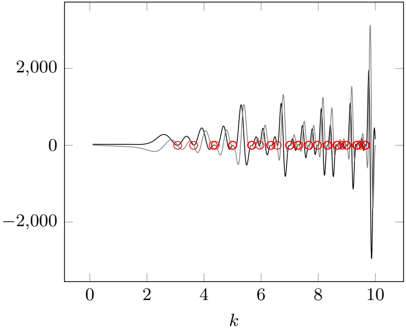
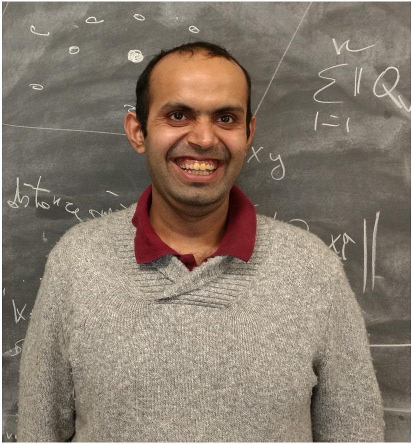
Fredholm Determinants: A Robust Approach to Computing Stokes Eigenvalues



Travis Askham (New Jersey Institute of Technology)
SIAM CSE 2019. Spokane, WA, USA.



Joint work with Manas Rachh
(Flatiron Institute)

Barnett, Greengard

Stokes Eigenvalues

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= k^2 \mathbf{u} \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \\
\mathbf{u} &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Stokes Eigenvalues

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= k^2 \mathbf{u} \text{ in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \\ \mathbf{u} &= 0 \text{ on } \partial\Omega \end{aligned}$$

- Stability of steady flows

Stokes Eigenvalues

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= k^2 \mathbf{u} \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \\
\mathbf{u} &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

- Stability of steady flows
- Decay of turbulent flows

Stokes Eigenvalues

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= k^2 \mathbf{u} \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \\
\mathbf{u} &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

- Stability of steady flows
- Decay of turbulent flows
- History of studying the spectrum

Related Problem: Buckling Eigenvalues

Related Problem: Buckling Eigenvalues

Stream function

$$\mathbf{u} = \nabla^{\perp}\Psi$$

Related Problem: Buckling Eigenvalues

Stream function

$$\mathbf{u} = \nabla^{\perp}\Psi$$

Stokes eigenvalue problem becomes

$$-\Delta^2\Psi = k^2\Delta\Psi \text{ in } \Omega$$
$$\nabla\Psi = 0 \text{ on } \partial\Omega$$

Related Problem: Buckling Eigenvalues

Stream function

$$\mathbf{u} = \nabla^{\perp}\Psi$$

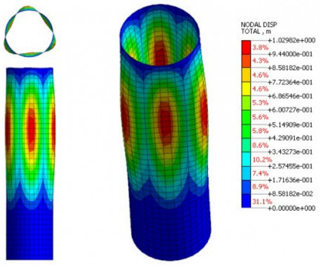
Stokes eigenvalue problem becomes

$$-\Delta^2\Psi = k^2\Delta\Psi \text{ in } \Omega$$
$$\nabla\Psi = 0 \text{ on } \partial\Omega$$

The buckling problem

$$-\Delta^2\Psi = k^2\Delta\Psi \text{ in } \Omega$$
$$\Psi = \partial_\nu\Psi = 0 \text{ on } \partial\Omega$$

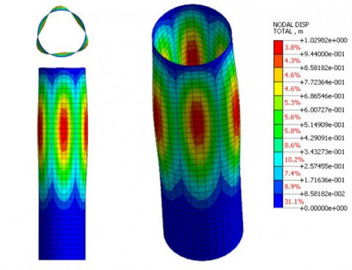
Buckling Eigenvalues



[fetraining.net]

Buckling Eigenvalues

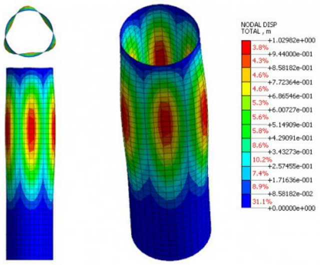
- First eigenvalue describes buckling load of an idealized elastic plate under compression



[fetraining.net]

Buckling Eigenvalues

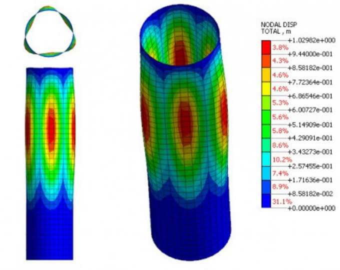
- First eigenvalue describes buckling load of an idealized elastic plate under compression
- Equivalent to Stokes eigenvalues on simply-connected domains



[fetraining.net]

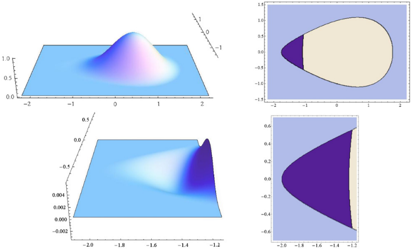
Buckling Eigenvalues

- First eigenvalue describes buckling load of an idealized elastic plate under compression
- Equivalent to Stokes eigenvalues on simply-connected domains
- Of pure mathematical interest:



[fetraining.net]

Buckling Eigenvalues

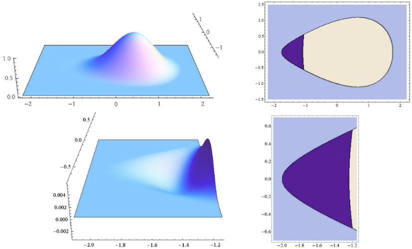


[Antunes 2011]

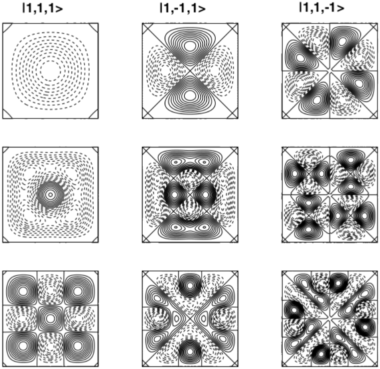
- First eigenvalue describes buckling load of an idealized elastic plate under compression
- Equivalent to Stokes eigenvalues on simply-connected domains
- Of pure mathematical interest:
    - Relation to Laplace (membrane) eigenvalues/ eigenfunctions

Buckling Eigenvalues

- First eigenvalue describes buckling load of an idealized elastic plate under compression
- Equivalent to Stokes eigenvalues on simply-connected domains
- Of pure mathematical interest:
  - Relation to Laplace (membrane) eigenvalues/ eigenfunctions
  - Intricate structure of eigenfunctions on domains with corners



[Antunes 2011]



[Leriche and Labrosse 2004]

