mle.tools: An R Package for Maximum Likelihood Bias Correction

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Abstract Recently, Mazucheli (2017) uploaded the package **mle.tools** to CRAN. It can be used for bias corrections of maximum likelihood estimates through the methodology proposed by Cox and Snell (1968). The main function of the package, `coxsnell.bc`, computes the bias corrected maximum likelihood estimates. Although in general, the bias corrected estimators may be expected to have better sampling properties than the uncorrected estimators, analytical expressions from the formula proposed by Cox and Snell (1968) are either tedious or impossible to obtain. The purpose of this paper is twofolded: to introduce the **mle.tools** package, especially the `coxsnell.bc` function; secondly, to compare, for thirty one continuous distributions, the bias estimates from the `coxsnell.bc` function and the bias estimates from analytical expressions available in the literature. We also compare, for five distributions, the observed and expected Fisher information. Our numerical experiments show that the functions are efficient to estimate the biases by the Cox-Snell formula and for calculating the observed and expected Fisher information.

Introduction

Since it was proposed by Fisher in a series of papers from 1912 to 1934, the maximum likelihood method for parameter estimation has been employed to several issues in statistical inference, because of its many appealing properties. For instance, the maximum likelihood estimators, hereafter referred to as MLEs, are asymptotically unbiased, efficient, consistent, invariant under parameter transformation and asymptotically normally distributed (Edwards, 1992; Lehmann, 1999). Most properties that make the MLEs attractive depend on the sample size, hence such properties as unbiasedness may not be valid for small samples or even moderate samples (Kay, 1995). Indeed, the maximum likelihood method produces biased estimators, i.e., expected values of MLEs differ from the real true parameter values providing systematic errors. In particular, these estimators typically have biases of order $\mathcal{O}\left(n^{-1}\right)$, thus these errors reduce as sample size increases (Cordeiro and Cribari-Neto, 2014).

Applying the corrective Cox-Snell methodology, many researchers have developed nearly unbiased estimators for the parameters of several probability distributions. Interested readers can refer to Cordeiro et al. (1997), Cribari-Neto and Vasconcellos (2002), Saha and Paul (2005), Lemonte et al. (2007), Giles and Feng (2009) Lagos-Álvarez et al. (2011), Lemonte (2011), Giles (2012b), Giles (2012a), Schwartz et al. (2013), Giles et al. (2013), Teimouri and Nadarajah (2013), Xiao and Giles (2014), Zhang and Liu (2015), Teimouri and Nadarajah (2016), Reath (2016), Giles et al. (2016), Schwartz and Giles (2016), Wang and Wang (2017), Mazucheli and Dey (2017) and references cited therein.

In general, the Cox-Snell methodology is efficient for bias corrections. However, obtaining analytical expressions for some probability distributions, mainly for those indexed by more than two parameters, can be notoriously cumbersome or impossible. Stočsić and Cordeiro (2009) presented *Maple* and *Mathematica* scripts that may be used to calculate closed form analytic expressions for bias corrections using the Cox-Snell formula. They tested the scripts for 20 two-parameter continuous probability distributions, and the results were compared with those published in earlier works. In the same direction, researchers from the University of Illinois, at Urbana-Champaign, have developed a *Mathematica* program, entitled "CSCK MLE Bias Calculation" (Johnson et al., 2012b) that enables the user to calculate the analytic Cox-Snell MLE bias vectors for various probability distributions with up to four unknown parameters. It is important to mention that both, *Maple* (Maple, 2017) and *Mathematica* (Wolfram Research, Inc., 2010), are commercial softwares.

In this paper, our objective is to introduce a new contributed R (R Core Team, 2016) package, namely **mle.tools** that computes the expected/observed Fisher information and the bias corrected estimates by the methodology proposed by Cox and Snell (1968). The theoretical background of the methodology is presented in Section Overview of the Cox-Snell Methodology. Details about the **mle.tools** package are described in Section The mle.tools Package Details. Closed form solutions of bias corrections are collected from the literature for a large number of distributions and compared to the output from the `coxsnell.bc` function, see Section Comparative Study. In Section Additional Applications, we compare various estimates of Fisher's information, considering a real application from the literature. Finally, Section Concluding Remarks contains some concluding remarks and directions for future research.

