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Assignment - 1 MFML Section 9

Question 1:

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Definitions:

1: $(V, +)$:

- V is a set with cardinality at least 2
- $(V, +)$ is an Abelian Group

2: $c \cdot v = 0$, for any $c \in R$ (real number)
Scalar Multiplication — where 0 is the identity element

Vector Space Requirements: A vector space over the feild must satisfy 10 axioms, b/w addition and scalar multiplication.

Verify Addition Axioms:

1: Closure: For all $u, v \in V$, $u + v \in V$

2: Associativity: $(u+v) + w = u + (v+w)$ for all $u, v, w \in V$

3: Inverse Element: For each $v \in V$, there exist $-v \in V$ such that $v + (-v) = 0$

4: Identity Element: There exists $0 \in V$ such that $v + 0 = v$ for all $v \in V$

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Commutativity: $u+v = v+u$ for all $u, v \in V$

Verify Scalar Multiplication Axioms

1. Distributivity of Scalar Mutiplication over vector:

$$c\cdot(u+v) = c\cdot u + c\cdot v \quad \text{for all } c \in R, u, v \in V$$

a. $c\cdot(u+v) = 0$ (by defination of scalar multiplication)

b. $c\cdot u + c\cdot v = 0 + 0 = 0$

Therefore, axiom holds.

2. Distributivity of Scalar Multiplication over field addition:

$$(c+d)\cdot v = c\cdot v + d\cdot v \quad \text{for all } c, d \in R, v \in V$$

a. $(c+d)\cdot v = 0$ (by definition)

b. $c\cdot v + d\cdot v = 0 + 0 = 0$

Therefore, axiom holds

3. Compatibility of scalar multiplication with field multiplication:

$$(cd)\cdot v = c\cdot(d\cdot v) \quad \text{for all } c, d \in R, v \in V$$

a. $(cd)\cdot v = 0$ and $c\cdot(d\cdot v) = c\cdot 0 = 0$

Therefore, axiom holds

4. Identity element of scalar multiplication

$$1 \cdot v = v \quad \text{for all } v \in V.$$

a. $1 \cdot v = 0$ (by defination), but this is not equal to v unless $v = 0$

This axiom fails

Since, the identity element axiom for scalar multiplication

$$1 \cdot v = v$$

fails, do not satisfy

$$\boxed{(V, +, \cdot) \text{ is not a vector space}}$$

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Question 2:

To express a vector x with respect to an orthogonal basis $\{b_1, b_2, \dots b_n\}$,

$$x = y_1 b_1 + y_2 b_2 + \dots\dots + y_n b_n.$$

Given:

Orthogonalty: The basis vectors $\{b_i\}$ are mutually orthogonal

$$b_i \cdot b_j = 0 \quad \text{for } i \neq j$$

Representation of x: Any vector x can be written as a linear combination of the basis vectors:

$$x = y_1 b_1 + y_2 b_2 + \dots + y_n b_n$$

where $y_1, y_2 \dots y_n$ are the components of x with respect to the basis

Objective: Finds the components $y_1, y_2 \dots y_n$

Compute Inner Products: For each basis vector, $\langle x, b_i \rangle$.

Components: Use the inner products

$$y_i = \frac{\langle x, b_i \rangle}{\langle b_i, b_i \rangle}.$$

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Assuming that the vectors x and b_i are n-dimensional

The inner product calculation $\langle x, b_i \rangle$ takes $O(n)$

Calculating $\langle b_i, b_i \rangle$ also takes $O(n)$.

Therefore, to find all components y_i (for $i = 1, 2, \ldots n$)

$$O(n^2)$$

Conclusion: Fastest algorithm to uncover the component $y_1, y_2 \ldots y_n$ of the vector x with respect to the orthogonal basis $\{b_1, b_2 \ldots b_n\}$

Divide the two for each i to get the component.

