

Study of Quantum Chromodynamics and Calculation of ρ Mass in Four Dimensions and Large N

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Spring 2008

ABSTRACT: Quantum Chromodynamics is the theory that explains with great accuracy the interaction between the fundamental quark particles and the interaction between them and the gluon field. The gluon field not only interacts with the quarks, but also interacts with itself. This leads to interesting results when studying a composite particle such as a meson which is made of two quarks. In addition to the additional energy picked up by the particle due to parallel transport theory, the gluon-gluon interaction also provides the particle with an additional source of internal energy. Taking this energy difference into consideration, our aim then is to determine the mass of the ρ meson at zero quark mass in a four dimensional space-time lattice where the number of color fields is taken to the limit of infinity.

KEYWORDS: Quantum Chromodynamics, Parallel Transport Theory, Quarks, Green's Function, Lattice.

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1. INTRODUCTION

The theory of protons, neutrons and electrons as being the fundamental building blocks of the universe pervaded the scientific community and the rest of the world until the introduction of the theory of quarks back in 1964 by Gell-Mann and Ne’eman. Their eightfold way theory in groups allowed for the construction of these new fundamental particles [1]. Advancements in accelerator physics led to the discovery of the up, down and charm quarks, each with its anti-particle complement. Later studies led to the discovery of the strange, top and bottom quarks. The theory, based on the $SU(3)$ theory of groups, gave rise to the color fields, which is analogous to the electromagnetic field in classical electromagnetism. However, unlike in electromagnetism, there exists three flavors of color fields duly named “red, blue and green.”

Now, unfortunately, quarks have never been seen in isolation. Quarks are always found in groups of three or two in such a way that their overall color charge is neutral. The reason for this isolation is explained via a non-abelian gauge field created by a particle known as the gluon. Quarks interact with other quarks via gluon exchange, similar to how charges in electromagnetism interact via photons. However the gluons interact with themselves resulting in a force between the quarks which is linear in distance “even for asymptotically large separations” [1]. This is the exact opposite of the properties exhibited by the coulomb and gravitational forces which are inversely proportional to the distance squared. This is what scientists call the Strong Force. From the Strong Force a new field in physics was developed dubbed “Quantum Chromodynamics: The study of quarks and their interaction between the strong force gluon carriers.”

From this a flux tube scenario can be utilized in order to confine the quarks. The tube carries a finite energy per unit length. The tube always begins with the quarks and ends at the quarks. This picture is similar to stretching a rubber band and watching it revert back to its original form; no matter how far you stretch (until the breaking point, which is non-existent in the world of quarks) the band always returns to its original form [1]. Thus, these groupings of quarks lead to the physical particles seen today. Baryons, such as protons and neutrons are due to three quark groupings and mesons are due to groups of two quarks.

Mesons, especially the ρ particle, which is composed of one up quark and one down anti-quark, will be studied in depth in this study. It becomes an interesting project to understand the relationship between the individual quarks and the system they create as a whole. In particular, the system mass dependence on the quark masses is where the main focus of this project delves into. One would think that as the internal mass of the system is taken to zero, the same would occur for the meson. This is the case for the π mass which has an iso-spin of 0 [2]. However this may not be the case for the ρ mass and by theory, it is predicted that the mass of this meson is non-zero when the mass of its quarks are taken to zero. The interesting aspect of this calculation would be to allow for not only 3 types of color gauge fields, but for an infinite amount of these colors. In order to properly solve this problem, methods such as the Feynman Path Integral method, and Green’s correlation method are required. Since these topics are studied at the graduate level, a large chunk

of the project involved a full understanding of these theoretical models and the methods before an actual calculation of the ρ mass was possible.

2. FEYNMAN PATH INTEGRAL FORMALISM

Rather than approach this as a matrix - operator calculation in quantum mechanics, we approach it through the famous path integral method developed by Dr. Richard P. Feynman in 1948 [3]. The time evolution of the particle through every possible path in existence is of interest, and consequently it is shown that the the particle follows a set of most probable paths governed by Hamilton's Least Action principle and Euler Lagrange equations of classical mechanics, and regulated by a factor of $\frac{1}{\hbar}$. By employing Green's function on the 1 dimensional solution to the Path Integral, our end will be to derive the energy of the particle in any state from only its ground state wavefunction.

2.1 Quantum Mechanics and Dirac Notation

Let us now set up the problem of learning about a particle moving from one point in space to another. We will denote the initial and final positions by q and q' respectively. We proceed under the assumption of the Heisenberg principle of quantum mechanics where the probability of finding the particle at q' after a certain time evolution is given by the Schrödinger equation:

$$\begin{aligned}
 H\Psi(x, t) &= -i\hbar \frac{\partial \Psi}{\partial t} \\
 \frac{i}{\hbar} \int_t^{t'} dt &= \int_t^{t'} \frac{d\Psi}{\Psi} \\
 \frac{i}{\hbar} H(t' - t) &= \ln \left(\frac{\Psi(t')}{\Psi(t)} \right) \\
 \Psi(t') &= e^{\frac{i}{\hbar} H(t' - t)} \Psi(t) \\
 \langle q' | &= e^{\frac{i}{\hbar} H(t' - t)} | q \rangle \\
 \langle q'; t' | q; t \rangle &= \langle q' | e^{-iH(t'-t)} | q \rangle [3]
 \end{aligned} \tag{2.1}$$

The term $U(t) = e^{-iH(t'-t)}$ is the unitary time evolution operator, allowing the wavefunction of the particle to evolve from some initial time t , to a final time t' . Before we can

proceed any further, we however need to understand the underlying meaning of 2.1. We can first think of each ‘ q ’ as an index of ψ^q . Thus ψ^q is a base function in some abstract functional space with indices $q = 1, 2, \dots, \infty$ and $\psi^q \in \mathcal{L}$, where \mathcal{L} is a special function space. From this we know that any operator operating on an element of \mathcal{L} will transform the original basis function through either a rotation, translation, etc. Thus the new functional basis can be represented as a linear combination of a basis functions in some new functional space \mathcal{L}^l .

$$\hat{H}\psi^q = \sum_{l=0}^{\infty} a_l^q \psi^l \quad (2.2)$$

By 2.2 we see that ψ^l is a new basis function in \mathcal{L}^l and a_l^q carries the weight of $\hat{H}\psi^q$ for a particular ψ^l . More specifically now, we can operate on ψ^q with the unitary operator:

$$e^{-iH(\Delta t)}\psi^q = \sum_{q'=0}^{\infty} a_{q'}^q(\Delta t) \psi^{q'} \quad (2.3)$$

The time dependence in the above equation is held by $a_{q'}^q$ since the new basis function needs to be time independent. Now we can express 2.3 more compactly using Dirac Notation

$$e^{-iH(\Delta t)}|q\rangle = \sum_{l=0}^{\infty} a_l^q(\Delta t)|l\rangle \quad (2.4)$$

which basically uses q' to denote all the basis functions of $\psi^{q'}$. So now we can determine how the particle will progress through time to its final point in space and determine how it’s route will appear to the observer. Since ψ^q is a set of basis functions, we can generalize and consider the space in which the particle travels as a general space of functions. Thus we can divide Δt into small time intervals in accordance with:

$$\delta t = \frac{(t' - t)}{n} \quad (2.5)$$

where n is the number of time segments. A pictorial representation of the scenario is shown in Figure 1. From this, the transition from $q \in \mathcal{L}$ at the initial time t to a second point $q_1 \in \mathcal{L}_1$ at a time $t + \delta t$, is determined by the following notation which determines the probability amplitude of the particle at q through a time shift to the point q_1 . By applying the rules of bra-ket notation, we see that:

$$\langle q_1 | e^{-iH\delta t} | q \rangle = \langle q_1 | \sum_{q_1=0}^{\infty} a_{q_1}^{(q)}(\delta t) | q \rangle \quad (2.6)$$

$$= \sum_{q_1=0}^{\infty} \langle q | q_1 \rangle a_{q_1}^{(q)}(\delta t) \quad (2.7)$$

$$= a_{q_1}^{(q)}(\delta t) \quad (2.8)$$

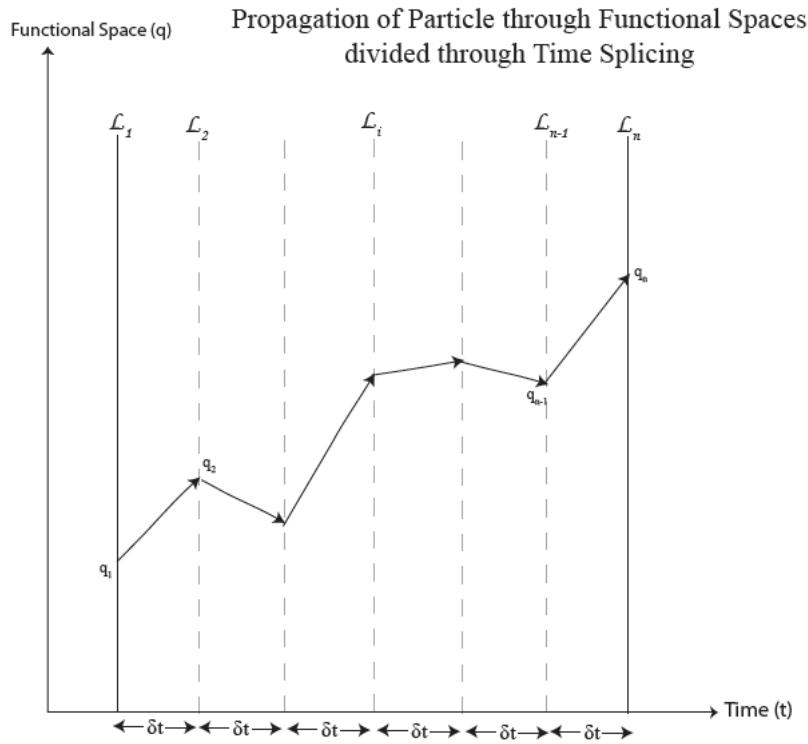


Figure 1: Particle amplitude at time spliced intervals in functional space.

Here, we have used $\langle q | q_1 \rangle = \delta_{qq_1}$. So we see that 2.8 gives the evolution of the amplitude of the wave function originally at a time t in the function space of \mathcal{L} and at some point q in that space, moved to a later time $t + \delta t$ into the function space of \mathcal{L}^1 at some point in that

space denoted by q_1 . In order to determine the total expectation value of the amplitude of the wave function from one space to another, we must take into account the probability of finding the particle at some other point in \mathcal{L}^1 . Thus if we continue by following 2.6, then we will need to add through all the possible paths taken by the particle from point q in \mathcal{L} to a point q_1 in \mathcal{L}^1 at any of the listed points of $q_1 = 1, 2, 3, \dots, \infty$. This will then lead to an integral over all time varied amplitudes produced from the transfer between the two spaces, with respect to dq_1 . Similarly the same process occurs when going from $t + \delta t$ to $t + 2\delta t$ where the same path integral over the new function spaces take place to determine the probability amplitude at some point in the new function space. This can be visualized graphically from Fig.1 above. Now, since probabilities add up by multiplying them together these integrals will then lead to:

$$\langle q' | e^{-iH(t-t')} | q \rangle = \int_{-\infty}^{\infty} dq_1 dq_2 \dots dq_{n-1} \langle q' | e^{-iH\delta t} | q_{n-1} \rangle \langle q_{n-1} | e^{-iH\delta t} | q_{n-2} \rangle \dots \langle q_1 | e^{-iH\delta t} | q \rangle \quad (2.9)$$

where the integration variables dq_1, \dots, dq_{n-1} refer to the Hilbertian spaces of $\mathcal{L}^1 \rightarrow \mathcal{L}^{n-1}$ thereby integrating over all the base functions in their corresponding spaces.

