

MATH 335 S2019

Practice Problems for Midterm Exam II

Exam II covers chapters 3-5 (∇ operators, suffix notation, and integral theorems). Here are a number of practice problems for the exam. The actual exam will be about the same length as the first exam.

1. Practice with the ∇ operator

- (a) What is the geometrical interpretation of ∇f ?

Solution

If we consider the level sets of f , i.e. the curves in 2D or surfaces in 3D defined by $f(\vec{x}) = c$, then ∇f points in the direction normal to the curve/surface and the magnitude of ∇f is the rate of change of f in that direction.

- (b) What are the definitions of the divergence and curl?

Solution

The divergence is defined as

$$\nabla \cdot \vec{u} = \lim_{|\delta V| \rightarrow 0} \frac{1}{|\delta V|} \iint_{\delta S} \vec{u} \cdot \hat{n} dS ,$$

where δS is the surface bounding the small volume δV . In standard coordinates, we have $\nabla \cdot \vec{u} = \partial_i u_i = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$

The curl is a vector valued quantity which we define in terms of its component in a given direction. Let \hat{n} be a unit vector. Then

$$\nabla \times \vec{u} = \lim_{|\delta S| \rightarrow 0} \frac{1}{|\delta S|} \int_{\delta C} \vec{u} \cdot d\vec{r} ,$$

where δS is a piece of surface which is perpendicular to \hat{n} and δC is the boundary curve of δS , oriented according to the right hand rule. In standard coordinates, we can compute the curl using the mnemonic

$$\nabla \times \vec{u} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

(c) Show that $\vec{F} = (2x + y, x, 2z)$ is conservative.

Solution

One way to show that \vec{F} is conservative is to find a function ϕ such that $\nabla\phi = \vec{F}$. Reading off each component of $\nabla\phi = \vec{F}$, we get $\partial_x\phi = 2x + y$, $\partial_y\phi = x$, and $\partial_z\phi = 2z$. Performing partial integration, we get $\phi = x^2 + xy + f_1(y, z)$, $\phi = xy + f_2(x, z)$, and $\phi = z^2 + f_3(x, y)$. Observe that setting $\phi = x^2 + xy + z^2$, we get such a ϕ . Thus, \vec{F} is conservative.

Alternatively, we can compute the curl of \vec{F} if $\nabla \times \vec{F} = 0$ over a domain with no holes, then \vec{F} is conservative on that domain. We have

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ 2x + y & x & 2z \end{vmatrix} \\ &= \vec{e}_1(0 - 0) - \vec{e}_2(0 - 0) + \vec{e}_3(1 - 1) \\ &= \vec{0} \end{aligned}$$

and \vec{F} and $\nabla \times \vec{F}$ are well-defined everywhere (no holes) so \vec{F} is conservative.

(d) Find the gradient and Laplacian of $\phi = \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z)$

Solution

We have

$$\begin{aligned} \partial_x\phi &= k \cos(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z) \\ \partial_{xx}\phi &= -k^2 \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z) \\ \partial_y\phi &= l \sin(kx) \cos(l y) \exp(\sqrt{k^2 + l^2} z) \\ \partial_{yy}\phi &= -l^2 \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z) \\ \partial_z\phi &= \sqrt{k^2 + l^2} \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z) \\ \partial_{zz}\phi &= (k^2 + l^2) \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z) \end{aligned}$$

so that

$$\nabla\phi = \begin{pmatrix} k \cos(kx) \sin(ly) \exp(\sqrt{k^2 + l^2}z) \\ l \sin(kx) \cos(ly) \exp(\sqrt{k^2 + l^2}z) \\ \sqrt{k^2 + l^2} \sin(kx) \sin(ly) \exp(\sqrt{k^2 + l^2}z) \end{pmatrix}, \quad \Delta\phi = \nabla^2\phi = \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0$$

Note that $\Delta\phi = 0$, i.e. ϕ is a solution of Laplace's equation (we call solutions of Laplace's equation *harmonic*).

- (e) Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$.

Solution

This surface is a level set of $f = x^2 + y^2 - z$ so we can use the gradient $\nabla f = (2x, 2y, -1)$, which is $(2, 2, -1)$ at $(1, 1, 2)$. To get a unit vector, we divide by the length $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, so we have $(2/3, 2/3, -1/3)$.

2. Practicing suffix notation.

- (a) Simplify the suffix expression $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni}$.

Solution

We recall the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. We then have $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mni}$. Using the substitution property, i.e. $\delta_{ij}a_j = a_i$, we get $(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mni} = \epsilon_{jnl} - \delta_{jl}\epsilon_{mnm} = \epsilon_{jnl}$.

