MATH 335 S2019
Practice Problems for Midterm Exam II

Exam II covers chapters 3-5 (∇ operators, suffix notation, and integral theorems). Here are a number of practice problems for the exam. The actual exam will be about the same length as the first exam.

1. Practice with the ∇ operator
   (a) What is the geometrical interpretation of ∇f?
   (b) What are the definitions of the divergence and curl?
   (c) Show that $\vec{F} = (2x + y, x, 2z)$ is conservative.
   (d) Find the gradient and Laplacian of $\phi = \sin(kx)\sin(ly)\exp(\sqrt{k^2 + l^2}z)$
   (e) Find the unit normal to the surface $x^2 + y^2 - z = 0$ at the point $(1, 1, 2)$.
2. Practicing suffix notation.
   (a) Simplify the suffix expression $\epsilon_{ijk}\epsilon_{klm}\epsilon_{mni}$.
   (b) Show that $\nabla \times (f\nabla f) = 0$.
   (c) Show that the vector field $\vec{u} = \nabla f \times \nabla g$ is solenoidal (divergence is zero)
   (d) Use suffix notation to show that $\nabla \cdot (\vec{u} \times \vec{v}) = \nabla \times \vec{u} \cdot \vec{v} - \nabla \times \vec{v} \cdot \vec{u}$.
3. Working with the divergence theorem.
   (a) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$\iint_S F \cdot \hat{n}\, dS$$

   (b) Use the divergence theorem to evaluate $\iint_S x^2+y+z\, dS$ where S is the sphere $x^2+y^2+z^2 = R^2$.
   (c) Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n}\, dS$ where $\vec{F} = (xy^2, x^2y, y)$ and S is the surface of the cylinder $x^2 + y^2 = R^2$ between $z = 1$ and $z = -1$, including the two disc shaped caps where $x^2 + y^2 \le R^2$ with $z = \pm 1$.

(d) Find the flux of the vector field $\vec{F} = (x - y^2, y, x^3)$ out of the rectangular solid $[0, 1] \times [1, 2] \times [1, 4]$.

(e) Suppose $\vec{F}$ is tangent to the closed surface S bounding a region V. Show that $\iiint_V \nabla \cdot \vec{F} \, dV = 0$.

4. Working with Stokes theorem.

(a) Let S be a surface and let $\vec{F}$ be perpendicular to the tangent to the boundary at S. Show that $\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS = 0$.

(b) For a surface S and a fixed vector $\vec{v}$, prove that $2 \iint_S \vec{v} \cdot \hat{n}, dS = \oint_C (\vec{v} \times \vec{r}) \cdot \hat{n} \, dS$, where C is the boundary ("rim") of S.

(c) Let $\vec{F} = (3y, -xz, -yz^2)$, and let S be the surface $2z = x^2 + y^2$ below the plane $z = 2$ (i.e. consider the parabaloid shape $z = (x^2 + y^2)/2$ between $z = 0$ and $z = 2$). Calculate $\iint_S \nabla \times \vec{F} \cdot \hat{n} \, dS$ both directly and by using Stokes theorem.

(d) Let $\vec{F} = (yze^x + xyze^x, xze^x, xye^x)$. Show that the circulation of $\vec{F}$ around an oriented simple curve C that is the boundary of a surface S is zero.

(e) Find the circulation of $\vec{F} = (x^2, y^2, -z)$ around the triangle with vertices $(0, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$, both directly and by using Stokes theorem.

5. Show that

$$\iiint_V (\nabla f) \cdot \vec{F} \, dV = \iint_S f\vec{F} \cdot \hat{n} \, dS - \iiint_V f \nabla \cdot \vec{F} \, dV$$

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