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MATH 335-002: Homework #3 Solutions

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Instructions

- This assignment is due in-class Thursday March 7th.
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

Outline

In this assignment, we practice working with the various del operators, i.e. the gradient, divergence, Laplacian, and curl. These are written in operator form as ∇, $\nabla\cdot$, ∇^2, and $\nabla\times$, respectively. A quick reference guide for these operators is below:

name	symbol	in coordinates	what-to-what	interpretation
grad	∇	$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$	scalar field to vector field	$\nabla\phi$ is the direction of greatest increase of ϕ, points perpendicular to level surfaces of ϕ
div	$\nabla\cdot$	$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$	vector field to scalar field	$\nabla \cdot \vec{F}$ is the "local flux density" of $\vec{F}$, indicates how much a fluid flowing like $\vec{F}$ would expand/contract at that point in an instant
Laplacian	∇^2	$\nabla^2\phi = \frac{\partial^2\phi}{\partial^2 x} + \frac{\partial^2\phi}{\partial^2 y} + \frac{\partial^2\phi}{\partial^2 z}$	scalar field to scalar field	$\nabla^2\phi$ (sometimes written $\Delta\phi$) at a given point denotes how much ϕ deviates from the average of ϕ at local neighboring points
curl	$\nabla\times$	$\nabla \times \vec{F} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$	vector field to vector field	$\hat{n} \cdot \nabla \times \vec{F}$ indicates how much $\vec{F}$ is rotating about the $\hat{n}$ axis

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1 Exercises from Matthews

Please complete exercises 3.4, 3.5, 3.9, 3.11, 3.13, and 3.15. from the textbook (2 pts each). They are provided below for convenience.

- 3.4: Find the angle between the surfaces of the sphere $x^2 + y^2 + z^2 = 2$ and the cylinder $x^2 + y^2 = 1$ at a point where they intersect. (Note that the angle between surfaces is defined to be the angle between the normals of each surface at a point where they intersect).
- 3.5: Find the gradient of the scalar field $f = yx^2 + y^3 - y$ and hence find the minima and maxima of f. Sketch the contours $f =$ constant and the vector field ∇f. (This is a difficult plot to come up with. It's worth trying to figure it out but don't spend too long before consulting the back of the book or asking a friend).
- 3.9: Find the gradient $\nabla\phi$ and the Laplacian $\nabla^2\phi$ for the scalar field $\phi = x^2 + xy + yz^2$.
- 3.11: Find the unit normal to the surface $xy^2 + 2yz = 4$ at the point $(-2, 2, 3)$.
- 3.13: Find the equation of the plane which is tangent to the surface $x^2 + y^2 - 2x^3 = 0$ at the point $(1, 1, 1)$.
- 3.15: Show that both the divergence and curl are linear operators, i.e. $\nabla \cdot (c\vec{u} + d\vec{v}) = c\nabla \cdot \vec{u} + d\nabla \cdot \vec{v}$ and $\nabla \times (c\vec{u} + d\vec{v}) = c\nabla \times \vec{u} + d\nabla \times \vec{v}$.

Solution

The solutions to these problems are available in the back of the textbook.

2 Other Exercises

Question 1 (5 pts) 1D Green's function

Consider the interval $[0, L]$. Let $G(x, \tilde{x})$ be the function of two variables defined by

$$G(x, \tilde{x}) = \begin{cases} -\frac{\tilde{x}}{L}(x - L) & \text{when} \quad x > \tilde{x} \\ \left(1 - \frac{\tilde{x}}{L}\right) x & \text{when} \quad x \leq \tilde{x} \end{cases} .$$

Let $f(x)$ be some function on $[0, L]$ and let

$$u(x) = \int_0^L G(x, \tilde{x}) f(\tilde{x})\, d\tilde{x} .$$

(a) (2 pts) Show that

$$u(x) = \left(1 - \frac{x}{L}\right) \int_0^x \tilde{x} f(\tilde{x})\, d\tilde{x} + x \int_x^L \left(1 - \frac{\tilde{x}}{L}\right) f(\tilde{x})\, d\tilde{x}$$

(b) (3 pts) Show that

$$-\frac{d^2}{dx^2} u(x) = f(x)$$

and that $u(0) = 0$ and $u(L) = 0$.

This property is what it means for G to be a Green's function for the operator $-\frac{d^2}{dx^2}$, i.e. integrating against G undoes the operator. The conditions that $u(0) = u(L) = 0$ are known as boundary conditions. Green's functions are powerful tools in the analysis and solution of differential equations. We'll learn more about them after Chapter 5.