Overview of the Cox-Snell Methodology

Let $X_1, \ldots, X_n$ be n be independent random variables with probability density function $f(x_i \mid \boldsymbol{\theta})$ depending on a p-dimensional parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$. Without loss of generality, let $l = l(\boldsymbol{\theta} \mid \boldsymbol{x})$ be the log-likelihood function for the unknown p-dimensional parameter vector $\boldsymbol{\theta}$ given a sample of n observations. We shall assume some regularity conditions on the behavior of $l(\boldsymbol{\theta} \mid \boldsymbol{x})$ (Cox and Hinkley, 1979).

The joint cumulants of the derivatives of l are given by:

$$\kappa_{ij} = \mathbb{E}\left[\frac{\partial^2 l}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j}\right], \tag{1}$$

$$\kappa_{ijl} = \mathbb{E}\left[\frac{\partial^3 l}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j \, \partial \boldsymbol{\theta}_l}\right], \tag{2}$$

$$\kappa_{ij,l} = \mathbb{E}\left[\left(\frac{\partial^2 l}{\partial \boldsymbol{\theta}_i \, \partial \boldsymbol{\theta}_j}\right)\left(\frac{\partial l}{\partial \boldsymbol{\theta}_l}\right)\right], \tag{3}$$

$$\kappa_{ij}^{(l)} = \frac{\partial \kappa_{ij}}{\partial \boldsymbol{\theta}_l} \tag{4}$$

for $i, j, l = 1, \ldots, p$.

The bias expression of the s-*th* element of $\widehat{\boldsymbol{\theta}}$, the MLEs of $\boldsymbol{\theta}$, when the sample data are independent, but not necessarily identically distributed, was proposed by Cox and Snell (1968):

$$\mathcal{B}\left(\widehat{\theta}_s\right) = \sum_{i=1}^{p} \sum_{j=1}^{p} \sum_{l=1}^{p} \kappa^{si} \kappa^{jl} \left[0.5\kappa_{ijl} + \kappa_{ij,l}\right] + \mathcal{O}\left(n^{-2}\right), \tag{5}$$

where $s = 1, \ldots, p$ and κ^{ij} is the (i, j)-*th* element of the inverse of the negative of the expected Fisher information.

Thereafter, Cordeiro and Klein (1994) noticed that equation (5) holds even if the data are non-independent, and it can be re-expressed as:

$$\mathcal{B}\left(\widehat{\theta}_s\right) = \sum_{i=1}^{p} \kappa^{si} \sum_{j=1}^{p} \sum_{l=1}^{p} \left[\kappa_{ij}^{(l)} - 0.5\kappa_{ijl}\right] \kappa^{jl} + \mathcal{O}\left(n^{-2}\right). \tag{6}$$

Defining $a_{ij}^{(l)} = \kappa_{ij}^{(l)} - 0.5\kappa_{ijl}$, $A^{(l)} = \left\{a_{ij}^{(l)}\right\}$ and $K = \left[-\kappa_{ij}\right]$, the expected Fisher information matrix for $i, j, l = 1, \ldots, n$, the bias expression for $\widehat{\boldsymbol{\theta}}$ in matrix notation is:

$$\mathcal{B}\left(\widehat{\boldsymbol{\theta}}\right) = K^{-1} A \operatorname{vec}\left(K^{-1}\right) + \mathcal{O}\left(n^{-2}\right), \tag{7}$$

where $\operatorname{vec}\left(K^{-1}\right)$ is the vector obtained by stacking the columns of K^{-1} and $A = \left\{A^1 \mid \cdots \mid A^p\right\}$.

