回归

七月算法　邹博

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对偶问题

- 一般优化问题的Lagrange乘子法

$$\begin{aligned} \text{minimize} \quad & f_0(x), \quad x \in \mathbf{R}^n \\ \text{subject to} \quad & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & h_j(x) = 0, \quad j = 1, \ldots, p \end{aligned}$$

- Lagrange函数

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

  - 对固定的x，Lagrange函数L(x, λ ,v)为关于λ和v的仿射函数

Lagrange对偶函数(dual function)

- Lagrange对偶函数

$$g(\lambda,\nu)=\inf_{x\in D}L(x,\lambda,\nu)=\inf_{x\in D}\left(f_0(x)+\sum_{i=1}^{m}\lambda_i f_i(x)+\sum_{i=1}^{p}\nu_i h_i(x)\right)$$

- 若没有下确界，定义：

$$g(\lambda,\nu)=-\infty$$

- 根据定义，显然有：对$\forall\ \lambda>0$，$\forall$v，若原优化问题有最优值p*，则

$$g(\lambda,\nu)\le p^*$$

- 进一步：Lagrange对偶函数为凹函数。

鞍点解释

- 为表述方便，假设没有等式约束，只考虑不等式约束，结论可方便的扩展到等式约束。
- 假设x0不可行，即存在某些i，使得$f_i(x)>0$。则选择 $\lambda_i \to \infty$，对于其他乘子，$\lambda_j = 0, j \neq i$
- 假设x0可行，则有$f_i(x) \leqslant 0, (i=1,2,...,m)$,选择

$$\lambda_i = 0, i = 1,2,\cdots,m$$

- 有：

$$\sup_{\lambda \geq 0} L(x,\lambda) = \sup_{\lambda \geq 0}\left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) \right) = \begin{cases} f_0(x) & f_i(x) \leq 0, i = 1,2\cdots,m \\ \infty & otherwise \end{cases}$$

鞍点：最优点

- 而原问题是：$\inf_x f_0(x)$
- 从而，原问题的本质为：$\inf_x \sup_{\lambda \geq 0} L(x, \lambda)$
- 而对偶问题，是求对偶函数的最大值，即：

$$\sup_{\lambda \geq 0} \inf_x L(x, \lambda)$$

- 而：

$$\sup_{\lambda \geq 0} \inf_x L(x, \lambda) \leq \inf_x \sup_{\lambda \geq 0} L(x, \lambda)$$

证明：$\max_x \min_y f(x,y) \le \min_y \max_x f(x,y)$

- 对于任意的(x,y) ∈ domf

$$f(x,y) \le \max_x f(x,y)$$

$$\Rightarrow \min_y f(x,y) \le \min_y \max_x f(x,y)$$

$$\Rightarrow \max_x \min_y f(x,y) \le \min_y \max_x f(x,y)$$

引理：向量的导数

- A为m×n的矩阵， x为n×1的列向量，
- 记 $\vec{y} = A \cdot \vec{x}$
- 思考： $\frac{\partial \vec{y}}{\partial \vec{x}} = ?$

向量导数的推导

- 令

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \vec{x}=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad A\cdot\vec{x}=\begin{pmatrix} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n \end{pmatrix}$$

- 从而，

$$\frac{\partial \vec{y}}{\partial \vec{x}}=\frac{\partial A\vec{x}}{\partial \vec{x}}=\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}=A^T$$

结论与直接推广

- 向量偏导公式：

$$\frac{\partial A\vec{x}}{\partial \vec{x}} = A^T$$

$$\frac{\partial A\vec{x}}{\partial \vec{x}^T} = A$$

$$\frac{\partial \left(\vec{x}^T A\right)}{\partial \vec{x}} = A$$

引理：标量对向量的导数

- A为n×n的矩阵， x为n×1的列向量，
- 记 $y = \vec{x}^T \cdot A \cdot \vec{x}$
- 同理可得：

$$\frac{\partial y}{\partial \vec{x}} = \frac{\partial\left(\vec{x}^T \cdot A \cdot \vec{x}\right)}{\partial \vec{x}} = \left(A^T + A\right) \cdot \vec{x}$$

- 若A为对称阵，则有 $\frac{\partial\left(\vec{x}^T A \vec{x}\right)}{\partial \vec{x}} = 2A\vec{x}$

注意

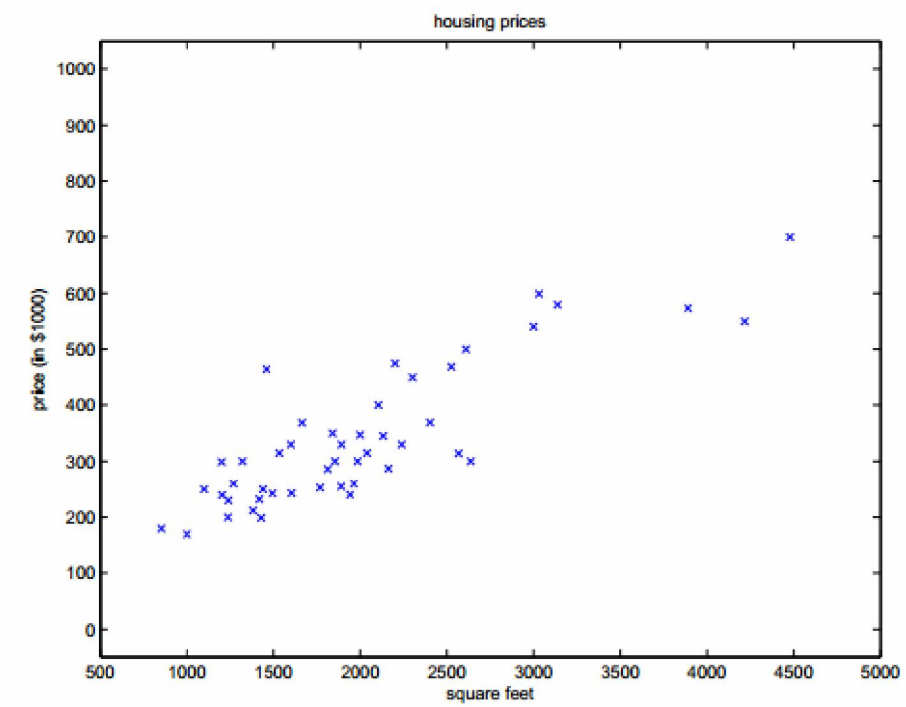
- 关于列向量求导，有文献给出如下方案：
- 记x为n×1的列向量，y为m×1的列向量，则：

$$\frac{\partial \vec{y}}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial \vec{x}} \\ \vdots \\ \vdots \\ \frac{\partial y_m}{\partial \vec{x}} \end{pmatrix} \qquad \frac{\partial y_1}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} \\ \vdots \\ \vdots \\ \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

