LECTURE 3

Vectors: the toolkit of products

We will briefly recap some ideas covered in your introductory vectors lectures. Then, we will turn to the triple scalar product.

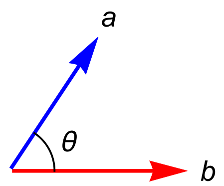
Scalar product

I'll use a geometric definition of the scalar product:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta \tag{3.1}$$

where $|\mathbf{a}|$ is the length of the vector $\mathbf{a}$ (and likewise for $\mathbf{b}$), and θ is the angle enclosed by the vectors if they are *drawn* starting from the same point.

Vectors only have magnitude and direction. **They do not have a 'starting point'**. So it is our *choice* to draw them starting from the same point.

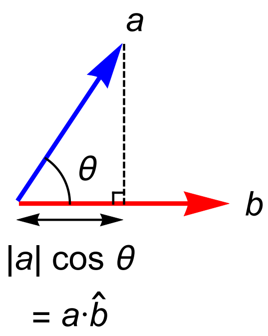


Some immediate consequences follow. You need to be fluent in using them:

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot \mathbf{b} = 0$ if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular

The raison d'être of the scalar product is to simplify finding vector *components*. The component of a vector is *how much* it points in a certain, specified direction.

We see this using some trigonometry. The component of $\mathbf{a}$ in the direction of $\mathbf{b}$, from the diagram below, is $|\mathbf{a}| \cos\theta$.



But using the definition of the scalar product, we can write this component as

$$\text{component of } \mathbf{a} \text{ along } \mathbf{b} = |\mathbf{a}|\cos\theta = \frac{|\mathbf{a}||\mathbf{b}|\cos\theta}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}.$$

If we recall that $\mathbf{b}/|\mathbf{b}|$ is the unit vector in the direction of $\mathbf{b}$, and denote it by $\hat{\mathbf{b}}$, then we simply have

$$\boxed{\text{component of } \mathbf{a} \text{ along } \mathbf{b} = \mathbf{a}\cdot\hat{\mathbf{b}}. \qquad (3.2)}$$

Of course, this relies on us having a way other than using (3.1) to calculate the scalar product. Otherwise we may as well just calculate $|\mathbf{a}|\cos\theta$ directly! Fortunately there is another way…

Expressing our vectors in the *Cartesian* basis, i.e.

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$
$$\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k},$$

it follows that

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})\cdot(b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})\\ &= a_xb_x\mathbf{i}\cdot\mathbf{i} + a_xb_y\mathbf{i}\cdot\mathbf{j} + \cdots + a_zb_z\mathbf{k}\cdot\mathbf{k}\\ &= a_xb_x + a_yb_y + a_zb_z\end{aligned}$$

I have tacitly assumed that the scalar product can be 'multiplied out'. It can; the proof is not difficult, but is too much of a distraction to put here.

In the last step, we have used that the basis vectors ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) are unit vectors that are all mutually perpendicular. The technical term for this is **orthonormal**. We also need the first and last 'consequence' following the definition of the scalar product.

The final result is familiar, of course; but note that it is a *consequence* of the geometric definition.

Example 1. Find the component of $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ in the direction of $\mathbf{b} = \mathbf{i} - \mathbf{j}$.

We need the unit vector in the direction of $\mathbf{b}$: this is clearly

$$\hat{\mathbf{b}} = \frac{\mathbf{i}-\mathbf{j}}{\sqrt{2}}.$$

Therefore, the component of $\mathbf{a}$ in that direction is

$$\mathbf{a}\cdot\hat{\mathbf{b}} = \frac{1-2+0}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

Note: the negative sign tells us that $\mathbf{a}$ points backwards along the direction of $\mathbf{b}$.

Unlike the scalar product, it is difficult to generalise the vector product to anything other than vectors in three-dimensional space. The idea almost works in seven-dimensional space, but even then there are issues…

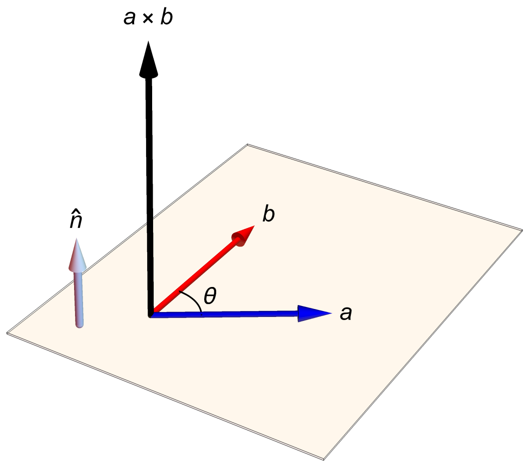
Vector product

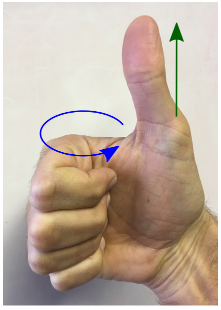
The vector product is defined geometrically as