Approximating Drum, Stokes, and Buckling Eigenvalues

Approximating Drum, Stokes, and Buckling Eigenvalues

- Pen and paper: Rayleigh, A. Weinstein, Pólya, Szegö, Payne, Rellich

Approximating Drum, Stokes, and Buckling Eigenvalues

- Pen and paper: Rayleigh, A. Weinstein, Pólya, Szegö, Payne, Rellich
- Finite elements and spectral methods dominate: Babuska, Osborn, Ciarlet, Q. Lin, Durán, J. Shen, Boyd

Approximating Drum, Stokes, and Buckling Eigenvalues

- Pen and paper: Rayleigh, A. Weinstein, Pólya, Szegö, Payne, Rellich
- Finite elements and spectral methods dominate: Babuska, Osborn, Ciarlet, Q. Lin, Durán, J. Shen, Boyd
- Method of fundamental solutions and other compact operator approaches: Kupradze and Aleksidze 1964; Kitahara 1985; Antunes 2011

Approximating Drum, Stokes, and Buckling Eigenvalues

- Pen and paper: Rayleigh, A. Weinstein, Pólya, Szegö, Payne, Rellich
- Finite elements and spectral methods dominate: Babuska, Osborn, Ciarlet, Q. Lin, Durán, J. Shen, Boyd
- Method of fundamental solutions and other compact operator approaches: Kupradze and Aleksidze 1964; Kitahara 1985; Antunes 2011
- Second-kind equations: Bäcker 2003, Bornemann 2010 (Nyström discretization of Fredholm determinant); Zhao and Barnett 2014 (drum); Lindsay, Quaife, and Wendelberger 2018 (mode elimination a la Farkas)

Computing Eigenvalues: 2 Approaches

$$
\begin{aligned}
-\Delta u &= k^2 u \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Computing Eigenvalues: 2 Approaches

$$\begin{aligned} -\Delta u &= k^2 u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

Eigenvalues of Discretization

$$\begin{aligned} -\Delta u &= k^2 u \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \\ &\downarrow \\ A_N u_N &= k^2 B_N u_N \\ &\downarrow \\ k_N &= \sqrt{\mathrm{eig}(A_N, B_N)} \end{aligned}$$

Computing Eigenvalues: 2 Approaches

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Eigenvalues of Discretization

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$
$$\downarrow$$
$$A_N u_N = k^2 B_N u_N$$
$$\downarrow$$
$$k_N = \sqrt{\mathrm{eig}(A_N, B_N)}$$

Discretization of Eigenvalue Indicator

$$u = -2\mathcal{D}(k)\mu$$
$$\downarrow$$
$$\dim(N(I - 2\mathcal{D}(k))) > 0 \iff k \text{ eval}$$
$$\downarrow$$
$$f(k) = \det(I - 2\mathcal{D}(k)) = 0$$
$$\downarrow$$
$$k = \mathrm{roots}(f_N)$$

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

- Direct PDE discretization suffers from high frequency pollution and ill-conditioning

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

- Direct PDE discretization suffers from high frequency pollution and ill-conditioning
- First kind equations and the method of fundamental solutions don't have good, convergent indicators of non-invertibility — the test of an eigenvalue is relative to the discretization

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

- Direct PDE discretization suffers from high frequency pollution and ill-conditioning
- First kind equations and the method of fundamental solutions don't have good, convergent indicators of non-invertibility — the test of an eigenvalue is relative to the discretization
- Fast direct methods behave well for second-kind equations

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

- Direct PDE discretization suffers from high frequency pollution and ill-conditioning
- First kind equations and the method of fundamental solutions don't have good, convergent indicators of non-invertibility — the test of an eigenvalue is relative to the discretization
- Fast direct methods behave well for second-kind equations
- Straightforward to make high order tools

Why use the Eigenvalue Indicator Approach? And Why Insist on Second-Kind Equations?

- Direct PDE discretization suffers from high frequency pollution and ill-conditioning
- First kind equations and the method of fundamental solutions don't have good, convergent indicators of non-invertibility — the test of an eigenvalue is relative to the discretization
- Fast direct methods behave well for second-kind equations
- Straightforward to make high order tools
- Down the line — for certain second kind kernels, corners can be handled robustly/efficiently a la Serkh et al. or Helsing

Zhao and Barnett Program

"drum eigenvalues"

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

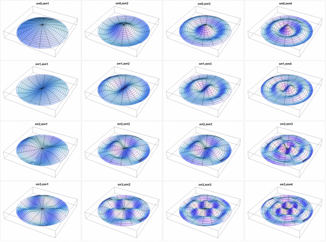


image: bio physics wiki

Zhao and Barnett Program

"drum eigenvalues"

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

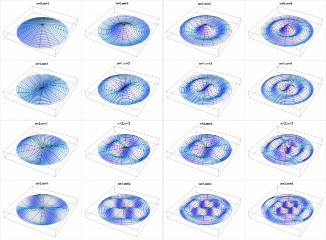


image: bio physics wiki

- Reformulate as second-kind integral equation $\mathcal{I} - 2\mathcal{D}(k)$

“drum eigenvalues”

Zhao and Barnett Program

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

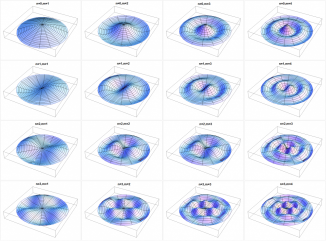


image: bio physics wiki

- Reformulate as second-kind integral equation $\mathcal{I} - 2\mathcal{D}(k)$
- Nyström discretize to high accuracy $\mathcal{I}_N - 2\mathcal{D}_N(k)$

Zhao and Barnett Program

"drum eigenvalues"

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

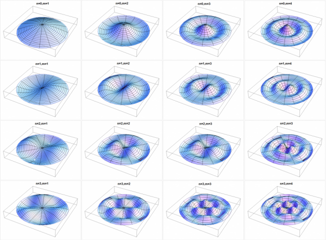


image: bio physics wiki

- Reformulate as second-kind integral equation $\mathcal{I} - 2\mathcal{D}(k)$
- Nyström discretize to high accuracy $\mathcal{I}_N - 2\mathcal{D}_N(k)$
- Approximate Fredholm determinant (an analytic function) $f_N(k) = \det(\mathcal{I}_N - 2\mathcal{D}_N(k))$

Zhao and Barnett Program

"drum eigenvalues"

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

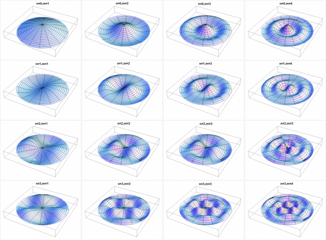


image: bio physics wiki

- Reformulate as second-kind integral equation $\mathcal{I} - 2\mathcal{D}(k)$
- Nyström discretize to high accuracy $\mathcal{I}_N - 2\mathcal{D}_N(k)$
- Approximate Fredholm determinant (an analytic function) $f_N(k) = \det(\mathcal{I}_N - 2\mathcal{D}_N(k))$
- Use high-order root finding on $f_N(k)$ to obtain eigenvalues

