Appendix: a sketch of a $\det(\mathsf{AB})$ proof, for those who are interested

The conventional way to prove that $\det(\mathsf{AB}) = (\det \mathsf{A})(\det \mathsf{B})$ is by using so-called 'elementary row operations'. Here's an alternative, if you're interested. It involves some rules of *block matrices*, and can be formalised using proof by induction.

It also gives me a reason to write this out properly, because I can never remember how to do it when I want it, and end up working it out from scratch every time.

First, we need to establish a preliminary result. Consider a square matrix, D. We will create a bigger square matrix by adding one extra row and column, as follows

$$\mathsf{M}_1 = \left(\begin{array}{c|c} a & \mathbf{v} \\ \hline \mathbf{0} & \mathsf{D} \end{array}\right). \tag{2.11}$$

a is a single number, $\mathbf{v}$ is an arbitrary row vector, and $\mathbf{0}$ is a column of zeros. For example, if

$$\mathsf{D} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

then M_1 might look like

$$\mathsf{M}_1 = \left(\begin{array}{c|cc} 5 & 6 & 7 \\ \hline 0 & 1 & 2 \\ 0 & 3 & 4 \end{array}\right)$$

(The dimensions of $\mathbf{v}$ and $\mathbf{0}$ are determined by the sizes of the other elements in the matrix.) By expanding (2.11) down column 1, we see that

$$\det \mathsf{M}_1 = a \det \mathsf{D}$$

Now consider

$$\mathsf{M}_2 = \left(\begin{array}{cc|c} a & b & \mathbf{u} \\ c & d & \mathbf{v} \\ \hline \mathbf{0} & \mathbf{0} & \mathsf{D} \end{array}\right).$$

Expanding this down column 1 gives

$$\det \mathsf{M}_2 = a \det \left(\begin{array}{c|c} d & \mathbf{v} \\ \hline 0 & \mathsf{D} \end{array}\right) - c \det \left(\begin{array}{c|c} b & \mathbf{u} \\ \hline 0 & \mathsf{D} \end{array}\right).$$

Using the earlier result for matrices of the form M_1, this becomes

$$\det \mathsf{M}_2 = ad \det \mathsf{D} - cb \det \mathsf{D},$$

which can be written as

$$\det \mathsf{M}_2 = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \mathsf{D}.$$

Hopefully you can now see the pattern. If we keep building matrices by adding a single row and column and column at a time, we can show in general that

$$\det \left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline 0 & \mathsf{D} \end{array}\right) = \det \mathsf{A} \det \mathsf{D} \tag{2.12}$$

where A and D are any square matrices. B is a rectangular matrix containing arbitrary elements, and 0 is a rectangular null matrix.

We now need a couple more facts about matrix multiplication. First, multiplying on the left by a *lower triangular* natrix adds multiples of one column to another. What I mean by this is, for example:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & \gamma & 1 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b+\alpha a & c+\beta a+\gamma b \\ d & e+\alpha d & f+\beta d+\gamma e \\ g & h+\alpha g & i+\beta g+\gamma h \end{pmatrix} \tag{2.13}$$

Second, matrices with the same block structure multiply as follows:

$$\left(\begin{array}{c|c} \mathsf{A} & \mathsf{B} \\ \hline \mathsf{C} & \mathsf{D} \end{array}\right)\left(\begin{array}{c|c} \mathsf{E} & \mathsf{F} \\ \hline \mathsf{G} & \mathsf{H} \end{array}\right) = \left(\begin{array}{c|c} \mathsf{AE}+\mathsf{BG} & \mathsf{AF}+\mathsf{BH} \\ \hline \mathsf{CE}+\mathsf{DG} & \mathsf{CF}+\mathsf{DH} \end{array}\right) \tag{2.14}$$

(where, for example, AE denotes the normal matrix multiplication of A and E.) You can check this result by writing down a couple of 4×4 matrices and multiplying them.

We're now ready to put all this together and construct the proof. Consider

$$\det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right)$$

We know that a determinant is unchanged by adding multiples of columns to each other. Therefore, from (2.13), a determinant of a matrix is unchanged if we multiply the matrix by a lower-triangular matrix on the left. Hence

There's a subtlety here. To make the proof watertight, we need to look a bit more closely at the order in which multiples of columns are added to each other…

$$\det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right) = \det\left[\left(\begin{array}{c|c} \mathsf{I} & 0 \\ \hline -\mathsf{B} & \mathsf{I} \end{array}\right)\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right)\right]$$

Carrying out the matrix multiplication using (2.14),

$$\det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right) = \det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline -\mathsf{AB} & 0 \end{array}\right)$$

Each block is $n\times n$, so we can take out n minus signs from the lower n rows of the determinant, to get

$$\det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right) = (-1)^n \det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline \mathsf{AB} & 0 \end{array}\right).$$

Then we can swap the first n columns with the second n columns. Each column swap introduces a factor of (-1), so overall this cancels out the $(-1)^n$ in front:

$$\det\left(\begin{array}{c|c} \mathsf{A} & \mathsf{I} \\ \hline 0 & \mathsf{B} \end{array}\right) = \det\left(\begin{array}{c|c} \mathsf{I} & \mathsf{A} \\ \hline 0 & \mathsf{AB} \end{array}\right).$$

Then, using (2.12) on both sides, we obtain

$$\det \mathsf{A} \det \mathsf{B} = \det \mathsf{AB}$$

as required.