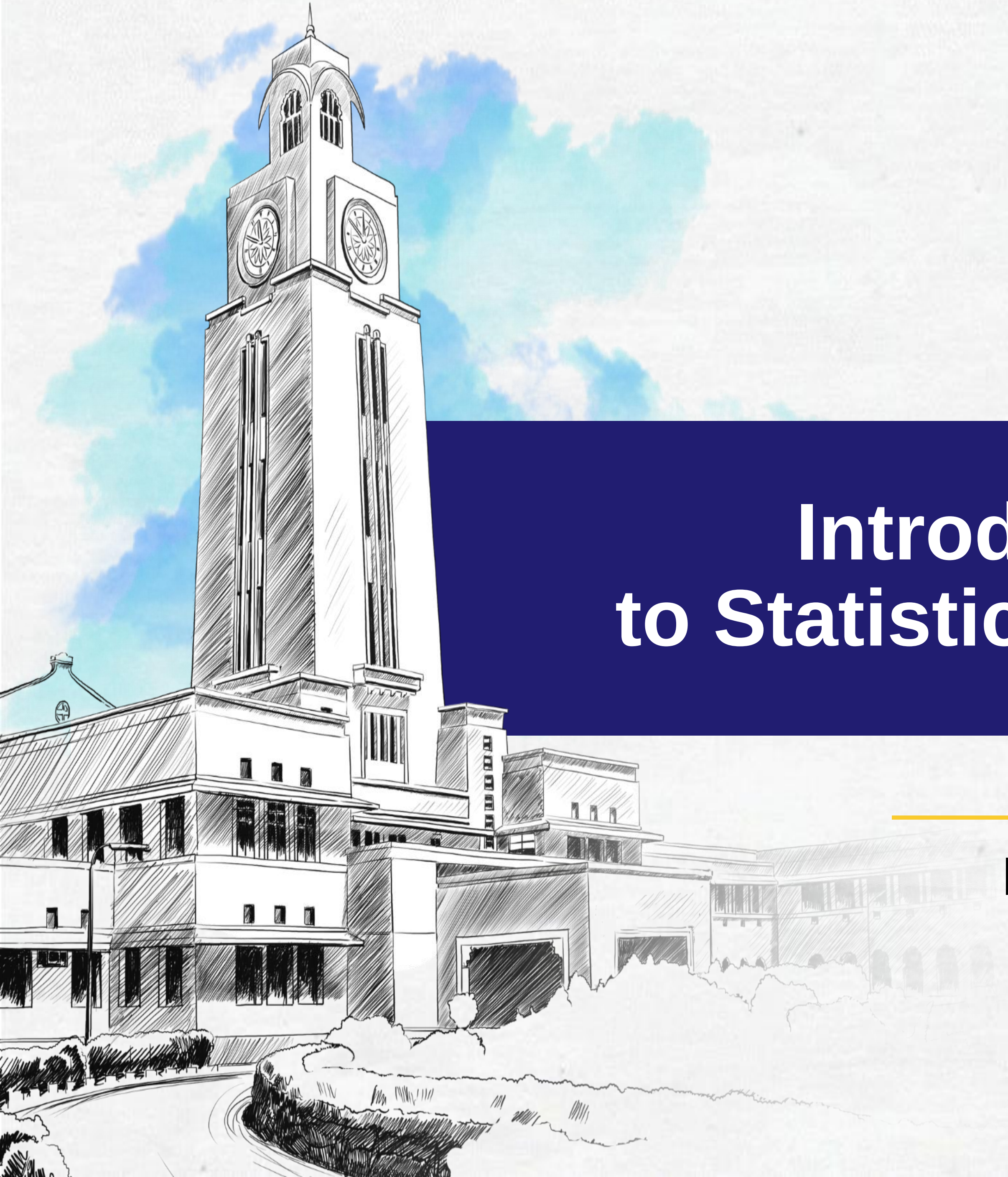

# Introduction to Statistical Methods

ISM Team

# IMP Note to Self

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BITS Pilani

# Session 2

## Axioms of Probability, Probability basics, Mutually exclusive and Independent events

## (30.11.2024 / 01.12.2024)

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## Contact Session 2: Module 1(Module 1:Basic Probability & Statisitcs)

| Contact Session | List of Topic Title | Reference |
|---|---|---|
| CS - 2 | Axioms of Probability,Mutually exclusive and independent events,Problem solving to understand basic probability concepts | T1 & T2 |
| HW | Problems on probability | T1 & T2 |
| Lab | | |

# Agenda

- Experiments, assignment of probabilities
- Events and their probability
- Some basic relationships of probability
- Basic problem solving

# Introduction to Probability

# Random Experiment

1. Flip a coin. Did it land with heads or tails facing up?

2. Walk to a bus stop. How long do you wait for the arrival of a bus?

3. Give a lecture. How many students are listerning?

4. Transmit one of a collection of waveforms over a channel. What waveform arrives at the receiver?

5. Transmit one of a collection of waveforms over a channel. Which waveform does the receiver identify as the transmitted waveform?

