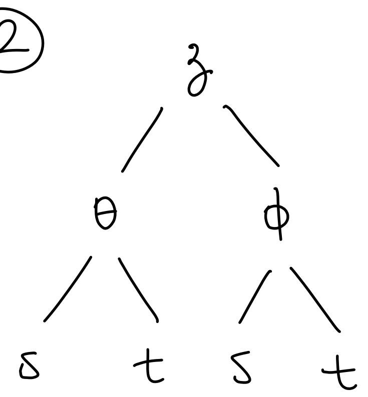
Avaliação P1

1) $f(x,y) = 1 + x \ln(xy-5)$

$$\frac{\partial f}{\partial x} = 1 \cdot \ln(xy-5) + x \cdot \frac{1}{xy-5} \cdot y = \ln(xy-5) + \frac{xy}{xy-5}$$

$$\frac{\partial f}{\partial y} = x \cdot \frac{1}{xy-5} \cdot x = \frac{x^2}{xy-5}.$$

2)



$$z = \operatorname{sen}\theta \cdot \cos\phi$$

$$\theta = st^2$$

$$\phi = s^2 t$$

$$\therefore \frac{\partial z}{\partial s} = \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} + \frac{\partial z}{\partial \phi} \cdot \frac{\partial \phi}{\partial s} = \cos\theta \cdot \cos\phi \cdot t^2 - \operatorname{sen}\theta \cdot \operatorname{sen}\phi \cdot 2st$$

e $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} + \frac{\partial z}{\partial \phi} \cdot \frac{\partial \phi}{\partial t} = \cos\theta \cdot \cos\phi \cdot 2st - \operatorname{sen}\theta \operatorname{sen}\phi \cdot s^2.$$

3) A taxa de var. máxima é dada por $\|\nabla f(4,1)\|$ e a direção em que ocorre é $\nabla f(4,1)$. Assim,

$$\nabla f = \left(\frac{2y}{\sqrt{x}}, 4\sqrt{x} \right) \Rightarrow \nabla f(4,1) = \left(\frac{2 \cdot 1}{\sqrt{4}}, 4 \cdot \sqrt{4} \right) = (1,8)$$

$$\Rightarrow \|\nabla f(4,1)\| = \sqrt{1^2 + 8^2} = \sqrt{65}.$$

4) Calculando os pontos críticos:

$$\frac{\partial f}{\partial x} = y(1-x-y) + xy(-1) = y(1-x-y) - xy$$

$$\frac{\partial f}{\partial y} = x(1-x-y) + xy(-1) = x(1-x-y) - xy$$

são polinomiais, logo estão definidas para todo $(x,y) \in \mathbb{R}^2$.

Assim, os pontos críticos são as soluções do sistema

$$\begin{cases} y(1-x-y) - xy = 0 \quad (1) \\ x(1-x-y) - xy = 0 \quad (2) \end{cases} \Rightarrow y(1-x-y) - xy = x(1-x-y) - xy$$

$(+xy)$

$$\Rightarrow y(1-x-y) = x(1-x-y) \Rightarrow y(1-x-y) - x(1-x-y) = 0 \Rightarrow (y-x)(1-x-y) = 0$$

$$\Rightarrow y-x=0 \quad \text{ou} \quad 1-x-y=0$$

Se $y-x=0$, temos $x=y$ e subst. em (1):

$$x(1-x-x) - x\cdot x = 0 \Rightarrow x(1-2x) - x^2 = 0 \Rightarrow x\left[(1-2x)-x\right] = 0$$

$$\Rightarrow x\left[1-3x\right] = 0 \Rightarrow x=0 \text{ ou } 1-3x=0 \Rightarrow x=0 \text{ ou } x=\frac{1}{3}.$$

$$\therefore \quad x=0 \Rightarrow y=0 \quad e \quad x=\frac{1}{3} \Rightarrow y=\frac{1}{3}.$$

Se $1-x-y=0$, temos $y=1-x$ e subst. em (1):

$$(1-x)\cdot 0 - x(1-x) = 0 \Rightarrow x(1-x) = 0 \Rightarrow x=0 \text{ ou } x=1.$$

$$\therefore \quad x=0 \Rightarrow y=1 \quad e \quad x=1 \Rightarrow y=0.$$

Portanto, os pontos críticos de f são $(0,0)$, $(0,1)$, $(1,0)$ e $\left(\frac{1}{3},\frac{1}{3}\right)$.