Finally, the bias corrected MLE for θ_s can be obtained as:

$$\widetilde{\theta}_s = \widehat{\theta}_s - \widehat{\mathcal{B}}\left(\widehat{\theta}_s\right). \tag{8}$$

Alternatively, using matrix notation the bias corrected MLEs can be expressed as Cordeiro and Klein (1994):

$$\widetilde{\boldsymbol{\theta}} = \widehat{\boldsymbol{\theta}} - \widehat{K}^{-1} \widehat{A} \operatorname{vec}\left(\widehat{K}^{-1}\right), \tag{9}$$

where $\widehat{K} = K\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$ and $\widehat{A} = A\big|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$.

The mle.tools Package Details

The current version of the **mle.tools** package, uploaded to CRAN in February 2017, has implemented three functions — `observed.varcov`, `expected.varcov` and `coxsnell.bc` — which are of great interest in data analysis based on MLEs. These functions calculate, respectively, the observed Fisher information, the expected Fisher information and the bias corrected MLEs using the bias formula in (5). The above mentioned functions can be applied to any probability density function whose terms are available in the derivatives table of the D function (see "deriv.c" source code for further details). Integrals, when required, are computed numerically via the integrate function. Below are some mathematical details of how the returned values from the three functions are calculated.

Let $X_1, \ldots, X_n$ be independent and identical random variables with probability density function $f(x_i \mid \boldsymbol{\theta})$ depending on a p-dimensional parameter vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_p)$. The *(j,k)-th* element of the observed, H_{jk}, and expected, I_{jk}, Fisher information are calculated, respectively, as

$$H_{jk} = -\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta_j \partial\theta_k} \log f(x_i \mid \boldsymbol{\theta}) \bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

and

$$I_{jk} = -n \times E\left(\frac{\partial^2}{\partial\theta_j \partial\theta_k} \log f(x \mid \boldsymbol{\theta})\right) = -n \times \int_{\mathcal{X}} \frac{\partial^2}{\partial\theta_j \partial\theta_k} \log f(x \mid \boldsymbol{\theta}) \times f(x \mid \boldsymbol{\theta}) \, \mathrm{d}x \bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}},$$

where $j, k = 1, \ldots, p$, $\widehat{\boldsymbol{\theta}}$ is the MLE of $\boldsymbol{\theta}$ and $\mathcal{X}$ denotes the support of the random variable X.

The observed.varcov function is as follows:

```
function (logdensity, X, parms, mle)
```

where logdensity is an R expression of the log of the probability density function, X is a numeric vector containing the observations, parms is a character vector of the parameter name(s) specified in the logdensity expression and mle is a numeric vector of the parameter estimate(s). This function returns a list with two components (i) mle: the inputed MLEs and (ii) varcov: the observed variance-covariance evaluated at the inputed MLE argument. The elements of the Hessian matrix are calculated analytically.

The functions expected.varcov and coxsnell.bc have the same arguments and are as follows:

```
function (density, logdensity, n, parms, mle, lower = "-Inf", upper = "Inf", ...)
```

where density and logdensity are R expressions of the probability density function and its logarithm, respectively, n is a numeric scalar of the sample size, parms is a character vector of the parameter names(s) specified in the density and log-density expressions, mle is a numeric vector of the parameter estimates, lower is the lower integration limit (-Inf is the default), upper is the upper integration limit (Inf is the default) and ... are additional arguments passed to the integrate function. The expected.varcov function returns a list with two components (i) mle: the inputed MLEs and (ii) varcov: the expected covariance evaluated at the inputed MLEs. The coxsnell.bc function returns a list with five components (i) mle: the inputed MLEs, (ii) varcov: the expected variance-covariance evaluated at the inputed MLEs, (iii) mle.bc: the bias corrected MLEs, (iv) varcov.bc: the expected variance-covariance evaluated at the bias corrected MLEs and (v) bias: the bias estimate(s).