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}.$$

The new quantity here, as compared to the scalar product, is $\hat{\mathbf{n}}$. It is defined as the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, in the direction of the right-hand corkscrew rule. The vector product gives another vector in that direction, of magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$.

If $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then $\hat{\mathbf{n}}$ is not defined. This is not an issue, because the vector product is itself obviously the zero vector, in this case, because the magnitudes of $\mathbf{a}$ and $\mathbf{b}$ are part of the definition.





Think of rotating the vector **a** onto the vector **b**. If your right-hand fingers curl in the direction of rotation, then your thumb will point in the direction of the cross product.

The vector product definition leads to two important consequences.

1. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
2. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

Again, you need to be familiar with both.

The purpose of the vector product is to simplify finding vectors perpendicular to a plane. If we know two vectors, **a** and **b**, that are parallel to a plane, then their vector product will automatically give us the perpendicular. But again, for this to be of any practical use, we need a way to calculate the vector product without going back to the definition.

A similar argument to the one used for the scalar product (which I'll leave as an exercise) shows that, in the Cartesian basis,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{3.3}$$

For this to work, the Cartesian basis vectors (**i**, **j** and **k**) have to form a *right-handed* coordinate system. This means that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ (rather than $-\mathbf{k}$).

Example 2. Find a vector perpendicular to the vectors **a** and **b** of Example 1.

This follows straightforwardly from the argument above.

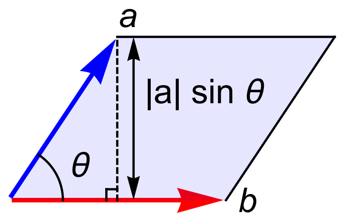
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

If we had taken $\mathbf{b} \times \mathbf{a}$, we would have got the negative of this; both vectors (and, indeed, any non-zero multiple of them) are all perpendicular to both **a** and **b**.

Notice that the two 'consequences' identified above are also obvious from the determinant form of the vector product, using properties of the determinant.

Area of a parallelogram

Using basic trigometry, the area of the parallelogram defined by **a** and **b** is



$$\begin{aligned}\text{area of parallelogram} &= \text{base} \times \text{perpendicular height}\\ &= |\mathbf{b}||\mathbf{a}|\sin\theta\\ &= |\mathbf{a}\times\mathbf{b}|.\end{aligned}$$

Example 3. Find the area of the parallelogram defined by vectors **a** and **b** of Example 1.

In Example 2 we showed that

$$\mathbf{a}\times\mathbf{b} = -\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

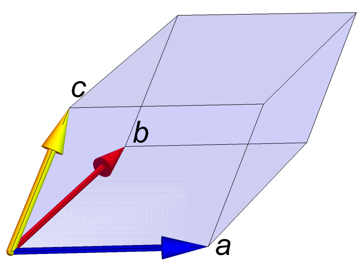
Hence the magnitude of this vector gives us the area:

$$\text{area of parallelogram} = |\mathbf{a}\times\mathbf{b}| = \sqrt{(-1)^2 + (-1)^2 + (-3)^2} = \sqrt{11}.$$

You've seen that before. But now, let's extend the principle to three dimensions.

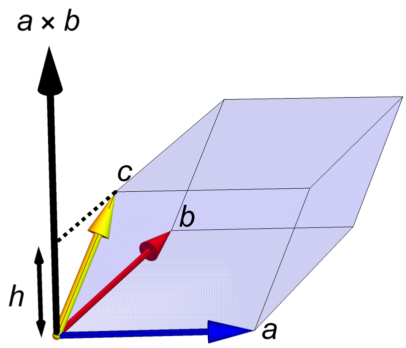
Triple scalar product

We can use three vectors to define a *parallelepiped*:



The volume of this object can be calculated as the area of its base, multiplied by the perpendicular height. The former is just the area of the parallelogram defined by vectors **a** and **b**, i.e. just $|\mathbf{a}\times\mathbf{b}|$.

