MATH 335 S2019
Final Exam

2019-05-14

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances. Good luck!

Reference

The following results and identities are provided for reference purposes. They may or may not be needed to complete the exam.

- The following identity may be used in the exam without the need to prove it:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

- Any rank-4 and isotropic tensor is of the form

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk},$$

  for some constants λ, μ, and ν.

- Spherical coordinates. A coordinate system in which the cartesian coordinates are given in terms of (ρ, ϕ, θ) as $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$. On a sphere of radius R, we have $0 \le \rho \le R$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

- You may use without proof that if A_{ij} and B_{ij} are tensors, then $C_{ij} = A_{ij} + B_{ij}$ is also a tensor.

Exam

1. (15 pts) Derive the following identities
   (a) $\nabla \times (\nabla f) = \vec{0}$.
   (b) $\nabla \cdot (\vec{u} \times \vec{v}) = (\nabla \times \vec{u}) \cdot \vec{v} - (\nabla \times \vec{v}) \cdot \vec{u}$.
   (c) $\nabla \cdot (\nabla f \times \nabla g) = 0$.

2. (15 pts) Let $\vec{a}$ be a constant vector and let $\vec{v}$ be a vector field. Let V be some volume and S its boundary surface. Show the following and explain each step.
   (a) $\nabla \cdot (a \times \vec{v}) = -(\nabla \times \vec{v}) \cdot \vec{a}$.
   (b) $\iiint_V -(\nabla \times \vec{v}) \cdot a \, dV = \oiint_S \vec{a} \cdot \vec{v} \times \hat{n} \, dS$.
   (c) $\iint_V -\nabla \times \vec{v} \, dV = \oiint_S \vec{v} \times \hat{n} \, dS$.

3. Let $\vec{v}$ be a differentiable vector field and let

$$E_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) .$$

Similarly, define A_{ij} to be

$$A_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) .$$

If L_{ij} is a given rotation and v'_i and x'_i denote the vector field and new coordinates, then E_{ij} in the new coordinates is

$$E'_{ij} = \frac{1}{2}\left(\frac{\partial v'_i}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_i}\right) .$$

(a) (5 pts) Recall that $x'_i = L_{ij}x_j$. Show why $x_k = L_{ik}x'_i$.

(b) (10 pts) Show that E_{ij} is a tensor (in the sense of the technical definition of a tensor).

(c) (5 pts) Show that E_{ij} is symmetric and A_{ij} is anti-symmetric.

(d) (5 pts) Let a_{ijkl} be an isotropic rank-4 tensor and let $P_{ij} = a_{ijkl}A_{kl}$. Suppose A_{kl} is non-zero. Show that P_{ij} is not symmetric.

4. (15 pts) A coordinate system (u, v, w) is related to cartesian coordinates by

$$x_1 = uvw\,, \quad x_2 = uv(1-w^2)^{1/2}\,, \quad x_3 = (u^2 - v^2)/2\,.$$

(a) Find the scale factors h_u, h_v, and h_w.

(b) Confirm that the (u, v, w) system is orthogonal.

(c) Find the volume element in the (u, v, w) coordinate system.

5. (20 pts) Verify Stokes' theorem by evaluating both the circulation (line integral around C) and the appropriate surface integral for the vector field $\vec{u} = (2x - y, -y^2, -y^2 z)$. Let the surface S be the flat disk given by $z = 0$ with $x^2 + y^2 \le 1$ so that C is the circle $z = 0$ with $x^2 + y^2 = 1$.

6. (20 pts) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$\iint_S F \cdot \hat{n}\, dS$$

using the divergence theorem.