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# Introduction to Statistical Methods

ISM Team

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# Session 14

# Time Series/ Forecasting

# (8th / 9th March 2025)

# IMP Note to Self

# Time Series

- Stationarity implies that the joint distribution of pair of observations $y_{t_1}$, $y_{t_2}$ viz.,

$$f(y_{t_1}, y_{t_2}) = f(y_{t_1+h}, y_{t_2+h})$$

for any integer h. That is, the joint distribution of any pair of observations on time points which differ by a constant quantity is the same for such pairs.

Thus

$$f(y_t, y_{t+h}) = f(y_{t_1}, y_{t_1+h}) = f(y_{t_2}, y_{t_2+h})$$

have the same distribution.

The form of $f(y_t, y_{t+h})$ can be inferred by plotting values of $(y_t, y_{t+h})$; $t = 1, 2, \ldots,$ i.e., values of $y_t$ separated by lag h.

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## Time Series ⇨ Autocovariance

- Let **$\{Y_t\}$** be a Time Series with E $\{Y_t^2\} < \infty$. The mean function of **$\{Y_t\}$** is $\mu_y(t) = E(Y_t)$.
- The covariance function of **$\{Y_t, Y_{t+h}\}$** is
- $\gamma_h = \mathrm{Cov}\,(Y_t, Y_{t+h}) = E[(Y_t - \mu)(Y_{t+h} - \mu)]$
- In general

$$\gamma_y(r, s) = \mathrm{Cov}\,(Y_r, Y_s) = E[\{Y_r - \mu_y(r)\}\{Y_s - \mu_y(s)\}]$$

for all integers r and s

## Time Series ➡ Autocovariance

- Clearly, with h=0, in $\gamma_h = \text{Cov}(Y_t, Y_{t+h}) = E[(Y_t-\mu)(Y_{t+h}-\mu)]$

$$\gamma_0 = \text{Cov}(Y_t, Y_{t+0}) = E[(Y_t-\mu)(Y_{t+0}-\mu)] = \sigma^2_y$$

$$|\gamma_h| \leq \gamma_0, \forall\, h = 1, 2, \ldots$$

because, $|Cov(Y_{t+h}, Y_t)| \leq \sqrt{V(Y_{t+h})V(Y_t)}$

## Time Series ➔ Autocovariance function

- Let $\{Y_t\}$ be a stationery Time Series.

- The Autocovariance Function (ACVF) of $\{Y_t\}$ at lag h is

$$\gamma_y(t+h, t) = \mathrm{Cov}\,(Y_{t+h}, Y_t).$$

The Autocorrelation Function (ACF) of $\{Y_t\}$ at lag h is

$$\rho_h = \frac{\gamma_h}{\gamma_0} = \frac{\mathrm{Cov}(Y_{t+h}, Y_t)}{\sigma^2}, \quad \boldsymbol{\rho_0 = 1}$$

## Time Series ⇨ Autocorrelation

- Correlation: measure of relationship between two variables are related
  - Direction of the relationship (Negative, Zero, and/ or positive
  - Degree/ Extent of relationship

## Time Series ⇨ Autocorrelation

- The autocorrelation measures the degree of relationship of the same variable between the observation at the current time period and the observations at the prior time periods.
- It measures how the lagged version of the value of a variable is related to the original version of it in time series.

The value of $r_k$ can be written as

$$r_k = \frac{\sum_{t=k+1}^{T} (y_t - \bar{y})(y_{t-k} - \bar{y})}{\sum_{t=1}^{T} (y_t - \bar{y})^2}$$

where T is the length of time series.

## Time Series ⇨ Autocorrelation

- Autocorrelation between ($Y_t$, $Y_{t-1}$)

$$r_1 = \frac{\sum(Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\sum(Y_t - \bar{Y})^2}$$

- Autocorrelation between ($Y_t$, $Y_{t-2}$)

$$r_2 = \frac{\sum(Y_t - \bar{Y})(Y_{t-2} - \bar{Y})}{\sum(Y_t - \bar{Y})^2}$$

– – – –

- Autocorrelation between ($Y_t$, $Y_{t-k}$)

$$r_k = \frac{\sum(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum(Y_t - \bar{Y})^2}$$

- There are several autocorrelation coefficients, corresponding to each panel in the lag plot.

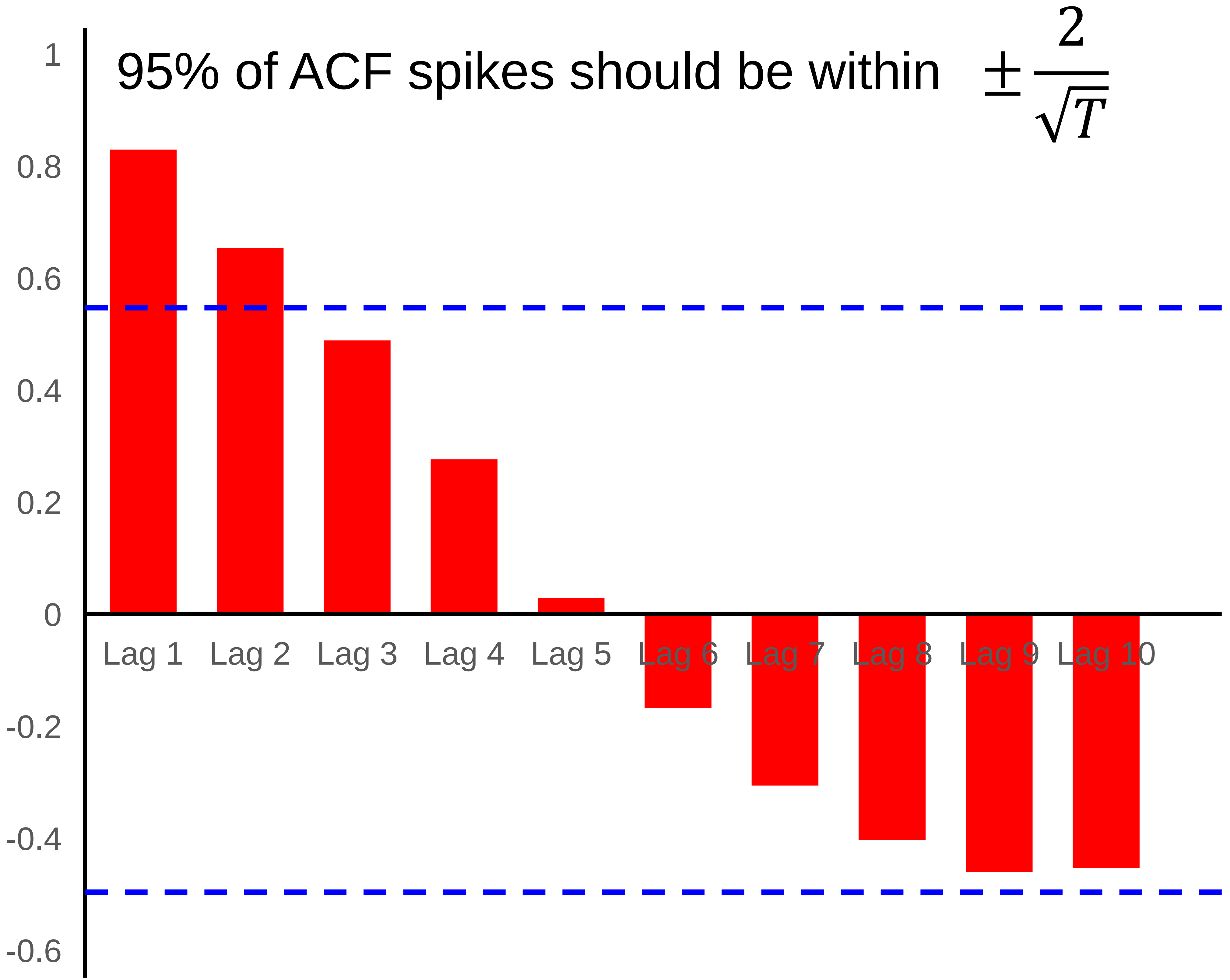
- For example, $r_1$ measures the relationship between $Y_t$ and $Y_{t-1}$, $r_2$ measures the relationship between $Y_t$ and $Y_{t-2}$, and so on..

- rk measures the relationship between $Y_t$ and $Y_{t-k}$.

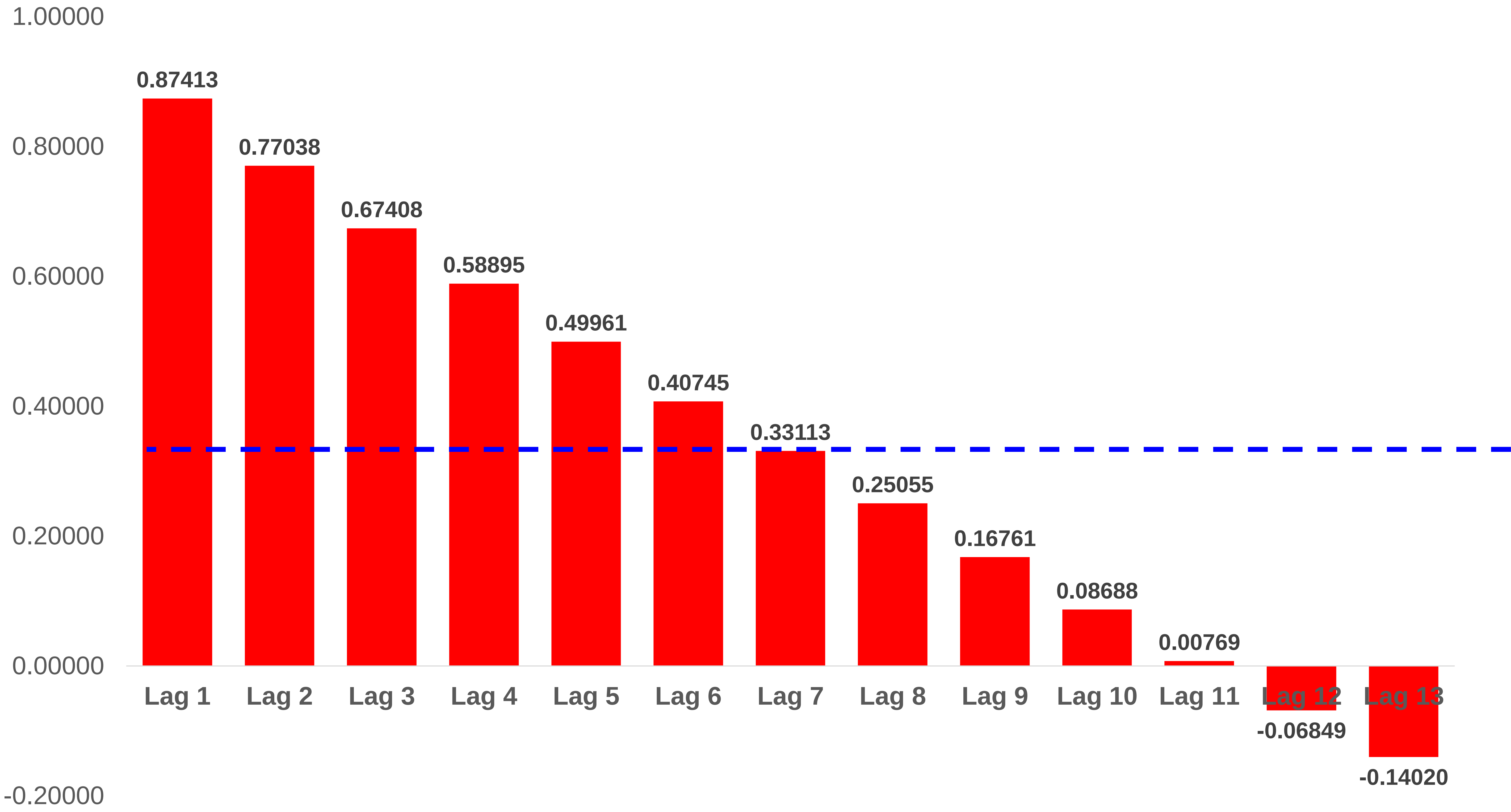
| Year | 2011 | | | | 2012 | | | |
|---|---|---|---|---|---|---|---|---|
| Quarter | Q1 | Q2 | Q3 | Q4 | Q1 | Q2 | Q3 | Q4 |
| Retail quarterly sale (million) ($Y_t$) | 147772 | 154400 | 166188 | 170202 | 173264 | 175371 | 184957 | 186395 |
| Year | 2013 | | | | 2014 | | | |
| Quarter | Q1 | Q2 | Q3 | Q4 | Q1 | Q2 | Q3 | Q4 |
| Retail quarterly sale (million) ($Y_t$) | 191130 | 191213 | 195749 | 198262 | 199980 | 209566 | 212529 | 213754 |
| Year | 2015 | | | | 2016 | | | |
| Quarter | Q1 | Q2 | Q3 | Q4 | Q1 | Q2 | Q3 | Q4 |
| Retail quarterly sale (million) ($Y_t$) | 222124 | 224372 | 229871 | 236260 | 238044 | 244076 | 244944 | 245799 |
| Year | 2017 | | | | 2018 | | | |
| Quarter | Q1 | Q2 | Q3 | Q4 | Q1 | Q2 | | |
| Retail quarterly sale (million) ($Y_t$) | 251125 | 254557 | 255223 | 264793 | 265064 | 277282 | | |

| Time | Value | Lag 1 | Lag 2 | Lag 3 | Lag 4 | Lag 5 | Lag 6 | Lag 7 | Lag 8 | Lag 9 | Lag 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 22 | | | | | | | | | | |
| 2 | 24 | 22 | | | | | | | | | |
| 3 | 25 | 24 | 22 | | | | | | | | |
| 4 | 25 | 25 | 24 | 22 | | | | | | | |
| 5 | 28 | 25 | 25 | 24 | 22 | | | | | | |
| 6 | 29 | 28 | 25 | 25 | 24 | 22 | | | | | |
| 7 | 34 | 29 | 28 | 25 | 25 | 24 | 22 | | | | |
| 8 | 37 | 34 | 29 | 28 | 25 | 25 | 24 | 22 | | | |
| 9 | 40 | 37 | 34 | 29 | 28 | 25 | 25 | 24 | 22 | | |
| 10 | 44 | 40 | 37 | 34 | 29 | 28 | 25 | 25 | 24 | 22 | |
| 11 | 51 | 44 | 40 | 37 | 34 | 29 | 28 | 25 | 25 | 24 | 22 |
| 12 | 48 | 51 | 44 | 40 | 37 | 34 | 29 | 28 | 25 | 25 | 24 |
| 13 | 47 | 48 | 51 | 44 | 40 | 37 | 34 | 29 | 28 | 25 | 25 |
| 14 | 50 | 47 | 48 | 51 | 44 | 40 | 37 | 34 | 29 | 28 | 25 |
| 15 | 51 | 50 | 47 | 48 | 51 | 44 | 40 | 37 | 34 | 29 | 28 |

| | Autocorrelation |
|---|---|
| Lag 1 | 0.83174 |
| Lag 2 | 0.65632 |
| Lag 3 | 0.49105 |
| Lag 4 | 0.27864 |
| Lag 5 | 0.03103 |
| Lag 6 | -0.1653 |
| Lag 7 | -0.3037 |
| Lag 8 | -0.401 |
| Lag 9 | -0.4582 |
| Lag 10 | -0.4505 |