Furthermore, the bias corrected MLE of θ_s, $s = 1, \ldots, p$ denoted by $\widetilde{\theta}_s$ is calculated as $\widetilde{\theta}_s = \widehat{\theta}_s - \widehat{\mathcal{B}}\left(\widehat{\theta}_s\right)$, where $\widehat{\theta}_s$ is the MLE of θ_s and

$$\widehat{\mathcal{B}}\left(\widehat{\theta}_s\right) = \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{l=1}^{p} \kappa^{sj} \kappa^{kl} \left[0.5\kappa_{jkl} + \kappa_{jk,l}\right] \bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}},$$

where κ^{jk} is the (j,k)-*th* element of the inverse of the negative of the expected Fisher information,

$$\kappa_{jkl} = n \int_{\mathcal{X}} \frac{\partial^3}{\partial\theta_j \partial\theta_k \partial\theta_l} \log f(x \mid \boldsymbol{\theta}) \, f(x \mid \boldsymbol{\theta}) \, \mathrm{d}x \bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}},$$

$$\kappa_{jk,l} = n \int_{\mathcal{X}} \frac{\partial^2}{\partial\theta_j \partial\theta_k} \log f(x \mid \boldsymbol{\theta}) \frac{\partial}{\theta_l} \log f(x \mid \boldsymbol{\theta}) \, f(x \mid \boldsymbol{\theta}) \, \mathrm{d}x \bigg|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}}$$

and $\mathcal{X}$ denotes the support of the random variable X.

It is important to emphasize that first, second and third-order partial log-density derivatives are analytically calculated via the D function while integrals are computed numerically, using the integrate function. Furthermore, if numerical integration fails and/or the expected/observed information is singular an error message is returned.

Comparative Study

In order to evaluate the robustness of the `coxsnell.bc` function, we compare, through real applications, the estimated biases obtained from the package and from the analytical expressions for a total of thirty one continuous probability distributions. The analytical expressions for each distribution, named as `distname.bc`, can be found in the supplementary file "analyticalBC.R". For example, the entry `lindley.bc(n,mle)` evaluates the bias estimates locally at `n` and `mle` values.

In the sequel, the probability density function, the analytical Cox-Snell expressions and the bias estimates are provided for: Lindley, inverse Lindley, inverse Exponential, Shanker, inverse Shanker, Topp-Leone, Lévy, Rayleigh, inverse Rayleigh, Half-Logistic, Half-Cauchy, Half-Normal, Normal, inverse Gaussian, Log-Normal, Log-Logistic, Gamma, inverse Gamma, Lomax, weighted Lindley, generalized Rayleigh, Weibull, inverse Weibull, generalized Half-Normal, inverse generalized Half-Normal, Marshall-Olkin extended Exponential, Beta, Kumaraswamy, inverse Beta, Birnbaum-Saunders and generalized Pareto distributions.

It is noteworthy that analytical bias corrected expressions are not reported in the literature for the Lindley, Shanker, inverse Shanker, Lévy, inverse Rayleigh, Half-Cauchy, inverse Weibull, inverse generalized Half-Normal and Marshall-Olkin extended Exponential distributions.

According to all the results presented below, we observe concordance between the bias estimates given by the `coxsnell.bc` function and the analytical expression(s) for 28 out the 31 distributions. The distributions which did not agree with the `coxsnell.bc` function were the Beta, Kumaraswamy and inverse Beta distributions. Perhaps, there are typos either, in our typing or in the analytical expressions reported by Cordeiro et al. (1997), Lemonte (2011) and Stočsić and Cordeiro (2009). Having this view, we recalculated the analytical expressions for the biases. For the Beta and inverse Beta distributions, our recalculated analytical expressions agree with the results returned by the `coxsnell.bc` function, so there are actually typos in the expression of Cordeiro et al. (1997) and Stočsić and Cordeiro (2009). For the Kumaraswamy, we could not evaluate the analytical expression given by the author but we compare the results from `coxsnell.bc` function with a numerical evaluation in *Maple* (Maple, 2017) and the results are exactly equals.

