Avaliação P1

① $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$

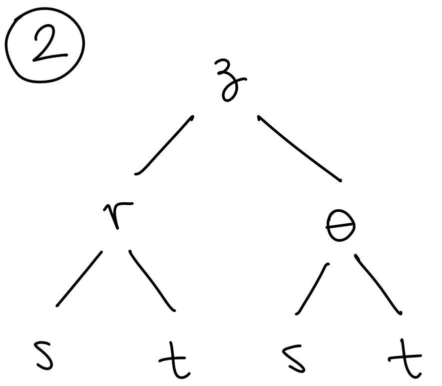
$$\frac{\partial f}{\partial x} = \frac{\frac{1}{2\sqrt{y-x^2}}\cdot(-2x)\cdot(1-x^2) - (-2x)\sqrt{y-x^2}}{(1-x^2)^2} = \frac{\frac{-x(1-x^2)}{\sqrt{y-x^2}} + 2x\sqrt{y-x^2}}{(1-x^2)^2}$$

$$= \frac{-x(1-x^2) + 2x(y-x^2)}{(1-x^2)^2\sqrt{y-x^2}} = \frac{-x+x^3+2xy-2x^3}{(1-x^2)^2\sqrt{y-x^2}}$$

$$= \frac{-x^3-x+2xy}{(1-x^2)^2\sqrt{y-x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1-x^2}\cdot\frac{1}{2\sqrt{y-x^2}} = \frac{1}{2(1-x^2)\sqrt{y-x^2}}$$

②




$z = e^r\cos\theta$

$r = st$

$\theta = \sqrt{s^2+t^2}$

Pela regra da cadeia:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + (-e^r\,\mathrm{sen}\,\theta)\cdot\frac{1}{2\sqrt{s^2+t^2}}\cdot 2s$$

$$= t\,e^r\cos\theta - \frac{s\,e^r\,\mathrm{sen}\,\theta}{\sqrt{s^2+t^2}}$$

e $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + (-e^r\,\mathrm{sen}\,\theta)\cdot\frac{1}{2\sqrt{s^2+t^2}}\cdot 2t$$

$$= s\,e^r\cos\theta - \frac{t\,e^r\,\mathrm{sen}\,\theta}{\sqrt{s^2+t^2}}$$

③ $f(x,y,z) = x^2yz - xyz^3$

a) $\nabla f = (2xyz - yz^3,\ x^2z - xz^3,\ x^2y - 3xyz^2)$

b) $\|u\| = \sqrt{\left(\frac{4}{5}\right)^2 + 0^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1 \Rightarrow u$ é unitário. Logo,

$\frac{\partial f}{\partial u}(1,2,1) = \nabla f(1,2,1) \cdot u = (2,0,-4) \cdot \left(\frac{4}{5}, 0, -\frac{3}{5}\right) = \frac{8}{5} + \frac{12}{5} = \frac{20}{5} = 4.$

④ Calculando os pontos críticos de $f$:

$\frac{\partial f}{\partial x} = 4x^3 - 4y$ e $\frac{\partial f}{\partial y} = 4y^3 - 4x$ estão def. p/ todo $(x,y) \in \mathbb{R}^2$.

Assim, os ptos críticos de $f$ são as sol. do sistema:

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \overset{(\div 4)}{\Rightarrow} \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 & (1) \\ y^3 - x = 0 & (2) \end{cases}$$

Substituindo (1) em (2):

$(x^3)^3 - x = 0 \Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0 \Rightarrow x = 0$ ou $x^8 - 1 = 0$

$\Rightarrow x = 0$ ou $x^8 = 1 \Rightarrow x = 0$ ou $x = 1$ ou $x = -1$.

$\therefore\ x = 0 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$ e $x = -1 \Rightarrow y = -1$.

Portanto, os ptos críticos de $f$ são $(0,0)$, $(1,1)$ e $(-1,-1)$.

Aplicando o teste da 2ª derivada:

$\frac{\partial^2 f}{\partial x^2} = 12x^2$, $\frac{\partial^2 f}{\partial y \partial x} = -4$, $\frac{\partial^2 f}{\partial x \partial y} = -4$ e $\frac{\partial^2 f}{\partial y^2} = 12y^2$

$$\Rightarrow D(x,y) = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16$$

$\therefore\ D(0,0) = -16 < 0 \Rightarrow (0,0)$ é pto de sela

$D(1,1) = 128 > 0$ e $\frac{\partial^2 f}{\partial x^2}(1,1) = 12 > 0 \Rightarrow (1,1)$ é pto de mín. local.