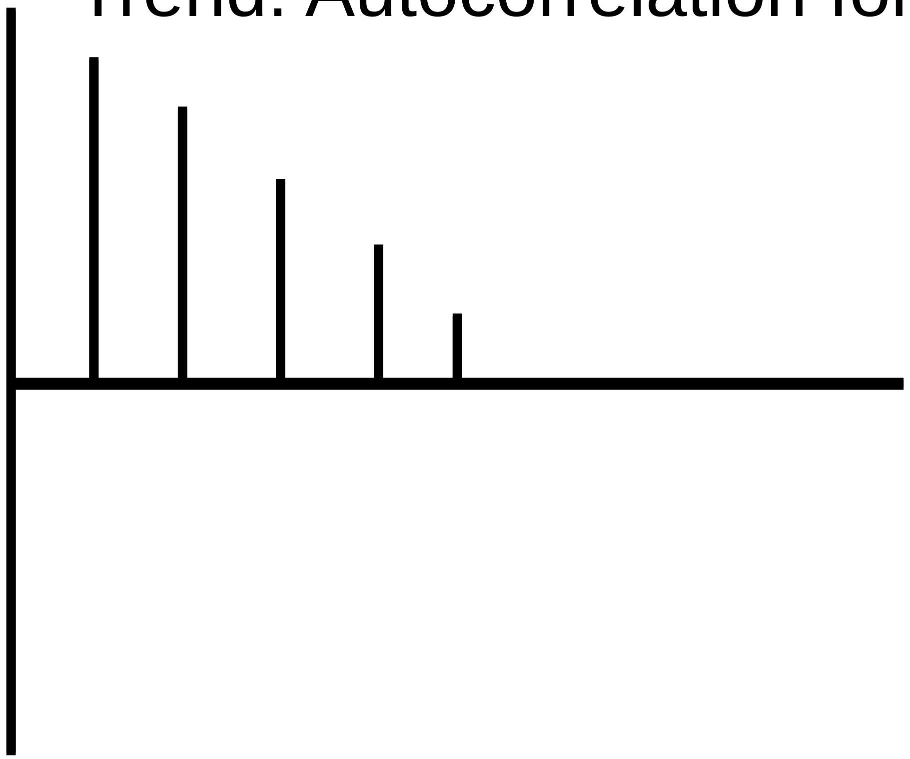


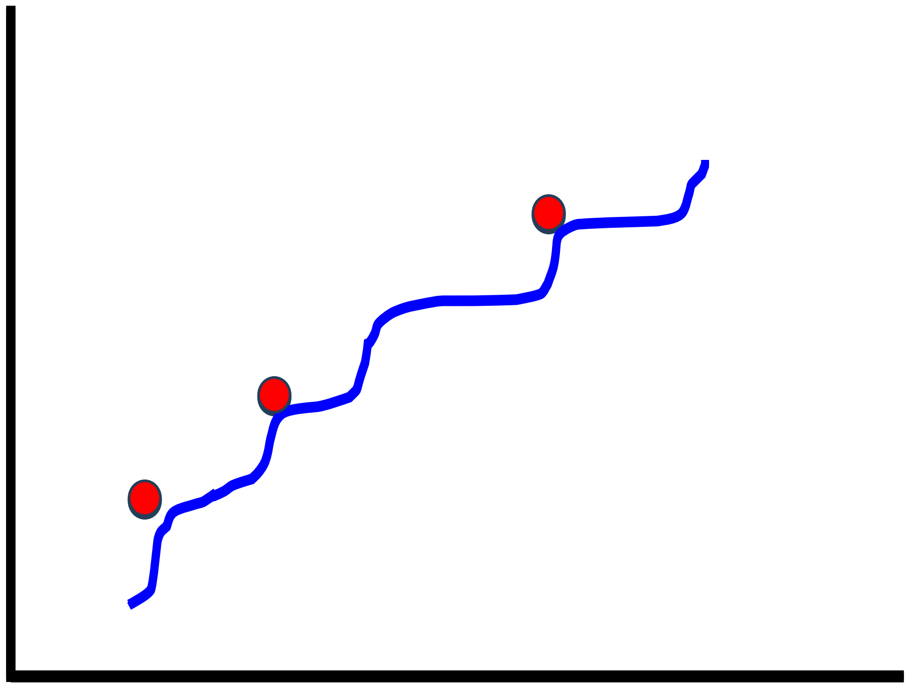
| Month/ Year | Retail quarterly sale (million) | Lag1 | Lag2 | Lag3 | Lag4 | Lag5 | Lag6 |
|---|---|---|---|---|---|---|---|
| Q1 - 2011 | 147772 | | | | | | |
| Q2 - 2011 | 154400 | 147772 | | | | | |
| Q3 - 2011 | 166188 | 154400 | 147772 | | | | |
| Q4 - 2011 | 170202 | 166188 | 154400 | 147772 | | | |
| Q1 - 2012 | 173264 | 170202 | 166188 | 154400 | 147772 | | |
| Q2 - 2012 | 175371 | 173264 | 170202 | 166188 | 154400 | 147772 | |
| Q3 - 2012 | 184957 | 175371 | 173264 | 170202 | 166188 | 154400 | 147772 |
| Q4 - 2012 | 186395 | 184957 | 175371 | 173264 | 170202 | 166188 | 154400 |
| Q1 - 2013 | 191130 | 186395 | 184957 | 175371 | 173264 | 170202 | 166188 |
| Q2 - 2013 | 191213 | 191130 | 186395 | 184957 | 175371 | 173264 | 170202 |
| Q3 - 2013 | 195749 | 191213 | 191130 | 186395 | 184957 | 175371 | 173264 |
| Q4 - 2013 | 198262 | 195749 | 191213 | 191130 | 186395 | 184957 | 175371 |
| Q1 - 2014 | 199980 | 198262 | 195749 | 191213 | 191130 | 186395 | 184957 |
| Q2 - 2014 | 209566 | 199980 | 198262 | 195749 | 191213 | 191130 | 186395 |
| Q3 - 2014 | 212529 | 209566 | 199980 | 198262 | 195749 | 191213 | 191130 |
| Q4 - 2014 | 213754 | 212529 | 209566 | 199980 | 198262 | 195749 | 191213 |
| Q1 - 2015 | 222124 | 213754 | 212529 | 209566 | 199980 | 198262 | 195749 |
| Q2 - 2015 | 224372 | 222124 | 213754 | 212529 | 209566 | 199980 | 198262 |
| Q3 - 2015 | 229871 | 224372 | 222124 | 213754 | 212529 | 209566 | 199980 |
| Q4 - 2015 | 236260 | 229871 | 224372 | 222124 | 213754 | 212529 | 209566 |
| Q1 - 2016 | 238044 | 236260 | 229871 | 224372 | 222124 | 213754 | 212529 |
| Q2 - 2016 | 244076 | 238044 | 236260 | 229871 | 224372 | 222124 | 213754 |
| Q3 - 2016 | 244944 | 244076 | 238044 | 236260 | 229871 | 224372 | 222124 |
| Q4 - 2016 | 245799 | 244944 | 244076 | 238044 | 236260 | 229871 | 224372 |
| Q1 - 2017 | 251125 | 245799 | 244944 | 244076 | 238044 | 236260 | 229871 |
| Q2 - 2017 | 254557 | 251125 | 245799 | 244944 | 244076 | 238044 | 236260 |
| Q3 - 2017 | 255223 | 254557 | 251125 | 245799 | 244944 | 244076 | 238044 |
| Q4 - 2017 | 264793 | 255223 | 254557 | 251125 | 245799 | 244944 | 244076 |
| Q1 - 2018 | 265064 | 264793 | 255223 | 254557 | 251125 | 245799 | 244944 |
| Q2 - 2018 | 277282 | 265064 | 264793 | 255223 | 254557 | 251125 | 245799 |



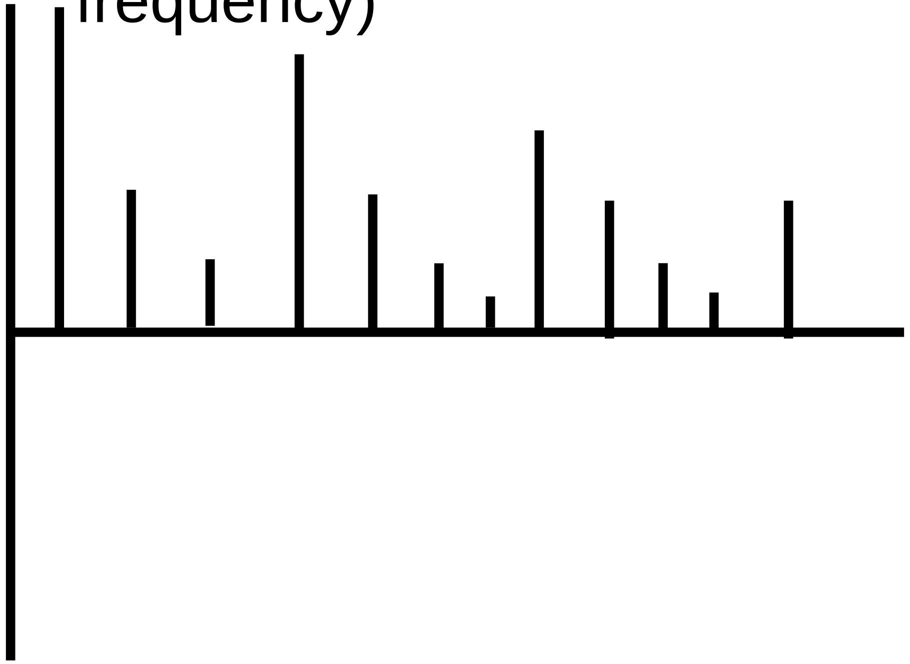
1.00000
0.80000
0.60000
0.40000
0.20000
0.00000
-0.20000
0.87413
0.77038
0.67408
0.58895
0.49961
0.40745
0.33113
0.25055
0.16761
0.08688
0.00769
-0.06849
-0.14020
Lag 1
Lag 2
Lag 3
Lag 4
Lag 5
Lag 6
Lag 7
Lag 8
Lag 9
Lag 10
Lag 11
Lag 12
Lag 13

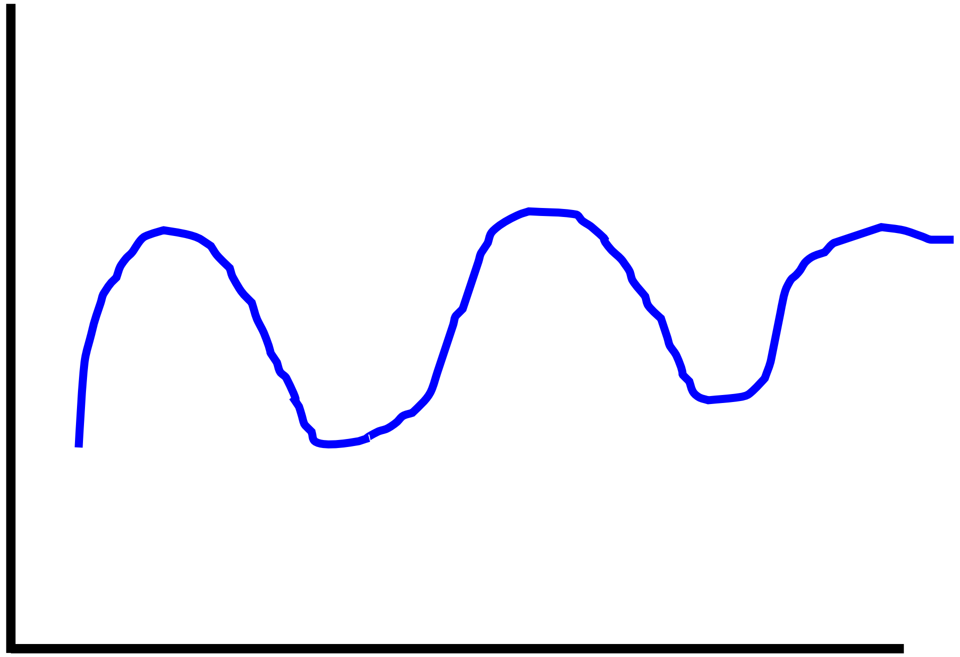
- Trend: Autocorrelation for small lags tend to be large and positive



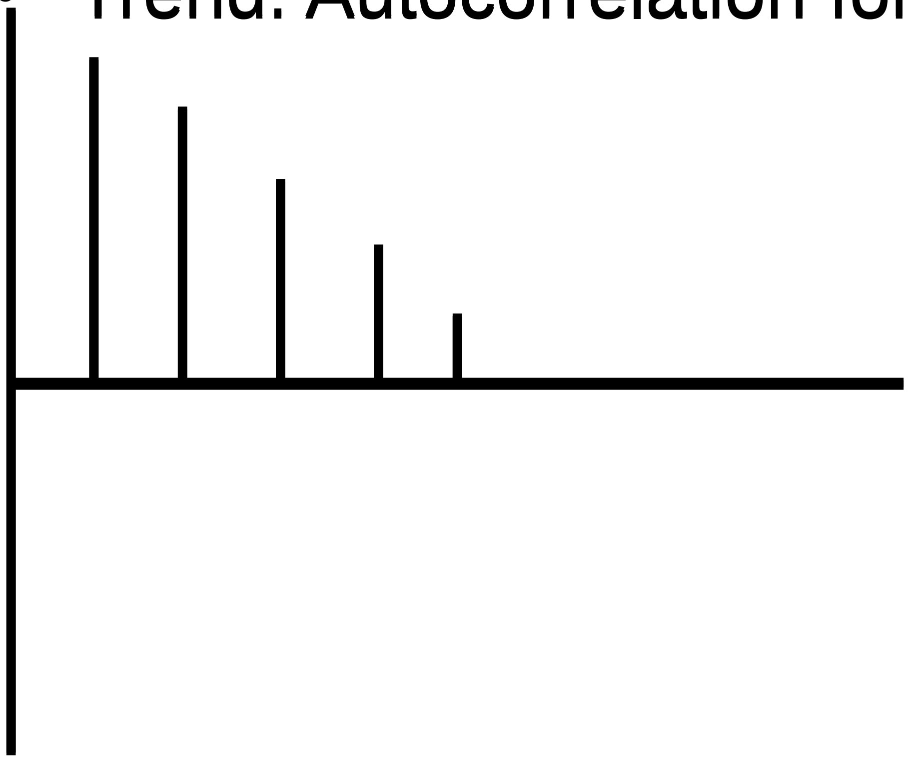


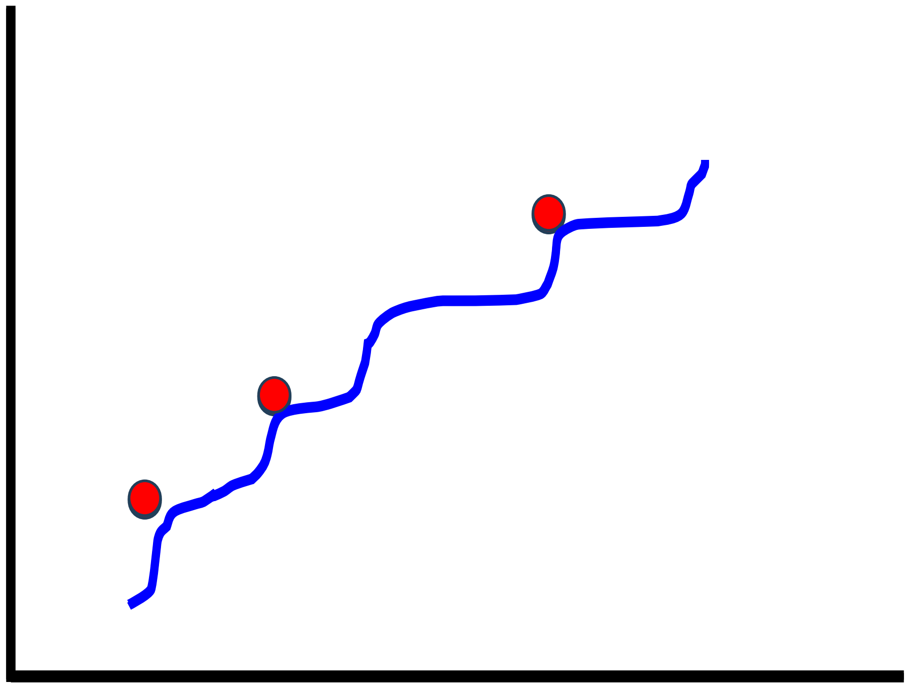
- Seasonal: Autocorrelation will be larger for smaller seasonal lags (at multiple seasonal frequency)



