MATH 335 S2019
Midterm Exam II Solutions

2019-03-28

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances.

Reference

- The following identity may be used in the exam without the need to prove it:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Exam

1. (Del and suffix notation)

   (a) (10 pts) Find the unit normal to the surface $z = x^2 + 2y^2$ at the point $(1, 1, 3)$.

   Solution

   The surface is an isosurface of the function $f = x^2 + 2y^2 - z$ where $f = 0$. Thus, the normal can be obtained via the gradient of f. We have $\nabla f = (2x, 4y, -1)$, which is $(2, 4, -1)$ at $(1, 1, 3)$. Normalizing, we get $(2, 4, -1)/\sqrt{21}$ (note that the negative of this is also acceptable).

   (b) (10 pts) Let $\vec{a}$ and $\vec{b}$ be vectors. Write $\vec{a} \cdot \vec{b}$ and $[\vec{a} \times \vec{b}]_i$ using suffix notation.

   Solution

   $\vec{a} \cdot \vec{b} = a_i b_i$ and $[\vec{a} \times \vec{b}]_i = \epsilon_{ijk} a_j b_k$

   (c) (10 pts) Let $\vec{u}$ be a vector field. Show that

   $$\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

   using suffix notation.

   Solution

   Note that $[\nabla \times (\nabla \times \vec{u})]_i = \epsilon_{ijk}\partial_j \epsilon_{klm}\partial_l u_m$. Using the identity above, $\epsilon_{ijk}\epsilon_{klm}\partial_j\partial_l u_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l u_m = \partial_i\partial_m u_m - \partial_j\partial_j u_i$. Finally, we note that $\partial_i\partial_m u_m = [\nabla(\nabla \cdot \vec{u})]_i$ and $\partial_j\partial_j u_i = [(\nabla \cdot \nabla)\vec{u}]_i$ so that we get the desired identity.

(d) (10 pts) Let $\vec{u}$ be a vector field. Show that $\nabla \cdot (\nabla \times \vec{u}) = 0$ using suffix notation.

Solution

In suffix notation, we have that $\nabla \cdot (\nabla \times \vec{u}) = \partial_i \epsilon_{ijk} \partial_j u_k = \epsilon_{ijk} \partial_i \partial_j u_k = -\epsilon_{jik} \partial_i \partial_j u_k = -\partial_j \epsilon_{jik} \partial_i u_k = -\nabla \cdot (\nabla \times \vec{u})$. Because the expression is equal to the negative of itself, it must be zero. Note that there are many acceptable variants of this argument.

(e) (2 pts) Extra credit. Use the physical definition of the divergence to explain why $\nabla \cdot (\nabla \times \vec{u})$ is zero.

Solution

Perhaps this was too deep of a call-back. By "physical definition" I meant what the book called the physical definition, i.e. the definition of the divergence as the scaled limit of the flux through smaller and smaller surfaces which enclose the given point.

Starting from that definition, we note that the flux of the curl over any of these closed surfaces must be zero, as a result of Stokes' theorem. Thus, the divergence is always zero.

I wouldn't worry about missing this problem, no one got it right. One point was awarded for writing down the flux definition of the divergence.

2. (30 pts) Evaluate the surface integral

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

where $\vec{F} = (xz, yz, z(x^2+y^2)^2)$ and S is the surface of the cylinder $x^2+y^2 \leq 1$ with $-1 \leq z \leq 1$ (this includes the tubular outer part and disc-shaped caps) using the divergence theorem.

Solution

The divergence theorem gives us that $\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$ where V is the volume enclosed by S. The divergence is $\nabla \cdot \vec{F} = 2z + (x^2 + y^2)^2$. Because V is a cylinder, and that expression looks nice in cylindrical coordinates, we use cylindrical coordinates. Thus,

$$\begin{aligned}
\iiint_V \nabla \cdot \vec{F} \, dV &= \iiint_V 2z + (x^2 + y^2)^2 \, dV \\
&= \int_{-1}^{1} \int_0^{2\pi} \int_0^1 (2z + r^4) r dr d\theta dz \\
&= \int_{-1}^{1} \int_0^{2\pi} z + \frac{1}{6} d\theta dz \\
&= 2\pi \int_{-1}^{1} z + \frac{1}{6} \, dz \\
&= \frac{2\pi}{3}
\end{aligned}$$

3. (30 pts) Let $\vec{F} = (x^2, 2xy + x, z)$. Let C be the circle $x^2 + y^2 = 1$ with $z = 0$ be oriented counter-clockwise (standard direction) with S the disc $x^2 + y^2 \leq 1$ with $z = 0$ so that the normal vector on S is $\hat{n} = (0, 0, 1)$ (the positive z direction). Validate Stokes' theorem by (1) computing the line integral of $\vec{F}$ around C directly **and** (2) computing the flux of $\nabla \times \vec{F}$ over S.

Solution

- We start with the line integral calculation.
  We can parameterize the circle in the usual way $x = \cos(t)$, $y = \sin(t)$, $z = 0$, with $0 \leq t \leq 2\pi$. In these coordinates, $\vec{F} = (\cos^2(t), 2\cos(t)\sin(t) + \cos(t), 0)$.
  In these coordinates $d\vec{r} = (-\sin t, \cos t, 0)dt$. Thus,

$$
\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (\cos^2 t, 2\cos t \sin t + \cos t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_0^{2\pi} -\cos^2 t \sin t + 2\cos^2 t \sin t + \cos^2 t\, dt \\
&= \left[-\frac{\cos^3 t}{3} \right]_0^{2\pi} + \int_0^{2\pi} \cos^2 t\, dt \\
&= \pi \ ,
\end{aligned}
$$

  where in the last line you can use any of a number of techniques for integrating $\cos^2 t$ (double angle formula, adding $\sin^2 t$, etc.)

- Next, we perform the flux integral of the curl.
  We can parameterize the disc in the usual way with $x = r\cos t$, $y = r\sin t$, and $z = 0$, where $0 \leq r \leq 1$ and $0 \leq t \leq 2\pi$. The normal $\hat{n} = (0, 0, 1)$ is given. Note that

$$
\nabla \times \vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2xy + x & z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2y + 1 \end{pmatrix}
$$

  Thus, on the disc, $\nabla \times \vec{F} \cdot \hat{n} = 2y + 1 = 2r\sin(t)$ so that the flux of the curl is

$$
\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS &= \int_0^1 \int_0^{2\pi} (2r\sin t + 1) r\, dt\, dr \\
&= 2\pi \int_0^1 r\, dr \\
&= \pi \ .
\end{aligned}
$$

By Stokes' theorem, these integrals should be equal, which is indeed the case.