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Introduction to Statistical Methods

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Session 10

Maximum likelihood & ANOVA

(Session 10: 8^{th} / 9^{th} February 2025)

IMP Note to Self

Agenda

- Maximum likelihood
- ANOVA

Maximum Likelihood Estimation (MLE)

- The method of maximum likelihood was first introduced by R. A. Fisher, a geneticist and statistician, in the 1920s.

- Most statisticians recommend this method, at least when the sample size is large, since the resulting estimators have certain desirable efficiency properties.

- Maximum likelihood estimation(MLE) is a method to find most likely density function, that would have generated data.

- MLE requires one to make distribution assumption first.

Maximum Likelihood Estimation (MLE)

Estimation is the process of estimating unknown true values of population parameters using their corresponding best sample statistics (good estimators) in an optimum manner.

An estimator is said to be a good if it is

- unbiased,
- consistent,
- efficient and
- sufficient while estimating its parameter.

Maximum Likelihood Estimation (MLE)

Method of Maximum Likelihood Estimation is the best and most popular one among all methods to obtain an almost good or best estimator for a population parameter.

It is a method of obtaining an estimator which most (maximum) likely estimates the true value of the parameter i.e., finding an estimator that can give most likely nearer value for the unknown true value of parameter.

The corresponding estimator is called maximum likelihood estimator (MLE).

MLE: Example

- A sample of ten new bike helmets manufactured by a certain company is obtained. Upon testing, it is found that the first, third, and tenth helmets are flawed, whereas the others are not.

- Let p = P(flawed helmet)
- Define (Bernoulli) random variables $X_1, X_2, \ldots, X_{10}$ by

$$X_1 = \begin{cases} 1 \text{ if 1st helmet is flawed} \\ 0 \text{ if 1st helmet isn't flawed} \end{cases} \quad \cdots \quad X_{10} = \begin{cases} 1 \text{ if 10th helmet is flawed} \\ 0 \text{ if 10th helmet isn't flawed} \end{cases}$$

Source: Probability and Statistics for Engineering and the Sciences, Jay L Devore, 8th Ed, Cengage

…MLE: Example

Then for the obtained sample, $X_1 = X_3 = X_{10} = 1$ and the other seven X's are all zero.

The probability mass function of any particular X, is $p^x .(1 – p)^{(1-x)}$,

which becomes p if x = 1 and 1-p when x, = 0

As the conditions of various helmets are independent of one another

This implies that the X's are independent, so their joint probability mass function is the product of the individual pmf's.

Joint pmf evaluated at the observed X's (likelihood function)is

$$f(x_1,..,...,x_{10}; p) = p(1 - p)p... p = p^3(1 - p)^7$$

...MLE: Example

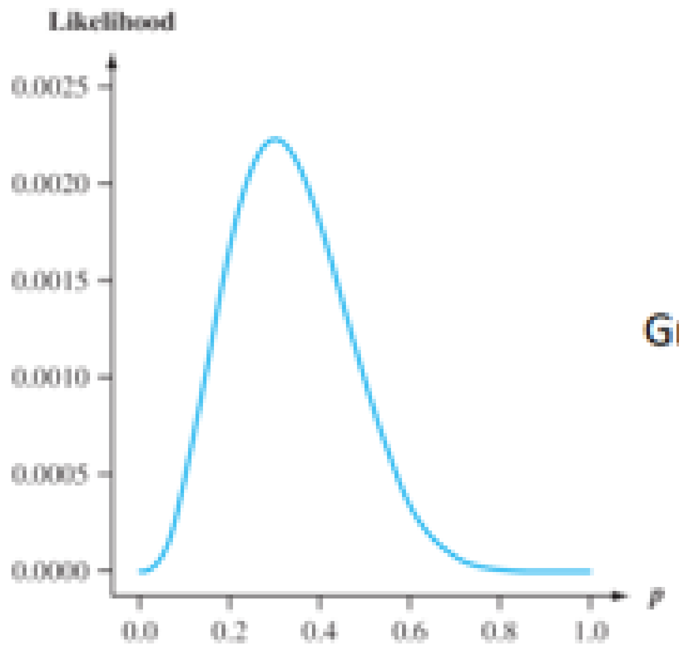
$f(x_1,..,...,x_{10}; p) = p(1 - p)p... p = p^3(1 - p)^7$

Suppose that $p = .25$. Then $f = (.25)^3(.75)^7 = .002086$.
If instead $p = .50$, then $f = (.50)^3(.50)^7 = .000977$.

For what value of p is the obtained sample most likely to have occurred?

That is, for what value of p is the joint pmf (likelihood function) as large as it can be?

…MLE: Example



Graph of the likelihood (joint pmf)

- Figure shows a graph of the likelihood as a function of p.
- It appears that the graph reaches its peak above p = 0.3 = the proportion of flawed helmets in the sample.

…MLE: Example

We can verify our visual impression by using calculus to find the value of p that maximizes likelihood.

Working with the natural log of the joint pmf is often easier than working with the joint pmf itself, since the joint pmf is typically a product so its logarithm will be a sum.

Here $\ln f = \ln[p^3(1 - p)^7]$
Gives $\ln f = 3\ln(p) + 7\ln(1 - p)$

Differentiating and equating this derivative to 0 and solving for p gives
$p = 3/10 = 0.30$ as conjectured.
That is, our point estimate is $p = .30$.

It is called the maximum likelihood estimate because it is the parameter value that maximizes the likelihood (joint pmf) of the observed sample.

MLE: Example

Suppose that rather than being told the condition of every helmet, we had only been informed that three of the ten were flawed.

Then we would have the observed value of a binomial random variable
X = the number of flawed helmets.

—The pmf of X becomes ${}^{10}C_3\, p^3(1 - p)^7$. The binomial coefficient (${}^{10}C_3$)is irrelevant to the maximization, so again $p = 0.30$.

