LECTURE 4

Vectors: applications of the toolkit

We will now use the ideas from the previous lecture to derive some results. While the final results are themselves useful, *the point of today's lecture is to show you how to derive them*, so that you can learn to derive similar results on your own.

Equation of a plane

There is simple relationship between the Cartesian components of a vector, and the equation of the plane to which it is normal.

Suppose we know a fixed point in the plane, at coordinates (x_0, y_0, z_0). Then the position vector of that point (with respect to the origin) is

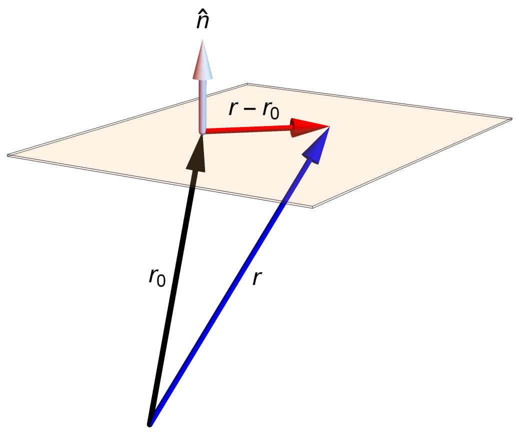
$$\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}.$$

Now let $\mathbf{r}$ be the position vector of an arbitrary point *also* in the plane, at coordinates (x, y, z):

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

From the diagram below, we see that $\mathbf{r} - \mathbf{r}_0$ is always perpendicular to the normal to the plane, $\mathbf{n}$, i.e.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0.$$



Now write the normal vector in the Cartesian basis. We'll call its components a, b and c for simplicity later.

$$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{z}.$$

Expressing the dot product in terms of components, we see that

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

i.e.

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

You might find it helpful to think of this equation as

$$\mathbf{r} \cdot \hat{\mathbf{n}} = \text{constant}.$$

In other words, the position vector of every point in the plane has the *same* component normal to the plane. The position vectors differ only in their components parallel to the plane.

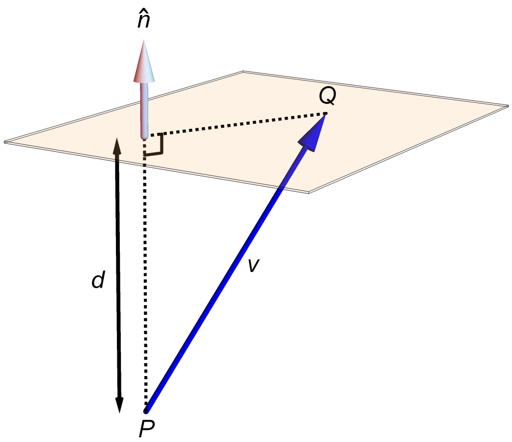
The right-hand side is a constant, which we can call d. Hence we've shown that

$$\boxed{\text{The plane } ax + by + cz = d \text{ has normal vector } a\mathbf{i} + b\mathbf{j} + c\mathbf{k}. \qquad (4.1)}$$

Distance from point to plane

In crystallography, where X-rays are scattered by planes of atoms to produce a diffraction pattern, we need to work out the distance between parallel planes. This can be done by finding the distance from a point in one plane to a point in the other.

Let's ask the simpler question: what is the shortest distance from point P to the plane (with normal $\hat{\mathbf{n}}$) in the diagram below?



To answer this, suppose we know a point in the plane, which we'll call Q. (It can be *any* point in the plane.) Let the vector $\overrightarrow{PQ}$ be called $\mathbf{v}$.

Then, very simply, the shortest distance from P to the plane is the (modulus of the) component of $\mathbf{v}$ along $\mathbf{n}$:

$$d = |\mathbf{v} \cdot \hat{\mathbf{n}}|.$$

To see this, look at the diagram above. I'll illustrate it with an example.

Example 1. Find the shortest distance from the point $(3, 2, 1)$ to the plane $x + 2y + 3z = 4$.