2.2 Conversion from Dirac Quantum Mechanics to Integral Formalism

So now we have a formula for determining the probability amplitude of the particle from point $q \rightarrow q'$ between an interval from time $t \rightarrow t'$. So we go ahead and make δt from 2.5 infinitesimally small, thereby allowing for the following approximation:

$$\langle q' | e^{-iH(t-t')} | q \rangle = \langle q' | [1 - iH(P, Q)\delta t] | q \rangle + O(\delta t)^2 \quad (2.10)$$

where $H(P, Q) = \frac{p^2}{2m} + V(Q)$ is just the Hamiltonian. So replacing this into 2.10 and ignoring some of the extra terms we find that we get:

$$\langle q | H(P, Q) | q \rangle = \langle q' | \frac{p^2}{2m} + V(Q) | q \rangle \quad (2.11)$$

$$= \langle q' | \frac{p^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \delta_{qq'} \quad (2.12)$$

where we taken $V(\frac{q+q'}{2})$ to be a constant and therefore $\langle q'|q\rangle = \delta_{qq'} = \frac{1}{2\pi} \int e^{ip(q'-q)} dp$. In addition, we can use the fact that $\int \frac{dp}{2\pi} |p\rangle \langle p| = 1$ and that $\langle q'|q\rangle = \frac{1}{2\pi} \int dp e^{ip(q'-q)}$. So now manipulating 12 using the new tricks we have:

$$\langle q' | \frac{p^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \delta_{qq'} = \int \frac{dp}{2\pi} \langle q' | p \rangle \langle p | \frac{p^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \int \frac{dp}{2\pi} e^{ip(q'-q)} \quad (2.13)$$

where we can now use the identity that $\langle q'|p\rangle = \delta_{q'p} = e^{ipq}$ to show that 2.13 can be written as:

$$\langle q' | \frac{p^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \delta_{qq'} = \int \frac{dp}{2\pi} e^{ip(q'-q)} \langle p | \frac{p^2}{2m} | q \rangle + V\left(\frac{q+q'}{2}\right) \int \frac{dp}{2\pi} e^{ip(q'-q)} \quad (2.14)$$

$$= \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[\frac{P^2}{2m} + V\left(\frac{q+q'}{2}\right) \right] \quad (2.15)$$

Now we notice that the above equation is merely missing an extra factor of $i\delta t$ and a subtraction from 1 in order to have the form of 2.10. Therefore we can safely say that:

$$\langle q' | e^{-iH(t-t')} | q \rangle = \int \frac{dp}{2\pi} e^{ip(q'-q)} \left[1 - i\delta t \left[\frac{P^2}{2m} + V\left(\frac{q+q'}{2}\right) \right] \right] \quad (2.16)$$

$$= \int \frac{dp}{2\pi} e^{ip(q'-q)} e^{i\delta t H(p, \frac{q+q'}{2})} \quad (2.17)$$

where we have basically used a reverse Taylor Expansion on the first equation (16) on the second exponential and by grouping the kinetic and potential into the Hamiltonian, this become a very nice and compact way to express the the expectation value of the particle a certain time δt away from $q \rightarrow q'$. Now plugging in 2.17 into 2.9 we obtain:

$$\begin{aligned} \langle q' | e^{-iH(t-t')} | q \rangle &\simeq \int \left(\frac{dp_1}{2\pi} \right) \dots \left(\frac{dp_n}{2\pi} \right) \int dq_1 \dots dq_{n-1} \times \\ &\times \exp \left[i \sum_{i=1}^n \left(p_i (q_i - q_{i-1}) - \delta t H \left(p_i, \frac{q_i + q_{i-1}}{2} \right) \right) \right] \end{aligned} \quad (2.18)$$

The left hand side can be formally equated by taking the limit as n goes to infinity of the right hand side, thereby eliminating the discontinuity between the time line and making an infinite number of functional spaces:

$$\begin{aligned}
\langle q' | e^{-iH(t-t')} | q \rangle &= \lim_{n \rightarrow \infty} \int \left(\frac{dp_1}{2\pi} \right) \dots \left(\frac{dp_n}{2\pi} \right) \int dq_1 \dots dq_{n-1} \times \\
&\quad \times \exp \left[i \sum_{i=1}^n \delta t p_i \left(\frac{q_i - q_{i-1}}{\delta t} \right) - H \left(p_i, \frac{q_i + q_{i-1}}{2} \right) \right] \\
&= \int \left[\frac{dp dq}{2\pi} \right] \exp \left[i \int_t^{t'} dt (p \dot{q} - H(p, q)) \right]
\end{aligned}$$

So now after manipulating the equations to our liking, we have been able to discern the correlation between the various functional spaces and the time dependent amplitude between t to t' into integrals which are dependent only on the position and momentum of the function. So from this we can try to determine if we can solve these integrals and understand what we can pull out from them.

2.3 Integral Formalism and its Dependence on Hamilton's Least Action Principle

From the last section, since it was stated that the equation on the right hand side was a function of two variables, we can also see that there are two main integrals; one which is a function of mainly the position and the other a function of momentum. Here we decide to compute the momentum part of the integral, taking only the real part into consideration. Now we will need to use the following integral in order to compute this:

$$\int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-ax^2+bx} = \frac{1}{\sqrt{4\pi a}} e^{b^2/4a} \quad (2.19)$$

We can begin by taking 2.10, and separating the Hamiltonian

$$H \left(p_i, \frac{q_i + q_{i-1}}{2} \right)$$

into its explicit equation of $H = \frac{p_i^2}{2m} + V(\frac{q_i + q_{i-1}}{2})$. Thus replacing this into 2.10 and taking $i\delta t$ to be real, we can write the integrand as:

$$e^{\left(i\delta t \left[p_i \left(\frac{q_i - q_{i-1}}{\delta t} \right) - \frac{p_i^2}{2m} - V \left(\frac{q_i - q_{i-1}}{2} \right) \right] \right)} = e^{\left(-i \frac{p_i^2}{2m} \delta t + ip_i (q_i - q_{i-1}) \right)} e^{\left(-iV \left(\frac{q_i - q_{i-1}}{2} \right) \right)} \quad (2.20)$$

So now we can write the momentum part of 2.10 as the following:

$$\int \frac{dp_i}{2\pi} \exp \left(-i \frac{p_i^2}{2m} \delta t + ip_i(q_i - q_{i-1}) \right) = \left(\frac{m}{2\pi i \delta t} \right)^{1/2} \exp \left(\frac{im(q_i - q_{i-1})^2}{2\delta t} \right) \quad (2.21)$$

Now taking 2.21, we can insert it back into 2.19:

$$\langle q' | e^{-iH(t-t')} | q \rangle = \lim_{n \rightarrow \infty} \left(\frac{m}{2\pi i \delta t} \right)^{1/2} \int \prod_{i=1}^{n-1} dq_i \exp \left[i \sum_{i=1}^n \delta t \frac{m}{2} \left(\frac{q_i - q_{i-1}}{\delta t} \right)^2 - V \right] \quad (2.22)$$

where the term in the exponential of 2.22 can be reduced to look like that of the well known Lagrangian equation:

$$i \sum_{i=1}^n \delta t \left[\frac{m}{2} \left(\frac{q_i - q_{i-1}}{\delta t} \right)^2 - V \right] = i \sum_{i=1}^n \delta t \left[\frac{m}{2} [\Delta \dot{q}]^2 - V \right] \quad (2.23)$$

$$= i \sum_{i=1}^n \delta t \left[\frac{m \dot{q}^2}{2} - V \right] \quad (2.24)$$

Now we see that if we take $n \rightarrow \infty$, the summation turns into an integral leaving us with:

$$\langle q' | e^{-iH(t-t')} | q \rangle = N \int [dq] e^{i \int_t^{t'} dt \left[\frac{m}{2} \dot{q}^2 - V(q) \right]} \quad (2.25)$$

$$\langle q'; t' | q; t \rangle = N \int [dq] e^{i \int_t^{t'} L(q, \dot{q}) dt} \quad (2.26)$$

where 2.26 is just a simplified form of 2.25 in which the substitution of $L = \frac{m}{2} \dot{q}^2 - V(q)$ gives us the Lagrangian from classical mechanics and N is the normalization factor.

The result obtained in 2.26 is what Feynman derived when formulating his theory of quantum electrodynamics. However we need to be able to understand this result if any extra information will ever be obtained. We see that the integral in the exponential is the mathematical interpretation of Hamilton's Principle which states that “*Of all the possible paths along which a dynamical system may move from one point to another within a specified time interval... the actual path followed is that which minimizes the time integral of the [Lagrangian].*”[4]

Since we have already introduced Hamilton's Principle, we can easily define the action as being:

$$iS = i \int_t^{t'} \left(\frac{m \dot{q}^2}{2} - V(q) \right) dt \quad (2.27)$$

A factor of $\frac{1}{\hbar}$ in the exponential and make the following substite $t = i\tau$, so we can take the exponential in 2.25 and manipulate it as such:

$$\begin{aligned}
 \frac{i}{\hbar} \int_t^{t'} dt \left[\frac{m}{2} \dot{q}^2 - V(q) \right] &= \frac{1}{\hbar} \int_t^{t'} idt \left[\frac{m}{2} \left(\frac{dq}{dt} \right)^2 - V(q) \right] \\
 &= -\frac{1}{\hbar} \int_{\tau}^{\tau'} i(id\tau) \left[\frac{m}{2} \left(\frac{dq}{id\tau} \right)^2 - V(q) \right] \\
 &= -\frac{1}{\hbar} \int_{\tau}^{\tau'} -d\tau \left[-\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 - V(q) \right] \\
 &= -\frac{1}{\hbar} \int_{\tau}^{\tau'} H d\tau
 \end{aligned} \tag{2.28}$$

producing a simplified and clearer version of the action.

2.4 Partition Function and Path Integral Method

So now we can introduce a factor of $-\frac{1}{\hbar} S_{min}$ to the exponential in 2.28:

$$e^{-\frac{1}{\hbar} S_{min}} \int [dq] e^{-\frac{1}{\hbar} (\int_{\tau_i}^{\tau_f} H d\tau - S_{min})}$$

where S_{min} is defined to be the minimum action able to satisfy Hamilton's principle. Thus it is the path most likely taken by the particle to go from $q \rightarrow q'$ in time $\Delta\tau$. But we see that for any significant deviations away from S_{min} , 2.4 is immediately nullified by the inverse \hbar factor. So the contribution of these out lier "paths" is removed due to the $\frac{1}{\hbar}$ which immediately removes the chances of the particle's amplitude to be found in any of the largely deviated paths.

Now there is obviously an S_{min} for which $\langle q'; t' | q; t \rangle$ has the greatest value. However, there are most certainly paths (S) for which the amplitude is not zero, and falls below the maximum at the same time. Thus there is no one path that the particle will take, but a probabilistic set of paths. We can then imagine the particle as fluctuating between a set of probabilistic paths. In turn a new function is defined Z and is known in thermodynamics as the partition function,

$$Z = \int [dq] e^{-\frac{1}{\hbar} \int_{\tau_i}^{\tau_f} H d\tau} \tag{2.29}$$

which is similar to the partition function in thermodynamics. The units of \hbar give rise to the same units of $k_b T$ where k_b is the Boltzmann's constant and T is the temperature, thereby indicating a similarity between thermal fluctuations and quantum fluctuations.

2.5 Green's Function and Path Integral Method

So now using our formulation we want to find the probability of the quantum particle in its ground state energy level. Well, this is denoted by $\langle q(0) \rangle$ defined as

$$\langle q(0) \rangle = \frac{\int [dq] e^{-\frac{1}{\hbar} \int_{\tau_i}^{\tau_f} H d\tau} q(0)}{Z} \quad (2.30)$$

In other words this is the expectation value of finding the particle in the ground state. However we wish to continue now with the logic of finding the particle at a certain position away from its starting position in a given time interval. For this we employ the correlation function and green's function of one dimensional quantum mechanics. Using the correlation function

$$\langle q(0)q(T) \rangle = \frac{\int [dq] e^{-\frac{1}{\hbar} \int_{\tau_i}^{\tau_f} H d\tau} q(0)q(T)}{Z} \quad (2.31)$$

we can find the expectation value of the particle at a certain time T and arbitrary position for a given ground state amplitude previously known. To solve 2.31 we first notice that $H(q)|\psi_n(q)\rangle = E_n\psi_n(q)$, where $H(q)$ is the Hamiltonian operator. We take part of 2.31

$$|q(T)\rangle = \sum_n a_n |\psi_n\rangle \quad (2.32)$$

giving us the wavefunction of the particle at some of other energy state at time T . The constants a_n hold the weight of the particle at time with respect to the various vector eigenfunctions:

$$a_n = \langle \psi_n | q(T) \rangle \quad (2.33)$$

and inserting 2.33 into 2.32 we obtain

$$|q(T)\rangle = \sum_n \langle \psi_n | q(T) \rangle |\psi_n\rangle \quad (2.34)$$

Leaving this to the side, we take the ground state wavefunction of the particle through a similar approach:

$$|q(0)\rangle = \sum_n b_n |\psi_n\rangle \quad (2.35)$$

The time evolution of the ground state wave function gives you the wave function of the particle at any other time T,

$$|q(T)\rangle = e^{HT}|q(0)\rangle = \sum_n a_n |\psi_n\rangle \quad (2.36)$$

and using the second part of 2.36 and solving for $|q(0)\rangle$,

$$|q(0)\rangle = \sum_n a_n e^{-HT} |\psi_n\rangle \quad (2.37)$$

$$= \sum_n a_n e^{-E_n T} |\psi_n\rangle \quad (2.38)$$

$$(2.39)$$

From this the wavefunction of the particle has been expressed in its ground state configuration. Thus from the path integral, we can basically produce the expectation value for the wavefunction at any time T from the ground state expectation value. What is even more important, is that we have found a way of determining the energy of the particle at that certain time T. This becomes a powerful tool, because from the energy of the particle many physical quantities are attainable.