The perpendicular height is the amount of **c** that is perpendicular to **a** and **b**. This is the (modulus of the) component of **c** *in the direction of their normal vector*, $\mathbf{a}\times\mathbf{b}$. See the diagram below:



i.e.

$$h = |\mathbf{c} \cdot \hat{\mathbf{n}}| \text{ where } \mathbf{n} = \mathbf{a} \times \mathbf{b}.$$

The modulus is needed because **c** may be pointing in the opposite direction to **n**, in which case $\mathbf{c} \cdot \hat{\mathbf{n}}$ would be negative. The perpendicular height, h, has to be a positive quantity.

Putting this all together, we have

$$h = \frac{|\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|}{|\mathbf{a} \times \mathbf{b}|}$$

and hence, by multiplying h by the area of the base, $|\mathbf{a} \times \mathbf{b}|$, we find that

$$\text{volume of parallelepiped } = |\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})|. \tag{3.4}$$

Example 4. Find the volume of the parallelepiped defined by the vectors **a** and **b** of Example 1, along with the vector

$$\mathbf{c} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We already calculated

$$\mathbf{a} \times \mathbf{b} = -\mathbf{i} - \mathbf{j} - 3\mathbf{k}.$$

So, very simply,

$$\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -1 - 1 - 3 = -5.$$

Hence the volume is $|-5| = 5$.

The quantity inside the modulus signs of (3.4), i.e. $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$, is called the *triple scalar product*. The brackets here are somewhat superfluous: if we write

$$\mathbf{c} \cdot \mathbf{a} \times \mathbf{b}$$

then it is obvious that the cross product has to be done first, because if one did the dot product $\mathbf{c} \cdot \mathbf{a}$ first, it would give a scalar which could not be crossed with **b**. So we will omit the brackets from now on.

In Example 4, we effectively calculated the triple scalar product in two stages, because we had already worked out $\mathbf{a} \times \mathbf{b}$. However, there is a much better way to calculate (and think about) the triple scalar product if we need to do it from scratch.

We can write, for any **a** and **b**,

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix},$$

Hence,

$$\mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = c_x\begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - c_y\begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + c_z\begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}.$$

This can be recombined to get a single determinant (if you can't see why, just multiply the next one out along the top).

$$\boxed{\mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = \begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad (3.5)}$$

Viewing the triple scalar product as a determinant allows us to generalise the 'volume of a parallelepiped' idea to n dimensions. An n-dimensional 'hyper-parallelepiped' is defined by n vectors in n dimensions, and its volume is the modulus of the determinant with those vectors as rows. You'll prove this for $n = 2$ in the exercises.

So the triple scalar product is just a determinant formed from the three vectors. It is usually easiest to calculate it in this way.

Coplanar vectors

We saw previously that determinants are zero when their rows are linearly dependent. Let's see what that implies in the context of the triple scalar product.

From (3.5), the properties of determinants imply that

$$\mathbf{c}\cdot\mathbf{a}\times\mathbf{b} = 0 \qquad \text{if } \mathbf{c} \text{ is linearly-dependent on } \mathbf{a} \text{ and } \mathbf{b}.$$

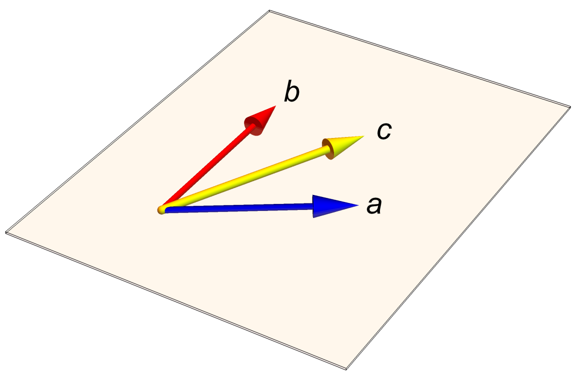
Linearly-dependent, as before, means that it can be written as the *linear combination*

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b},$$

for a certain choice of coefficients α and β.

Remember: vectors do not have starting points! When we say that they *are* in the same plane, we really mean that they *would be* in the same plane if we drew them originating from a common point.

But if **c** can be written in this way, it must be in the same plane as **a** and **b**. This is easiest to see by drawing a diagram:



So a vanishing triple scalar product implies that the vectors are coplanar. Another way to argue this is as follows. The cross product $\mathbf{a}\times\mathbf{b}$ produces a vector normal to **a** and **b**. If **c** is in the same plane as **a** and **b**, it will also be perpendicular to that normal. Hence $\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) = 0$.