# Random Experiment :

The term "**random experiment**" is used to describe any action whose outcome is not known in advance.

Examples:

- Counting how many times a certain word or a combination of words appears in the text of the "King Lear" or in a text of Confucius.
- Pulling a card from the deck.
- Monitor three consecutive phone calls going through a telephone switching office. Classify each one as a voice call (v) if someone is speaking, or a data call (d) if the call is carrying a modemorfax signal.

# Sample space :

The sample space of a random experiment is a set S that includes all possible outcomes of the experiment.

Example:

- If the experiment is to throw a die and record the outcome, then, sample space is $S = \{ 1,2,3,4,5,6\}$
- Manufacture an integrated circuit and test it to determine whether it meets quality objectives. The possible outcomes are "accepted" (a) and "rejected" (r). The sample space is $S = \{a,r\}$.

# Event

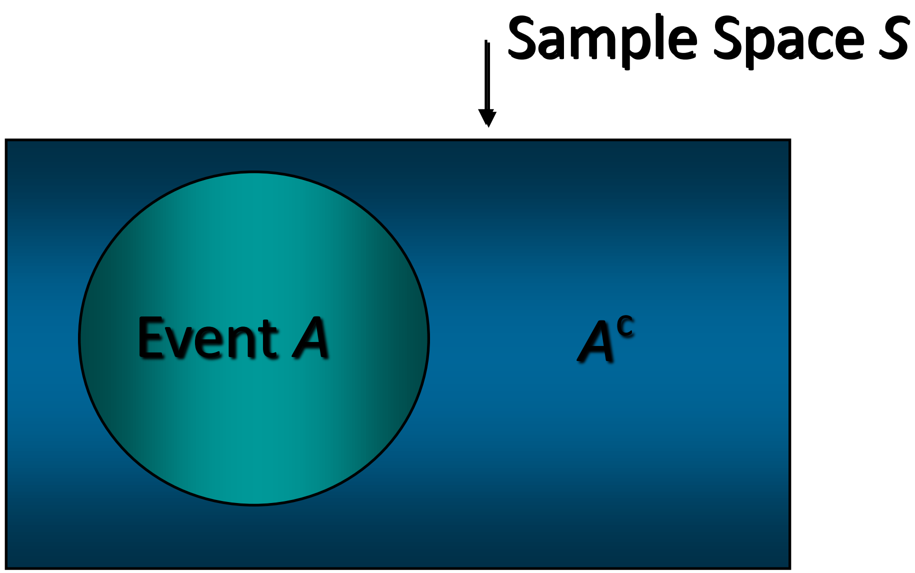
**An event is a subset of the sample space of a random experiment.**

- An event is a set of outcomes of the experiment. This includes the *null* (empty) set of outcomes and the set of *all* outcomes.

- Each time the experiment is run, a given event *A* either *occurs*, if the outcome of the experiment is an element of *A*, or *does not occur*, if the outcome of the experiment is not an element of *A*.

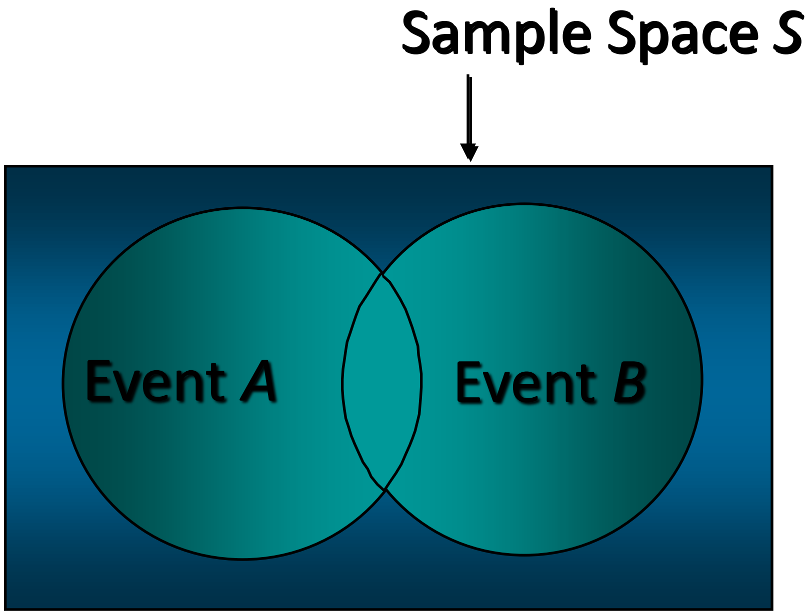
# Complement of an Event

- The complement of event *A* is defined to be the event consisting of all sample points that are not in *A.*
- The complement of *A* is denoted by $A^c$.
- The Venn diagram below illustrates the concept of a complement.