1. One-parameter Lindley distribution with scale parameter θ

$$f(x \mid \theta) = \frac{\theta^2}{1+\theta}(1+x)\exp(-\theta x), \quad x > 0.$$

• Bias expression (not previously reported in the literature):

$$\mathcal{B}\left(\widehat{\theta}\right) = \frac{\left(\theta^3 + 6\,\theta^2 + 6\,\theta + 2\right)(\theta+1)\,\theta}{n\left(\theta^2 + 4\,\theta + 2\right)^2}. \tag{10}$$

Using the data set from Ghitany et al. (2008) we have $n = 100$, $\widehat{\theta} = 0.1866$ and $\widehat{se}\left(\widehat{\theta}\right) = 0.0133$. Evaluating the analytical expression (10) and the `coxsnell.bc` function, we have, respectively,

```
R> lindley.bc(n = 100, mle = 0.1866)

    theta
0.0009546

R> pdf <- quote(theta^2 / (theta + 1) * (1 + x) * exp(-theta * x))
R> lpdf <- quote(2 * log(theta) - log(1 + theta) - theta * x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 100,
+   parms = c("theta"), mle = 0.1866, lower = 0)$bias

    theta
0.0009546
```

2. Inverse Lindley distribution with scale parameter θ

$$f(x \mid \theta) = \frac{\theta^2}{1+\theta}\left(\frac{1+x}{x^3}\right)\exp\left(-\frac{\theta}{x}\right), \quad x > 0.$$

• Bias expression (Wang, 2015):

$$\mathcal{B}\left(\widehat{\theta}\right) = \frac{(\theta+1)\,\theta\,\left(\theta^3 + 6\,\theta^2 + 6\,\theta + 2\right)}{n\left(\theta^2 + 4\,\theta + 2\right)^2}. \tag{11}$$

Using the data set from Sharma et al. (2015) we have $n = 58$, $\widehat{\theta} = 60.0016$ and $\widehat{se}\left(\widehat{\theta}\right) = 7.7535$. Evaluating the analytical expression (11) and the `coxsnell.bc` function, we have, respectively,

```
R> invlindley.bc(n = 58, mle =  60.0016)

theta
1.017

R> pdf <- quote(theta^2 / (theta + 1) * ((1 + x) / x^3) *
+   exp(-theta / x))
R> lpdf <- quote(2 * log(theta) - log(1 + theta) - theta / x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 58,
+   parms = c("theta"), mle =  60.0016, lower = 0)$bias

theta
1.017
```

3. Inverse Exponential distribution with rate parameter θ

$$f(x \mid \theta) = \frac{\theta}{x^2} \exp\left(-\frac{\theta}{x}\right), \quad x > 0.$$

• Bias expression (Johnson et al., 2012b):

$$\mathcal{B}\left(\widehat{\theta}\right) = \frac{\theta}{n}. \tag{12}$$

Using the data set from Lawless (2011), we have $n = 30$, $\widehat{\theta} = 11.1786$ and $\widehat{se}\left(\widehat{\theta}\right) = 2.0409$. Evaluating the analytical expression (12) and the `coxsnell.bc` function, we have, respectively,

```
R> invexp.bc(n = 30, mle = 11.1786)

 theta
0.3726

R> pdf <- quote(theta / x^2 * exp(- theta / x))
R> lpdf <- quote(log(theta) - theta / x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 30,
+   parms = c("theta"), mle = 11.1786, lower = 0)$bias

 theta
0.3726
```

4. Shanker distribution with scale parameter θ

$$f(x \mid \theta) = \frac{\theta^2}{\theta^2 + 1} (\theta + x) \exp(-\theta x), \quad x > 0.$$

• For bias expression (not previously reported in the literature, see the "analyticalBC.R" file.