3. ENERGY AND GEOMETRY: PARALLEL TRANSPORT THEORY

Particles at the quantum level are subject to many physical phenomena, which are beyond the scope of the macroscopic particle to feel. As massive objects are subject the rules of General Relativity and the curvature produced by a gravitational field, the quantum particle is subject to similar physical phenomena governed by Non-Euclidean geometry resulting from the presence of a electromagnetic potential. The curvature of space-time affects the particle by producing a change in its total potential energy. So we can try to understand how this happens by a simple example of a particle moving around in a plane with an electromagnetic potential

3.1 Introduction to Parallel Transport Theory

Let's describe a particle by its wave function $\psi(x)$ in one dimensional quantum mechanics. To describe a change in its position, the derivative of the function is taken.

$$\frac{d\psi(x)}{dx} = \frac{\psi(x+dx) - \psi(x)}{dx} \quad (3.1)$$

Now since the wave function can be complex it can be expressed by its *Real Component* and its *Complex Phase*. If one were to graph these components the wave function could be

translated such that its real component was kept constant while it's imaginary component was the sole component being translated. If these components were formed into a vector representation of the wave function then a real analysis of the change in the wave function could be analyzed. We can graph a three dimensional representation with two axes as the real and imaginary components and the third being a position axis. Keeping the wave function constant while moving it through flat space, does not change it. Thus the translation from $x \rightarrow x + dx$ uses the derivative in 3.1. A simple diagram of what this may look like can be seen in figure 2.

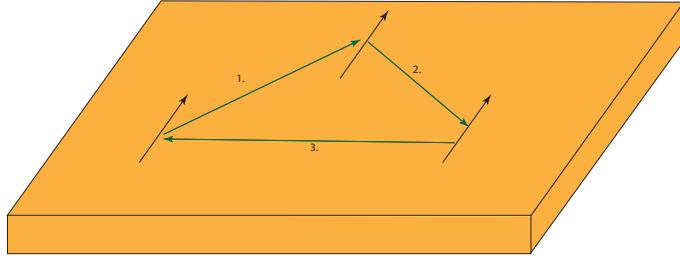


Figure 2: Particle propagating through space with no potential field felt in the surrounding area. Particle (Denoted by arrow) keeps same orientation through the entire cycle.

Now suppose that same particle moves through a given electromagnetic potential denoted by $A_{\mu\nu}$ from a position $x \rightarrow x + dx$ (The subscripts μ and ν represent directions). Well, this potential changes the geometry of the local space around it, similar to gravitational potential curving space-time (see figure 3). Thus we say that a certain potential A_μ allows for the curvature on which the particle traverses, changing the wave function of that particle by a phase given by:

$$e^{i \int_x^{x+dx} A_\mu(x') dx'} \quad (3.2)$$

This then takes the electromagnetic potential in the x direction and integrates its contributions along that direction. Equation 3.2 operated on $\psi(x)$ will rotate $\psi(x)$ after it moves to $x + dx$, thereby allowing for a comparison between the translation and rotation. So, the change in the wave function from when the particle moves to a different location (namely $x + dx$) in the presence of the electromagnetic potential is:

$$\psi(x + dx) - e^{i \int_x^{x+dx} A_\mu(x') dx'} \psi(x) \quad (3.3)$$

Thus the total derivative in the presence of the electromagnetic potential is:

$$\frac{D\psi(x)}{dx} = \frac{\psi(x + dx) - e^{i \int_x^{x+dx} A_\mu(x') dx'} \psi(x)}{dx} \quad (3.4)$$

This change, tells us that the particle carries some extra information with it, dealing with the actual curvature of the space time around itself. This process is known as the “Parallel Transport Theory.”

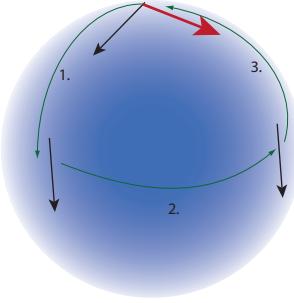


Figure 3: Curvature of space-time due to presence of gauge potential field. The particle (denoted by the arrow) does not remain in the same orientation after one full cycle. Orientation may refer to characteristics of the particle, and not necessarily the physical description of the particle in real space.

3.2 Parallel Transport in 4 Dimensions: EM Potential

Elaborating on the previous section, we wish to observe the effects on a quantum particle in the presence of an electromagnetic field if it traverses around a closed loop. We take a rectangular loop on a μ, ν plane where the center is located at the coordinates (x_μ, x_ν) , and is also where the electromagnetic potential $A_{\mu\nu}$ passes through. Starting the particle at the bottom left hand corner the wave function at those coordinates is $\psi(1) = \psi(x_\mu - \frac{dx_\mu}{2}, x_\nu - \frac{dx_\nu}{2})$, and going counter-clockwise around the loop, each corner is given a label from $\psi(1) \rightarrow \psi(4)$ denoting the wavefunctions of the particle at each corner. A new operator \hat{C} is introduced and takes the particle back to its starting position at $(x_\mu - \frac{dx_\mu}{2}, x_\nu - \frac{dx_\nu}{2})$ after one full rotation in the counterclockwise direction, thereby determining how the original wave function changes. This new wave function will be denoted as $\psi'(1)$. The loop can be seen in Figure 2.

So now moving from $\psi(1)$ to $\psi(2)$ gives the following transformation:

$$\psi(2) = e^{iA_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})dx_\mu} \psi(1) \quad (3.5)$$

where we have approximated that the box is infinitesimally small such that $A_{\mu\nu}$ remains the same throughout the integration in 3.3. Continuing where we left off in 3.5 the rest of the transformations are as follows:

$$\psi(3) = e^{iA_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)dx_\nu} \psi(2) \quad (3.6)$$

$$= e^{iA_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)dx_\nu} e^{iA_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})dx_\mu} \psi(1) \quad (3.7)$$

$$\psi(4) = e^{-iA_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2})dx_\mu} \psi(3) \quad (3.8)$$

$$= e^{-iA_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2})dx_\mu} e^{iA_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)dx_\nu} e^{iA_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})dx_\mu} \psi(1) \quad (3.9)$$

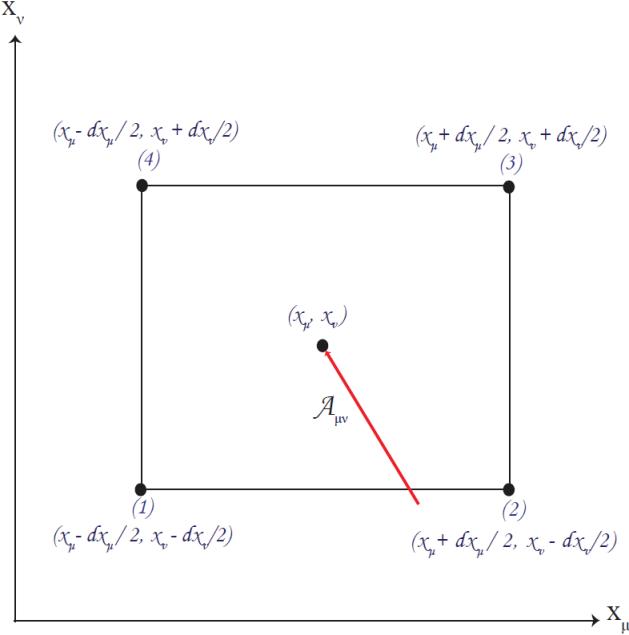


Figure 4: Set-up of four points in space with electromagnetic potential $A_{\mu\nu}$ through (x_μ, x_ν)

where each equation is put in terms of $\psi(1)$. Now all that needs to be done is to complete the loop and to send the particle back to its original position at $\psi(1)$. Thus the new wave function of the particle when returned to its starting position is simply the wave function at $(x_\mu - \frac{dx_\mu}{2}, x_\nu + \frac{dx_\nu}{2})$ multiplied by the change in phase. Therefore this new wave function at the starting point when one full rotation in the counterclockwise direction is completed is denoted by $\psi'(1)$:

$$\begin{aligned}\psi'(1) &= e^{-iA_\nu(x_\mu - \frac{dx_\mu}{2}, x_\nu)} dx_\nu \psi(4) \\ &= e^{-iA_\nu(x_\mu - \frac{dx_\mu}{2}, x_\nu)} dx_\nu e^{-iA_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2})} dx_\mu e^{iA_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu e^{iA_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})} dx_\mu \psi(1) \\ &= e^{i[A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu) - A_\nu(x_\mu - \frac{dx_\mu}{2}, x_\nu)]} dx_\nu + i[A_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2}) - A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})] dx_\mu \psi(1)\end{aligned}$$

We can further simplify the previous expression by noticing that

$$\left[A_\nu \left(x_\mu + \frac{dx_\mu}{2}, x_\nu \right) - A_\nu \left(x_\mu - \frac{dx_\mu}{2}, x_\nu \right) \right] dx_\nu$$

is similar to a change in A_ν in the x_μ direction. Therefore, by definition:

$$\frac{\partial A_\nu}{\partial x_\mu} dx_\nu dx_\mu = (\partial_\mu A_\nu) dx_\nu dx_\mu \quad (3.10)$$

where:

$$\partial_\mu = \frac{\partial}{\partial x_\mu}. \quad (3.11)$$

So now using 3.10, we can say:

$$\left(A_\nu \left(x_\mu + \frac{dx_\mu}{2}, x_\nu \right) - A_\nu \left(x_\mu - \frac{dx_\mu}{2}, x_\nu \right) \right) dx_\nu = (\partial_\mu A_\nu) dx_\nu dx_\mu \quad (3.12)$$

Using this expression, we can express the particle's trip around the loop as:

$$\hat{C}\psi(1) = \psi'(1) \quad (3.13)$$

$$= e^{i[\partial_\mu A_\nu] dx_\nu dx_\mu - i[\partial_\nu A_\mu] dx_\mu dx_\nu} \psi(1) \quad (3.14)$$

$$= e^{i[\partial_\mu A_\nu - \partial_\nu A_\mu] dx_\nu dx_\mu} \psi(1) \quad (3.15)$$

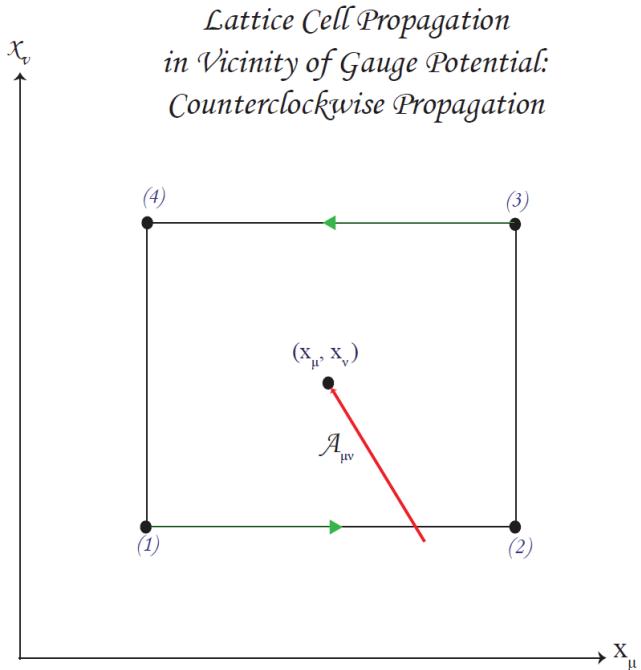


Figure 5: Particle traversing in the counter-clockwise direction through gauge potential $A_{\mu\nu}$.

So now that we have seen how the initial wave function changes with a trip in one direction around a closed loop with a certain electromagnetic field, let's see how it changes with a trip around the closed loop in the opposite direction. This operation is given by \hat{C}^{-1} on $\psi(1)$, where we start at point 1 and move in the clockwise direction. Therefore the wave function at each point given by the change in phase is:

$$\psi(4) = e^{A_\nu \left(x_\mu - \frac{dx_\mu}{2}, x_\nu \right) dx_\nu} \psi(1) \quad (3.16)$$

$$\psi(3) = e^{A_\mu \left(x_\mu, x_\nu + \frac{dx_\nu}{2} \right) dx_\mu} \psi(4) \quad (3.17)$$

$$\psi(2) = e^{-A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu \psi(3) \quad (3.18)$$

$$\psi''(1) = e^{-A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})} dx_\mu \psi(2) \quad (3.19)$$

From this we can replace $\psi(2)$ in 3.19 and so on and so forth to get $\psi''(1)$, (which is the wave function of the particle once it has completed one full rotation in the clockwise direction), in terms of $\psi(1)$.

$$\begin{aligned} \hat{C}^{-1}\psi(1) &= \psi''(1) \\ &= e^{i[A_\nu(x_\mu - \frac{dx_\mu}{2}, x_\nu) - A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)] dx_\nu + i[A_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2}) - A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})] dx_\mu} \psi(1) \\ &= e^{i[-\partial_\mu A_\nu dx_\nu dx_\mu + \partial_\nu A_\mu dx_\nu dx_\mu]} \psi(1) \\ &= e^{i[\partial_\mu A_\nu - \partial_\nu A_\mu] dx_\nu dx_\mu} \psi(1) \end{aligned}$$

Given this definition for the change in the the particle's wave function after a round trip around the loop with a curvature due to the electromagnetic potential, it can be expected that this round trip produces an irreversible change from the original wave function of the particle before it ever moved from its starting point. In order to confirm this, we subtract the original wave function by the average of $\psi'(1)$ and $\psi''(1)$.