Zhao and Barnett Program

"drum eigenvalues"

$$-\Delta u = k^2 u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

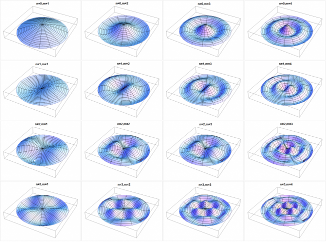
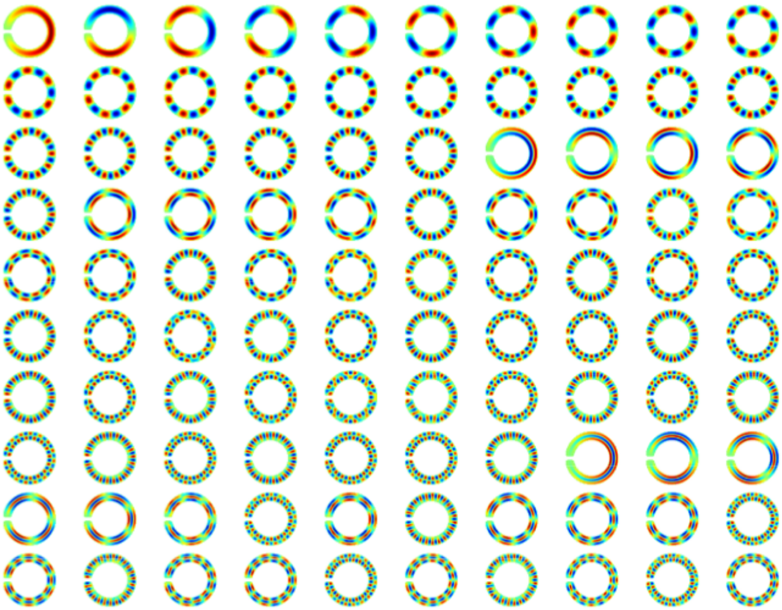


image: bio physics wiki

- Reformulate as second-kind integral equation $\mathcal{I} - 2\mathcal{D}(k)$
- Nyström discretize to high accuracy $\mathcal{I}_N - 2\mathcal{D}_N(k)$
- Approximate Fredholm determinant (an analytic function) $f_N(k) = \det(\mathcal{I}_N - 2\mathcal{D}_N(k))$
- Use high-order root finding on $f_N(k)$ to obtain eigenvalues
- On multiply-connected / exterior resonance, replace $\mathcal{D}$ with combined field



The ZB Program for Stokes

The ZB Program for Stokes

- Develop a second kind representation for Stokes EV problem, $\mathbf{u} = \mathbf{K}(k)\boldsymbol{\mu}$

The ZB Program for Stokes

- Develop a second kind representation for Stokes EV problem, $\mathbf{u} = \mathbf{K}(k)\boldsymbol{\mu}$
- Establish that $\{\text{ eigenvalues }\} = \{k : \dim(N(\mathcal{I} - \mathcal{K}(k))) > 0\}$

The ZB Program for Stokes

- Develop a second kind representation for Stokes EV problem, $\mathbf{u} = \mathbf{K}(k)\boldsymbol{\mu}$
- Establish that $\{$ eigenvalues $\} = \{k : \dim(N(\mathcal{I} - \mathcal{K}(k))) > 0\}$
- Discretization and solution is "off-the-shelf"

Oscillatory Stokes BVPs

Oscillatory Stokes BVPs

Interior Dirichlet Problem

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= k^2 \mathbf{u} \text{ in } \Omega \\
\nabla \cdot \mathbf{u} &= 0 \text{ in } \Omega \\
\mathbf{u} &= \mathbf{f} \text{ on } \Gamma
\end{aligned}
$$

Compatibility condiiton $\int_\Gamma \boldsymbol{\nu} \cdot \mathbf{f}\, dS = 0$.

Oscillatory Stokes BVPs

Interior Dirichlet Problem

$$-\Delta \mathbf{u} + \nabla p = k^2 \mathbf{u} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$
$$\mathbf{u} = \mathbf{f} \text{ on } \Gamma$$

Compatibility condiiton $\int_\Gamma \boldsymbol{\nu} \cdot \mathbf{f} \, dS = 0$.

Interior Neumann Problem

$$-\Delta \mathbf{u} + \nabla p = k^2 \mathbf{u} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$
$$\boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{g} \text{ on } \Gamma$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

$$G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) = -\frac{\log|\mathbf{x}-\mathbf{y}|}{2\pi} \Rightarrow \Delta G^{\mathrm{L}} = \delta_y(\mathbf{x})$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

$$G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) = -\frac{\log|\mathbf{x}-\mathbf{y}|}{2\pi} \Rightarrow \Delta G^{\mathrm{L}} = \delta_y(\mathbf{x})$$

$$\downarrow$$

$$\Rightarrow p = \nabla G^{\mathrm{L}} \cdot \mathbf{f}$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

$$G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) = -\frac{\log|\mathbf{x}-\mathbf{y}|}{2\pi} \Rightarrow \Delta G^{\mathrm{L}} = \delta_y(\mathbf{x})$$

$$\downarrow$$

$$\Rightarrow p = \nabla G^{\mathrm{L}} \cdot \mathbf{f}$$

$$\downarrow$$

$$-(\Delta + k^2)\mathbf{u} = \Delta G^L \mathbf{f} - \nabla(\nabla G^L \cdot \mathbf{f})$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

$$G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) = -\frac{\log|\mathbf{x}-\mathbf{y}|}{2\pi} \Rightarrow \Delta G^{\mathrm{L}} = \delta_y(\mathbf{x})$$
$$\downarrow$$
$$\Rightarrow p = \nabla G^{\mathrm{L}} \cdot \mathbf{f}$$
$$\downarrow$$
$$-(\Delta + k^2)\mathbf{u} = \Delta G^L \mathbf{f} - \nabla(\nabla G^L \cdot \mathbf{f})$$
$$\downarrow$$
$$\mathbf{u} = ((\nabla \otimes \nabla - \Delta I)G^{\mathrm{BH}})\mathbf{f}$$
$$= -(\nabla^{\perp} \otimes \nabla^{\perp} G^{\mathrm{BH}})\mathbf{f}$$

Oscillatory Stokeslets

$$-(\Delta + k^2)\mathbf{u} + \nabla p = \delta_y(\mathbf{x})\mathbf{f} \text{ in } \Omega$$
$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

$$G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) = -\frac{\log|\mathbf{x}-\mathbf{y}|}{2\pi} \Rightarrow \Delta G^{\mathrm{L}} = \delta_y(\mathbf{x})$$
$$\downarrow$$
$$\Rightarrow p = \nabla G^{\mathrm{L}} \cdot \mathbf{f}$$
$$\downarrow$$
$$-(\Delta + k^2)\mathbf{u} = \Delta G^L \mathbf{f} - \nabla(\nabla G^L \cdot \mathbf{f})$$
$$\downarrow$$
$$\begin{aligned}\mathbf{u} &= ((\nabla \otimes \nabla - \Delta I)G^{\mathrm{BH}})\mathbf{f} \\ &= -(\nabla^\perp \otimes \nabla^\perp G^{\mathrm{BH}})\mathbf{f}\end{aligned}$$

where

$$G^{\mathrm{BH}}(\mathbf{x},\mathbf{y};k) = \frac{1}{k^2}\left(\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}| + \frac{i}{4}H_0^1(k|\mathbf{x}-\mathbf{y}|)\right)$$

$$\Delta(\Delta + k^2)G^{\mathrm{BH}} = \delta$$

Stresslet

Stresslet

Let $\mathbf{G}^{(k)}(\mathbf{x},\mathbf{y}) = (\nabla \otimes \nabla - \Delta I) G^{\mathrm{BH}}(\mathbf{x},\mathbf{y};k)$.