- 以上公式将会导致向量间求导得到“超越矩阵”——矩阵的每个元素仍然是一个矩阵。在实践层面，不利于公式推导，故本课程未采纳。

线性回归

- y=ax+b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price (in $1000)
square feet
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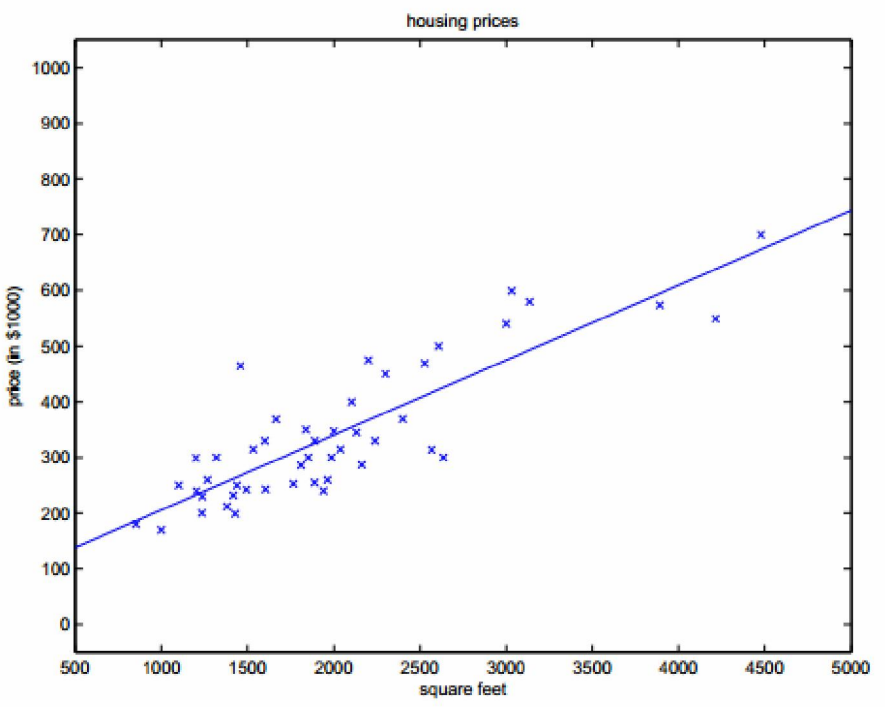
housing prices
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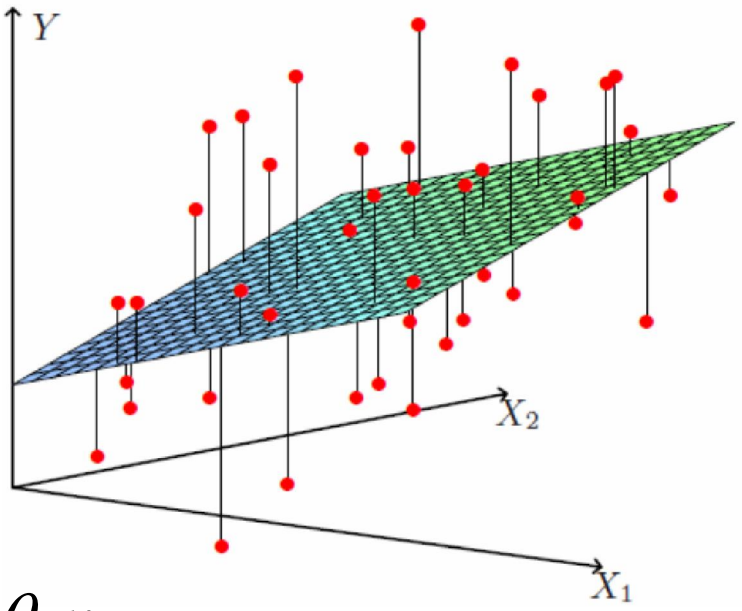
3000
3500
4000
4500
5000

多个变量的情形

考虑两个变量

Living area (feet^2)	#bedrooms	Price (1000$s)
2104	3	400
1600	3	330
2400	3	369
1416	2	232
3000	4	540
⋮	⋮	⋮



$$h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2$$

$$h_\theta(x) = \sum_{i=0}^{n} \theta_i x_i = \theta^T x$$

使用极大似然估计解释最小二乘

$$y^{(i)} = \theta^T x^{(i)} + \varepsilon^{(i)}$$

the $\epsilon^{(i)}$ are distributed IID (independently and identically distributed) according to a Gaussian distribution (also called a Normal distribution) with mean zero and some variance σ^2

- 误差 $\varepsilon^{(i)}(1 \leqslant i \leqslant m)$是独立同分布的，服从均值为0，方差为某定值 σ^2的高斯分布。
  - 原因：中心极限定理

中心极限定理的意义

- 实际问题中，很多随机现象可以看做众多因素的独立影响的综合反应，往往近似服从正态分布。
  - 城市耗电量：大量用户的耗电量总和
  - 测量误差：许多观察不到的、微小误差的总和
  - 注：应用前提是多个随机变量的和，有些问题是乘性误差，则需要鉴别或者取对数后再使用。

似然函数 $y^{(i)} = \theta^T x^{(i)} + \varepsilon^{(i)}$

$$p(\epsilon^{(i)}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\epsilon^{(i)})^2}{2\sigma^2}\right)$$

$$p(y^{(i)}|x^{(i)};\theta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right)$$

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)};\theta) \\ &= \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \end{aligned}$$

高斯的对数似然与最小二乘

$$
\begin{aligned}
\ell(\theta) &= \log L(\theta) \\
&= \log \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= \sum_{i=1}^{m} \log \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y^{(i)} - \theta^T x^{(i)})^2}{2\sigma^2}\right) \\
&= m \log \frac{1}{\sqrt{2\pi}\sigma} - \frac{1}{\sigma^2} \cdot \frac{1}{2} \sum_{i=1}^{m} (y^{(i)} - \theta^T x^{(i)})^2
\end{aligned}
$$

$$
J(\theta) = \frac{1}{2} \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)})^2
$$

θ的解析式的求解过程

- 将M个N维样本组成矩阵X：
  - 其中，X的每一行对应一个样本，共M个样本
  - X的每一列对应样本的一个维度，共N维
    - 其实还有额外的一维常数项，全为1
- 目标函数 $J(\theta)=\frac{1}{2}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right)-y^{(i)}\right)^2=\frac{1}{2}(X\theta-y)^T(X\theta-y)$
- 梯度：

$$\nabla_\theta J(\theta)=\nabla_\theta\left(\frac{1}{2}(X\theta-y)^T(X\theta-y)\right)=\nabla_\theta\left(\frac{1}{2}\left(\theta^T X^T-y^T\right)(X\theta-y)\right)$$

$$=\nabla_\theta\left(\frac{1}{2}\left(\theta^T X^T X\theta-\theta^T X^T y-y^T X\theta+y^T y\right)\right)$$

$$=\frac{1}{2}\left(2X^T X\theta-X^T y-\left(y^T X\right)^T\right)=X^T X\theta-X^T y\xrightarrow{\text{求驻点}}0$$

