Q Answer the following questions with justifications.

(1) Given the characteristic equation of a matrix A, can we compute the characteristic equation of cA where c is a non-zero scalar without knowing the entries of A? If so, show how to do it using detailed calculations. Otherwise explain why it is not possible. Clearly state all your assumptions

(2 Marks)

(2) Consider an inner product space with an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x^T A y}$ defined with the help of matrix $\mathbf{A}$ defined below.

$$\mathbf{A} = \begin{bmatrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{bmatrix}$$

Consider two vectors $\mathbf{a} = [1 \quad 5]^T$ and $\mathbf{b} = [2 \quad 7]^T$ in the inner product space.

(a) Find the distance $d(\mathbf{a}, \mathbf{b}) = \|a - b\|$ between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space where $\| \cdot \|$ is the norm induced by the inner product.

(2 marks)

(b) Find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ in the above inner product space.

(2 marks)

A Answers

(1) The characteristic equation of the matrix cA is $det(cA - \lambda I) = 0$. We can rewrite this as $det(c(A - \frac{\lambda}{c} I)) = 0$ which can then be written as $c^n det(A - \frac{\lambda}{c} I) = 0$ or $det(A - \frac{\lambda}{c} I) = 0$. Let $\mu = \frac{\lambda}{c}$, and assume that the characteristic equation $det(A - \mu I) = 0$ can be written in terms of the polynomial $\mu^n + a_{n-1}\mu^{n-1} + \ldots a_1\mu + a_0 = 0$. Substituting $\lambda/c = \mu$ in this polynomial we get $(\frac{\lambda}{c})^n + a_{n-1}(\frac{\lambda}{c})^{n-1} + \ldots a_1(\frac{\lambda}{c}) + a_0 = 0$. Multiplying throught by c^n we finally get $\lambda^n + ca_{n-1}\lambda^{n-1} + \ldots a_1 c^{n-1}\lambda + a_0 c^n = 0$. Thus we see that the characteristic equation of cA can be obtained by taking the coefficient a_k of the kth term of the characteristic equation of A and multiplying it by c^{n-k}. Thus there is no need to look at the entries of the matrix A.

Marking Scheme: 1 Mark $\rightarrow$ expanding the determinant $det(A - \frac{\lambda}{c} I)$, and obtaining the equation $\lambda^n + ca_{n-1}\lambda^{n-1} + \ldots a_1 c^{n-1}\lambda + a_0 c^n = 0$. 1 Mark $\rightarrow$ remaining argument.

(2) (a)

$$\mathbf{(d(a, b))^2} = \begin{bmatrix} 1-2 & 5-7 \end{bmatrix} \begin{bmatrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{bmatrix} \begin{bmatrix} 1-2 \\ 5-7 \end{bmatrix} = \mathbf{21.5}$$

$d(\mathbf{a}, \mathbf{b}) = 4.63$

(b)

$$\cos(\theta) = \frac{\mathbf{a^T A b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\mathbf{a^T A b}}{\sqrt{\mathbf{a^T A a}}\sqrt{\mathbf{b^T A b}}}$$

$$\cos(\theta) = \frac{178}{\sqrt{128}\sqrt{249.5}} = 0.005573$$

$\theta = 89.6$ deg

Marking Scheme: 1 mark each for proper construction of equation. 1 mark for correct numerical answer

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(A) Using elementary row operations, write the matrix in its row echelon form

(2 marks)

(B) Let $\mathbf{V}$ be a vector subspace spanned by the columns of matrix $\mathbf{A}$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(C) Let $\mathbf{V}$ be a vector subspace spanned by vectors $\mathbf{x}$, such that $\mathbf{Ax} = 0$. Find the basis and dimension of $\mathbf{V}$.

(2 marks)

(D) Give the set of linearly independent rows of $\mathbf{A}$. What is the number of vectors in this set?

(2 marks)

A Answers

(A)

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 + 3*R_1, R_3 \leftarrow R_3 - 2*R_1, R_2 \leftarrow R_2/5, R_3 \leftarrow R_3 - R_2, R_1 \leftarrow R_1 - 2*R_2}$$

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

NOTE: REF is not unique.

1 Mark for correct elementary row operations, 1 Mark for getting correct pivot columns

(B) column number 1 and column number 3 have pivot.

Basis of column space is:

$$\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(1 mark)

Rank of col space = dimension of $\mathbf{V} = 2$ (1 mark)

(C) From REF, the null space is:

$$\begin{bmatrix} x \\ r \\ y \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} t$$

basis =

$$\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}$$

(1 mark)

Rank of null space = dimension of $\mathbf{V} = 3$ (1 mark)

(D) From REF, row 1 and row 2 form basis of row space (1 mark)

Rank of row space = 2 (1 mark)

Q Answer the following for the given matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(A) Obtain the left-singular vectors of $\mathbf{A}$.

