MATH 335 S2019
Electrostatics and the Fundamental Theorem of Vector Calculus

1 Electrostatics and a Green's function for the Laplacian

Let μ be a charge density contained in a set V. Recall that the electric field induced by this charge density is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3}\mu(\vec{p})\,dV(\vec{p}).$$

We saw previously that this field is the gradient of a potential ϕ given by

$$\phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\iiint_V \frac{1}{|\vec{r}-\vec{p}|}\mu(\vec{p})\,dV(\vec{p}).$$

Note that this means that $\vec{E}$ is conservative.

What is the divergence of $\vec{E}$? When $\vec{r} \neq \vec{p}$, we have that

$$\nabla_r \cdot \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3} = 3\frac{1}{|\vec{r}-\vec{p}|^3} - \frac{3}{2}\frac{2(\vec{r}-\vec{p})}{|\vec{r}-\vec{p}|^5}\cdot(\vec{r}-\vec{p}) = 0$$

where we have used the vector identity $\nabla\cdot(\vec{f}g) = (\nabla\cdot\vec{f})g + \vec{f}\cdot\nabla g$. In some sense, all of the action happens when $\vec{p} = \vec{r}$ (where the potential is not defined). We can use the divergence theorem to figure out the value of $\nabla_r \cdot \vec{E}$. First, we bring the divergence inside the integral, i.e.

$$\nabla_r \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\iiint_V \nabla_r \cdot \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3}\mu(\vec{p})\,dV(\vec{p}) = -\frac{1}{4\pi\epsilon_0}\iiint_V \nabla_p \cdot \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3}\mu(\vec{p})\,dV(\vec{p})$$

where we have used the fact that $\frac{d}{dx}f(x-y) = -\frac{d}{dy}f(x-y)$. From the result above, we see that if $\vec{r}$ is outside of V, then the integrand is always zero. Indeed, the divergence is equal to the same integral taken over any ball which contains $\vec{r}$. Let $B_\delta(\vec{r})$ be the ball of radius δ around the point $\vec{r}$ and let $S_\delta(\vec{r})$ be its boundary (a sphere). If we take δ to be small, then we can approximate μ by a constant, namely $\mu(\vec{r})$ inside. Then,

$$\nabla_r \cdot \vec{E}(\vec{r}) \approx -\frac{\mu(\vec{r})}{4\pi\epsilon_0}\iiint_{B_\delta(\vec{r})} \nabla_p \cdot \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3}\,dV(\vec{p})\ .$$

We can now apply the divergence theorem to this expression. We get

$$\nabla_r \cdot \vec{E}(\vec{r}) \approx \frac{\mu(\vec{r})}{4\pi\epsilon_0} \iint_{S_\delta(\vec{r})} \frac{\vec{p}-\vec{r}}{|\vec{p}-\vec{r}|^3} \cdot \hat{n}\, dS(\vec{p}) \ .$$

Setting $\vec{u} = \vec{p} - \vec{r}$, we get

$$\nabla_r \cdot \vec{E}(\vec{r}) \approx \frac{\mu(\vec{r})}{4\pi\epsilon_0} \iint_{S_\delta(0)} \frac{\vec{u}}{|\vec{u}|^3} \cdot \frac{\vec{u}}{|\vec{u}|}\, dV(\vec{p}) = \frac{\mu(\vec{r})}{4\pi\epsilon_0\delta^2} \iint_{S_\delta(0)} dS(\vec{p}) = \frac{\mu(\vec{r})}{\epsilon_0} \ .$$

In the limit as $\delta \to 0$ this expression is exact for a continuous μ.

We now have that $\nabla \cdot \vec{E} = \nabla^2 \phi = \mu/\epsilon_0$, i.e. ϕ solves Poisson's equation. If we set $G(\vec{r}, \vec{p}) = -\frac{1}{4\pi} \frac{\vec{r}-\vec{p}}{|\vec{r}-\vec{p}|^3}$ we see that

$$\nabla^2 \iiint_V G(\vec{r}, \vec{p}) \mu(\vec{p})\, dV(\vec{p}) = \mu(\vec{r}) \ ,$$

where the derivatives are taken with respect to $\vec{r}$, which we will denote by ∇_r. As in the 1D case from HW3, we call such a function a Green's function for the Laplacian. Because of this, we sometimes write $\nabla_r^2 G(\vec{r}, \vec{p}) = \delta_{\vec{r}}(\vec{p})$. The δ function simply means that $\iiint_V \delta_{\vec{r}}(\vec{p}) f(\vec{p})\, dV(\vec{p}) = f(\vec{r})$ for sufficiently smooth f.