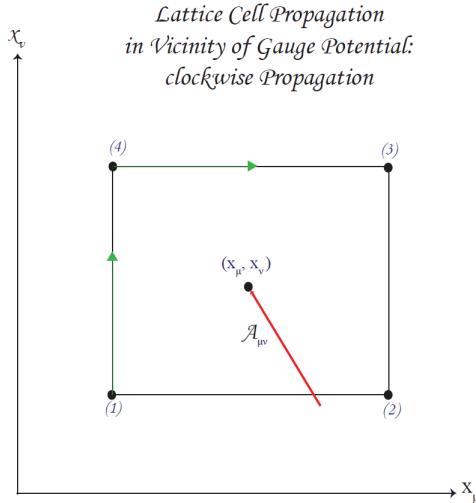


Figure 6: Particle traversing in the clockwise direction through gauge potential $A_{\mu\nu}$.

$$\psi(1) - \left[\frac{\hat{C}\psi(1) + \hat{C}^{-1}\psi(1)}{2} \right] = \psi(1) - \left[\frac{\psi'(1) + \psi''(1)}{2} \right]$$

$$\begin{aligned}
&= \left[1 - \frac{e^{i[\partial_\mu A_\nu - \partial_\nu A_\mu]dx_\nu dx_\mu} + e^{i[\partial_\mu A_\nu - \partial_\nu A_\mu]dx_\nu dx_\mu}}{2} \right] \psi(1) \\
&= [1 - \cos((\partial_\nu A_\mu - \partial_\mu A_\nu)dx_\mu dx_\nu)] \psi(1)
\end{aligned}$$

Here we make a substitution where $F_{\mu\nu} = [\partial_\nu A_\mu - \partial_\mu A_\nu]$ thus changing 3.20 to

$$[1 - \cos(F_{\mu\nu}dx_\nu dx_\mu)] \psi(1) \quad (3.20)$$

Now if we take a Taylor expansion of $\cos(F_{\mu\nu}dx_\nu dx_\mu)$ and keep only up to the second term, we can approximate 3.20 to be:

$$\left[1 - \left[1 - \frac{(F_{\mu\nu}dx_\mu dx_\nu)^2}{2!} \right] \right] \psi(1) = (F_{\mu\nu}^2 dx_\mu^2 dx_\nu^2) \psi(1) \quad (3.21)$$

which gives us the total transformation of the particle's wave function once it makes a complete round trip around the closed loop in the presence of an electromagnetic field. From this non-zero value, we can try to extract various types of information from the propagation.

3.2.1 Energy Derivation from Parallel Transport in 4 Dimensions

In the previous section we came to the conclusion that the non zero difference between the particle's wave function and the new wave function after a round trip around the loop was an indication of some extra information being produced from the curvature due to the electromagnetic field. So in order to find out exactly what type of information we are dealing with, we take the simplest case for our loop and assume a symmetric case where $dx_\mu = dx_\nu$. In addition, we restrict the values for μ and ν to be between values of $[0,1,2,3]$, where $\mu \neq \nu$. These values correspond to the three space coordinates and one time coordinate:

$$x_0 = t, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z.$$

For simplicity we say that $dx_\nu = dx_\mu = a$. So in order to figure out what we're dealing with, we try to solve 3.21. Well, we know that $F_{\mu\nu} = [\partial_\nu A_\mu - \partial_\mu A_\nu]$, and if we take various combinations of μ and ν we can tabulate the results for $F_{\mu\nu}^2$:

Thus the above table gives all the combinations for all the various orientations in space and time for the function $F_{\mu\nu}$. Now we can add up all the various combinations seen in the last column of the table giving us:

μ	ν	$F_{\mu\nu}$	$F_{\mu\nu}^2$
1	2	$\partial_2 A_1 - \partial_1 A_2$	$\partial_2^2 A_1^2 + \partial_1^2 A_2^2 - 2\partial_2 A_1(\partial_1 A_2)$
1	3	$\partial_3 A_1 - \partial_1 A_3$	$\partial_3^2 A_1^2 + \partial_1^2 A_3^2 - 2\partial_3 A_1(\partial_1 A_3)$
2	3	$\partial_3 A_2 - \partial_2 A_3$	$\partial_3^2 A_2^2 + \partial_2^2 A_3^2 - 2\partial_3 A_2(\partial_2 A_3)$
0	1	$\partial_1 A_0 - \partial_0 A_1$	$\partial_1^2 A_0^2 + \partial_0^2 A_1^2 - 2\partial_1 A_0(\partial_0 A_1)$
0	2	$\partial_2 A_0 - \partial_0 A_2$	$\partial_2^2 A_0^2 + \partial_0^2 A_2^2 - 2\partial_2 A_0(\partial_0 A_2)$
0	3	$\partial_3 A_0 - \partial_0 A_3$	$\partial_3^2 A_0^2 + \partial_0^2 A_3^2 - 2\partial_3 A_0(\partial_0 A_3)$

Table 1: Results for higher order Gauge field Calculation of $F_{\mu\nu}$

$$\sum_{Loops} [1 - \cos(F_{\mu\nu} a^2)] = \sum \frac{F_{\mu\nu}^2 a^4}{2} \psi(1) \quad (3.22)$$

$$= \frac{a^4}{2} \sum F_{\mu\nu}^2 \psi(1) \quad (3.23)$$

So instead of calculating everything all at once, we can take the first three rows of the last column in the table and we see that this is nothing more than the square of the cross product of $\vec{A}_{\mu\nu}$.

$$(\nabla \times \vec{A})^2 = (\partial_2 A_1 - \partial_1 A_2)^2 + (\partial_3 A_1 - \partial_1 A_3)^2 + (\partial_3 A_2 - \partial_2 A_3)^2 \quad (3.24)$$

Now we take the last three rows of the last column and try to simplify those equations.

$$\begin{aligned}
 (\partial_1 A_0 - \partial_0 A_1)^2 + (\partial_2 A_0 - \partial_0 A_2)^2 + (\partial_3 A_0 - \partial_0 A_3)^2 &= (\partial_1^2 + \partial_2^2 + \partial_3^2) A_0^2 + \\
 &\quad + \partial_0^2 (A_1^2 + A_2^2 + A_3^2) - 2\partial_0 \vec{A}(\nabla A_0) \\
 &= \nabla^2 A_0^2 + \partial_0^2 (A_1^2 + A_2^2 + A_3^2) - \\
 &\quad - \partial_0 \vec{A}(\nabla A_0) - \nabla A_0 \partial_0 \vec{A} \\
 &= (\nabla A_0 - \partial_0 \vec{A})^2.
 \end{aligned} \quad (3.25)$$

From this, the various A'_n s are replaced with the following:

$$A_0 = -\phi \quad (3.26)$$

$$A_1 = A_x \quad (3.27)$$

$$A_2 = A_y \quad (3.28)$$

$$A_3 = A_z \quad (3.29)$$

where ϕ is electric potential and the rest are the spatial components of \vec{A} , which is the magnetic potential. The extra negative sign comes from the contra-variance nature of tensor mathematics. So 3.25 becomes $(-\nabla\phi - \frac{\partial\vec{A}}{\partial t})^2$. Thus 3.24 and 3.25 become:

$$(\nabla \times \vec{A})^2 = |\vec{B}|^2 \quad (3.30)$$

$$(\nabla A_0 - \partial_0 \vec{A})^2 = |\vec{E}|^2 \quad (3.31)$$

where \vec{E} and \vec{B} are the electric and magnetic fields of respectively. From this we can finally complete the summation in 3.23 by replacing $F_{\mu\nu}^2$ by 3.30 and 3.31 thus producing:

$$\frac{a^4}{2} \sum F_{\mu\nu}^2 \psi(1) = \left[|\vec{E}|^2 + |\vec{B}|^2 \right] \frac{a^4}{2} \psi(1). \quad (3.32)$$

From this we separate a^4 into its various components where $a^4 = \Delta x \Delta y \Delta z \Delta(ct)$ and if we take a summation of 3.32 over discrete space and time we can produce the following:

$$\lim_{N \rightarrow \infty} \frac{1}{2} \sum_{x,y,z,ct}^N (\vec{E}^2 + \vec{B}^2) \Delta x \Delta y \Delta z \Delta(ct) \psi(1) = \int \frac{(\vec{E}^2 + \vec{B}^2)}{2} dx dy dz d(ct) \psi(1) \quad (3.33)$$

where:

$$\int \frac{(\vec{E}^2 + \vec{B}^2)}{2} dx dy dz d(ct) \psi(1) = U \frac{\mu}{\epsilon} \psi(1) \quad (3.34)$$

where U is the total energy. Well, this energy is produced solely from the presence of the electromagnetic field. This then concludes that the non-zero difference between the original wave function of the particle and the average of the round trip wave functions of the particle is a result of the curvature of the space-time in the vicinity of the electromagnetic potential, which then gives the particle an extra amount of energy.

4. PARTICLE PROPAGATION IN NON-ABELIAN GAUGE FIELD

We had naively assumed in the last section that the electromagnetic potential, produces a non-euclidean curvature of the space-time around it. This in turn influences the particle

by giving it an additional amount of energy. If this potential were instead made to be multi-dimensional, then along with the additional energy, comes an additional property for the particle which allows the particle to interact with others like it, as seen with gluons and the strong interaction within the nucleus of an atom. This “multi-dimensional potential” is usually called a non-Abelian Gauge field which is field with an internal symmetry [1].

4.1 Parallel Transport in multiple Dimensions

Now instead of treating the original electromagnetic potential $A_{\mu\nu}$ in a one dimensional form, we take it to be an $N \times N$ matrix. From this we can use the same square loop used in the previous chapter with the center at (x_μ, x_ν) and where the particle starting at the bottom left hand corner with an initial wave function of $\psi(1) = \psi(x_\mu - \frac{dx_\mu}{2}, x_\nu - \frac{dx_\nu}{2})$. From here we allow it to move from corner to corner in both counter-clockwise and clockwise directions by means of the \hat{C} and \hat{C}^{-1} operator respectively. However due to the fact that the potential is now a matrix, there are certain properties which need to be handled with care before proceeding to simplify the solutions to a more decipherable form.

4.1.1 Transport Along the Loop: Counter-clockwise Rotation

Now from the chapter on “Parallel Transport”, we know that if the particle moves from point 1 to point to point 2, the particles wave function is shifted by a factor of $e^{A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2}) dx_\mu}$. As it continues to make its way around the loop, and back to the starting position, the original wave function is rotated by the operator \hat{C} giving it the form:

$$\hat{C}\psi(1) = e^{-iA_\nu(x_\mu - \frac{dx_\mu}{2}, x_\nu) dx_\nu} e^{-iA_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2}) dx_\mu} e^{iA_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu) dx_\nu} e^{iA_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2}) dx_\mu} \psi(1). \quad (4.1)$$

One can refer back to Figure 5 for reference. Each phase is now labeled accordingly to simplify further calculations:

$$M_1 = A_\nu \left(x_\mu - \frac{dx_\mu}{2}, x_\nu \right) \quad (4.2)$$

$$N_1 = A_\mu \left(x_\mu, x_\nu + \frac{dx_\nu}{2} \right) \quad (4.3)$$

$$M_2 = A_\nu \left(x_\mu + \frac{dx_\mu}{2}, x_\nu \right) \quad (4.4)$$

$$N_2 = A_\mu \left(x_\mu, x_\nu - \frac{dx_\nu}{2} \right) \quad (4.5)$$

$$a = i dx_\mu \text{ or } \nu \quad (4.6)$$

which turns 4.1 into

$$\hat{C}\psi(1) = e^{-aM_1}e^{-aN_1}e^{-aM_2}e^{-aN_2} \psi(1)$$

Now using the following identity for multiplying exponential matrices,

$$e^{aA}e^{aB} = e^{aP_1+a^2P_2+a^3P_3+O(a^4)} \quad (4.7)$$

we use the first two exponentials in 4.1 and replace those with the expression on the left of 4.7. Thus, 4.7 becomes:

$$e^{-aM_1}e^{-aN_1} = e^{aP_1+a^2P_2+a^3P_3+O(a^4)} \quad (4.8)$$

To simplify and make any sense of this expression, we need to taylor expand both sides of 4.8. So first we expand the left hand side and keeping the first three terms in the expansion, and cross multiplying both exponentials, we produce:

$$\begin{aligned} e^{-aM_1}e^{-aN_1} &= \left(1 - aM_1 + \frac{a^2M_1^2}{2!} - \frac{a^3M_1^3}{3!} + \dots\right) \left(1 - aN_1 + \frac{a^2N_1^2}{2!} - \frac{a^3N_1^3}{3!} + \dots\right) \\ &= 1 + a(-N_1 - M_1) + a^2 \left(\frac{N_1^2}{2} + M_1N_1 + \frac{M_1^2}{2}\right) + O(a^3). \end{aligned} \quad (4.9)$$

