MATH 335 S2019
Notes 2019-01-22

1 Implicit description vs parametric description of a plane

In class today I presented two ways of describing a plane. Let $\vec{r}_0$ be a given point on the plane and let $\hat{n}$ be a vector which is perpendicular to the surface of the plane.

For any point on the plane, $\vec{r}$, we have that

$$(\vec{r} - \vec{r}_0) \cdot \hat{n} = 0 \; , \tag{1}$$

by the definition of $\hat{n}$ and the properties of the dot product.

Setting $c_0 = \vec{r}_0 \cdot \hat{n}$ (which is a constant), we obtain

$$r_1 n_1 + r_2 n_2 + r_3 n_3 = c_0 \; , \tag{2}$$

an equation which any point on the plane satisfies. As a sanity check, we note that points in 3d have 3 degrees of freedom, i.e. r_1, r_2, r_3 can take any value, but the one constraint above removes a degree of freedom, leaving 2 (as we would expect for a 2d object like a plane).

This connects to another natural description of the plane. Let $\hat{a}$ and $\hat{b}$ be unit vectors which are parallel to the plane and are perpendicular to each other. Then, with $\vec{r}_0$ as above, any point $\vec{r}$ can be written as

$$\vec{r} = \vec{r}_0 + \alpha \hat{a} + \beta \hat{b} \tag{3}$$

for some scalars α and β.

Equation (2) is often referred to as an implicit equation for the plane because the location of points on the plane is implied by finding solutions to this equation. By contrast, (3) gives you the values of a point on the plane for any given α and β. The values of α and β are often referred to as parameters and (3) a parametric description of the plane.

In class, someone asked how we find vectors $\hat{a}$ and $\hat{b}$ (a good question!) I stumbled with that a bit. The answer is basically: you just find them. We know that $\hat{a} \cdot \hat{n} = 0$. To be concrete, say that $\hat{n} = (1, 1, 1)/\sqrt{3}$. Then we just need to find (non-zero) values such that

$$\frac{1}{\sqrt{3}}(a_1 + a_2 + a_3) = 0 \; . \tag{4}$$

Setting $a_1 = 0$, we get that $a_2 = -a_3$. We can set $a_2 = 1/\sqrt{2}$ and $a_3 = -1/\sqrt{2}$ (the square roots are just there to get a normal vector). Now that we have one vector which is perpendicular to $\hat{n}$, it is easy to obtain a third which is perpendicular to both using the cross product, i.e. let $\hat{b} = \hat{a} \times \hat{n}$.

Something to think about: why is $\hat{b}$ given by the cross product above automatically unit length?