Name:

MATH 335-002: Homework #2 Solutions

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Instructions

- This assignment is due in-class Tuesday February 12th.
- Please put your full name in the upper right hand corner of each page of your solutions.
- Please show your work and be as neat as possible.
- Submitting typed/ LaTeX-based solutions is encouraged (but not required!) The LaTeX source for this homework is available on the course website.
- Note that for the exercises from P.C. Matthews' "Vector Calculus" (Corrected Edition, 2000), the answers are in the back of the book but you are still expected to write out an answer. For these problems, it is best to attempt the problem first and then check against the solution.

Outline

In this assignment, we practice evaluating integrals along curves, over surfaces, and throughout volumes by hand. We will also explore some of the physical and geometrical interpretations of these quantities.

1 Exercises from Matthews

Please complete exercises 2.1, 2.2, 2.5, 2.7, 2.9, and 2.11 from the textbook (2 pts each). They are provided below for convenience.

- 2.1: Consider the vector field $\vec{F}$ defined by $\vec{F}(x, y, z) = (5z^2, 2x, x + 2y)$. Evaluate the line integral

  $$\int_C \vec{F} \cdot d\vec{r} \ ,$$

  where C is the curve $\vec{r}(t) = (t, t^2, t^2)$ for $0 \leq t \leq 1$.
- 2.2: Evaluate the line integral for the same field as question 2.1 but over the curve C given by the straight line joining $(0, 0, 0)$ with $(1, 1, 1)$. Is $\vec{F}$ a conservative vector field?
- 2.5: Evaluate the flux of $\vec{u} = (xy, x, x + y)$ over the surface S defined by $z = 0$ with $0 \leq x \leq 1$ and $0 \leq y \leq 2$, with the normal $\vec{n}$ take in the positive z direction.
- 2.7: The surface S is defined to be that part of the plane $z = 0$ lying between the curves $y = x^2$ and $x = y^2$. Find the flux of $\vec{u} = (z, xy, x^2)$ with $\vec{n} = (0, 0, 1)$.
- 2.9: Find the volume integral of the scalar field $\phi = x^2 + y^2 + z^2$ over the region V specified by $0 \leq x \leq 1$, $1 \leq y \leq 2$, $0 \leq z \leq 3$.
- 2.11: A circular pond with radius 1m and a maximum depth of 1m has the shape of a paraboloid, so that its depth z is $z = 1 - x^2 - y^2$. What is the total volume of the pond? How does this compare with the case where the pond has the same radius and depth but has the shape of a hemisphere?

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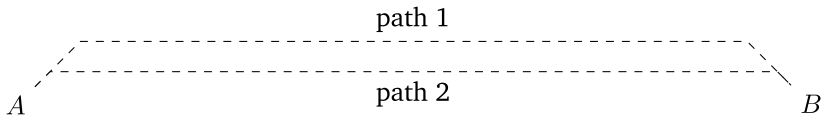
Solution

Solutions to these problems are available in the back of the textbook.

2 Other Exercises

Question 1 (5 pts)

Consider the motion of an "airplane" between two points A at $(0, 0, 0)$ and B at $(500, 0, 1)$. This is drawn in the (x, z) plane below.



As a simple model, we will take the external forces acting on the airplane to be the force due to gravity and the force of drag. The airplane works against these forces using thrust and (indirectly) lift. We take the force due to gravity to be $\vec{F}_g(x, y, z) = (0, 0, -\frac{1}{100})$ (Note: gravity is a little more complicated than this). Because the plane is flying primarily in one direction and the effect of drag depends on the density of the air, we model the drag as a force in the opposite direction with strength which depends on the height of the airplane. We set $\vec{F}_d(x, y, z) = (-\frac{1}{100}(1 - z/60), 0, 0)$ (Note: drag is a lot more complicated than this).

Compute the work done by each of the gravitational and drag forces separately for both of the trajectories below

- $(0, 0, 0)$ to $(30, 0, 30)$, then $(30, 0, 30)$ to $(471, 0, 30)$, then $(471, 0, 30)$ to $(500, 0, 1)$
- $(0, 0, 0)$ to $(10, 0, 10)$, then $(10, 0, 10)$ to $(491, 0, 10)$, then $(491, 0, 10)$ to $(500, 0, 1)$

Is drag conservative?

Solution

We first parameterize each of the paths. The first segment of path 1 is given by $\vec{r}(t) = (t, 0, t)$ for $0 \leq t \leq 30$, the second segment by $\vec{r}(t) = (30 + t, 0, 30)$ for $0 \leq t \leq 441$, and the third segment by $\vec{r}(t) = (471 + t, 0, 30 - t)$ for $0 \leq t \leq 29$. We have that $d\vec{r}/dt = (1, 0, 1)$, $(1, 0, 0)$, and $(1, 0, -1)$ for each of the first, second, and third segments, respectively. The first segment of path 2 is given by $\vec{r}(t) = (t, 0, t)$ for $0 \leq t \leq 10$, the second segment by $\vec{r}(t) = (10 + t, 0, 10)$ for $0 \leq t \leq 481$, and the third segment by $\vec{r}(t) = (491 + t, 0, 10 - t)$ for $0 \leq t \leq 9$. We have that $d\vec{r}/dt = (1, 0, 1)$, $(1, 0, 0)$, and $(1, 0, -1)$ for each of the first, second, and third segments, respectively.

