MATH 335 S2019
Final Exam Solutions

2019-05-14

Read the problems carefully and be sure to show your work. No cell phones or calculators are allowed. Please turn off your phone to avoid any disturbances. Good luck!

Reference

The following results and identities are provided for reference purposes. They may or may not be needed to complete the exam.

- The following identity may be used in the exam without the need to prove it:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

- Any rank-4 and isotropic tensor is of the form

$$\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk},$$

  for some constants λ, μ, and ν.

- Spherical coordinates. A coordinate system in which the cartesian coordinates are given in terms of (ρ, ϕ, θ) as $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, and $z = \rho\cos\phi$. On a sphere of radius R, we have $0 \le \rho \le R$, $0 \le \phi \le \pi$, and $0 \le \theta \le 2\pi$.

- You may use without proof that if A_{ij} and B_{ij} are tensors, then $C_{ij} = A_{ij} + B_{ij}$ is also a tensor.

Exam

1. (15 pts) Derive the following identities

   (a) $\nabla \times (\nabla f) = \vec{0}$.

   Solution

$$\begin{aligned}
[\nabla \times (\nabla f)]_i &= \epsilon_{ijk}\partial_j\partial_k f \\
&= \epsilon_{ikj}\partial_k\partial_j f \quad \text{renaming, swap } j \text{ and } k \\
&= \epsilon_{ikj}\partial_j\partial_k f \quad \text{order of diff. doesn't matter} \\
&= -\epsilon_{ijk}\partial_j\partial_k f \quad \text{def. of alt. tensor}
\end{aligned}$$

$$= -[\nabla \times (\nabla f)]_i$$

Because each entry is the negative of itself, all entries must be zero.

(b) $\nabla \cdot (\vec{u} \times \vec{v}) = (\nabla \times \vec{u}) \cdot \vec{v} - (\nabla \times \vec{v}) \cdot \vec{u}$.

Solution

$$\begin{aligned}
\nabla \cdot (\vec{u} \times \vec{v}) &= \partial_i(\epsilon_{ijk} u_j v_k) \\
&= \epsilon_{ijk} \left(v_k \partial_i u_j + u_j \partial_i v_k \right) \\
&= v_k \epsilon_{ijk} \partial_i u_j + u_j \epsilon_{ijk} \partial_i v_k \\
&= v_k \epsilon_{kij} \partial_i u_j - u_j \epsilon_{jik} \partial_i v_k \\
&= (\nabla \times \vec{u}) \cdot \vec{v} - (\nabla \times \vec{v}) \cdot \vec{u}
\end{aligned}$$

(c) $\nabla \cdot (\nabla f \times \nabla g) = 0$.

Solution

From part (b)

$$\nabla \cdot (\nabla f \times \nabla g) = (\nabla \times \nabla f) \cdot \nabla g - (\nabla \times \nabla g) \cdot \nabla f$$

Then the result follows from part (a).

2. (15 pts) Let $\vec{a}$ be a constant vector and let $\vec{v}$ be a vector field. Let V be some volume and S its boundary surface. Show the following and explain each step.

   (a) $\nabla \cdot (\vec{a} \times \vec{v}) = -(\nabla \times \vec{v}) \cdot \vec{a}$.

   Solution

   From 1(b), we have

$$\nabla \cdot (\vec{a} \times \vec{v}) = (\nabla \times \vec{a}) \cdot \vec{v} - (\nabla \times \vec{v}) \cdot \vec{a}$$

   The identity then follows because $\nabla \times \vec{a} = \vec{0}$ for the constant vector $\vec{a}$.

   (b) $\iiint_V -(\nabla \times \vec{v}) \cdot \vec{a} \, dV = \oiint_S \vec{a} \cdot \vec{v} \times \hat{n} \, dS$.

   Solution

   From part (a)

$$\iiint_V -(\nabla \times \vec{v}) \cdot \vec{a} \, dV = \iiint_V \nabla \cdot (\vec{a} \times \vec{v}) \, dV$$

   Then applying the divergence theorem

$$\iiint_V \nabla \cdot (\vec{a} \times \vec{v}) \, dV = \oiint_S (\vec{a} \times \vec{v}) \cdot \hat{n} \, dS$$

   Finally, we recall that you can interchange the order of the $\times$ and $\cdot$ in the triple product, giving the result.

(c) $\iint_V -\nabla \times \vec{v}\, dV = \oiint_S \vec{v} \times \hat{n}\, dS$.