最小二乘意义下的参数最优解

- 参数的解析式

$$\theta = \left(X^T X\right)^{-1} X^T y$$

- 若X^TX不可逆或防止过拟合，增加λ扰动

$$\theta = \left(X^T X + \lambda I\right)^{-1} X^T y$$

- “简便”方法记忆结论

$$X\theta = y \Rightarrow X^T X\theta = X^T y$$

$$\Rightarrow \theta = \left(X^T X\right)^{-1} X^T y$$

加入λ扰动后

- X^TX半正定：对于任意的非零向量u

$$uX^TXu = (Xu)^TXu \xrightarrow{令v=Xu} v^Tv \geq 0$$

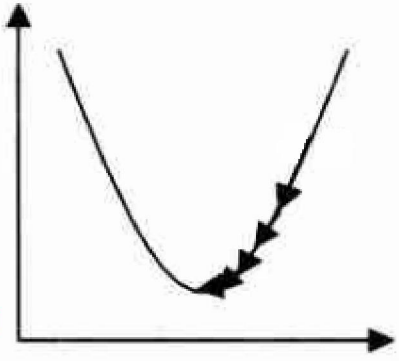
- 所以，对于任意的实数$\lambda>0$，$X^TX+\lambda I$正定，从而可逆。保证回归公式一定有意义。

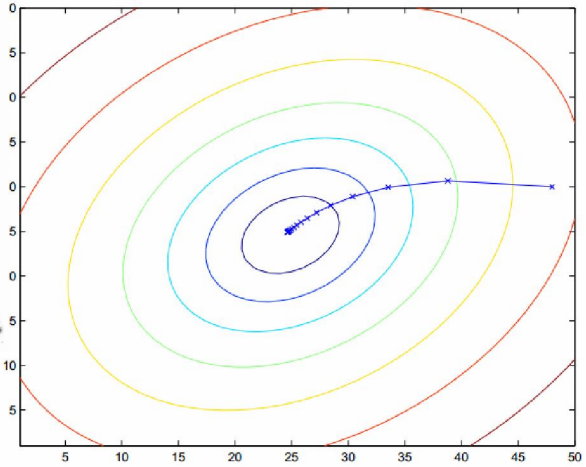
$$\theta = (X^TX+\lambda I)^{-1}X^Ty$$

广义逆矩阵（伪逆）$A^+ = \left(A^T A\right)^{-1} A^T$

- 若A为非奇异矩阵,则线性方程组Ax=b的解为$x = A^{-1}b$ 其中A的A的逆矩阵A^{-1}满足$A^{-1}A = AA^{-1} = I$ (I为单位矩阵)。若A是奇异阵或长方阵, x=A^+ b。A^+叫做A的伪逆阵。
- 1955年R.Penrose证明了对每个m×n阶矩阵A,都存在惟一的n×m阶矩阵X，满足：①AXA=A；②XAX=X；③(AX)* = I；④(XA)* = I。通常称X为A的穆尔-彭罗斯广义逆矩阵,简称M-P逆，记作A^+。
- 在矛盾线性方程组Ax = b的最小二乘解中, x=A^+b是范数最小的一个解。
  - 在奇异值分解SVD的问题中，将继续该话题的讨论。

梯度下降算法





- 初始化θ(随机初始化)
- 迭代，新的θ能够使得J(θ)更小
- 如果J(θ)能够继续减少，返回(2)

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

- α称为学习率/步长

梯度方向

$$
\begin{aligned}
\frac{\partial}{\partial\theta_j}J(\theta) &= \frac{\partial}{\partial\theta_j}\frac{1}{2}\left(h_\theta(x)-y\right)^2 \\
&= 2\cdot\frac{1}{2}\left(h_\theta(x)-y\right)\cdot\frac{\partial}{\partial\theta_j}\left(h_\theta(x)-y\right) \\
&= \left(h_\theta(x)-y\right)\cdot\frac{\partial}{\partial\theta_j}\left(\sum_{i=0}^{n}\theta_i x_i-y\right) \\
&= \left(h_\theta(x)-y\right)x_j
\end{aligned}
$$

批处理梯度下降算法

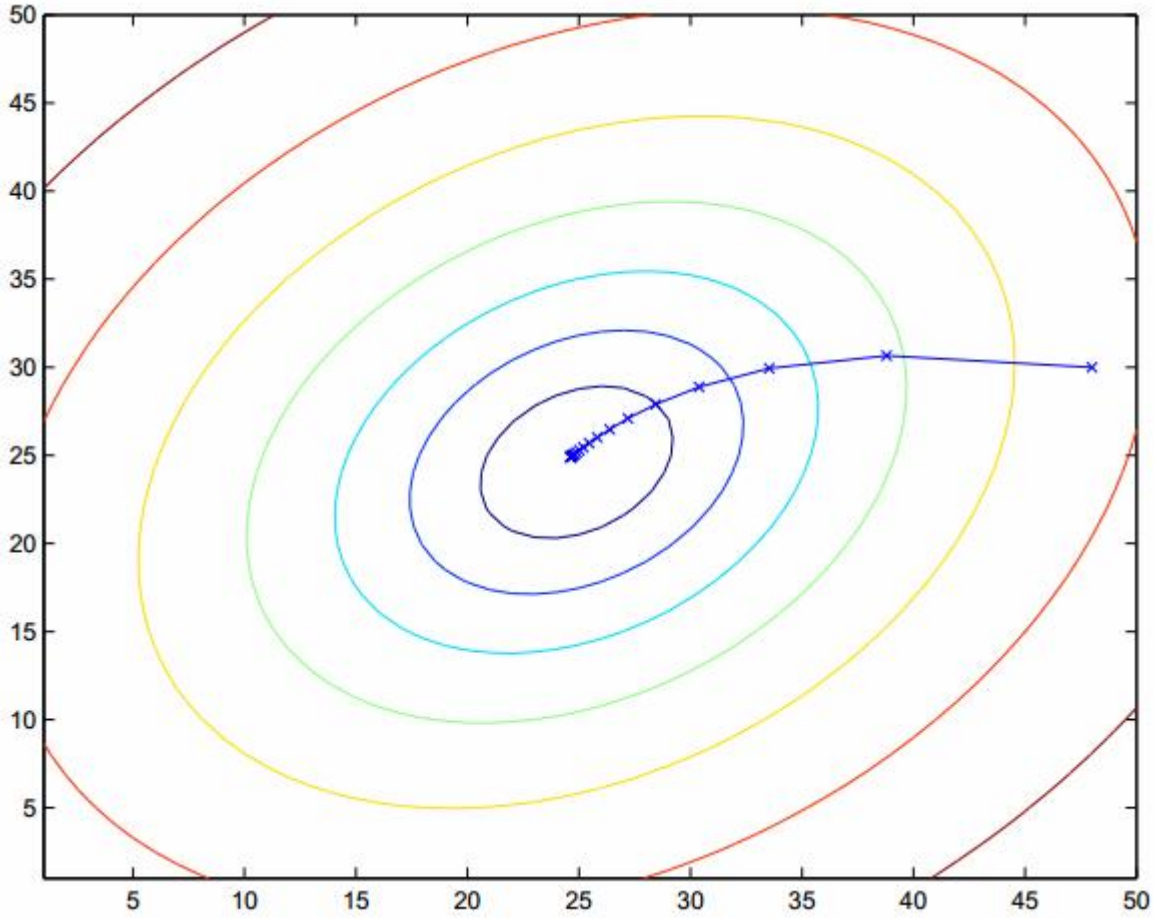
Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}$$

