MATH 335 S2019
Midterm Exam I Solutions

2019-02-19

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances.

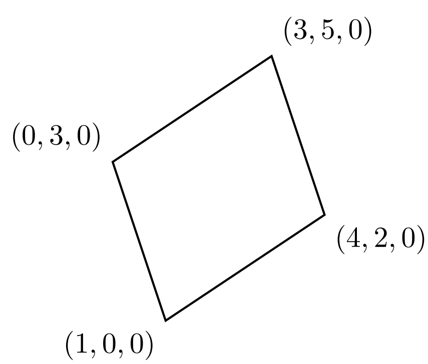
1. Complete the following

   (a) (10 pts) Find the implicit equation for the plane which includes the points $(1,0,1)$, $(0,1,1)$, and $(-1,1,0)$.

   Solution

   By taking the difference of these points, we can obtain 2 linearly independent vectors which are parallel to the plane. Let $\vec{u} = (1,-1,0)$ and $\vec{v} = (-1,0,-1)$. Then we can get a normal vector using the cross product $\vec{n} = \vec{u} \times \vec{v} = (1,1,-1)$. In the usual way, we can get the equation for the plane by $(\vec{r} - (1,0,1)) \cdot (1,1,-1) = 0$. Simplifying, we get $r_1 + r_2 - r_3 = 0$.

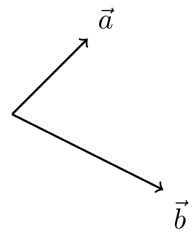
   (b) (10 pts) Find the area of the parallelogram in the figure below (drawn in the (x, y) plane) using your preferred method



Solution

The area of the parallelogram can be computed using the cross product. Let $\vec{u} = (4,2,0) - (1,0,0) = (3,2,0)$ and $\vec{v} = (0,3,0) - (1,0,0) = (-1,3,0)$. Then $|\vec{u} \times \vec{v}| = |(0,0,11)| = 11$.

(c) (10 pts) Does $\vec{a} \times \vec{b}$ point into the paper or out of the paper?



Solution

From the right hand rule, we see that a screw turning from $\vec{a}$ to $\vec{b}$ would be tightening (righty-tighty), so going into the page.

(d) (10 pts) Find the component of $(1,2,3)$ in the direction $(-3,0,4)$.

Solution

The component is given by $(\vec{a} \cdot \vec{b})\vec{b}/|\vec{b}|^2$, which gives

$$\frac{(1,2,3) \cdot (-3,0,4)}{|(-3,0,4)|^2} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} = \frac{9}{25} \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix} \tag{1}$$

2. (20 pts) Consider the vector field $\vec{F}(x, y, z) = (-y, x, z)$. Compute the integral

$$\oint_C \vec{F} \cdot d\vec{r} \,, \tag{2}$$

where C is the unit circle $x^2 + y^2 = 1$ with $z = 0$ traced out in the counter-clockwise direction in the (x, y) plane. Is it possible that $\vec{F}$ is a conservative vector field?

Solution

We parameterize the circle in θ as $\vec{r}(\theta) = (\cos\theta, \sin\theta, 0)$ for $0 \le \theta \le 2\pi$. To compute the work, we need $d\vec{r}/d\theta(\theta) = (-\sin\theta, \cos\theta, 0)$. The force $\vec{F}$ is $\vec{F} = (-\sin\theta, \cos\theta, 0)$ in these coordinates. The work is then

$$\int_0^{2\pi} \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix} d\theta = \int_0^{2\pi} d\theta = 2\pi \,. \tag{3}$$

3. Consider a point charge with strength q located at the origin so that the electric field at any given point (x, y, z) is

$$\vec{E}(x, y, z) = \frac{q}{4\pi |(x, y, z)|^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \,, \tag{4}$$

where $|(x, y, z)|$ is the length of the vector (x, y, z). Consider the outer surface of the cylinder $x^2 + y^2 \le 1$ with $-1 \le z \le 1$. This consists of a tubular part given by $x^2 + y^2 = 1$ with $-1 \le z \le 1$ and the two disc-shaped caps $x^2 + y^2 \le 1$ with $z = -1$ for one and $z = 1$ for the other.

