Appendix: why geometric multiplicity equals algebraic multiplicity for symmetric matrices

For clarity, we will show the result for a special case. The generalisation is not difficult. Let's take A to be a real, symmetric 5×5 matrix, and suppose that λ_* is an eigenvalue of A with geometric multiplicity 2: i.e. that all the eigenvectors that satisfy

$$\mathsf{A}\mathbf{v} = \lambda_* \mathbf{v}$$

span a two-dimensional plane. Then choose two eigenvectors in this plane, called $\mathbf{v}_1$ and $\mathbf{v}_2$, that are orthonormal.

Construct three further unit vectors ($\mathbf{v}_3$, $\mathbf{v}_4$, and $\mathbf{v}_5$) that are all orthogonal to these eigenvectors, and to each other. Then the five vectors $\mathbf{v}_i$ are orthonormal. Use this to form a matrix S:

$$\mathsf{S} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}.$$

Now consider the effect of premultiplying S by A. We can use a similar approach to that used at the beginning of Lecture 5. However, since only $\mathbf{v}_1$ and $\mathbf{v}_2$ are known to be eigenvectors, it follows that

$$\mathsf{AS} = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \lambda_*\mathbf{v}_1 & \lambda_*\mathbf{v}_2 & \mathbf{x}_3 & \mathbf{x}_4 & \mathbf{x}_5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

where nothing can be said about the columns denoted $\mathbf{x}_i$, at this point. Then, again following the idea of Lecture 5, this can be written as

$$\mathsf{AS} = \mathsf{S} \begin{pmatrix} \lambda_* & 0 & c_{13} & c_{14} & c_{15} \\ 0 & \lambda_* & c_{23} & c_{24} & c_{25} \\ 0 & 0 & c_{33} & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} \end{pmatrix}$$

(where each c_{ij} represents some arbitrary element). Noting that S is orthogonal, because it was constructed from orthonormal columns, we can then pre-multiply both sides by S^T to get

$$\mathsf{S}^\mathsf{T}\mathsf{AS} = \begin{pmatrix} \lambda_* & 0 & c_{13} & c_{14} & c_{15} \\ 0 & \lambda_* & c_{23} & c_{24} & c_{25} \\ 0 & 0 & c_{33} & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} \end{pmatrix}$$

From the properties of the transpose, we have $(\mathsf{S}^\mathsf{T}\mathsf{AS})^\mathsf{T} = \mathsf{S}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{S}$. Given that A is symmetric, this means that

$$(\mathsf{S}^\mathsf{T}\mathsf{AS})^\mathsf{T} = \mathsf{S}^\mathsf{T}\mathsf{AS}$$

i.e. $\mathsf{S}^\mathsf{T}\mathsf{AS}$ is itself symmetric. Therefore we can conclude that some of the c_{ij} elements must actually be zero:

$$\mathsf{S}^\mathsf{T}\mathsf{AS} = \begin{pmatrix} \lambda_* & 0 & 0 & 0 & 0 \\ 0 & \lambda_* & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} \end{pmatrix}.$$

Let's denote by C the 3×3 block in the bottom-right corner.

Now, we will look at the characteristic polynomial of A. Since the determinant is unaffected by a similarity transformation, we have

$$\begin{aligned}
\det(\mathsf{A} - \lambda \mathrm{I}) &= \det\left[\mathsf{S}^{-1}(\mathsf{A} - \lambda \mathrm{I})\mathsf{S}\right] \\
&= \det(\mathsf{S}^{-1}\mathsf{A}\mathsf{S} - \lambda \mathrm{I}) \\
&= \det\begin{pmatrix} \lambda_* - \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda_* - \lambda & 0 & 0 & 0 \\ 0 & 0 & c_{33} - \lambda & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} - \lambda & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} - \lambda \end{pmatrix} \\
&= (\lambda_* - \lambda)^2 \det\begin{pmatrix} c_{33} - \lambda & c_{34} & c_{35} \\ c_{43} & c_{44} - \lambda & c_{45} \\ c_{53} & c_{54} & c_{55} - \lambda \end{pmatrix} \\
&= (\lambda_* - \lambda)^2 \det(\mathsf{C} - \lambda \mathrm{I}).
\end{aligned}$$

We want to show that the algebraic multiplicity of the λ_* eigenvalue is 2 (i.e. the same as the geometric multiplicity). That is,

$$\det(\mathsf{C} - \lambda \mathrm{I})$$

can have no factors of $(\lambda - \lambda_*)$, i.e. λ_* must not be an eigenvalue of C.

It is easy to prove this by contradiction. If $\mathsf{C}\mathbf{u} = \lambda_* \mathbf{u}$, then

$$\begin{pmatrix} \lambda_* & 0 & 0 & 0 & 0 \\ 0 & \lambda_* & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} & c_{35} \\ 0 & 0 & c_{43} & c_{44} & c_{45} \\ 0 & 0 & c_{53} & c_{54} & c_{55} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \lambda_* \begin{pmatrix} 0 \\ 0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

i.e. $\begin{pmatrix} 0 & 0 & u_1 & u_2 & u_3 \end{pmatrix}^{\mathsf{T}}$, which we shall denote by $\mathbf{v}'$, satisfies

$$\mathsf{S}^{\mathsf{T}}\mathsf{A}\mathsf{S}\mathbf{v}' = \lambda_* \mathbf{v}'$$

i.e.

$$\mathsf{A}\left(\mathsf{S}\mathbf{v}'\right) = \lambda_* \left(\mathsf{S}\mathbf{v}'\right).$$

Thus, $\mathsf{S}\mathbf{v}'$ is an eigenvector of A with eigenvalue λ_*. But from the definitions of S and $\mathbf{v}'$, the vector $\mathsf{S}\mathbf{v}'$ is a linear combination of $\mathbf{v}_3$, $\mathbf{v}_4$ and $\mathbf{v}_5$. It is therefore orthogonal to the plane of $\mathbf{v}_1$ and $\mathbf{v}_2$, which contradicts our initial assumption that all eigenvectors with eigenvalue λ_* are in that plane.

Therefore, we have shown that the geometric multiplicity of 2 (the fact that the eigenvectors span a two-dimensional plane) corresponds exactly to an algebraic multiplicity of two [the fact that exactly $(\lambda_* - \lambda)^2$ appears in the characteristic polynomial]. The proof clearly relied on A being symmetric.