

MATH 335 S2019

Practice Problems for Midterm Exam II

Exam II covers chapters 3-5 (∇ operators, suffix notation, and integral theorems). Here are a number of practice problems for the exam. The actual exam will be about the same length as the first exam.

1. Practice with the ∇ operator

- (a) What is the geometrical interpretation of ∇f ?
- (b) What are the definitions of the divergence and curl?
- (c) Show that $\vec{F} = (2x + y, x, 2z)$ is conservative.
- (d) Find the gradient and Laplacian of $\phi = \sin(kx) \sin(l y) \exp(\sqrt{k^2 + l^2} z)$
- (e) Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$.

2. Practicing suffix notation.

- (a) Simplify the suffix expression $\epsilon_{ijk} \epsilon_{klm} \epsilon_{mni}$.
- (b) Show that $\nabla \times (f \nabla f) = 0$.
- (c) Show that the vector field $\vec{u} = \nabla f \times \nabla g$ is solenoidal (divergence is zero)
- (d) Use suffix notation to show that $\nabla \cdot (\vec{u} \times \vec{v}) = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$.

3. Working with the divergence theorem.

- (a) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

- (b) Use the divergence theorem to evaluate $\iint_S x^2 + y^2 + z^2 dS$ where S is the sphere $x^2 + y^2 + z^2 = R^2$.
- (c) Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n} dS$ where $\vec{F} = (xy^2, x^2y, y)$ and S is the surface of the cylinder $x^2 + y^2 = R^2$ between $z = 1$ and $z = -1$, including the two disc shaped caps where $x^2 + y^2 \leq R^2$ with $z = \pm 1$.

- (d) Find the flux of the vector field $\vec{F} = (x - y^2, y, x^3)$ out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.
- (e) Suppose \vec{F} is tangent to the closed surface S bounding a region V . Show that $\iiint_V \nabla \cdot \vec{F} dV = 0$.

4. Working with Stokes theorem.

- (a) Let S be a surface and let \vec{F} be perpendicular to the tangent to the boundary at S . Show that $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS = 0$.
- (b) For a surface S and a fixed vector \vec{v} , prove that $2 \iint_S \vec{v} \cdot \hat{n} dS = \oint_C (\vec{v} \times \vec{r}) \cdot d\vec{r}$, where C is the boundary (“rim”) of S .
- (c) Let $\vec{F} = (3y, -xz, -yz^2)$, and let S be the surface $2z = x^2 + y^2$ below the plane $z = 2$ (i.e. consider the paraboloid shape $z = (x^2 + y^2)/2$ between $z = 0$ and $z = 2$). Calculate $\iint_S \nabla \times \vec{F} \cdot \hat{n} dS$ both directly and by using Stokes theorem.
- (d) Let $\vec{F} = (yze^x + x y z e^x, x z e^x, x y e^x)$. Show that the circulation of \vec{F} around an oriented simple curve C that is the boundary of a surface S is zero.
- (e) Find the circulation of $\vec{F} = (x^2, y^2, -z)$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$, both directly and by using Stokes theorem.

5. Show that

$$\iiint_V (\nabla f) \cdot \vec{F} dV = \iint_S f \vec{F} \cdot \hat{n} dS - \iiint_V f \nabla \cdot \vec{F} dV$$