Aplicando o teste da 2ª derivada:

$$\frac{\partial^2 f}{\partial x^2} = -2y, \quad \frac{\partial^2 f}{\partial y \partial x} = (1-x-y) + y(-1) - x = 1-2x-2y,$$

$$\frac{\partial^2 f}{\partial y^2} = -2x, \quad \frac{\partial^2 f}{\partial x \partial y} = (1-x-y) + x(-1) - y = 1-2x-2y$$

$$\Rightarrow D(x,y) = \begin{vmatrix} -2y & 1-2x-2y \\ 1-2x-2y & -2x \end{vmatrix} = 4xy - (1-2x-2y)^2$$

$$\therefore \quad D(0,0) = -1 < 0 \Rightarrow (0,0) \text{ é pto de sela}$$

$$D(0,1) = -1 < 0 \Rightarrow (0,1) \text{ é pto de sela}$$

$$D(1,0) = -1 < 0 \Rightarrow (1,0) \text{ é pto de sela}$$

$$D\left(\tfrac{1}{3},\tfrac{1}{3}\right) = \frac{4}{9} - \left(1-\frac{2}{3}-\frac{2}{3}\right)^2 = \frac{4}{9} - \left(\frac{3-2-2}{3}\right)^2 = \frac{4}{9} - \left(-\frac{1}{3}\right)^2 = \frac{4}{9} - \frac{1}{9}$$

$$= \frac{1}{3} > 0 \quad e \quad \frac{\partial^2 f}{\partial x^2}\left(\tfrac{1}{3},\tfrac{1}{3}\right) = -\frac{2}{3} < 0 \Rightarrow \left(\tfrac{1}{3},\tfrac{1}{3}\right) \text{ é pto de máx. local.}$$

⑤ Queremos $x, y, z > 0$ tais que $x+y+z=12$ e que $x^2+y^2+z^2$ tenha o menor valor possível. Aplicando o método dos Mult. de Lagrange, sejam

$$f(x,y,z) = x^2+y^2+z^2 \quad \text{e} \quad g(x,y,z) = x+y+z$$

$$\Rightarrow \nabla f = (2x, 2y, 2z) \quad \text{e} \quad \nabla g = (1,1,1)$$

$$\therefore \begin{cases} 2x = \lambda & (1) \\ 2y = \lambda & (2) \\ 2z = \lambda & (3) \\ x+y+z=12 & (4) \end{cases}$$

De (1) e (2), temos: $2x = 2y \Rightarrow x = y$.

De (2) e (3), temos: $2y = 2z \Rightarrow y = z$

Substituindo em (4):

$$x+x+x = 12 \Rightarrow 3x = 12 \Rightarrow x = 4 \Rightarrow y = 4 \Rightarrow z = 4.$$

Avaliação P2

① Dado $D = \{(x,y) \mid 0 \leq x \leq \pi,\ 0 \leq y \leq \cos x\}$

$$\iint_D x\, dA = \int_0^{\pi}\int_0^{\cos x} x\, dy\, dx = \int_0^{\pi} x\left(y\Big|_0^{\cos x}\right) dx = \int_0^{\pi} x(\cos x - 0)\, dx$$

$$= \int_0^{\pi} x\cos x\, dx$$

$$\left(\begin{array}{ll} \text{Por partes:} & \\ u = x & \Rightarrow\ du = dx \\ dv = \cos x\, dx & \quad v = \operatorname{sen} x \end{array}\right)$$

$$= x \operatorname{sen} x\Big|_0^{\pi} - \int_0^{\pi} \operatorname{sen} x\, dx = \pi \cdot \operatorname{sen}\pi - 0 \cdot \operatorname{sen} 0 + \cos x\Big|_0^{\pi} = \cos\pi - \cos 0 = -2$$

② Temos que $D = \{(\rho, \theta, \phi) \mid 0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi,\ 0 \leq \phi \leq \pi\}$ descreve, em coord. esféricas, a metade de uma esféra sólida de centro na origem e raio 1 que está localizada a direita do plano xz (semi-espaço $y \geq 0$).

Note que a integral calcula o volume de E, ou seja,

$$\frac{\frac{4}{3}\pi \cdot 1^2}{2} = \frac{2}{3}\pi.$$

Calculando a integral:

$$\int_0^{\pi}\int_0^{\pi}\int_0^{1} \rho^2 \operatorname{sen}\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi} \operatorname{sen}\phi\, d\phi \cdot \int_0^{\pi} d\theta \cdot \int_0^{1} \rho^2\, d\rho$$

$$= \left(-\cos\phi\Big|_0^{\pi}\right) \cdot \left(\theta\Big|_0^{\pi}\right) \cdot \left(\frac{\rho^3}{3}\Big|_0^{1}\right) = (-\cos\pi + \cos 0) \cdot (\pi - 0) \cdot \left(\frac{1}{3} - 0\right) = \frac{2\pi}{3}$$