Stresslet

Let $\mathbf{G}^{(k)}(\mathbf{x},\mathbf{y}) = (\nabla \otimes \nabla - \Delta I)G^{\mathrm{BH}}(\mathbf{x},\mathbf{y};k)$.

$$\mathbf{u}(\mathbf{x}) = \mathbf{G}^{(k)}(\mathbf{x},\mathbf{y})\mathbf{f}\,,\; p(\mathbf{x}) = \nabla G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) \cdot \mathbf{f}$$

Stresslet

Let $\mathbf{G}^{(k)}(\mathbf{x},\mathbf{y}) = (\nabla \otimes \nabla - \Delta I)G^{\mathrm{BH}}(\mathbf{x},\mathbf{y};k)$.

$$\mathbf{u}(\mathbf{x}) = \mathbf{G}^{(k)}(\mathbf{x},\mathbf{y})\mathbf{f}\,,\; p(\mathbf{x}) = \nabla G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) \cdot \mathbf{f}$$

The stress tensor is

$$\boldsymbol{\sigma}(\mathbf{x}) = -p(\mathbf{x})I + \nabla\mathbf{u}(\mathbf{x}) + (\nabla\mathbf{u}(\mathbf{x}))^{\intercal} =: \mathbf{T}^{(k)}\mathbf{f}$$

Stresslet

Let $\mathbf{G}^{(k)}(\mathbf{x},\mathbf{y}) = (\nabla \otimes \nabla - \Delta I)G^{\mathrm{BH}}(\mathbf{x},\mathbf{y};k)$.

$$\mathbf{u}(\mathbf{x}) = \mathbf{G}^{(k)}(\mathbf{x},\mathbf{y})\mathbf{f}\,,\; p(\mathbf{x}) = \nabla G^{\mathrm{L}}(\mathbf{x},\mathbf{y}) \cdot \mathbf{f}$$

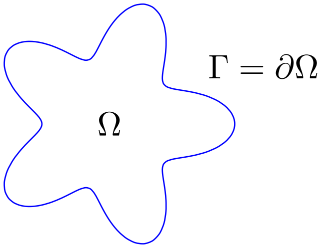
The stress tensor is

$$\boldsymbol{\sigma}(\mathbf{x}) = -p(\mathbf{x})I + \nabla\mathbf{u}(\mathbf{x}) + (\nabla\mathbf{u}(\mathbf{x}))^{\intercal} =: \mathbf{T}^{(k)}\mathbf{f}$$

Stresslet

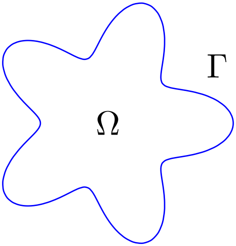
$$T_{ij\ell} = -\partial_{x_j} G^{\mathrm{L}}\delta_{i\ell} + \partial_{x_\ell}\left(-\Delta G^{\mathrm{BH}}\delta_{ij} + \partial_{x_i}\left(\partial_{x_j} G^{\mathrm{BH}}\right)\right)$$

Layer Potentials



$\boldsymbol{\nu}$ is normal to boundary

Layer Potentials



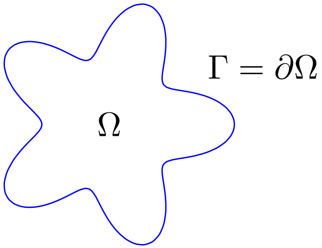
$\Gamma = \partial\Omega$

$\boldsymbol{\nu}$ is normal to boundary

Single

$$\mathbf{S}^{(k)}[\boldsymbol{\mu}](\mathbf{x}) = \int_\Gamma \mathbf{G}^{(k)}(\mathbf{x},\mathbf{y})\boldsymbol{\mu}(\mathbf{y})\, dS(\mathbf{y})$$

Layer Potentials



$\boldsymbol{\nu}$ is normal to boundary

Single

$$\mathbf{S}^{(k)}[\boldsymbol{\mu}](\mathbf{x}) = \int_\Gamma \mathbf{G}^{(k)}(\mathbf{x},\mathbf{y})\boldsymbol{\mu}(\mathbf{y})\, dS(\mathbf{y})$$

Double

$$\mathbf{D}^{(k)}[\boldsymbol{\mu}](\mathbf{x}) = \int_\Gamma \left(\mathbf{T}^{(k)}_{\cdot,\cdot,\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})\right)^\mathsf{T} \boldsymbol{\mu}(\mathbf{y})\, dS(\mathbf{y})$$

- $\mathbf{S}^{(k)}$ single layer
- $\boldsymbol{\sigma}_{\mathbf{S}}^{(k)}$ stress of single layer off boundary
- $\mathbf{D}^{(k)}$ double layer off boundary
- $\mathcal{D}^{(k)}$ double layer on boundary
- $\mathcal{N}^{(k)} = \mathcal{D}^{(k)\mathsf{T}}$ stress of single layer on boundary

- $\mathbf{S}^{(k)}$ single layer
- $\boldsymbol{\sigma}_{\mathbf{S}}^{(k)}$ stress of single layer off boundary
- $\mathbf{D}^{(k)}$ double layer off boundary
- $\mathcal{D}^{(k)}$ double layer on boundary
- $\mathcal{N}^{(k)} = \mathcal{D}^{(k)\mathsf{T}}$ stress of single layer on boundary

Lemma (Jump conditions)

