MATH 335 S2019
Practice Problems for Midterm Exam II

Exam II covers chapters 3-5 (∇ operators, suffix notation, and integral theorems). Here are a number of practice problems for the exam. The actual exam will be about the same length as the first exam.

1. Practice with the ∇ operator

   (a) What is the geometrical interpretation of ∇f?

   Solution

   If we consider the level sets of f, i.e. the curves in 2D or surfaces in 3D defined by $f(\vec{x}) = c$, then ∇f points in the direction normal to the curve/surface and the magnitude of ∇f is the rate of change of f in that direction.

   (b) What are the definitions of the divergence and curl?

   Solution

   The divergence is defined as

   $$\nabla \cdot \vec{u} = \lim_{|\delta V| \to 0} \frac{1}{|\delta V|} \iint_{\delta S} \vec{u} \cdot \hat{n} \, dS \ ,$$

   where δS is the surface bounding the small volume δV. In standard coordinates, we have $\nabla \cdot \vec{u} = \partial_i u_i = \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$
   The curl is a vector valued quantity which we define in terms of its component in a given direction. Let $\hat{n}$ be a unit vector. Then

   $$\nabla \times \vec{u} = \lim_{|\delta S| \to 0} \frac{1}{|\delta S|} \int_{\delta C} \vec{u} \cdot d\vec{r} \ ,$$

   where δS is a piece of surface which is perpendicular to $\hat{n}$ and δC is the boundary curve of δS, oriented according to the right hand rule. In standard coordinates, we can compute the curl using the mnemonic

$$\nabla \times \vec{u} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix}.$$

(c) Show that $\vec{F} = (2x + y, x, 2z)$ is conservative.

Solution

One way to show that $\vec{F}$ is conservative is to find a function ϕ such that $\nabla\phi = \vec{F}$. Reading off each component of $\nabla\phi = \vec{F}$, we get $\partial_x\phi = 2x + y$, $\partial_y\phi = x$, and $\partial_z\phi = 2z$. Performing partial integration, we get $\phi = x^2 + xy + f_1(y, z)$, $\phi = xy + f_2(x, z)$, and $\phi = z^2 + f_3(x, y)$. Observe that setting $\phi = x^2 + xy + z^2$, we get such a ϕ. Thus, $\vec{F}$ is conservative.

Alternatively, we can compute the curl of $\vec{F}$ if $\nabla \times \vec{F} = 0$ over a domain with no holes, then $\vec{F}$ is conservative on that domain. We have

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ 2x + y & x & 2z \end{vmatrix} \\ &= \vec{e_1}\,(0 - 0) - \vec{e}_2\,(0 - 0) + \vec{e}_3\,(1 - 1) \\ &= \vec{0} \end{aligned}$$

and $\vec{F}$ and $\nabla \times \vec{F}$ are well-defined everywhere (no holes) so $\vec{F}$ is conservative.

(d) Find the gradient and Laplacian of $\phi = \sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z)$

Solution

We have

$$\begin{aligned} \partial_x\phi &= k\cos(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z) \\ \partial_{xx}\phi &= -k^2\sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z) \\ \partial_y\phi &= l\sin(kx)\cos(ly)\exp(\sqrt{k^2 + l^2}z) \\ \partial_{yy}\phi &= -l^2\sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z) \\ \partial_z\phi &= \sqrt{k^2 + l^2}\sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z) \\ \partial_{zz}\phi &= (k^2 + l^2)\sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z) \end{aligned}$$

so that

$$\nabla\phi = \begin{pmatrix} k\cos(kx)\sin(ly)\exp(\sqrt{k^2+l^2}z) \\ l\sin(kx)\cos(ly)\exp(\sqrt{k^2+l^2}z) \\ \sqrt{k^2+l^2}\sin(kx)\sin(ly)\exp(\sqrt{k^2+l^2}z) \end{pmatrix}, \qquad \Delta\phi = \nabla^2\phi = \partial_{xx}\phi + \partial_{yy}\phi + \partial_{zz}\phi = 0$$

Note that $\Delta\phi = 0$, i.e. ϕ is a solution of Laplace's equation (we call solutions of Laplace's equation *harmonic*).

(e) Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$.

Solution

This surface is a level set of $f = x^2 + y^2 - z$ so we can use the gradient $\nabla f = (2x, 2y, -1)$, which is $(2, 2, -1)$ at $(1, 1, 2)$. To get a unit vector, we divide by the length $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, so we have $(2/3, 2/3, -1/3)$.

2. Practicing suffix notation.

(a) Simplify the suffix expression $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni}$.

Solution

We recall the identity $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. We then have $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mni}$. Using the substitution property, i.e. $\delta_{ij}a_j = a_i$, we get $(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\epsilon_{mni} = \epsilon_{jnl} - \delta_{jl}\epsilon_{mnm} = \epsilon_{jnl}$.