Expanding the right hand side of 4.8 we get;

$$\begin{aligned} e^{aP_1+a^2P_2+a^3P_3+O(a^4)} &= 1 + (aP_1 + a^2P_2 + a^3P_3) + \frac{1}{2} (aP_1 + a^2P_2 + a^3P_3)^2 + \dots \\ &= 1 + aP_1 + a^2 \left(P_2 + \frac{P_1^2}{2}\right) + O(a^3) \end{aligned} \quad (4.10)$$

Thus we have expanded expressions for both sides of 4.8, and as such we can then equate the terms both sides of the expansions and produce expressions for P_1 and P_2 . Obtaining expressions for P_3 and on is not necessary as those extra terms are of no consequence in the expansion. Thus once equated, we see that P_1 and P_2 have the following forms:

$$P_1 = (N_1 + M_1) \quad (4.11)$$

and

$$P_2 = \frac{[M_1, N_1]}{2}. \quad (4.12)$$

Replacing 4.11 and 4.12 into 4.8 we obtain the following expression for multiplying the first two exponentials:

$$e^{-aM_1}e^{-aN_1} = e^{a(-N_1 - M_1) + \frac{a}{2}[M_1, N_1]}. \quad (4.13)$$

Taking the next two exponentials from 4.1 and replacing the variables in the exponentials with their M and N counterparts, we see that the same kind of result will occur

$$e^{aM_2}e^{aN_2} = e^{aQ_1+a^2Q_2+a^3Q_3+O(a^4)}. \quad (4.14)$$

Taking 4.14 and expanding,

$$e^{aM_2}e^{aN_2} = 1 + a(N_2 + M_2) + a^2 \left(\frac{N_2^2}{2} + M_2 N_2 + \frac{M_2^2}{2} \right) + O(a^3)$$

and

$$e^{aQ_1+a^2Q_2+a^3Q_3+O(a^4)} = 1 + aQ_1 + a^2 \left(Q_2 + \frac{Q_1^2}{2} \right) + O(a^3).$$

Thus equating these two equations, we produce expressions for Q_1 and Q_2 :

$$Q_1 = (M_2 + N_2) \quad (4.15)$$

and

$$Q_2 = \frac{[M_2, N_2]}{2}. \quad (4.16)$$

From this, second half of the right hand side expression of 4.1 becomes:

$$e^{aM_2}e^{aN_2} = e^{a(M_2+N_2)+\frac{a^2}{2}[M_2,N_2]} \quad (4.17)$$

So now putting 4.13 and 4.17 together,

$$e^{-aM_1}e^{-aN_1}e^{aM_2}e^{aN_2} = e^{a(-M_1-N_1)+\frac{a^2}{2}[M_1,N_1]} e^{a(M_2+N_2)+\frac{a^2}{2}[M_2,N_2]}. \quad (4.18)$$

We see that this is the matrix operator expression for the wave function going around the square loop once in a counter clockwise direction. Now this last equation can still be simplified into something with more physical meaning. By using the proven equation in 4.13 and 4.17 we can say that in general,

$$e^{\alpha\alpha}e^{\alpha\beta} = e^{a(\alpha+\beta)+\frac{a^2}{2}[\alpha,\beta]} \quad (4.19)$$

is true when α and β are represented as matrices. So, the right hand side of 4.18 can be seen as the left hand side of 4.19 where

$$\alpha = (-M_1 - N_1) + \frac{a}{2}[M_1, N_1]$$

and

$$\beta = (M_2 + N_2) + \frac{a}{2}[M_2, N_2].$$

So taking the right hand side of 4.19 we add α and β , thereby obtaining

$$\alpha + \beta = -M_1 - N_1 + M_2 + N_2 + \frac{a}{2}([M_1, N_1] + [M_2, N_2])$$

Replacing the M 's and N 's by the potential expressions, we see that the addition can be rewritten as:

$$\begin{aligned} -M_1 - N_1 + M_2 + N_2 &= A_\nu \left(x + \frac{a'}{2}\hat{\mu} \right) - A_\nu \left(x - \frac{a'}{2}\hat{\mu} \right) + A_\mu \left(x - \frac{a'}{2}\hat{\nu} \right) - A_\mu \left(x + \frac{a'}{2}\hat{\nu} \right) \\ &\quad + \frac{a}{2}([M_1, N_1] + [M_2, N_2]) \\ &= (\partial_\mu A_\nu) dx_\mu - (\partial_\nu A_\mu) dx_\mu + \frac{a}{2}(2[A_\nu, A_\mu]) \\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu)a' + a[A_\nu, A_\mu] \end{aligned} \quad (4.20)$$

where $a' = dx_{(\nu \text{ or } \mu)}$ as opposed to a which has an extra factor of i . However, the tricky part comes in calculating $[\alpha, \beta]$. Well, without having to explicitly write it out, it can be shown, through extended calculation that $[\alpha, \beta] = 0$. Thus, exponentiating 4.20 and equating that to 4.18 we finally come to the conclusion which gives us an expression for the particle going counter-clockwise around the loop with a multi-dimensional potential.

$$\begin{aligned} e^{a(-M_1-N_1)+\frac{a^2}{2}[M_1,N_1]} e^{a(M_2+N_2)+\frac{a^2}{2}[M_2,N_2]} \psi(1) &= e^{a(a'(\partial_\mu A_\nu - \partial_\nu A_\mu) + a[A_\nu, A_\mu])} \psi(1) \\ &= e^{ia^2(\partial_\mu A_\nu - \partial_\nu A_\mu) - a^2[A_\nu, A_\mu]} \psi(1) \\ &= e^{ia^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + a^2[A_\mu, A_\nu]} \psi(1) \end{aligned} \quad (4.21)$$

4.2 Transport along the loop: Clockwise Rotation

So now that we have completed one full rotation in one direction, we can refer to figure 6 and allow the particle to travel in the clockwise direction of the loop with the same potential field as before, remembering that it is still a matrix. So from the previous chapter we use the same expression for \hat{C}^{-1} ,

$$\begin{aligned} \hat{C}^{-1}\psi(1) &= e^{-A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})} dx_\mu e^{-A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu e^{A_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2})} dx_\mu e^{A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu \psi(1) \\ &= e^{-aN_2} e^{-aM_2} e^{-aN_1} e^{-aM_2} \psi(1) \end{aligned} \quad (4.22)$$

where we have used the same M and N substitutions from the counter-clockwise rotation section. So, as before, we take 4.22 and use the matrix expansion identity on the first two exponentials;

$$e^{-aN_2} e^{-aM_2} = e^{aS_1 + a^2S_2 + a^3S_3 + O(a^4)} \quad (4.23)$$

where we need to expand both sides of the equation using a Taylor expansion. Thus the left hand side is similar to what we did before,

$$\begin{aligned} e^{-aN_2} e^{-aM_2} &= \left(1 - aN_2 + \frac{a^2 N_2^2}{2!} - \frac{a^3 N_2^3}{3!} + \dots\right) \left(1 - aM_2 + \frac{a^2 M_2^2}{2!} - \frac{a^3 M_2^3}{3!} + \dots\right) \\ &= 1 + a(-N_2 - M_2) + a^2 \left(\frac{N_2^2}{2} + M_2 N_2 + \frac{M_2^2}{2}\right) + O(a^3) \end{aligned} \quad (4.24)$$

and the right hand side can be simply expressed as

$$e^{aS_1 + a^2S_2 + a^3S_3 + O(a^4)} = 1 + aS_1 + a^2 \left(S_2 + \frac{S_1^2}{2}\right) + O(a^3) \quad (4.25)$$

So, equating both 4.25 and 4.24 we group similar coefficients of a and find that

$$S_1 = -(M_2 + N_2) \quad S_2 = \frac{[N_2, M_2]}{2} \quad (4.26)$$

therefore

$$e^{-aN_2} e^{-aM_2} = e^{a(-M_2 - N_2) + \frac{a^2}{2}[N_2, M_2]}. \quad (4.27)$$

Using the second set of exponentials in 4.22 we can see that

$$e^{aN_1} e^{aM_1} = e^{aT_1 + a^2 T_2 + a^3 T_3 + O(a^4)} \quad (4.28)$$

and by now we can easily relate this format to previous expressions of this kind and state that

$$T_1 = (N_1 + M_1) \quad T_2 = \frac{[N_1, M_1]}{2} \quad (4.29)$$

and so

$$e^{aN_1} e^{aM_1} = e^{a(N_1 + M_1) + \frac{a^2}{2}[N_1, M_1]}. \quad (4.30)$$

Now combining 4.27 and 4.30 we come to the conclusion that

$$e^{-aN_2} e^{-aM_2} e^{-aN_1} e^{-aM_1} = e^{a(-M_2 - N_2) + \frac{a^2}{2}[N_2, M_2]} e^{a(N_1 + M_1) + \frac{a^2}{2}[N_1, M_1]} \quad (4.31)$$

From this we use 4.19 and say that $\alpha = -M_2 - N_2 + \frac{a}{2}[N_2, M_2]$ and $\beta = N_1 + M_1 + \frac{a}{2}[N_1, M_1]$. So,

$$\begin{aligned} \alpha + \beta &= -M_2 - N_2 + N_1 + M_1 + \frac{a}{2}[N_2, M_2] + \frac{a}{2}[N_1, M_1] \\ &= -A_\nu \left(x + \frac{a'}{2}\hat{\mu} \right) - A_\mu \left(x - \frac{a'}{2}\hat{\nu} \right) + A_\mu \left(x + \frac{a'}{2}\hat{\nu} \right) + A_\nu \left(x - \frac{a'}{2}\hat{\mu} \right) \\ &\quad + \frac{a}{2}([N_2, M_2] + [N_1, M_1]) \\ &= -\partial_\mu A_\nu dx_\nu + \partial_\nu A_\mu dx_\nu + \frac{a}{2}([N_2, M_2] + [N_1, M_1]) \\ &= (-\partial_\mu A_\nu + \partial_\nu A_\mu)a' - \frac{a}{2}([M_2, N_2] + [M_1, N_1]) \\ &= (-\partial_\mu A_\nu + \partial_\nu A_\mu)a' - a[A_\nu, A_\mu] \end{aligned} \quad (4.32)$$

and from the previous section we know that $[\alpha, \beta] = 0$. So now all we need to do is multiply 4.32 by a giving us

$$\begin{aligned} a(\alpha + \beta) &= aa'(-\partial_\mu A_\nu + \partial_\nu A_\mu) - a^2[A_\nu, A_\mu] \\ &= ia^2(-\partial_\mu A_\nu + \partial_\nu A_\mu) + a^2[A_\nu, A_\mu] \\ &= -ia^2((- \partial_\mu A_\nu + \partial_\nu A_\mu) + [A_\nu, A_\mu]). \end{aligned} \quad (4.33)$$

So we can explicitly state 4.22 by replacing it with 4.33

$$\begin{aligned}\hat{C}^{-1}\psi(1) &= e^{-A_\mu(x_\mu, x_\nu - \frac{dx_\nu}{2})} dx_\mu e^{-A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu e^{A_\mu(x_\mu, x_\nu + \frac{dx_\nu}{2})} dx_\mu e^{A_\nu(x_\mu + \frac{dx_\mu}{2}, x_\nu)} dx_\nu \psi(1) \\ &= e^{a(-M_2 - N_2) + \frac{a^2}{2}[N_2, M_2]} e^{a(N_1 + M_1) + \frac{a^2}{2}[N_1, M_1]} \psi(1) \\ &= e^{-ia^2((-A_\mu A_\nu + A_\nu A_\mu) + [A_\mu, A_\nu])} \psi(1)\end{aligned}\quad (4.34)$$

giving us the final expression for the particle moving in the clockwise direction around the square loop.

4.3 Effect Non-Abelian Guage Fields: Self Interaction of Potential Fields

Now in order for us to see how the wave function of the particle changes when rotated in both direction from its original position, we do the same manipulations as in the last chapter; by subtracting the average of the rotations from the original wave functions. So by doing this we see that

$$\psi(1) - \left(\frac{\hat{C}\psi(1) + \hat{C}^{-1}\psi(1)}{2} \right) = \quad (4.35)$$

$$\begin{aligned}&\left[1 - \left(\frac{e^{ia^2((-A_\mu A_\nu + A_\nu A_\mu) + [A_\mu, A_\nu])} + e^{-ia^2((-A_\mu A_\nu + A_\nu A_\mu) + [A_\mu, A_\nu])}}{2} \right) \right] \psi(1) \\ &= [1 - \cos(a^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu])] \psi(1)\end{aligned}\quad (4.36)$$

and can thus we can define a new F :

$$F'_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (4.37)$$

which is different from the linear F found in the last chapter by the extra commutator. Now the potential, which is an $N \times N$ hermitian matrix has a total of $N^2 - 1$ degrees of freedom because of the degeneracy due to the complex elements and by its hermitian properties. Thus there are $N^2 - 1$ functions and each one is a vector. Therefore we can define A in one arbitrary direction as $A_\nu^a(x)$ where $a \in [0, N^2 - 1]$. From this we can state that N determines the type of potential field, whether it is an electromagnetic potential, the weak-force potential or the strong force potential. For instance, $N = 2$ gives us $A_\mu^1, A_\mu^2, A_\mu^3$ where each vector potential describes a weak field. From the commutator expression we can determine that this commutator is merely to show the interaction between the field with itself. If now we had $N = 3$, there would be 8 different vector potentials which constitute the definition of the strong force field.