For a given density $\boldsymbol{\mu}$ defined on Γ, $\mathbf{S}\boldsymbol{\mu}$ is continuous across Γ, the exterior and interior limits of the surface traction of $\mathbf{D}\boldsymbol{\mu}$ are equal, and for each $\mathbf{x}_0 \in \Gamma$,

$$\begin{aligned} \lim_{h \downarrow 0} \boldsymbol{\sigma}_{\mathbf{S}}^{(k)}[\boldsymbol{\mu}](\mathbf{x}_0 \pm h\boldsymbol{\nu}(\mathbf{x}_0)) \cdot \boldsymbol{\nu}(\mathbf{x}_0) &= \mp \frac{1}{2}\boldsymbol{\mu}(\mathbf{x}_0) + \mathcal{N}^{(k)}[\boldsymbol{\mu}](\mathbf{x}_0) \\ \lim_{h \downarrow 0} \mathbf{D}^{(k)}[\boldsymbol{\mu}](\mathbf{x}_0 \pm h\boldsymbol{\nu}(\mathbf{x}_0)) &= \pm \frac{1}{2}\boldsymbol{\mu}(\mathbf{x}_0) + \mathcal{D}^{(k)}[\boldsymbol{\mu}](\mathbf{x}_0) . \end{aligned}$$

Setting

$$\mathbf{u}(\mathbf{x}) = \mathbf{D}^{(k)}[\boldsymbol{\mu}](\mathbf{x})$$

the Dirichlet problem becomes

$$-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} = \mathbf{f} \text{ on } \Gamma$$

Setting

$$\mathbf{u}(\mathbf{x}) = \mathbf{D}^{(k)}[\boldsymbol{\mu}](\mathbf{x})$$

the Dirichlet problem becomes

$$-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} = \mathbf{f} \text{ on } \Gamma$$

Note: $\dim(N(-\frac{1}{2} + \mathcal{D}^{(k)})) > 0$ for any k

$(\boldsymbol{\mu}, (-\frac{1}{2} + \mathcal{N}^{(k)})\boldsymbol{\nu}) = (-\frac{1}{2} + \mathcal{D}^{(k)})\boldsymbol{\mu}, \boldsymbol{\nu}) = 0$ for all $\boldsymbol{\mu}$, i.e. $(-\frac{1}{2} + \mathcal{N}^{(k)})\boldsymbol{\nu} = 0$.

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|} \int_\Gamma \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}))\, dS(\mathbf{y})$$

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|}\int_{\Gamma} \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}))\, dS(\mathbf{y})$$

Properties

- $\mathcal{W}[\mathcal{W}[\boldsymbol{\mu}]] = \mathcal{W}[\boldsymbol{\mu}]$
- $\mathcal{W}[1/2 \pm \mathcal{D}^{(k)}] = 0$
- $\mathcal{W}[\mathcal{S}^{(k)}] = 0$

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}))\, dS(\mathbf{y})$$

Properties

- $\mathcal{W}[\mathcal{W}[\boldsymbol{\mu}]] = \mathcal{W}[\boldsymbol{\mu}]$
- $\mathcal{W}[1/2 \pm \mathcal{D}^{(k)}] = 0$
- $\mathcal{W}[\mathcal{S}^{(k)}] = 0$

Adding $\mathcal{W}$ Doesn't change equation for compatible $\mathbf{f}$

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y})\cdot\boldsymbol{\mu}(\mathbf{y}))\,dS(\mathbf{y})$$

Properties

- $\mathcal{W}[\mathcal{W}[\boldsymbol{\mu}]] = \mathcal{W}[\boldsymbol{\mu}]$
- $\mathcal{W}[1/2 \pm \mathcal{D}^{(k)}] = 0$
- $\mathcal{W}[\mathcal{S}^{(k)}] = 0$

Adding $\mathcal{W}$ Doesn't change equation for compatible $\mathbf{f}$

$$-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} + \mathcal{W}\boldsymbol{\mu} = \mathbf{f}$$

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|} \int_\Gamma \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y}) \cdot \boldsymbol{\mu}(\mathbf{y}))\, dS(\mathbf{y})$$

Properties

- $\mathcal{W}[\mathcal{W}[\boldsymbol{\mu}]] = \mathcal{W}[\boldsymbol{\mu}]$
- $\mathcal{W}[1/2 \pm \mathcal{D}^{(k)}] = 0$
- $\mathcal{W}[\mathcal{S}^{(k)}] = 0$

Adding $\mathcal{W}$ Doesn't change equation for compatible $\mathbf{f}$

$$-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} + \mathcal{W}\boldsymbol{\mu} = \mathbf{f}$$

$$\mathcal{W}(-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} + \mathcal{W}\boldsymbol{\mu}) = \mathcal{W}[\mathbf{f}]$$

Nullspace Correction

Definition

$$\mathcal{W}[\boldsymbol{\mu}](\mathbf{x}) = \frac{1}{|\Gamma|}\int_\Gamma \boldsymbol{\nu}(\mathbf{x})(\boldsymbol{\nu}(\mathbf{y})\cdot\boldsymbol{\mu}(\mathbf{y}))\,dS(\mathbf{y})$$

Properties

- $\mathcal{W}[\mathcal{W}[\boldsymbol{\mu}]] = \mathcal{W}[\boldsymbol{\mu}]$
- $\mathcal{W}[1/2 \pm \mathcal{D}^{(k)}] = 0$
- $\mathcal{W}[\mathcal{S}^{(k)}] = 0$

Adding $\mathcal{W}$ Doesn't change equation for compatible $\mathbf{f}$

$$-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} + \mathcal{W}\boldsymbol{\mu} = \mathbf{f}$$

$$\mathcal{W}(-\frac{1}{2}\boldsymbol{\mu} + \mathcal{D}^{(k)}\boldsymbol{\mu} + \mathcal{W}\boldsymbol{\mu}) = \mathcal{W}[\mathbf{f}]$$

$$\mathcal{W}\boldsymbol{\mu} = 0$$

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

Proof outline

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

Proof outline

- *Derive radiation condition for exterior BVPs*

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

---

Proof outline

- *Derive radiation condition for exterior BVPs*
- *Show that layer potentials satisfy this radiation condition*

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

---

Proof outline

- *Derive radiation condition for exterior BVPs*
- *Show that layer potentials satisfy this radiation condition*
- *Show that exterior problems are uniquely solvable subject to the radiation condition*

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

---

Proof outline

- *Derive radiation condition for exterior BVPs*
- *Show that layer potentials satisfy this radiation condition*
- *Show that exterior problems are uniquely solvable subject to the radiation condition*
- *Apply the Fredholm alternative*

Theorem

For simply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + \mathcal{W}$ is not invertible if and only if k^2 is an eigenvalue

Proof outline

- *Derive radiation condition for exterior BVPs*
- *Show that layer potentials satisfy this radiation condition*
- *Show that exterior problems are uniquely solvable subject to the radiation condition*
- *Apply the Fredholm alternative*

Theorem

For multiply-connected Ω, $-\frac{1}{2} + \mathcal{D}^{(k)} + i\eta\mathcal{S}^{(k)} + \mathcal{W}$, with η real and positive, is not invertible if and only if k^2 is an eigenvalue

Fredholm Determinant

Definition of Trace Class

An operator $\mathcal{K}$ defined on a Banach space is trace-class if the sum of its singular values is absolutely convergent. We write $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$ to denote this class.

Fredholm Determinant

Definition of Trace Class

An operator $\mathcal{K}$ defined on a Banach space is trace-class if the sum of its singular values is absolutely convergent. We write $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$ to denote this class.