}

gradient descent. Note that, while gradient descent can be susceptible to local minima in general, the optimization problem we have posed here for linear regression has only one global, and no other local, optima; thus gradient descent always converges (assuming the learning rate α is not too large) to the global minimum. Indeed, J is a convex quadratic function.

批处理梯度下降图示



随机梯度下降算法

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}$$

}

}

This algorithm is called **stochastic gradient descent** (also **incremental gradient descent**). Whereas batch gradient descent has to scan through the entire training set before taking a single step—a costly operation if m is large—stochastic gradient descent can start making progress right away, and continues to make progress with each example it looks at. Often, stochastic gradient descent gets θ "close" to the minimum much faster than batch gradient descent. (Note however that it may never "converge" to the minimum, and the parameters θ will keep oscillating around the minimum of $J(\theta)$; but in practice most of the values near the minimum will be reasonably good approximations to the true minimum.[2]) For these reasons, particularly when the training set is large, stochastic gradient descent is often preferred over batch gradient descent.

mini-batch

- 如果不是每拿到一个样本即更改梯度，而是若干个样本的平均梯度作为更新方向，则是mini-batch梯度下降算法。

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)}$$

}

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)})\right) x_j^{(i)}$$

}

}

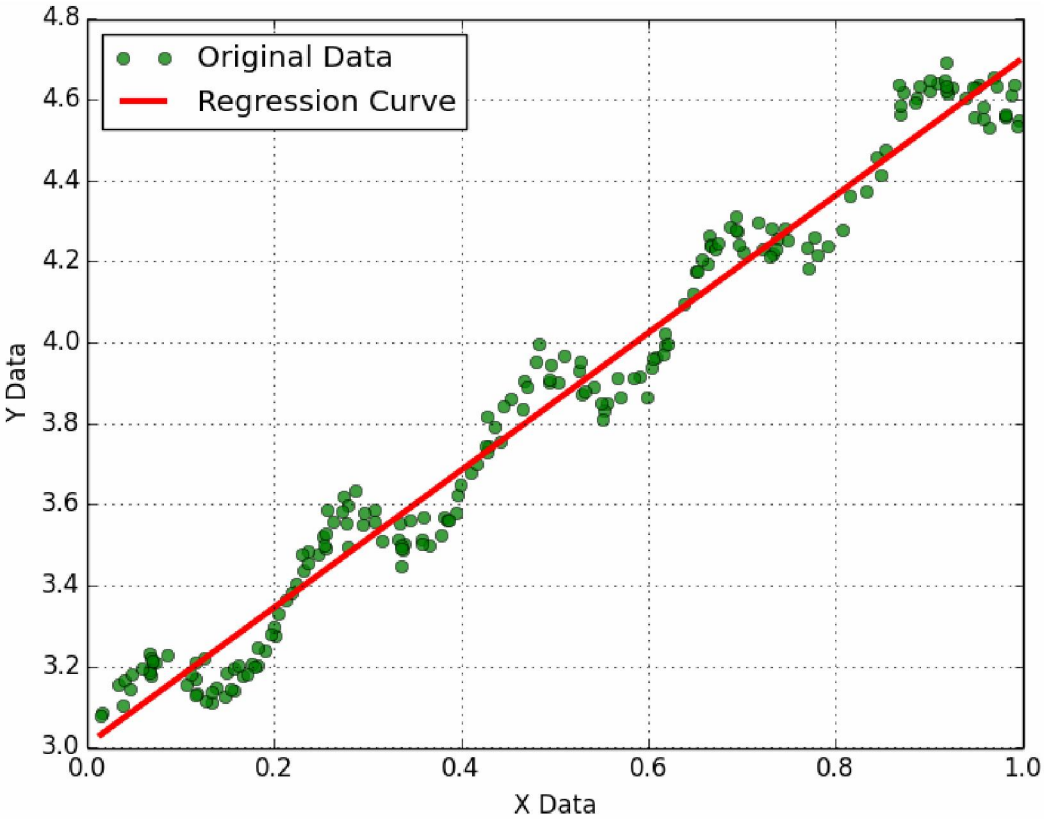
回归Code

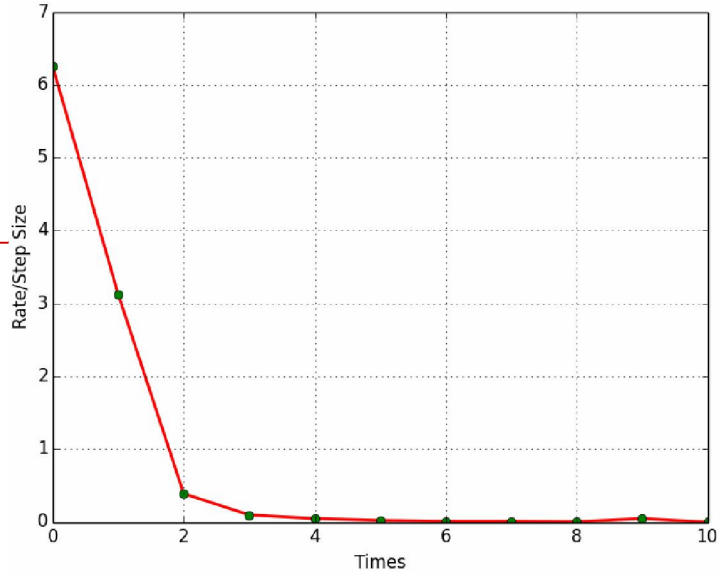
```
def calcCoefficient(data, listA, listW, listLostFunction):
    N = len(data[0])  # 维度
    w = [0 for i in range(N)]
    wNew = [0 for i in range(N)]
    g = [0 for i in range(N)]

    times = 0
    alpha = 100.0  # 学习率随意初始化
    while times < 10000:
        j = 0
        while j < N:
            g[j] = gradient(data, w, j)
            j += 1
        normalize(g)  # 正则化梯度
        alpha = calcAlpha(w, g, alpha, data)
        numberProduct(alpha, g, wNew)

        print "times,alpha,fw,w,g:\t", times, alpha, fw(w, data), w, g
        if isSame(w, wNew):
            break
        assign2(w, wNew)  # 更新权值
        times += 1

        listA.append(alpha)
        listW.append(assign(w))
        listLostFunction.append(fw(w, data))
    return w
```