The likelihood function tells us how likely the observed sample is as a function of the possible parameter values.

Maximizing the likelihood gives the parameter values for which the observed sample is most likely to have been generated—that is, the parameter values that "agree most closely" with the observed data.

Maximum Likelihood Estimation (MLE)

Suppose we have a random sample $x_1, x_2, .., x_n$ whose assumed probability distribution depends on some unknown parameter θ.

Ex:

1) For Bernoulli unknown parameter are p.
2) For Binomial unknown parameters are n, p.
3) For Poisson unknown parameter is λ.
4) For Normal unknown parameters are $\mu \ and \ \sigma^2$.

Our goal is to find good estimate of θ(population parameter) using sample and which can be done with the help of MLE.

Maximum Likelihood function

It is observed that a good estimate of unknown parameter θ would be the value of θ that maximizes the probability

i.e. the likelihood of getting the data we observed (this is reason, why we called as likelihood function)

Let $x_1, x_2, .., x_n$ be i.i.d. random variables drawn from some probability distribution that depends on some unknown parameter θ.

The goal of MLE to maximize likelihood function

$$L(\theta) = f(x_1, x_2, \ldots, x_n | \theta)$$

$$= f(x_1|\theta) * f(x_2|\theta) * \cdots * f(x_n|\theta)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i / \theta)$$

Maximum Likelihood Estimation (MLE)

The maximum likelihood estimate (MLE) of θ is that value of θ that maximizes likelihood(θ).

It is defined as

$$L(\theta) = \prod_{i=1}^{n} f(x_i / \theta)$$

$$\log L(\theta) = \sum_{i=1}^{n} \log f(x_i / \theta)$$

For maximization, we have

$$\frac{dL}{d\theta} = 0 \;;\quad \frac{d^2L}{d\theta^2} < 0$$

Maximum Likelihood Estimation (MLE)

If L and $\log_e L$ are not differentiable or integrable or principle of maxima-minima fails then in such case direct method of finding the estimator of the parameter which maximizes L or $\log_e L$ is applied using order statistic principle empirically.

MLE for a Binomial distribution parameter

Suppose we wish to find the MLE of θ for a Binomial distribution with k successes in n trials then,

$$p_k(k,\theta) = nC_k \theta^k (1-\theta)^{n-k}$$

$$\log p_k(k,\theta) = \log(nC_k) + k\log(\theta) + (n-k)\log((1-\theta)$$

$$\frac{\partial \log p_k(k,\theta)}{\partial \theta} = 0 \Rightarrow 0 + \frac{k}{\theta} - \frac{n-k}{1-\theta} = 0$$

$$k - k\theta = n\theta - k\theta \Rightarrow \theta = \frac{k}{n}$$

MLE for Poisson Distribution Parameter

- Suppose we have data generated from a Poisson distribution. We want to estimate the parameter of the distribution
- The probability of observing a particular random variable is $P(X;\mu) = \dfrac{e^{-\mu}\mu^{X}}{X!}$
- Joint likelihood by multiplying the individual probabilities together

$$P(X_1, X_2, \ldots, X_n; \mu) = \frac{e^{-\mu}\mu^{X_1}}{X_1!} \times \frac{e^{-\mu}\mu^{X_2}}{X_2!} \times \ldots \times \frac{e^{-\mu}\mu^{X_n}}{X_n!}$$

$$L(\mu; \mathbf{X}) = \prod_i e^{-\mu}\mu^{X_i}$$

$$L(\mu; \mathbf{X}) = e^{-n\mu}\mu^{n\overline{X}}$$

MLE for Poisson Distribution Parameter

- Note in the likelihood function the factorials have disappeared.
- This is because they provide a constant that does not influence the relative likelihood of different values of the parameter
- It is usual to work with the **log likelihood** rather than the likelihood.
- Note that maximising the log likelihood is equivalent to maximising the likelihood.

$$L(\mu;\mathbf{X}) = e^{-n\mu}\mu^{n\bar{X}}$$

Take the natural log of the likelihood function

$$\ell(\mu;\mathbf{X}) = -n\mu + n\bar{X}\log\mu$$

Find where the derivative of the log likelihood is zero

$$\frac{d\ell}{d\mu} = -n + \frac{n\bar{X}}{\mu}$$

$$\hat{\mu} = \bar{X}$$

Note that here the MLE is the same as the moment estimator

MLEs for Normal Distribution Parameters

- Let $X_1, \ldots, X_n$ be a random sample from a normal distribution.
- The likelihood function is

$$f(x_1, \ldots, x_n; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_1-\mu)^2/(2\sigma^2)} \cdot \ldots \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_n-\mu)^2/(2\sigma^2)}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\Sigma(x_i-\mu)^2/(2\sigma^2)}$$

- so

$$\ln[f(x_1, \ldots, x_n; \mu, \sigma^2)] = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \Sigma(x_i - \mu)^2$$

MLEs for Normal Distribution Parameters

- To find the maximizing values of μ and σ^2, we must take the partial derivatives of $\ln(f)$ with respect to μ and σ^2, equate them to zero, and solve the resulting two equations.

