MATH 335 S2019
Practice Final Exam Solutions

2019-04-29

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances.

Reference

The following results and identities are provided for reference purposes (this is the same list as will be provided in the final). They may or may not be needed to complete the exam.

- The following identity may be used in the exam without the need to prove it:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

- Any rank-4 and isotropic tensor is of the form

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk},$$

for some constants λ, μ, and ν.

Exam

1. Write the "physical" definition of the divergence of a vector field $\vec{F}$ (given as the limit of some integral over a shrinking region) and compute the divergence of $\vec{F} = (x, z - y, z)$.

   Solution

   The "physical" definition is given as the scaled limit of the flux around a point. For a small volume δV, let δS denote its boundary surface. Then

$$\nabla \cdot \vec{F} = \lim_{|\delta V| \to 0} \frac{1}{|\delta V|} \oiint_{\delta S} \vec{F} \cdot \hat{n}\, dS$$

   We can use the cartesian coordinate formula to evaluate the divergence of $\vec{F}$. We have $\nabla \cdot \vec{F} = 1 - 1 + 1 = 1$.

2. Show that $\nabla \cdot (\phi \nabla \phi) = \phi \nabla^2 \phi + \nabla \phi \cdot \nabla \phi$. Then show that

$$\int_V \phi \nabla^2 \phi \, dV = \oiint_S \phi \nabla \phi \cdot \hat{n} \, dS - \iiint_V |\nabla \phi|^2 \, dV \ .$$

Suppose further that $\nabla^2 \phi = 0$ in V and $\phi = 0$ on S. In that case, explain why $\phi = 0$ throughout V.

Solution

We can show this using suffix notation.

$$\nabla \cdot (\phi \nabla \phi) = \partial_{x_i}(\phi \partial_{x_i} \phi) = \partial_{x_i} \phi \partial_{x_i} \phi + \phi \partial_{x_i} \partial) x_i \phi = |\nabla \phi|^2 + \phi \nabla^2 \phi$$

From the divergence theorem and the result above, we have

$$\iiint_V \nabla \cdot (\phi \nabla \phi) \, dV = \oiint_S \phi \nabla \phi \cdot \hat{n} \, dS$$
$$\iint_V |\nabla \phi|^2 + \phi \nabla^2 \phi = \oiint_S \phi \nabla \phi \cdot \hat{n} \, dS$$

so that we get the result after rearranging.

If $\nabla^2 \phi = 0$ in V and $\phi = 0$ on S, then the above simplifies to

$$\iint_V |\nabla \phi|^2 = 0 \ .$$

Because $|\nabla \phi|$ is nonnegative, it must then be zero throughout the domain. Thus, ϕ is constant. Because $\phi = 0$ on S, that constant must be zero.

3. Is $\vec{F} = (z, z, x + y)$ conservative? Show why it is or isn't.

Solution

Here are two approaches:

(a) We show that $\vec{F} = \nabla \phi$ so that $\vec{F}$ is conservative. We can anti-differentiate each of the three equations implied by $\nabla \phi = (z, z, x + y)$ to get $\phi = xz + h_1(y, z)$, $\phi = yz + h_2(x, z)$ and $\phi = z(x + y) + h_3(x, y)$ for some unspecified functions h_i. Note that $\phi = z(x + y)$ satisfies all of these and $\vec{F} = \nabla \phi$.

(b) Because $\vec{F}$ has no singularities (and neither does its curl) we can show that $\vec{F}$ is conservative by showing that its curl is zero. We have

$$\nabla \times \vec{F} = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ z & z & x + y \end{vmatrix} = (1 - 1, -(1 - 1), 0 - 0) = (0, 0, 0)$$