④ a) Queremos f tal que $\nabla f = F$:

$$\begin{cases} \dfrac{\partial f}{\partial x} = 3 + 2xy & (1) \\[2ex] \dfrac{\partial f}{\partial y} = x^2 - 3y^2 & (2) \end{cases}$$

$(1) \Rightarrow f(x,y) = 3x + x^2 y + C(y)$

Subst. em (2):

$$x^2 - 3y^2 \overset{(2)}{=} \frac{\partial f}{\partial y} = x^2 + C'(y) \quad \Rightarrow C'(y) = -3y^2 \Rightarrow C(y) = -y^3$$

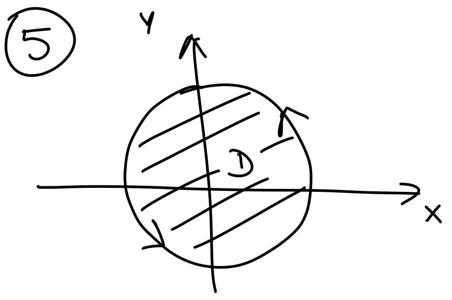
Portanto, $f(x,y) = 3x + x^2y - y^3$ é uma função potencial de F e F é conservativo.

b) Pelo Teo. Fund. das Int. de Linha:

$$\int F\cdot dr = \int \nabla f\cdot dr = f(r(1)) - f(r(0)) = f(e,e) - f(0,0)$$

$$= (3e + e^2\cdot e - e^3) - (3\cdot 0 + 0^2\cdot 0 - 0^3) = 3e$$

⑤

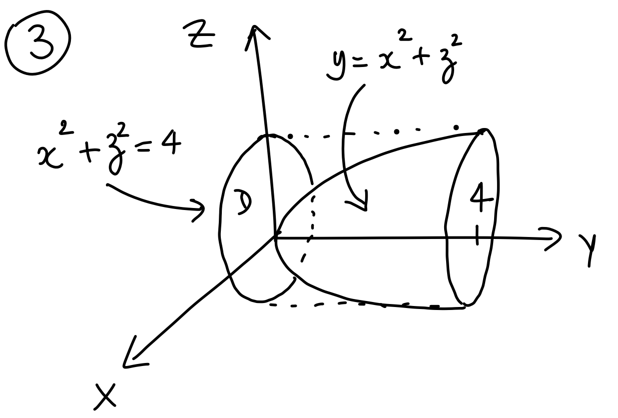


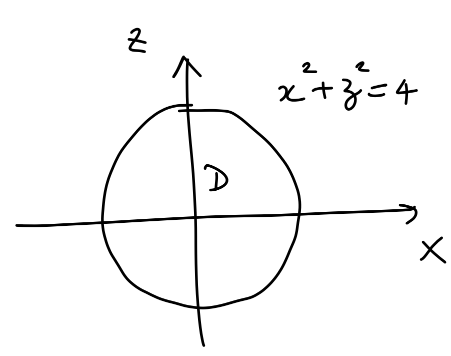
Temos que $C: r(t) = (r\cos t, r\,\mathrm{sen}\,t)$, $0 \le t \le 2\pi$.

$$\therefore \text{ área} = \iint_D dA \overset{\text{(Green)}}{=} \frac{1}{2}\int_0^{2\pi} (-r\,\mathrm{sen}\,t)(-r\,\mathrm{sen}\,t) + (r\cos t)(r\cos t)\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} r^2\mathrm{sen}^2 t + r^2\cos^2 t\, dt = \frac{1}{2}\int_0^{2\pi} r^2\, dt = \frac{1}{2} r^2\cdot\left(t\Big|_0^{2\pi}\right) = \frac{1}{2} r^2\, 2\pi = \pi r^2.$$

③





Temos que:

$$E = \{(x,y,z) \mid x^2+z^2 \le y \le 4 \text{ e } (x,y)\in D\}$$

onde

$$D = \{(x,y) \mid x^2+z^2 \le 4\}$$

$\Rightarrow D = \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$ em coord. polares

Logo,

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy\right] dA$$

$$= \iint_D \sqrt{x^2+z^2}\cdot\left(y\Big|_{x^2+z^2}^{4}\right) dA = \iint_D 4\sqrt{x^2+z^2} - (x^2+z^2)\sqrt{x^2+z^2}\, dA$$

$$= \int_0^{2\pi}\int_0^2 \left(4\sqrt{r^2} - r^2\sqrt{r^2}\right) r\, dr\, d\theta = \int_0^{2\pi}\int_0^2 4r^2 - r^4\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^2 4r^2 - r^4\, dr = 2\pi\cdot\left(\frac{4r^3}{3} - \frac{r^5}{5}\Big|_0^2\right) = 2\pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{128\pi}{15}$$