(b) Show that $\nabla \times (f\nabla f) = 0$.

Solution

$$[\nabla \times (f\nabla f)]_i = \epsilon_{ijk}\partial_j\,(f\partial_k f) = \epsilon_{ijk}(\partial_j f\partial_k f + f\partial_{jk} f)$$

Because the order of the derivatives and products on the right doesn't matter, this is equal to

$$[\nabla \times (f\nabla f)]_i = \epsilon_{ikj}(\partial_j f\partial_k f + f\partial_{jk} f) = -\epsilon_{ijk}(\partial_j f\partial_k f + f\partial_{jk} f) = -[\nabla \times (f\nabla f)]_i$$

Because $[\nabla \times (f\nabla f)]_i$ is the negative of itself, it must be 0.

(c) Show that the vector field $\vec{u} = \nabla f \times \nabla g$ is solenoidal (divergence is zero)

Solution

$$\nabla \cdot \vec{u} = \partial_i \epsilon_{ijk} \partial_j f \partial_k g = \epsilon_{ijk} \partial_{ij} f \partial_k g + \epsilon_{ijk} \partial_j f \partial_{ik} g$$

Again, changing the order of i and j doesn't change the derivative of f in the first term and the order of i and k doesn't change the derivative of g in the second, so

$$\nabla \cdot \vec{u} == \epsilon_{ikj} \partial_{ij} f \partial_k g + \epsilon_{kji} \partial_j f \partial_{ik} g = -\epsilon_{ijk} \partial_{ij} f \partial_k g + -\epsilon_{ijk} \partial_j f \partial_{ik} g = -\nabla \cdot \vec{u}$$

Because the divergence is the negative of itself, it must be 0.

(d) Use suffix notation to show that $\nabla \cdot (\vec{u} \times \vec{v}) = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$.

Solution

$$\nabla \cdot (\vec{u} \times \vec{v}) = \partial_i \epsilon_{ijk} u_j v_k = \epsilon_{ijk} \partial_i u_j v_k + \epsilon_{ijk} u_j \partial_i v_k = v_k \epsilon_{kij} \partial_i u_j - u_j \epsilon_{jik} \partial_i v_k = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$$

3. Working with the divergence theorem.

(a) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

Solution

$$\iint_S F \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$

Note that $\nabla \cdot \vec{F} = 2 + 2y + 2z$. We will use spherical coordinates, i.e. $x = \rho \cos(\theta) \sin(\phi)$, $y = \rho \sin(\theta) \sin(\phi)$, and $z = \rho \cos(\phi)$ with $dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$.

$$\begin{aligned}
\iint_S F \cdot \hat{n} \, dS &= \iiint_V \nabla \cdot \vec{F} \, dV \\
&= \int_0^R \int_0^{2\pi} \int_0^{\pi} (2 + 2\rho \sin(\theta) \sin(\phi) + 2\rho \cos(\phi)) \; \rho^2 \sin(\phi) d\rho d\theta d\phi \\
&= 2\pi \int_0^R \rho^2 \, d\rho \int_0^{\pi} \sin\phi \, d\phi + 2 \int_0^R \int_0^{\pi} \rho^3 \sin^2\phi \, d\rho d\phi \int_0^{2\pi} \sin(\theta) \, d\theta \\
&\quad + 2\pi \int_0^R \rho^3 \, d\rho \int_0^{\pi} \sin(2\phi) \, d\phi \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

(b) Use the divergence theorem to evaluate $\iint_S x^2+y+z\,dS$ where S is the sphere $x^2+y^2+z^2 = R^2$.

Solution

This one is tricky. We can figure out the normal by taking the gradient of $f = x^2 + y^2 + z^2 - R^2$ and dividing by the length. This gives $\hat{n} = (x/R, y/R, z/R)$. To use the divergence theorem, we need a vector field $\vec{F}$ such that $\vec{F} \cdot \hat{n} = x^2 + y + z$. Note that $\vec{F} = (Rx, R, R)$ works. The divergence of $\vec{F}$ is simple, i.e. $\nabla \cdot \vec{F} = R$, a constant. Then

$$\iint_S x^2 + y + z\,dS = \iint_S \vec{F} \cdot \hat{n}\,dS = \iiint_V R\,dV = R\iiint_V dV = \frac{4\pi R^4}{3}$$

(c) Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n}\,dS$ where $\vec{F} = (xy^2, x^2y, y)$ and S is the surface of the cylinder $x^2 + y^2 = R^2$ between $z = 1$ and $z = -1$, including the two disc shaped caps where $x^2 + y^2 \leq R^2$ with $z = \pm 1$.