It is then straightforward to compute the work done over each path by adding the work over each segment. We have

$$
\begin{aligned}
\int_{\text{path 1}} \vec{F}_g \cdot d\vec{r} &= \int_0^{30} -\frac{1}{100}\,dt + \int_0^{441} 0\,dt + \int_0^{29} \frac{1}{100}\,dt \\
&= -\frac{1}{100} \\
\int_{\text{path 1}} \vec{F}_d \cdot d\vec{r} &= \int_0^{30} -\frac{1}{100}(1-t/60)\,dt + \int_0^{441} -\frac{1}{100}(1-1/2)\,dt + \int_0^{29} -\frac{1}{100}(1-(30-t)/60)\,dt \\
&= -\frac{1}{100}(30 - 900/120 + 441/2 + 29/2 + 841/120) \\
&= -\frac{31741}{12000} \approx -2.64 \\
\int_{\text{path 2}} \vec{F}_g \cdot d\vec{r} &= \int_0^{10} -\frac{1}{100}\,dt + \int_0^{481} 0\,dt + \int_0^{9} \frac{1}{100}\,dt \\
&= -\frac{1}{100} \\
\int_{\text{path 2}} \vec{F}_d \cdot d\vec{r} &= \int_0^{10} -\frac{1}{100}(1-t/60)\,dt + \int_0^{481} -\frac{1}{100}(1-1/6)\,dt + \int_0^{9} -\frac{1}{100}(1-(10-t)/60)\,dt \\
&= -\frac{1}{100}(10 - 100/120 + 2405/6 + 15/2 + 81/120) \\
&= -\frac{16727}{4000} \approx -4.18
\end{aligned}
$$

Note that drag is not conservative because we can obtain two different values for the work by taking different paths between the same two points.

Question 2 (5 pts)

A uniform fluid that flows vertically downward is described by the vector field $\vec{u} = (0, 0, -1)$. Find the total flux through the surface with height $z = \sqrt{x^2 + y^2}$ where x and y range through the points in the unit square $0 \leq x, y \leq 1$ (you may take either convention for the normal direction $\vec{n}$).

Solution

Note that we can parameterize the surface by $\vec{r}(x, y) = (x, y, \sqrt{x^2 + y^2})$ over $0 \leq x, y \leq 1$. Then $\partial\vec{r}/\partial x = (1, 0, x/\sqrt{x^2 + y^2})$ and $\partial\vec{r}/\partial y = (0, 1, y/\sqrt{x^2 + y^2})$. We need the cross product of these to use the formula for flux given a parameterization. We have

$$
\frac{\partial\vec{r}}{\partial x} \times \frac{\partial\vec{r}}{\partial y} = \begin{vmatrix} \vec{e_1} & \vec{e_2} & \vec{e_3} \\ 1 & 0 & x/\sqrt{x^2+y^2} \\ 0 & 1 & y/\sqrt{x^2+y^2} \end{vmatrix} = \begin{pmatrix} -x/\sqrt{x^2+y^2} \\ -y/\sqrt{x^2+y^2} \\ 1 \end{pmatrix} \tag{1}
$$

The flux is then

$$
\begin{aligned}
\iint_S \vec{u} \cdot \vec{n}\,dS &= \int_0^1 \int_0^1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -x/\sqrt{x^2+y^2} \\ -y/\sqrt{x^2+y^2} \\ 1 \end{pmatrix} dx\,dy \\
&= \int_0^1 \int_0^1 -1\,dx\,dy \\
&= -1
\end{aligned}
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Question 3 (5 pts)

Find the volume of the tetrahedron with vertices $(0,0,0)$, $(1,1,0)$, $(0,2,0)$, and $(0,1,1)$. You may use any method we've covered so far but show your work.

Solution (scalar triple product formula)

The three vectors $(1,1,0)$, $(0,2,0)$, and $(0,1,1)$ define the tetrahedron. We can obtain its volume from the scalar triple product

$$V = \frac{1}{6}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\1\end{pmatrix}\right| = \frac{1}{6}\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\0\end{pmatrix}\right| = \frac{1}{3}$$

Solution (integration)

It is much trickier to obtain the volume by integration. We can integrate over the tetrahedron by iterated integration. In the $z=0$ plane, we have that x runs from 0 to 1 and y runs from the line $y=x$ to the line $y=2-x$. In z, the upper bound is determined by one of two planes. For $y\leq 1$, the plane is the one through $(0,1,1)$, $(1,1,0)$, and $(0,0,0)$. This plane has normal $(0,1,1)\times(1,1,0)=(-1,1,-1)$ so that the equation is $-x+y-z=0$ or $z=y-x$. For $y>1$, the plane is the one through $(0,1,1)$, $(1,1,0)$, and $(0,2,0)$. This plane has normal $(-1,0,1)\times(-1,1,0)=(-1,-1,-1)$ so that the equation for the plane is $-x-y-z=-2$ or $z=2-x-y$. The integral is then straightforward

$$\begin{aligned}
\iiint_V 1\,dV &= \int_0^1\int_x^1\int_0^{y-x} 1\,dz\,dy\,dx + \int_0^1\int_1^{2-x}\int_0^{2-x-y} 1\,dz\,dy\,dx \\
&= \int_0^1\int_x^1 (y-x)\,dy\,dx + \int_0^1\int_1^{2-x}(2-x-y)\,dy\,dx \\
&= \int_0^1 \frac{x^2}{2} - x + \frac{1}{2}\,dx + \int_0^1 (2-x)(1-x)\frac{1}{2} - \frac{(2-x)^2}{2}\,dx \\
&= 2\left[\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$