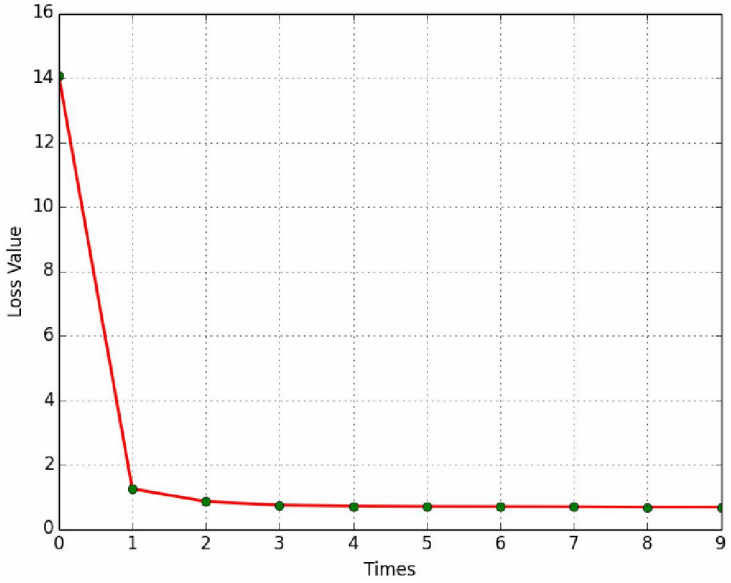
附：学习率Code

```
# w当前值；g当前梯度方向；a当前学习率；data数据
def calcAlpha(w, g, a, data):
    c1 = 0.3
    now = fw(w, data)
    wNext = assign(w)
    numberProduct(a, g, wNext)
    next = fw(wNext, data)
    # 寻找足够大的a，使得h(a)>0
    count = 30
    while next < now:
        a *= 2
        wNext = assign(w)
        numberProduct(a, g, wNext)
        next = fw(wNext, data)
        count -= 1
        if count == 0:
            break

    # 寻找合适的学习率a
    count = 50
    while next > now - c1*a*dotProduct(g, g):
        a /= 2
        wNext = assign(w)
        numberProduct(a, g, wNext)
        next = fw(wNext, data)

        count -= 1
        if count == 0:
            break
    return a
```