Solution

The divergence of $\vec{F}$ is $\nabla \cdot \vec{F} = y^2 + x^2 + 0$. We will integrate in cylindrical coordinates where $\nabla \cdot \vec{F} = r^2$. Thus

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\,dS &= \iiint_V \nabla \cdot \vec{F}\,dV \\
&= \int_0^R \int_0^{2\pi} \int_{-1}^1 r^2 r\,dr d\theta dz \\
&= \frac{R^4}{4} 4\pi = \pi R^4
\end{aligned}$$

(d) Find the flux of the vector field $\vec{F} = (x - y^2, y, x^3)$ out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.

Solution

We can again use the divergence theorem. The divergence of $\vec{F}$ is $\nabla \cdot \vec{F} = 1 + 1 + 0 = 2$. Then

$$\iint_S \vec{F} \cdot \hat{n}\,dS = \iiint_V \nabla \cdot \vec{F}\,dV = \iiint_V 2\,dV = 6$$

where in the last equality we used that the volume of the rectangular solid is 3.

(e) Suppose $\vec{F}$ is tangent to the closed surface S bounding a region V. Show that $\iiint_V \nabla \cdot \vec{F}\, dV = 0$.

Solution

If $\vec{F}$ is tangent to the surface, then $\vec{F} \cdot \hat{n} = 0$ along the surface. Thus $\iiint_V \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS = 0$.

4. Working with Stokes theorem.

(a) Let S be a surface and let $\vec{F}$ be perpendicular to the tangent to the boundary of S. Show that $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = 0$.

Solution

By Stokes' theorem $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot\, d\vec{r}$. The second integral is zero because $\vec{F}$ is perpendiculaar to the tangent to the boundary of S (which is the direction of $d\vec{r}$).

(b) For a surface S and a fixed vector $\vec{v}$, prove that $2 \iint_S \vec{v} \cdot \hat{n}\, dS = \oint_C (\vec{v} \times \vec{r}) \cdot\, d\vec{r}$, where C is the boundary ("rim") of S.

Solution

Another tricky one.

To apply Stokes' theorem, we need the curl of $\vec{v} \times \vec{r}$ with $\vec{v}$ a constant.

$$\begin{aligned}
[\nabla \times (\vec{v} \times \vec{r})]_i &= \epsilon_{ijk} \partial_j \epsilon_{klm} v_l r_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) v_l \partial_j r_m \\
&= v_i \delta_{jm} \partial_j r_m - v_j \partial_j r_i
\end{aligned}$$

Note that $\partial_i r_j = \delta_{ij}$ for $\vec{r}$ the position vector. Further, note that $\delta_{ij}\delta_{ij} = 3$ by taking the implied sums. Thus,

$$\begin{aligned}
[\nabla \times (\vec{v} \times \vec{r})]_i &= v_i \delta_{jm} \partial_j r_m - v_j \partial_j r_i \\
&= v_i \delta_{jm} \delta_{jm} - v_j \delta_{ji} \\
&= 3v_i - v_i = 2v_i
\end{aligned}$$

Then Stokes' theorem implies

$$\oint_C (\vec{v} \times \vec{r}) \cdot\, d\vec{r} = \iint_S \nabla \times (\vec{v} \times \vec{r}) \cdot \hat{n}\, dS = 2 \iint_S \vec{v} \cdot \hat{n}\, dS$$

(c) Let $\vec{F} = (3y, -xz, -yz^2)$, and let S be the surface $2z = x^2 + y^2$ below the plane $z = 2$ (i.e. consider the parabaloid shape $z = (x^2 + y^2)/2$ between $z = 0$ and $z = 2$). Calculate $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS$ both directly and by using Stokes theorem.

Solution

First, we use Stokes' theorem. This gives that the flux is $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$. The boundary curve of S is the circle $x^2 + y^2 = 4$ with $z = 2$. We can discretize this as $r(t) = (2\cos(t), 2\sin(t), 2)$ so that $d\vec{r} = (-2\sin(t), 2\cos(t), 0)dt$.

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS &= \oint_C \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} (6\sin(t), -4\cos(t), -8\sin(t)) \cdot (-2\sin(t), 2\cos(t), 0)\, dt \\
&= \int_0^{2\pi} -12\sin^2(t) - 8\cos^2(t)\, dt \\
&= -8\int_0^{2\pi} dt - 4\int_0^{2\pi} \sin^2(t)\, dt \\
&= -16\pi - 4\int_0^{2\pi} \sin^2(t)\, dt
\end{aligned}$$

Note that $\int_0^{2\pi} \sin^2(t)\, dt = \int_0^{2\pi} \cos^2(t)\, dt$ so that $2\int_0^{2\pi} \sin^2(t)\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$. Thus

