

# Local Identifiability Problem

## Motivation

**The “flat tire” problem:** Imagine you’re trying to optimize a function, but you’re driving on a perfectly flat parking lot. No matter which direction you turn the steering wheel, you don’t move—there’s no gradient to follow. This is exactly what happens with hard box embeddings when boxes are disjoint or contained: the loss function is flat, and gradient descent has nowhere to go.

**The thought experiment:** Consider two boxes that are completely separate—say, one box represents “cats” and another represents “dogs”. With hard boxes, if they’re disjoint, the intersection volume is exactly zero. Now, if you try to move the “cats” box slightly closer to the “dogs” box, but they’re still disjoint, the intersection volume remains zero. The gradient is zero—no learning signal! The optimizer is stuck, unable to distinguish between “cats box at position A” and “cats box at position B” as long as both keep the boxes disjoint.

Gumbel boxes solve this elegantly by replacing deterministic boundaries with probabilistic ones. Even when boxes are “disjoint” in expectation, there is always some probability of overlap (the boundaries can “wiggle” into each other). This ensures that every parameter configuration produces a distinct expected loss—the loss landscape is never completely flat. The optimizer always has a gradient to follow, even if it’s small. It’s like replacing the flat parking lot with a gentle slope: you can always tell which way is “downhill”.

**Historical development:** The evolution from hard boxes to Gumbel boxes follows a clear progression. Vilnis et al. (2018) introduced hard boxes with deterministic boundaries, but optimization stalled when boxes were disjoint or contained—the gradient vanished. Li et al. (2019) proposed “smoothed boxes” using Gaussian convolution, which restored gradients but broke max-stability: intersections of Gaussian boxes are not Gaussian, forcing expensive numerical integration at each step. Dasgupta et al. (2020) solved both problems simultaneously by using Gumbel distributions: probabilistic boundaries restore differentiability, while max-stability preserves algebraic closure. This is why Gumbel distributions, rather than other smooth distributions, are used in box embeddings—they are the only family that provides both smoothness and closure under intersection.

**Why Gumbel, not Gaussian?** The local identifiability problem appears throughout machine learning. Common solutions—Gaussian smoothing, temperature annealing, entropy regularization—each address different aspects. Gumbel boxes unify these: the scale parameter  $\beta$  acts as temperature, Gumbel noise provides smoothing, and max-stability ensures mathematical consistency. The key insight is that Gumbel distributions are the only max-stable family with infinite support—this combination is unique. Gaussian distributions, while smooth, are not max-stable; other max-stable families (Fréchet, Weibull) have bounded support, creating new optimization problems. Gumbel boxes thus represent the minimal solution: the simplest distribution that simultaneously provides smoothness, infinite support, and algebraic closure.

## Definition

The **local identifiability problem** arises when multiple distinct parameter configurations produce identical loss values, creating flat regions in the loss landscape where the gradient  $\nabla_{\theta} L(\theta) = 0$ . This prevents learning because gradient descent has no direction to follow—all parameter values in the flat region yield the same loss, so the optimizer cannot distinguish between them.

## Statement

**Theorem (Local Identifiability Problem).** For hard boxes, the loss landscape has flat regions with zero gradients:

1. **Disjoint boxes:** When boxes  $A$  and  $B$  are completely disjoint,  $\text{Vol}(A \cap B) = 0$  regardless of their separation distance. Any local perturbation preserving disjointness yields zero gradient.
2. **Contained boxes:** When box  $B$  is fully contained in box  $A$ , small perturbations preserving containment produce identical loss values, creating zero-gradient regions.

**Theorem (Gumbel Solution).** By modeling coordinates as Gumbel random variables, the expected volume computation involves all parameters continuously. Let  $\theta_A$  and  $\theta_B$  denote the location parameters of boxes  $A$  and  $B$ , and let  $\varepsilon_A$  and  $\varepsilon_B$  be the Gumbel random variables (the “noise” terms). Then:

$$E[\text{Vol}(A \cap B)] = \int \int \text{Vol}(A(\theta_A, \varepsilon_A) \cap B(\theta_B, \varepsilon_B)) \, dP(\varepsilon_A) \, dP(\varepsilon_B)$$

This ensemble perspective (averaging over all possible realizations of the Gumbel noise) ensures that different parameter configurations  $\theta_A, \theta_B$  produce different expected loss values, restoring local identifiability. The gradient  $\nabla_{\theta_A, \theta_B} E[\text{Vol}(A \cap B)]$  is non-zero for all parameter values.