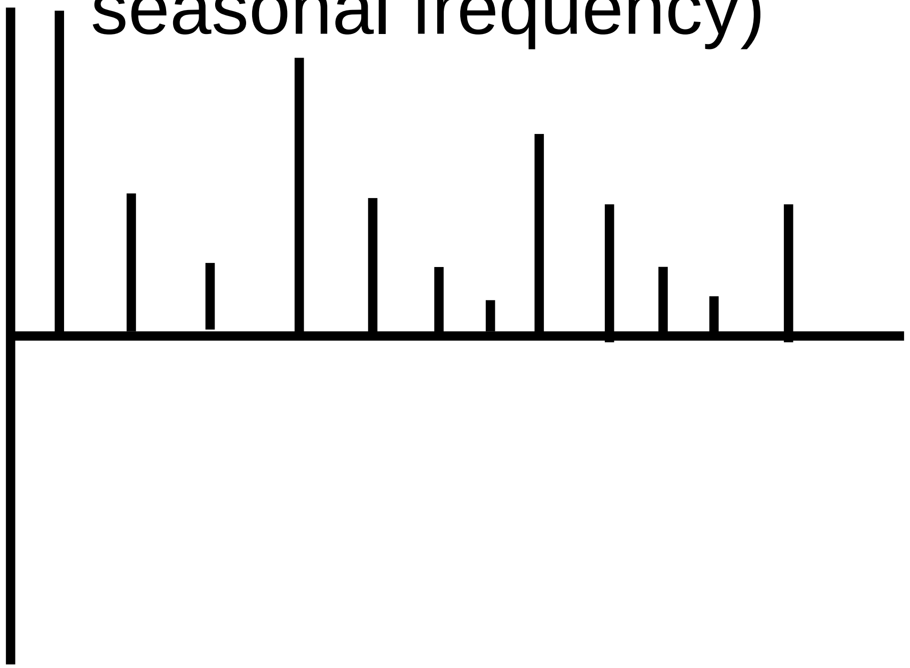


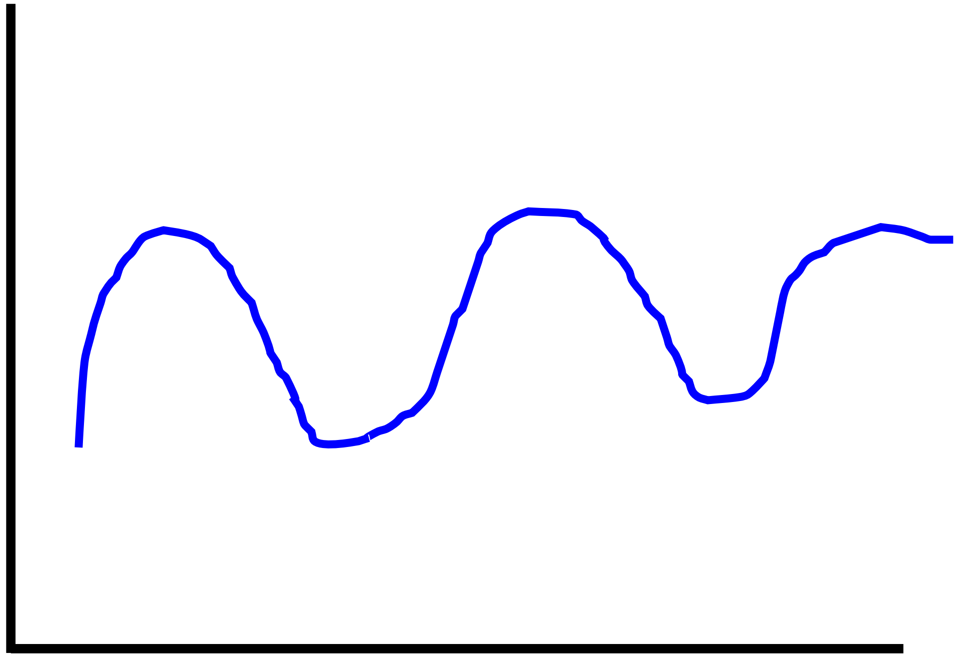
- Trend: Autocorrelation for small lags tend to be large and positive



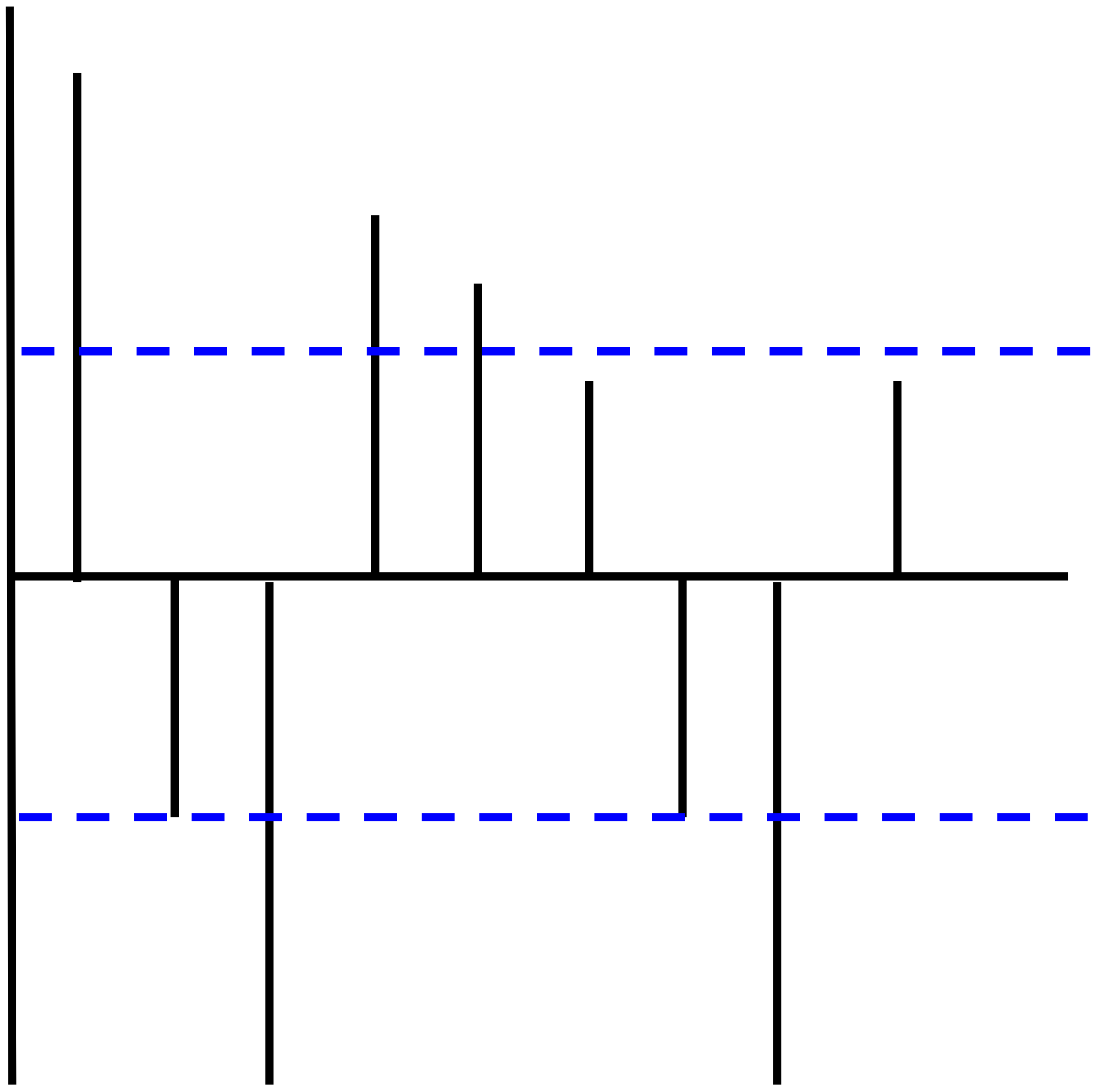


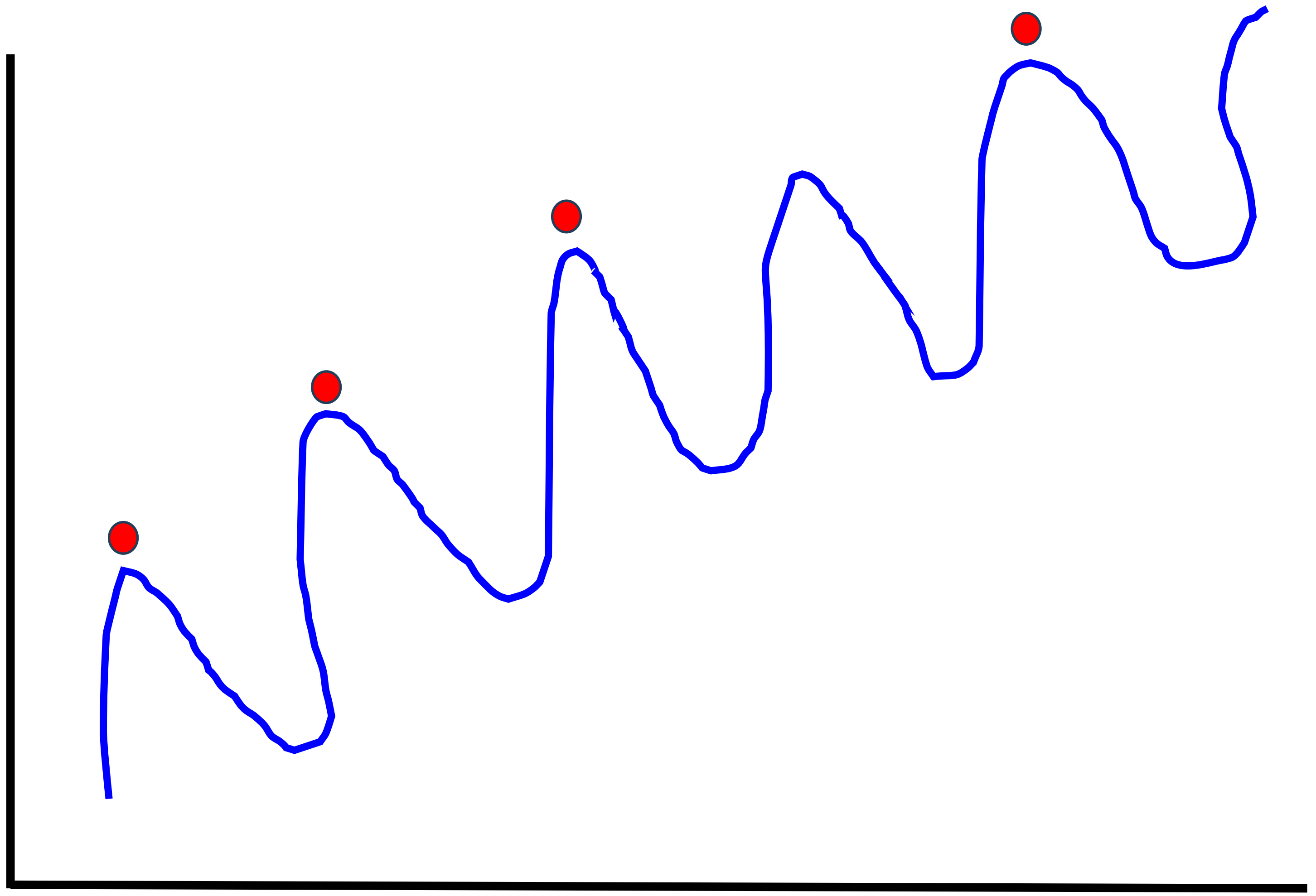
- Seasonal: Autocorrelation will be larger for smaller seasonal lags (at multiple seasonal frequency)





# Seasonal series





## Time Series → White Noise

Time series that show no autocorrelation are called **white noise.** That is, if the variables are independent and identically distributed with a mean of zero. This means that all variables have the same variance ($\sigma^2$) and each value has a zero correlation with all other values in the series.