(3 marks)

(B) Obtain the right-singular vectors of $\mathbf{A}$.

(3 marks)

(C) Obtain the singular value matrix $\mathbf{\Sigma}$. What is the spectral norm of $\mathbf{A}$?

(2 marks)

A Answers

(A)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{A^T} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{AA^T} = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

eigenvalues and eigenvectors of AA^T :

$$\lambda_1 = \sigma_1^2 = 6, u_1 = \begin{bmatrix} 5 & 2 & 1 \end{bmatrix}^T, \|u_1\| = \sqrt{30}$$

$$\lambda_2 = \sigma_2^2 = 1, u_2 = \begin{bmatrix} 0 & \frac{-1}{2} & 1 \end{bmatrix}^T, \|u_2\| = \frac{\sqrt{5}}{2}$$

$$\lambda_3 = \sigma_3^2 = 0, u_3 = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}^T, \|u_3\| = \sqrt{6}$$

Singular value matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

Left singular vectors

$$\mathbf{U} = \begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} \end{bmatrix}$$

Marking Scheme: 1 mark for getting eigenvalues, 1 mark for eigenvectors, 1 mark for construction of U

(B) Right Singular vectors:

$$v_1 = \frac{1}{\sigma_1}\mathbf{A}^T u_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \frac{1}{\sqrt{30}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \frac{6}{\sqrt{6}\sqrt{30}}$$

$$v_2 = \frac{1}{\sigma_2}\mathbf{A}^T u_2 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \frac{2}{\sqrt{5}} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}}$$

Marking Scheme: 1 mark for getting eigenvalues, 1 mark for eigenvectors, 1 mark for construction of U

(C) Singular value matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{1} \\ 0 & 0 \end{bmatrix}$$

Spectral norm = $\sqrt{6}$ Marking Scheme: 1 mark for matrix, 1 mark for spectral norm

Q Answer the following

(1) Compute the following for the function $f(x_1, x_2) = e^{x_1} + x_1x_2 - \log(1 + x_2)$.

(A) The expression for gradient, its dimension and its value at $(1, 2)$.

(2 marks)

(B) The expression for the Hessian matrix, its dimension and its value at $(1, 2)$.

(2 marks)

(C) The derivative $\frac{df}{dt}$ using chain rule of differentiation when $x_1 = t^2 + 2at,\ x_2 = \sin(t)$.

(2 marks)

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(\boldsymbol{x}) = (\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})$ where $\boldsymbol{A} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. Find Taylor's polynomial of degree 1 of f at $[1, 1]$.

(2 Marks)

A Answers

(1) (A)

$$\nabla_{\boldsymbol{x}} f = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{x_1} + x_2 & x_1 - \frac{1}{1+x_2} \end{bmatrix}$$

$$\nabla_{\boldsymbol{x}} f(1, 2) = \begin{bmatrix} e^1 + 2 & \frac{2}{3} \end{bmatrix}$$

Marking Scheme: 1 mark for expression, 1 mark for value at (1,2)

(B)

$$\nabla^2_{\boldsymbol{x}} f = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} e^{x_1} & 1 \\ 1 & \frac{1}{(1+x_2)^2} \end{bmatrix}$$

$$\nabla^2_{\boldsymbol{x}} f(1, 2) = \begin{bmatrix} e^1 & 1 \\ 1 & \frac{1}{9} \end{bmatrix}$$

Marking Scheme: 1 mark for expression, 1 mark for value at (1,2)

(C)

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t}$$

$$= (e^{x_1} + x_2)(2t + 2a) + (x_1 - \frac{1}{1 + x_2})(\cos t)$$

$$= (e^{(t^2+2at)} + \sin t)(2t + 2a) + (t^2 + 2at - \frac{1}{1 + \sin(t)})\cos t$$

Marking Scheme: 1 mark for chain rule expression, 1 mark for final solution

(2) Using the identity, we get $\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) = 2(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b})^T \boldsymbol{A}$.
Therefore, we have

$$\begin{aligned} f([1,1]^T) &= 2 \\ \nabla_{\boldsymbol{x}} f([1,1]^T) &= [10, 10] \end{aligned}$$

Therefore, if $\boldsymbol{x} = [x_1, x_2]^T$, Taylor's Polynomial is defined as

$$\begin{aligned} T_1 f(\boldsymbol{x}) &= f([1,1]^T) + \nabla_{\boldsymbol{x}} f([1,1]^T)[(x_1 - 1), (x_2 - 1)]^T, \\ &= 10x_1 + 10x_2 - 18 \end{aligned}$$

Marking scheme: 0.5 Marks for computing $f([1,1]^T)$, 2 Marks for computing $\nabla_{\boldsymbol{x}} f$ and 0.5 Marks for $\nabla_{\boldsymbol{x}} f([1,1]^T)$. 1 Mark for the computing Taylor's polynomial.