- (b) Show that $\nabla \times (f\nabla f) = 0$.

Solution

$$[\nabla \times (f\nabla f)]_i = \epsilon_{ijk}\partial_j (f\partial_k f) = \epsilon_{ijk}(\partial_j f\partial_k f + f\partial_{jk}f)$$

Because the order of the derivatives and products on the right doesn't matter, this is equal to

$$[\nabla \times (f\nabla f)]_i = \epsilon_{ikj}(\partial_j f\partial_k f + f\partial_{jk}f) = -\epsilon_{ijk}(\partial_j f\partial_k f + f\partial_{jk}f) = -[\nabla \times (f\nabla f)]_i$$

Because $[\nabla \times (f\nabla f)]_i$ is the negative of itself, it must be 0.

- (c) Show that the vector field $\vec{u} = \nabla f \times \nabla g$ is solenoidal (divergence is zero)

Solution

$$\nabla \cdot \vec{u} = \partial_i \epsilon_{ijk} \partial_j f \partial_k g = \epsilon_{ijk} \partial_{ij} f \partial_k g + \epsilon_{ijk} \partial_j f \partial_{ik} g$$

Again, changing the order of i and j doesn't change the derivative of f in the first term and the order of i and k doesn't change the derivative of g in the second, so

$$\nabla \cdot \vec{u} = \epsilon_{ikj} \partial_{ij} f \partial_k g + \epsilon_{kji} \partial_j f \partial_{ik} g = -\epsilon_{ijk} \partial_{ij} f \partial_k g + -\epsilon_{ijk} \partial_j f \partial_{ik} g = -\nabla \cdot \vec{u}$$

Because the divergence is the negative of itself, it must be 0.

(d) Use suffix notation to show that $\nabla \cdot (\vec{u} \times \vec{v}) = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$.

Solution

$$\nabla \cdot (\vec{u} \times \vec{v}) = \partial_i \epsilon_{ijk} u_j v_k = \epsilon_{ijk} \partial_i u_j v_k + \epsilon_{ijk} u_j \partial_i v_k = v_k \epsilon_{kij} \partial_i u_j - u_j \epsilon_{jik} \partial_i v_k = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$$

3. Working with the divergence theorem.

(a) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

Solution

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV$$

Note that $\nabla \cdot \vec{F} = 2 + 2y + 2z$. We will use spherical coordinates, i.e. $x = \rho \cos(\theta) \sin(\phi)$, $y = \rho \sin(\theta) \sin(\phi)$, and $z = \rho \cos(\phi)$ with $dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$.

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n} dS &= \iiint_V \nabla \cdot \vec{F} dV \\ &= \int_0^R \int_0^{2\pi} \int_0^\pi (2 + 2\rho \sin(\theta) \sin(\phi) + 2\rho \cos(\phi)) \rho^2 \sin(\phi) d\rho d\theta d\phi \\ &= 4\pi \int_0^R \rho^2 d\rho \int_0^\pi \sin \phi d\phi + 2 \int_0^R \int_0^\pi \rho^3 \sin^2 \phi d\rho d\phi \int_0^{2\pi} \sin(\theta) d\theta \\ &\quad + 2\pi \int_0^R \rho^3 d\rho \int_0^\pi \sin(2\phi) d\phi \\ &= \frac{8\pi R^3}{3} \end{aligned}$$

- (b) Use the divergence theorem to evaluate $\iint_S x^2 + y + z \, dS$ where S is the sphere $x^2 + y^2 + z^2 = R^2$.

Solution

This one is tricky. We can figure out the normal by taking the gradient of $f = x^2 + y^2 + z^2 - R^2$ and dividing by the length. This gives $\hat{n} = (x/R, y/R, z/R)$. To use the divergence theorem, we need a vector field \vec{F} such that $\vec{F} \cdot \hat{n} = x^2 + y + z$. Note that $\vec{F} = (Rx, R, R)$ works. The divergence of \vec{F} is simple, i.e. $\nabla \cdot \vec{F} = R$, a constant. Then

$$\iint_S x^2 + y + z \, dS = \iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V R \, dV = R \iiint_V dV = \frac{4\pi R^4}{3}$$

- (c) Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n} \, dS$ where $\vec{F} = (xy^2, x^2y, y)$ and S is the surface of the cylinder $x^2 + y^2 = R^2$ between $z = 1$ and $z = -1$, including the two disc shaped caps where $x^2 + y^2 \leq R^2$ with $z = \pm 1$.

