MATH 335 S2019
Practice Midterm Exam I

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances.

1. Compute the area between the curves $y = x^3$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$.

   Solution

   We can compute this using iterated integration.

$$\begin{aligned} A &= \iint_S dS \\ &= \int_0^1 \int_{x^3}^{\sqrt{x}} dy\,dx \\ &= \int_0^1 \sqrt{x} - x^3\,dx \\ &= \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^4\right]_0^1 \\ &= \frac{5}{12} \end{aligned}$$

2. What is the work done by the force $\vec{F}(\vec{r}) = (0, 0, r_3^2)$ in moving from $(1, 0, 0)$ to $(0, 2, 0)$?

   Solution

   First we parameterize the line. $(0, 2, 0) - (1, 0, 0) = (-1, 2, 0)$, so that $\vec{r}(t) = (1, 0, 0) + t(-1, 2, 0)$ traces out the line when $0 \leq t \leq 1$. Note that $d\vec{r}/dt = (-1, 2, 0)$. Then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (0, 0, 0) \cdot (-1, 2, 0)\,dt = 0\;,$$

which is not that interesting.

Let $\vec{F_*}(\vec{r}) = (0, r_2^2, 0)$. Then the work done by $\vec{F_*}$ over the same curve is

$$W = \int_C \vec{F_*} \cdot d\vec{r} = \int_0^1 (0, (2t)^2, 0) \cdot (-1, 2, 0)\, dt = \frac{8}{3} t^3|_0^1 = \frac{8}{3} \ ,$$

3. Complete the following

 (a) Find an implicit equation for the plane containing both of the lines $\vec{l_1}(t) = (1, 0, 1) + t(-1, -1, 1)$ and $\vec{l_2}(t) = (0, 2, 0) + t(-1, -1, 1)$.

 Solution

 In general, there may or may not be a plane containing any 2 lines. These lines are parallel so it is possible. Note that $(-1, -1, 1)$ is certainly parallel to the plane, because it follows along the line. To get a second, linearly independent vector parallel to the plane, we can take the difference between two points, one on each line. Consider $(1, 0, 1) - (0, 2, 0) = (1, -2, 1)$, which is linearly independent of $(-1, -1, 1)$.
 Taking the cross product of these gives a normal vector

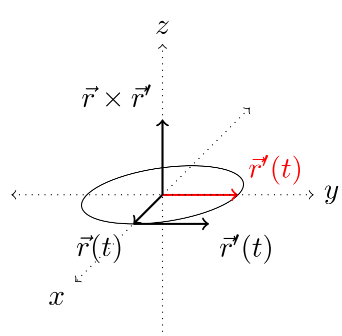
 $$\vec{n} = (1, -2, 1) \times (-1, -1, 1) = (-1, -2, -3) \ .$$

 We can then obtain the implicit equation using the fact that $(\vec{r} - (0, 2, 0)) \cdot (-1, -2, -3) = 0$ for any $\vec{r}$ in the plane. Simplifying, we get $r_1 + 2r_2 + 3r_3 = 4$.

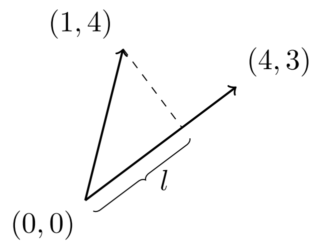
 (b) Let the curve C be the unit circle in the (x, y) plane. If $\vec{r}(t)$ is a parameterization of the circle which traces it counter-clockwise, does $\vec{r} \times d\vec{r}/dt$ point in the positive or negative z direction?

 Solution

 In the image below we draw a discretization of the curve and what the tangent looks like going counter-clockwise. The tangent, $\vec{r}'(t)$ is translated to the origin in red. Using the right hand rule, we see that the cross product $\vec{r} \times \vec{r}'$ points up.



(c) Compute the length l below



Solution

The length is $\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{16}{25}$

(d) Compute the integral $\int_0^{2\pi} x \sin x \, dx$.

Solution

Using integration by parts, we have

$$\begin{aligned}\int_0^{2\pi} x \sin x \, dx &= -x\cos x|_0^{2\pi} + \int_0^{2\pi} \cos x \, dx \\ &= -2\pi + 0 = -2\pi\end{aligned}$$

4. (a) Consider the surface S where $x^2 + y^2 = 1$ and $0 \leq z \leq 1$ (a tube). Compute the flux

$$\iint_S \vec{F} \cdot \hat{n} \, dS \ , \tag{1}$$

where $\vec{F}(x, y, z) = (y, z^3 y, x)$.

Solution

It is natural to discretize the cube in cylindrical coordinates. At each fixed z, the x and y coordinates just trace out a unit circle. We can then parameterize the curve as $\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z)$ where θ runs from 0 to 2π and z runs from 0 to 1.
We can then use the formula for the flux given a parameterization of the surface. For this, we need $\partial_\theta \vec{r}$ and $\partial_z \vec{r}$, which are $(-\sin\theta, \cos\theta, 0)$ and $(0, 0, 1)$, respectively. We also need $\partial_\theta \vec{r} \times \partial_z \vec{r} = (\cos\theta, \sin\theta, 0)$. Then,

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \int_0^{2\pi} \int_0^1 \vec{F} \cdot \partial_\theta \vec{r} \times \partial_z \vec{r} \, dz \, d\theta$$

$$
\begin{aligned}
&= \int_0^{2\pi} \int_0^1 (\sin\theta, z^3 \sin\theta, \cos\theta) \cdot (\cos\theta, \sin\theta, 0)\, dz\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \sin\theta\cos\theta + z^3 \sin^2\theta\, dz\, d\theta \\
&= \int_0^{2\pi} \sin\theta\cos\theta + \frac{1}{4}\sin^2\theta\, dz\, d\theta \\
&= \frac{1}{4}\int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\, d\theta \\
&= \frac{\pi}{4}
\end{aligned}
$$

(b) Compute the volume of the tube above $z = 0$ and below the plane through the points $(1, 0, 1)$, $(-1, 1, 0)$, $(-1, -1, 0)$.

Solution

We can compute this volume by integrating the height of the plane over the base of the tube, which is the unit disc. It is natural to integrate over the disc using polar coordinates, so we need to obtain z as a function of (r, θ). We can find the height by first finding the equation of the plane. We can subtract the vectors for the points on the plane to get parallel vectors. This gives $\vec{u} = (2, -1, 1)$ and $\vec{v} = (2, 1, 1)$. A normal vector can then be obtained from the cross product $\vec{n} = \vec{u} \times \vec{v} = (-2, 0, 4)$. The equation of the plane can then be obtained in the usual way, $(\vec{r} - (1, 0, 1)) \cdot (-2, 0, 4) = 0$. Simplifying, we get $z = (x + 1)/2 = (r\cos\theta + 1)/2$ (we can check that each point satisfies this to be sure). This gives the height z as a function of (r, θ). Then, for the volume, we get the iterated integral

$$
\begin{aligned}
\iiint_V dV &= \int_0^{2\pi} \int_0^1 \int_0^{(r\cos\theta+1)/2} r\, dz\, dr\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \int_0^1 r^2\cos\theta + r\, dr\, d\theta \\
&= \frac{1}{2}\int_0^{2\pi} \frac{1}{3}\cos\theta + \frac{1}{2}\, d\theta \\
&= \frac{\pi}{2}
\end{aligned}
$$

You might recognize that this is half the volume of the tube itself, which makes sense because the plane cuts it in half diagonally.