From (4.1), the unit normal to the plane is

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{14}} (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}).$$

We are told that P is $(3, 2, 1)$. A point Q in the plane is just some (x, y, z) that satisfies the equation: for example, $(4, 0, 0)$ works. Then

$$\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

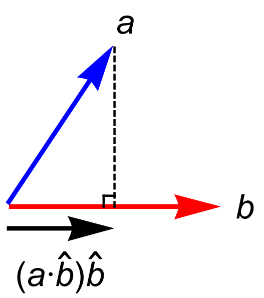
Now we just find the component of $\mathbf{v}$ along $\hat{\mathbf{n}}$, and take its modulus:

$$d = |\mathbf{v} \cdot \hat{\mathbf{n}}| = \frac{6}{\sqrt{14}}.$$

Projecting vectors onto vectors and planes

The previous section leads nicely into the idea of orthogonal projection. Given a vector $\mathbf{v}$ and a plane normal to $\hat{\mathbf{n}}$, we often want to extract the contribution from $\mathbf{v}$ that is parallel to the plane. We'll see how to do that in due course.

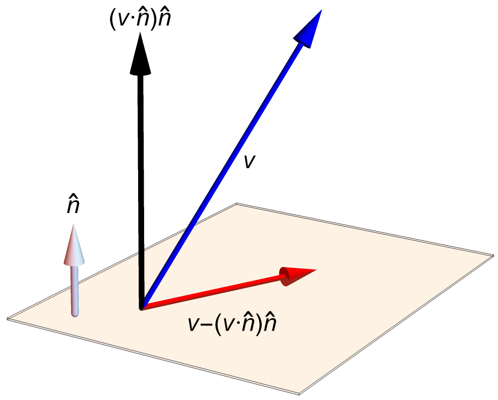
But first, we'll look at the projection of a vector in the direction of another vector. This is shown in the diagram below:



The *component* of $\mathbf{a}$ along $\mathbf{b}$, i.e. *how much* $\mathbf{a}$ points in the direction of $\mathbf{b}$, is $\mathbf{a} \cdot \hat{\mathbf{b}}$. Multiplying this component by the unit vector $\hat{\mathbf{b}}$ gives us a vector which is the 'shadow' cast by $\mathbf{a}$ along $\mathbf{b}$:

$$\text{vector projection of } \mathbf{a} \text{ onto } \mathbf{b} \text{ is } (\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}. \qquad (4.2)$$

Now we find the projection of a vector onto a plane. Look at the diagram below.



The projection of $\mathbf{v}$ in the direction *perpendicular* to the plane is (from the argument above)

$$\mathbf{v}_\perp = (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}.$$

Therefore, since $\mathbf{v}$ is the sum of its contributions perpendicular to, and parallel to, the plane, we can find the parallel contribution by subtraction. Specifically,

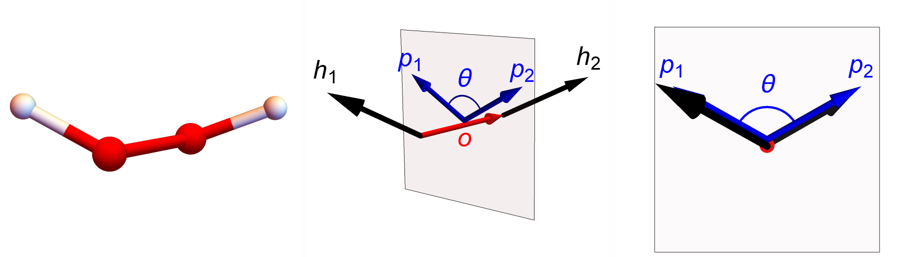
$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel,$$

so $\mathbf{v}_\parallel = \mathbf{v} - \mathbf{v}_\perp$, and hence

projection of $\mathbf{v}$ onto plane with normal $\hat{\mathbf{n}}$ is $\mathbf{v} - (\mathbf{v} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}$.

Example 2. Let $\mathbf{h}_1$ and $\mathbf{h_2}$ be vectors along the OH bonds of H_2O_2 (from O to H). Let $\mathbf{o}$ be a vector along the OO bond. Find the dihedral angle.

One method is to project the $\mathbf{h}_1$ and $\mathbf{h}_2$ vectors onto a plane perpendicular to $\mathbf{o}$. See the diagrams below.



The centre figure shows the projected $\mathbf{h}_1$ and $\mathbf{h}_2$ vectors, which we call $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Viewed along the OO bond (right-hand figure), we see that the angle θ between these is the desired dihedral angle.

Hence,

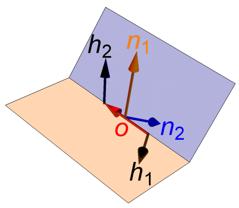
$$\cos\theta = \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{|\mathbf{p}_1||\mathbf{p}_2|}$$

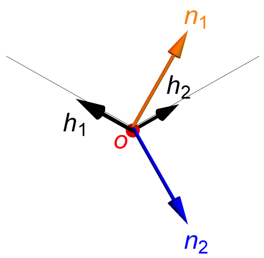
where

$$\mathbf{p}_i = \mathbf{h}_i - (\mathbf{h}_i \cdot \hat{\mathbf{o}})\hat{\mathbf{o}}.$$

Example 3. Do Example 2 again, this time using the vector product.