4. Show that $\nabla \times (f\nabla f) = 0$.

Solution

We can show this using suffix notation. We have

$$
\begin{aligned}
[\nabla \times (f\nabla f)]_i &= \epsilon_{ijk}\partial_{x_j}(f\partial_{x_k} f) \\
&= \epsilon_{ijk}\left(\partial_{x_j} f\partial_{x_k} f + f\partial_{x_j}\partial_{x_k} f\right) \\
&= \epsilon_{ikj}\left(\partial_{x_k} f\partial_{x_j} f + f\partial_{x_k}\partial_{x_j} f\right) \quad \text{rename} \\
&= -\epsilon_{ijk}\left(\partial_{x_k} f\partial_{x_j} f + f\partial_{x_k}\partial_{x_j} f\right) \quad \text{use identity } \epsilon_{ikj} = -\epsilon_{ijk} \\
&= -\epsilon_{ijk}\left(\partial_{x_j} f\partial_{x_k} f + f\partial_{x_j}\partial_{x_k} f\right) \quad \text{order of diff. doesn't matter}
\end{aligned}
$$

Note that in the last line we have the negative of the first line so that the ith entry is the negative of itself. Thus, it must be zero.

5. Let $\vec{F} = (3y, -xz, -yz^2)$, and let S be the surface $2z = x^2 + y^2$ below the plane $z = 2$ (i.e. consider the parabaloid shape $z = (x^2 + y^2)/2$ between $z = 0$ and $z = 2$). Calculate $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS$ both directly and by using Stokes theorem.

Solution

First, we use Stokes' theorem. This gives that the flux is $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$. The boundary curve of S is the circle $x^2 + y^2 = 4$ with $z = 2$. We can discretize this as $r(t) = (2\cos(t), 2\sin(t), 2)$ so that $d\vec{r} = (-2\sin(t), 2\cos(t), 0)dt$.

$$
\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS &= \oint_C \vec{F} \cdot d\vec{r} \\
&= \int_0^{2\pi} (6\sin(t), -4\cos(t), -8\sin(t)) \cdot (-2\sin(t), 2\cos(t), 0)\, dt \\
&= \int_0^{2\pi} -12\sin^2(t) - 8\cos^2(t)\, dt \\
&= -8\int_0^{2\pi} dt - 4\int_0^{2\pi} \sin^2(t)\, dt \\
&= -16\pi - 4\int_0^{2\pi} \sin^2(t)\, dt
\end{aligned}
$$

Note that $\int_0^{2\pi} \sin^2(t)\, dt = \int_0^{2\pi} \cos^2(t)\, dt$ so that $2\int_0^{2\pi} \sin^2(t)\, dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = 2\pi$. Thus

$$
\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS &= -16\pi - 4\int_0^{2\pi} \sin^2(t)\, dt \\
&= -16\pi - 4\pi = -20\pi
\end{aligned}
$$

Second, we compute the flux directly. We can parameterize the surface over r and t where $x = r\cos t$, $y = r\sin t$, and $z = r^2/2$ with $0 \le r \le 2$ and $0 \le t \le 2\pi$. It is straightforward to compute $\nabla\times\vec{F}=(x-z^2,0,-3-z)$. In these coordinates, $\nabla\times\vec{F}=(r\cos t-r^4/4,0,-3-r^2/2)$. We will use the formula for the flux over a parameterized surface which requires $\partial_r\vec{p}\times\partial_t\vec{p}$ with $\vec{p}=(r\cos t, r\sin t, r^2/2)$. This is

$$\partial_r\vec{p}\times\partial_t\vec{p}=\begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \cos t & \sin t & r \\ -r\sin t & r\cos t & 0 \end{vmatrix}=(-r^2\cos t,-r^2\sin t,r).$$

Then, we have

$$\begin{aligned}
\iint_S \nabla\times\vec{F}\cdot\hat{n}\,dS &= \int_0^2\int_0^{2\pi}(r\cos t-r^4/4,0,-3-r^2/2)\cdot(-r^2\cos t,-r^2\sin t,r)\,drdt \\
&= \int_0^2\int_0^{2\pi}\left(-r^3\cos^2 t+(r^6\cos t)/4-3r-r^3/2\right)drdt \\
&= \int_0^2 -r^3dr\int_0^{2\pi}\cos^2 t\,dt+\int_0^2 r^6/4dr\int_0^{2\pi}\cos t\,dt+2\pi\int_0^2 -3r-r^3/2\,dr \\
&= -\frac{2^4\pi}{4}+\frac{2^7\cdot 0}{28}-2\pi\left(3\cdot 2^2/2+2^4/8\right) \\
&= -4\pi-12\pi-4\pi=-20\pi
\end{aligned}$$

