Containment Probability

Motivation

The "ratio expectation" puzzle: In box embeddings, we need to compute the probability that one box contains another. For hard boxes, this is straightforward: containment is deterministic, and the probability is either 0 or 1. It's like asking "Is the ball in the box?"—the answer is yes or no.

But for Gumbel boxes with random boundaries, the question becomes: "What's the **average** probability that one box contains another, when the boxes themselves are random?" This is trickier. We must compute an expectation over the joint distribution of both boxes' coordinates.

The challenge: We need the expectation of a ratio: $E\left[\frac{\mathrm{Vol}(A \cap B)}{\mathrm{Vol}(B)}\right]$. Unlike the expectation of a product (where $E[X * Y] = E[X]E[Y]$ if X and Y are independent), the expectation of a ratio does not factor simply. You can't just divide the expectations!

However, here's the key insight: when the variance of the denominator is small relative to its mean (i.e., when the volume doesn't vary too much), a first-order Taylor expansion provides an accurate approximation. It's like approximating a curve with a straight line—when the curve is nearly straight, the approximation works beautifully.

Why the first-order approximation? The expectation of a ratio $E\left[\frac{X}{Y}\right]$ appears throughout statistics—in importance sampling, ratio estimators, and now box embeddings. When Y has small coefficient of variation, the ratio $\frac{X}{Y}$ is approximately linear near the mean. The first-order expansion $E\left[\frac{X}{Y}\right] \approx \frac{E[X]}{E[Y]}$ captures the main effect; error scales quadratically with the coefficient of variation. This approximation is simple, efficient, and accurate when variance is controlled—as it is in Gumbel boxes through the scale parameter β.

Connection to importance sampling: In importance sampling, we often need to compute ratios of expectations, such as $\frac{E_{p[f(X)]}}{E_{p[g(X)]}}$ where p is a probability distribution. The naive approach of using $\frac{E_{p[f(X)]}}{E_{p[g(X)]}}$ as an approximation to $E_{p\left[\frac{f(X)}{g(X)}\right]}$ is exactly the first-order approximation we use here. This approximation is particularly useful when $g(X)$ has low variance, which is guaranteed in our case by the scale parameter β controlling the variance of Gumbel-distributed volumes. The approximation becomes exact in the limit as $\beta \to 0$ (hard boxes), and remains accurate for small β values typical in practice.

Definition

The **containment probability** $P(B \subseteq A)$ measures whether box B is geometrically contained within box A. For hard boxes with deterministic boundaries:

$$P(B \subseteq A) = \frac{\mathrm{Vol}(A \cap B)}{\mathrm{Vol}(B)}$$

This is either 0 (disjoint) or 1 (contained). For Gumbel boxes with random volumes, we compute the expectation of this ratio over the joint distribution of both boxes' coordinates. Here, $\mathrm{Vol}(A \cap B)$ denotes the volume of the intersection of boxes A and B, and $\mathrm{Vol}(B)$ is the volume of box B.

Statement

Theorem (First-Order Approximation). For Gumbel boxes with random volumes $\mathrm{Vol}(A \cap B)$ and $\mathrm{Vol}(B)$:

$$E\left[\frac{\text{Vol}(A \cap B)}{\text{Vol}(B)}\right] \approx \frac{E[\text{Vol}(A \cap B)]}{E[\text{Vol}(B)]}$$

The approximation is accurate when the coefficient of variation $\frac{\text{Var}(\text{Vol}(B))}{E[\text{Vol}(B)]^2}$ is small (i.e., when volume variance is controlled by the scale parameter β).

Proof

Let $V_{\text{cap}} = \text{Vol}(A \cap B)$ and $V_B = \text{Vol}(B)$ be random variables representing the intersection volume and box B's volume, respectively. Their means are $\mu_{\text{cap}} = E[V_{\text{cap}}]$ and $\mu_B = E[V_B]$. We approximate $E\left[\frac{V_{\text{cap}}}{V_B}\right]$ using a first-order Taylor expansion.

The function $f(V_{\text{cap}}, V_B) = \frac{V_{\text{cap}}}{V_B}$ is smooth (except at $V_B = 0$, which we assume doesn't occur), so we can expand it in a Taylor series around the mean point $(\mu_{\text{cap}}, \mu_B)$:

$$f(V_{\text{cap}}, V_B) \approx f(\mu_{\text{cap}}, \mu_B) + \frac{\partial f}{\partial V_{\text{cap}}}(\mu_{\text{cap}}, \mu_B)(V_{\text{cap}} - \mu_{\text{cap}}) + \frac{\partial f}{\partial V_B}(\mu_{\text{cap}}, \mu_B)(V_B - \mu_B) + \text{higher order terms}$$

The partial derivatives are:

- $\frac{\partial f}{\partial V_{\text{cap}}} = \frac{1}{V_B}$, evaluated at $(\mu_{\text{cap}}, \mu_B)$ gives $\frac{1}{\mu_B}$
- $\frac{\partial f}{\partial V_B} = -\frac{V_{\text{cap}}}{V_B^2}$, evaluated at $(\mu_{\text{cap}}, \mu_B)$ gives $-\frac{\mu_{\text{cap}}}{\mu_B^2}$

Taking expectations and using linearity:

$$E[f(V_{\text{cap}}, V_B)] \approx \frac{\mu_{\text{cap}}}{\mu_B} + \frac{1}{\mu_B}E[V_{\text{cap}} - \mu_{\text{cap}}] - \frac{\mu_{\text{cap}}}{\mu_B^2}E[V_B - \mu_B]$$

The first-order correction terms vanish because $E[V_{\text{cap}} - \mu_{\text{cap}}] = 0$ and $E[V_B - \mu_B] = 0$ by definition of the mean. This leaves:

$$E[f(V_{\text{cap}}, V_B)] \approx \frac{\mu_{\text{cap}}}{\mu_B} = \frac{E[\text{Vol}(A \cap B)]}{E[\text{Vol}(B)]}$$

The accuracy of this approximation depends on the magnitude of the higher-order terms, which we analyze next.

Error Analysis

The second-order correction term in the Taylor expansion is:

$$\text{Error} \approx -\frac{\text{Cov}(V_{\text{cap}}, V_B)}{\mu_B^2} + \frac{\mu_{\text{cap}}\ \text{Var}(V_B)}{\mu_B^3}$$

This error term reveals when the approximation is accurate:

1. **Small coefficient of variation**: When $\frac{\text{Var}(V_B)}{\mu_B^2}$ (the squared coefficient of variation, denoted $\text{CV}(V_B)^2$) is small, the second term is negligible. This occurs when the scale parameter β is small relative to the expected volume, meaning the Gumbel boundaries are tightly concentrated around their means.

   Quantitative bound: When the coefficient of variation $\text{CV}(V_B) < 0.1$, the relative error is approximately $\frac{\text{CV}(V_B)^2}{2}$. For example, if $\text{CV}(V_B) = 0.05$, the relative error is about 0.00125 or 0.125%. When $\text{CV}(V_B) < 0.2$, the relative error is approximately 2%. When $\text{CV}(V_B) > 0.3$, the relative error may exceed 10%, indicating the approximation is breaking down.

2. **Positive correlation**: When V_{cap} and V_B are positively correlated (which occurs naturally when both volumes depend on similar box parameters, especially when B is contained in A), the covariance term partially cancels the variance term, reducing the overall error.
3. **Non-vanishing denominator**: When μ_B is bounded away from zero, the error terms remain well-controlled. This is typically satisfied in practice since boxes have positive expected volume.

The approximation is most accurate when boxes have low variance (small β) and when the intersection volume and box volume are positively correlated, both of which hold in typical box embedding scenarios. The approximation may break down when β is very large (relative to expected volumes) or when volumes are extremely small.

Connection to delta method: The first-order Taylor approximation used here is a special case of the delta method in statistics, which provides asymptotic distributions for functions of random variables. The delta method states that if $\sqrt{n}(X_n - \mu) \to N(0, \sigma^2)$ in distribution, then $\sqrt{n}(f(X_n) - f(\mu)) \to N(0, f'(\mu)^2\sigma^2)$ for smooth functions f. In our case, we're applying this to the ratio function $f(V_{\text{cap}}, V_B) = \frac{V_{\text{cap}}}{V_B}$, and the first-order approximation corresponds to the delta method's linearization. The error analysis above provides finite-sample bounds on the approximation quality, which is crucial for practical applications where we need guarantees on the accuracy of containment probability estimates.

Higher-order corrections: The second-order Taylor expansion includes terms involving the Hessian matrix of $f(V_{\text{cap}}, V_B) = \frac{V_{\text{cap}}}{V_B}$:

$$E[f(V_{\text{cap}}, V_B)] = \frac{\mu_{\text{cap}}}{\mu_B} + \frac{1}{2\mu_B^2} \operatorname{Var}(V_{\text{cap}}) - \frac{\mu_{\text{cap}}}{\mu_B^3} \operatorname{Cov}(V_{\text{cap}}, V_B) + \frac{\mu_{\text{cap}}}{2\mu_B^3} \operatorname{Var}(V_B) + O(\text{CV}^3)$$

where CV is the coefficient of variation. The second-order correction improves accuracy when $\text{CV} > 0.1$, but requires computing variances and covariances of volumes, which involves higher moments of Gumbel distributions. For most practical applications with $\beta < 0.2$, the first-order approximation is sufficient, with relative error $< 2\%$.