This time, look at the diagrams below.





Vectors $\mathbf{o}$ and $\mathbf{h}_1$ define a plane (coloured orange) with normal $\mathbf{n}_1$. Likewise, $\mathbf{o}$ and $\mathbf{h}_2$ define a plane (coloured blue) with normal $\mathbf{n}_2$.

Viewed from the side (right-hand figure), we see that the angle between the normals is the angle between the planes, since both normals are rotated by $\pi/2$ from their planes.

The normals can be calculated using the vector product. However, we need to ensure that we take these the correct way round, using the right-hand corkscrew rule. Looking at the diagram above, it follows that

$$\mathbf{n}_1 = \mathbf{o} \times \mathbf{h}_1$$

and

$$\mathbf{n}_2 = \mathbf{o} \times \mathbf{h}_2.$$

Then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}.$$

Homogeneous linear equations: redux

A couple of lectures ago, we saw that homogeneous linear equations like the following

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \quad (4.3)$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \quad (4.4)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0. \quad (4.5)$$

have only trivial solutions when $D \neq 0$. This is because we can use Cramer's rules when $D \neq 0$, and those obviously imply $x_1 = x_2 = x_3 = 0$. So non-trivial solutions can only exist when $D = 0$.

Now we can show that non-trivial solutions *always* exist when $D = 0$.

The homogeneous system of equations can be expressed as a set of scalar products, each equal to zero:

$$\mathbf{a}_1 \cdot \mathbf{x} = 0 \tag{4.6a}$$
$$\mathbf{a}_2 \cdot \mathbf{x} = 0 \tag{4.6b}$$
$$\mathbf{a}_3 \cdot \mathbf{x} = 0 \tag{4.6c}$$

where

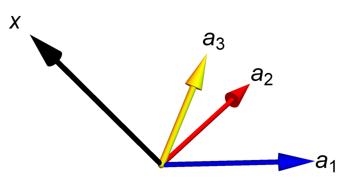
$$\mathbf{a}_1 = a_{11}\mathbf{i} + a_{12}\mathbf{j} + a_{13}\mathbf{k},$$

etc., and

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}.$$

We want to determine x_1, x_2 and x_3 given a particular set of coefficients a_{ij}. Viewed in terms of scalar products, we want to find the vector $\mathbf{x}$ that is orthogonal to the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.

Clearly this is only possible if the three $\mathbf{a}_i$ vectors are coplanar; if they are not, it is impossible for a fourth vector to be orthogonal to them all in three-dimensional space. Convince yourself of this by looking at the diagram below.



So, for a non-trivial solution to be possible, we need $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3 = 0$, which is precisely the condition $D = 0$ we established before!

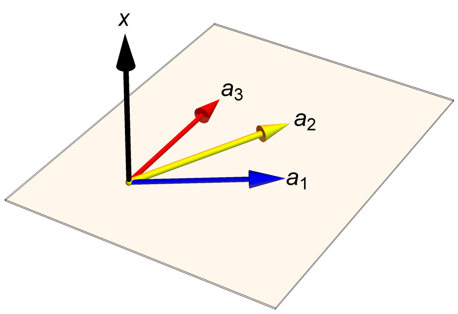
And if the three vectors are coplanar, it is clearly possible to make $\mathbf{x}$ orthogonal to them all, by making it orthogonal to their mutual plane.

All the arguments here generalise to n dimensional vectors. In 3 dimensions, there is an extra trick that we can exploit, though. Since $\mathbf{x}$ is perpendicular to the $\mathbf{a}_i$ vectors, we can cross two of the latter to find a non-trivial solution, $\mathbf{x}$. Then all non-trivial solutions are just multiples of this one.

To illustrate this, look back at Example 4 of Lecture 2. We find

$$\mathbf{a}_1 \times \mathbf{a}_3 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}.$$

So the non-trivial solutions are of the form $(-\lambda, \lambda, 0)$, as we obtained by a longer method earlier.



Indeed, *any* x that is in a direction orthogonal to this plane will satisfy (4.6), so there are infinitely-many non-trivial solutions, as we saw before.