- Omitting the details, the resulting MLE's are

$$\hat{\mu} = \overline{X} \qquad \hat{\sigma}^2 = \frac{\sum(X_i - \overline{X})^2}{n}$$

- The MLE of σ^2 is not the unbiased estimator, so two different principles of estimation (unbiasedness and maximum likelihood) yield two different estimators

MLEs

- If θ denotes the parameter to be estimated, then estimators will be denoted by $\hat{\theta}$
.
- For Bernoulli $\hat{p} = \dfrac{\sum_{i=1}^{n} x_i}{n} = \overline{X}$ For Exponential $\hat{\lambda} = 1/\overline{X}$
- For Poisson's $\hat{\lambda} = \dfrac{\sum_{i=1}^{n} x_i}{n}$
- For Normal

$$\hat{\mu} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \hat{\mu})^2}{n}}$$

Fisher’s F – test for ratio of variances

F table: http://www.socr.ucla.edu/Applets.dir/F_Table.html

Hypothesis Tests for Two Variances

Goal: Test hypotheses about two population variances

Tests for Two Population Variances

F test statistic

$H_0: \sigma_1^2 = \sigma_2^2$
$H_1: \sigma_1^2 < \sigma_2^2$

Lower-tail test

$H_0: \sigma_1^2 = \sigma_2^2$
$H_1: \sigma_1^2 > \sigma_2^2$

Upper-tail test

$H_0: \sigma_1^2 = \sigma_2^2$
$H_1: \sigma_1^2 \neq \sigma_2^2$

Two-tail test

The two populations are assumed to be independent and normally distributed

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1 State null and alternative hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \text{ vs } H_1 : \sigma_1^2 < \sigma_2^2$$
$$\text{or } H_1 : \sigma_1^2 > \sigma_2^2$$
$$\text{or } H_1 : \sigma_1^2 \neq \sigma_2^2$$

2 Specify the level of significance ‘α’

3 Fisher’s F - Distribution

4 Compute the test statistic

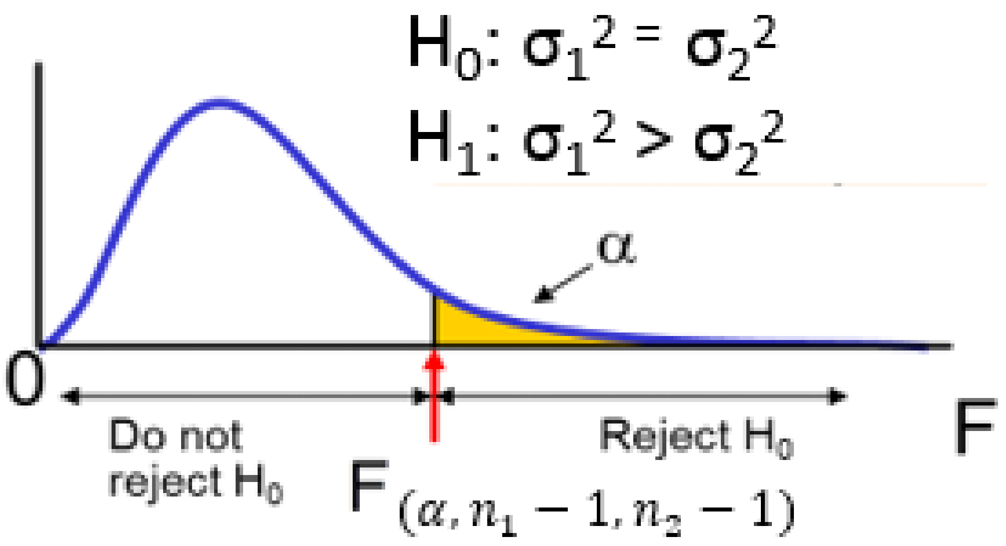
$$F = \frac{S_1^2}{S_2^2} \cong F_{(\alpha, n_1 - 1, n_2 - 1)}$$

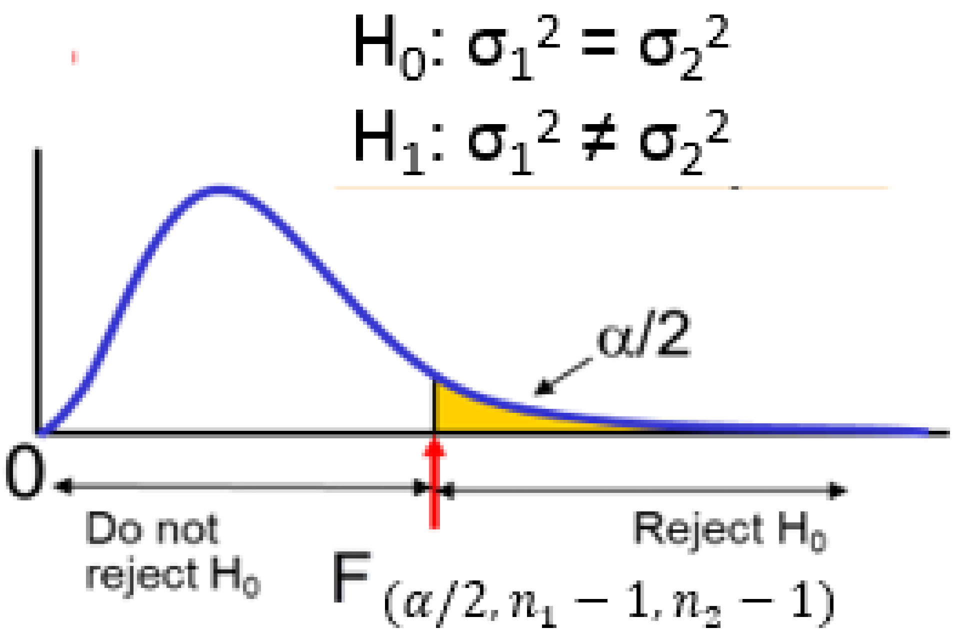
5 Define the critical region/ rejection criteria

6 Conclusion

Note : s_1^2 is the larger of the two sample variances

Decision Rules: Two Variances





The variability in the amount of impurities present in a batch of chemicals used for a particular process depends on the length of time that the process is in operation.

Suppose a sample of size 25 is drawn from the normal process which is to be compared to a sample of a new process that has been developed to reduce the variability of impurities. Test at 5%, whether the variability in the new process is less as compared to the original process.

