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MATH 335-002: Homework #6

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Instructions

- This assignment is due in-class Thursday May 2nd (this is the last homework).
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

Outline

In this assignment, we practice working with tensors and outline some applications of vector calculus.

1 Exercises from Matthews

Please complete exercises 7.2, 7.3, 7.14, 7.17, 8.4, 8.5, 8.10, and 8.11 from Matthews (2 pts each). They are provided below for convenience.

- 7.2: If $\vec{u}$ is a vector field, show that $\nabla \cdot \vec{u}$ is a scalar field.
- 7.3: Given that $\vec{a}$ and $\vec{b}$ are vectors, show that $a_i b_j$ is a second-rank tensor.
- 7.14: A quantity A_{ij} is related to a vector $\vec{B}$ by $A_{ij} = \epsilon_{ijk} B_k$.
  (a) Show that A_{ij} is a second-rank tensor and describe its symmetry property.
  (b) Find an equation for $\vec{B}$ in terms of A_{ij}.
- 7.17: Show that the kinetic energy E of a body rotating with angular velocity $\vec{\Omega}$ is related to its inertia tensor I_{jk} by $E = I_{jk}\Omega_j\Omega_k/2$.
- 8.4: Show from Maxwell's equations in a vacuum that the magnetic field $\vec{B}$ obeys the wave equation. (Hint: you can mimic the argument for $\vec{E}$).
- 8.5: For the electromagnetic wave in which the $\vec{E}$ field is given by $\vec{E} = \vec{E}_0 f(\vec{k}\cdot\vec{x} - \omega t)$, calculate the corresponding magnetic field $\vec{B}$. What can be deduced about the directions of the vectors $\vec{B}$ and $\vec{E}$? (are they parallel? perpendicular? something else?) (Hint: use eq (8.11) in the book)
- 8.10: An isotropic elastic solid with Lamé constants λ and μ is subjected to a deformation $v_1 = ax_1x_2$, $v_2 = b(x_1^2 - x_2^2)$, and $v_3 = 0$.

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(a) Find the strain tensor E_{ij}.

(b) Find the stress tensor P_{ij}.

(c) Determine the values of a and b for which it is possible for the material to be in equilibrium.

- 8.11: A compressible fluid with negligible viscosity ($\mu = 0$) is initially at rest with uniform density ρ_0 and pressure p_0, with no body forces. A small perturbation is then introduced so that there is a velocity $\vec{u}(\vec{r}, t)$ and the density becomes $\rho_0 + \rho_1(\vec{r}, t)$.

  (a) Assuming that the products of the small quantities $\vec{u}$ and ρ_1 can be neglected, show that the equation for conservation of mass (5.9) becomes

  $$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \, .$$

  (b) Assuming that the perturbation p_1 to the pressure is related to ρ_1 by $p_1 = a\rho_1$ where a is a constant, show that the Navier-Stokes equation reduces to

  $$\rho_0 \frac{\partial \vec{u}}{\partial t} = -a \nabla \rho_1 \, .$$

  (c) Combine the previous two equations to show that the perturbation ρ_1 obeys the wave equation and interpret this result physically.

Solution

The solutions to these exercises can be found in the textbook.

2 Other Exercises

Question 1

(5 pts) Let A_{ij} and B_{ij} be tensors. Show that $C_{ij} = A_{ik}B_{kj}$ is also a tensor.

Solution

Let L_{ij} be a linear transformation. Because A_{ij} and B_{ij} are tensors, we have $C'_{ij} = A'_{ik}B'_{kj} = L_{il}L_{km}A_{lm}L_{kp}L_{jq}B_{pq}$. Observe that $L_{km}L_{kp} = \delta_{mp}$ so that $C'_{ij} = L_{il}L_{jq}A_{lm}B_{mq}$. But this is the same as $C'_{ij} = L_{il}L_{jq}C_{lq}$ by the definition of C_{ij}. Therefore, C_{ij} is a tensor.

Question 2

(5 pts) Let $u(x, t)$ be a solution of the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u \, .$$

Show that

- $u(x, t - \tau)$, for some constant τ, is also a solution of the heat equation.
- $u(x, t) + l(x)$, where $l(x) = ax + b$, is also a solution of the heat equation.

These two points highlight why we need both initial and boundary conditions for such problems.

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Solution

- We have
$$\frac{\partial u(x,t-\tau)}{\partial t} - \nabla^2(u(x,t-\tau)) = \frac{\partial u}{\partial t}|_{(x,t-\tau)} - \nabla^2 u|_{(x,t-\tau)} = 0$$
because u is itself a solution of the heat equation.

- Note that $\frac{\partial l(x)}{\partial t} = 0$ and $\nabla^2(l(x)) = 0$, so that the heat equation holds for $u(x,t) + l(x)$.

(Notes, not required for solution): Both of the results above show that, without further specification, the heat equation can have many solutions. By specifying the value of u at some specific time, we eliminate the first kind of non-uniqueness (this is called specifying an initial condition). Usually, this takes the form $u(x,t_0) = f(x)$ in 1D. If we are solving the equation on an interval $[a,b]$, we can specify the values of u at the endpoints to eliminate the second kind of non-uniqueness above. Note that specifying the value at one end is not enough, because there are infinitely many lines through one point. By specifying the value at both ends, we eliminate ambiguity. Usually, this takes the form $u(a,t) = g_a(t)$ and $u(b,t) = g_b(t)$. Alternatively, some combination of the x derivative and value of u at either end can be specified. What happens when you specify the derivative at both ends?