Now that we have an enlightened understanding of the extra commutator let's try to see what else we can determine. Well, we know that $F_{\mu\nu}$ is a matrix since $A_{\mu\nu}$ is a matrix. This matrix then can be commuted with itself and then its trace can be obtained giving us

$\text{Tr}[F_{\mu\nu}, F_{\mu\nu}]$ which is basically the curvature the field produces. If we revert our attention back to 2.26 on the chapter for Feynman's Path Integral, we can state that the minimum action, $S = \text{Tr}[F_{\mu\nu}, F_{\mu\nu}]$ and plugging this into the path integral formulation,

$$\int [dA_\mu] e^{-S} = \int [dA_\mu] e^{-\text{Tr}[F^2]dx_\mu} \quad (4.38)$$

So here we have a general expression for the path integral with respect to all gauge potentials. So from here, we can try to understand the role of matrix elements of the gauge potentials and its effects on quantum particles of various gauge fields.

5. PROPAGATION OF THE ρ MESON

The theory of the past sections need to be applied to the specific case of a subatomic particle, namely the ρ meson, the particle in question in this project. Green's theorem, the path integral method and use of parallel transport will be employed to extract the mass of the ρ meson at zero quark mass.

5.1 Propagation through Space-time Lattice

After much ambiguity, it would be wise to define the matrix A_μ and understand the role it plays in the physics of the project. We know that A_μ is an “element of the Lie algebra of the gauge group”[1]. Now this matrix is such that all the elements of its diagonals add up to zero, thus making it traceless. In addition, A_μ is hermitian, where it is equal to its inverse-transpose ($A_\mu = A_\mu^\dagger$). We take the “exponential matrix” from the previous chapter, and order to simplify matters we take the various transport exponential terms and denote them by a simpler notation. We realize that if $HR = \Lambda R$, where H is a hermitian operator and R is any state, then Λ is the diagonalized matrix of eigenvalues of H . From this we see that $H = R\Lambda R^\dagger$. So we can easily prove that phase exponential of a hermitian matrix results in a unitary matrix such that

$$e^{iA_\mu(x)} = U_\mu(x)[8] \quad (5.1)$$

where we still keep the left hand term to be a matrix thereby keeping the right hand term to be a matrix as well. Now since we stated that the gauge potential matrix is hermitian, this implies that the matrix U_μ is a special unitary matrix. It is unitary because $U_\mu U_\mu^\dagger = \mathcal{I}$ where \mathcal{I} is the identity matrix, and special because the $\det(U_\mu) = 1$. Thus it is an NxN special unitary matrix and is denoted by saying that it is an $SU(N)$ matrix.

We redefine the action or path used in section 3.2.1. We replace each path transformation exponential matrix by its appropriate unitary matrix notation. Thus taking the counterclockwise rotation, we define our original operator \hat{C} to be the trace of the product of all path unitary matrices. This then is denoted by U_{\square} :

$$U_{\square} = \text{Tr} \left[U_{\nu}^{\dagger}(x) U_{\mu}^{\dagger}(x + \hat{\nu}) U_{\nu}(x + \hat{\mu}) U_{\mu}(x) \right]. \quad (5.2)$$

Traversing around the loop in the opposite direction, the particle in question is then acted upon by \hat{C}^{-1} . Thus this operator, in terms of the unitary matrices, is denoted by \tilde{U}_{\square} and equates to

$$\tilde{U}_{\square} = \text{Tr} \left[U_{\mu}^{\dagger}(x) U_{\nu}^{\dagger}(x + \hat{\mu}) U_{\mu}(x + \hat{\nu}) U_{\nu}(x) \right]. \quad (5.3)$$

We wish to see, as before, what the total change in the particle's wavefunction will look like. So we subtract out the original wavefunction of the particle from the average of the two new unitary operators on the original wavefunction from 5.2 and 5.3. We also notice that if you have two matrices A and B , then $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$. Thus we take all the various orientation of loops the change in the particle's wavefunction is denoted by S and is then summed over all types of "loops":

$$\begin{aligned} S &= \beta \sum_{\square} \left(1 - \frac{1}{2} \left(\text{Tr}(U_{\square}) + \text{Tr}(\tilde{U}_{\square}) \right) \right) \\ &= \beta \sum_{\square} \text{Tr} \left(1 - \frac{1}{2} \left(U_{\square} + \tilde{U}_{\square} \right) \right) \\ &= \beta \sum_{\square} (1 - \text{ReTr}(U_{\square})) \quad [1] \end{aligned} \quad (5.4)$$

Then 5.4 can be thought of as the action seen in 4.38 and 2.26. Now we know that each unitary operator, is defined at each point in space in the direction of its prorogation as stated in 5.1. Thus at each point in space, there exists an SU(N) matrix. The dimensions of the matrix depends on the physics involved and for strong interactions $N = 3$. Whatever we choose our N to be, we now have an action for the particle propagating through the space-time lattice in the form of 5.4. Therefore the partition function for this propagation is denoted by:

$$Z = \int [dU] e^{-\beta \sum_{\square} (1 - \text{ReTr}(U_{\square}))} \quad (5.5)$$

5.2 Propagation through Quark Field

The generalization of a particle propagating through a gauge potential allows one to determine the expectation value of the particle at some other point in the lattice from its initial starting point. However we now specify the particle in question and determine how to use its properties to determine how its propagation differs from the generalized particle.

5.2.1 Dirac Operator

We start out with the relativistic Dirac equation in quantum mechanics. The original Schrödinger equation does not take into the effects of relativity. Thus, Dirac's equation is given by:

$$i(\vec{\gamma} \cdot \vec{p} + i\gamma_\mu E)\psi + i\psi = 0 \quad (5.6)$$

[5]

$$\gamma_\mu = \begin{pmatrix} 0 & -i\vec{\sigma}_\mu \\ i\vec{\sigma}_\mu & 0 \end{pmatrix} \quad (5.7)$$

where the index on σ_μ has μ in the range from $1 \rightarrow 4$ denoting the four Pauli Matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \sigma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (5.8)$$

Another special matrix – γ_5 matrix – is defined by a matrix multiplication of all the successive gamma matrices:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_5 \quad (5.9)$$

We can rewrite 5.6 by stating that the momentum in a certain direction is related to the derivative where $P_\mu \rightarrow -i\partial_\mu = -i(\frac{\partial}{\partial_\mu})$. Using this, the Dirac operator can be rewritten as follows for a spin (1/2) particle: a

$$(i\gamma_\mu \partial_\mu + m)\psi(x)[5] \quad (5.10)$$

As mentioned above, the previous equation is for spin (1/2) fermions, and so we need to generalize it to encompass all sorts of particles, which may not be fermions of the said spin. For these fermions γ_μ is a 4×4 matrix and ∂_μ is taken to be the same size for the 3 spatial and 1 temporal variables. However we now extend the idea of the simple fermions and consider ∂_μ denoted by D_μ , to be an $N \times N$ matrix. The particle we wish to examine are the quarks which make up the ρ meson. The wavefunction of these quarks $\psi_{s,c}$ now has a color index (c) and a spatial index (s). Thus the particle is both a function of position

and the gauge color field. Since both γ_μ and D_μ are matrices of different sizes, the only way to multiply them is to take a direct product between the two operators:

$$\gamma_\mu D_\mu \rightarrow \gamma_\mu \otimes D_\mu \quad (5.11)$$

Thus operating on a particular wavefunction with the said color and space-time indices, we can write the operation as a sum thus creating a new wavefunction:

$$\sum_{\beta,j} (\gamma_\mu)^{\alpha\beta} (D_\mu)^{i,j} \Psi_{\beta,j} = \chi_{\alpha,i} \quad (5.12)$$

Thus γ acts on ψ independently of how D_μ acts on ψ . However, we want to look at the ρ meson not as a whole, but a dual quark system. With this in mind, propagation of said system is to be propagated through a given lattice configuration. Taking our gauge color field of N colors, we ignore the ρ altogether. The field warps the geometry of the space around it. As we saw from 4.37, the additional commutator deals with the interaction between the field to itself, and as N increases, this term will become significant. So, in addition to the lattice field, which was accounted for by 5.5, we need to describe a partition function for the propagation of the quarks through the color field.

5.2.2 Quark Propagator for a Vector Meson

What is of interest in this study is mass of the system (ρ) when the quarks are massless. Thus the mass in 5.10 is set to zero leaving us with a massless dirac operator:

$$(i\gamma_\mu \partial_\mu) \psi = \not{D} \psi \quad (5.13)$$

where the right hand side of 5.13 denotes the full massless Dirac operator. However this equation itself is used for any light like particle and does not specify what type of particle we are dealing with. So we first require the lagrangian in order to obtain our action which describes how the particle travels, and then the partition function in order to tell us the state of the particle after its propagation.

The Lagrangian is simple enough to formulate; by taking the expectation value of the massless dirac operator, we have

$$\begin{aligned} \tilde{\psi} \not{D} \psi &= i \int [d\tilde{\psi}] [d\psi] \tilde{\psi} \gamma_\mu D_\mu \psi \\ &= i \int L d\tau. \end{aligned} \quad (5.14)$$

Thus the Lagrangian from this is seen to be:

$$L = \tilde{\psi} \not{D} \psi. \quad (5.15)$$

With the Lagrangian, the partition function can be defined. Using the template from 2.31, we can define our total partition function for the system to be a combination of the propagation through the lattice and the quark color field:

$$Z_q = \int [du] \left[d\tilde{\psi} \right] [d\psi] e^{S_g(U) + \tilde{\psi} \not{D} \psi} \quad (5.16)$$

The state of the particle at some position $\psi(y)$ from it's starting position $\psi(x)$ is given by now using the partition function in 5.16:

$$\langle \tilde{\psi}(x)\psi(y) \rangle = \frac{\int [du] \left[d\tilde{\psi} \right] [d\psi] e^{S_g(U) + \tilde{\psi} \not{D} \psi} \tilde{\psi}(x)\psi(y)}{Z_q} \quad (5.17)$$

In order to solve the above equation, we perform a special trick where we add an extra factor of $e^{\tilde{\psi} J + \tilde{J}\psi}$ in 5.17. So let's consider what the consequences would be

$$\langle \tilde{\psi}(x)\psi(y) \rangle = \frac{\int [du] e^{S_g(U)} \int \left[d\tilde{\psi} \right] [d\psi] e^{\tilde{\psi} \not{D} \psi} e^{\tilde{\psi} J + \tilde{J}\psi}}{\int d\tilde{\psi} d\psi e^{\tilde{\psi} \not{D} \psi}} \quad (5.18)$$

Since we are interested in how the quarks propagate through the color field, we ignore the lattice portion of the partition function. Working with the exponential terms in 5.18 we determine the following relationship between the Dirac operator and the mysterious J wavefunctions:

$$\begin{aligned} \tilde{\psi} \not{D} \psi + \tilde{\psi} J + \tilde{J}\psi \\ = \left[\tilde{\psi} + \tilde{J} \not{D}^{-1} \right] \left[\not{D} \psi + \tilde{\psi} \right] - \tilde{J} \not{D}^{-1} J \\ = \left[\tilde{\psi} + \tilde{J} \not{D}^{-1} \right] \not{D} \left[\psi + \not{D}^{-1} J \right] - \tilde{\psi} \not{D}^{-1} J \end{aligned} \quad (5.19)$$

Now, if we take our modified correlation function from 5.18 we can replace the modification with what was discovered above giving us the following:

$$\langle \tilde{\psi}(x)\psi(y) \rangle = e^{-\tilde{J} \not{D}^{-1} J} \frac{\int [d\tilde{\psi}] [d\psi] e^{[\tilde{\psi} + \tilde{J} \not{D}^{-1}] \not{D} [\psi + \not{D}^{-1} J]}}{\int [d\tilde{\psi}] [d\psi] e^{\tilde{\psi} \not{D} \psi}} \quad (5.20)$$

Here, the following substitution is made to simplify the equations, where $\tilde{\chi} = (\tilde{\psi} + \tilde{J} \not{D}^{-1})$ and $\chi = (\psi + \not{D}^{-1} J)$. From this we see that everything from the correlation function except for the exponential taken out of the integral reduces to unity.

$$\begin{aligned} \langle \tilde{\psi}(x)\psi(y) \rangle &= e^{-\tilde{J} \not{D}^{-1} J} \frac{\int d\tilde{\chi} d\chi e^{\tilde{\chi} \not{D} \chi}}{\int d\tilde{\chi} d\chi e^{\tilde{\chi} \not{D} \chi}} \\ &= e^{-\tilde{J} \not{D}^{-1} J} \end{aligned} \quad (5.21)$$

By taking the derivative of 5.18 with respect to $J(x)$ and $J(y)$ we get the following result:

$$\begin{aligned}
& \frac{\partial}{\partial J(x)} \frac{\partial}{\partial J(y)} \left(\frac{\int [du] e^{S_g(U)} \int [d\tilde{\psi}] [d\psi] e^{\tilde{\psi} \not{D} \psi} e^{\tilde{\psi} J + \tilde{J} \psi}}{\int d\tilde{\psi} d\psi e^{\tilde{\psi} \not{D} \psi}} \right) \\
&= e^{-\tilde{J}_x \not{D}_{xy}^{-1} J_y} \\
&= \not{D}_{yx}^{-1} J_x \\
&= \not{D}_{yx}^{-1} \\
&= Z
\end{aligned} \tag{5.22}$$

[6]

So from this we see that by differentiating we get back a partition function. Therefore it is safe to say that the correlation function, relating two different states is a product of the partition function of the lattice propagation divided by the same propagation and the inverse dirac operator.

$$\langle \tilde{\psi}(x)\psi(y) \rangle = \frac{\int [du] e^{S_g(U)} [\not{D}^{-1}]}{\int [du] e^{S_g(U)}} \tag{5.23}$$

However we have not yet given a description of the constraints for any particular particle. We know that the general equation for the Lagrangian is given by 5.15. We can extend this to the case of a two quark system, where we have the wavefunction of a down anti-quark denoted by $\tilde{d}(\vec{x})$ and one up-quark $u(\vec{x})$. Performing a sum over all color fields at a specific spin we obtain the Lagrangian to be:

$$\rho = \sum_i \tilde{d}(\vec{x})_{\beta,j} \not{D} u(\vec{x})_{\alpha,i} \tag{5.24}$$

In 5.24, the label α refers to the spin of the particle and i refers to the color. In physics, there are various composite particles which have different properties and mathematically they must be dealt with in different ways. For instance, there exist scalar particles which are particles such as the electron. The π meson is a pseudovector particle while the ρ meson is a pseudovector particle. The definition for each type of particle is listed in table 2.