$D(-1,-1) = 128 > 0$ e $\frac{\partial^2 f}{\partial x^2}(-1,-1) = 12 > 0 \Rightarrow (-1,-1)$ é pto de mín. local.

⑤ Temos que:

$$d[(x,y,z),(2,0,-3)] = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$$

$$\Rightarrow d[(x,y,z),(2,0,-3)]^2 = (x-2)^2 + y^2 + (z+3)^2.$$

Aplicando o método dos multiplicadores de Lagrange, sejam

$$f(x,y,z) = (x-2)^2 + y^2 + (z+3)^2$$

$$g(x,y,z) = x+y+z-1$$

$$\Rightarrow \nabla f = \big(2(x-2),\ 2y,\ 2(z+3)\big)$$

$$\nabla g = (1,1,1)$$

$$\therefore \begin{cases} 2(x-2) = \lambda & (1) \\ 2y = \lambda & (2) \\ 2(z+3) = \lambda & (3) \\ x+y+z-1=0 & (4) \end{cases}$$

De (1) e (2), temos: $2(x-2) = 2y \Rightarrow x-2 = y \Rightarrow x = y+2.$

De (2) e (3), temos: $2y = 2(z+3) \Rightarrow y = z+3 \Rightarrow z = y-3.$

Substituindo em (4):

$$(y+2) + y + (y-3) - 1 = 0 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

$$\Rightarrow x = \frac{2}{3} + 2 = \frac{8}{3}$$

$$\Rightarrow z = \frac{2}{3} - 3 = -\frac{7}{3}$$

Avaliação P2

① $\iint_R x \operatorname{sen}(x+y)\, dA = \int_0^{\pi/3}\int_0^{\pi/6} x \operatorname{sen}(x+y)\, dx\, dy$

$\left( \begin{array}{ll} \text{Por partes:} & \\ u = x & \Rightarrow \quad du = dx \\ dv = \operatorname{sen}(x+y)\,dx & \quad\;\; v = -\cos(x+y) \end{array} \right)$

$= \int_0^{\pi/3} \left[ -x\cos(x+y) \Big|_{x=0}^{x=\pi/6} + \int_0^{\pi/6} \cos(x+y)\,dx \right] dy$

$= \int_0^{\pi/3} \left[ -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + 0\cdot\cos(0+y) + \operatorname{sen}(x+y) \Big|_{x=0}^{x=\pi/6} \right] dy$

$= \int_0^{\pi/3} -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \operatorname{sen}\left(\frac{\pi}{6}+y\right) - \operatorname{sen} y \;\; dy$

$= -\frac{\pi}{6}\left[ \operatorname{sen}\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} \right] - \left[ \cos\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} \right] + \left[ \cos y \Big|_0^{\pi/3} \right]$

$= -\frac{\pi}{6}\left[ \operatorname{sen}\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \operatorname{sen}\left(\frac{\pi}{6}\right) \right] - \left[ \cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{6}\right) \right] + \left[ \cos\frac{\pi}{3} - \cos 0 \right]$

$= -\frac{\pi}{6}\left[ \operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{6} \right] - \left[ \cos\frac{\pi}{2} - \cos\frac{\pi}{6} \right] + \left[ \cos\frac{\pi}{3} - \cos 0 \right]$

$= -\frac{\pi}{6}\left[ 1 - \frac{1}{2} \right] - \left[ 0 - \frac{\sqrt{3}}{2} \right] + \left[ \frac{1}{2} - 1 \right] = \frac{6\sqrt{3} - 6 - \pi}{12}$

② O conjunto $E = \{ (r,\theta,z) \mid 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi,\ 0 \leq z \leq 1 \}$ descreve em coord. cilíndricas a metade de um cilindro de raio 1 contido no semi-espaço onde $y \geq 0$ e limitado pelos planos $z=0$ e $z=1$.