# Time Series ⇒ White Noise

A white noise called **white** when it has the same intensity at every frequency. Its name is derived by analogy to light, which is called "**white**" when it contains all visible frequencies **noise.**

The term **noise**, in this context, came from signal processing where it was used to refer to unwanted electrical or electromagnetic energy that degrades the quality of signals and data

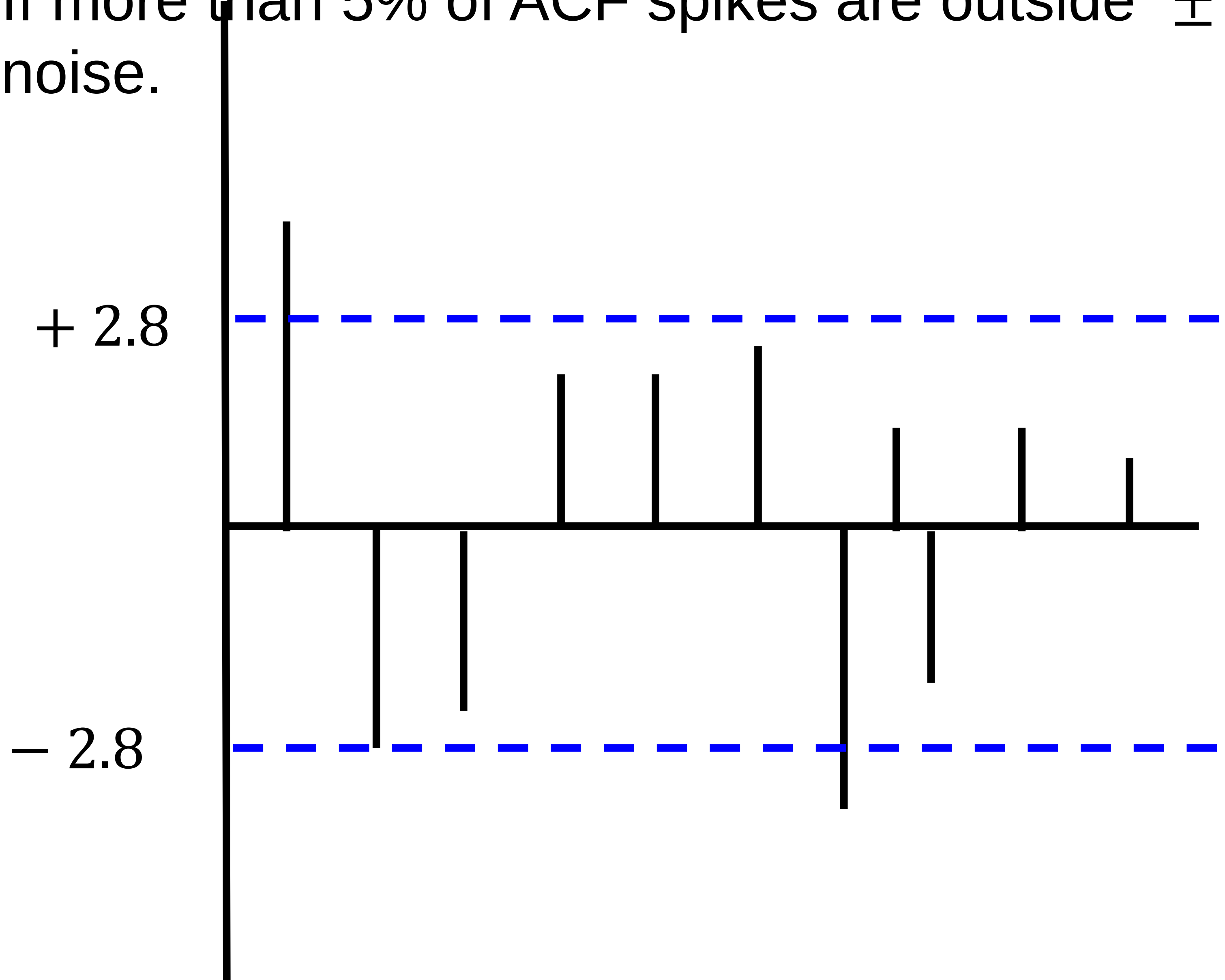
## Time Series ⇨ White Noise

For white noise series, we expect each autocorrelation to be close to zero. Of course, they will not be exactly equal to zero as there is some random variation. For a white noise series, we expect 95% of the spikes in the ACF to lie within $\pm 2/\sqrt{T}$ where T is the length of the time series.

## Time Series ⇨ White Noise

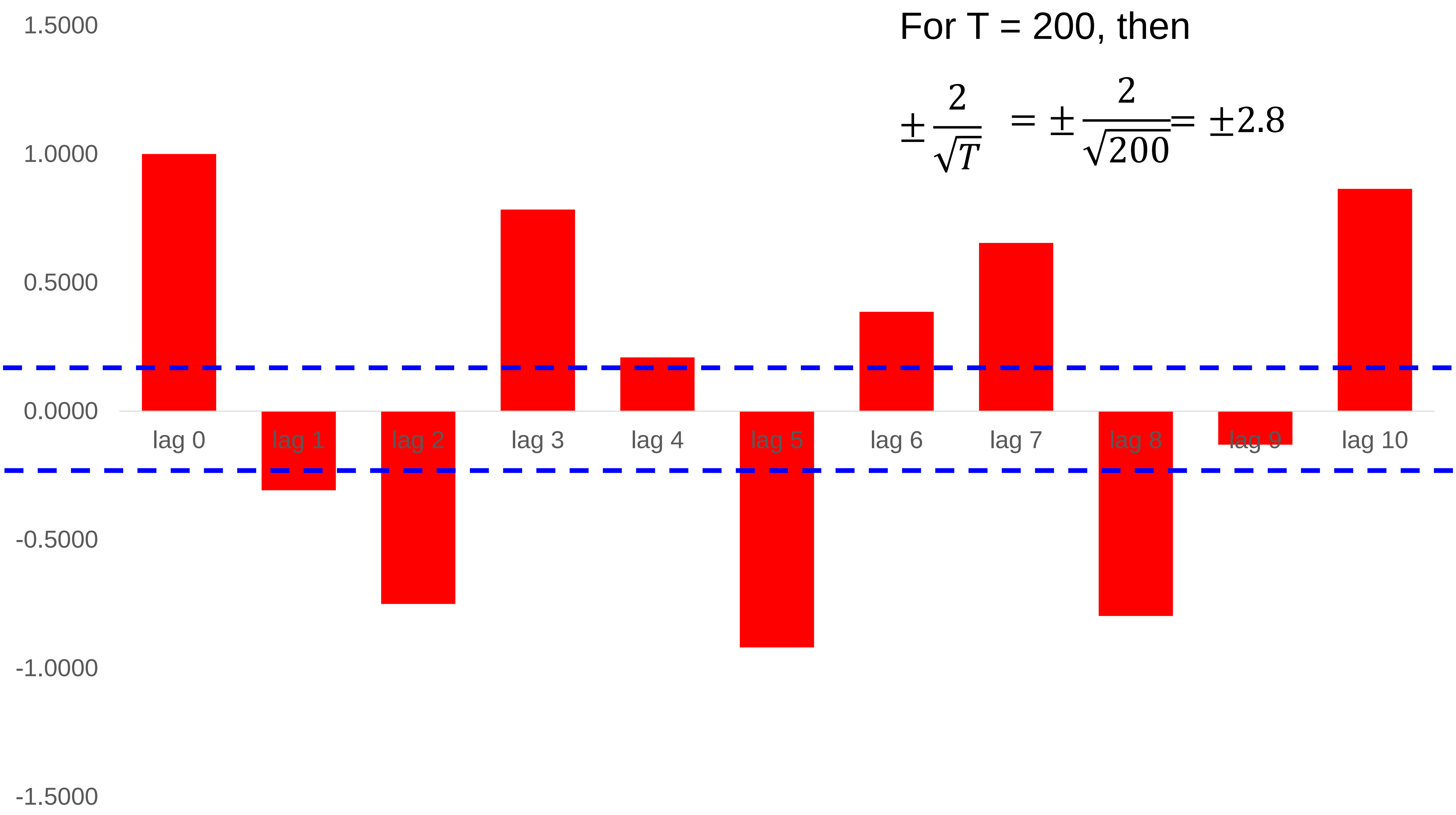
It is common to plot these bounds on a graph of the ACF (the blue dashed lines above). If one or more large spikes are outside these bounds, or if substantially more than 5% of spikes are outside these bounds, then the series is probably not white noise

- White Noise series
- Time series that shows no autocorrelation
- 95% of ACF spikes should be within $\pm \frac{2}{\sqrt{T}}$
- If more than 5% of ACF spikes are outside $\pm \frac{2}{\sqrt{T}}$, then the series is not white noise.

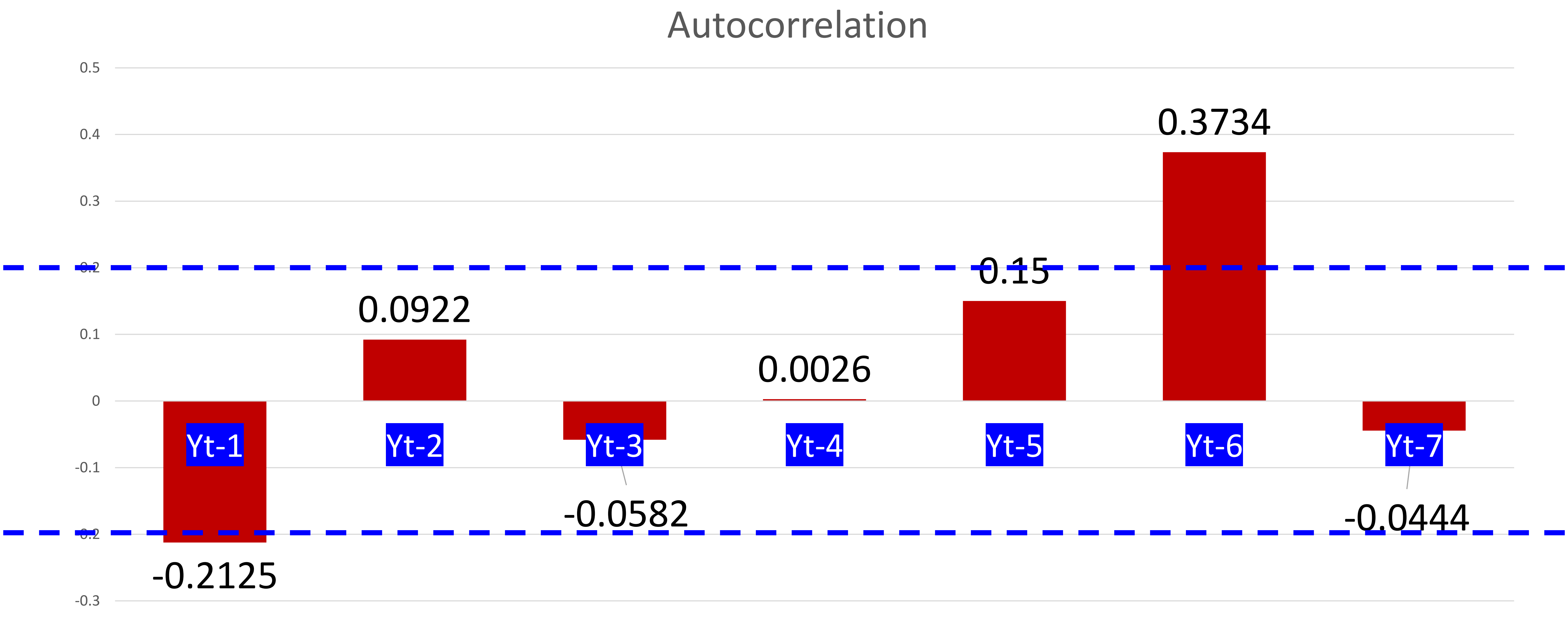



Example: If T = 50, then

$$\pm \frac{2}{\sqrt{T}} = \pm \frac{2}{\sqrt{50}} = \pm 2.8$$


For T = 200, then
$\pm\frac{2}{\sqrt{T}} = \pm\frac{2}{\sqrt{200}} = \pm 2.8$
1.5000
1.0000
0.5000
0.0000
-0.5000
-1.0000
-1.5000
lag 0
lag 1
lag 2
lag 3
lag 4
lag 5
lag 6
lag 7
lag 8
lag 9
lag 10

## Time Series ➡ Autocorrelation



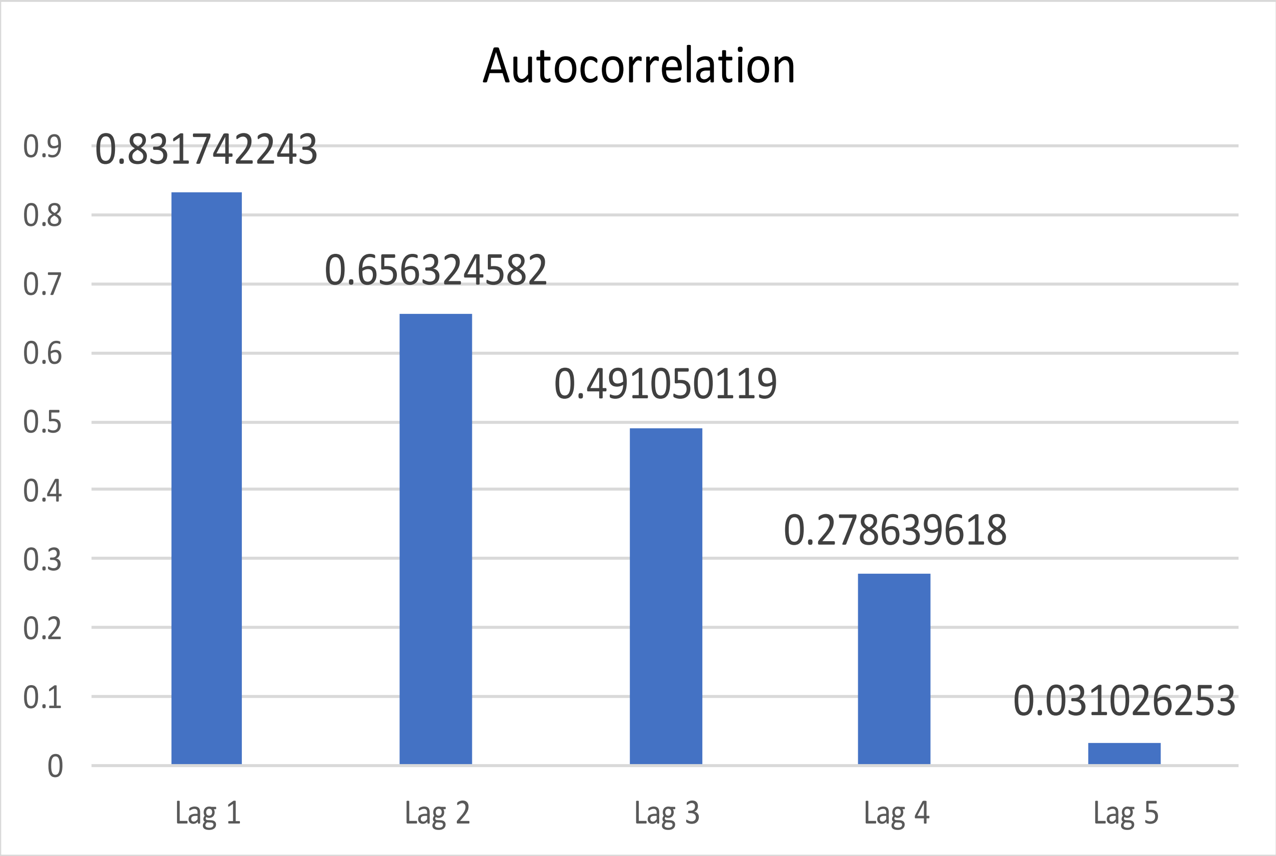
## Time Series ⇨ Test for Autocorrelation: Ljung-Box test