Note that in the above, because we are using the formula for any parameterization, the area in the integral is $drdt$ not $rdrdt$. Though, you can see that there is an r in the cross product of $\partial_r\vec{p}\times\partial_t\vec{p}$.

6. Conservation of mass for a fluid can be expressed in integral form as

$$\frac{d}{dt}\iiint_V \rho\,dV=-\oiint_S \rho\vec{u}\cdot\hat{n}\,dS\ ,$$

which holds for any volume V. Turn this into a differential equation in the usual way. Write the simplified expression you get when ρ is constant in space and time.

Solution

We can change the right hand side to a volume integral using the divergence theorem. This gives

$$-\oiint_S \rho\vec{u}\cdot\hat{n}\,dS=-\iiint_V \nabla\cdot(\rho\vec{u})\,dV$$

Then, rearranging, we get

$$\iiint_V \partial_t\rho+\nabla\cdot(\rho\vec{u})\,dV=0$$

for any V. Thus, the integrand itself must be zero, giving the differential equation $\partial_t\rho+\nabla\cdot(\rho\vec{u})=0$.

When ρ is a constant, this simplifies to $\nabla\cdot\vec{u}=0$ (known as the incompressibility or divergence-free condition).

7. A set of “conical coordinates” are defined by

$$
\begin{aligned}
x_1 &= \frac{rvw}{2} \\
x_2 &= \frac{r}{2}\sqrt{\frac{(4-v^2)(4-w^2)}{3}} \\
x_3 &= r\sqrt{\frac{(v^2-1)(1-w^2)}{3}}
\end{aligned}
$$

for $0 \le w^2 \le 1 \le v^2 \le 4$.

- Show that $x_1^2 + x_2^2 + x_3^2 = r^2$. (This means that constant r gives a sphere).
- Find the scale factors and unit vectors of this coordinate system. Show that the system is orthogonal.
- What is the area element for a surface with constant r?

Solution

- We can show this by direct computation:

$$
\begin{aligned}
x_1^2 + x_2^2 + x_3^2 &= \frac{r^2v^2w^2}{4} + \frac{r^2}{4}\frac{(4-v^2)(4-w^2)}{3} + r^2\frac{(v^2-1)(1-w^2)}{3} \\
&= \frac{r^2v^2w^2}{4} + \frac{r^2}{12}\left((4-v^2)(4-w^2) + 4(v^2-1)(1-w^2)\right) \\
&= \frac{r^2v^2w^2}{4} + \frac{r^2}{12}\left(16 - 4v^2 - 4w^2 + v^2w^2 + 4v^2 - 4v^2w^2 - 4 + 4w^2\right) \\
&= \frac{r^2v^2w^2}{4} + \frac{r^2}{12}\left(12 - 3v^2w^2\right) \\
&= r^2
\end{aligned}
$$

- We have

$$
\partial_r\vec{x} = \begin{pmatrix} vw/2 \\ \sqrt{\frac{(4-v^2)(4-w^2)}{3}}/2 \\ \sqrt{\frac{(v^2-1)(1-w^2)}{3}} \end{pmatrix} \qquad \partial_v\vec{x} = r\begin{pmatrix} w/2 \\ -v\sqrt{(4-w)^2/(3(4-v^2))}/2 \\ v\sqrt{(1-w^2)/(3(v^2-1))} \end{pmatrix}
$$

$$
\partial_w\vec{x} = r\begin{pmatrix} v/2 \\ -w\sqrt{(4-v)^2/(3(4-w^2))}/2 \\ -w\sqrt{(v^2-1)/(3(1-w^2))} \end{pmatrix}
$$