	Sample 1	Sample 2
n	25	25
S^2	1.04	0.51

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At 5% (0.05) level of significance with critical value is 1.98 for (24, 24) degrees of freedom

Hypothesis to test

$$F = \frac{S_1^2}{S_2^2} = \frac{1.04}{0.51} = 2.04$$

$H_0: \sigma_1^2 = \sigma_2^2$

Vs

$H_1: \sigma_1^2 > \sigma_2^2$

Critical value for $\alpha = 0.05$ is 1.98. Since $F = 2.04 > 1.98$, Reject H_0 & Accept H_1

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A company manufactures impellers for use in jet-turbine engines. One of the operations involves grinding a particular surface finish of a titanium alloy component. Two different grinding processes can be used and both processes can produce parts at identical mean surface roughness. The manufacturing engineer would like to select the process having the least variability in surface roughness. A random sample of n_1= 12 parts from the first process results in a sample standard of s_1 = 5.1 microinches of n_2= 15 parts from the second process results in sample standard deviation of s_2 = 4.7 microinches. Test at 5%, is there a sufficient evidence that the first process vary more than the second process?

At 5% (0.05) level of significance with critical value is 2.53 for (11, 14) degrees of freedom

Hypothesis to test

$$F = \frac{S_1^2}{S_2^2} = 1.18$$

$H_0: \sigma_1^2 = \sigma_2^2$

Vs

$H_1: \sigma_1^2 > \sigma_2^2$

Critical value for $\alpha = 0.05$ is 2.53 Since cal F < Tab F , Accept H_0 and Reject H_1

Analysis of Variance (ANOVA)

Testing of Hypothesis → Analysis of Variance (ANOVA)

Types of ANOVA

- One-way
- Two-way
- Three-way (LSD)
- Repeated measures of ANOVA (Rm ANOVA)

Note: In this unit we discuss only One-way ANOVA and Two-way ANOVA in detail

- Analysis of Variance (ANOVA) can be used to test for the equality of three or more population means

- We want to use the sample results to test the following hypotheses:

$$H_0: \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_k$$

$$H_1: \text{Not all population means are equal}$$

- If H_0 is rejected, we cannot conclude that all population means are not equal
- Rejecting H_0 means that at least two population means have different values

Assumptions for Analysis of Variance

- For each population, the response (dependent) variable is normally distributed

- The variance of the response variable, denoted σ^2, is the same for all of the populations

- The observations must be independent

One-way Analysis of Variance

- In one-way ANOVA the effect of one factor on the mean is tested. It is based on independent random samples drawn from k – different levels of a factor, also called treatments(groups).

The data and the notations used in one-way ANOVA are represented in the following tabular structure.

One-way Analysis of Variance

Treatment-1	Treatment-2	Treatment-3	. . .	Treatment-k
X_{11}	X_{21}	X_{31}	. . .	X_{k1}
X_{12}	X_{22}	X_{32}	. . .	X_{k2}
X_{13}	X_{23}	X_{33}	. . .	X_{k3}
. . .	. . .	. . .	. . .	. . .
X_{1n_1}	X_{1n_2}	X_{1n_3}	. . .	X_{1n_k}
C_1	C_2	C_3	. . .	C_k

$$n_1 + n_2 + n_3 + \ldots + n_k = n$$

$$C_1 + C_2 + C_3 + \ldots + C_k = G$$

The hypotheses to be tested are

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$$

vs

$$H_1 : \mu_1 \neq \mu_2 \neq \ldots \neq \mu_k$$

One-way Analysis of Variance

- Assume that a simple random sample of size n_j has been selected from each of the *k* populations or treatments. For the resulting sample data, let

  x_{ij} = value of observation i for treatment *j*

  n_j = number of observations for treatment *j*

  x_j = sample mean for treatment *j*

  s_j = sample standard deviation for treatment *j*

Between-Treatments Estimate of Population Variance σ^2

- The estimate of σ^2 based on the variation of the sample means is called the mean square due to treatments and is denoted by MSTR

$$\text{MSTR} = \frac{\sum_{j=1}^{k} n_j(\bar{x}_j - \bar{\bar{x}})^2}{k-1}$$

Denominator is the degrees of freedom associated with SSTR

Numerator is called the sum of squares due to treatments (SSTR)

Within-Treatments Estimate of Population Variance σ^2

- The estimate of σ^2 based on the variation of the sample observations within each sample is called the mean square error and is denoted by MSE