Using the data set from Shanker (2015), we have $n = 31$, $\widehat{\theta} = 0.0647$ and $\widehat{se}\left(\widehat{\theta}\right) = 0.0082$. Evaluating the analytical expression and the `coxsnell.bc` function, we have, respectively,

```
R> shanker.bc(n = 31, mle = 0.0647)

   theta
0.001035

R> pdf <- quote(theta^2 / (theta^2 + 1) *  (theta + x) *
+   exp(-theta * x))
R> lpdf <- quote(2*log(theta) - log(theta^2 + 1) + log(theta + x) -
+   theta * x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 31,
+   parms = c("theta"), mle = 0.0647, lower = 0)$bias

   theta
0.001035
```

5. Inverse Shanker distribution with scale parameter θ

$$f(x \mid \theta) = \frac{\theta^2}{1+\theta^2}\left(\frac{1+\theta\, x}{x^3}\right)\exp\left(-\frac{\theta}{x}\right), \quad x > 0.$$

• Bias expression (not previously reported in the literature):

$$\mathcal{B}\left(\widehat{\theta}\right) = \frac{\theta^3 + 2\,\theta}{n\,\left(\theta^2+1\right)}. \tag{13}$$

Using the data set from Sharma et al. (2015), we have $n = 58$, $\widehat{\theta} = 59.1412$ and $\widehat{se}\left(\widehat{\theta}\right) = 7.7612$. Evaluating the analytical expression (13) and the coxsnell.bc function, we have, respectively,

```
R> invshanker.bc(n = 58, mle = 59.1412)

theta
 1.02

R> pdf <- quote(theta^2 / (theta^2 + 1) * (theta * x + 1) /
+   x^3 * exp(-theta / x))
R> lpdf <- quote(log(theta) - 2 * log(x) - theta / x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 58,
+   parms = c("theta"), mle = 59.1412, lower = 0)$bias

theta
 1.02
```

6. Topp-Leone distribution with shape parameter ν

$$f(x \mid \nu) = 2\,\nu\,(1-x)\,x^{\nu-1}\,(2-x)^{\nu-1}, \quad 0 < x < 1.$$

• Bias expression (Giles, 2012a):

$$\mathcal{B}\left(\widehat{\nu}\right) = \frac{\nu}{n}. \tag{14}$$

Using the data set from Cordeiro and dos Santos Brito (2012), we have $n = 107$, $\widehat{\nu} = 2.0802$ and $\widehat{se}\left(\widehat{\nu}\right) = 0.2011$. Evaluating the analytical expression (14) and the coxsnell.bc function, we have, respectively,

```
R> toppleone.bc(n = 107, mle = 2.0802)

     nu
0.01944

R> pdf <- quote( 2 * nu * x^(nu - 1) * (1 - x) * (2 - x)^(nu - 1))
R> lpdf <- quote(log(nu) + nu * log(x) + log(1 - x) + (nu - 1) *
+   log(2 - x))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 107,
+   parms = c("nu"), mle = 2.0802, lower = 0, upper = 1)$bias

     nu
0.01944
```

7. One-parameter Lévy distribution with scale parameter σ

$$f(x \mid \sigma) = \sqrt{\frac{\sigma}{2\,\pi}}\, x^{-\frac{3}{2}}\, \exp\left(-\frac{\sigma}{2\,x}\right), \quad x > 0.$$

• Bias expression (not previously reported in the literature):

$$\mathcal{B}\left(\widehat{\sigma}\right) = \frac{2\,\sigma}{n}. \tag{15}$$

Using the data set from Achcar et al. (2013), we have $n = 361$, $\widehat{\sigma} = 4.4461$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.3309$. Evaluating the analytical expression (15) and the coxsnell.bc function, we have, respectively,

```
R> levy.bc(n = 361, mle = 4.4460)
```

```
  sigma
0.02463

R> pdf <- quote(sqrt(sigma / (2 * pi)) * exp(-0.5 * sigma / x) /
+   x^(3 / 2))
R> lpdf <- quote(0.5 * log(sigma) - 0.5 * sigma / x - (3 / 2) * log(x))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 361,
+   parms = c("sigma"), mle = 4.4460, lower = 0)$bias

  sigma
0.02463
```

8. Rayleigh distribution with scale parameter σ

$$f(x \mid \sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\,\sigma^2}\right), \quad x > 0.$$

