MATH 335 S2019
Practice Final Exam

2019-04-29

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances.

Reference

The following results and identities are provided for reference purposes (this is the same list as will be provided in the final). They may or may not be needed to complete the exam.

- The following identity may be used in the exam without the need to prove it:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

- Any rank-4 and isotropic tensor is of the form

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk},$$

  for some constants λ, μ, and ν.

Exam

1. Write the "physical" definition of the divergence of a vector field $\vec{F}$ (given as the limit of some integral over a shrinking region) and compute the divergence of $\vec{F} = (x, z - y, z)$.

2. Show that $\nabla \cdot (\phi\nabla\phi) = \phi\nabla^2\phi + \nabla\phi \cdot \nabla\phi$. Then show that

$$\int_V \phi\nabla^2\phi \, dV = \oiint_S \phi\nabla\phi \cdot \hat{n} \, dS - \iiint_V |\nabla\phi|^2 \, dV \ .$$

   Suppose further that $\nabla^2\phi = 0$ in V and $\phi = 0$ on S. In that case, explain why $\phi = 0$ throughout V.

3. Is $\vec{F} = (z, z, x + y)$ conservative? Show why it is or isn't.

4. Show that $\nabla \times (f\nabla f) = 0$.

5. Let $\vec{F} = (yze^x + xyze^x, xze^x, xye^x)$. Show that the circulation of $\vec{F}$ around an oriented simple curve C that is the boundary of a surface S is zero.

6. Conservation of mass for a fluid can be expressed in integral form as

$$\frac{d}{dt}\iiint_V \rho\, dV = -\oiint_S \rho\vec{u}\cdot\hat{n}\, dS\ ,$$

which holds for any volume V. Turn this into a differential equation in the usual way. Write the simplified expression you get when ρ is constant in space and time.

7. A set of "conical coordinates" are defined by

$$\begin{aligned} x_1 &= \frac{rvw}{2} \\ x_2 &= \frac{r}{2}\sqrt{\frac{(4-v^2)(4-w^2)}{3}} \\ x_3 &= r\sqrt{\frac{(v^2-1)(1-w^2)}{3}} \end{aligned}$$

for $0 \le w^2 \le 1 \le v^2 \le 4$.

- Show that $x_1^2 + x_2^2 + x_3^2 = r^2$. (This means that constant r gives a sphere).
- Find the scale factors and unit vectors of this coordinate system. Show that the system is orthogonal.
- What is the area element for a surface with constant r?

8. Stokes flow describes incompressible, creeping flow (high viscosity flow). The equations are

$$\mu\nabla^2\vec{u} - \nabla p = 0 \quad \text{in } V \tag{1}$$
$$\nabla\cdot\vec{u} = 0 \quad \text{in } V \tag{2}$$

where V is some region, S is its boundary, $\vec{u}$ is the velocity of the fluid, and p is the pressure (we have left out the boundary condition).

There is an equivalent of a charge (as in electrical charge) for the Stokes equations. The "charge" strength is a vector, which we'll denote by $\vec{h}$. The field $\vec{u}$ induced by this charge is defined to be $u_i = S_{ij}h_j$, where

$$S_{ij} = \frac{1}{8\pi\mu}\left(\delta_{ij}\frac{1}{r} + r_i r_j\frac{1}{r^3}\right),$$

$\vec{r} = (x, y, z)$, and $r = |\vec{r}|$. The pressure associated with $\vec{u}$ is given by $p = P_i h_i$ where

$$P_i = r_i\frac{1}{4\pi r^3}\ .$$

(a) Show that setting $u_i = S_{ij}h_j$ and $p = P_i h_i$ gives a solution of (1) (away from the origin).

(b) Recall that for an orthogonal transformation L_{ij} we have $L_{ij}L_{kj} = \delta_{ik}$. For any vector v_i, let $v_i' = L_{ij}v_j$, Show that $|\vec{v}\,'| = |\vec{v}|$.

(c) Confirm that P_i is a vector and S_{ij} is a tensor.