Containment Probability

Motivation

In box embeddings, we need to compute the probability that one box contains another. For hard boxes, this is straightforward: containment is deterministic, and the probability is either 0 or 1. For Gumbel boxes with random boundaries, we must compute an expectation over the joint distribution of both boxes' coordinates.

The challenge is that we need the expectation of a ratio: $E\left[\frac{\text{Vol}(A \cap B)}{\text{Vol}(B)}\right]$. Unlike the expectation of a product, the expectation of a ratio does not factor simply. However, when the variance of the denominator is small relative to its mean, a first-order Taylor expansion provides an accurate approximation.

Definition

The **containment probability** $P(B \subseteq A)$ measures whether box B is geometrically contained within box A. For hard boxes with deterministic boundaries:

$$P(B \subseteq A) = \frac{\text{Vol}(A \cap B)}{\text{Vol}(B)}$$

This is either 0 (disjoint) or 1 (contained). For Gumbel boxes with random volumes, we compute the expectation of this ratio over the joint distribution of both boxes' coordinates.

Statement

Theorem (First-Order Approximation). For Gumbel boxes with random volumes $\text{Vol}(A \cap B)$ and $\text{Vol}(B)$:

$$E\left[\frac{\text{Vol}(A \cap B)}{\text{Vol}(B)}\right] \approx \frac{E[\text{Vol}(A \cap B)]}{E[\text{Vol}(B)]}$$

The approximation is accurate when the coefficient of variation $\frac{\text{Var}(\text{Vol}(B))}{E[\text{Vol}(B)]^2}$ is small (i.e., when volume variance is controlled by the scale parameter β).

Proof

We approximate $E\left[\frac{X}{Y}\right]$ where $X = \text{Vol}(A \cap B)$ and $Y = \text{Vol}(B)$ are random variables with means $\mu_X = E[X]$ and $\mu_Y = E[Y]$. The function $f(X, Y) = \frac{X}{Y}$ is smooth, so we can expand it in a Taylor series around the mean point (μ_X, μ_Y):

$$f(X, Y) \approx f(\mu_X, \mu_Y) + \frac{\partial f}{\partial X}(\mu_X, \mu_Y)(X - \mu_X) + \frac{\partial f}{\partial Y}(\mu_X, \mu_Y)(Y - \mu_Y) + \text{higher order terms}$$

The partial derivatives are:

- $\frac{\partial f}{\partial X} = \frac{1}{Y}$, evaluated at (μ_X, μ_Y) gives $\frac{1}{\mu_Y}$
- $\frac{\partial f}{\partial Y} = -\frac{X}{Y^2}$, evaluated at (μ_X, μ_Y) gives $-\frac{\mu_X}{\mu_Y^2}$

Taking expectations and using linearity:

$$E[f(X, Y)] \approx \frac{\mu_X}{\mu_Y} + \frac{1}{\mu_Y} E[X - \mu_X] - \frac{\mu_X}{\mu_Y^2} E[Y - \mu_Y]$$

The first-order correction terms vanish because $E[X - \mu_X] = 0$ and $E[Y - \mu_Y] = 0$ by definition of the mean. This leaves:

$$E[f(X, Y)] \approx \frac{\mu_X}{\mu_Y} = \frac{E[\mathrm{Vol}(A \cap B)]}{E[\mathrm{Vol}(B)]}$$

The accuracy of this approximation depends on the magnitude of the higher-order terms, which we analyze next.

Error Analysis

The second-order correction term in the Taylor expansion is:

$$\text{Error} \approx -\frac{\mathrm{Cov}(X, Y)}{\mu_Y^2} + \frac{\mu_X \ \mathrm{Var}(Y)}{\mu_Y^3}$$

This error term reveals when the approximation is accurate:

1. **Small coefficient of variation**: When $\frac{\mathrm{Var}(Y)}{\mu_Y^2}$ is small, the second term is negligible. This occurs when the scale parameter β is small relative to the expected volume, meaning the Gumbel boundaries are tightly concentrated around their means.
2. **Positive correlation**: When X and Y are positively correlated (which occurs naturally when both volumes depend on similar box parameters), the covariance term partially cancels the variance term, reducing the overall error.
3. **Non-vanishing denominator**: When μ_Y is bounded away from zero, the error terms remain well-controlled. This is typically satisfied in practice since boxes have positive expected volume.

The approximation is most accurate when boxes have low variance (small β) and when the intersection volume and box volume are positively correlated, both of which hold in typical box embedding scenarios.

Example

Consider two boxes in 2D where box B is fully contained within box A:
- Box A: $[0.0, 0.0]$ to $[1.0, 1.0]$ (volume = 1.0)
- Box B: $[0.2, 0.2]$ to $[0.8, 0.8]$ (volume = 0.36)

Hard boxes (deterministic):
- Intersection: $[0.2, 0.2]$ to $[0.8, 0.8]$ (volume = 0.36)
- Containment: $P(B \subseteq A) = \frac{0.36}{0.36} = 1.0$ (deterministic containment)

Gumbel boxes (with $\beta = 0.1$, introducing small randomness):
- Expected intersection volume ≈ 0.35 (slightly reduced due to probabilistic boundaries)
- Expected volume of B ≈ 0.35 (similarly affected)
- Containment: $P(B \subseteq A) \approx \frac{0.35}{0.35} = 1.0$

The first-order approximation is accurate here because:
- The coefficient of variation $\frac{\mathrm{Var}(\mathrm{Vol}(B))}{E[\mathrm{Vol}(B)]^2}$ is small (controlled by $\beta = 0.1$)
- The intersection and box volumes are highly correlated (both depend on box B's parameters)
- The expected volumes are well-separated from zero

This demonstrates that the approximation works well in the regime where Gumbel boxes behave similarly to hard boxes, with small probabilistic perturbations.