After some lengthy calculations, we get

$$
\begin{aligned}
h_r &= 1 \\
h_v &= r\sqrt{\frac{v^2-w^2}{(4-v^2)(v^2-1)}}
\end{aligned}
$$

$$h_w = r\sqrt{\frac{v^2 - w^2}{(4-w^2)(1-w^2)}}$$

The unit vectors can then be determined from

$$\vec{e}_r = \partial_r \vec{x}\,, \quad \vec{e}_v = \frac{1}{h_v}\partial_v \vec{x}\,, \quad \vec{e}_w \frac{1}{h_w}\partial_w \vec{x}$$

Then, we can check for orthogonality by taking inner products. For simplicity, I look at the dot products of $\partial_r \vec{x}$, $(\partial_v \vec{x})/r$, and $(\partial_w \vec{x})/r$ with each other. We have

$$\begin{aligned}
\partial_r \vec{x} \cdot (\partial_v \vec{x})/r &= \frac{vw^2}{4} - \frac{v(4-w^2)}{12} + \frac{v(1-w^2)}{3} = 0 \\
\partial_r \vec{x} \cdot (\partial_w \vec{x})/r &= \frac{v^2 w}{4} - \frac{w(4-v^2)}{12} - \frac{w(v^2-1)}{3} = 0 \\
(\partial_v \vec{x})/r \cdot (\partial_w \vec{x})/r &= \frac{vw}{4} + \frac{vw}{12} - \frac{vw}{3} = 0
\end{aligned}$$

- The area element on a constant r surface is given by

$$dS = h_v h_w \, dv \, dw = \frac{r^2(v^2 - w^2)}{\sqrt{(4-v^2)(v^2-1)(4-w^2)(1-w^2)}} \, dv \, dw$$

8. Stokes flow describes incompressible, creeping flow (high viscosity flow). The equations are

$$\mu \nabla^2 \vec{u} - \nabla p = 0 \quad \text{in } V \tag{1}$$
$$\nabla \cdot \vec{u} = 0 \quad \text{in } V \tag{2}$$

where V is some region, S is its boundary, $\vec{u}$ is the velocity of the fluid, and p is the pressure (we have left out the boundary condition).

There is an equivalent of a charge (as in electrical charge) for the Stokes equations. The "charge" strength is a vector, which we'll denote by $\vec{h}$. The field $\vec{u}$ induced by this charge is defined to be $u_i = S_{ij} h_j$, where

$$S_{ij} = \frac{1}{8\pi\mu}\left(\delta_{ij}\frac{1}{r} + r_i r_j \frac{1}{r^3}\right),$$

$\vec{r} = (x, y, z)$, and $r = |\vec{r}|$. The pressure associated with $\vec{u}$ is given by $p = P_i h_i$ where

$$P_i = r_i \frac{1}{4\pi r^3}\,.$$

(a) Show that setting $u_i = S_{ij} h_j$ and $p = P_i h_i$ gives a solution of (1) (away from the origin).

(b) Recall that for an orthogonal transformation L_{ij} we have $L_{ij} L_{kj} = \delta_{ik}$. For any vector v_i, let $v_i' = L_{ij} v_j$, Show that $|\vec{v}\,'| = |\vec{v}|$.

(c) Confirm that P_i is a vector and S_{ij} is a tensor.

Solution

(a) To check that $u_i = S_{ij}h_j$ and $p = P_j h_j$ gives a solution, we first substitute them into (1). This gives

$$\begin{aligned}
\mu\partial_{x_k}\partial_{x_k}(S_{ij}h_j) - \partial_{x_i}(P_j h_j) &= h_j\left(\mu\partial_{x_k}\partial_{x_k}S_{ij} - \partial_{x_i}P_j\right) \\
&= \frac{h_j}{8\pi}\left(\partial_{x_k}\partial_{x_k}(\delta_{ij}/r + r_i r_j/r^3) - 2\partial_{x_i}(r_j/r^3)\right) \qquad (3)
\end{aligned}$$

To simplify, we look at some of these terms individually. We have

$$\begin{aligned}
\partial_{x_k}\partial_{x_k}\frac{1}{r} &= -\partial_{x_k}\frac{r_k}{r^3} \\
&= -\frac{3}{r^3} - r_k\left(-\frac{3r_k}{r^5}\right) \\
&= -\frac{3}{r^3} + \frac{3r_k r_k}{r^5} = 0 \\
\partial_{x_k}\partial_{x_k}\left(\frac{r_i r_j}{r^3}\right) &= \partial_{x_k}\left(\frac{\delta_{ik}r_j}{r^3} + \frac{\delta_{jk}r_i}{r^3} - 3\frac{r_i r_j r_k}{r^5}\right) \\
&= \partial_{x_i}\left(\frac{r_j}{r^3}\right) + \partial_{x_j}\left(\frac{r_i}{r^3}\right) - 3\left(\frac{\delta_{ik}r_j r_k}{r^5} + \frac{\delta_{jk}r_i r_k}{r^5} + 3\frac{r_i r_j}{r^5} - 5\frac{r_i r_j r_k r_k}{r^7}\right) \\
&= \partial_{x_i}\left(\frac{r_j}{r^3}\right) + \partial_{x_j}\left(\frac{r_i}{r^3}\right) \\
\partial_{x_i}\left(\frac{r_j}{r^3}\right) &= \frac{\delta_{ij}}{r^3} - 3\frac{r_i r_j}{r^5} \quad \text{(note that this is symmetric in } i \text{ and } j\text{)}
\end{aligned}$$

Plugging these formulas into (3) shows that the pair $(\vec{u}, p)$ satisfy the first equation (1). We must also check that $\vec{u}$ is divergence free. We have

$$\begin{aligned}
\partial_{x_i}u_i &= \frac{h_j}{8\pi\mu}\partial_{x_i}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) \\
&= \frac{h_j}{8\pi\mu}\left(-\frac{r_j}{r^3} + 3\frac{r_j}{r^3} + \frac{\delta_{ij}r_i}{r^3} - 3\frac{r_i r_j r_i}{r^5}\right) \qquad = \frac{h_j}{8\pi\mu}0 = 0
\end{aligned}$$

as desired. Note that this is true for any h_j.

(b) This follows from the definitions. For convenience, we check that $|v'|^2 = |v|^2$ which is equivalent (by taking square roots). We have

$$\begin{aligned}
|v'|^2 &= v'_i v'_i \\
&= L_{ij}v_j L_{ik}v_k \\
&= v_j \delta_{jk} v_k \\
&= v_j v_j \\
&= |v|^2
\end{aligned}$$

(c) The previous problem makes it easier to check that these are indeed tensors. For S_{ij} to be a tensor, we need that $S'_{ij} = L_{ik}L_{jl}S_{kl}$ and that $P'_i = L_{ij}P_j$ for any rotation L_{ij}. We have

$$
\begin{aligned}
8\pi\mu S'_{ij} &= \frac{\delta'_{ij}}{r'} + \frac{r'_i r'_j}{r'^3} \\
&= \frac{\delta_{ij}}{r} + \frac{r'_i r'_j}{r^3} \quad \text{using previous problem} \\
&= \frac{L_{ik}L_{jk}}{r} + \frac{L_{ik}r_k L_{jl} r_l}{r^3} \quad \text{using def. and proprerties of change of coordinates} \\
&= L_{ik}L_{jl}\frac{\delta_{kl}}{r} + L_{ik}L_{jl}\frac{r_k r_l}{r^3} \\
&= L_{ik}L_{jl} 8\pi\mu S_{kl} \\
4\pi P'_i &= \frac{r'_i}{r'^3} \\
&= \frac{L_{ij} r_j}{r^3} \quad \text{using def. of change of coordinates and previous problem} \\
&= L_{ij} 4\pi P_j
\end{aligned}
$$