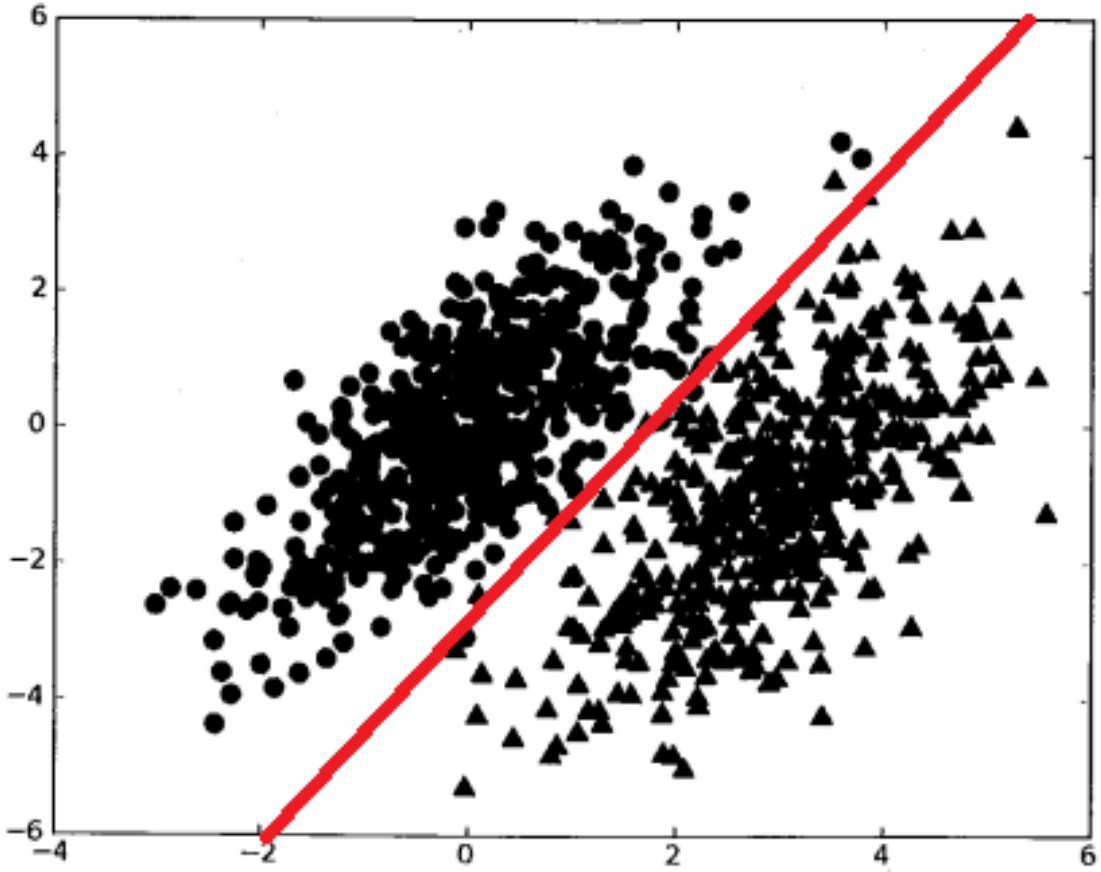
线性回归、rate、Loss



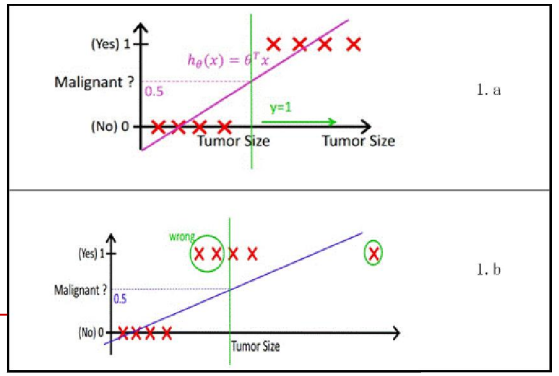




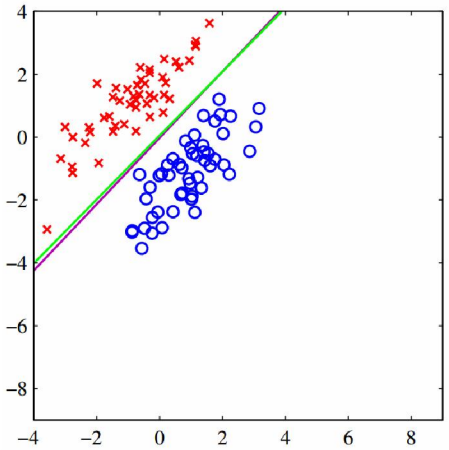
用回归解决分类问题，如何？

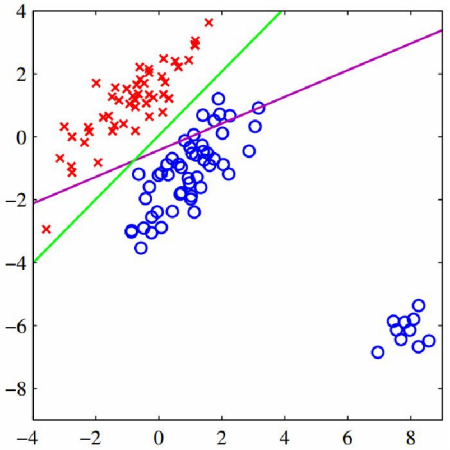


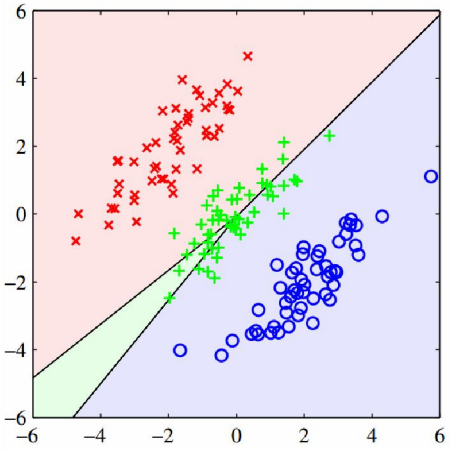
线性回归-Logistic回归

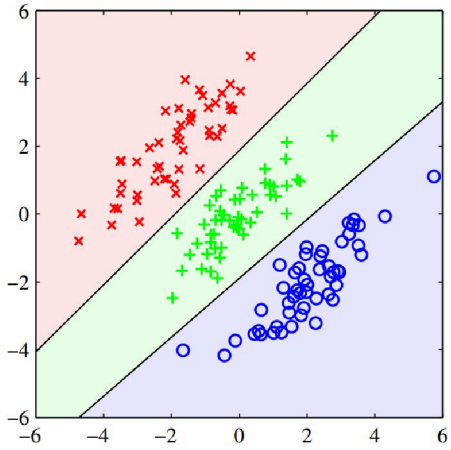


- 紫色：
  - 线性回归
- 绿色：
  - Logistic回归
- 左侧：
  - 线性回归
- 右侧：
  - Softmax回归



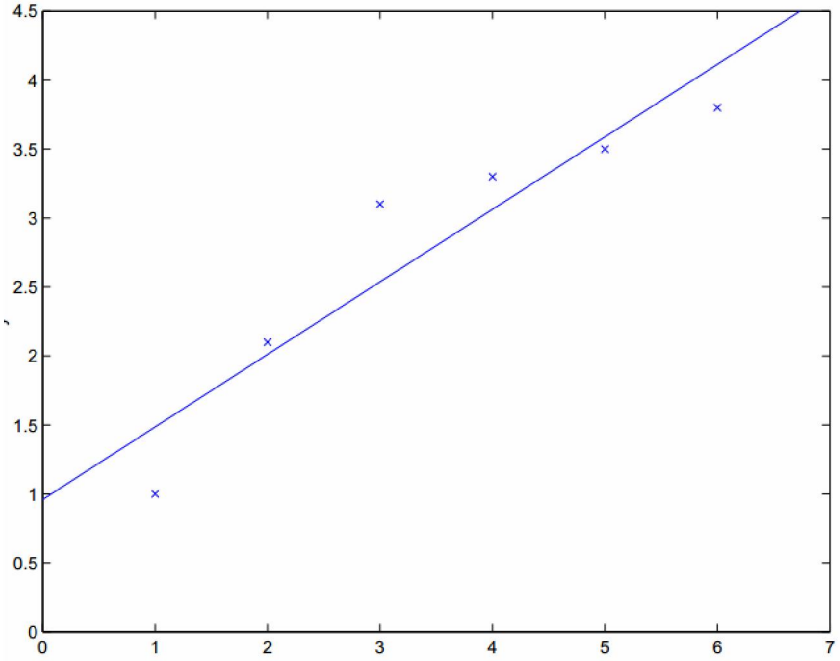


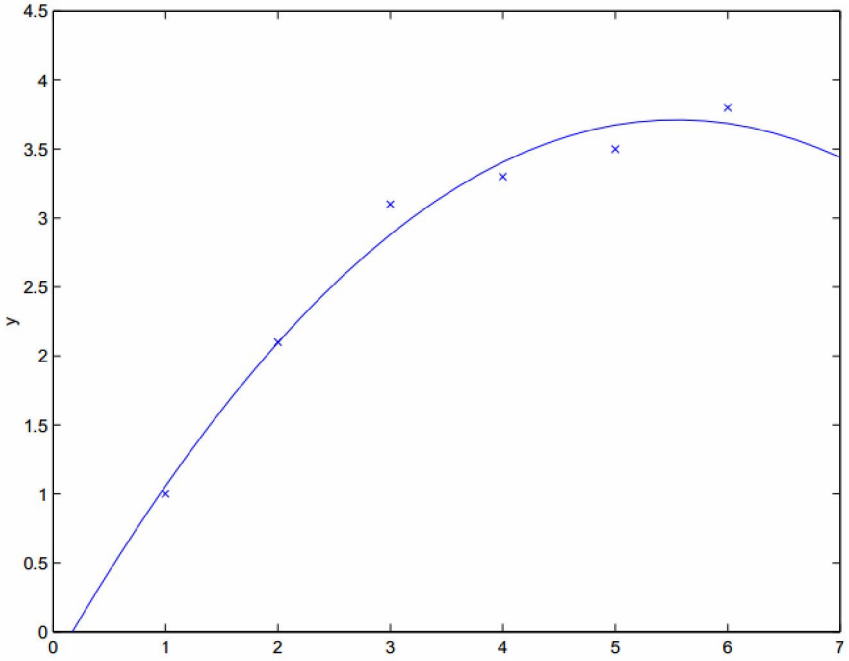




线性回归的进一步分析

- 可以对样本是非线性的，只要对参数θ线性

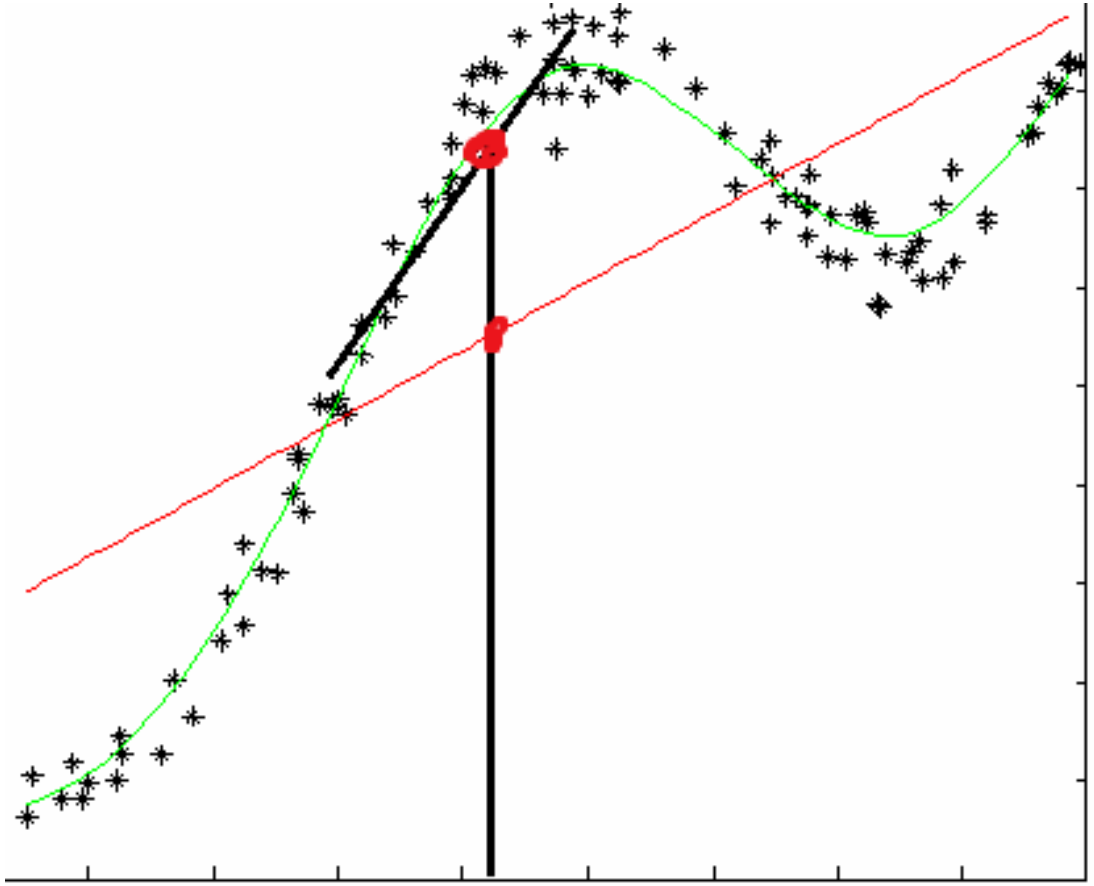




$$y = \theta_0 + \theta_1 x + \theta_2 x^2$$

局部加权回归

- 黑色是样本点
- 红色是线性回归曲线
- 绿色是局部加权回归曲线



局部加权线性回归

- LWR：Locally Weighted linear Regression

1. Fit θ to minimize $\sum_i (y^{(i)} - \theta^T x^{(i)})^2$