For $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$, can define the Fredholm determinant

$$\det(I - \mathcal{K}) = \prod_{j=1}^{\infty} (1 - \lambda_j(\mathcal{K}))$$

Fredholm Determinant

Definition of Trace Class

An operator $\mathcal{K}$ defined on a Banach space is trace-class if the sum of its singular values is absolutely convergent. We write $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$ to denote this class.

For $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$, can define the Fredholm determinant

$$\det(I - \mathcal{K}) = \prod_{j=1}^{\infty}(1 - \lambda_j(\mathcal{K}))$$

- If $\mathcal{K}$ trace-class, $\det(I - \mathcal{K}) = 0$ if and only if $I - \mathcal{K}$ is not invertible

Fredholm Determinant

Definition of Trace Class

An operator $\mathcal{K}$ defined on a Banach space is trace-class if the sum of its singular values is absolutely convergent. We write $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$ to denote this class.

For $\mathcal{K} \in \mathcal{J}_1(L_2(\Gamma))$, can define the Fredholm determinant

$$\det(I - \mathcal{K}) = \prod_{j=1}^{\infty} (1 - \lambda_j(\mathcal{K}))$$

- If $\mathcal{K}$ trace-class, $\det(I - \mathcal{K}) = 0$ if and only if $I - \mathcal{K}$ is not invertible
- $\mathcal{D}^{(k)}$ is trace-class, but $\mathcal{S}^{(k)}$ is not!

Theory Recap

Theory Recap

- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues

Theory Recap

- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues
- $f(k) = \det(I - 2\mathcal{D}^{(k)} - 2\mathcal{W})$ is a good, convergent (and analytic) indicator of eigenvalues

Theory Recap

- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues
- $f(k) = \det(I - 2\mathcal{D}^{(k)} - 2\mathcal{W})$ is a good, convergent (and analytic) indicator of eigenvalues
- From Bornemann and Zhao-Barnett

Theory Recap

- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues
- $f(k) = \det(I - 2\mathcal{D}^{(k)} - 2\mathcal{W})$ is a good, convergent (and analytic) indicator of eigenvalues
- From Bornemann and Zhao-Barnett
  - $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2\mathcal{W}_N)$ given by computing determinant of Nyström discretization of operator converges to order of accuracy of quadrature

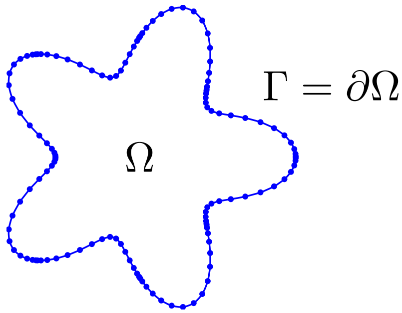
Theory Recap

- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues
- $f(k) = \det(I - 2\mathcal{D}^{(k)} - 2\mathcal{W})$ is a good, convergent (and analytic) indicator of eigenvalues
- From Bornemann and Zhao-Barnett
  - $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2\mathcal{W}_N)$ given by computing determinant of Nyström discretization of operator converges to order of accuracy of quadrature
  - $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2i\mathcal{S}^{(k)}{}_N - 2\mathcal{W}_N)$ works ok

Theory Recap

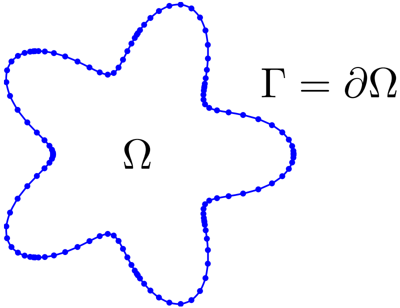
- Invertibility of $I - 2\mathcal{D}^{(k)} - 2\mathcal{W}$ or $I - 2\mathcal{D}^{(k)} - 2i\mathcal{S}^{(k)} - 2\mathcal{W}$ indicates eigenvalues
- $f(k) = \det(I - 2\mathcal{D}^{(k)} - 2\mathcal{W})$ is a good, convergent (and analytic) indicator of eigenvalues
- From Bornemann and Zhao-Barnett
  - $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2\mathcal{W}_N)$ given by computing determinant of Nyström discretization of operator converges to order of accuracy of quadrature
  - $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2i\mathcal{S}^{(k)}{}_N - 2\mathcal{W}_N)$ works ok
  - high-order root finding on f_N produces high accuracy eigenvalues efficiently

Computational Tools



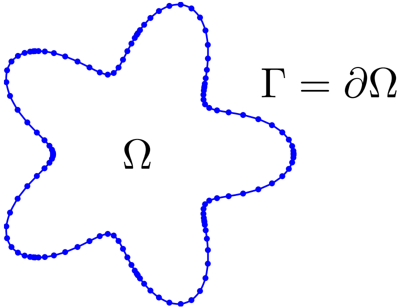
Computational Tools

- Discretization of curves in panels (O'Neil)



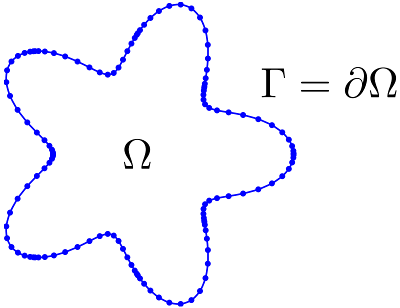
Computational Tools

- Discretization of curves in panels (O'Neil)
- Singular integrals with generalized Gaussian quadrature (Bremer)



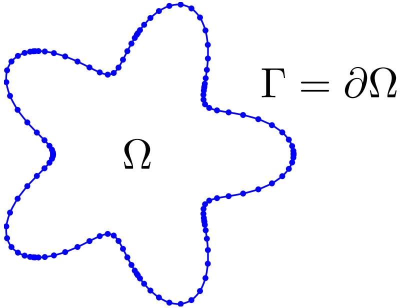
Computational Tools

- Discretization of curves in panels (O'Neil)
- Singular integrals with generalized Gaussian quadrature (Bremer)
- Fast determinant computation using recursive skeletonization (`FLAM` Ho)



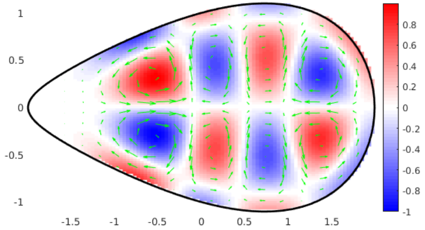
Computational Tools

- Discretization of curves in panels (O'Neil)
- Singular integrals with generalized Gaussian quadrature (Bremer)
- Fast determinant computation using recursive skeletonization (`FLAM` Ho)
- High order root finding with Chebyshev polynomials (`chebfun` Trefethen et al.)