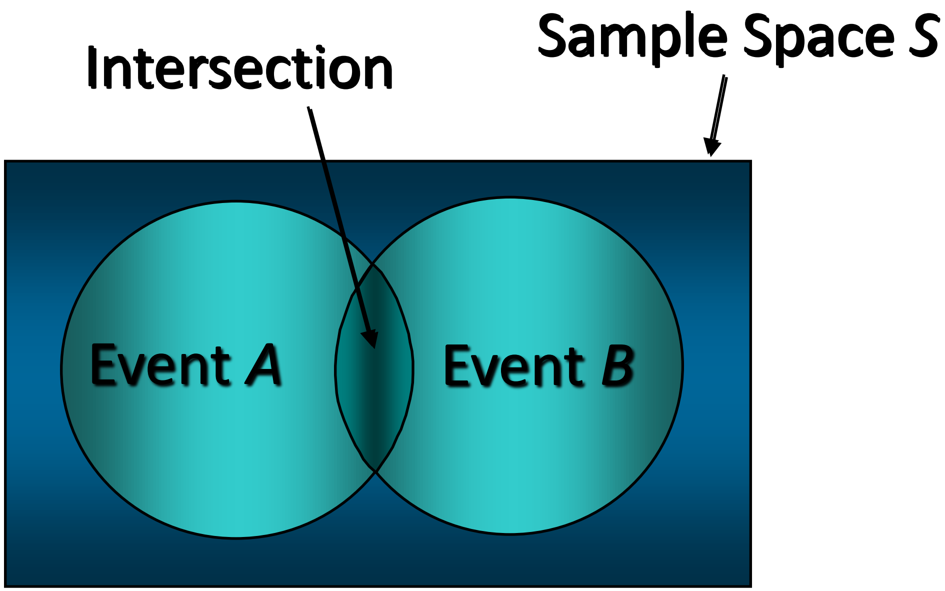
# Union of Two Events

- The union of events *A* and *B* is the event containing all sample points that are in *A* or *B* or both.
- The union is denoted by *A* U *B*
- The union of *A* and *B* is illustrated below.



# Intersection of Two Events

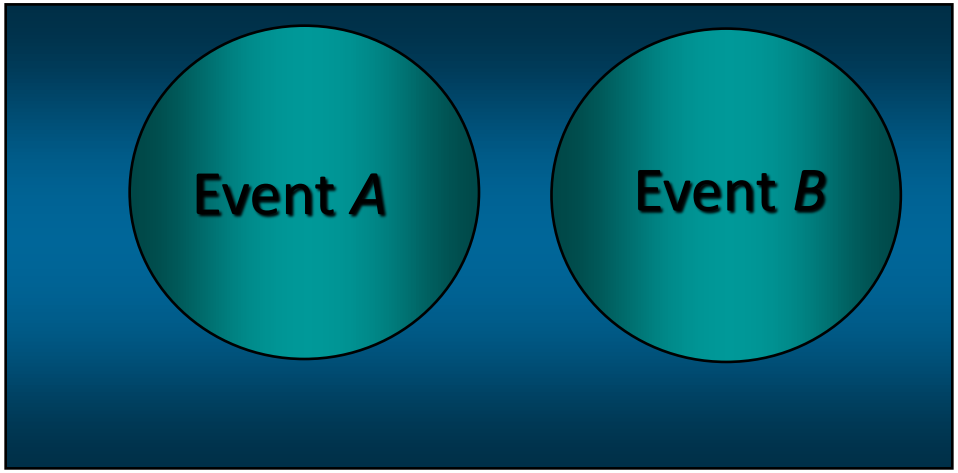
- The intersection of events *A* and *B* is the set of all sample points that are in both *A* and *B*.
- The intersection of *A* and *B* is the area of overlap in the illustration below.



# Mutually Exclusive Events

- Two events are said to be <u>mutually exclusive</u> if the events have no sample points in common. That is, two events are mutually exclusive if, when one event occurs, the other cannot occur.

Sample Space $S$



# Definition of Probability

## Classical approach:

**CLASSICAL PROBABILITY**

$$\text{Probability of an event} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

# Empirical approach:

**Empirical** or **relative frequency** is the second type of objective probability. It is based on the number of times an event occurs as a proportion of a known number of trials.