The Green's function idea can be used to establish some powerful results.

2 Warm up — a Representation Theorem

Recall Green's 2nd I.D., i.e.

$$\iiint_V g\nabla^2 f - f\nabla^2 g\, dV = \iint_S g\partial_n f - f\partial_n g\, dS \ .$$

If we set $g(\vec{p}) = G(\vec{r}, \vec{p})$ (where G is the Green's function above) and let f be any twice continuously differentiable function, then

$$\iiint_V G\nabla^2 f - f\nabla^2 G\, dV = \iint_S G\partial_n f - f\partial_n G\, dS \ .$$

Rearranging, we get

$$f(\vec{r}) = \iiint_V G\nabla^2 f\, dV + \iint_S f\partial_n G - G\partial_n f\, dS.$$

A result known as Green's third identity. Note that this is rather remarkable: any twice continuously differentiable function can be determined by its Laplacian in the domain and its value and normal derivative on the boundary.

If $\nabla^2 f = 0$, then

$$f(\vec{r}) = \iint_S f\partial_n G - G\partial_n f\, dS \ ,$$

Which shows that a harmonic function f can be represented by an integral over the boundary. A similar result holds for vector fields, which we will show next.

3 Fundamental Theorem of Vector Calculus

Before we prove the main theorem, we require the following lemma, which is like the divergence theorem for the curl operator.

Lemma 1. *Let $\vec{v}$ be a vector field defined on a domain D with boundary S. Then*

$$\iiint_V \nabla \times \vec{v}\, dV = \iint_S \hat{n} \times \vec{v}\, dS$$

Proof. We can establish the following identity for a constant vector $\vec{a}$ and vector field $\vec{v}$ using suffix notation. We have

$$\begin{aligned}
\nabla \cdot (\vec{a} \times \vec{v}) &= \partial_i \epsilon_{ijk} a_j v_k \\
&= \epsilon_{ijk} \partial_i (a_j v_k) \\
&= a_j \epsilon_{ijk} \partial_i v_k \\
&= -a_j \epsilon_{jik} \partial_i v_k \\
&= -\vec{a} \cdot \nabla \times \vec{v}
\end{aligned}$$

Applying the divergence theorem to $\vec{u} = \vec{a} \times \vec{v}$, we obtain

$$\begin{aligned}
\iiint_V \nabla \cdot (\vec{a} \times \vec{v})\, dV &= \iint_S \vec{a} \times \vec{v} \cdot \hat{n}\, dS \\
\iiint_V -\vec{a} \cdot \nabla \times \vec{v}\, dV &= \iint_S \vec{a} \cdot \vec{v} \times \hat{n}\, dS \\
\iiint_V \vec{a} \cdot \nabla \times \vec{v}\, dV &= \iint_S \vec{a} \cdot \hat{n} \times \vec{v}\, dS
\end{aligned}$$

Because this holds for any $\vec{a}$, we can read off the desired result one entry at a time by setting $\vec{a} = e_1, e_2, e_3$. □

In the following theorem, we will make use of some of the vector identities from Chapter 4:

- $\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \nabla \times \vec{u}$
- $\nabla \cdot (f\vec{u}) = f\nabla \cdot \vec{u} + \vec{u} \cdot \nabla f$
- $\nabla \times (f\vec{u}) = f\nabla \times \vec{u} + \nabla f \times \vec{u} \quad .$

We will also make use of the fact that $\nabla_p G(\vec{p},\vec{r}) = -\nabla_r G(\vec{p},\vec{r})$. Finally, we will use the divergence theorem and the fact that G is a Green's function for the Laplacian.