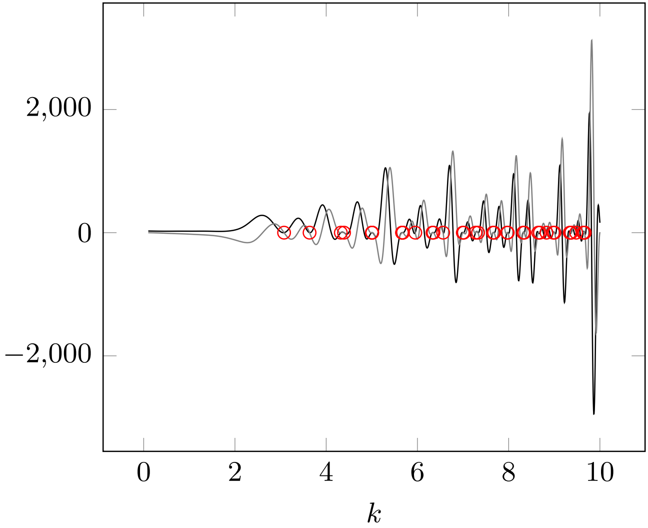
Simply Connected Example

- $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2\mathcal{W}_N)$
- 96 panels
- 16th order Legendre nodes
- approximate $f_N(k)$ by a global chebfun on $[0.1, 10]$ of order 295 (used 513 function evals).
- basic post-processing on roots



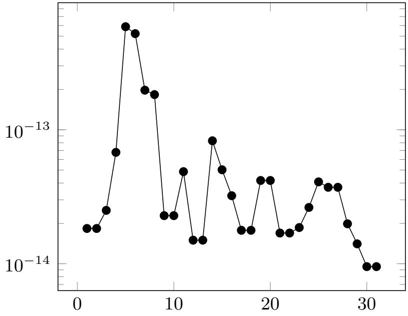
10th eigenfield with vorticity

Determinant

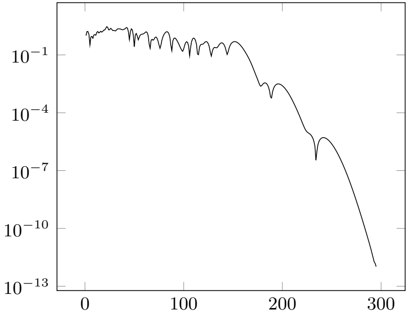


Diagnostics

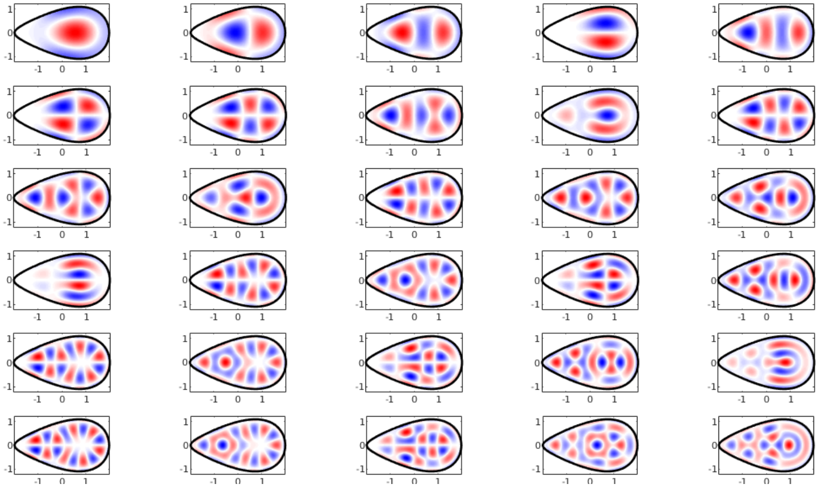
Smallest singular value per root



Chebyshev coefficients

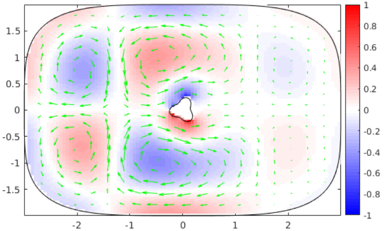


First 30 eigenfunctions (plotting vorticity)



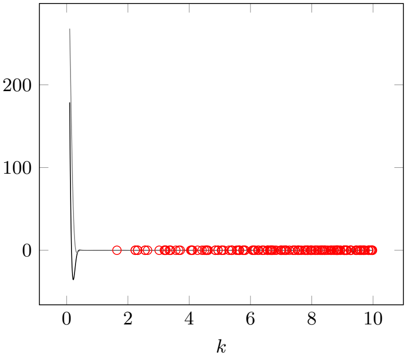
Multiply Connected Example

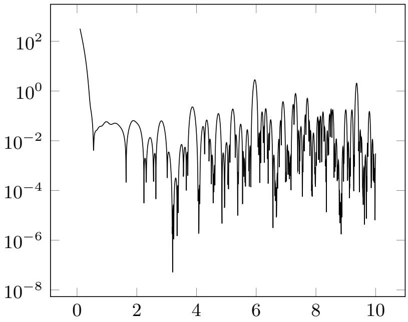
- $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2i\mathcal{S}^{(k)}{}_N - 2\mathcal{W}_N)$
- 192 panels
- 16th order Legendre nodes
- approximate $f_N(k)$ by a global chebfun on $[0.1, 10]$ of order 1024.



10th eigenfield with vorticity

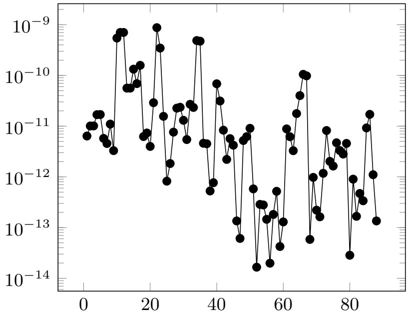
Determinant



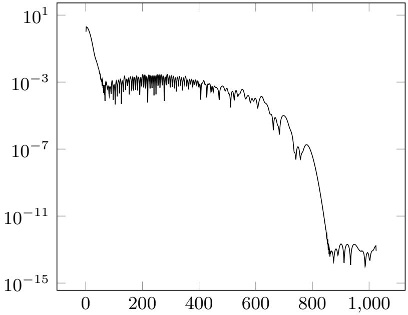


Diagnostics

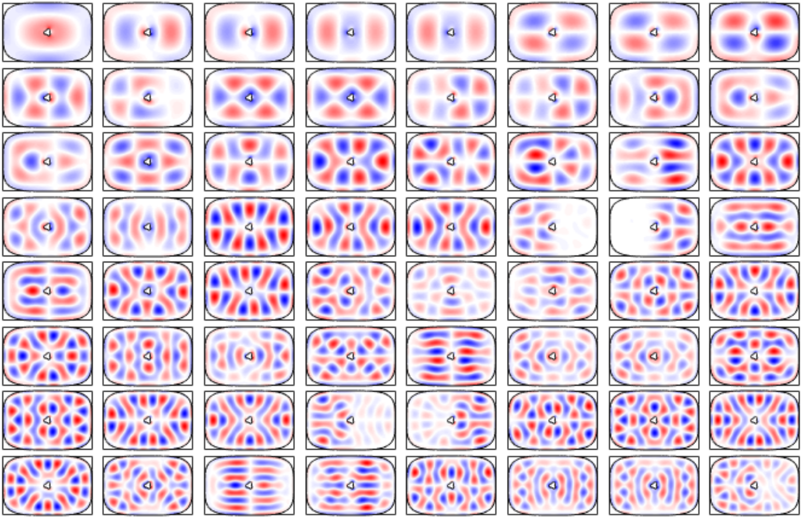
Smallest singular value per root



Chebyshev coefficients

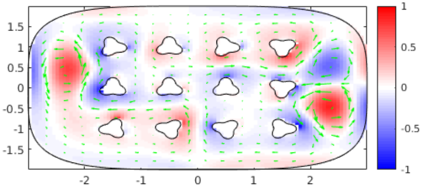


First 64 eigenfunctions (plotting vorticity)