Solution

The divergence of \vec{F} is $\nabla \cdot \vec{F} = y^2 + x^2 + 0$. We will integrate in cylindrical coordinates where $\nabla \cdot \vec{F} = r^2$. Thus

$$\begin{aligned} \iint_S \vec{F} \cdot \hat{n} \, dS &= \iiint_V \nabla \cdot \vec{F} \, dV \\ &= \int_0^R \int_0^{2\pi} \int_{-1}^1 r^2 r \, dr d\theta dz \\ &= \frac{R^4}{4} 4\pi = \pi R^4 \end{aligned}$$

- (d) Find the flux of the vector field $\vec{F} = (x - y^2, y, x^3)$ out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.

Solution

We can again use the divergence theorem. The divergence of \vec{F} is $\nabla \cdot \vec{F} = 1 + 1 + 0 = 2$. Then

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV = \iiint_V 2 \, dV = 6$$

where in the last equality we used that the volume of the rectangular solid is 3.

- (e) Suppose \vec{F} is tangent to the closed surface S bounding a region V . Show that $\iiint_V \nabla \cdot \vec{F} dV = 0$.

Solution

If \vec{F} is tangent to the surface, then $\vec{F} \cdot \hat{n} = 0$ along the surface. Thus $\iiint_V \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} dS = 0$.

4. Working with Stokes theorem.

- (a) Let S be a surface and let \vec{F} be perpendicular to the tangent to the boundary of S . Show that $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$.

Solution

By Stokes' theorem $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = \oint_C \vec{F} \cdot d\vec{r}$. The second integral is zero because \vec{F} is perpendicular to the tangent to the boundary of S (which is the direction of $d\vec{r}$).

- (b) For a surface S and a fixed vector \vec{v} , prove that $2 \iint_S \vec{v} \cdot \hat{n} dS = \oint_C (\vec{v} \times \vec{r}) \cdot d\vec{r}$, where C is the boundary ("rim") of S .

Solution

Another tricky one.

To apply Stokes' theorem, we need the curl of $\vec{v} \times \vec{r}$ with \vec{v} a constant.

$$\begin{aligned} [\nabla \times (\vec{v} \times \vec{r})]_i &= \epsilon_{ijk} \partial_j \epsilon_{klm} v_l r_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) v_l \partial_j r_m \\ &= v_i \delta_{jm} \partial_j r_m - v_j \partial_j r_i \end{aligned}$$

Note that $\partial_i r_j = \delta_{ij}$ for \vec{r} the position vector. Further, note that $\delta_{ij} \delta_{ij} = 3$ by taking the implied sums. Thus,

$$\begin{aligned} [\nabla \times (\vec{v} \times \vec{r})]_i &= v_i \delta_{jm} \partial_j r_m - v_j \partial_j r_i \\ &= v_i \delta_{jm} \delta_{jm} - v_j \delta_{ji} \\ &= 3v_i - v_i = 2v_i \end{aligned}$$

Then Stokes' theorem implies

$$\oint_C (\vec{v} \times \vec{r}) \cdot d\vec{r} = \iint_S \nabla \times (\vec{v} \times \vec{r}) \cdot \hat{n} dS = 2 \iint_S \vec{v} \cdot \hat{n} dS$$

- (c) Let $\vec{F} = (3y, -xz, -yz^2)$, and let S be the surface $2z = x^2 + y^2$ below the plane $z = 2$ (i.e. consider the paraboloid shape $z = (x^2 + y^2)/2$ between $z = 0$ and $z = 2$). Calculate $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS$ both directly and by using Stokes theorem.

Solution

First, we use Stokes' theorem. This gives that the flux is $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = \oint_C \vec{F} \cdot d\vec{r}$. The boundary curve of S is the circle $x^2 + y^2 = 4$ with $z = 2$. We can discretize this as $r(t) = (2 \cos(t), 2 \sin(t), 2)$ so that $d\vec{r} = (-2 \sin(t), 2 \cos(t), 0)dt$.

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot \hat{n} dS &= \oint_C \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} (6 \sin(t), -4 \cos(t), -8 \sin(t)) \cdot (-2 \sin(t), 2 \cos(t), 0) dt \\ &= \int_0^{2\pi} -12 \sin^2(t) - 8 \cos^2(t) dt \\ &= -8 \int_0^{2\pi} dt - 4 \int_0^{2\pi} \sin^2(t) dt \\ &= -16\pi - 4 \int_0^{2\pi} \sin^2(t) dt \end{aligned}$$

Note that $\int_0^{2\pi} \sin^2(t) dt = \int_0^{2\pi} \cos^2(t) dt$ so that $2 \int_0^{2\pi} \sin^2(t) dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = 2\pi$. Thus