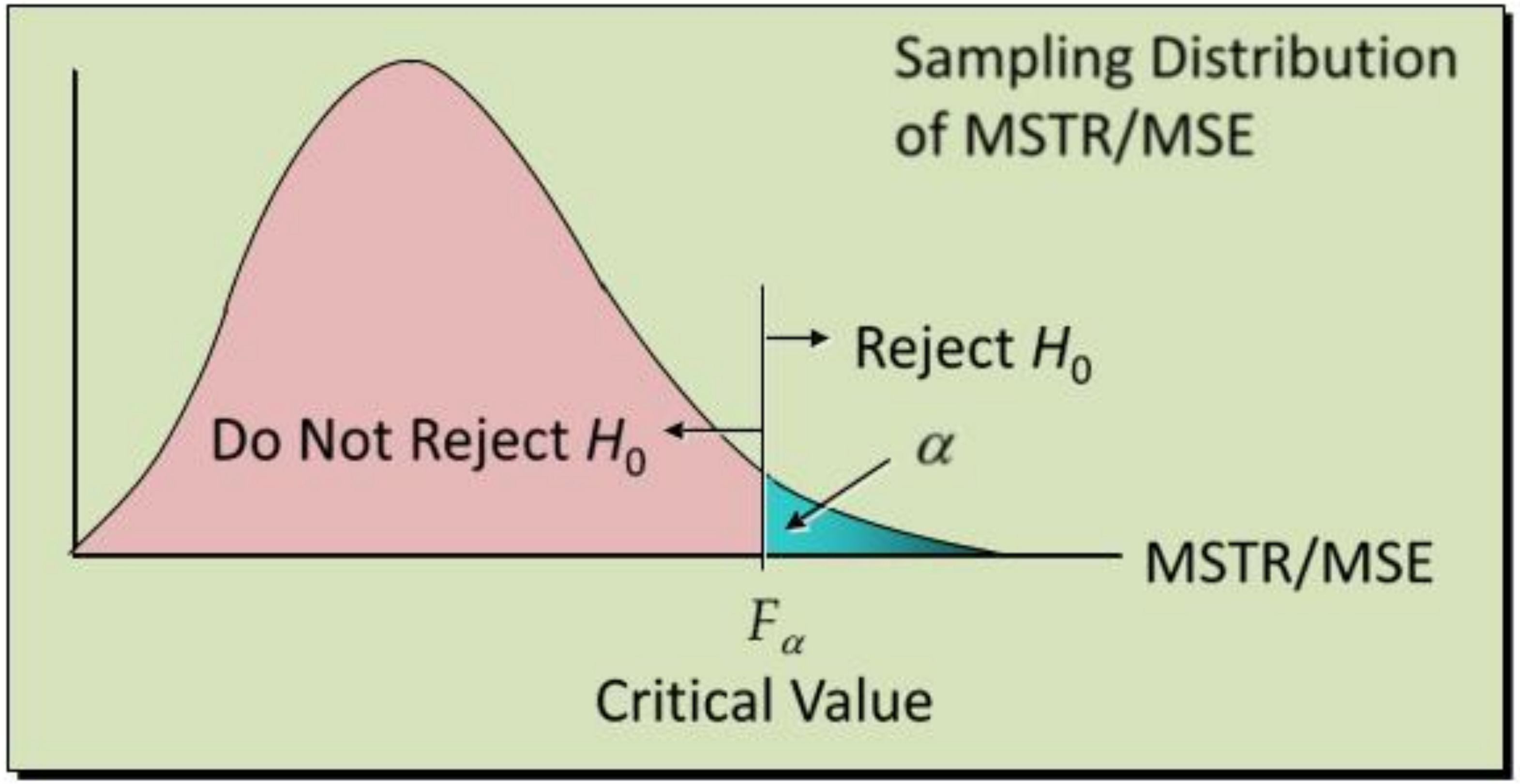
$$\text{MSE} = \frac{\sum_{j=1}^{k}(n_j - 1)s_j^2}{n - k}$$

Denominator is the degrees of freedom associated with SSE

Numerator is called the sum of squares due to error (SSE)

F -Test

- If the null hypothesis is true and the ANOVA assumptions are valid, the sampling distribution of MSTR/MSE is an *F* distribution with MSTR d.f equal to *k* - 1 and MSE d.f. equal to *n* - *k*.



ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Treatments	SSTR	k - 1	$\text{MSTR} = \dfrac{\text{SSTR}}{k\text{-}1}$	$\dfrac{\text{MSTR}}{\text{MSE}}$
Error	SSE	n - k	$\text{MSE} = \dfrac{\text{SSE}}{n\text{-}k}$	
Total	SST	n - 1		

SST is partitioned into SSTR and SSE.

SST's degrees of freedom (d.f.) are partitioned into SSTR's d.f. and SSE's d.f.

- SST divided by its degrees of freedom $n - 1$ is the overall sample variance that would be obtained if we treated the entire set of observations as one data set.
- With the entire data set as one sample, the formula for computing the total sum of squares, SST, is:

$$\mathrm{SST} = \sum_{j=1}^{k} \sum_{i=1}^{n_j} (x_{ij} - \bar{\bar{x}})^2 = \mathrm{SSTR} + \mathrm{SSE}$$

One-way ANOVA example

Test at 5% level of significance is there any significant difference in mean of the three samples obtained from the Normal population of equal variances

Group 1	Group 2	Group 3
8	7	12
10	5	9
7	10	13
14	9	12
11	9	14

$H_0 : \mu_1 = \mu_2 = \mu_3$

$H_1 : \mu_1 \neq \mu_2 \neq \mu_3$

For α= 0.05 F tab value for (k-1, n-k) =(2, 12) d.o.f. is 3.89

Solution Method-1:

	Group 1	Group 2	Group 3
Grand total, G = 150	8	7	12
	10	5	9
	7	10	13
	14	9	12
	11	9	14
Total C_i	50 $\bar{x}_1$=10	40 $\bar{x}_2$=8	60 $\bar{x}_3$=12

Grand total , G =150 and $\bar{x} = 10$

Step 1: H_0: $\mu_1 = \mu_2 = \mu_3$, H_1: $\mu_1 \neq \mu_2 \neq \mu_3$

Step 2: SSTR: Sum of sq. of variations between treatments

$=r\sum(\bar{x}_j - \bar{x})^2=5[(10-10)^2+(8-10)^2+(12-10)^2]=5(0+4+4)=40$

Step 3: SSE: Sum of sq. of variations due to error $=\sum\sum(x_{i\,j} - \bar{x}_j)^2$

$=(8-10)^2+(10-10)^2+(7-10)^2+(14-10)^2+(11-10)^2+$
$(7-8)^2+(5-8)^2+(10-8)^2+(9-8)^2+(9-8)^2+$
$(12-12)^2+(9-12)^2+(13-12)^2+(12-12)^2+(14-12)^2=60$

Step 4: SST Total variation $=$ SSTR $+$ SSE $= 100$

Step 5: ANOVA Table

One-way ANOVA table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between groups	k – 1 =2	SSTR =40	$MSTR = \frac{SSTR}{k-1} = 20$	$F = \frac{MSTR}{MSE} = 4$
Within groups	n – k =12	SSE =60	$MSE = \frac{SSSSE}{n-k} = 5$	
Total		SST =100	$F \approx F$ - distribution with k - 1 and n - k df	

For α= 0.05 F tab value for (k-1, n-k) =(2, 12) d.o.f. is 3.89   Thus F tab 3.89 < F cal 4, Reject H_0 and accept H_1.

