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MATH 335-002: Homework #5 Solutions

Instructor: Travis Askham
askham@njit.edu

NJIT — April 10th, 2019

Instructions

WARNING: in spherical coordinates, the book uses a different convention for the names of the angles, i.e. the roles of ϕ and θ are switched from our discussion in class.

- This assignment is due in-class Tuesday April 23rd.
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

Outline

In this assignment, we practice working with curvilinear coordinate systems. The key to these is the use of scale factors which describe the relative size of a change in cartesian space as a change is made in the curvilinear coordinates.

1 Exercises from Matthews

Please complete exercises 6.2, 6.3, 6.4, 6.5, 6.8, and 6.9 from the textbook (2 pts each). They are provided below for convenience.

- 6.2: A coordinate system (u, v, w) is related to cartesian coordinates by

$$x_1 = uvw\,, \quad x_2 = uv(1-w^2)^{1/2}\,, \quad x_3 = (u^2 - v^2)/2\,.$$

  (a) Find the scale factors h_u, h_v, and h_w.
  (b) Confirm that the (u, v, w) system is orthogonal.
  (c) Find the volume element in the (u, v, w) coordinate system.

- 6.3: Find the scale factors and hence the volume element for the coordinate system (u, v, θ) defined by

$$x_1 = uv\cos\theta\,, \quad x_2 = uv\sin\theta\,, \quad x_3 = (u^2 - v^2)/2\,,$$

  in which u and v are positive and $0 \le \theta < 2\pi$. Find the volume of the region enclosed by the surfaces $u = 1$ and $v = 1$.

Name:

- 6.4: Re-derive the formula for ∇f in a general orthogonal curvilinear coordinate system by writing ∇f in Cartesian coordinates and then finding the component of ∇f in the $\vec{e}_1$ direction.
- 6.5: A cylindrical apple corer of radius a cuts through a spherical apple of radius b. How much of the apple does it remove? Assume $a < b$.
- 6.8: Find $(\vec{u} \cdot \nabla)\vec{u}$ where $\vec{u} = \vec{e}_\theta$ in cylindrical polar coordinates.
- 6.9: Find a formula for the R component of the Laplacian of a velocity field in cylindrical polar coordinates (start with the definition $\nabla^2\vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times (\nabla \times \vec{u})$). Verify that this does not equal the Laplacian of the R component of the velocity field.

Solution

The solutions to these exercises can be found in the back of the book.

2 Other Exercises

Name:

Question 1

(15 pts) Bessel functions and the vibration of a circular drum

In polar coordinates, the Laplacian is just like the Laplacian for the cylinder, but with the z part removed:

$$\nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}f(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2 f(r,\theta)}{\partial^2\theta} \ .$$

The structure of the Laplacian is what we call separable because the r and θ terms are separate — this allows us to solve certain physics problems on the disc by searching for solutions of the form $f(r,\theta) = a(r)b(\theta)$.

The vibration of a circular drum head is described by

$$\frac{\partial^2 u(r,\theta,t)}{\partial^2 t} = c^2\nabla^2 u(r,\theta,t) \ ,$$

where u is the displacement of the membrane. Because the membrane is fastened to the edge of the drum, which we take to have radius 1, we have the boundary condition $u(1,\theta,t) = 0$ for all θ and t.

- (2 pts) Suppose that u is radially symmetric and that it has a constant frequency ω in time, i.e. that $u(r,\theta,t) = a(r)\cos(\omega t - \psi)$ where ψ is some phase.

  Show that

$$\frac{d^2}{dr^2}a + \frac{1}{r}\frac{d}{dr}a + \frac{\omega^2}{c^2}a = 0 \qquad (1)$$

- (2 pts) The solutions of

$$\frac{d^2}{dr^2}y + \frac{1}{r}\frac{d}{dr}y + y = 0 \qquad (2)$$

  are known as Bessel functions of order 0. The solution with $y(0) = 1$ and $y'(0) = 0$ has a special name and is written as $J_0(r)$. It is the only bounded solution in the disc (up to multiplication by a constant). Show that $a(r) = \alpha J_0\left(\frac{\omega r}{c}\right)$ solves (1).

- (1 pt) Suppose that z is a root of J_0, i.e. $J_0(z) = 0$. Find a value of ω so that $a(r) = J_0(\omega r/c)$ satisfies the boundary condition $a(1) = 0$. (Note that this means that only certain frequencies give radially symmetric vibrating modes, which you can determine based on the roots of J_0!)

- (5 pts) Show that

$$J_0(r) = \sum_{n=0}^{\infty}(-1)^n r^{2n}\frac{1}{2^{2n}n!n!}$$

  satisfies (2) and that $J_0(0) = 1$ and $J_0'(0) = 0$. (You are allowed evaluate the derivative of the sum by differentiating term-by-term.)