Solution

From part (b)

$$\iiint_V -(\nabla \times \vec{v}) \cdot \vec{a}\, dV = \oiint_S \vec{a} \cdot \vec{v} \times \hat{n}\, dS$$

holds for any constant vector $\vec{a}$. If we set $\vec{a} = \vec{e}_i$, the ith coordinate vector, we get

$$\iiint_V -[\nabla \times \vec{v}]_i\, dV = \oiint_S [\vec{v} \times \hat{n}]_i\, dS$$

for each i, which is equivalent to the desired result.

3. Let $\vec{v}$ be a differentiable vector field and let

$$E_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) .$$

Similarly, define A_{ij} to be

$$A_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) .$$

If L_{ij} is a given rotation and v'_i and x'_i denote the vector field and new coordinates, then E_{ij} in the new coordinates is

$$E'_{ij} = \frac{1}{2}\left(\frac{\partial v'_i}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_i}\right) .$$

(a) (5 pts) Recall that $x'_i = L_{ij}x_j$. Show why $x_k = L_{ik}x'_i$.

Solution

We have $x'_i = L_{ij}x_j$. Multiplying by L_{ik}, we get $L_{ik}x'_i = L_{ik}L_{ij}x_j$. Because L_{ij} is a rotation, $L_{ik}L_{ij} = \delta_{kj}$. Thus, $L_{ik}x'_i = \delta_{kj}x_j = x_k$.

(b) (10 pts) Show that E_{ij} is a tensor (in the sense of the technical definition of a tensor).

Solution

We focus on the pieces of E_{ij} separately. We have

$$\begin{aligned}
\frac{1}{2}\frac{\partial v'_i}{\partial x'_j} &= \frac{1}{2}\frac{\partial L_{ik}v_k}{\partial x'_j} \quad \text{because } v_i \text{ is a vector}\\
&= L_{ik}\frac{1}{2}\frac{\partial v_k}{\partial x_l}\frac{\partial x_l}{\partial x'_j} \quad \text{because } L_{ik} \text{ constant and using chain rule}\\
&= L_{ik}L_{jl}\frac{1}{2}\frac{\partial v_k}{\partial x_l} \quad \text{using part (a)}
\end{aligned}$$

Thus $\frac{1}{2}\frac{\partial v_k}{\partial x_l}$ is a tensor. Because E_{ij} is the sum of $\frac{1}{2}\frac{\partial v_i}{\partial x_j}$ and $\frac{1}{2}\frac{\partial v_j}{\partial x_i}$, and each is a tensor, we have that E_{ij} is a tensor.

(c) (5 pts) Show that E_{ij} is symmetric and A_{ij} is anti-symmetric.

Solution

For E_{ij}, we have

$$\begin{aligned} E_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\ &= \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) \\ &= E_{ji} \end{aligned}$$

For A_{ij}, we have

$$\begin{aligned} A_{ij} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \\ &= -\frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) \\ &= -A_{ji} \end{aligned}$$

(d) (5 pts) Let a_{ijkl} be an isotropic rank-4 tensor and let $P_{ij} = a_{ijkl}A_{kl}$. Suppose A_{kl} is not identically zero. Show that P_{ij} is not symmetric in general.

Solution

The isotropic rank-4 tensor must be of the form $a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$ for some constants λ, μ, and ν. Thus,

$$\begin{aligned} P_{ij} &= (\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk})A_{kl} \\ &= \lambda\delta_{ij}A_{kk} + \mu A_{ij} + \nu A_{ji} \quad \text{using substitution properties of } \delta_{ij} \\ &= (\mu - \nu)A_{ij} \quad \text{because } A_{kk} = 0 \text{ for } A \text{ antisymmetric} \end{aligned}$$

In particular, P_{ij} is a multiple of A_{ij} and is therefore not symmetric (except when $\mu = \nu$)

4. (15 pts) A coordinate system (u, v, w) is related to cartesian coordinates by

$$x_1 = uvw\,, \quad x_2 = uv(1-w^2)^{1/2}\,, \quad x_3 = (u^2 - v^2)/2\,.$$

(a) Find the scale factors h_u, h_v, and h_w.