Solution Method-2: using Correction Factor

	Group 1	Group 2	Group 3
Grand total, G = 150	8	7	12
	10	5	9
	7	10	13
	14	9	12
	11	9	14
Total C_i	50	40	60

Step 1: Correction Factor (CF)= G^2 / n = $(150)^2$ /15 =1500

Step 2: SST =$\sum\sum x_{i,j}^2 - CF = 1600 - 1500 = 100$

Step 3: SSTR = $\frac{\sum C_i^2}{n_i} - CF = \frac{(50)^2}{5} + \frac{(40)^2}{5} + \frac{(60)^2}{5} - 1500 = 40$

Step 4: SSE =$SST - SSTR = 100 - 40 = 60$

Step 5: ANOVA Table

Grand total , G =150

One-way ANOVA table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between groups	k – 1 =2	SSTR =40	$MSTR = \frac{SSTR}{k-1} = 20$	$F = \frac{SSTR}{SSE} = 4$
Within groups	n – k =12	SSE =60	$MSE = \frac{SSE}{n-k} = 5$	
Total		SST =100	$F \approx F$ - distribution with k - 1 and n - k df	

For $\alpha = 0.05$ F tab value for (k-1, n-k) =(2, 12) d.o.f. is 3.89 Thus F tab 3.89 < F cal 4, Reject H_0 and accept H_1.

One-way ANOVA Example-2:

Test at 5% level of significance is there any significant difference in mean iron intake among four groups of patients? Also test if significant, which group means have contributed to the difference in means using LSD test

A clinical trial Iron intake of four groups of patients (mg)

Group 1	Group 2	Group 3	Group 4
11.5	19.5	18.5	30.0
12.5	18.5	16.5	26.5
18.5	16.0	24.5	27.0
21.0	22.0	30.0	34.0
28.0	30.0	28.5	20.0
26.0	24.5	14.0	22.5
14.0	19.0	19.0	28.0
22.0	24.0	17.0	32.0
20.0	19.5	18.0	27.0
22.0	15.0	29.0	25.5

Solution:

Grand total , G =981

Step 1: Correction Factor (CF)= G^2 / n = $(981)^2$ /40 =19847.03

Step 2: SST =$\sum\sum x_{i,j}^2 - CF = 21119.5 - 19847.03 = 1272.47$

Step 3: SSTR = $\frac{\sum C_i^2}{n_i} - CF$ =20196.6-19847.03=349.52

Step 4: SSE =$SST - SSTR$ =1272.47-349.52=922.95

Step 5: ANOVA Table

One-way ANOVA table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between groups	3	349.52	116.51	F=4.455
Within groups	36	922.95	25.64	
Total	39	1272.47	$F(4.46) > F_{(0.05;\ 3,\ 36)} = 4.38$	

H_0 rejected and **H_1 accepted**

F-table of Critical Values of α = 0.05 for F(df1, df2)

	DF1=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
DF2=1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.37
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

F-table of Critical Values of α = 0.05 for F(df1, df2)

	DF1=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30
DF2=1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.10
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46

F-table of Critical Values of α = 0.05 for F(df1, df2)

	DF1=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
DF2=1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.10	251.14	252.20	253.25	254.31

F-table of Critical Values of α = 0.05 for F(df1, df2)

	DF1=1	2	3	4	5	6	7	8	9	10	12	15	20	24	30
DF2=1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.10

Two-way Analysis of Variance
(Two-way ANOVA)

TWO-WAY ANOVA

- In two-way ANOVA, a study variable is compared over three or more groups, controlling for another variable.
- The grouping is taken as one factor and the control is taken as another factor. The grouping factor is usually known as Treatment.
- The control factor is usually called Block.
- The accuracy of the test in two-way ANOVA is considerably higher than that of one-way ANOVA, as the additional factor, block is used to reduce the error variance.

Treatments

Blocks

	1	**2**	**…**	**k**	Total
1	x_{11}	x_{12}	…	x_{1k}	$x_{.1}$
2	x_{21}	x_{22}	…	x_{2k}	$x_{.2}$
3	x_{31}	x_{32}	…	x_{3k}	$x_{.3}$
…	…	…	…	…	…
m	x_{m1}	x_{m2}	…	x_{mk}	$x_{.m}$
Total	$x_{1.}$	$x_{2.}$	…	$x_{k.}$	G

Steps involved in two-way ANOVA

Step 1: In two-way ANOVA we have two pairs of hypotheses, one for treatments and one for the blocks.

Null Hypotheses

$H_0$1: There is no significant difference among the population means of different Treatments(groups)

$H_0$2: There is no significant difference among the population means of different Blocks

Alternative Hypotheses

$H_1$1: Atleast one pair of treatment means differs significantly

$H_1$2: Atleast one pair of block means differs significantly

Steps involved in two-way ANOVA

Step 2: **Data is presented in a rectangular table form as described in one-way ANOVA.**

Step 3: **Level of significance α.**

Step 4: **Test Statistic**

For treatments F= MSTR/MSE

For blocks = MSBL/MSE

Steps involved in two-way ANOVA

To find the test statistic we have to find the following intermediate values.

i) Correction Factor: $C.F = \frac{G^2}{n}\ where\ G = \sum_{j=1}^{m} \sum_{i=1}^{k} x_j$

ii) Total Sum of Squares: $SST = \sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}^2 - C.F$

iii) Sum of Squares between Treatments: $SSTR = \sum_{i=1}^{k} \frac{x_i^2}{m} - C.F$

iv) Sum of squares between blocks:$SSBL = \sum_{j=1}^{m} \frac{x_j^2}{k} - C.F$

v) Sum of Squares due to Error: *SSE = SST - SSTR - SSBL*

vi) ANOVA Table

Sources of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F
Treatments	SSTR	k-1	MSTR= SSTR/k-1	F= MSTR/MSE
Blocks	SSBL	(m-1)	MSBL = SSBL/m- 1	And
Error	SSE	(k-1)(m-1)	MSE= SSE/(k-1)(m-1)	F= MSBL/MSE
Total	SST	n-1		