- A statistical test of whether any group of autocorrelation of a time series are different from zero
- A test for the overall randomness based on a number of lags
- **$H_0$:** The series is random or white noise or independent and identically distributed (iid)
- **$H_1$:** The series exhibits serial/ autocorrelation (non-random)
- A small P-value ($P < 0.05$) ⇨ Reject $H_0$ ⇨ the series not white noise
- A large P-value ($P > 0.05$) ⇨ Fail to Reject $H_0$ ⇨ the series white noise
- **Note: Normally, P-value will be very large for white noise series**

# Time Series → Autocorrelation

| Time | Value | lag 1 | lag 2 | lag 3 | lag 4 | lag 5 |
|---|---|---|---|---|---|---|
| 1 | 22 | | | | | |
| 2 | 24 | 22 | | | | |
| 3 | 25 | 24 | 22 | | | |
| 4 | 25 | 25 | 24 | 22 | | |
| 5 | 28 | 25 | 25 | 24 | 22 | |
| 6 | 29 | 28 | 25 | 25 | 24 | 22 |
| 7 | 34 | 29 | 28 | 25 | 25 | 24 |
| 8 | 37 | 34 | 29 | 28 | 25 | 25 |
| 9 | 40 | 37 | 34 | 29 | 28 | 25 |
| 10 | 44 | 40 | 37 | 34 | 29 | 28 |
| 11 | 51 | 44 | 40 | 37 | 34 | 29 |
| 12 | 48 | 51 | 44 | 40 | 37 | 34 |
| 13 | 47 | 48 | 51 | 44 | 40 | 37 |
| 14 | 50 | 47 | 48 | 51 | 44 | 40 |
| 15 | 51 | 50 | 47 | 48 | 51 | 44 |

| | |
|---|---|
| Lag 1 | 0.8317 |
| Lag 2 | 0.6563 |
| Lag 3 | 0.4911 |
| Lag 4 | 0.2786 |
| Lag 5 | 0.0310 |



# Time Series ⇨ Autocorrelation Function

=(SUMPRODUCT((B2:B13)-AVERAGE(B2:B16), B5:B16-AVERAGE(B2:B16))/COUNT(B2:B16))/VAR.P(B2:B16)

# Time Series ⇨ Partial Autocorrelation Function (PACF)

- The correlation between the observation at two time points given that we consider both observations are correlated to observations at other time periods.
  - Example: Today's stock price can be correlated to the day before yesterday and yesterday can also be correlated to the day before yesterday. Then PACF of yesterday is the "real" correlation between today and yesterday after taking out the influence of the day before yesterday.

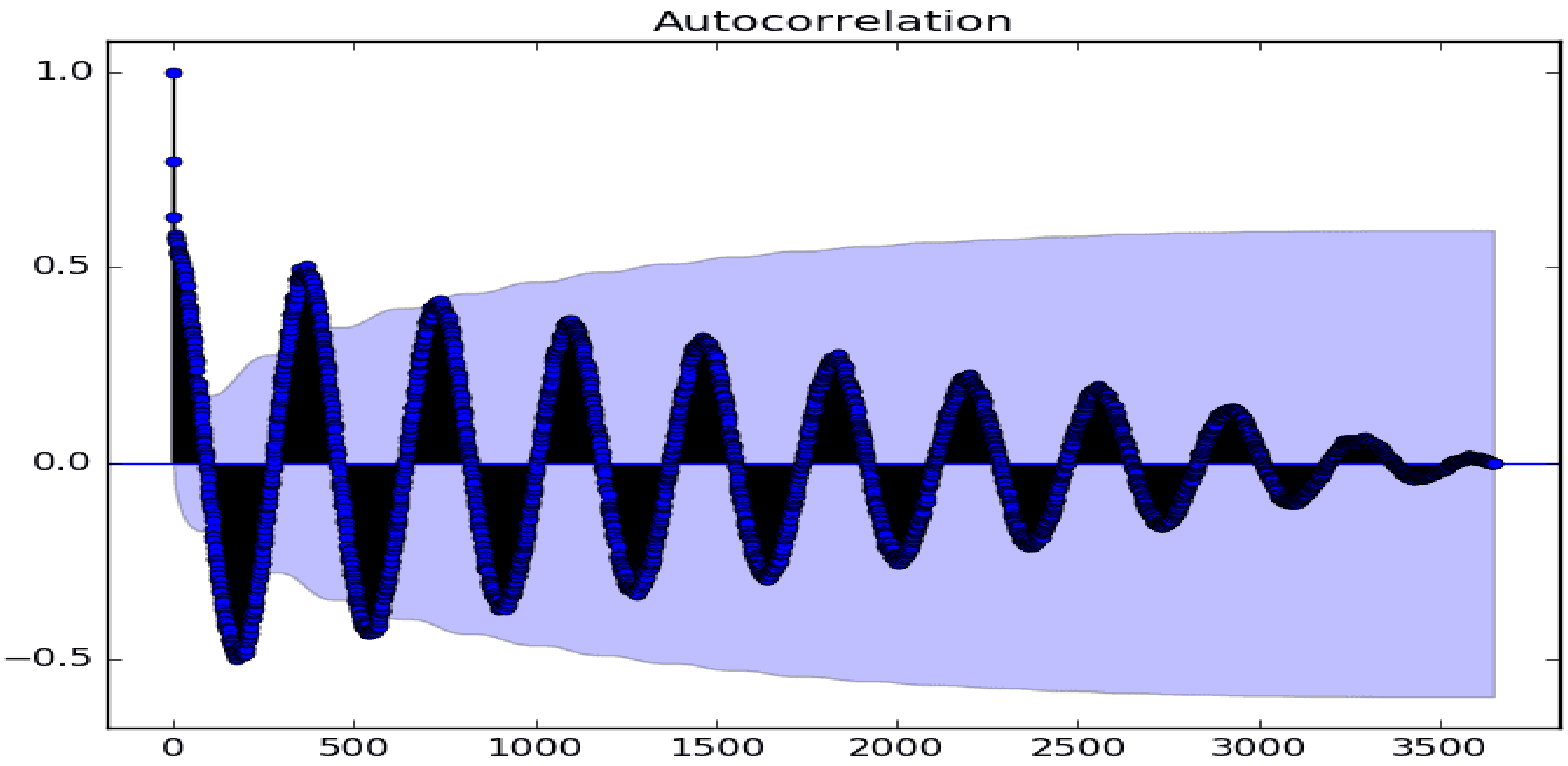
## Time Series ⇨ Partial – Autocorrelation Function (PACF)

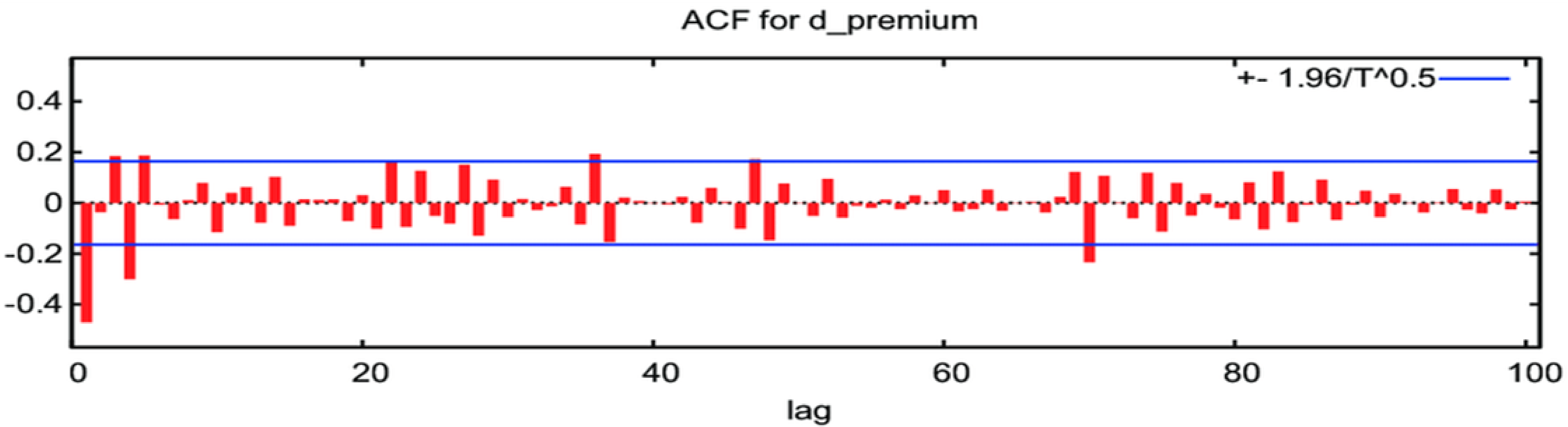
- The partial correlation between two variables is a conditional correlation taking into account their dependence on all other remaining variables
  - Eg. A third order (lag) partial autocorrelation is

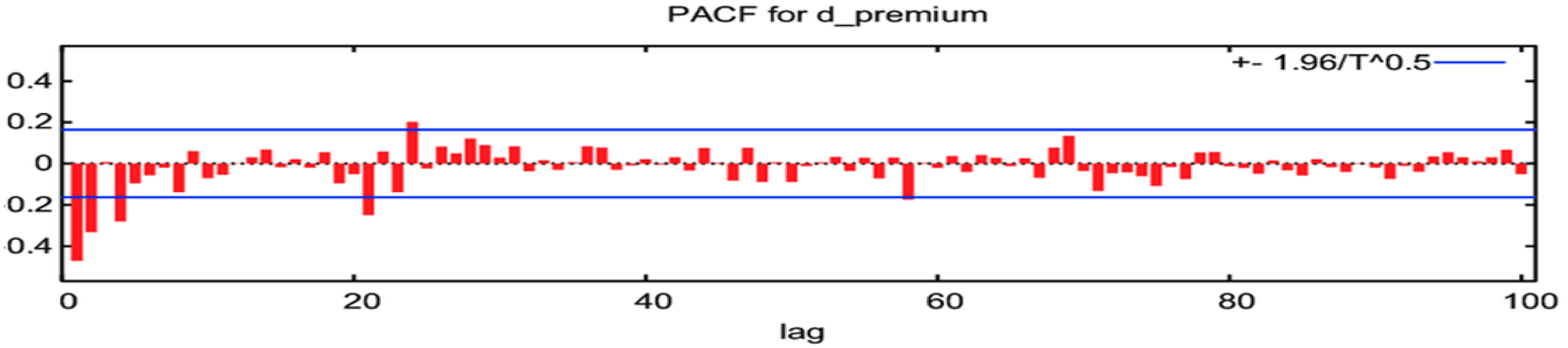
$$\frac{Cov(X_t, X_{t-3} | X_{t-1}, X_{t-2})}{\sqrt{Var(X_t | X_{t-1}, X_{t-2}) Var(X_{t-3} | X_{t-1}, X_{t-2})}}$$

- The first order PACF & ACF are same.

## Time Series ➡ Correlogram

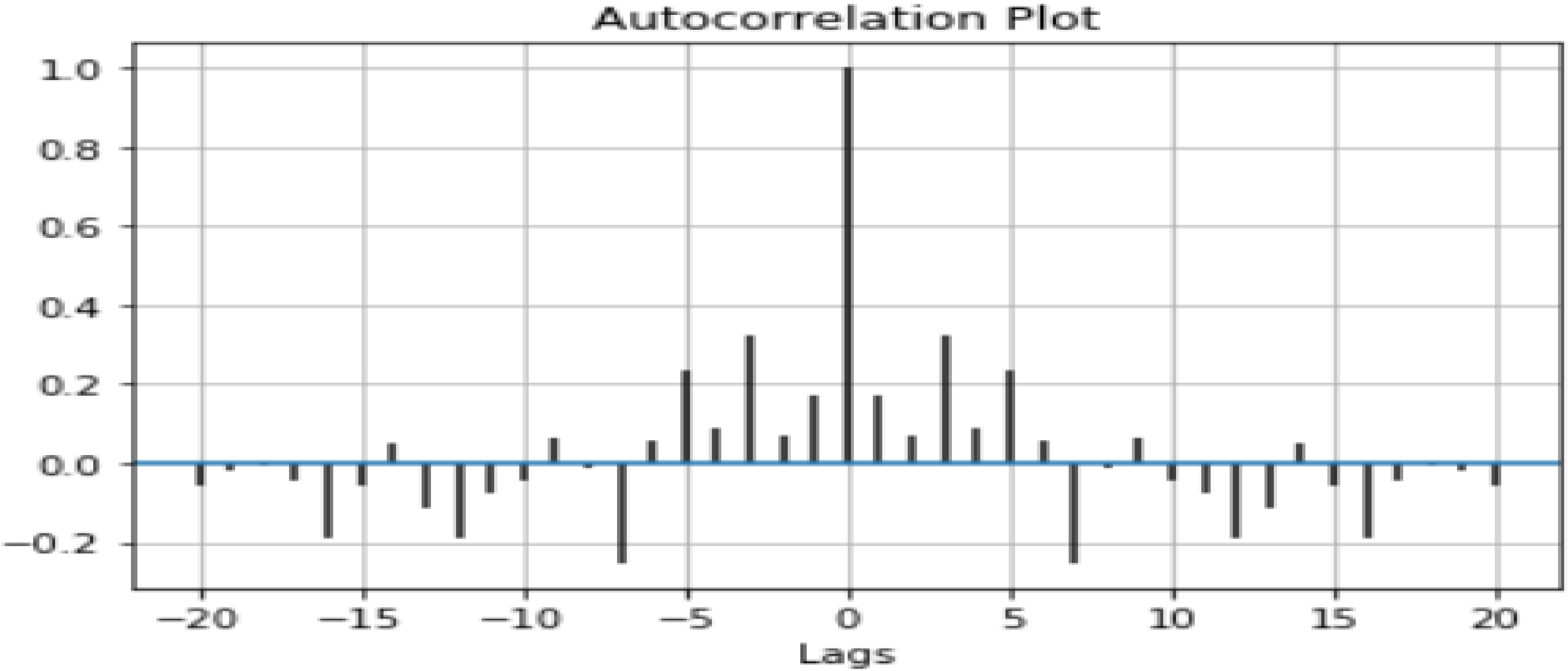



ACF for d_premium
+- 1.96/T^0.5
0.4
0.2
0
-0.2
-0.4
0
20
40
60
80
100
lag


PACF for d_premium
+- 1.96/T^0.5
0.4
0.2
0
0
20
40
60
80
100
lag

# Time Series ➡ Correlogram

The Autocorreleation plot for the data is:



## Time Series ➡ Autoregressive (AR) models

- An **autoregressive (AR) model** is a representation of some type of random process; used to describe certain time-varying processes. The autoregressive model specifies that the output variable depends linearly on its own previous values and on a stochastic (random) error term; thus the model is in the form of a stochastic difference equation.