**EMPIRICAL PROBABILITY** The probability of an event happening is the fraction of the time similar events happened in the past.

In terms of a formula:

$$\text{Empirical probability} = \frac{\text{Number of times the event occurs}}{\text{Total number of observations}}$$

The empirical approach to probability is based on what is called the law of large numbers. The key to establishing probabilities empirically is that more observations will provide a more accurate estimate of the probability.

**LAW OF LARGE NUMBERS** Over a large number of trials, the empirical probability of an event will approach its true probability.

# Axiomatic approach:

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If $S$ is the sample space and $E$ is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

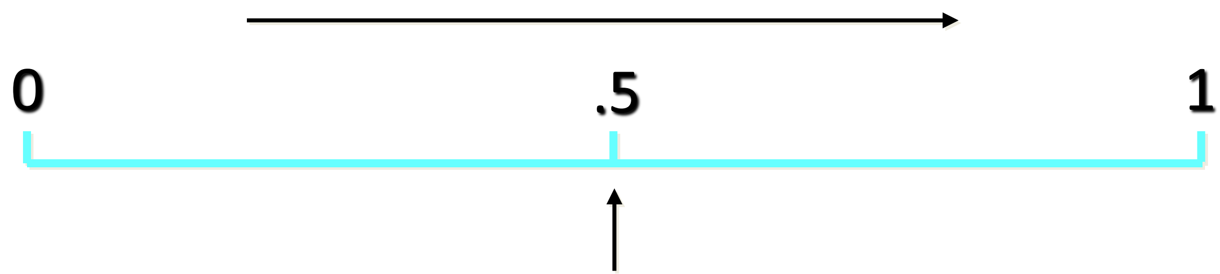
(3) For two events $E_1$ and $E_2$ with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

# Probability as a Numerical Measure of the Likelihood of Occurrence

Increasing Likelihood of Occurrence

Probability:




The occurrence of the event is just as likely as it is unlikely.

# THE ADDITION RULE

- The probability that event *A* or *B* will occur is given by

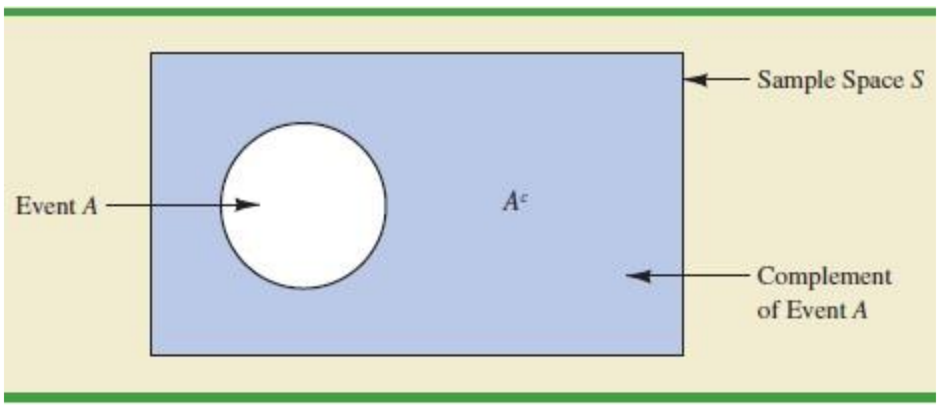
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

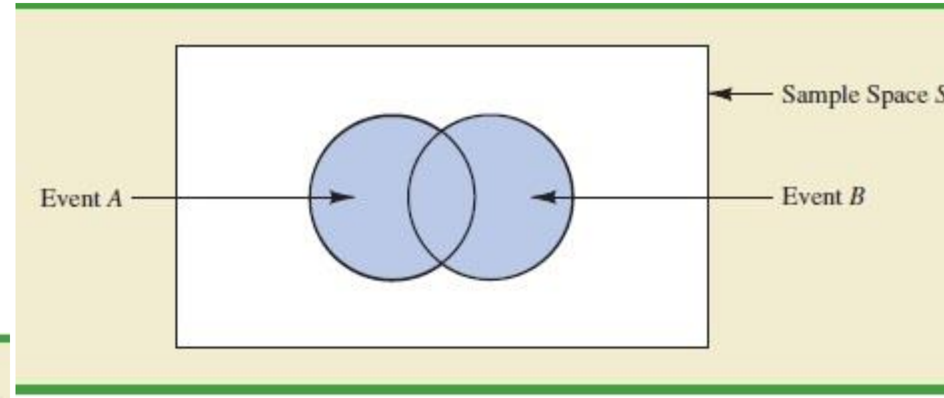
- If events *A* and *B* are **mutually exclusive**, then the rule can be simplified to

  $P(A \cup B) = P(A) + P(B)$.