• Bias expression (Xiao and Giles, 2014):

$$\mathcal{B}\left(\widehat{\sigma}\right) = -\frac{\sigma}{8\,n}. \tag{16}$$

Using the data set from Bader and Priest (1982), we have $n = 69$, $\widehat{\sigma} = 1.2523$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.0754$. Evaluating the analytical expression (16) and the coxsnell.bc function, we have, respectively,

```
R> rayleigh.bc(n = 69, mle = 1.2522)

    sigma
-0.002268

R> pdf <- quote(x / sigma^2 * exp(- 0.5 * (x / sigma)^2))
R> lpdf <- quote(- 2 * log(sigma) - 0.5 * x^2 / sigma^2)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 69,
+   parms = c("sigma"), mle = 1.2522, lower = 0)$bias

    sigma
-0.002268
```

9. Inverse Rayleigh distribution with scale parameter σ

$$f(x \mid \sigma) = \frac{2\,\sigma^2}{x^3} \exp\left(-\frac{\sigma}{x^2}\right), \quad x > 0.$$

• Bias expression (not previously reported in the literature):

$$\mathcal{B}\left(\widehat{\sigma}\right) = \frac{3\sigma}{8\,n}. \tag{17}$$

Using the data set from Bader and Priest (1982), we have $n = 63$, $\widehat{\sigma} = 2.8876$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.1819$. Evaluating the analytical expression (17) and the coxsnell.bc function, we have, respectively,

```
R> invrayleigh.bc(n = 63, mle = 2.8876)

  sigma
0.01719

R> pdf <- quote(2 * sigma^2 / x^3 * exp(-sigma^2 / x^2))
R> lpdf <- quote(2 * log(sigma) - sigma^2 / x^2)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 63,
+   parms = c("sigma"), mle = 2.8876, lower = 0)$bias

  sigma
0.01719
```

10. Half-Logistic distribution with scale parameter σ

$$f(x \mid \sigma) = \frac{2\,\exp\left(-\frac{x}{\sigma}\right)}{\sigma\left[1 + \exp\left(-\frac{x}{\sigma}\right)\right]^2}, \quad x > 0.$$

• Bias expressions (Giles, 2012b):

$$\mathcal{B}(\widehat{\sigma}) = -\frac{0.05256766607\,\sigma}{n}. \tag{18}$$

Using the data set from Bhaumik et al. (2009), we have $n = 34$, $\widehat{\sigma} = 1.3926$ and $\widehat{se}(\widehat{\sigma}) = 0.2056$. Evaluating the analytical expression (17) and the coxsnell.bc function, we have, respectively,

```
R> halflogistic.bc(n = 34, mle = 1.3925)

    sigma
-0.002153

R> pdf <- quote((2/sigma) * exp(-x / sigma) / (1 + exp(-x / sigma))^2)
R> lpdf <- quote(-log(sigma) - x / sigma - 2 * log(1 + exp(-x / sigma)))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 34,
+   parms = c("sigma"), mle = 1.3925, lower = 0)$bias

    sigma
-0.002153
```

11. Half-Cauchy distribution with scale parameter σ

$$f(x \mid \sigma) = \frac{2}{\pi}\,\frac{\sigma}{\sigma^2 + x^2}, \quad x > 0.$$

• Bias expression (not previously reported in the literature):

$$\mathcal{B}(\widehat{\sigma}) = -\frac{\sigma}{n}. \tag{19}$$

Using the data set from Alzaatreh et al. (2016), we have $n = 64$, $\widehat{\sigma} = 28.3345$ and $\widehat{se}(\widehat{\sigma}) = 4.4978$. Evaluating the analytical expression (19) and the coxsnell.bc function, we have, respectively,

```
R> halfcauchy.bc(n = 64, mle = 28.3345)

 sigma
0.4427

R> pdf <- quote( 2 / pi * sigma / (x^2 + sigma^2))
R> lpdf <- quote(log(sigma) - log(x^2 + sigma^2))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 64,
+   parms = c("sigma"), mle = 28.3345, lower = 0)$bias

 sigma
0.4456
```