- (2 pts) If we allow vibrations that oscillate in the θ direction, we can have solutions of the form $u(r,\theta,t) = a(r)\cos(\omega t - \psi)\sin(n\theta)$. Show that such an a satisfies

$$\frac{d^2}{dr^2}a + \frac{1}{r}\frac{d}{dr}a + \left(\frac{\omega^2}{c^2} - \frac{n^2}{r^2}\right)a = 0 \qquad (3)$$

- (2 pts) Show that if y solves (2), its derivative, which we'll call $v = dy/dr$, satisfies

$$\frac{d^2}{dr^2}v + \frac{1}{r}\frac{d}{dr}v + \left(1 - \frac{1}{r^2}\right)v = 0 \qquad (4)$$

- (1 pt) Let v solve (4). Show that $a(r) = v\left(\frac{\omega r}{c}\right)$ solves (3) with $n = 1$.

Name:

Solution

- If we plug $u = a(r)\cos(\omega t - \psi)$ into the wave equation, we get

$$\begin{aligned}
\frac{\partial^2}{\partial^2 t}\left(a(r)\cos(\omega t-\psi)\right) &= c^2\nabla^2\left(a(r)\cos(\omega t-\psi)\right)\\
-\omega^2 a(r)\cos(\omega t-\psi) &= c^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right)\left(a(r)\cos(\omega t-\psi)\right)\\
-\frac{\omega^2}{c^2}a(r)\cos(\omega t-\psi) &= \frac{d^2}{dr^2}a(r)\cos(\omega t-\psi)+\frac{1}{r}\frac{d}{dr}a(r)\cos(\omega t-\psi)
\end{aligned}$$

  Because this holds for all t, we can divide through by $\cos(\omega t-\psi)$ for some t where it is non-zero. Doing so gives the desired equation.

- If $a(r) = \alpha J_0\left(\frac{\omega r}{c}\right)$, then

$$\begin{aligned}
\frac{d^2}{dr^2}a+\frac{1}{r}\frac{d}{dr}a+\frac{\omega^2}{c^2}a &= \frac{\omega^2}{c^2}J_0''\left(\frac{\omega r}{c}\right)+\frac{1}{r}\frac{\omega}{c}J_0'\left(\frac{\omega r}{c}\right)+\frac{\omega^2}{c^2}J_0\left(\frac{\omega r}{c}\right)\\
&= \frac{\omega^2}{c^2}\left(J_0''\left(\frac{\omega r}{c}\right)+\frac{1}{\frac{\omega r}{c}}J_0'\left(\frac{\omega r}{c}\right)+J_0\left(\frac{\omega r}{c}\right)\right)
\end{aligned}$$

  Because J_0 solves (2), the last expression above is zero.

- If z is a root then setting $\omega = cz$ implies that $J_0(\omega/c) = 0$. Thus, $a(r) = J_0(\omega r/c)$ satisfies the boundary condition $a(1) = 0$.

- If we differentiate the power series for J_0, we get

$$\begin{aligned}
J_0'(r) &= \sum_{n=1}^{\infty}(-1)^n r^{2n-1}\frac{1}{2^{2n-1}n!(n-1)!}\\
J_0''(r) &= \sum_{n=1}^{\infty}(-1)^n r^{2n-2}\frac{1}{2^{2n-2}(n-1)!(n-1)!}-\sum_{n=1}^{\infty}(-1)^n r^{2n-2}\frac{1}{2^{2n-1}n!(n-1)!}\\
&= -\sum_{n=1}^{\infty}(-1)^{n-1} r^{2n-2}\frac{1}{2^{2n-2}(n-1)!(n-1)!}-\frac{1}{r}\sum_{n=1}^{\infty}(-1)^n r^{2n-1}\frac{1}{2^{2n-1}n!(n-1)!}\\
&= -\sum_{n=0}^{\infty}(-1)^n r^{2n}\frac{1}{2^{2n}n!n!}-\sum_{n=1}^{\infty}(-1)^n r^{2n-2}\frac{1}{2^{2n-1}n!(n-1)!}\\
&= -J_0(r)-\frac{1}{r}J_0'(r)\,,
\end{aligned}$$

  so that J_0 solves (2). If we plug in $r = 0$ in the formulas for J_0 and J_0', we get that $J_0(0)$ has only the constant term, 1, and J_0', which doesn't have a constant term, is 0.

- The only part that is different from the other example above is the $\sin(n\theta)$ term. Noting that $\frac{\partial^2\sin(n\theta)}{\partial^2\theta} = -n^2\sin(n\theta)$ gives the result.

- Let y solve

$$\frac{d^2}{dr^2}y+\frac{1}{r}\frac{d}{dr}y+y = 0\,.$$

  Differentiating this equation, we get

$$\frac{d^3}{dr^3}y+\frac{1}{r}\frac{d^2}{dr^2}y-\frac{1}{r^2}\frac{d}{dr}y+\frac{d}{dr}y = 0.$$

Name:

Making the substitution $v = dy/dr$, we get

$$\frac{d^2}{dr^2}v + \frac{1}{r}\frac{d}{dr}v + \left(1 - \frac{1}{r^2}\right)v = 0 \ .$$

- This follows the same reasoning as the example above. Note that J_0' is such a solution ($J_1 = -J_0'$ is another Bessel function).