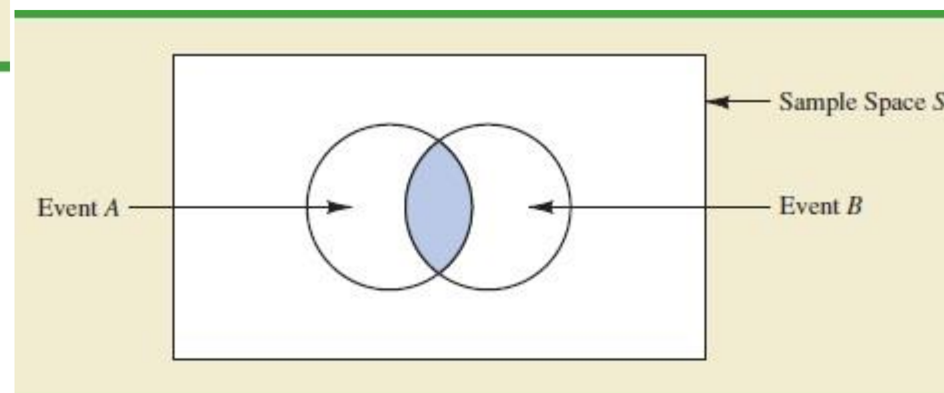
# Probability and Venn Diagram

$$P(A) = 1 - P(A^c)$$




$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



# Independent & Dependent

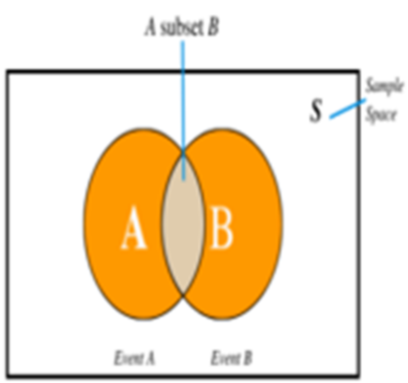
Events are either

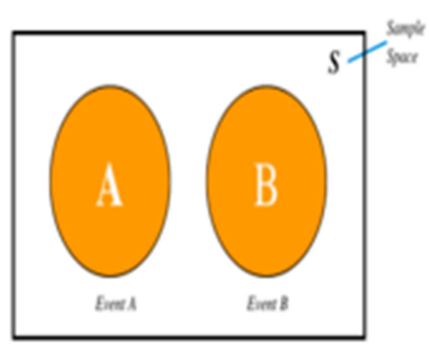
- *Independent* (the occurrence of one event has no effect on the probability of occurrence of the other) or
- *Dependent* (the occurrence of one event gives information about the occurrence of the other)

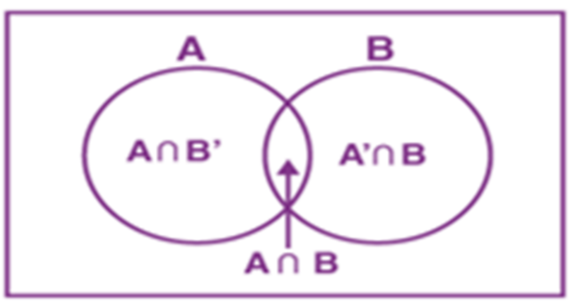
$A$ and $B$ are independent if and only if (iff)

$$P(A \cap B) = P(A) \cdot P(B)$$

# Comparison between Mutually Exclusive events and Independent Events

<table>
<tr><th>Comparison on the Basis</th><th>Mutually Exclusive Event</th><th>Independent Event</th></tr>
<tr><td>1. Definition</td><td>Here the events cannot happen simultaneously.</td><td>The occurrence and outcome of one event don’t affect the occurrence and outcome of the other event.</td></tr>
<tr><td>2.Dependency</td><td>Occurrence of event A results in non-occurrence of event B.</td><td>Occurrence of event A does not affect event B in any manner.</td></tr>
<tr><td>3. Occurrence of both events</td><td>The mathematical formula for the representation of the mutually exclusive event is $P(A \cap B) = 0$</td><td>The mathematical formula for the representation of the mutually exclusive event is $P(A \cap B) = P(A).P(B)$</td></tr>
<tr><td>4. Venn Diagram Representation</td><td>



Here the events A and B do not overlap.</td><td>

Here the events do overlap.</td></tr>
</table>

# Example: 1

An experiment has the four possible mutually exclusive outcomes A, B, C, D. Check whether the following assignments of probability ar e permissible:

(a) $P(A)= 0.38$, $P(B) = 0.16$, $P(C) =0.11$, $P(D) = 0.35$ Permissible
(b) $P(A)= 0.31$, $P(B) = 0.27$, $P(C) =0.28$, $P(D) = 0.16$ NOT
(c) $P(A)= 0.32$, $P(B) = 0.27$, $P(C) = -0.06$, $P(D) = 0.47$ NOT
(d) $P(A)= 1/2$, $P(B) = 1/4$, $P(C) = 1/8$, $P(D) = 1/16$ NOT
(e) $P(A)= 5/8$, $P(B) = 1/6$, $P(C) = 1/3$, $P(D) = 2/9$ NOT

# Example: 2

If two dice are thrown , what is the probability that the sum is

a) Greater than 8

b) Less than 6

c) Neither 7 nor 11

# Solution:

Probability of outcome, when a pair of die is thrown:

$$P(X = 2) = P\{(1, 1)\} = \frac{1}{36}$$

$$P(X = 3) = P\{(1, 2), (2, 1)\} = \frac{2}{36}$$

$$P(X = 4) = P\{(1, 3), (2, 2), (3, 1)\} = \frac{3}{36}$$

$$P(X = 5) = P\{(1, 4), (2, 3), (3, 2), (4, 1)\} = \frac{4}{36}$$

$$P(X = 6) = P\{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} = \frac{5}{36}$$

$$P(X = 7) = P\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} = \frac{6}{36}$$

$$P(X = 8) = P\{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\} = \frac{5}{36}$$

$$P(X = 9) = P\{(3, 6), (4, 5), (5, 4), (6, 3)\} = \frac{4}{36}$$

$$P(X = 10) = P\{(4, 6) (5, 5), (6, 4)\} = \frac{3}{36}$$

$$P(X = 11) = P\{(5, 6), (6, 5)\} = \frac{2}{36}$$

$$P(X = 12) = P\{(6, 6)\} = \frac{1}{36}$$

# Contd..

If two dice are thrown , what is the probability that the sum is

a) Greater than 8

b)Less than 6:

a) $P(\text{sum} > 8)$

$$= P(9) + P(10) + P(11) + P(12)$$

$$= \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36}$$

b) $P(\text{sum} < 6)$

$$= P(5) + P(4) + P(3) + P(2)$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} = \frac{10}{36}$$

# Contd..

c) Neither 7 nor 11

$$P(\text{neither } 7 \text{ nor } 11)$$

$$= 1 - [P(7) + P(11)]$$

$$= 1 - \left[\frac{6}{36} + \frac{2}{36}\right]$$

$$= {}^{28}/_{36} = {}^{7}/_{9}$$

# Example: 3

The probability that a student passes in statistics examination is 2/3 and the probability that he /she will not pass in mathematics examination is 5/9. The probability that he/she will pass in at least one of the examination is 4/5. Find the probability that he /she will pass in both the examinations

# Solution:

The probability of passing in statistics P(S)=2/3

The probability of passing in Mathematics P(M)=1-5/9 =4/9

Probability of Passing at least one of these examination

= P(SUM) = 4/5

$\boldsymbol{P\ (S \cup M) = P\ (S) + P\ (M) - P\ (S \cap M\ ).}$

$\frac{4}{5} = \frac{2}{3} + \frac{4}{9} - P\ (S \cap M\ ).$

$P\ (S \cap M\ ) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}\ .$

The probability of passing both examinations $P\ (S \cap M\ ) = \frac{14}{45}$

# Example: 4

Suppose that 75% of all investors invest in traditional annuities and 45% of them invest in the stock market. If 85% invest in the stock market and/or traditional annuities, what percentage invest in both?

# Solution:

Let A be the event that a randomly selected investor invests in traditional annuities.

Let B be the event that he or she invests in the stock market.

Then P (A) = 0.75, P (B) = 0.45, and P (A U B) = 0.85

Since,

$$
\begin{aligned}
P(AB) &= P(A) + P(B) - P(A \cup B) \\
&= 0.75 + 0.45 - 0.85 \\
&= 0.35.
\end{aligned}
$$

# Example: 5

Suppose the manufacturer's specifications for the length of a certain type of computer cable are $2000 \pm 10$ millimeters. In this industry, it is known that small cable is just as likely to be defective (not meeting specifications) as large cable. That is, the probability of randomly producing a cable with length exceeding 2010 millimeters is equal to the probability of producing a cable with length smaller than 1990 millimeters. The probability that the production procedure meets specifications is known to be 0.99.

(a) What is the probability that a cable selected randomly is too large?

(b) What is the probability that a randomly selected cable is larger than 1990 millimeters?