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS &= -16\pi - 4\int_0^{2\pi} \sin^2(t)\, dt \\
&= -16\pi - 4\pi = -20\pi
\end{aligned}$$

Second, we compute the flux directly. We can parameterize the surface over r and t where $x = r\cos t$, $y = r\sin t$, and $z = r^2/2$ with $0 \le r \le 2$ and $0 \le t \le 2\pi$. It is straightforward to compute $\nabla \times \vec{F} = (x - z^2, 0, -3 - z)$. In these coordinates, $\nabla \times \vec{F} = (r\cos t - r^4/4, 0, -3 - r^2/2)$. We will use the formula for the flux over a parameterized surface which requires $\partial_r \vec{p} \times \partial_t \vec{p}$ with $\vec{p} = (r\cos t, r\sin t, r^2/2)$. This is

$$\partial_r \vec{p} \times \partial_t \vec{p} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos t & \sin t & r \\ -r\sin t & r\cos t & 0 \end{vmatrix} = (-r^2\cos t, -r^2\sin t, r).$$

Then, we have

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \int_0^2 \int_0^{2\pi} (r\cos t - r^4/4, 0, -3 - r^2/2) \cdot (-r^2\cos t, -r^2\sin t, r)\, drdt$$

$$
\begin{aligned}
&= \int_0^2 \int_0^{2\pi} \left(-r^3 \cos^2 t + (r^6 \cos t)/4 - 3r - r^3/2\right) drdt \\
&= \int_0^2 -r^3 dr \int_0^{2\pi} \cos^2 t \, dt + \int_0^2 r^6/4dr \int_0^{2\pi} \cos t \, dt + 2\pi \int_0^2 -3r - r^3/2 \, dr \\
&= -\frac{2^4\pi}{4} + \frac{2^7 \cdot 0}{28} - 2\pi \left(3 \cdot 2^2/2 + 2^4/8\right) \\
&= -4\pi - 12\pi - 4\pi = -20\pi
\end{aligned}
$$

Note that in the above, because we are using the formula for any parameterization, the area in the integral is $drdt$ not $rdrdt$. Though, you can see that there is an r in the cross product of $\partial_r \vec{p} \times \partial_t \vec{p}$.

(d) Let $\vec{F} = (yze^x + xyze^x, xze^x, xye^x)$. Show that the circulation of $\vec{F}$ around an oriented simple curve C that is the boundary of a surface S is zero.

Solution

The curl of $\vec{F}$ is zero (on the test, show your work). Thus, by Stokes' theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = 0$$

(e) Find the circulation of $\vec{F} = (x^2, y^2, -z)$ around the triangle with vertices $(0,0,0)$, $(0,2,0)$ and $(0,0,3)$, both directly and by using Stokes theorem.

Solution

As with the previous problem, the curl of $\vec{F}$ is zero (on test, show your work). So, using Stokes' theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = 0$$

To compute the circulation directly, we discretize the three segments that make up the triangle C. Let C_1 be $\vec{r}_1(t) = (0, 2t, 0)$, C_2 be $\vec{r}_2(t) = (0, 2-2t, 3t)$, and C_3 be $\vec{r}_3(t) = (0, 0, 3-3t)$ where t ranges from 0 to 1 over each piece. This gives

$$
\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 4t^2, 0) \cdot (0, 2, 0) \, dt \\
&= 8/3 \\
\int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 4 - 8t + 4t^2, -3t) \cdot (0, -2, 3) \, dt
\end{aligned}
$$

$$
\begin{aligned}
&= -8 + 8 - 8/3 - 9/2 \\
\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_0^1 (0, 0, 3t - 3) \cdot (0, 0, -3)\, dt \\
&= -9/2 + 9
\end{aligned}
$$

Adding all 3 contributions we see we again get 0.

5. Show that

$$\iiint_V (\nabla f) \cdot \vec{F}\, dV = \iint_S f\vec{F} \cdot \hat{n}\, dS - \iiint_V f\nabla \cdot \vec{F}\, dV$$

.

Solution

We can use suffix notation to expand $\nabla \cdot (f\vec{F})$:

$$\nabla \cdot (f\vec{F}) = \partial_i [f\vec{F}]_i = \partial_i f\vec{F}_i + f\partial_i \vec{F}_i = \nabla f \cdot \vec{F} + f\nabla \cdot \vec{F}$$

Thus, by the divergence theorem

$$
\begin{aligned}
\iint_S f\vec{F} \cdot \hat{n}\, dS &= \iiint_V \nabla \cdot (f\vec{F})\, dV \\
&= \iiint_V (\nabla f) \cdot \vec{F}\, dV + \iiint_V f\nabla \cdot \vec{F}\, dV
\end{aligned}
$$

which we can re-arrange to get the desired result.