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Solution

(a) This follows from the definition of G. Split the integral at x, i.e.

$$u(x) = \int_0^1 G(x,\tilde{x})\, d\tilde{x} = \int_0^x G(x,\tilde{x})\, d\tilde{x} + \int_x^1 G(x,\tilde{x})\, d\tilde{x}\ .$$

In first part of the integral $x > \tilde{x}$ and in the second $x \le \tilde{x}$. Thus

$$\begin{aligned} u(x) = \int_0^1 G(x,\tilde{x})\, d\tilde{x} &= \int_0^x -\frac{\tilde{x}}{L}(x-L)\, d\tilde{x} + \int_x^1 \left(1-\frac{\tilde{x}}{L}\right) x\, d\tilde{x} \\ &= \left(1-\frac{x}{L}\right)\int_0^x \tilde{x} f(\tilde{x})\, d\tilde{x} + x\int_x^L \left(1-\frac{\tilde{x}}{L}\right) f(\tilde{x})\, d\tilde{x}\ , \end{aligned}$$

where the last line follows because x is a constant in the integrals (which are taken over $\tilde{x}$).

(b) Plugging in $x = 0$ into the equation for $u(x)$ from part (a), we get

$$u(0) = \left(1-\frac{0}{L}\right)\int_0^0 \tilde{x} f(\tilde{x})\, d\tilde{x} + 0\int_0^L \left(1-\frac{\tilde{x}}{L}\right) f(\tilde{x})\, d\tilde{x} = 0\ ,$$

because in both terms we have that either the integral is 0 or it is multiplied by 0. Similarly, we have

$$u(L) = \left(1-\frac{L}{L}\right)\int_0^L \tilde{x} f(\tilde{x})\, d\tilde{x} + L\int_L^L \left(1-\frac{\tilde{x}}{L}\right) f(\tilde{x})\, d\tilde{x} = 0\ .$$

For the rest, we make repeated use of the fact that $\frac{d}{dx}\int_0^x g(\tilde{x})\, d\tilde{x} = g(x)$. The first derivative of u is

$$\begin{aligned} u'(x) &= -\frac{1}{L}\int_0^x \tilde{x} f(\tilde{x})\, d\tilde{x} + (1-x/L)xf(x) + \int_x^L (1-\tilde{x}/L) f(\tilde{x})\, d\tilde{x} - x(1-x/L)f(x) \\ &= -\frac{1}{L}\int_0^x \tilde{x} f(\tilde{x})\, d\tilde{x} + \int_x^L (1-\tilde{x}/L) f(\tilde{x})\, d\tilde{x}\ . \end{aligned}$$

Then

$$\begin{aligned} u''(x) &= -xf(x)/L - (1-x/L)f(x) \\ &= -f(x)\ . \end{aligned}$$

Question 2 (5 pts)

(a) (2 pts) Suppose w satisfies that $w(0) = 0$ and $w(1) = 0$ and that $\frac{d^2}{dx^2}w(x) = 0$ on $[0,1]$. Explain why $w(x) = 0$ for all x in $[0,1]$.

(b) (3 pts) Let f be some function on $[0,1]$. Suppose that $u(0) = u(1) = v(0) = v(1) = 0$ and that $\frac{d^2}{dx^2}u(x) = f(x)$ and $\frac{d^2}{dx^2}v(x) = f(x)$ on $[0,1]$. Explain why $u(x) = v(x)$ for all $x \in [0,1]$. (If you're stuck, email me for a hint `askham@njit.edu`)

Solution

(a) There are at least two ways to look at this. One way is to say that by integrating twice, we have that $w(x) = ax + b$ for some constants a and b. Then the boundary conditions ($w(0) = w(1) = 0$) imply that $0 = w(0) = a \cdot 0 + b = b$ and $0 = w(1) = a \cdot 1 + b$ so that $a = b = 0$ and $w(x) = 0$.

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Another way is to use integration by parts. Note that $w''(x)w(x) = 0$ for all x. We have

$$\begin{aligned} 0 &= \int_0^1 w(x)w''(x)\,dx \\ &= w(1)w'(1) - w(0)w'(0) + \int_0^1 (w'(x))^2\,dx \\ &= \int_0^1 (w'(x))^2\,dx \end{aligned}$$

As we argued in class, if the integral of a non-negative function is zero, the function is zero. Therefore, $w'(x) = 0$ for all x. Thus, w is a constant. The boundary conditions imply that that constant is 0.

(b) If we consider the function $w(x) = u(x) - v(x)$, then $w''(x) = u''(x) - v''(x) = f(x) - f(x) = 0$ for all x. Likewise, $w(0) = w(1) = 0$ so that part (a) applies. Then $w(x) = 0$ or $u(x) = v(x)$ for all x.