Example

Consider two boxes in 2D where box B is fully contained within box A:
- Box A: $[0.0, 0.0]$ to $[1.0, 1.0]$ (volume = 1.0)
- Box B: $[0.2, 0.2]$ to $[0.8, 0.8]$ (volume = 0.36)

Hard boxes (deterministic):
- Intersection: $[0.2, 0.2]$ to $[0.8, 0.8]$ (volume = 0.36)
- Containment: $P(B \subseteq A) = \frac{0.36}{0.36} = 1.0$ (deterministic containment)

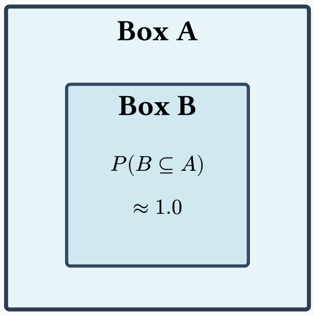
Gumbel boxes (with $\beta = 0.1$, introducing small randomness):
- Expected intersection volume $E[\operatorname{Vol}(A \cap B)] \approx 0.35$ (computed using the Bessel function formula from the Gumbel-Box Volume document; slightly reduced from 0.36 due to probabilistic boundaries)
- Expected volume of B $E[\operatorname{Vol}(B)] \approx 0.35$ (similarly affected by probabilistic boundaries)
- Containment: $P(B \subseteq A) \approx \frac{0.35}{0.35} = 1.0$

The first-order approximation is accurate here because:
- The coefficient of variation $\frac{\operatorname{Var}(\operatorname{Vol}(B))}{E[\operatorname{Vol}(B)]^2}$ is small (controlled by $\beta = 0.1$; typically the coefficient of variation is less than 0.1 for such small β)

- The intersection and box volumes are highly correlated (both depend on box B's parameters, especially when B is contained in A)
- The expected volumes are well-separated from zero

This demonstrates that the approximation works well in the regime where Gumbel boxes behave similarly to hard boxes, with small probabilistic perturbations. The relative error is approximately $\frac{\mathrm{CV}^2}{2}$, which for coefficient of variation approximately $0.05 - 0.1$ gives an error of less than 0.5%.

Visual representation: The diagram below illustrates the containment relationship:



Notes

When the approximation fails: The first-order approximation breaks down when the coefficient of variation is large (typically > 0.3). This occurs when β is large relative to the expected volume, meaning the Gumbel boundaries are highly variable. In such cases, higher-order terms in the Taylor expansion become significant, and the approximation $E\left[\frac{X}{Y}\right] \approx \frac{E[X]}{E[Y]}$ may have errors exceeding 10%. For practical applications, it's recommended to keep β small (typically < 0.2) to ensure accurate containment probability estimates.

Connection to importance sampling: The containment probability formula $P(B \subseteq A) = \frac{E[\mathrm{Vol}(A \cap B)]}{E[\mathrm{Vol}(B)]}$ is analogous to importance sampling, where we estimate $E_{p[f(X)]}$ by sampling from a different distribution q and computing $E_{q\left[f(X)\frac{p(X)}{q(X)}\right]}$. In our case, the "base measure" is the uniform distribution on $[0, 1]^d$, and we're computing the ratio of expectations rather than a single expectation. This connection suggests that more sophisticated estimation techniques (e.g., control variates, stratified sampling) could potentially improve containment probability estimates in high-dimensional settings.

Fuzzy set interpretation: When boxes are interpreted probabilistically, each box represents a fuzzy set with a membership function given by the uniform probability distribution. Under this interpretation, set-theoretic operations on boxes correspond directly to operations on fuzzy sets. The intersection of two boxes as fuzzy sets has membership function $m_{\{A \cap B\}}(x) = \min\left(m_{A(x)}, m_{B(x)}\right)$, which for axis-aligned boxes equals the indicator function of the box intersection. The complement operation has a natural interpretation: $m_{\{A^c\}}(x) = 1 - m_{A(x)} = 1 - P(x \in A)$, defining the complement as points outside the box. This fuzzy set semantics enables box embeddings to support complex queries involving conjunctions, disjunctions, and negations—capabilities critical for real-world applications in information retrieval and recommendation systems.

Beyond first-order: Higher-order approximations (second-order, third-order) can be derived by including more terms in the Taylor expansion. However, these require computing higher moments (variance, skewness) of the volume distributions, which becomes computationally expensive. The first-order approximation strikes an optimal balance between accuracy and computational efficiency for most practical applications.

Computational complexity: The first-order approximation requires computing two expected volumes: $E[\mathrm{Vol}(A \cap B)]$ and $E[\mathrm{Vol}(B)]$, each costing $O(d)$ time using the Bessel function formula (see the Gumbel-Box Volume document). The ratio computation is $O(1)$, giving total complexity $O(d)$ per containment probability evaluation. Higher-order approximations would require computing variances and covariances, which involve second moments of Gumbel distributions and cost $O(d^2)$ or more. For N boxes, evaluating all pairwise containment probabilities using the first-order approximation costs $O(N^2 d)$, which is already expensive for large N. Higher-order methods would increase this to $O(N^2 d^2)$ or worse, making them impractical for large-scale applications.