LECTURE 2

Systems of *n* linear equations in *n* unknowns

In the previous lecture, we examined the properties of determinants. Our motivation for doing this will now become clear: determinants *determine the number of solutions* of a set of simultaneous equations. When there is a *unique* solution, they can also be used to find the solution.

Systems of simultaneous linear equations

A system of simultaneous linear equations looks like the following:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = c_1 \tag{2.1}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = c_2 \tag{2.2}$$
$$\vdots \tag{2.3}$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = c_n. \tag{2.4}$$

The unknowns are x_1, x_2, …, x_m: i.e. there are m unknowns in the system. The values of a_{ij} and c_j are known, constant coefficients. There are n equations in total.

Using matrix methods, as introduced later in this course, one can tackle the general question of how to solve systems of n equations in m unknowns. Today we'll focus on the special case of n equations in n unknowns: i.e. the same number of equations and unknowns. This is the case that you will meet most often in chemistry.

Even in this special case, there are some important questions to ask. Can such systems always be solved? Is there only one solution, or can there be multiple solutions? You may have some intuition on this already: we'll see today how to formalise it mathematically.

Systems of 3 equations in 3 unknowns

Let's make life a little simpler by considering the specific case of 3 equations in 3 unknowns. The method is easily extended to the case of general n.

Our system of equations becomes

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 \tag{2.5}$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 \tag{2.6}$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3. \tag{2.7}$$

The 9 coefficients on the LHS can be put into a determinant, which we'll call D:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.8}$$

Then, by multiplying both sides by the unknown x_1, we have

$$x_1 D = x_1 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} \\ a_{21}x_1 & a_{22} & a_{23} \\ a_{31}x_1 & a_{32} & a_{33} \end{vmatrix}, \tag{2.9}$$

where we have used the property (from last lecture) that common factors can be taken in or out of a single column.

Now we'll use another property: that any multiple of one column can be added to another. We'll add x_2 lots of column 2 to column 1. Then we'll add x_3 lots of column 3 to column 1:

$$x_1 D = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 & a_{12} & a_{13} \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 & a_{22} & a_{23} \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 & a_{32} & a_{33} \end{vmatrix}.$$

The first column of $x_1 D$ now contains the LHSs of our system of equations! We can therefore replace each element in column 1 by the corresponding RHS of the system of equations, to get

$$x_1 D = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}.$$

We'll call this determinant D_1, i.e. we've shown that

$$\boxed{x_1 D = D_1 \qquad \text{where} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \quad D_1 = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}. \qquad (2.10)}$$

This is our key result.

The values of D and D_1 depend on the coefficients of the system of equations. Depending on these coefficients, there are three possibilities:

1. $D \neq 0$. In this case, (2.10) can be rearranged to get $x_1 = D_1/D$.
2. $D = 0$ and $D_1 \neq 0$. In this case, (2.10) is true for *no* value of x_1, since it becomes $0 = D_1$ (which, dividing by D_1, becomes $0 = 1$).
3. $D = 0$ and $D_1 = 0$. In this case, (2.10) tells us nothing about x_1, since it becomes $0 = 0$.

We'll consider them each in turn.

$D \neq 0$: Cramer's rules

There is nothing special about the unknown variable x_1. By repeating the argument above, it is easy to show, for example, that

$$x_2 D = D_2 \qquad \text{where} \quad D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \quad D_2 = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}.$$

Notice that the *same* determinant D appears on the LHS, and that the RHS determinant D_2 is obtained from D by replacing column 2 by the numbers on the RHS of our system of equations, (2.5). It will not surprise you to hear that $x_3 D = D_3$, with D_3 obtained in an obvious way.

So in the case of $D \neq 0$, therefore, we can divide all three equations by D to get

$$x_1 = \frac{D_1}{D} \qquad x_2 = \frac{D_2}{D} \qquad x_3 = \frac{D_3}{D}.$$

These are Cramer's rules. They give us a systematic way to solve 3 equations in 3 unknowns, in the case of $D \neq 0$, simply by working out 4 determinants.

Working out all the determinants, however, is generally tedious and time-consuming. There are better methods to use if you want to solve lots of these systems, or if you want to program a computer to do it…

Since Cramer's rules lead to specific values for x_1, x_2, and x_3, the system of equations has a *single* solution.

Example 1. Check that the following system has a single solution, and find it.

$$\begin{aligned} x + y &= 2 \\ x - y &= 2. \end{aligned}$$

It is overkill to use Cramer's rules here, but this example illustrates the method. First we construct the determinant D:

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 1 \times (-1) - 1 \times 1 = -2.$$

Since $D \neq 0$, there is a single solution, calculable by Cramer's rules.