Solution

These can be computed by taking the partial derivatives of $\vec{x}$ with respect to each curvilinear coordinate. We have

$$\partial_u\vec{x} = (vw, v(1-w^2)^{1/2}, u)$$

$$\begin{aligned}
h_u &= |\partial_u \vec{x}| = (u^2 + v^2)^{1/2} \\
\partial_v \vec{x} &= (uw, u(1-w^2)^{1/2}, -v) \\
h_v &= |\partial_v \vec{x}| = (u^2 + v^2)^{1/2} \\
\partial_w \vec{x} &= (uv, -uvw(1-w^2)^{-1/2}, 0) \\
h_w &= |\partial_w \vec{x}| = uv/(1-w^2)^{1/2}
\end{aligned}$$

(b) Confirm that the (u, v, w) system is orthogonal.

Solution

Because the scale factors are just scalars, we can check orthogonality of the vectors $\partial_u \vec{x}$, $\partial_v \vec{x}$, and $\partial_w \vec{x}$. We have

$$\begin{aligned}
\partial_u \vec{x} \cdot \partial_v \vec{x} &= uvw^2 + uv(1-w^2) - uv = 0 \\
\partial_u \vec{x} \cdot \partial_w \vec{x} &= uv^2 w - uv^2 w + 0 = 0 \\
\partial_v \vec{x} \cdot \partial_w \vec{x} &= u^2 vw - u^2 vw + 0 = 0
\end{aligned}$$

(c) Find the volume element in the (u, v, w) coordinate system.

Solution

The volume element can be found using the scale factors via $dV = h_u h_v h_w \, du \, dv \, dw$. We have

$$dV = \frac{(u^2 + v^2)uv}{(1-w^2)^{1/2}} \, du \, dv \, dw$$

5. (20 pts) Verify Stokes' theorem by evaluating both the circulation (line integral around C) and the appropriate surface integral for the vector field $\vec{u} = (2x - y, -y^2, -y^2 z)$. Let the surface S be the flat disk given by $z = 0$ with $x^2 + y^2 \le 1$ so that C is the circle $z = 0$ with $x^2 + y^2 = 1$.

Solution

To validate Stokes' theorem, we must show that

$$\oint_C \vec{u} \cdot d\vec{r} = \iint_S \nabla \times \vec{u} \cdot \hat{n} \, dS$$

We compute the line integral using the parameterization $\vec{r}(t) = (\cos t, \sin t, 0)$ for $0 \le t \le 2\pi$. We have

$$\oint_C \vec{u} \cdot d\vec{r} = \int_0^{2\pi} \vec{u}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$
\begin{aligned}
&= \int_0^{2\pi} (2\cos t - \sin t, -\sin^2 t, 0) \cdot (-\sin t, \cos t, 0)\, dt \\
&= \int_0^{2\pi} -2\cos t \sin t + \sin^2 t - \sin^2 t \cos t\, dt \\
&= \int_0^{2\pi} -\sin(2t) + (1-\cos(2t))/2 - \sin^2 t \cos t\, dt \\
&= \pi
\end{aligned}
$$

For the surface integral, we have

$$
\nabla \times \vec{u} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_x & \partial_y & \partial_z \\ 2x - y & -y^2 & -y^2 z \end{vmatrix} = (-2yz, 0, 1)
$$

So that $\nabla \times \vec{u} \cdot \hat{n} = 1$ here. Thus

$$
\iint_S \nabla \times \vec{u} \cdot \hat{n}\, dS = \iint_S dS = \pi
$$

6. (20 pts) Let $\vec{F} = (2x, y^2, z^2)$ and S be the sphere defined by $x^2 + y^2 + z^2 = R^2$. Evaluate

$$
\iint_S \vec{F} \cdot \hat{n}\, dS
$$

using the divergence theorem.

Solution

$$
\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{F}\, dV
$$

Note that $\nabla \cdot \vec{F} = 2 + 2y + 2z$. We will use spherical coordinates, i.e. $x = \rho\cos(\theta)\sin(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, and $z = \rho\cos(\phi)$ with $dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$.

$$
\begin{aligned}
\iint_S F \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{F}\, dV \\
&= \int_0^R \int_0^{2\pi} \int_0^{\pi} (2 + 2\rho\sin(\theta)\sin(\phi) + 2\rho\cos(\phi))\ \rho^2 \sin(\phi) d\rho d\theta d\phi \\
&= 4\pi \int_0^R \rho^2\, d\rho \int_0^{\pi} \sin\phi\, d\phi + 2\int_0^R \int_0^{\pi} \rho^3 \sin^2\phi\, d\rho d\phi \int_0^{2\pi} \sin(\theta)\, d\theta \\
&\quad + 2\pi \int_0^R \rho^3\, d\rho \int_0^{\pi} \sin(2\phi)\, d\phi \\
&= \frac{8\pi R^3}{3}
\end{aligned}
$$