2. Output $\theta^T x$.

1. Fit θ to minimize $\sum_i w^{(i)} (y^{(i)} - \theta^T x^{(i)})^2$
2. Output $\theta^T x$.

权值的设置

- ω的一种可能的选择方式(高斯核函数):

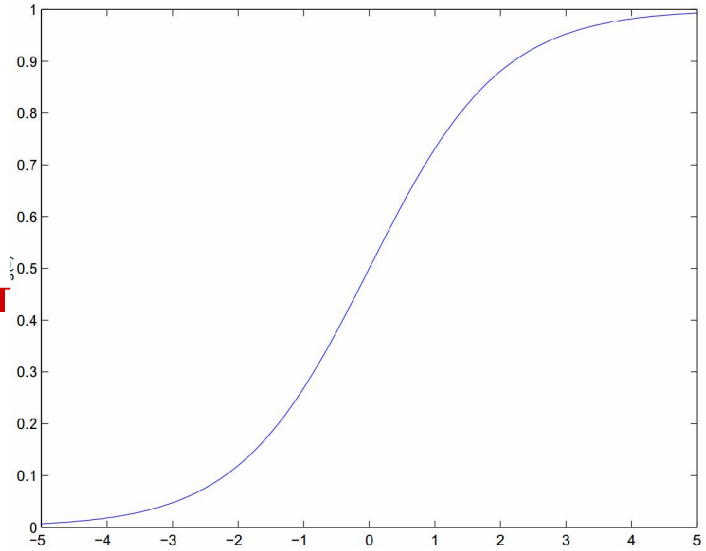
$$w^{(i)} = \exp\left(-\frac{(x^{(i)} - x)^2}{2\tau^2}\right)$$

- τ称为带宽，它控制着训练样本随着与$x^{(i)}$距离的衰减速率。
- 多项式核函数

$$\kappa(\boldsymbol{x}_1, \boldsymbol{x}_2) = (\langle \boldsymbol{x}_1, \boldsymbol{x}_2 \rangle + R)^d$$

Logistic回归

Logistic函数



$$g(z) = \frac{1}{1+e^{-z}}$$

$$h_\theta(x) = g(\theta^T x) = \frac{1}{1+e^{-\theta^T x}}$$

$$\begin{aligned} g'(z) &= \frac{d}{dz}\frac{1}{1+e^{-z}} \\ &= \frac{1}{(1+e^{-z})^2}(e^{-z}) \\ &= \frac{1}{(1+e^{-z})} \cdot \left(1 - \frac{1}{(1+e^{-z})}\right) \\ &= g(z)(1-g(z)) \end{aligned}$$