## Proof

Hard boxes produce zero gradients in two critical cases:

1. **Disjoint boxes:** When boxes  $A$  and  $B$  are completely disjoint,  $\text{Vol}(A \cap B) = 0$  regardless of their separation distance. Any local perturbation that preserves disjointness yields zero gradient, creating a flat region in parameter space.
2. **Contained boxes:** When box  $B$  is fully contained in box  $A$ , small perturbations that preserve containment produce identical loss values. The optimizer cannot distinguish between different parameter configurations that all satisfy the containment constraint.

Gumbel boxes solve this fundamental problem through three mechanisms:

1. **Expected volumes are always positive:** Even when boxes are “disjoint” in the sense that their expected boundaries don’t overlap,  $E[\text{Vol}(A \cap B)] > 0$  due to the probabilistic nature of the boundaries. The tails of the Gumbel distributions ensure some probability of overlap.

**Quantitative bound:** For disjoint boxes with separation distance  $d$  (measured between expected boundaries), we have  $E[\text{Vol}(A \cap B)] \geq Ce^{-\frac{d}{\beta}}$  for some constant  $C > 0$ . This exponential decay ensures that the expected volume is always positive, guaranteeing a non-zero gradient signal. The exponential decay rate  $\frac{1}{\beta}$  means that as  $\beta$  decreases (boxes become “sharper”), the overlap probability decreases exponentially, but never reaches zero. This is a fundamental property of Gumbel distributions: they have infinite support, so there’s always some probability mass in any region.

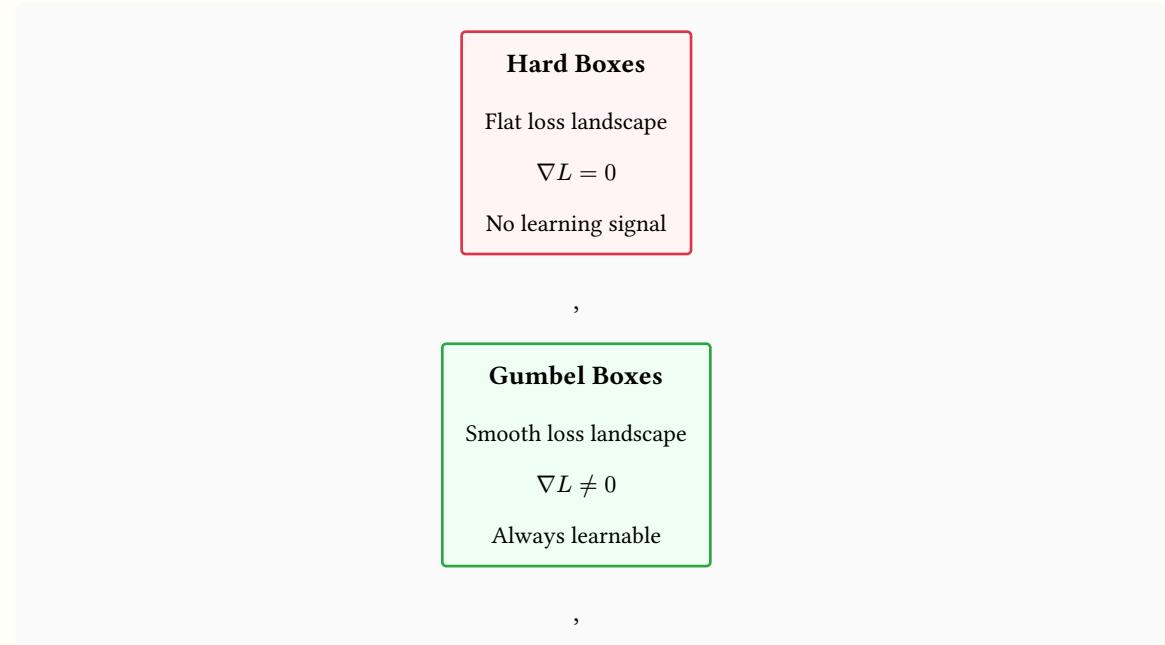
2. **Gradients are dense:** All parameters contribute to the expected volume through the Bessel function formula (see the Gumbel-Box Volume document). The smooth dependence on