- The Autoregressive model of order p denoted by AR (p) is given by

$$Y_t = c_t + \emptyset_1 Y_{t-1} + \emptyset_2 Y_{t-2} + \cdots + \emptyset_p Y_{t-p} + \varepsilon_t$$

where $\phi_1$, $\phi_2$, $\phi_3$, … $\phi_p$ are the parameter of the model; c is constant and $\varepsilon_t$ is the **white noise**.

| Time | Deflection | lag 1 | lag 2 | lag 3 | lag 4 | lag 5 | lag 6 | lag 7 | lag 8 | lag 9 | lag 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.2001 | -213 | | | | | | | | | | |
| 2.2001 | -564 | -213 | | | | | | | | | |
| 3.2001 | -35 | -564 | -213 | | | | | | | | |
| 4.2001 | -15 | -35 | -564 | -213 | | | | | | | |
| 5.2001 | 141 | -15 | -35 | -564 | -213 | | | | | | |
| 6.2001 | 115 | 141 | -15 | -35 | -564 | -213 | | | | | |
| 7.2001 | -420 | 115 | 141 | -15 | -35 | -564 | -213 | | | | |
| 8.2001 | -360 | -420 | 115 | 141 | -15 | -35 | -564 | -213 | | | |
| 9.2001 | 203 | -360 | -420 | 115 | 141 | -15 | -35 | -564 | -213 | | |
| 10.2001 | -338 | 203 | -360 | -420 | 115 | 141 | -15 | -35 | -564 | -213 | |
| 11.2001 | -431 | -338 | 203 | -360 | -420 | 115 | 141 | -15 | -35 | -564 | -213 |
| 12.2001 | 194 | -431 | -338 | 203 | -360 | -420 | 115 | 141 | -15 | -35 | -564 |
| 1.2002 | -220 | 194 | -431 | -338 | 203 | -360 | -420 | 115 | 141 | -15 | -35 |
| 2.2002 | -513 | -220 | 194 | -431 | -338 | 203 | -360 | -420 | 115 | 141 | -15 |
| 3.2002 | 154 | -513 | -220 | 194 | -431 | -338 | 203 | -360 | -420 | 115 | 141 |
| 4.2002 | -125 | 154 | -513 | -220 | 194 | -431 | -338 | 203 | -360 | -420 | 115 |
| 5.2002 | -559 | -125 | 154 | -513 | -220 | 194 | -431 | -338 | 203 | -360 | -420 |
| 6.2002 | 92 | -559 | -125 | 154 | -513 | -220 | 194 | -431 | -338 | 203 | -360 |
| 7.2002 | -21 | 92 | -559 | -125 | 154 | -513 | -220 | 194 | -431 | -338 | 203 |
| 8.2002 | -579 | -21 | 92 | -559 | -125 | 154 | -513 | -220 | 194 | -431 | -338 |
| 9.2002 | -52 | -579 | -21 | 92 | -559 | -125 | 154 | -513 | -220 | 194 | -431 |
| 10.2002 | 99 | -52 | -579 | -21 | 92 | -559 | -125 | 154 | -513 | -220 | 194 |
| 11.2002 | -543 | 99 | -52 | -579 | -21 | 92 | -559 | -125 | 154 | -513 | -220 |
| 12.2002 | -175 | -543 | 99 | -52 | -579 | -21 | 92 | -559 | -125 | 154 | -513 |
| 1.2003 | 162 | -175 | -543 | 99 | -52 | -579 | -21 | 92 | -559 | -125 | 154 |
| 2.2003 | -457 | 162 | -175 | -543 | 99 | -52 | -579 | -21 | 92 | -559 | -125 |
| 3.2003 | -346 | -457 | 162 | -175 | -543 | 99 | -52 | -579 | -21 | 92 | -559 |
| 4.2003 | 204 | -346 | -457 | 162 | -175 | -543 | 99 | -52 | -579 | -21 | 92 |
| 5.2003 | -300 | 204 | -346 | -457 | 162 | -175 | -543 | 99 | -52 | -579 | -21 |
| 6.2003 | -474 | -300 | 204 | -346 | -457 | 162 | -175 | -543 | 99 | -52 | -579 |
| 7.2003 | 164 | -474 | -300 | 204 | -346 | -457 | 162 | -175 | -543 | 99 | -52 |
| 8.2003 | -107 | 164 | -474 | -300 | 204 | -346 | -457 | 162 | -175 | -543 | 99 |
| 9.2003 | -572 | -107 | 164 | -474 | -300 | 204 | -346 | -457 | 162 | -175 | -543 |
| 10.2003 | -8 | -572 | -107 | 164 | -474 | -300 | 204 | -346 | -457 | 162 | -175 |
| 11.2003 | 83 | -8 | -572 | -107 | 164 | -474 | -300 | 204 | -346 | -457 | 162 |
| 12.2003 | -541 | 83 | -8 | -572 | -107 | 164 | -474 | -300 | 204 | -346 | -457 |
| 1.2004 | -224 | -541 | 83 | -8 | -572 | -107 | 164 | -474 | -300 | 204 | -346 |
| 2.2004 | 180 | -224 | -541 | 83 | -8 | -572 | -107 | 164 | -474 | -300 | 204 |
| 3.2004 | -420 | 180 | -224 | -541 | 83 | -8 | -572 | -107 | 164 | -474 | -300 |
| 4.2004 | -374 | -420 | 180 | -224 | -541 | 83 | -8 | -572 | -107 | 164 | -474 |
| 5.2004 | 201 | -374 | -420 | 180 | -224 | -541 | 83 | -8 | -572 | -107 | 164 |
| 6.2004 | -236 | 201 | -374 | -420 | 180 | -224 | -541 | 83 | -8 | -572 | -107 |
| 7.2004 | -531 | -236 | 201 | -374 | -420 | 180 | -224 | -541 | 83 | -8 | -572 |
| 8.2004 | 83 | -531 | -236 | 201 | -374 | -420 | 180 | -224 | -541 | 83 | -8 |
| 9.2004 | 27 | 83 | -531 | -236 | 201 | -374 | -420 | 180 | -224 | -541 | 83 |
| 10.2004 | -564 | 27 | 83 | -531 | -236 | 201 | -374 | -420 | 180 | -224 | -541 |
| 11.2004 | -112 | -564 | 27 | 83 | -531 | -236 | 201 | -374 | -420 | 180 | -224 |
| 12.2004 | 131 | -112 | -564 | 27 | 83 | -531 | -236 | 201 | -374 | -420 | 180 |
| 1.2005 | -507 | 131 | -112 | -564 | 27 | 83 | -531 | -236 | 201 | -374 | -420 |
| 2.2005 | -254 | -507 | 131 | -112 | -564 | 27 | 83 | -531 | -236 | 201 | -374 |
| 3.2005 | 199 | -254 | -507 | 131 | -112 | -564 | 27 | 83 | -531 | -236 | 201 |
| 4.2005 | -311 | 199 | -254 | -507 | 131 | -112 | -564 | 27 | 83 | -531 | -236 |
| 5.2005 | -495 | -311 | 199 | -254 | -507 | 131 | -112 | -564 | 27 | 83 | -531 |
| 6.2005 | 143 | -495 | -311 | 199 | -254 | -507 | 131 | -112 | -564 | 27 | 83 |
| 7.2005 | -46 | 143 | -495 | -311 | 199 | -254 | -507 | 131 | -112 | -564 | 27 |
| 8.2005 | -579 | -46 | 143 | -495 | -311 | 199 | -254 | -507 | 131 | -112 | -564 |
| 9.2005 | -90 | -579 | -46 | 143 | -495 | -311 | 199 | -254 | -507 | 131 | -112 |
| 10.2005 | 136 | -90 | -579 | -46 | 143 | -495 | -311 | 199 | -254 | -507 | 131 |
| 11.2005 | -472 | 136 | -90 | -579 | -46 | 143 | -495 | -311 | 199 | -254 | -507 |
| 12.2005 | -338 | -472 | 136 | -90 | -579 | -46 | 143 | -495 | -311 | 199 | -254 |
| 1.2006 | 202 | -338 | -472 | 136 | -90 | -579 | -46 | 143 | -495 | -311 | 199 |

| Regression Statistics | Results |
|---|---|
| Multiple R | 0.972056 |
| R Square | 0.944892 |
| Adjusted R Square | 0.941813 |
| Standard Error | 67.17728 |
| Observations | 190 |

| | Coefficients | Standard Error | t Stat | P-value |
|---|---|---|---|---|
| Intercept | -301.643 | 57.0362 | -5.28862 | 3.56E-07 |
| lag 1 | -0.27739 | 0.074602 | -3.71825 | 0.000268 |
| lag 2 | -0.55811 | 0.076787 | -7.2683 | 1.09E-11 |
| lag 3 | 0.356679 | 0.086535 | 4.12177 | 5.74E-05 |
| **lag 4** | **0.013823** | **0.090524** | **0.152697** | **0.878809** |
| lag 5 | -0.14453 | 0.089015 | -1.62366 | 0.106209 |
| **lag 6** | -0.02001 | 0.078551 | -0.25476 | 0.7992 |
| **lag 7** | 0.048186 | 0.074732 | 0.644783 | 0.519894 |
| **lag 8** | -0.01276 | 0.072079 | -0.17699 | 0.859713 |
| **lag 9** | -0.04807 | 0.057523 | -0.83569 | 0.404441 |
| **lag 10** | -0.04469 | 0.057081 | -0.78286 | 0.434743 |

## ANOVA

| SV | df | SS | MS | F-value | P-value |
|---|---|---|---|---|---|
| Regression | 10 | 13850518 | 1385052 | 306.9172 | 5.2E-107 |
| Residual | 179 | 807788.9 | 4512.787 | | |
| Total | 189 | 14658307 | | | |

| Regression Statistics | Results |
|---|---|
| Multiple R | 0.965633 |
| R Square | 0.932447 |
| Adjusted R Square | 0.93066 |
| Standard Error | 73.23764 |
| Observations | 195 |

## ANOVA

| SV | df | SS | MS | F-value | P-value |
|---|---|---|---|---|---|
| Regression | 5 | 13992985 | 2798597 | 521.7611 | 1.6E-108 |
| Residual | 189 | 1013749 | 5363.752 | | |
| Total | 194 | 15006735 | | | |

| | Coefficients | Standard Error | t Stat | P-value |
|---|---|---|---|---|
| Intercept | -288.389 | 34.1692 | -8.44002 | 8.17E-15 |
| lag 1 | -0.11143 | 0.06172 | -1.80543 | 0.072598 |
| lag 2 | -0.66093 | 0.061539 | -10.7399 | 2.62E-21 |
| lag 3 | 0.388821 | 0.072508 | 5.362419 | 2.38E-07 |
| lag 5 | -0.12262 | 0.061361 | -1.99838 | 0.047109 |

$Y_t = b_0+b_1X_{t-1}+b_2X_{t-2}+b_3X_{t-3}+b_5X_{t-5}$

$Y_t = -288.389 - 0.11143 *X_{t-1} - 0.66093*X_{t-2} + 0.388821*X_{t-3} - 0.12262*X_{t-5}$

AR $p=4$

| Regression Statistics | Results |
|---|---|
| Multiple R | 0.965633 |
| R Square | 0.932447 |
| Adjusted R Square | 0.93066 |
| Standard Error | 73.23764 |
| Observations | 195 |

## ANOVA

| SV | df | SS | MS | F-value | P-value |
|---|---|---|---|---|---|
| Regression | 5 | 13992985 | 2798597 | 521.7611 | 1.6E-108 |
| Residual | 189 | 1013749 | 5363.752 | | |
| Total | 194 | 15006735 | | | |

| | Coefficients | Standard Error | t Stat | P-value |
|---|---|---|---|---|
| Intercept | -288.389 | 34.1692 | -8.44002 | 8.17E-15 |
| Lag 1 | -288.389 | 34.1692 | -8.44002 | 8.17E-15 |
| Lag 2 | -0.11143 | 0.06172 | -1.80543 | 0.072598 |
| Lag 3 | -0.66093 | 0.061539 | -10.7399 | 2.62E-21 |
| Lag 4 | 0.388821 | 0.072508 | 5.362419 | 2.38E-07 |
| Lag 5 | -0.11281 | 0.061297 | -1.84034 | 0.067286 |

$Y_t = b_0+b_1X_{t-1}+b_2X_{t-2}+b_3X_{t-3}+b_4X_{t-4}+b_5X_{t-5}$

$Y_t = -288.389 - 0.11143 *X_{t-1} - 0.66093*X_{t-2} + 0.388821*X_{t-3} - 0.11281*X_{t-4} - 0.12262*X_{t-5}$

| Period (t) | Sales ($Y_t$) | Lag 1 | Lag 2 | Lag 3 | Lag 4 | Lag 5 | Lag 6 | Lag 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 5.30 | | | | | | | |
| 2 | 4.40 | 5.30 | | | | | | |
| 3 | 5.40 | 4.40 | 5.30 | | | | | |
| 4 | 5.80 | 5.40 | 4.40 | 5.30 | | | | |
| 5 | 5.60 | 5.80 | 5.40 | 4.40 | 5.30 | | | |
| 6 | 4.80 | 5.60 | 5.80 | 5.40 | 4.40 | 5.30 | | |
| 7 | 5.60 | 4.80 | 5.60 | 5.80 | 5.40 | 4.40 | 5.30 | |
| 8 | 5.60 | 5.60 | 4.80 | 5.60 | 5.80 | 5.40 | 4.40 | 5.30 |
| 9 | 5.40 | 5.60 | 5.60 | 4.80 | 5.60 | 5.80 | 5.40 | 4.40 |
| 10 | 6.50 | 5.40 | 5.60 | 5.60 | 4.80 | 5.60 | 5.80 | 5.40 |
| 11 | 5.10 | 6.50 | 5.40 | 5.60 | 5.60 | 4.80 | 5.60 | 5.80 |
| 12 | 5.80 | 5.10 | 6.50 | 5.40 | 5.60 | 5.60 | 4.80 | 5.60 |
| 13 | 5.00 | 5.80 | 5.10 | 6.50 | 5.40 | 5.60 | 5.60 | 4.80 |
| 14 | 6.20 | 5.00 | 5.80 | 5.10 | 6.50 | 5.40 | 5.60 | 5.60 |
| 15 | 5.60 | 6.20 | 5.00 | 5.80 | 5.10 | 6.50 | 5.40 | 5.60 |
| 16 | 6.70 | 5.60 | 6.20 | 5.00 | 5.80 | 5.10 | 6.50 | 5.40 |
| 17 | 5.20 | 6.70 | 5.60 | 6.20 | 5.00 | 5.80 | 5.10 | 6.50 |
| 18 | 5.50 | 5.20 | 6.70 | 5.60 | 6.20 | 5.00 | 5.80 | 5.10 |
| 19 | 5.80 | 5.50 | 5.20 | 6.70 | 5.60 | 6.20 | 5.00 | 5.80 |
| 20 | 5.10 | 5.80 | 5.50 | 5.20 | 6.70 | 5.60 | 6.20 | 5.00 |
| 21 | 5.80 | 5.10 | 5.80 | 5.50 | 5.20 | 6.70 | 5.60 | 6.20 |
| 22 | 6.70 | 5.80 | 5.10 | 5.80 | 5.50 | 5.20 | 6.70 | 5.60 |
| 23 | 5.20 | 6.70 | 5.80 | 5.10 | 5.80 | 5.50 | 5.20 | 6.70 |
| 24 | 6.00 | 5.20 | 6.70 | 5.80 | 5.10 | 5.80 | 5.50 | 5.20 |
| 25 | 5.80 | 6.00 | 5.20 | 6.70 | 5.80 | 5.10 | 5.80 | 5.50 |
| | | 5.80 | 6.00 | 5.20 | 6.70 | 5.80 | 5.10 | 5.80 |
| | | | 5.80 | 6.00 | 5.20 | 6.70 | 5.80 | 5.10 |
| | | | | 5.80 | 6.00 | 5.20 | 6.70 | 5.80 |
| | | | | | 5.80 | 6.00 | 5.20 | 6.70 |
| | | | | | | 5.80 | 6.00 | 5.20 |
| | | | | | | | 5.80 | 6.00 |
| | | | | | | | | 5.80 |

| Lag | Intercept | Coefficient |
|---|---|---|
| 1 | 8.502 | -0.486 |
| 2 | 4.814 | 0.161 |
| 3 | 6.408 | -0.121 |
| 4 | 6.617 | -0.159 |
| 5 | 6.246 | -0.094 |
| 6 | 3.381 | 0.421 |
| 7 | 5.816 | -0.017 |

## Time Series → Test for stationarity using Dickey-Fuller test

- **Consider the first order autoregressive model**

$Y_t = \mu + \phi_1 Y_{t-1} + \epsilon_t$ ---------- (1)

where $\mu$ is constant, $\phi_1$ is the regression coefficient and $\epsilon_t$ is the random error with zero mean and constant variance

If $Y_t$ is a non-stationary process, then it is essential to test $\phi_1$ whether it has unitary root or not.