Transformation type	Transformation Name	Number of components
$\tilde{\psi}\psi$	Scalar Particle	1
$\tilde{\psi}\gamma_\mu\psi$	Vector Particle	4
$\tilde{\psi}\Sigma_{\mu\nu}\psi$	Tensor Particle	6
$\tilde{\psi}\gamma_\mu\gamma_5\psi$	Pseudovector Particle	4
$\tilde{\psi}\gamma_5\psi$	Pseudoscalar particle	1

Table 2: Transformation types for various particle types [5].

Now for the particle which we are studying, the ρ meson, we define it as a vector particle. Thus the γ_μ in the massless dirac operator defined in 5.13 remains as it is, making the lagrangian in 5.24 to be,

$$L = i \sum_i \tilde{d}(\vec{x})_{\beta,j} \gamma_\mu D_\mu u(\vec{x})_{\alpha,i} \quad (5.25)$$

Now that we have the Lagrangian of both the lattice field propagation of the composite ρ meson and the Lagrangian quark propagation through the strong gluon field, the total partition function can be defined in 5.16 with the appropriate form for a vector meson as shown in 2. The partition function, which we can also label as the propagator, bears nothing particularly interesting in the form it is in. We wish to find the mass of the system, when the mass of the quarks are allowed to be set to zero. Thus the partition function which tells us the amplitude in another state in position space relative to a previous state the particle was in, does not give any information on the energy of the particle itself. Thus we can perform a fourier transform on the inverse dirac operator which we proved to be the propagator from 5.22. So in general we have that the propagator for the particle in momentum space is given by the following fourier transformation:

$$G(\vec{p}) = \int [dx] [dy] (\not{D})_{y,x}^{-1} e^{-p(x-y)} \quad (5.26)$$

Ignoring the lattice propagation, the interesting calculation is that of the quark propagation; how the two quarks behave when interacting with the gluon strong field and how the self interaction of the gluon field affects the final results. We imagine the system of the ρ meson moving through the sliced space-time lattice with some initial momentum \vec{p}_i . We can then zoom into the inner workings of this crude “ball” model of the meson and observe the ever elusive quarks in their tango with the gluon field. We apply the gluon flux tube model and using confinement, imagine the ρ particle to split and two quarks are created. At the end of the tube, the two quarks merge and annihilate each other leaving only the remaining energy. We can use Green’s contour model and think of one quark moving along the upper contour and the other quark along the lower contour, meeting up at the ends, making it seem the particle split into two parts. The initial momentum can be divided equally amongst the two quarks. Now we let the quark on the lower contour become the down anti-quark as stated in 5.24. The strange properties of the anti-quark come to light when one looks at Dirac’s hole theory. Unable to reconcile the negative energies found in the Klein-Gordon equation, Dirac proposed the idea of a sea of negative energy, where in empty space, there exists all states where the negative energy states are filled [5]. Now if a particle were to be raised from the negative energy state to a positive state, the space would show a “hole exhibiting positive energy and the same mass as the [particle] but [with] a opposite charge”[5]. Thus we can think of an anti-quark as a backward time propagating quark exhibiting a momentum negative to that of the regular up quark. The sign issue, so that the propagation is not null with respect to both quarks, is taken care of by the additional γ_μ matrices which switches the chirality (spin). Thus the overall propagation value is

$$G_\rho^{\mu\nu} = \text{Tr} \left(G \left(q + \frac{p}{2} \right) \gamma_\nu G \left(q - \frac{p}{2} \right) \gamma_\mu \right) \quad (5.27)$$

$$= \frac{f(p)}{p^2 + m_\rho^2} \quad (5.28)$$

where p is the momentum and m_ρ is the mass of the system after propagation. G is what we would have called $F_{\mu\nu}$ back in section 3. Thus we see that through this calculation the mass of the system is obtainable. Figure 7 shows what the entire process will look like through a pictorial representation.

6. NUMERICAL CALCULATION OF THE ρ MASS

The calculation of the mass of the ρ meson when taking the quarks to be light-like (in other words massless), involved using an algorithm created by Dr. Rajamani Narayanan at the Physics Department of Florida International University. The algorithm takes into account the lattice calculation and the propagation of the quarks through the gluon field.

6.1 Thermalization of the Lattice Field

The initial set up is to define the space by a seven dimensional array which contains information about the position of the particle in the 4-dimensional space time lattice, the direction it is moving in (μ or ν) and the information as to where in the matrix we are when considering the actual calculation. Thus the unitary matrix is defined by $U_{\mu,x,y,z,ct}(i,j) = U(i,j,\mu,n_1,n_2,n_3,n_4)$.

Now the lattice itself needs to be thermalized so that the fluctuations in the lattice are minimal and won't affect the overall calculation. The thermalization process works similar to that the maxwell's demon thermalization process. By visiting each site in the $N \times N$ matrix, a test is done to see whether the temperature in that cell is above a certain limit. If so, then reduction of the temperature begins, where the "demon" takes away the temperature value allotted to the cell and moves on. If the value in the cell is below the limit, then there is a certain probability for the temperature to be altered. By moving across the entire lattice this way, the action is then fully defined, using the definition from 5.4. Performing this calculation repetitively, allows for a successful thermalization and creates a lattice which is nearly devoid of any fluctuation points. The lattice size L , is set to different values. By starting with a small L , we work our way up to higher L 's taking different measurements. By allowing for large value of L , it becomes a good

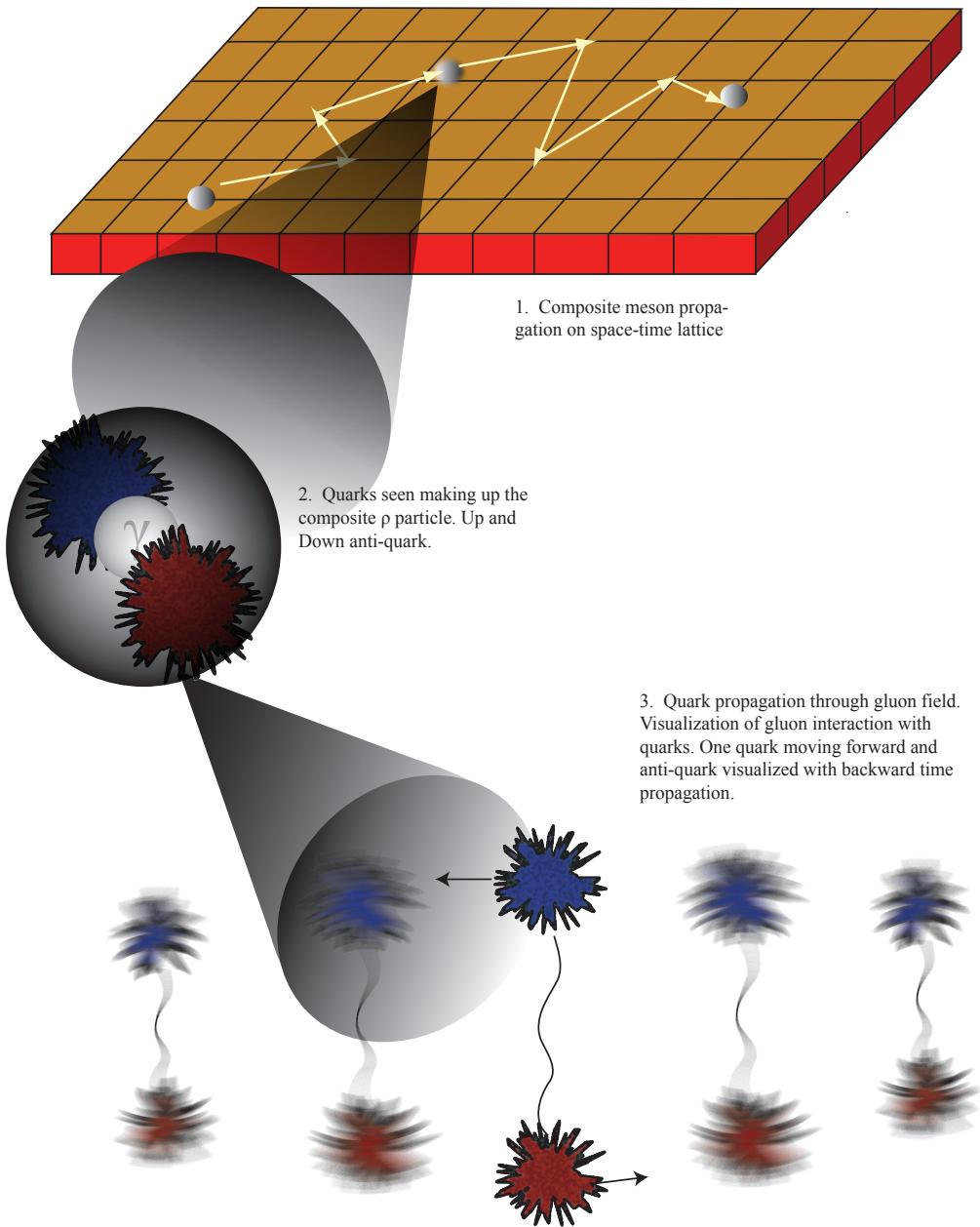


Figure 7: Full picture: ρ propagation through lattice and flux tube propagation of quarks.

enough approximation for the lattice spacing approaching zero, thereby eliminating the discreteness of space-time and making it continuous once again.

6.2 ρ Propagation Calculation

The calculation of the ρ mass was separated into two parts. Each part calculated the final amplitude of the two quarks of the particle. An initial wavefunction or source $|b(ich)\rangle$ was set with a positive chirality. This then is acted on by “the Smear Operator” S which is the “inverse of the laplacian” [2],

$$S^{-1} = \frac{1}{2} \sum_{\mu=1}^{d-1} \left(2 - T_\mu - T_\mu^\dagger \right) [2] \quad (6.1)$$

where T_μ is the translation operator of the lattice. The smearing of the source makes physical sense since the wavefunction of any particle is not confined to the space of the potential. Thus this spreads the wavefunction over many lattice spaces. Then is where the smeared source function, given a momentum which is the complex conjugate of the original initial momentum, is operated on by a particular γ_μ and then acted on by $|G(e^{\frac{-ip_m u}{2}})\rangle$ [2] giving an overall wavefunction denoted by $|gm\rangle_\mu$. Therefore this entire process can be described as follows:

$$|gm\rangle_\mu = |G(e^{\frac{-ip_m u}{2}})\rangle \gamma_\mu S |b\rangle \quad (6.2)$$

A similar process occurs with the other quark but in the reverse order with a positive chiral propagator $|G(e^{\frac{ip_m u}{2}})\rangle$ and a momentum which was left unchanged. Thus the final outcome is a wavefunction denoted by $|t_2\rangle_\nu$

$$|t_2\rangle_\nu = \gamma_\nu S |G(e^{\frac{ip_m u}{2}})\rangle |b\rangle \quad (6.3)$$

The entire process is broken down in figure 8 where the red path calculates the translated wavefunction for the up quark and the green path is the calculated wavefunction for the down anti-quark. It is important to mention that in order to make calculations much more efficient, we used only the momentum in one of the space-time directions. This is because only the absolute value of the momentum was considered. Thus the momentum chosen was put in terms of the the lattice size (L) and the number of color fields (N)

$$p_\mu = \begin{cases} 0 & \text{if } \mu = 1, 2, 3 \\ \frac{2\pi n}{NL} & \text{if } \mu = 4. \end{cases} \quad (6.4)$$

for values of $n = 2, 3, 4, 5, 6$.

So using 6.3 and 6.2, we can take the inner product of the two by calculating the complex conjugate of $|gm\rangle_\mu$ and then multiplying them together. From this the value for the inverse energy is calculated by

$$\begin{aligned} G_\rho^{\mu\nu} &= \langle b | S \gamma_\mu G(ichg) \gamma_\nu S G(ich) | b \rangle \\ &= \langle b | S \gamma_\mu G(e^{\frac{ip_\mu}{2}}) \gamma_\nu S G(e^{\frac{-ip_\mu}{2}}) | b \rangle \end{aligned} \quad (6.5)$$

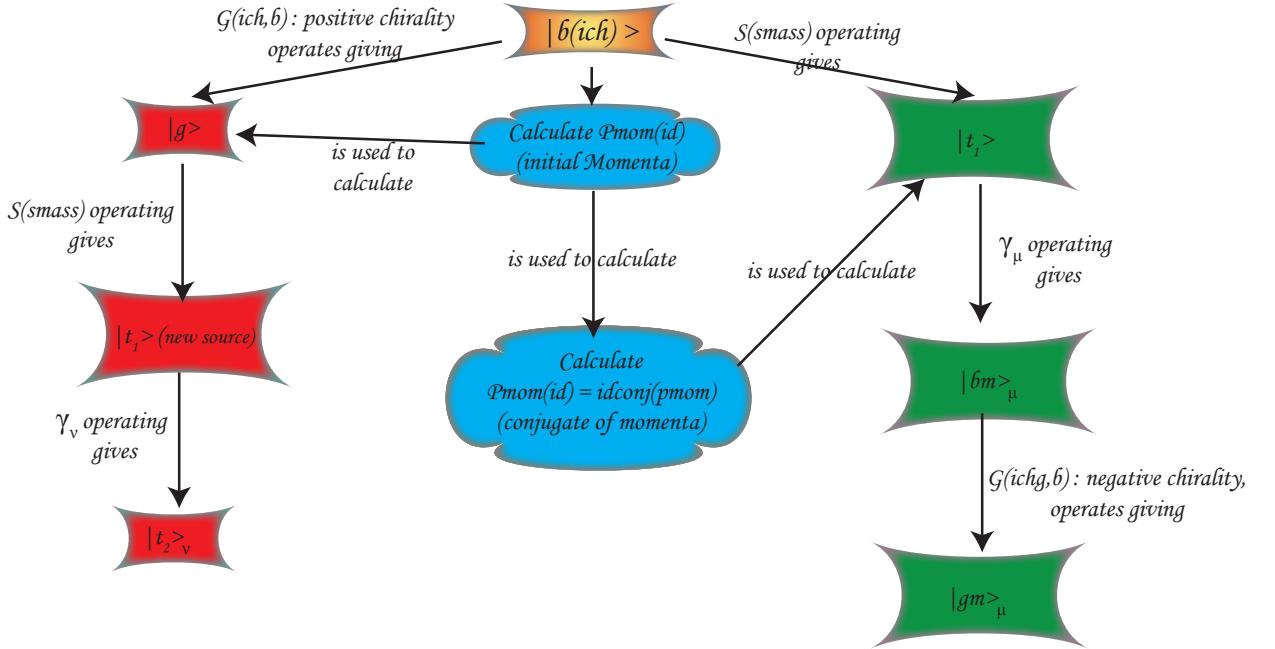


Figure 8: Calculation of two quark state amplitude propagation.

$$= \frac{f(p)}{p^2 + m_\rho^2} \quad (6.6)$$

where 6.6 gives a numerical answer for the expectation value of the two quark system amplitude, coupled with the value of the mass of the ρ meson. This mass is extracted in the following way: $G_\rho^{\mu\nu}$ is supposed to be a combination of a constant and a function $f(p)$ which is inversely proportional to the energy,

$$G_\rho^{\mu\nu} = C_{\mu\nu} + f(p) (p_\mu p_\nu - p^2 \delta^{\mu\nu}) \quad (6.7)$$

$$= C_{\mu\nu} + \frac{(p_\mu p_\nu - p^2 \delta^{\mu\nu})}{p^2 + m_\rho^2} \quad (6.8)$$

and $\delta^{\mu\nu}$ is the kroenecker delta function. Since μ and ν goes from $1 \rightarrow 4$ (indicative of the 3 space and 1 time directions) the output will be that of a 4×4 matrix where only the diagonal terms will be non-zero, because only one direction for the momentum is initially used. The only thing left is the extra constant $C_{\mu\nu}$ which hinders the final calculation. Thus a calculation of zero momenta is needed in order to subtract out this additional constant. From this an average over the diagonal entries are determined and a plot of $f(p)$ versus the intial momentum ($p_{\mu\nu}$) is created. From this the ρ meson mass squared value is obtained which is proportional to the intercept of the plot.

The calculation was done, not by setting the value of the initial mass to be zero, but by extrapolation of a ρ quark mass vs. meson mass graph. For the time being, we only used

b (thermalization factor)	Chiral Condensate	L (Lattice size)	N (color fields)	$L_c(b)$	$\Sigma^{\frac{1}{3}}(b)$
0.345		8	19	4.77	0.1675
0.350		8	23	5.97	0.1420
0.355		10	19	6.96	0.1265
0.360		11	17	8.01	0.1130

Table 3: Parameters used in calculation of meson mass[2].

quark masses which were of equal value to each other. Six different masses were used for the calculation, along with five initial momenta given to the meson particle. An average of the propagational expectation values were taken for each mass using the five different momenta. These were then used to extract the mass of the particle using a linear extrapolation where the mass was equivalent to the root of the ratio between the y-intercept and the slope of the graph, ignoring some physical constants of course. One can see that only two plots were done in figure 9. This is because only two of the lattice coupling constants were used as the other constants were too strong, in addition to the excessive mass of the meson as opposed to the lighter pion in [2]. The first two lattice couplings were discarded from 3. The two that were used were $b = 0.355$ and $b = 0.360$. Plots of the numerical computation using the two different lattice couplings are shown in 9, where linear regressions are taken in order to determine the mass of the quark-less meson.

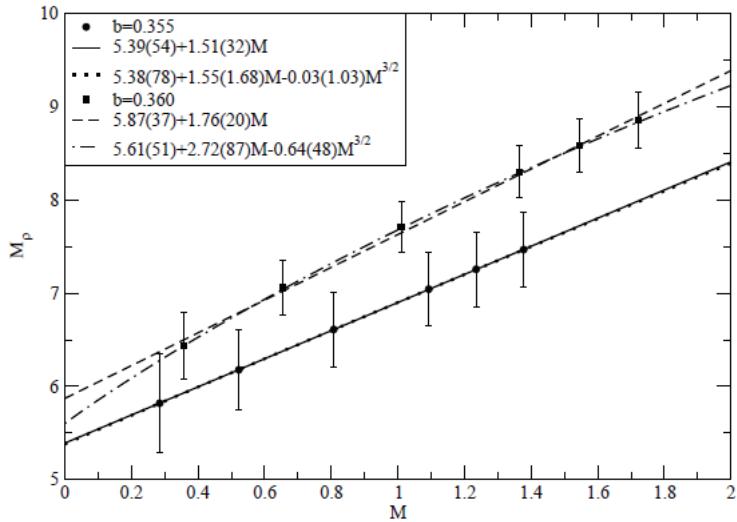


Figure 9: A plot of ρ mass as a function of the renormalization group invariant quark mass in dimensionless units [9].

Since this calculation was done for large N , we try to make sure that reality is left intact. Therefore, letting $N = 3$ and using the largest coupling constant (0.360), we see that the mass of the meson comes out to be $1082 \pm 70 MeV$ [9]. The actual experimental

value of the ρ meson mass comes out to be $775.5 \pm 0.4 MeV$.

6.3 Vector vs. Pseudovector Vector Meson Calculation

To diverge a little, we hit upon an important note in the calculation. The original plan of setting the mass of the quarks to zero did not bode well with the calculations, as will be explained in the following paragraphs. We start by simplifying our lives by calling 6.2 and 6.3, C_1 and C_2 respectively. From this, the propagational amplitude can be defined by the sum of these two values.

$$C_1 + C_2 = \langle b | G^\dagger(p, m_1) S \gamma_\nu G(-p, m_2) \gamma_\mu S | b \rangle. \quad (6.9)$$

A new property is also introduced here where 5.9 can be produce the adjoint of the dirac operator,

$$G^\dagger = \gamma_5 G \gamma_5 \quad (6.10)$$

through the employ of 5.13 and the fact that

$$\gamma_5 \not{D} \gamma_5 = \gamma_5 (i \gamma_\mu \partial_\mu + m) \gamma_5 \quad (6.11)$$

$$= i \gamma_5 \gamma_\mu \gamma_\mu \partial_\mu + m \quad (6.12)$$

$$= -i \gamma_\mu \partial_\mu + m \quad (6.13)$$

$$= \not{D}^\dagger \quad (6.14)$$

We can use the property seen in 6.10 in order to rearrange 6.9.

$$C_1 + C_2 = \langle b | \gamma_5 G^\dagger(p, m_1) S \gamma_\nu G(-p, m_2) \gamma_\mu S \gamma_5 | b \rangle. \quad (6.15)$$

$$= \langle b | \gamma_5 G^\dagger(p, m_1) S \gamma_\nu \gamma_5 \gamma_5 G(-p, m_2) \gamma_\mu S \gamma_5 | b \rangle. \quad (6.16)$$

$$= \langle b | G(p, m_1) S \gamma_\nu G^\dagger(-p, m_2) \gamma_\mu S | b \rangle. \quad (6.17)$$

However, instead of simply placing a γ_5 into the propagational result. Therefore the decision was to split the propagational amplitude of the quarks in order to leave some freedom into whether one would wish to calculate the mass of either a vector or pseudovector. The values of 6.2 and 6.3 are now to be denoted by the vectors $|\alpha\rangle$ and $|\beta\rangle$ respectively. We know from table 2 that in order to obtain a vector particle, one must perform $\langle\alpha|\gamma_5|\beta\rangle$ and $\langle\alpha|\beta\rangle$ in order to obtain a pseudovector particle. Instead of this straightforward calculation, we go ahead and perform the following calculations:

$$C_1 = \frac{1}{2} \langle \alpha | 1 + \gamma_5 | \beta \rangle \quad (6.18)$$

$$C_2 = \frac{1}{2} \langle \alpha | 1 - \gamma_5 | \beta \rangle \quad (6.19)$$

By adding the extra $\frac{1 \pm \gamma_5}{2}$, the matrix values either remain non-zero and zero for the top half of the matrix when it is plus and the bottom half when it is negative. Thus, only half of the expectation value is ever attained at any time. Therefore, by subtracting or adding, one can attain either the pseudovector form or the vector form of the answer. Since our goal is to calculate the mass of the a ρ meson, we therefore choose to do the calculation where we would subtract 6.18 and 6.19.

7. CONCLUSION

Our study of Quantum Chromodynamics led to a greater understanding to the fundamental nature of the universe and the particles which reside in it. Quarks, which are one of the fundamental particles in the universe, are the most elusive of all particles due to their confined nature. The confinement created by the strong force interaction via the gluons, results in very interesting phenomena for the composite particles. By performing a stochastic calculation of the ρ meson, we implemented Feynman's path integral method to determine how the particle and its quarks propagated through their respective fields, and how the least action principle played a role in that particular behavior. Green's function gave us a way to determine the state of the particle from an initial ground state. Parallel transport theory hints at the changes felt by a particle in the vicinity of a gauge potential and how that potential, depending on whether you deal with an EM field, weak field or strong field, interacts with itself and the particle in question. In the case of $N = 3$ and above, the strong force dominates and the self interaction term in the extra commutator takes precedence. The calculation was done using a code created by Dr. Rajamani Narayanan, running simultaneously on a 32 node cluster. Two separate lattice couplings were used to perform these calculations, along with six different initial masses and five separate momenta. The final propagational expectation value was recorded and an average was taken over the various momenta. A linear regression was performed on the graph of the initial quark mass against the final meson mass in order to determine the mass of the vector meson at initial zero quark mass. Using the lattice coupling value of 0.360, the calculated meson mass regressed back to $N = 3$ colors produced a vector meson with a theoretically calculated mass close to 25% less than experimental results. This discrepancy, although noteworthy, is most likely due to the size of the coupling and the massiveness of the particle itself, thereby causing induced lattice spacing effects.

Acknowledgments

I would like to acknowledge the partnership of Carlos Prays, who also participated in this research endeavor. I would also like to acknowledge Dr. Jorge Rodriguez of the Physics Department at Florida International University (FIU) for his help with using the Tier 3 cluster at FIU. Also to Ricardo Leante for his aid in helping us understand the process of grid computing, Ari Heitenan and Richard Galvez for their support and to Dr. Rajamani Narayanan who guided us through this project.

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