Two-way ANOVA Example-1:

A Farmer applies three types of fertilizers on four separate plots. The figures on yield per acre are tabulated. Test at 5% level of significance is there any significant difference in the yield due to fertilizers and also due to the plots

Plots → Fertilizers	A	B	C	D
Nitrogen	6	4	8	6
Potash	7	6	6	9
Phosphate	8	5	10	9

H_0 : Plots do not differ the yield and Fertilizers do not differ the yield

H_1 : Plots differ the yield and Fertilizers differ the yield

Solution:

Plots(Treatments) Fertilizers(Blocks)	A	B	C	D	Total
Nitrogen	6	4	8	6	24
Potash	7	6	6	9	28
Phosphate	8	5	10	9	32
Total	21	15	24	24	G=84

Step 1: CF= G^2 / n = $(84)^2$ /12 =588

Step 2: SST =$\sum\sum x_{i,j}^2 - CF = 624 - 588 = 36$

Step 3: For column SSTR = $\frac{\sum C_i^2}{n_i} - CF = \frac{(21)^2}{3} + \frac{(15)^2}{3} + \frac{(24)^2}{3} + \frac{(24)^2}{3} - 588$ =18

Step 4: For row SSBL = $\frac{\sum R_i^2}{n_j} - CF = \frac{(24)^2}{4} + \frac{(28)^2}{4} + \frac{(32)^2}{4} - 588$ =8

Step 5: SSE =$SST - SSTR - SSBL = 36 - 18 - 8 = 10$

Step 6: ANOVA Table

F table: http://www.socr.ucla.edu/Applets.dir/F_Table.html

Two-way ANOVA Table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between Treatments (Plots)	k – 1 =3	SSTR =18	**MSTR = SSTR/ k-1 =6**	For Treatment F =MSTR/MSE =6/1.67=3.6
Between Blocks (Fertilizers)	m– 1 =2	SSBL = 8	**MSBL = SSBL/ m-1 =4**	For Block F =MSBL/MSE =4/1.67 =2.4
Within groups	(k-1)*(m-1)=6	SSE =10	**MSE = SSE/ (k-1)*(m-1) =1.67**	

F tab value for treatment =(3, 6) d.o.f. is 4.76 and thus Fcal < F tab hence Accept H_0

F tab value for block =(2, 6) d.o.f. is 5.14 and thus Fcal < F tab hence Accept H_0

Two-way ANOVA Example-1:

Investigate the effect of two factors Teaching Method and Study Time on test scores for the following data:

Teaching Method	Study Time	Test Scores
Traditional	Less than 2 hours	60, 65, 70
Traditional	More than 2 hours	68, 70, 75
Online	Less than 2 hours	55, 58, 60
Online	More than 2 hours	65, 67, 70
Hybrid	Less than 2 hours	62, 64, 66
Hybrid	More than 2 hours	70, 72, 75

H_0 : There is no effect of Teaching Method on test scores and
There is no effect of Study Time on test scores.

H_1 : There is effect of Teaching Method on test scores and
There is effect of Study Time on test scores.

Solution:

Calculation of Mean Test Scores:

Traditional:

Less than 2 hours: (60 + 65 + 70) / 3 = 65

More than 2 hours: (68 + 70 + 75) / 3 = 71

Online:

Less than 2 hours: (55 + 58 + 60) / 3 = 57.67

More than 2 hours: (65 + 67 + 70) / 3 = 67.33

Hybrid:

Less than 2 hours: (62 + 64 + 66) / 3 = 64

More than 2 hours: (70 + 72 + 75) / 3 = 72.33

Table of Average Test Scores

Teaching Method	Less than 2 hours	More than 2 hours
Traditional	65	71
Online	57.67	67.33
Hybrid	64	72.33

Solution:

Teaching Method(Treatments) Study Time(Blocks)	Traditional	Online	Hybrid	Total
Less than 2 hours	65	57.67	64	186.67
More than 2 hours	71	67.33	72.33	210.66
Total	136	125	136.33	G=397.33

Step 1: CF= G^2 / n = $(397.33)^2$ /6 =26311.85

Step 2: SST =$\sum\sum x_{i,j}^2 - CF = 26452.79 - 26311.85 = 140.94$

Step 3: For column SSTR = $\frac{\sum C_i^2}{n_i} - CF = \frac{(136)^2}{2} + \frac{(125)^2}{2} + \frac{(136.33)^2}{2} - 26311.85$ =41.58

Step 4: For row SSBL = $\frac{\sum R_i^2}{n_j} - CF = \frac{(186.67)^2}{3} + \frac{(210.66)^2}{3} - 26311.85 = 95.92$

Step 5: SSE =$SST - SSTR - SSBL = 140.94 - 41.58 - 95.92$ =3.44

Step 6: ANOVA Table

F table: http://www.socr.ucla.edu/Applets.dir/F_Table.html

Two-way ANOVA Table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between Treatments (Teaching Method)	k – 1 =2	SSTR =41.58	**MSTR = SSTR/ k-1 =20.79**	For Treatment F =MSTR/MSE =12.08
Between Blocks (Study Time)	m– 1 =1	SSBL = 95.92	**MSBL = SSBL/ m-1 =95.92**	For Block F =MSBL/MSE =55.77
Within groups	(k-1)*(m-1)=2	SSE =3.44	**MSE = SSE/ (k-1)*(m-1) =1.72**	

F tab value for treatment =(2, 2) d.o.f. is 19 and thus Fcal < F tab hence Accept H_0

F tab value for block =(1, 2) d.o.f. is 18.51 and thus Fcal > F tab hence Do not Accept H_0

Practice Problem 1:

The number of defective hard drives produced daily by a production line can be modeled as Poisson's distribution. The counts for ten days are 7 3 1 2 4 1 2 3 1 2. Obtain max likelihood estimate of probability of 0 or 1 defectives on one day.