Hence, the hypothesis to be tested is

**$H_0$: $\phi_1=1$ (Unitary root) against $H_1$: $\phi_1 < 1$ (No unitary root)**

## Time Series ⇨ Test for stationarity using Dickey-Fuller test

- **By rewriting the model (1)**

$Y_t - Y_{t-1} = \mu + (\phi_1 - 1)\, Y_{t-1} + \epsilon_t$. ---------(2)

Denoting $\Delta Yt = Y_t - Y_{t-1}$ the model (2) may be expressed as

$\Delta Y_t = \mu + \delta Y_{t-1} + \epsilon_t$ ---------- (3)

Hence, the hypothesis to be tested is

**$H_0$: $\delta$= 0 against $H_1$: $\delta$< 0**

$$\textit{The test} - \textit{statistic}:\ t_{\hat{\delta}} = \frac{\hat{\delta}}{SE(\hat{\delta})}$$

$$\textit{The test} - \textit{statistic}:\ t_{\hat{\delta}} = \frac{\hat{\delta}}{SE(\hat{\delta})}$$

Will follow Dickey – Fuller distribution

**Reject H0, if $t_{\hat{\delta}}$ < DF – CV (P ≤ 0.05) and**

**Fail to reject H0, if $t_{\hat{\delta}}$ > DF – CV (P > 0.05)**

**Time Series** → **Test for stationarity using Dickey-Fuller test**

- **Consider the 'p' order autoregressive model**

$Y_t = \mu + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + \cdots + \phi_p Y_{t-p} + \epsilon_t$ ---------- (1)

where $\mu$ is constant, $\phi_1$, $\phi_2$ , $\phi_3$ , …, $\phi_p$ are the regression coefficients and $\epsilon_t$ is the random error with zero mean and constant variance

If $Y_t$ is a non-stationary process, then it is essential to test $\phi_1$, $\phi_2$ , $\phi_3$ , …, $\phi_p$ whether they haveunitary root or not.

Hence, the hypothesis to be tested is

**$H_0$: $\phi_i$=1 (Unitary root) against $H_1$: $\phi_i$ < 1 (No unitary root) for all i = 1, 2, 3, …, p**

## Time Series → Test for stationarity using Dickey-Fuller test

- **By rewriting the model (1)**

$$Y_t - Y_{t-1} = \mu + (\phi_1 - 1)\, Y_{t-1} + \phi_2 Y_{t-2} + \phi_3 Y_{t-3} + \cdots + \phi_p Y_{t-p} + \epsilon_t \text{ -------(2)}$$

Denoting $\Delta Yt = Y_t - Y_{t-1}$ the model (2) may be expressed as

$$\Delta Y_t = \mu + \delta Y_{t-1} + \sum_{i=2}^{p} \beta_i Y_{t-i} + \epsilon_t \text{ --------}$$

Hence, the hypothesis to be tested is

**$H_0$: $\delta = 0$ against $H_1$: $\delta < 0$**

$$\textit{The test} - \textit{statistic}:\ t_{\hat{\delta}} = \frac{\hat{\delta}}{SE(\hat{\delta})}$$

$$\textit{The test} - \textit{statistic}:\ t_{\hat{\delta}} = \frac{\hat{\delta}}{SE(\hat{\delta})}$$

Will follow Dickey – Fuller distribution

**Reject H0, if $t_{\hat{\delta}}$ < DF (CV) (P ≤ 0.05) and**

**Fail to reject H0, if $t_{\hat{\delta}}$ > DF (CV) (P > 0.05)**

innovate achieve lead

## Time Series → Test for stationarity using Dickey-Fuller test

$$\Delta Y_t = \mu + \delta Y_{t-1} + \sum_{i=2}^{p} \beta_i Y_{t-i} + \epsilon_t \text{ ---------- } (3)$$

Like testing for δ, there is a need to test other parameters $\beta_i$

**$H_0$: $\beta_i = 0$ against $H_1$: $\beta_i < 0$ for all i = 1, 2, 3, …, p**

$$The\ test - statistic: t_{\hat{\beta}_i} = \frac{\hat{\beta}_i}{SE(\hat{\beta}_i)}$$

Hence, the hypothesis to be tested is

**$H_0$: $\delta = 0$ against $H_1$: $\delta < 0$**

$$The\ test - statistic: t_{\hat{\beta}_i} = \frac{\hat{\beta}_i}{SE(\hat{\beta}_i)}$$

Will follow Student's t – distribution with p-1 degrees of freedom

**Reject H0, if $t_{\hat{\beta}_i}$ > t (CV) (P ≤ 0.05) and**

**Fail to reject H0, if $t_{\hat{\delta}}$ < t (CV) (P > 0.05)**

## Time Series ➡ Moving Average (MA) Model

- **Moving-average model**: The **moving-average model** specifies that the output variable depends linearly on the **current and various past** values of a stochastic term.

- Moving average model of order q (MA(q)):

$$Y_t = w_t + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots + \theta_q Y_{t-q} + \varepsilon_t$$

## Time Series ⇨ Moving Average (MA) Model

- **Moving-average model** of order q (MA(q)):

$$X_t = w_t + \sum_{i=1}^{q} \theta_i w_{t-q} + \varepsilon_t$$

where: $\theta_1, \theta_2, \ldots, \theta_q$ are constants with $\theta_q \neq 0$; and $w_t$ is **Gaussian white noise** $w_t\,(0, \sigma^2_w)$.

**Note:** **Gaussian noise**, named after Carl Friedrich **Gauss**, is **statistical noise** having a probability density function (PDF) equal to that of the normal distribution, which is also known as the **Gaussian** distribution. In other words, the values that the **noise** can take on are **Gaussian** distributed.

## Time Series ⇨ Autoregressive Moving Average (ARMA) Model

- **Autoregressive–moving-average model**. In the statistical analysis of time series Auto-Regressive Moving Average (ARMA) models provide a parsimonious description of a (weakly) stationary stochastic process in terms of two polynomials, one for the **autoregression** (AR) and the second for the **moving average** (MA).

## Time Series ⇨ Autoregressive Moving Average (ARMA) Model

- The AR and MA models dynamics can be combined into what is called an *autoregressive moving-average (ARMA) model*.
- The ARMA (1, 1) is $Y_t = \phi_0 + \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t$
  - $\phi_0$ – Constant term in AR model
  - $\phi_1$ – Coefficient associated with $Y_{t-1}$ in AR model
  - $\theta_1$ – Coefficient associated with $\epsilon_{t-1}$ in MA model
  - $\epsilon_{t-1}$ – Error while measuring $Y_{t-1}$ in MA model
  - $\epsilon_t$ – Error while measuring $Y_t$ in MA model

## Time Series ➡ Autoregressive Moving Average (ARMA) Model

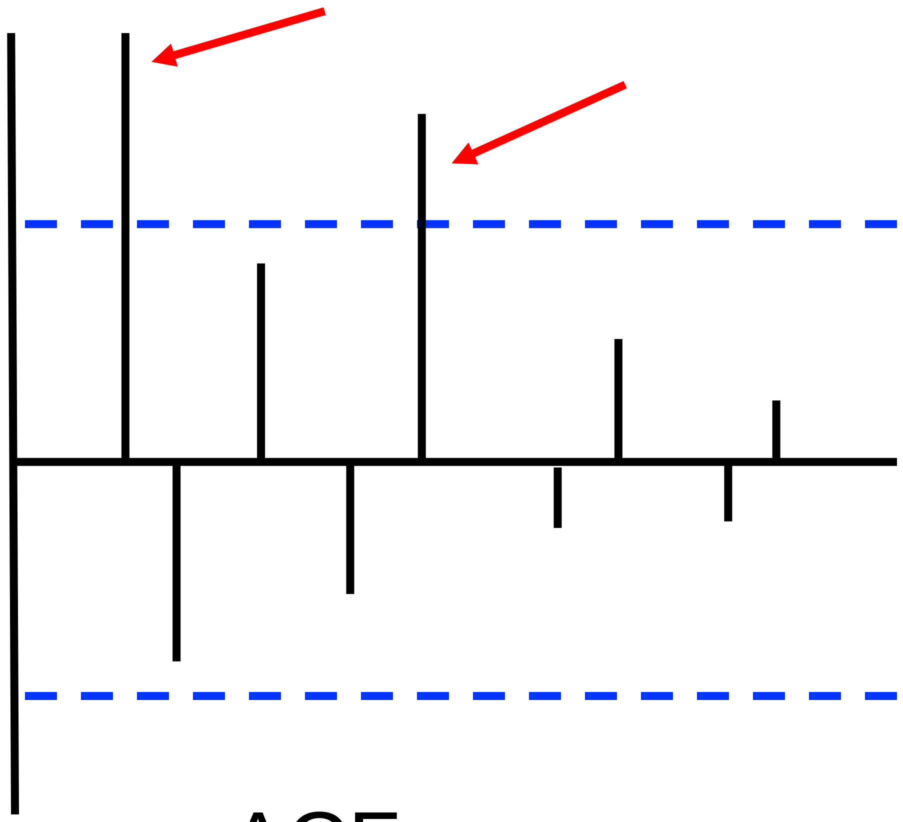
- The predicted value for the ARMA (1, 1) is

$$\hat{Y}_t = \phi_0 + \phi_1 Y_{t-1} + \theta_1 \epsilon_{t-1}$$

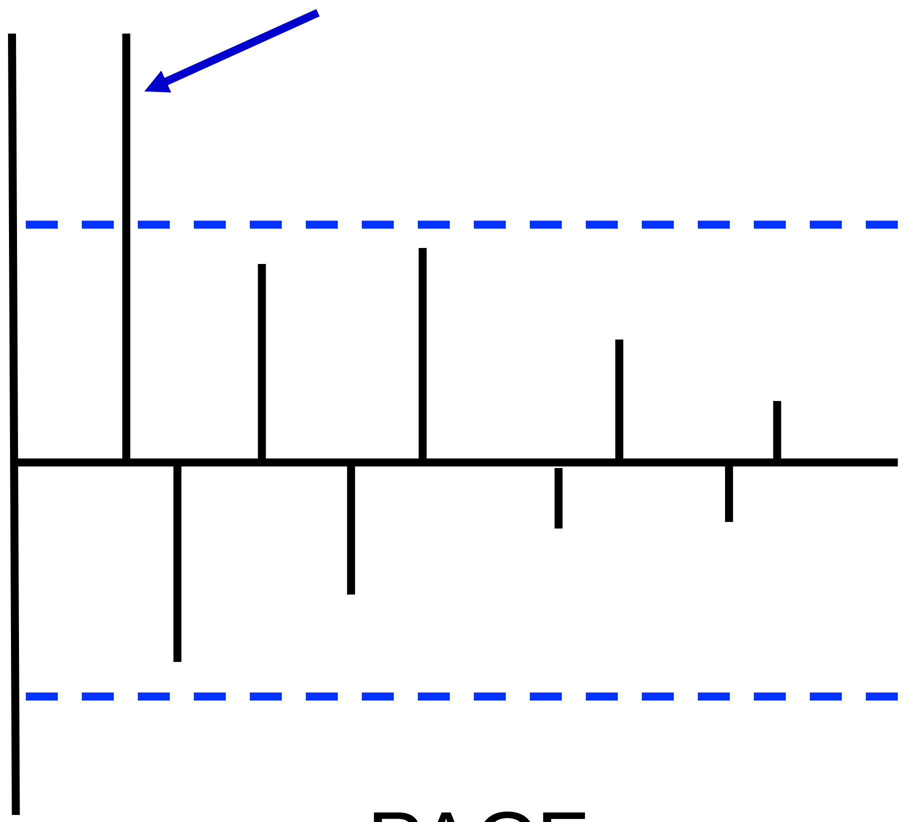
- Here $\phi_0$, $\phi_{1,}$ and $\epsilon_{t\text{-}1}$ are all known except $\epsilon_t$ – the error of the current time
- With the help of ACF which helps to identify the MA order and PCAF helps to identify the AR model the ARMA (p, q) model is given by

## Time Series ⇨ Autoregressive Moving Average (ARMA) Model

- The ACF and PACF are one of the many ways to use to decide the order of ARMA model



ACF
MA order is 2



PACF
AR order is 1

## Time Series ⇨ Autoregressive Moving Average (ARMA) Model

- Higher order ARMA processes involve additional lags of X and epsilon.
- The ARMA (p, q) is