A integral calcula o volume de E, logo deve ser igual a

$$\frac{\pi r^2 \cdot h}{2} = \frac{\pi \cdot 1^2 \cdot 1}{2} = \frac{\pi}{2}.$$

Calculando a integral:

$$\int_0^1\int_0^{\pi}\int_0^1 r\, dr\, d\theta\, dz = \int_0^1 r\,dr \cdot \int_0^{\pi} d\theta \cdot \int_0^1 dz = \left( \frac{r^2}{2}\Big|_0^1 \right) \cdot \pi \cdot 1 = \frac{\pi}{2}.$$

③ Usando coord. esféricas, temos:

$$B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

$$\therefore \iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} e^{(\rho^2)^{3/2}} \rho^2 \operatorname{sen}\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^1 e^{\rho^3} \cdot \rho^2 \, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \operatorname{sen}\phi \, d\phi \qquad \left( \begin{array}{l} u = \rho^3 \Rightarrow du = 3\rho^2 d\rho \\ \rho = 0 \Rightarrow u = 0 \text{ e } \rho = 1 \Rightarrow u = 1 \end{array} \right)$$

$$= \frac{1}{3} \int_0^1 e^u \, du \cdot (2\pi) \cdot \left( -\cos\phi \Big|_0^{\pi} \right)$$

$$= \frac{2\pi}{3} \left( e^u \Big|_0^1 \right) \cdot \left( -\cos\pi + \cos 0 \right) = \frac{2\pi}{3} (e-1) \cdot 2 = \frac{4\pi}{3}(e-1).$$

④ a) Queremos $f$ tal que $\nabla f = F$, logo

$$\begin{cases} \dfrac{\partial f}{\partial x} = 1 - ye^{-x} & (1) \\[2mm] \dfrac{\partial f}{\partial y} = e^{-x} & (2) \end{cases}$$

Integrando (2): $\displaystyle\int \frac{\partial f}{\partial y}\, dy = \int e^{-x} dy \Rightarrow f(x,y) = e^{-x} \cdot y + C(x).$

Substituindo em (1):

$$1 - ye^{-x} \overset{(1)}{=} \frac{\partial f}{\partial x} = -e^{-x} y + C'(x) \Rightarrow C'(x) = 1 \Rightarrow C(x) = x.$$

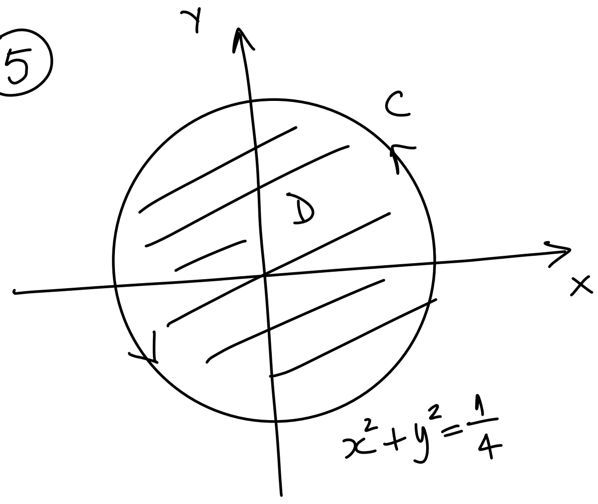
Portanto, $f(x,y) = x + ye^{-x}$ é uma função potencial de $F$ e $F$ é conservativo.

b) Pelo Teo. Fund. das Int. de Linha:

$$\int F \cdot dr = \int \nabla f \cdot dr = f(r(1)) - f(r(0)) = f(1,2) - f(0,0)$$

$$= (1 + 2e^{-1}) - (0 + 0 \cdot e^0) = 1 + 2e^{-1}.$$

⑤




A curva C é simples, fechada, suave e pode ser orientada positivamente.

$\frac{\partial P}{\partial y} = \frac{1}{1+y}$ e $\frac{\partial Q}{\partial x} = -\frac{y}{1+y}$ são contínuas em D,

Pelo Teo. de Green:

$$\int_C \ln(1+y)\,dx - \frac{xy}{1+y}\,dy = \iint_D -\frac{y}{1+y} - \frac{1}{1+y}\,dA = \iint_D -\frac{y+1}{1+y}\,dA$$

$$= \iint_D -1\,dA = -\iint_D dA = -\text{área}(D) = -\pi\left(\frac{1}{2}\right)^2 = -\frac{\pi}{4}.$$