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***Solution*:** Let $M$ be the event that a cable meets specifications. Let $S$ and $L$ be the events that the cable is too small and too large, respectively. Then

(a) $P(M) = 0.99$ and $P(S) = P(L) = (1 - 0.99)/2 = 0.005$.

(b) Denoting by $X$ the length of a randomly selected cable, we have

$$P(1990 \leq X \leq 2010) = P(M) = 0.99.$$

Since $P(X \geq 2010) = P(L) = 0.005$,

$$P(X \geq 1990) = P(M) + P(L) = 0.995.$$

This also can be solved

$$P(X \geq 1990) + P(X < 1990) = 1.$$

Thus, $P(X \geq 1990) = 1 - P(S) = 1 - 0.005 = 0.995$.

# Example: 6

Let S be a sample space and A and B are two mutually exclusive events such that AUB = S. If P(.) denotes the probability of the event, then what is the maximum value of P(A).P(B)?

**Solution:**

$P(A) + P(B) = 1$, since both are mutually exclusive and $A \cup B = S$.

When sum is a constant, product of two numbers becomes maximum when they are equal.

So P(A) = P(B) = ½

Hence P(A).P(B) = 1/4

# Example: 7

Throw a die twice. Let A = {max is 2} and B = {min is 2}. Are A and B independent?

Solution: First list out A and B:

A = {(1, 2), (2, 1), (2, 2)},

B = {(2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (4, 2), (5, 2), (6, 2)}.

Therefore, the probabilities are

$$P[A] = \frac{3}{36}, \; P[B] = \frac{9}{36}, \quad \text{and } P[A \cap B] = P[(2, 2)] = \frac{1}{36},$$

Clearly, $P[A \cap B] \neq P[A]P[B]$ and so A and B are dependent.

# Example: 8

The computers of six faculty members in a certain department are to be replaced. Two of the faculty members have selected laptop machines and the other four have chosen desktop machines. Suppose that only two of the setups can be done on a particular day, and the two computers to be set up are randomly selected from the six (implying 15 equally likely outcomes; if the computers are numbered 1, 2, . . . , 6, then one outcome consists of computers 1 and 2, another consists of computers 1 and 3, and so on).

# Example: 8

**a.** What is the probability that both selected setups are for laptop computers?

**b.** What is the probability that both selected setups are desktop machines?

**c.** What is the probability that at least one selected setup is for a desktop computer?

**d.** What is the probability that at least one computer of each type is chosen for setup?

## Solution:

In the exercise, it is described that the computers are numbered from one to six, and that the first two (computer 1 and computer 2) are the laptops. All possible outcomes are

{1,2},{1,3},{1,4},{1,5},{1,6},{2,3},{2,4},{2,5},{2,6},{3,4},{3,5},{3,6},{4,5},{4,6},{5,6},{1,2},{1,3},{1,4},{1,5},{1,6},{2,3},{2,4},{2,5},{2,6},{3,4},{3,5},{3,6},{4,5},{4,6},{5,6},

because we select only two computers out of six, randomly.

All the outcomes have the same probability which is 1/15.

(we have 15 outcomes).

# Contd..

(a) Probability that both selected setups are for laptop computers is the probability that we select computer 1 and computer 2 is $\frac{1}{15}$.

(b) Probability that both selected setups are desktop machines can be calculated in two ways: adding the probabilities of outcomes where we do not have numbers 1 or 2 or subtracting from 1 the probabilities of outcomes that do have numbers 1 or 2.

$$P[\{\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}] = \frac{6}{15}$$

# Contd..

(c)Probability that at least one selected setup is for a desktop computer is complement of event that the both selected computers are Laptops .

$$1 - P(\{\text{both are laptops}\}) \quad = \frac{14}{15}$$

(d) Probability that at least one computer of each type is chosen for setup is harder to calculate than the probability of the complement of the event (usually when there is at least in a sentence you would like to find the complement). The complement is that both computers are same type (desktop or laptop)

$$1 - \frac{1}{15} - \frac{6}{15} = \frac{8}{15}$$

# Example: 9

A committee of 5 is chosen from a group of 8 men and 4 women. What is the probability that the group contains a majority of women?

Solution:

There are totally 12 people. Among them we have to choose 5 such that majority would be women. Hence the possible selection of men and women will be either:

1 men and 4 women or 2 men and 3 women.

$$P(1M\ and\ 4\ W) = \frac{C_1^8 \times C_4^4}{C_5^{12}} = \frac{8}{792}$$

$$P(2M\ and\ 3\ W) = \frac{C_2^8 \times C_3^4}{C_5^{12}} = \frac{112}{792}$$

Therefore the probability of selecting 5 people is $\frac{8}{792} + \frac{112}{792} = \frac{5}{33}$

# Practice problems:

**Q1:** A Survey conducted by a bank revealed that 40% of the accounts are savings accounts and 35% of the accounts are current accounts and the balance are loan accounts.