This approch keeps the calculation efficiet & straightforward, leveraging the population of orgonaccly

Therefore

$O(n^2)$ algorithm to compute each component

$$y_i = \frac{\langle x, b_i \rangle}{\langle b_i . b_i \rangle} \text{ or } y_i = \frac{x . b_i}{\|b_i\|^2}$$

for $i = 1, 2, \ldots n$

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Question 3:

Given: $A = bb^T$, where $b = [1, p, q]^T$

(i) Nullspace of A and its dimension

1) Matrix A: $A = bb^T$

$$A = \begin{bmatrix} 1 \\ p \\ q \end{bmatrix} \begin{bmatrix} 1 & p & q \end{bmatrix} = \begin{bmatrix} 1.1 & 1.p & 1.q \\ p.1 & p.p & p.q \\ q.1 & q.p & q.q \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & p & q \\ p & p^2 & pq \\ q & pq & q^2 \end{bmatrix}$$

Nullspace of A $\quad x \in R^3$

$$Ax = 0$$

$$\begin{bmatrix} 1 & p & q \\ p & p^2 & pq \\ q & pq & q^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_1 + px_2 + qx_3 = 0 \quad -①$$
$$px_1 + p^2x_2 + pqx_3 = 0 \quad -②$$
$$qx_1 + pqx_2 + q^2x_3 = 0 \quad -③$$

Notice that the second & third equations are scalar multiples of the first equation

Therefore, all 3 equations reduce to:

$$x_1 + px_2 + qx_3 = 0$$

$$x_1 = -px_2 - qx_3$$

Let $x_2 = t, x_3 = s$

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$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -pt-qs \\ t \\ s \end{bmatrix} = t\begin{bmatrix} -p \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -q \\ 0 \\ 1 \end{bmatrix}$$

$$Null(A) = span\left\{ \begin{bmatrix} -p \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -q \\ 0 \\ 1 \end{bmatrix} \right\}$$

Dimension of Nullspace

$$\dim(Null(A)) = 2$$

ii) Eigenvalues of A:

1) Eigenvalues of the Outer Product Matrix: The matrix $A = bb^T$ has a special structure. The rank of A is 1, meaning it has one nonzero eigenvalue, which corresponding to the magnitude of the vector b, and the rest of the eigenvalue are zero

2) Non zero Eigenvalue: The nonzero eigenvalue is given by the trace of A, since the trace of a matrix equals the sum of its eigenvalue

$$trace(A) = b^Tb = 1^2 + p^2 + q^2$$

$$\lambda_1 = 1 + p^2 + q^2$$

3) Zero Eigenvalue: The remaining two eigenvectors are zero

$$\lambda_1 = 1+p^2+q^2, \lambda_2 = 0, \lambda_3 = 0$$

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Question 4:

Given:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ \alpha \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

equation $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ \alpha \end{bmatrix}$$

For $Ax=b$, to have a solution, b must lie in the coloumn space (range) of A.

$$\text{Rank}(A) = \text{Rank}([A|B])$$

a) Compute A rank(A): $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

The second row is a multiple of the first row $2 \times row_1 = row_2$ therefore

$$\text{Rank}(A) = 1$$

b) Augmented Matrix: $[A|b]$

$$[A|b] = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & \alpha \end{bmatrix}$$ performing row reduction

subtract $2 \times row_1$ from row_2

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & \alpha - 6 \end{bmatrix}$$, For system to be consitant

$$\alpha - 6 = 0 \qquad \boxed{\alpha = 6}$$

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Question 5 :

Given: Three vectors in $\mathbb{R}^3$:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

to proof:

$$\text{span}\{u, v, w\} = \mathbb{R}^3 \text{ if \& only if } \det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right) \neq 0$$

1) Span & Linear Independence : Span of $\{u, v, w\}$ is $\mathbb{R}^3$ if & only if vectors are linear independent

If they were linearly dependent, one could be expressed as a linear combination of the others, which would restrict the span to a plane

2) Determinant Condition : The condition for 3 vectors to be linearly independent is equivalent to the determinant of a matrix,

$$\det\left(\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}\right) = 0$$

The determinant is non-zero,

$$c_1 u + c_2 v + c_3 w = 0 \quad \text{is } c_1 = c_2 = c_3 = 0$$

Hence Profed:

$$\det(M) \neq 0, \ \text{span}\{u, v, w\} \text{ equal } \mathbb{R}^3$$