BITS Pilani
Pilani Campus

# Mathematical Foundations

MFDS Team

# * ZC416,  Lecture 0

# Agenda

- Matrices and their types
- REF and RREF
- Rank, its computation and properties
- Determinant, its computation and properties
- Consistency and inconsistency of linear systems
- Nature of solutions of linear systems

# Matrices



- A **matrix** is a rectangular array of numbers or functions which we will enclose in brackets. For example,

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad \textbf{(1)}$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

- The numbers (or functions) are called **entries** or, less commonly, *elements* of the matrix.
- The first matrix in (1) has two **rows**, which are the horizontal lines of entries.

# Matrix – Notations



- We shall denote matrices by capital boldface letters **A**, **B**, **C**, … , or by writing the general entry in brackets; thus **A** = $[a_{jk}]$, and so on.
- By an ***m*** × ***n*** **matrix** (read *m by n matrix*) we mean a matrix with $m$ rows and $n$ columns—rows always come first! $m \times n$ is called the **size** of the matrix. Thus an $m \times n$ matrix is of the form

$$\mathbf{A} = \left[a_{jk}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}. \qquad \textbf{(2)}$$

# Vectors

- A **vector** is a matrix with only one row or column. Its entries are called the **components** of the vector.
- We shall denote vectors by ***lowercase*** boldface letters **a**, **b**, … or by its general component in brackets, $\mathbf{a} = [a_j]$, and so on. Our special vectors in (1) suggest that a (general) **row vector** is of the form

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{a} = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}.$$

A **column vector**

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}. \quad \text{For instance,} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}.$$

# Equality of Matrices

- Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if (1) they have the same size and (2) the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on.

- Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

# Algebra of Matrices

## 1. Addition of Matrices

- The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

## 2. Scalar Multiplication (Multiplication by a Number)

- The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** $c$ (number $c$) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by $c$.

(a) $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(b) $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$ (written $\mathbf{A}+\mathbf{B}+\mathbf{C}$)

(c) $\mathbf{A}+\mathbf{0}=\mathbf{A}$

(d) $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$.

(a) $c(\mathbf{A}+\mathbf{B})=c\mathbf{A}+c\mathbf{B}$

(b) $(c+k)\mathbf{A}=c\mathbf{A}+k\mathbf{A}$

(c) $c(k\mathbf{A})=(ck)\mathbf{A}$ (written $ck\mathbf{A}$)

(d) $1\mathbf{A}=\mathbf{A}$.

- Here $\mathbf{0}$ denotes the **zero matrix** (of size $m \times n$), that is, the $m \times n$ matrix with all entries zero.

# Matrix Multiplication

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## <u>Multiplication of a Matrix by a Matrix</u>

•The **product C** = **AB** (in this order) of an $m \times n$ matrix **A** = $[a_{jk}]$ times an $r \times p$ matrix **B** = $[b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix **C** = $[c_{jk}]$ with entries

**(3)**
$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + a_{j2} b_{2k} + \cdots + a_{jn} b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array}$$

- The condition $r = n$ means that the second factor, **B**, must have as many rows as the first factor has columns, namely $n$. A diagram of sizes that shows when matrix multiplication is possible is as follows:

$$\begin{array}{ccccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p]. \end{array}$$

# Matrix Multiplication

## EXAMPLE 1

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & \boxed{14} & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

- Here $c_{11} = 3 \cdot 2 + 5 \cdot 5 + (-1) \cdot 9 = 22$, and so on. The entry in the box is $c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$.

- The product **BA** is not defined.

# Matrix Multiplication

**Matrix Multiplication Is *Not Commutative*, AB ≠ BA in General**

- This is illustrated by Example 1, where one of the two products is not even defined. But it also holds for square matrices. For instance,

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{but} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}.$$

- It is interesting that this also shows that $\mathbf{AB} = \mathbf{0}$ does *not* necessarily imply $\mathbf{BA} = \mathbf{0}$ or $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

# Transposition of Matrices & Vectors

- The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of **A** as its first *column*, the second *row* of **A** as its second *column*, and so on. Thus the transpose of **A** in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = \left[a_{kj}\right] = \begin{bmatrix} a_{11} & \mathrm{a21} & \cdots & a_{m1} \\ \mathrm{a12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

- As a special case, transposition converts row vectors to column vectors and conversely.

# Transposition of Matrices

- Rules for transposition are

$$
\begin{aligned}
&\text{(a)} \quad (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A} \\
&\text{(b)} \quad (\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}} \\
&\text{(c)} \quad (c\mathbf{A})^{\mathsf{T}} = c\mathbf{A}^{\mathsf{T}} \\
&\text{(d)} \quad (\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}.
\end{aligned}
\qquad \textbf{(5)}
$$

**CAUTION!** Note that in (5d) the transposed matrices are ***in reversed order.***

# Special Matrices

- Symmetric: $a_{ij} = a_{ji}$ Eg: $\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$
- Skew Symmetric : $a_{ij} = -\, a_{ji}$ Eg: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$
- Triangular: Upper Triangular → $a_{ij} = 0$ for all $i > j$
  Lower Triangular → $a_{ij} = 0$ for all $i < j$

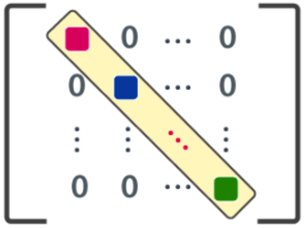
Upper triangular matrix: U

$$\begin{bmatrix} 1 & 1/2 & 3 & 0 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Lower triangular matrix: L

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 1 & -2 & 0 \end{bmatrix}$$

- Diagonal Matrix: $a_{ij} = 0$ for all $i \neq j$ Eg:



- Sparse Matrix: Many zeroes and few non-zero entities

$$\begin{bmatrix} 0 & 0 & 3 & 0 & 4 \\ 0 & 0 & 5 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 6 & 0 & 0 \end{bmatrix}$$

# Positive Definite matrix

- Let **A** be a real symmetric matrix. Then **A** is positive definite if for any x ≠ 0,

$$\boldsymbol{x^T A x > 0}$$

- Example:
- A = $\begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix}$ $x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$=2x_1^2 + 12x_1x_2 + 20x_2^2 = 2(x_1 + 3x_2)^2 + 2x_2^2 > 0$$

- **A** is positive semi-definite **if** $\boldsymbol{x^T A x \geq 0}$
- A = $\begin{bmatrix} 2 & 6 \\ 6 & 18 \end{bmatrix}$ ⇒ $x^T A x = 2(x_1 + 3x_2)^2 = 0$ when $x_1$=3 and $x_2 = -1$

# Elementary Row Operations



Given a matrix A, the following operations are called Elementary Row Operations

- *Interchange of two rows*
- *Addition of a constant multiple of one row to another row*
- *Multiplication of a row by a* ***non-zero*** *constant c*

**CAUTION!** These operations are for rows, ***not for columns***!

# Row Echelon Form (REF) of a matrix

- Any rows of all zeros are below any other non zero rows.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros
- Example

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 9 \\ 0 & 4 & 5 & 10 & 0 \\ 0 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Reduced Rpw Echelon Form (RREF)

- We say that a matrix is in Reduced Row Echelon Form if it is in Echelon form and additionally,
  1. The leading entry in each row is 1.
  2. Each leading 1 is the only non zero entry in its column

- Example

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 9 \\ 0 & 1 & 4 & 0 & -6 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Uniqueness of Row Reduced Echelon Form

- We can transform any matrix into a matrix in reduced row echelon form by using elementary row operations.

- No matter what sequence of row operations we use each matrix is row equivalent to one and only one reducedrow echelon matrix

# REF of a Matrix

$$\begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ -2 & 2 & 0 & 3 & 2 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix}$$

Swap rows 1 and 2

$$\begin{bmatrix} \color{red}{-2} & 2 & 0 & 3 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 2 & 8 & 20 & 0 & 6 \end{bmatrix}$$

Replace R3 by R3+1.R1

$$\begin{bmatrix} \color{red}{-2} & 2 & 0 & 3 & 2 \\ 0 & \color{red}{1} & 2 & 0 & -1 \\ 0 & 10 & 20 & 3 & 8 \end{bmatrix}$$

Replace R3 by R3+(-10).R2

$$\begin{bmatrix} \color{red}{-2} & 2 & 0 & 3 & 2 \\ 0 & \color{red}{1} & 2 & 0 & -1 \\ 0 & 0 & 0 & \color{red}{3} & 18 \end{bmatrix}$$

Row Echelon Form

# Rank of a matrix

- The number of nonzero rows, $r$, in the reduced row (or row echelon form) coefficient matrix **R** is called the **rank of R** and also the **rank of A**.

- The rank is **invariant** under elementary row operations:

# Determination of Rank

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \text{Row 2} + 2\text{ Row 1} \\ \text{Row 3} - 7\text{ Row 1} \end{matrix}$$

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{Row 3} + \frac{1}{2}\text{ Row 2.}$$

- The last matrix is in row-echelon form and has two nonzero rows. Hence rank $\mathbf{A} = 2$.

# Minor

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor.

Delete first row and column from **A**. The determinant of the remaining 2x2 submatrix is the minor of $a_{11}$ which is $a_{22}a_{33} - a_{23}a_{32}$

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

# Minor

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for $a_{13}$ is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

# Cofactor



The cofactor $C_{ij}$ of an element $a_{ij}$ is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

# Determinant

The determinant of an n x n matrix **A** can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \ldots + a_{1n}c_{1n}$$

The determinant of **A** is therefore the sum of the products of the elements of the first row of **A** and their corresponding cofactors.

(It is possible to define |**A**| in terms of any other row or column but for simplicity, the first row only is used)

# Determinant -Example



Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

# Determinant -Example

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=a_{22}a_{33}-a_{23}a_{32}$$

$$c_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=-(a_{21}a_{33}-a_{23}a_{31})$$

$$c_{13}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=a_{21}a_{32}-a_{22}a_{31}$$

# Determinants – Example



$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2-0) - (0)(0+3) + (1)(0+2) = 4$$

# Adjoint

The adjoint matrix of **A**, denoted by adj **A**, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

**A**(adj **A**) = (adj**A**) **A** = |**A**| **I**

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

# Adjoint

$$A(adjA)=\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

$$(adjA)A=\begin{bmatrix}4 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & 4\end{bmatrix}=\begin{bmatrix}10 & 0\\ 0 & 10\end{bmatrix}=10I$$

# Properties of Determinants

1. det(AB) =det(A) * det (B)
2. det(A) nonzero implies there exists a matrix B such that AB=BA=I
3. Two Rows Equal➔ det = 0(Singular)
4. $R_i$ and $R_j$ swapped ➔ det gets a minus sign ($i \neq j$)
5. det(A) = det($A^T$)
6. $R_i \leftarrow cR_j$ ➔det A ←c detA

# Inverse

- $A^{-1} = \dfrac{adj(A)}{det(A)}$ where $det(A) \neq 0$

Reiterate $det(A) \neq 0 \rightarrow$ A is Non singular

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

# Inverse – 2x2 Example

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Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^{-1}A = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

# Inverse – 3x3 Example

Example 2

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of **A** is

$$|\mathbf{A}| = (3)(-1-0)-(-1)(-2-0)+(1)(4-1) = -2$$

The elements of the cofactor matrix are

$$c_{11} = +(-1), \quad c_{12} = -(-2), \quad c_{13} = +(3),$$
$$c_{21} = -(-1), \quad c_{22} = +(-4), \quad c_{23} = -(7),$$
$$c_{31} = +(-1), \quad c_{32} = -(-2), \quad c_{33} = +(5),$$

# Inverse – 3x3 Example

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

so

$$adjA = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

and

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

# Inverse

The result can be checked using

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants

# Inverse -Simple 2 x 2 case

So that for a 2 x 2 matrix the inverse can be constructed in a simple fashion as

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Exchange elements of main diagonal
- Change sign in elements off main diagonal
- Divide resulting matrix by the determinant

# Inverse -Simple 2 x 2 case

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

$$A^{-1} = -\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.3 \\ 0.4 & -0.2 \end{bmatrix}$$

Check inverse

**$A^{-1}$ A=I**

$$-\frac{1}{10}\begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

# Linear System

- A **linear system of *m* equations in *n* unknowns** $x_1, \ldots,$ $x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \qquad \textbf{(1)}$$

- The system is called *linear* because each variable $x_j$ appears in the first power only, just as in the equation of a straight line.

# Linear System

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2 \qquad \textbf{(1)}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m.$$

•$a_{11}, \ldots , a_{mn}$ are given numbers, called the **coefficients** of the system.
•$b_1, \ldots , b_m$ on the right are also given numbers.
•If all the *bj* are zero, then **(1)** is called a **homogeneous system**.
•If at least one *bj* is not zero, then **(1)** is called a **non-homogeneous system**.

# Coefficient Matrix



**Matrix Form of the Linear System (1).**

•From the definition of matrix multiplication we see that the $m$ equations of (1) may be written as a single vector equation

$$\mathbf{Ax} = \mathbf{b} \qquad \textbf{(2)}$$

where the **coefficient matrix A** = $[a_{jk}]$ is the $m \times n$ matrix are column vectors.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

# Augmented Matrix



**Matrix Form of the Linear System (1). (continued)**

- We assume that the coefficients $a_{jk}$ are not all zero, so that **A** is not a zero matrix. Note that **x** has $n$ components, whereas **b** has $m$ components. The matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1).

- The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix **A** but came from vector **b**. Thus, we *augmented* the matrix **A**.

# Solution to System of Linear Equations

- A linear system (1) is called **overdetermined** if it has more equations than unknowns, **determined** if $m = n$, and **underdetermined** if it has fewer equations than unknowns.

- A linear system is **consistent** if rank(**A**) = rank($\tilde{\mathbf{A}}$)

- A **consistent** system has at least one solution (thus, one solution or infinitely many solutions), but **inconsistent** has no solutions at all, as $x_1 + x_2 = 1$, $x_1 + x_2 = 0$.

# Solution

The method for determining whether **Ax** = **b** has solutions and what they are:

**(a) No solution.** If $r$ is less than $m$ (meaning that **R** actually has at least one row of all 0s) *and* at least one of the numbers $f_{r+1}, f_{r+2}, \ldots, f_m$ is not zero, then the system **Rx** = **f** is inconsistent: No solution is possible. Therefore the system **Ax** = **b** is inconsistent as well.

# Solution

If the system is consistent (either $r = m$, or $r < m$ *and* all the numbers $f_{r+1}, f_{r+2}, \dots, f_m$ are zero), then there are solutions.

**(b) Unique solution.** If the system is consistent and $r = n$, there is exactly one solution, which can be found by back substitution.

**(c) Infinitely many solutions.** To obtain any of these solutions, choose values of $x_{r+1}, \dots, x_n$ arbitrarily. Then solve the $r$th equation for $x_r$ (in terms of those arbitrary values), then the $(r - 1)$st equation for $x_{r-1}$, and so on up the line.