz}{dt}$ using the chain rule.

[5 Marks]

Solution

We can write $\frac{dz}{dt} = [\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}][\frac{dx}{dt}\frac{dy}{dt}]^T$ using the chain rule. Now $\frac{\partial z}{\partial x} = y + 2x$, $\frac{\partial z}{\partial y} = x$. Also $\frac{dx}{dt} = e^t \sin t + e^t \cos t$ and $\frac{dy}{dt} = -e^{-t}$. Substituting these values into the expression for $\frac{dz}{dt} = (e^{-t}+2e^t \sin t)(e^t \sin t+e^t \cos t)+(e^t * \sin t)(-e^{-t}) = (\sin t + \cos t + 2e^{2t}\sin^2 t + 2e^{2t}\sin t \cos t - \sin t) = \cos t + 2e^{2t}\sin^2 t + 2e^{2t}\sin t \cos t$.

Suggested Marking Scheme: Chain rule expression for $\frac{dz}{dt} \to 1$ Mark, partial derivatives $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \to 2$ Marks, final answer $\to 2$ Marks.

(6) We are given the equation $Ax = b$ in the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where all the entries in the matrices A and b are real.

(a) If we only know that $a_{11} \neq 0$, $a_{22} \neq 0$, and $a_{34} \neq 0$, what restriction would you have to place on the vector $[b_1, b_2, b_3]^T$ to ensure that the system has a solution and why? [3 Marks]

(b) Is the information given enough to calculate the dimension of the nullspace of the matrix A? In case you can calculate it what is the dimension of the nullspace? Otherwise explain why you do not have enough information to calculate the dimension of the nullspace of the matrix. [2 Marks]

Solution

(a) The given matrix has a structure that allows us to note that the third column can be expressed as a linear combination of the first two regardless of the value of a_{23}, while the fourth column is linearly independent of the first two columns. Thus the rank of the matrix is 3, and the linearly independent columns of the matrix form a basis for $\mathbb{R}^3$. Some linear combination of these columns will be able to generate any right hand side, so there is no restriction on the values taken by $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

(b) The rank nullity theorem says that $Rank(\boldsymbol{A}) + dim(N_{\boldsymbol{A}}) = n$ where n is the number of columns in the matrix $\boldsymbol{A}$. Since the rank of $\boldsymbol{A}$ is shown to be 3, we can conclude that its nullspace is of dimension 1.