Name:

# MATH 335-002: Homework #6

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## Instructions

- This assignment is due in-class Thursday May 2nd (this is the last homework).
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

## Outline

In this assignment, we practice working with tensors and outline some applications of vector calculus.

## 1 Exercises from Matthews

Please complete exercises 7.2, 7.3, 7.14, 7.17, 8.4, 8.5, 8.10, and 8.11 from Matthews (2 pts each). They are provided below for convenience.

- 7.2: If $\vec{u}$ is a vector field, show that $\nabla \cdot \vec{u}$ is a scalar field.
- 7.3: Given that $\vec{a}$ and $\vec{b}$ are vectors, show that $a_i b_j$ is a second-rank tensor.
- 7.14: A quantity $A_{ij}$ is related to a vector $\vec{B}$ by $A_{ij} = \epsilon_{ijk} B_k$.
  (a) Show that $A_{ij}$ is a second-rank tensor and describe its symmetry property.
  (b) Find an equation for $\vec{B}$ in terms of $A_{ij}$.
- 7.17: Show that the kinetic energy $E$ of a body rotating with angular velocity $\vec{\Omega}$ is related to its inertia tensor $I_{jk}$ by $E = I_{jk}\Omega_j\Omega_k/2$.
- 8.4: Show from Maxwell's equations in a vacuum that the magnetic field $\vec{B}$ obeys the wave equation. (Hint: you can mimic the argument for $\vec{E}$).
- 8.5: For the electromagnetic wave in which the $\vec{E}$ field is given by $\vec{E} = \vec{E}_0 f(\vec{k} \cdot \vec{x} - \omega t)$, calculate the corresponding magnetic field $\vec{B}$. What can be deduced about the directions of the vectors $\vec{B}$ and $\vec{E}$? (are they parallel? perpendicular? something else?) (Hint: use eq (8.11) in the book)
- 8.10: An isotropic elastic solid with Lamé constants $\lambda$ and $\mu$ is subjected to a deformation $v_1 = ax_1x_2$, $v_2 = b(x_1^2 - x_2^2)$, and $v_3 = 0$.

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(a) Find the strain tensor $E_{ij}$.

(b) Find the stress tensor $P_{ij}$.

(c) Determine the values of $a$ and $b$ for which it is possible for the material to be in equilibrium.

- 8.11: A compressible fluid with negligible viscosity ($\mu = 0$) is initially at rest with uniform density $\rho_0$ and pressure $p_0$, with no body forces. A small perturbation is then introduced so that there is a velocity $\vec{u}(\vec{r}, t)$ and the density becomes $\rho_0 + \rho_1(\vec{r}, t)$.

  (a) Assuming that the products of the small quantities $\vec{u}$ and $\rho_1$ can be neglected, show that the equation for conservation of mass (5.9) becomes

  $$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \vec{u} = 0 \; .$$

  (b) Assuming that the perturbation $p_1$ to the pressure is related to $\rho_1$ by $p_1 = a\rho_1$ where $a$ is a constant, show that the Navier-Stokes equation reduces to

  $$\rho_0 \frac{\partial \vec{u}}{\partial t} = -a \nabla \rho_1 \; .$$

  (c) Combine the previous two equations to show that the perturbation $\rho_1$ obeys the wave equation and interpret this result physically.

## 2 Other Exercises

**Question 1**

(5 pts) Let $A_{ij}$ and $B_{ij}$ be tensors. Show that $C_{ij} = A_{ik}B_{kj}$ is also a tensor.

**Question 2**

(5 pts) Let $u(x, t)$ be a solution of the heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u \; .$$

Show that

- $u(x, t - \tau)$, for some constant $\tau$, is also a solution of the heat equation.
- $u(x, t) + l(x)$, where $l(x) = ax + b$, is also a solution of the heat equation.

These two points highlight why we need both initial and boundary conditions for such problems.