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot \hat{n} dS &= -16\pi - 4 \int_0^{2\pi} \sin^2(t) dt \\ &= -16\pi - 4\pi = -20\pi \end{aligned}$$

Second, we compute the flux directly. We can parameterize the surface over r and t where $x = r \cos t$, $y = r \sin t$, and $z = r^2/2$ with $0 \leq r \leq 2$ and $0 \leq t \leq 2\pi$. It is straightforward to compute $\nabla \times \vec{F} = (x - z^2, 0, -3 - z)$. In these coordinates, $\nabla \times \vec{F} = (r \cos t - r^4/4, 0, -3 - r^2/2)$. We will use the formula for the flux over a parameterized surface which requires $\partial_r \vec{p} \times \partial_t \vec{p}$ with $\vec{p} = (r \cos t, r \sin t, r^2/2)$. This is

$$\partial_r \vec{p} \times \partial_t \vec{p} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos t & \sin t & r \\ -r \sin t & r \cos t & 0 \end{vmatrix} = (-r^2 \cos t, -r^2 \sin t, r).$$

Then, we have

$$\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = \int_0^2 \int_0^{2\pi} (r \cos t - r^4/4, 0, -3 - r^2/2) \cdot (-r^2 \cos t, -r^2 \sin t, r) dr dt$$

$$\begin{aligned}
&= \int_0^2 \int_0^{2\pi} (-r^3 \cos^2 t + (r^6 \cos t)/4 - 3r - r^3/2) dr dt \\
&= \int_0^2 -r^3 dr \int_0^{2\pi} \cos^2 t dt + \int_0^2 r^6/4 dr \int_0^{2\pi} \cos t dt + 2\pi \int_0^2 -3r - r^3/2 dr \\
&= -\frac{2^4\pi}{4} + \frac{2^7 \cdot 0}{28} - 2\pi (3 \cdot 2^2/2 + 2^4/8) \\
&= -4\pi - 12\pi - 4\pi = -20\pi
\end{aligned}$$

Note that in the above, because we are using the formula for any parameterization, the area in the integral is $drdt$ not $rdrdt$. Though, you can see that there is an r in the cross product of $\partial_r \vec{p} \times \partial_t \vec{p}$.

- (d) Let $\vec{F} = (yze^x + xye^x, xze^x, xye^x)$. Show that the circulation of \vec{F} around an oriented simple curve C that is the boundary of a surface S is zero.

Solution

The curl of \vec{F} is zero (on the test, show your work). Thus, by Stokes' theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$$

- (e) Find the circulation of $\vec{F} = (x^2, y^2, -z)$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$, both directly and by using Stokes theorem.

Solution

As with the previous problem, the curl of \vec{F} is zero (on test, show your work). So, using Stokes' theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$$

To compute the circulation directly, we discretize the three segments that make up the triangle C . Let C_1 be $\vec{r}_1(t) = (0, 2t, 0)$, C_2 be $\vec{r}_2(t) = (0, 2 - 2t, 3t)$, and C_3 be $\vec{r}_3(t) = (0, 0, 3 - 3t)$ where t ranges from 0 to 1 over each piece. This gives

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 4t^2, 0) \cdot (0, 2, 0) dt \\
&= 8/3 \\
\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 4 - 8t + 4t^2, -3t) \cdot (0, -2, 3) dt
\end{aligned}$$

$$\begin{aligned}
&= -8 + 8 - 8/3 - 9/2 \\
\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 0, 3t - 3) \cdot (0, 0, -3) dt \\
&= -9/2 + 9
\end{aligned}$$

Adding all 3 contributions we see we again get 0.

5. Show that

$$\iiint_V (\nabla f) \cdot \vec{F} dV = \iint_S f \vec{F} \cdot \hat{n} dS - \iiint_V f \nabla \cdot \vec{F} dV$$

Solution

We can use suffix notation to expand $\nabla \cdot (f\vec{F})$:

$$\nabla \cdot (f\vec{F}) = \partial_i [f\vec{F}]_i = \partial_i f \vec{F}_i + f \partial_i \vec{F}_i = \nabla f \cdot \vec{F} + f \nabla \cdot \vec{F}$$

Thus, by the divergence theorem

$$\begin{aligned}
\iint_S f \vec{F} \cdot \hat{n} dS &= \iiint_V \nabla \cdot (f\vec{F}) dV \\
&= \iiint_V (\nabla f) \cdot \vec{F} dV + \iiint_V f \nabla \cdot \vec{F} dV
\end{aligned}$$

which we can re-arrange to get the desired result.