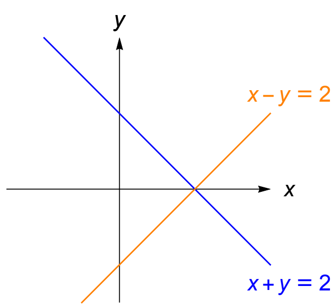
So we find D_1 and D_2, by replacing the appropriate columns of D by the RHS of the system:

$$D_1 = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} = -4;$$

$$D_2 = \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

Hence $x = D_1/D = 2$ and $y = D_2/D = 0$.

It is helpful to draw a graph. The lines $x + y = 2$ and $x - y = 2$ meet at a single point, hence the simultaneous equations have one solution.



$D = 0$ and $D_i \neq 0$: no solution

If $D = 0$ and $D_1 \neq 0$, it follows that (2.10) reduces to $0 = 1$, for any value of x_1. This tells us that the system of equations is *inconsistent*, i.e. there is no solution. The following simple example illustrates this:

$$\begin{aligned} x + y &= 1 \\ x + y &= 2. \end{aligned}$$

It is obviously impossible to find x and y that add up to 1 and 2 at the same time. Let's analyse this using our determinant method. We have

$$D = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0.$$

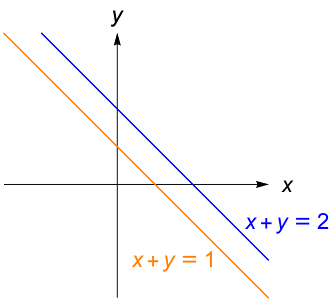
whereas

$$D_1 = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

(and I'll leave you to show that $D_2 = 1$). Hence (2.10) gives us

$$0x = -1 \qquad \text{and} \qquad 0y = 1$$

which clearly have no solutions for x and y.



Of course: in this example it is obvious without using determinants that the equations are inconsistent. The lines $x + y = 1$ and $x + y = 2$ are parallel and therefore never meet. *The point is that the general method works for any number of n equations in n unknowns.*

Example 2. Show that

$$\begin{aligned} 3x_1 + 2x_2 + 5x_3 &= 1 \\ 2x_1 + 5x_2 + 7x_3 &= 0 \\ 4x_1 + 4x_2 + 8x_3 &= 0 \end{aligned}$$

is an inconsistent system.

It's not obvious that the equations are inconsistent, just by looking at them. But using our determinant method, we have

$$D = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 5 & 7 \\ 4 & 4 & 8 \end{vmatrix}.$$

Expanding this gives zero. (A quicker method is to notice that column 3 is the sum of columns 1 and 2, so $D = 0$ by the linear dependence rule.) Moving on to D_1, we have

$$D_1 = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 5 & 7 \\ 0 & 4 & 8 \end{vmatrix}.$$

We can evaluate this easily by expanding down column 1: it comes out to be 12. Hence $D = 0$ and $D_1 \neq 0$, so the equations are inconsistent.

The method also gives us some insight into *why* the equations are inconsistent.

Recall that *a determinant is zero if, and only if, its rows are linearly dependent*.

So, if $D = 0$, at least one of the LHSs of the system can be written as a linear combination of the others. In the example above, after a little head-scratching, we find that 12 times the first LHS, plus 4 times the second, gives us 11 times the third.

$$\begin{aligned} & 12 \times (3x_1 + 2x_2 + 5x_3) \\ + \quad & 4 \times (2x_1 + 5x_2 + 7x_3) \\ = \quad & 44x_1 + 44x_2 + 88x_3 \end{aligned}$$

(I'd strongly advise checking that yourself, to help your understanding.)

Now look at what happens if we apply the same linear combination to *both* sides of the first two equations, not just the LHSs:

$$\begin{aligned} & 12 \times (3x_1 + 2x_2 + 5x_3 = 1) \\ + \quad & 4 \times (2x_1 + 5x_2 + 7x_3 = 0) \\ = \quad & 44x_1 + 44x_2 + 88x_3 = 12 \end{aligned}$$

i.e. we find

$$4x_1 + 4x_2 + 8x_3 = \tfrac{12}{11}.$$

This shows us why the first two equations of the system are inconsistent with the third. The latter has zero on the RHS, not $\frac{12}{11}$.

There is a subtlety. The implication only works one way round. $D = 0$, $D_i \neq 0$ is sufficient for inconsistency, but not necessary. This is demonstrated in the next section.

In other words, inconsistency arises when the LHSs are linearly dependent, but the RHSs do not share the same linear dependence. This leads to $D = 0$ but $D_i \neq 0$ (because substituting the RHSs into D to get D_i breaks the linear dependence of rows). Hence $D = 0$ and $D_i \neq 0$ implies inconsistency, as argued from (2.10).

