Binary Search Algorithm Solution


```
def binary_search(arr, target):
    low = 0
    high = len(arr) - 1

    while low <= high:
       mid = (low + high) // 2
        if arr[mid] == target:
            return mid
        elif arr[mid] < target:
            low = mid + 1
        else:
            high = mid - 1
    return -1
```

Binary Search Proof of Correctness with Induction

Statement F(n): Binary Search correctly returns the index of the array of size n, or -1 if it is not present

Base Case: F(1)

- Array has one element low = 0, high = 0
- The loop runs. mid = 0
- It checks if arr[0] == target
  - If yes, returns 0.
  - If no, it updates low or high and the loop ends, returning -1.

Inductive Hypothesis: Assume F(k) is true for all k < n

- We trust that binary_search works perfectly on any list smaller than the one we have now. This is our foundation.

Inductive Step: Prove F(n) for n > 1

- The algorithm computes mid. Now, three things can happen →

Binary Search Proof of Correctness with Induction

Case 1
arr[mid] == target

- It returns mid immediately. This is trivial but correct.

Case 2
arr[mid] > target

- Because the list is sorted by, say, zip code, every element from mid to the end is too large
- So, it sets high = mid - 1. The new search space is arr[low...mid-1].
- The size of this subarray is mid - low. Since n > 1 and mid is a valid index, this size is less than n
- By our Inductive Hypothesis, the search on this smaller subarray is correct

Case 3
arr[mid] < target

- Symmetric logic. Every element from the start to mid is too small
- It sets low = mid + 1. The new search space is arr[mid+1...high], which has size high - mid, which is also less than n
- The Inductive Hypothesis guarantees correctness here too

Conclusion

- By the principle of mathematical induction, since it's true for n=1 and truth for all k < n implies truth for n, then F(n) is true for all positive integers n.

The Master Theorem gives us a solution. For recurrences of form:
T(n) = aT(n/b) + f(n)

- a ≥ 1: Number of recursive calls
- b > 1: Factor by which problem shrinks
- f(n): Work to divide/combine

Taking an equation of the form:

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where, $a \geq 1$, $b > 1$ and $f(n) > 0$

The Master's Theorem states:

- **CASE 1** - if $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
- **CASE 2** - if $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$
- **CASE 3** - if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.

Three Cases:

1. If f(n) grows slower than n^(log_b a) -> Case 1
2. If f(n) grows same as n^(log_b a) -> Case 2
3. If f(n) grows faster than n^(log_b a) -> Case 3

Apply to Binary Search:

- a = 1, b = 2, f(n) = c -
- n^(log_b a) = n^(log_2 1) = n^0 = 1
- f(n) = c = **Θ**(1) = **Θ**(n^0) -> Case 2
- Solution: T(n) ∈ **Θ**(log n)

Apply to Merge Sort:

- a = 2, b = 2, f(n) = cn
- n^(log_b a) = n^(log_2 2) = n^1 = n
- f(n) = cn = **Θ**(n) -> Case 2
- Solution: T(n) ∈ **Θ**(n log n)

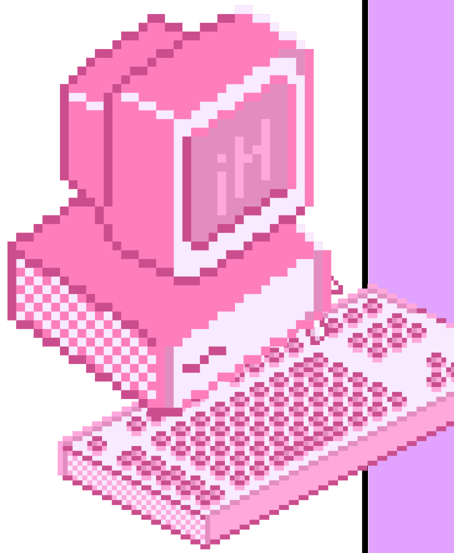
Solving Recurrences - Subsitution - Binary Search

For recurrences where the problem size shrinks quickly, repeated substitution is very effective like in Binary Search.

1. $T(n) = T(n/2) + c$
2. Substitute: $T(n) = [T(n/4) + c\ ] + c = T(n/4) + 2c$
3. Substitute: $T(n) = [T(n/8) + c] + 2c = T(n/8) + 3c$
4. After k substitutions: $T(n) = T(n\ /\ 2\text{^}k) + k*c$

Now, we hit the base case. Let $T(1) = a$ (a constant).
Set $n/2\text{^}k = 1 \rightarrow n = 2\text{^}k \rightarrow k = \log_2(n)$

Substitute back:
$T(n) = T(1) + log\_2(n) * c$ $T(n) = a + c*log\_2(n)$
Therefore, $T(n) \in O(\log n)$.



Solving Recurrences - Recursion Tree - Binary Search

Recursion Tree Analysis:

- Level 0: 1 problem of size n -> Work = c
  - Searching all shelters in the state
- Level 1: 1 problem of size n/2 -> Work = c
  - Eliminated half the state, searching one region
- Level 2: 1 problem of size n/4 -> Work = c
  - Eliminated half again, searching a county

  ...
- Level k: 1 problem of size n/2^k -> Work = c
  - Now searching a specific neighborhood

How many levels? → The search stops when the problem size is 1 (when we're looking at a single shelter): n/2^h = 1 → 2^h = n -> h = log_2(n)

Total Work: We have $\log_2(n)$ levels, each doing constant c work. T(n) = c + c + c + ... + c [for log_2(n) terms] T(n) = c * log_2(n)

Therefore, $T(n) \in O(\log n)$.

Proving Runtime - The Recurrence Relations

Home Content Contact

Binary Search

$T(n) = T(n/2) + c$

- c represents the constant work: calculating mid, comparing arr[mid] to the target, and updating pointers

THE RECURRENCE RELATIONS

Merge Sort

$T(n) = 2T(n/2) + cn$

- $2T(n/2)$ is the cost of solving the two subproblems.
- cn is the cost of merging two sublists of size $n/2$. This is $O(n)$.