Logistic回归参数估计

- 假定：

$$P(y=1 \mid x;\theta) = h_\theta(x)$$
$$P(y=0 \mid x;\theta) = 1 - h_\theta(x)$$

$$p(y \mid x;\theta) = (h_\theta(x))^y \, (1 - h_\theta(x))^{1-y}$$

$$\begin{aligned} L(\theta) &= p(\vec{y} \mid X;\theta) \\ &= \prod_{i=1}^{m} p(y^{(i)} \mid x^{(i)};\theta) \\ &= \prod_{i=1}^{m} \left(h_\theta(x^{(i)})\right)^{y^{(i)}} \left(1 - h_\theta(x^{(i)})\right)^{1-y^{(i)}} \end{aligned}$$

对数似然函数

$$\begin{aligned}\ell(\theta) &= \log L(\theta) \\ &= \sum_{i=1}^{m} y^{(i)} \log h(x^{(i)}) + (1-y^{(i)}) \log(1-h(x^{(i)}))\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\theta_j}\ell(\theta) &= \left(y\frac{1}{g(\theta^T x)} - (1-y)\frac{1}{1-g(\theta^T x)}\right)\frac{\partial}{\partial\theta_j}g(\theta^T x) \\ &= \left(y\frac{1}{g(\theta^T x)} - (1-y)\frac{1}{1-g(\theta^T x)}\right)g(\theta^T x)(1-g(\theta^T x)\frac{\partial}{\partial\theta_j}\theta^T x \\ &= \left(y(1-g(\theta^T x)) - (1-y)g(\theta^T x)\right)x_j \\ &= (y-h_\theta(x))\,x_j\end{aligned}$$

参数的迭代

- Logistic回归参数的学习规则：

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}$$

- 比较上面的结果和线性回归的结论的差别：
  - 它们具有相同的形式！

Repeat until convergence {

$$\theta_j := \theta_j + \alpha \sum_{i=1}^{m} \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}$$

}

Loop {

for i=1 to m, {

$$\theta_j := \theta_j + \alpha \left(y^{(i)} - h_\theta(x^{(i)}) \right) x_j^{(i)}$$

}

}

对数线性模型

- 一个事件的几率odds，是指该事件发生的概率与该事件不发生的概率的比值。
- 对数几率：logit函数

$$
\begin{aligned}
P(y=1 \mid x;\theta) &= h_\theta(x) \\
P(y=0 \mid x;\theta) &= 1-h_\theta(x)
\end{aligned}
$$

$$
\log it(p) = \log \frac{p}{1-p} = \log \frac{h_\theta(x)}{1-h_\theta(x)} = \log \left(\frac{\frac{1}{1+e^{-w^T x}}}{\frac{e^{-w^T x}}{1+e^{-w^T x}}} \right) = w^T x
$$

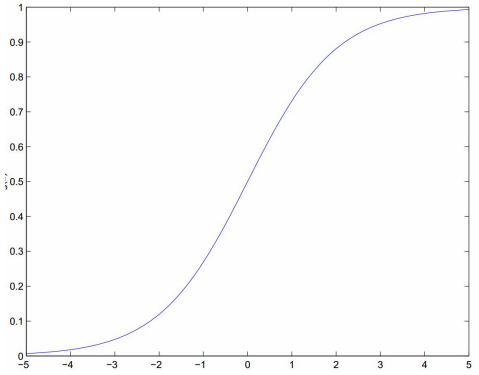
复习：指数族

- 指数族概念的目的，是为了说明广义线性模型Generalized Linear Models
- 凡是符合指数族分布的随机变量，都可以用GLM回归分析

广义线性模型GLM

- 因变量y不再只是正态分布，而是扩大为指数族中的任一分布；
- 解释变量x的线性组合不再直接用于解释因变量y的均值u，而是通过一个联系函数g来解释g(u)；这里，要求g连续单调可导
  - 如Logistic回归中的 $g(z)=\frac{1}{1+e^{-z}}$
  - 拉伸变换：

$$g(z)=\frac{1}{1+e^{-\lambda z}}$$



线性方程的最小二乘问题

- 原问题 $\text{minimize} \quad x^T x, \quad x \in \mathbf{R}^n$

  $\text{subject to} \quad Ax = b$

- Lagrange函数

$$L(x, \nu) = x^T x + \nu^T (Ax - b)$$

- Lagrange对偶函数

$$g(\nu) = -\frac{1}{4}\nu^T A A^T \nu - b^T \nu$$

  - 对L求x的偏导，带入L，得到g
  - 对g求v的偏导，求g的极大值，作为原问题的最小值

求L的对偶函数 $L(x,\nu)=x^Tx+\nu^T(Ax-b)$

$$\frac{\partial L}{\partial x}=\frac{\partial\left(x^Tx+\nu^T(Ax-b)\right)}{\partial x}=2x+A^T\nu\overset{令}{=}0\Rightarrow x^*=-\frac{1}{2}A^T\nu$$

$$L(x,\nu)=x^Tx+\nu^T(Ax-b)$$

$$=\left(-\frac{1}{2}A^T\nu\right)^T\left(-\frac{1}{2}A^T\nu\right)+\nu^T\left(A\left(-\frac{1}{2}A^T\nu\right)-b\right)$$

$$=\frac{1}{4}\nu^TAA^T\nu-\frac{1}{2}\nu^TAA^T\nu-\nu^Tb$$

$$=-\frac{1}{4}\nu^TAA^T\nu-\nu^Tb$$

$$\overset{\Delta}{=}g(\nu)$$

求对偶函数的极大值 $g(\nu)=-\frac{1}{4}\nu^T AA^T\nu-\nu^T b$

$$\frac{\partial g}{\partial \nu}=\frac{\partial\left(-\frac{1}{4}\nu^T AA^T\nu-\nu^T b\right)}{\partial \nu}=-\frac{1}{2}AA^T\nu-b\overset{令}{=}0$$

$$\Rightarrow AA^T\nu=-2b$$

$$\Rightarrow A^T AA^T\nu=-2A^T b$$

$$\Rightarrow A^T\nu=-2\left(A^T A\right)^{-1}A^T b$$

$$\Rightarrow -\frac{1}{2}A^T\nu=\left(A^T A\right)^{-1}A^T b$$

$$\Rightarrow x^*=\left(A^T A\right)^{-1}A^T b$$

极小值点 $x^* = \left(A^T A\right)^{-1} A^T b$

- 极小值：$\min\left(x^T x\right)$

$$= \left(\left(A^T A\right)^{-1} A^T b\right)^T \left(\left(A^T A\right)^{-1} A^T b\right)$$

$$= b^T A\left(A^T A\right)^{-1}\left(A^T A\right)^{-1} A^T b$$

$$= b^T A\left(A^T A\right)^{-2} A^T b$$

- 极小值点的结论，和通过线性回归计算得到的结论是完全一致的。
  - 线性回归问题具有强对偶性。

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恳请大家批评指正！