(a) (5 pts) Compute the derivative of $\frac{x}{\sqrt{1+x^2}}$, which may be useful below.

Solution

Using the product rule, we obtain

$$\frac{d}{dx} \frac{x}{\sqrt{1+x^2}} = \frac{\sqrt{1+x^2} - x^2/\sqrt{1+x^2}}{1+x^2} = \frac{1}{(1+x^2)^{3/2}}$$

(b) (20 pts) Compute the flux

$$\iint_S \vec{E} \cdot \hat{n}\, dS \,, \tag{5}$$

where S is the tubular part of the cylinder, i.e. the surface $x^2 + y^2 = 1$ with $-1 \leq z \leq 1$. Take the normal vector to be pointing out of the cylinder.

Solution

We parameterize the surface as $\vec{r}(\theta, z) = (\cos\theta, \sin\theta, z)$ where θ runs from 0 to 2π and z runs from -1 to 1. Note that

$$\vec{E}(\cos\theta, \sin\theta, z) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \frac{q}{4\pi(1+z^2)^{3/2}}$$

To use the formula for the flux through a parameterized surface, we need $\partial_\theta \vec{r} = (-\sin\theta, \cos\theta, 0)$ and $\partial_z \vec{r} = (0, 0, 1)$ as well as the cross product $\partial_\theta \vec{r} \times \partial_z \vec{r} = (\cos\theta, \sin\theta, 0)$.
Then, we have

$$\begin{aligned}
\iint_S \vec{E} \cdot d\vec{r} &= \frac{q}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \frac{1}{(1+z^2)^{3/2}} \begin{pmatrix} \cos\theta \\ \sin\theta \\ z \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} dz\, d\theta \\
&= \frac{q}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \frac{1}{(1+z^2)^{3/2}} dz\, d\theta \\
&= \frac{q}{4\pi} \int_0^{2\pi} \left[\frac{z}{\sqrt{1+z^2}} \right]_{-1}^{1} d\theta \\
&= \frac{q}{2\pi\sqrt{2}} \int_0^{2\pi} d\theta \\
&= \frac{q}{\sqrt{2}}
\end{aligned}$$

(c) (15 pts) Compute the flux

$$\iint_S \vec{E} \cdot \hat{n}\, dS\ , \tag{6}$$

where S is the top circular cap (a disc), i.e. the surface $x^2 + y^2 \leq 1$ with $z = 1$. Here you may take for granted that the normal vector is $\vec{n} = (0, 0, 1)$. (Hint: $\vec{E} \cdot \hat{n}$ is a scalar function of the disc, so that standard techniques for integrating over a disc may be used). Using symmetry considerations, the flux through the bottom is the same as the top. What is the total flux through the surface of the cylinder?

Solution

We will parameterize the points on the cap by $(r\cos\theta, r\sin\theta, 1)$ for $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Note that $\vec{E}\cdot\hat{n} = \frac{q}{4\pi(r^2+1)^{3/2}}$. We can then integrate

$$\begin{aligned}\iint_S \vec{E}\cdot\hat{n}\,dS &= \frac{q}{4\pi}\int_0^{2\pi}\int_0^1 \frac{r}{(r^2+1)^{3/2}}\,dr\,d\theta \\ &= \frac{q}{4\pi}\int_0^{2\pi}\int_1^2 \frac{1}{2u^{3/2}}\,du\,d\theta \\ &= \frac{q}{4\pi}\int_0^{2\pi} \frac{-1}{\sqrt{u}}\Big|_1^2\,d\theta \\ &= \frac{q}{2}\left(1-\frac{1}{\sqrt{2}}\right)\end{aligned}$$

The total is $q/\sqrt{2} + 2(q/2(1-1/\sqrt{2})) = q$. This is a special case of Gauss' law.