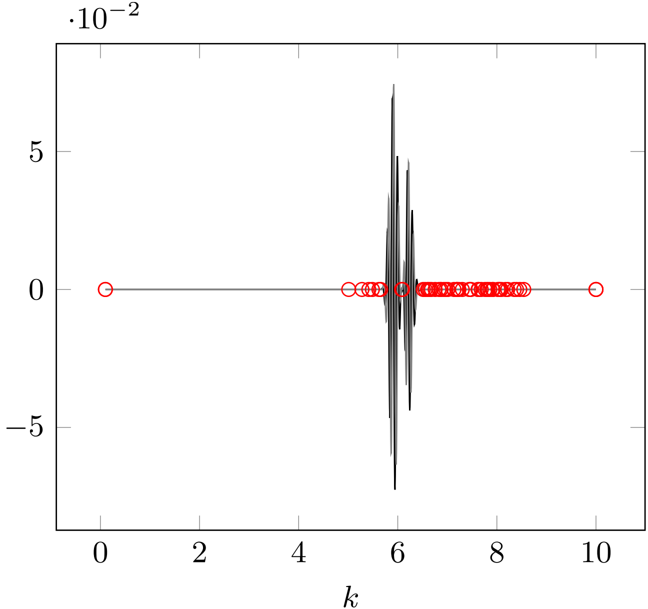
Example with More Holes

- $f_N(k) = \det(I_N - 2\mathcal{D}^{(k)}{}_N - 2i\mathcal{S}^{(k)}{}_N - 2\mathcal{W}_N)$
- 368 panels
- 16th order Legendre nodes
- approximate $f_N(k)$ by a piecewise chebfuns on $[j, j+1]$ for $j = 1, \ldots, 8$ of order 51-256 (used 65 to 257 function evals).
- basic post-processing on roots

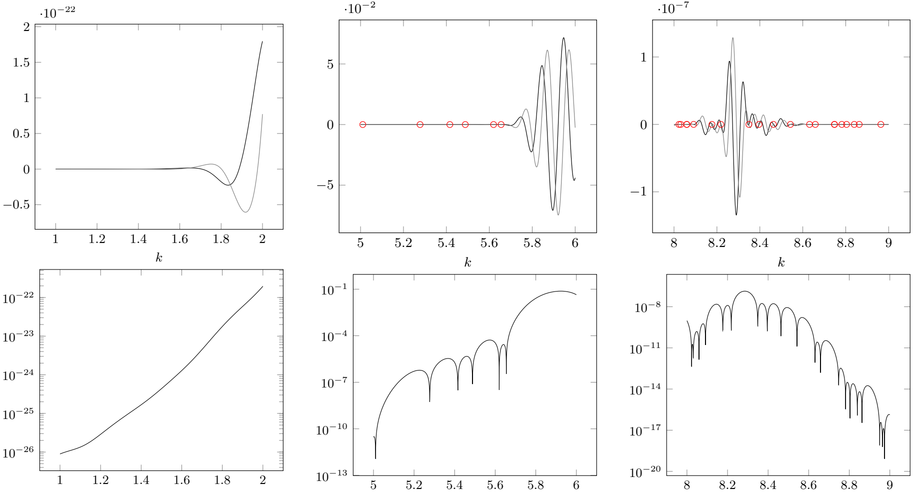


10th eigenfield with vorticity

Determinant (global fit — bad idea)

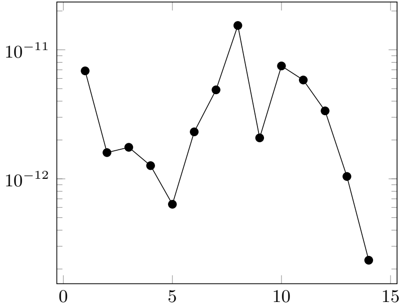


Determinant (piecewise — works ok)

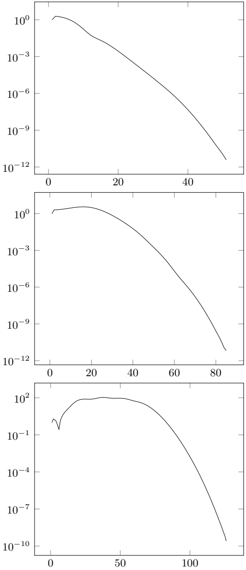


Diagnostics

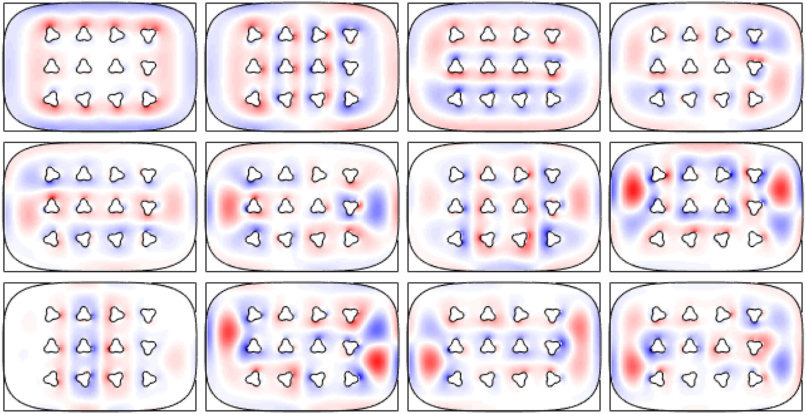
Smallest singular value per root



Chebyshev coefficients



First 12 eigenfunctions (plotting vorticity)



Conclusions and future directions

- The Fredholm determinant framework is a robust approach to computing eigenvalues for the Stokes equation
- Integral equation tools are reasonably mature
- Can be extended easily to the "buckling" problem
- Look into $I - 2\mathcal{D}^{(k)} - 2i(\mathcal{S}^{(k)})^2 - 2\mathcal{W}$ formulation (implementation issue)
- Work on the corner problem
- Generalizations?