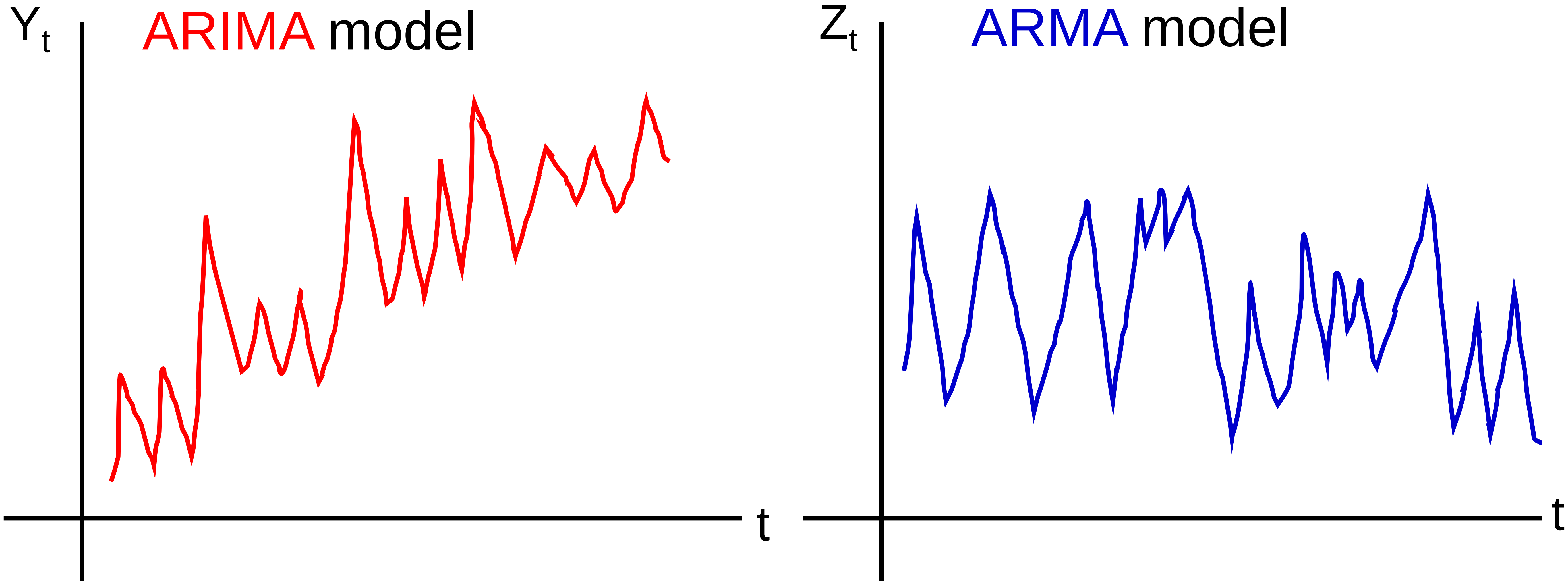
$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p}$$

$$+ \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q}, \quad or$$

$$\Phi(L) y_t = \Theta(L) \varepsilon_t$$

## Time Series ➡ Autoregressive Integrated Moving Average (ARIMA) Model

- ARIMA is an acronym that stands for Auto-Regressive Integrated Moving Average. Specifically,
  - AR ➡ Auto-regression: A model that uses the dependent relationship an observation and some number of **lagged observations**
  - I ➡ Integrated: The use of **differencing** of raw observations in order to make the time series stationary
  - MA ➡ Moving Average: A model that uses the dependency between an observation and **residual error** from a moving average model applied to lagged observations




Non-stationary – Mean not same

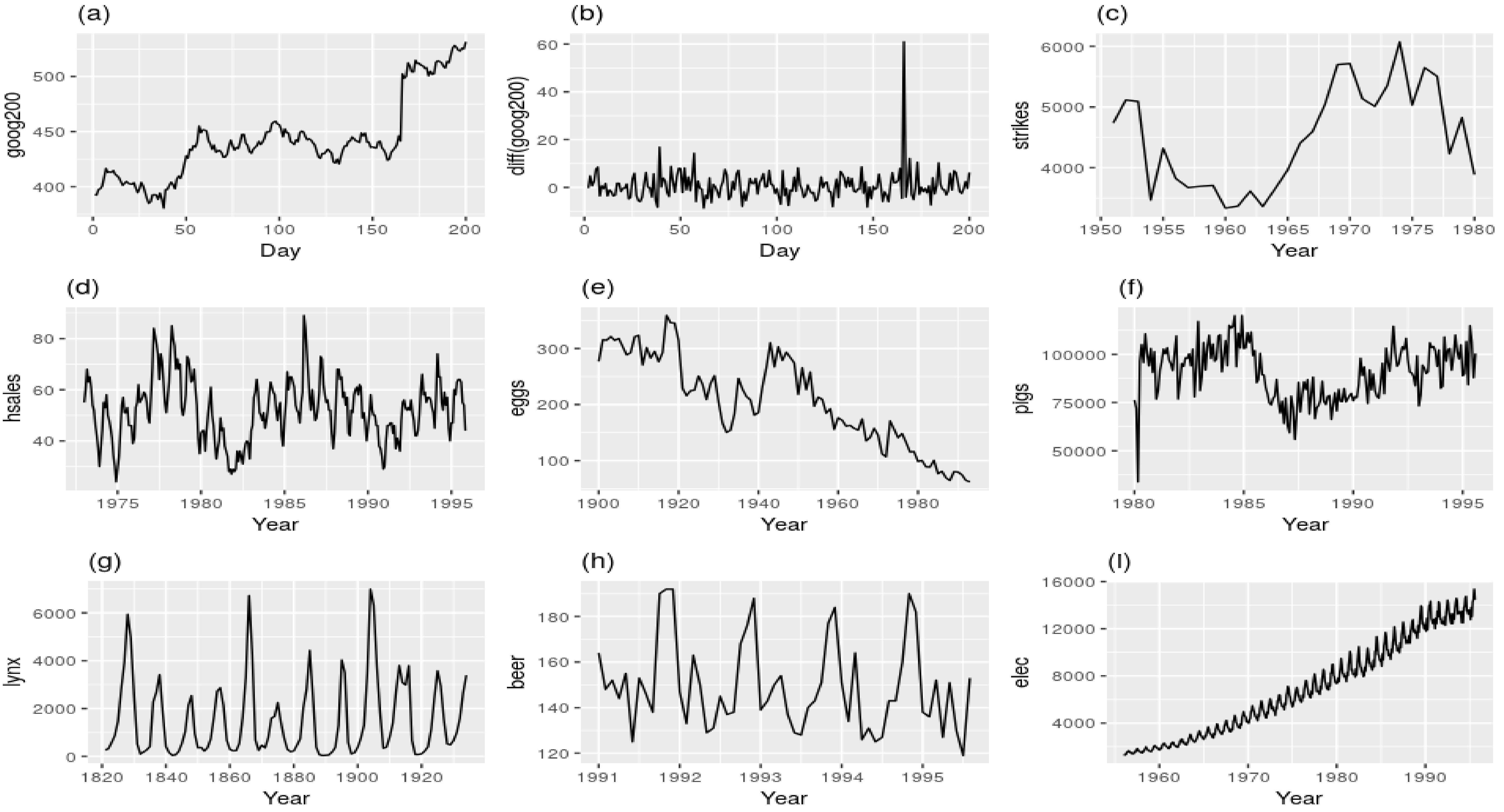
Stationary – Mean same

The Stationary model now is $$Y_k = \sum_{i=1}^{k-l} Z_{k-i} + a_l$$

## Time Series ⇨ Autoregressive Integrated Moving Average (ARIMA) Model

- Let $Y_t$, $Y_{t-1}$, $Y_{t-2}$, …, $Y_{t-p}$ are the ‘p’ non-stationary time series data.
- Define $Z_t = Y_{t+1} – Y_t$ be the first order differencing between two successive values.
- The $Z_t$ will now be transformed into stationary data

- Each of these are explicitly specified in the model as a parameter.
- Note that AR and MA are two widely used **linear models** that work on stationary time series and I is a **pre-processing to stationarize** time series if needed.



(a)
goog200
500
450
400
0
50
100
150
200
Day
(b)
diff(goog200)
60
40
20
0
0
50
100
150
200
Day
(c)
strikes
6000
5000
4000
1950
1955
1960
1965
1970
1975
1980
Year
(d)
hsales
80
60
40
1975
1980
1985
1990
1995
Year
(e)
eggs
300
200
100
1900
1920
1940
1960
1980
Year
(f)
pigs
100000
75000
50000
1980
1985
1990
1995
Year
(g)
lynx
6000
4000
2000
0
1820
1840
1860
1880
1900
1920
Year
(h)
beer
180
160
140
120
1991
1992
1993
1994
1995
Year
(i)
elec
16000
12000
8000
4000
1960
1970
1980
1990
Year

## Time Series ⇨ Autoregressive Integrated Moving Average (ARIMA) Model

- **Rationale** - The first task is to provide a reason why we're interested in a particular model, as quants. Why are we introducing the time series model? What effects can it capture? What do we gain (or lose) by adding in extra complexity?
- **Definition** - We need to provide the full mathematical definition (and associated notation) of the time series model in order to minimise any ambiguity

## Time Series ⇨ Choice of p, d and q

- **Second Order Properties** - We will discuss (and in some cases derive) the second order properties of the time series model, which includes its mean, its variance and its autocorrelation function
- **Correlogram** - We will use the second order properties to plot a correlogram of a realisation of the time series model in order to visualise its behaviour.

## Time Series ➡ Choice of p, d and q

- **Simulation** - We will simulate realisations of the time series model and then fit the model to these simulations to ensure we have accurate implementations and understand the fitting process.

## Time Series ⇨ Choice of p, d and q

- Look at autocorrelation graph of data (will help if MA model is appropriate).
- Look at partial autocorrelation graph of data (will help if MR model is appropriate).
- Look at extended autocorrelation chart of data (will help if a combination of MA and AR models is needed).

## Time Series ➡ Choice of p, d and q

- Try Akaike's information criterion (AIC) on a set of models and investigate the models with the lowest AIC values
- Try the Schwartz Bayesian information criterion (BIC) and investigate the models with the lowest BIC values
- All the above criterion to choose p, d and q are available as package in R.

## Time Series → SARIMA model

- Autoregressive Integrated Moving Average, or ARIMA, is one of the most widely used forecasting methods for univariate time series data forecasting.
- Although the method can handle data with a trend, it does not support time series with a seasonal component.
- An extension to ARIMA that supports the direct modelling of the seasonal component of the series is called Seasonal Autoregressive Integrated Moving Average model (SARIMA).

## Time Series ⇨ SARIMAX model

- SARIMAX(Seasonal Auto-Regressive Integrated Moving Average with eXogenous factors) is **an updated version of the ARIMA model**. ARIMA includes an autoregressive integrated moving average, while SARIMAX includes seasonal effects and eXogenous factors with the autoregressive and moving average component in the model.

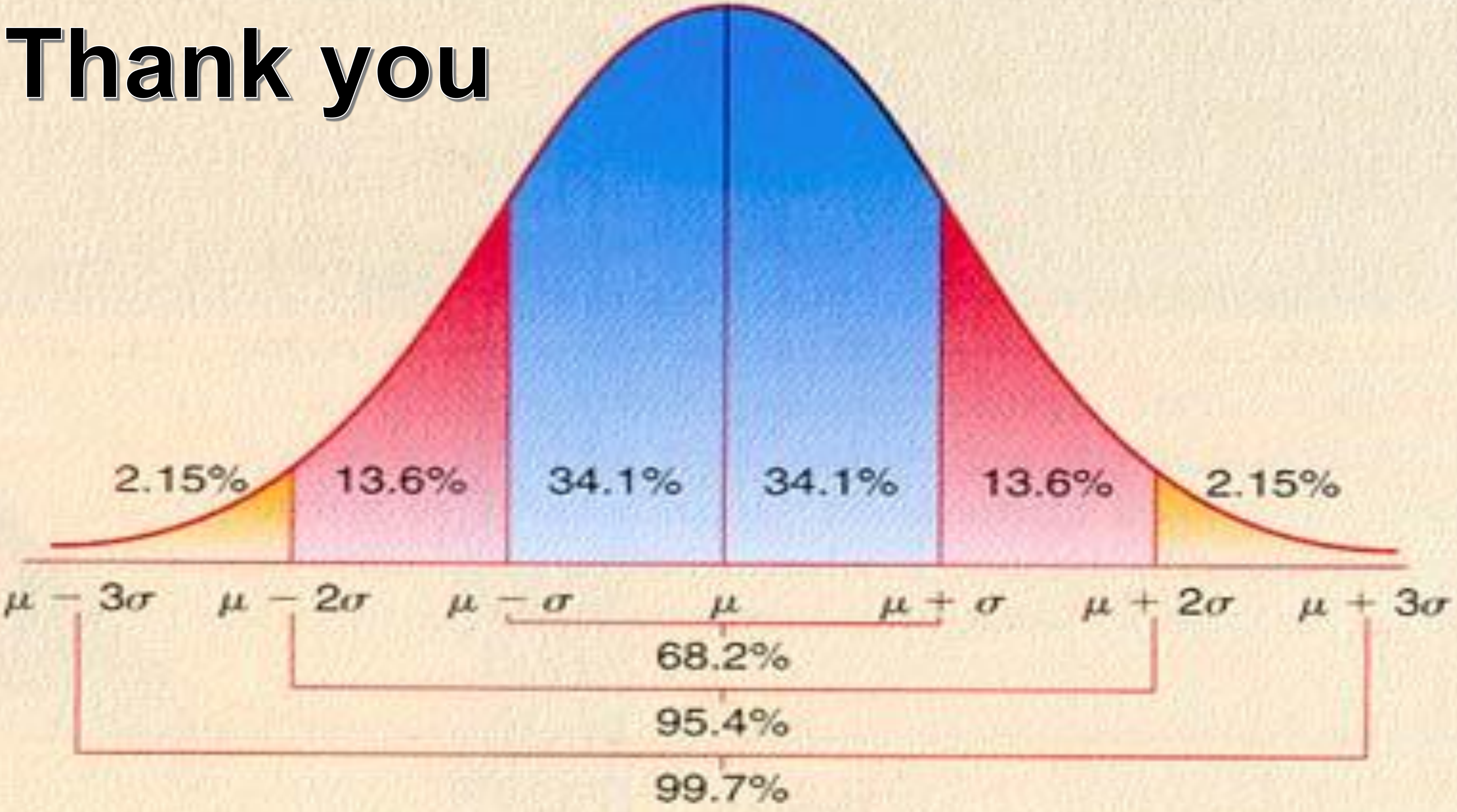
## Time Series → VAR and VARMAX models

- VAR models (vector autoregressive models) are used for multivariate time series. The structure is that each variable is a linear function of past lags of itself and past lags of the other variables in the system.
- The VARMAX procedure **enables you to model the dynamic relationship both between the dependent variables and also between the dependent and independent variables**. VARMAX models are defined in terms of the orders of the autoregressive or moving-average process (or both).

# Reference


- Peter J Brockwell and Richard A Davis. Introduction to Time Series and Forecasting, 2/e, Springer
- Douglas C. Montgomery, and Cheryl L. Jennings, Murat Kulahcin. *Introduction to* Time Series Analysis and Forecasting, 2/e, Wiley

# Thank you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