- What is the probability that an account taken at random is a loan account ? Ans: 0.25
- What is the probability that an account taken at random is **NOT** savings account ? Ans: 0.60
- What is the probability that an account taken at random is **NOT** a current account ? Ans: 0.65
- What is the probability that an account taken at random is a current account or a loan account? Ans: 0.60

# Practice problems:

**Q2**. A speaks truth in 80% cases and B speaks in 60% cases. What percentage of cases are they likely to contradict each other in stating the same fact.

Hint / Ans:

*P*(Contradiction)=P(A truth and B false)+P(A false and B truth)=0.32+0.12=0.44

**Q3**. In a certain residential suburb, 60% of all households get Internet service from the local cable company, 80% get television service from that company, and 50% get both services from that company. If a household is randomly selected,

i) What is the probability that it gets at least one of these two services from the company?

*Ans: P*(Internet or Television)=0.90

ii) What is the probability that it gets exactly one of these services from the company?

Ans:

*P*(Exactly one service)=*P*(Internet but not Television)+*P*(Television but not Internet)=0.10+0.30=0.40

# Practice problems:

**Q4:** The next generation of miniaturised wireless capsules with active locomotion will require two miniature electric motors to manoeuvre each capsule. Suppose 10 motors have been fabricated but that, in spite of test performed on the individual motors 2 will not operate satisfactorily when placed into capsule, to fabricate a new capsule,2 motors will be randomly selected(that is, each pair of motors has the same chance of being selected ) find the probability that

a) Both motors will operate satisfactorily in the capsule.
b) One motor will operate satisfactorily and other will not.

# Practice problems:

**Q5.** Suppose that 55% of all adults regularly consume coffee, 45% regularly consume carbonated soda, and 70% regularly consume at least one of these two products.

**i)** What is the probability that a randomly selected adult regularly consumes both coffee and soda?

**ii)** What is the probability that a randomly selected adult doesn't regularly consume at least one of these two products?

# Practice problems:

**Q6.** Suppose a student is selected at random from 80 students where 30 are taking mathematics, 20 are taking chemistry and 10 are taking both. Find the probability 'p' that the student is taking Mathematics or chemistry?.

**Q 7:** If A and B are events with $P(A \cup B) = 7/8$,

$P(A \cap B) = 1/4$ and $P(A') = 5/8$, find $P(A)$, $P(B)$ and $P(A \cap B')$.

# Practice problems:

**Q8.** In a Sample space, events A and B are such that P(A)=P(B),

$P(\bar{A} \cap \bar{B}) = P(A \cap B) = \frac{1}{6}. \, Find$

$a) \; P(A),$

$b) P(\bar{A} \cup \bar{B})$

$c) \; P(exactly \; one \; of \; the \; events \; A \; or \; B)$

**Q9.** An insurance company offers four different deductible levels—none, low, medium, and high—for its homeowner's policyholders and three different levels—low, medium, and high—for its automobile policyholders. The accompanying table gives proportions for the various categories of policyholders who have both types of insurance.

# Practice problems:

For example, the proportion of individuals with both low homeowner's deductible and low auto deductible is .06 (6% of all such individuals.

**Homeowner's**

| Auto | N | L | M | H |
| --- | --- | --- | --- | --- |
| L | 0.04 | 0.06 | 0.05 | 0.03 |
| M | 0.07 | 0.10 | 0.20 | 0.10 |
| H | 0.02 | 0.03 | 0.15 | 0.1 |

Suppose an individual having both types of policies is randomly selected.

**a.** What is the probability that the individual has a medium auto deductible and a high homeowner's deductible?

**b.** What is the probability that the individual has a low auto deductible? A low homeowner's deductible?

**c.** What is the probability that the individual is in the same category for both auto and homeowner's deductibles?

**d.** Based on your answer in part (c), what is the probability that the two categories are different?

**e.** What is the probability that the individual has at least one low deductible level ?

**f.** Using the answer in part (e), what is the probability that neither deductible level is low ?

# Glossary

- **Random experiment-** an experiment whose outcome is uncertain before performing it.
- **Sample space**-set of all possible outcomes of an experiment.
- **Event**- outcome of an experiment.
- **Equally Likely Outcomes –** Outcomes which are having same probability
- **Mutually exclusive events**-events that do not occur at the same time.
- **Independent events**- the events which occur freely of each other.
- **Operations on events**
  - Union of events and its probability
  - Intersection of events and its probability
  - Complement of an event and its probability

# IMP Note to Self

# Thank you