Theorem 1 (Fundamental Theorem of Vector Calculus)**.** *Let $\vec{F}$ be a twice continuously differentiable vector field defined on a domain V with boundary S. Then $\vec{F}$ can be decomposed into a curl-free component and a divergence-free component: $\vec{F} = \nabla\Phi + \nabla\times\vec{\Psi}$, where*

$$\Phi = \iiint_V G(\vec{p},\vec{r})\nabla\cdot\vec{F}(\vec{r})\,dV(\vec{r}) - \iint_S G(\vec{p},\vec{r})\hat{n}\cdot\vec{F}(\vec{r})\,dS(\vec{r}) \tag{1}$$

$$\vec{\Psi} = -\iiint_V G(\vec{p},\vec{r})\nabla\times\vec{F}(\vec{r})\,dV(\vec{r}) + \iint_S G(\vec{p},\vec{r})\hat{n}\times\vec{F}(\vec{r})\,dS(\vec{r})\,. \tag{2}$$

Proof. To start, we note that

$$\vec{F} = \nabla_p^2 \iiint_V G(\vec{p},\vec{r})\vec{F}(\vec{r})\,dV(\vec{r})\,. \tag{3}$$

Using the identity $\nabla^2\vec{u} = \nabla(\nabla\cdot\vec{u}) - \nabla\times\nabla\times\vec{u}$, we have that

$$\vec{F} = \nabla_p\nabla_p\cdot\iiint_V G(\vec{p},\vec{r})\vec{F}(\vec{r})\,dV(\vec{r}) - \nabla_p\times\nabla_p\times\iiint_V G(\vec{p},\vec{r})\vec{F}(\vec{r})\,dV(\vec{r})\,. \tag{4}$$

Note that

$$\begin{aligned}
\nabla_p\cdot(G(\vec{p},\vec{r})\vec{F}(\vec{r})) &= \nabla_p G(\vec{p},\vec{r})\cdot\vec{F}(\vec{r}) + 0\\
&= -\nabla_r G(\vec{p},\vec{r})\cdot\vec{F}(\vec{r})\\
&= G(\vec{p},\vec{r})\nabla_r\cdot\vec{F}(\vec{r}) - \nabla_r\cdot(G(\vec{p},\vec{r})\vec{F}(\vec{r}))
\end{aligned}$$

and similarly

$$\begin{aligned}
\nabla_p\times(G(\vec{p},\vec{r})\vec{F}(\vec{r})) &= \nabla_p G(\vec{p},\vec{r})\times\vec{F}(\vec{r}) + 0\\
&= -\nabla_r G(\vec{p},\vec{r})\times\vec{F}(\vec{r})\\
&= G(\vec{p},\vec{r})\nabla_r\times\vec{F}(\vec{r}) - \nabla_r\times(G(\vec{p},\vec{r})\vec{F}(\vec{r}))\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{F} = \nabla_p &\iiint_V G(\vec{p},\vec{r})\nabla_r\cdot\vec{F}(\vec{r}) - \nabla_r\cdot(G(\vec{p},\vec{r})\vec{F}(\vec{r}))\,dV(\vec{r})\\
&- \nabla_p\times\iiint_V G(\vec{p},\vec{r})\nabla_r\times\vec{F}(\vec{r}) - \nabla_r\times(G(\vec{p},\vec{r})\vec{F}(\vec{r}))\,dV(\vec{r})\,.
\end{aligned}$$

Then, we can apply the divergence theorem and the lemma above to obtain

$$\begin{aligned}\vec{F} = \nabla_p &\left(\iiint_V G(\vec{p},\vec{r}) \nabla_r \cdot \vec{F}(\vec{r})\, dV(\vec{r}) - \iint_S G(\vec{p},\vec{r}) \vec{F}(\vec{r}) \cdot \hat{n}\, dS(\vec{r}) \right) \\ &- \nabla_p \times \left(\iiint_V G(\vec{p},\vec{r}) \nabla_r \times \vec{F}(\vec{r})\, dV(\vec{r}) - \iint_S G(\vec{p},\vec{r}) \hat{n} \times \vec{F}(\vec{r})\, dS(\vec{r}) \right) \ .\end{aligned}$$

The above means we have explicit formulas for Φ and $\vec{\Psi}$ given $\vec{F}$, i.e.

$$\begin{aligned}\Phi(\vec{p}) &= \iiint_V G(\vec{p},\vec{r}) \nabla_r \cdot \vec{F}(\vec{r})\, dV(\vec{r}) - \iint_S G(\vec{p},\vec{r}) \vec{F}(\vec{r}) \cdot \hat{n}\, dS(\vec{r}) \\ \vec{\Psi}(\vec{p}) &= -\iiint_V G(\vec{p},\vec{r}) \nabla_r \times \vec{F}(\vec{r})\, dV(\vec{r}) + \iint_S G(\vec{p},\vec{r}) \hat{n} \times \vec{F}(\vec{r})\, dS(\vec{r}) \ .\end{aligned}$$

$\square$