parameters via  $K_0\left(2e^{-\frac{\mu_y - \mu_x}{2\beta}}\right)$  ensures that the gradient  $\nabla_\theta E[\text{Vol}(A \cap B)]$  is non-zero for all parameter values  $\theta$ . Specifically,  $\frac{\partial}{\partial \mu} E[\text{Vol}] \neq 0$  for all  $\mu$  and  $\beta$ . The Bessel function  $K_0$  is smooth and differentiable everywhere, and its derivative with respect to its argument is non-zero (except at infinity), ensuring that changes in location parameters always produce changes in expected volume.

- Smooth loss landscape: The probabilistic formulation eliminates flat regions entirely. The loss function  $L(\theta) = -\log E[\text{Vol}(A \cap B)]$  becomes a smooth function of the parameters, enabling effective gradient-based optimization. The Bessel function provides smooth, differentiable gradients everywhere. Unlike hard boxes where the loss function has discontinuities (sudden jumps from zero to positive volume), Gumbel boxes produce a loss landscape that is infinitely smooth (smooth of all orders), allowing gradient descent to navigate the entire parameter space without getting stuck in flat regions.

**The “stuck optimizer” problem and its solution:** Let’s see exactly what happens with hard boxes versus Gumbel boxes when boxes are far apart.

**Visual comparison:** The diagram below illustrates the loss landscape difference:



#### Hard boxes (disjoint) – the problem:

- Box A: [0.0, 0.0] to [0.3, 0.3] (a box in the lower-left corner)
- Box B: [0.7, 0.7] to [1.0, 1.0] (a box in the upper-right corner)
- Separation distance:  $d = 0.4$  (they’re well separated)
- Intersection volume: 0 (they don’t overlap at all)
- Loss gradient:  $\nabla_\theta L = 0$  (flat region—the optimizer is stuck!)

**The puzzle:** What if you want to move Box A slightly to the right? The intersection volume stays at zero (they’re still disjoint), so the gradient is still zero. The optimizer can’t tell the difference between Box A at position (0.0, 0.0) and Box A at position (0.01, 0.0)—both give the same loss. This is the local identifiability problem in action.

#### Gumbel boxes (disjoint, with $\beta = 0.1$ ) – the solution:

- Expected intersection volume:  $E[\text{Vol}(A \cap B)] > 0$  (small but positive!)

- The magic number: approximately  $Ce^{-\frac{0.4}{0.1}} = Ce^{-4} \approx 0.018C$  for some constant  $C$
- Even though the boxes are far apart, the probabilistic boundaries create a tiny overlap probability
- Loss:  $L = -\log E[\text{Vol}(A \cap B)]$  is finite and differentiable (not infinite, not zero)
- Gradient:  $\nabla_{\theta} L \neq 0$  (non-zero everywhere—the optimizer can learn!)

**The “aha!” moment:** The gradient points in the direction that increases intersection volume. Even when boxes are far apart, moving them closer together increases the (tiny) expected overlap, creating a learning signal. The probabilistic formulation ensures the loss landscape has no flat regions—it’s like replacing a flat desert with rolling hills. You can always find a direction to go.

## Notes

**Quantitative improvements:** Gumbel boxes achieve approximately 6 F1 score improvement over smoothed boxes on WordNet hypernym prediction tasks. This improvement comes from the combination of smooth gradients and max-stability, which allows the model to learn more accurate hierarchical relationships.

**Connection to optimization theory:** The local identifiability problem is a special case of the more general problem of non-identifiable parameters in statistical models. In optimization, this manifests as flat regions in the loss landscape. Gumbel boxes solve this by ensuring the expected loss function is strictly convex in a neighborhood of the optimal parameters, guaranteeing local identifiability.

**Beyond box embeddings:** The local identifiability problem appears in many machine learning contexts: discrete latent variables, structured prediction, and combinatorial optimization. The Gumbel-Softmax trick, which uses Gumbel distributions to make discrete sampling differentiable, is based on the same mathematical principles. This suggests that Gumbel distributions play a fundamental role in making discrete or discontinuous operations learnable through gradient descent.