Solution From the maximum likelihood estimate of λ is $\hat{\lambda} = \bar{x} = 26/10 = 2.6$. Consequently, by the invariance property, the maximum likelihood estimate of

$$P(X = 0 \quad \text{or} \quad 1) = e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!}$$

is

$$e^{-\hat{\lambda}} + \frac{\hat{\lambda} e^{-\hat{\lambda}}}{1!} = e^{-2.6} + \frac{2.6 \cdot e^{-2.6}}{1!} = 0.267$$

There will 1 or fewer defectives on just over one-quarter of the days.

Practice Problem 2:

Consider a sample 0,1,0,0,1,0 from a binomial distribution, with the form P[X=0]=(1-p), P[X=1]=p. Find the maximum likelihood estimate of p.

Solution :

L(p)=P[X=0] P[X=1] P[X=0] P[X=0] P[X=1] P[X=0]

=(1-p) p (1-p) (1-p) p (1-p)

$=(1-p)^3p^2$.

Log L(p)=log[$(1-p)^3p^2$.]=log[$(1-p)^3$]+log[(p^2).]=3log(1-p)+2logp

$$\frac{\partial LogL(p)}{\partial p}=0 \quad \textbf{means,} \quad \frac{-3}{1-p}+\frac{2}{p}=0 \Rightarrow \frac{-3p+2-2p}{p(1-p)}=0 \Rightarrow p=1/3$$

That is , there is 1/3 chance to observe this sample if we believe the population to be Binomial distributed .

Practice Problem 3:

Linda is using ANOVA to measure whether there is a difference between the average weekly sales of her 3 salespeople. The test will be at the 0.05 level of significance.

	Weekly Sales(x) in Thousands of Dollars for 3 Treatments(T)				
Salesperson	L	7	6	7	4
	M	6	8	6	6
	N	9	8	7	10

Practice Problem 4:

Construct an analysis of variance table, and test the equality of mean weights with $\alpha = 0.05$. Each laboratory measures the tin-coating weights of 12 disks and that the results are as follows:

Laboratory A	Laboratory B	Laboratory C	Laboratory D
0.25	0.18	0.19	0.23
0.27	0.28	0.25	0.30
0.22	0.21	0.27	0.28
0.30	0.23	0.24	0.28
0.27	0.25	0.18	0.24
0.28	0.20	0.26	0.34
0.32	0.27	0.28	0.20
0.24	0.19	0.24	0.18
0.31	0.24	0.25	0.24
0.26	0.22	0.20	0.28
0.22	0.29	0.21	0.22
0.28	0.16	0.19	0.21

Practice Problem 5:

A clinical trial on Iron intake of four groups of patients (mg). Test at 5% level of significance is there any significant difference in mean iron intake among ten groups of patients as well as between three trimesters?

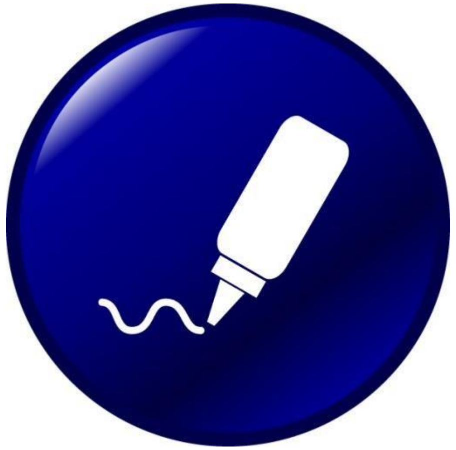
Trimester	Group means for iron intake									
	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9	G_{10}
I	11.5	19.5	18.5	12.5	18.5	16.5	26.5	18.5	16.0	24.5
II	27.0	28.0	22.0	21.0	15.0	19.5	20.0	26.0	30.0	28.5
III	28.0	30.0	26.0	30.0	24.5	28.5	26.0	30.0	27.0	25.5

Practice Problem 6:

- An experiment was performed to determine the effect of four different chemicals on the strength of a fabric.
- These chemicals are used as part of the permanent press finishing process.
- Five fabric samples were selected, and a randomized complete block design was run by testing each chemical type once in random order on each fabric sample.
- The data are shown in Table.
- Test for differences in means using an ANOVA with alpha = 0.01.

	Fabric Sample				
Chemical Type	1	2	3	4	5
1	1.3	1.6	0.5	1.2	1.1
2	2.2	2.4	0.4	2.0	1.8
3	1.8	1.7	0.6	1.5	1.3
4	3.9	4.4	2.0	4.1	3.4

Problem 7



Three drying formulas for curing glue are studied

Formula A	13	10	8	11	8	
Formula B	13	11	14	14		
Formula C	17	14	13	10	11	12

Test at 5% level of significance whether is any difference in the mean curing time of glue?

Solution:

Grand total , G =179

Step 1: Correction Factor (CF)= G^2 / n = $(179)^2$ /15 =2136.067

Step 2: SST =$\sum\sum x_{i,j}^2 - CF = 2219.5 - 2136.067 = 82.93$

Step 3: SSTR = $\frac{\sum C_i^2}{n_i} - CF$ =2164.167-2136.067=28.1

Step 4: SSE =$\mathrm{SST} - \mathrm{SSTR}$ =82.93-28.1=54.83

Step 5: ANOVA Table

F table: http://www.socr.ucla.edu/Applets.dir/F_Table.html

innovate achieve lead

One-way ANOVA table

Source of variation	df	Sum of squares	Mean sum of squares	F-ratio
Between groups	2	28.1	14.05	F=3.078
Within groups	12	54.83	4.569	
Total	14	82.93	$F(3.078) < F_{(0.05;\ 2,\ 12)} = 3.885$	

H_0 not rejected and **H_1 rejected**

P – value = 0.084

IMP Note to Self

Thank You