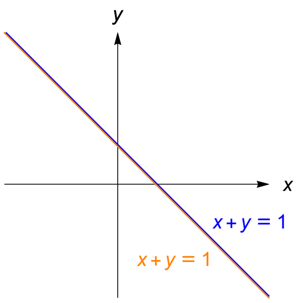
$D = 0$ and $D_i = 0$: no information about x_i

Now suppose that we have a system where the whole equations (i.e. not just the LHSs) are linearly dependent. Following the logic of the previous section, $D = 0$ *and* $D_i = 0$, so (2.10) reads $0 = 0$ for *any* x_1. So (2.10) tells us nothing; we would need to do more work to determine how many solutions the system has.

In fact, when $D = 0$ and $D_i = 0$, we can't use determinants to understand the equations. Instead, a matrix method called Gaussian elimination (a.k.a. 'row reduction') could be used.

To illustrate how $D = 0$ and $D_i = 0$ is inconclusive, consider the trivial case

$$\begin{aligned} x + y &= 1 \\ x + y &= 1. \end{aligned}$$



This has $D = D_1 = D_2 = 0$. In this case the linear dependence is obvious, and we see that there are *infinitely* many solutions: all pairs of x and y values that add to 1.

And in the following case

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ 2x + 2y + 2z &= 3, \end{aligned}$$

all the determinants are zero too ($D = D_1 = D_2 = D_3 = 0$) because the third equation is the sum of the first two. But in this case there are *no* solutions, because the first two equations are inconsistent with each other.

Homogeneous systems of equations

Before we finish, there is one special system of equations that we need to know about: the *homogeneous* system. In this case, all the c_j coefficients on the RHS of (2.1) are identically zero.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0 \tag{2.11}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0 \tag{2.12}$$
$$\vdots \tag{2.13}$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = 0. \tag{2.14}$$

Homogeneous systems are special because *they always have a solution*. It is always possible to solve them by setting the unknowns all to zero. So we call this type of solution the *trivial solution*:

$$x_1 = x_2 = \cdots = x_m = 0 \qquad \text{trivial solution.}$$

Note the 'at least one': the non-trivial solution does not have to have non-zero values for *all* the unknowns.

It is possible for some homogeneous systems to have non-trivial solutions, in which at least one of the unknowns is non-zero. These non-trivial solutions are very important for quantum mechanics (amongst other things): they lead to the concept of an *eigenvalue*.

We can see how non-trivial solutions arise by returning to (2.10). Since the RHSs of (2.11) are all zero, it is easy to see that all the D_i determinants will be zero. Hence (2.10) reduces to

$$x_i D = 0.$$

If $D \neq 0$, then we can divide both sides of this by D and obtain $x_i = 0$, for all i. In other words, if $D \neq 0$ then we obtain only the trivial solution.

On the other hand, if $D = 0$ then we know that one of the equations must be a linear combination of the others. Hence we have fewer equations than unknowns, which (it turns out) is what is needed for there to be non-trivial solutions.

We've shown above that $D = 0$ is *necessary* for non-trivial solutions, but we've not shown that it is sufficient. That will come in two lectures' time, when we show that non-trivial solutions *always* arise when $D = 0$.

Example 3. Show that the system of homogeneous equations

$$x + y = 0$$
$$x - y = 0$$

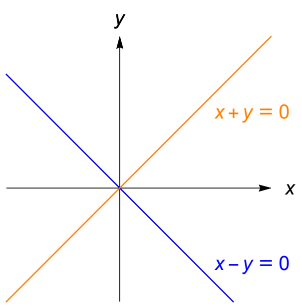
has only trivial solutions.

We calculate D:

$$D = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2.$$

Since $D \neq 0$, the equations have only the trivial solution.

It is useful to look at the graph of the system. The LHSs are not linearly dependent, so the lines are not parallel. But both lines must go through the origin, because the RHSs are zero. Therefore, the system has only one solution, at $(0, 0)$.



Example 4. Show that the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 - x_3 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

has non-trivial solutions, and find them.

We examine D:

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix}.$$

Since the first two columns are identical, it follows from the rules of determinants that $D = 0$. (Of course, we could have shown this by cofactor expansion too.)

So there must be non-trivial solutions. To find them, we can substitute the equations into each other. We see from the third equation that

$$x_2 = -x_1$$

and therefore, from the first equation that

$$x_3 = 0.$$

Notice that we have not used the second equation. This is because *we don't need it*. Since $D = 0$, the second equation must be a linear combination of the other two, and is therefore redundant information.

Therefore, we only really have two equations and three unknowns, and the system is said to be 'under-determined'. We don't get specific values of x_1, x_2 and x_3 as solutions: we are left with just the conditions above:

$$x_2 = -x_1 \qquad \text{and} \qquad x_3 = 0.$$

The system of equations provides only two pieces of information, because one equation is redundant. We would need three pieces of information to assign a specific values to each of the three unknowns.

In other words, there are infinitely-many solutions, of the form

$$(x_1, x_2, x_3) = (\lambda, -\lambda, 0).$$

It may help to view the solutions as the line of intersection of three planes: