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MATH 335-002: Homework #4 Solutions

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Instructions

- This assignment is due in-class Thursday March 14th.
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

Outline

In this assignment, we practice using suffix notation in vector identities and working with the divergence and Stokes theorems.

1 Exercises from Matthews

Please complete exercises 4.9, 4.12, 4.16, 5.1, 5.3, 5.5, 5.8, and 5.12 from the textbook (2 pts each). They are provided below for convenience.

- 4.9: Write in suffix notation the vector equation $\vec{a} \times \vec{b} + \vec{c} = (\vec{a} \cdot \vec{b})\vec{b} - \vec{d}$.
- 4.12:
  (a) Show that $\nabla \times (f \nabla f) = 0$
  (b) Evaluate $\nabla \cdot (f \nabla f)$
- 4.16: The vector fields $\vec{u}$ and $\vec{w}$ and the scalar field ϕ are related by the equation
$$\vec{u} + \nabla \times \vec{w} = \nabla \phi + \nabla^2 \vec{u}$$
- 5.1: Use the divergence theorem to evaluate the surface integral
$$\oiint_S \vec{u} \cdot \hat{n} \, dS$$
where $\vec{u} = (x \sin y, \cos^2 x, y^2 - z \sin y)$ and S is the surface of the sphere $x^2 + y^2 + (z-2)^2 = 1$.

Name:

- 5.3: An incompressible fluid is contained within a volume V with surface S and $\vec{u}\cdot\hat{n} = 0$ on S. Using the divergence theorem, show that

$$\iiint_V \vec{u}\cdot\nabla\phi\, dV = 0$$

  for any differentiable scalar field ϕ.

- 5.5: Use the divergence theorem to evaluate the surface integral

$$\iint_S \vec{v}\cdot\hat{n}\, dS$$

  where $\vec{v} = (x+y, z^2, x^2)$ and S is the surface of the hemisphere $x^2+y^2+z^2 = 1$ with $z > 0$ and $\hat{n}$ is the upward pointing normal. Note that S is not closed.

- 5.8: Show that $\oint_C \vec{r}\cdot d\vec{r} = 0$ for any closed curve C.

- 5.12: The magnetic field $\vec{B}$ in an electrically conducting fluid moving with velocity $\vec{u}$ obeys the magnetic induction equation

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}).$$

  Show that the total flux of magnetic field through a surface enclosed by a streamline of the flow (a closed curve which is everywhere parallel to $\vec{u}$ is independent of time.

Solution

The solutions to these exercises can be found in the back of the book.

2 Other Exercises

Question 1 (5 pts) Some properties of viscous fluid equations

A highly viscous and incompressible fluid is described by the Stokes equations. In a volume V with boundary surface S, they are

$$-\nu\nabla^2\vec{u} + \nabla p = \vec{F} \text{ in } V\ , \tag{1}$$
$$\nabla\cdot\vec{u} = 0 \text{ in } V\ , \tag{2}$$
$$\vec{u} = \vec{f} \text{ on } S\ , \tag{3}$$

where $\vec{u}$ denotes the velocity field, p denotes the pressure, $\vec{F}$ denotes forces which act on the body of the fluid (like gravity), $\vec{f}$ is the Dirichlet boundary condition (specifies what the velocity of the fluid is on the boundary), and ν denotes the viscosity.

Looking at these equations, you can see that vector calculus is essential to understanding what's going on!

- (2 pts) Show that $\iint_S \vec{f}\cdot\hat{n}\, dS = 0$. (This is known as a compatibility condition on $\vec{f}$.)
- (3 pts) Suppose that $\vec{F} = \nabla\phi + \nabla\times\vec{\psi}$ for some scalar field ϕ and some vector field $\vec{\psi}$. Show that if $\phi = 0$, then p is harmonic (i.e. $\nabla^2 p = 0$).

Name:

Solution

- Note that because $\vec{f} = \vec{u}$ on the boundary, we have that $\iint_S \vec{f} \cdot \hat{n}\, dS = \iint_S \vec{u} \cdot \hat{n}\, dS$. Applying the divergence theorem, we get that $\iint_S \vec{f} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{u}\, dV = 0$, where the last equality follows because $\nabla \cdot \vec{u} = 0$ inside the domain.

- Plugging this expression for $\vec{F}$ into (1), we get

$$-\nu \nabla^2 \vec{u} + \nabla p = \nabla \times \vec{\psi}$$

Taking the divergence of both sides, we get that

$$\nabla^2 p = 0$$

because $\nabla \cdot \vec{u} = 0$ and $\nabla \cdot \nabla \times \vec{\psi} = 0$ for any $\vec{\psi}$.

Question 2 (5 pts)

Evaluate the surface integral

$$\iint_S \vec{F} \cdot \hat{n}\, dS$$

where $\vec{F} = (1, 1, z(x^2 + y^2)^2)$ and S is the surface of the cylinder $x^2 + y^2 \leq 1$ with $-1 \leq z \leq 1$ (this includes the tubular outer part and disc-shaped caps) using the divergence theorem.

Solution

If we apply the divergence theorem, we get the integral $\iiint_V \nabla \cdot \vec{F}\, dV$, where V is the cylinder. From the definition of the divergence, we get that $\nabla \cdot \vec{F} = (x^2 + y^2)^2$. Integrating in cylindrical coordinates, we obtain

$$\begin{aligned}
\iint_S \vec{F} \cdot \hat{n}\, dS &= \iiint_V \nabla \cdot \vec{F}\, dV \\
&= \int_{-1}^{1} \int_0^{2\pi} \int_0^1 r^4 r\, dr\, d\theta\, dz \\
&= \frac{1}{6} \int_{-1}^{1} \int_0^{2\pi} d\theta\, dz \\
&= \frac{2\pi}{3}
\end{aligned}$$