12. Half-Normal distribution with scale parameter σ

$$f(x \mid \sigma) = \sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma}\,\exp\left(-\frac{x^2}{2\,\sigma^2}\right), \quad x > 0.$$

• Bias expressions (Xiao and Giles, 2014):

$$\mathcal{B}(\widehat{\sigma}) = -\frac{\sigma}{4\,n}. \tag{20}$$

Using the data set from Raqab et al. (2008), we have $n = 69$, $\widehat{\sigma} = 1.5323$ and $\widehat{se}(\widehat{\sigma}) = 0.1304$. Evaluating the analytical expression (20) and the coxsnell.bc function, we have, respectively,

```
R> halfnormal.bc(n = 69, mle = 1.5323)

    sigma
-0.005552

R> pdf <- quote(sqrt(2) / (sqrt(pi) * sigma) * exp(-x^2 / (2 * sigma^2)))
R> lpdf <- quote(-log(sigma) - x^2 / sigma^2 / 2 )
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 69,
+   parms = c("sigma"), mle = 1.5323, lower = 0)$bias

    sigma
-0.005552
```

13. Normal distribution with mean μ and standard deviation σ

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\,\sigma^2}\right], \quad x \in (-\infty, \infty).$$

• Bias expressions (Stočsić and Cordeiro, 2009):

$$\mathcal{B}\left(\widehat{\mu}\right) = 0 \text{ and } \mathcal{B}\left(\widehat{\sigma}\right) = -\frac{3\,\sigma}{4\,n}. \tag{21}$$

Using the data set from Kundu (2005), we have $n = 23$, $\widehat{\mu} = 4.1506$, $\widehat{\sigma} = 0.5215$, $\widehat{se}\left(\widehat{\mu}\right) = 0.1087$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.0769$. Evaluating the analytical expressions (21) and the coxsnell.bc function, we have, respectively,

```
R> normal.bc(n = 23, mle = c(4.1506, 0.5215))

      mu    sigma
 0.00000 -0.01701

R> pdf <- quote(1 / (sqrt(2 * pi) * sigma) *
+   exp(-0.5 / sigma^2 * (x - mu)^2))
R> lpdf <- quote(-log(sigma) - 0.5 / sigma^2 * (x - mu)^2)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 23,
+   parms = c("mu", "sigma"), mle = c(4.1506, 0.5215))$bias

        mu      sigma
-4.071e-13 -1.701e-02
```

14. Inverse Gaussian distribution with mean μ and shape λ

$$f(x \mid \mu, \lambda) = \sqrt{\frac{\lambda}{2\,\pi\,x^3}} \exp\left[-\frac{\lambda\,(x-\mu)^2}{2\,x\,\mu^2}\right], \quad x > 0.$$

• Bias expressions (Stočsić and Cordeiro, 2009):

$$\mathcal{B}\left(\widehat{\mu}\right) = 0 \text{ and } \mathcal{B}\left(\widehat{\lambda}\right) = \frac{3\lambda}{n}. \tag{22}$$

Using the data set from Chhikara and Folks (1977), we have $n = 46$, $\widehat{\mu} = 3.6067$, $\widehat{\lambda} = 1.6584$, $\widehat{se}\left(\widehat{\mu}\right) = 0.7843$ and $\widehat{se}\left(\widehat{\lambda}\right) = 0.3458$. Evaluating the analytical expressions (22) and the coxsnell.bc function, we have, respectively,

```
R> invgaussian.bc(n = 46, mle =  c(3.6065, 1.6589))

    mu lambda
0.0000 0.1082

R> pdf <- quote(sqrt(lambda / (2 * pi * x^3)) *
+   exp(-lambda * (x - mu)^2 / (2 * mu^2 * x)))
R> lpdf <- quote(0.5 * log(lambda) - lambda * (x - mu)^2 /
+   (2 * mu^2 * x))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 46,
+   parms = c("mu", "lambda"), mle = c(3.6065, 1.6589), lower = 0)$bias

       mu    lambda
3.483e-07 1.082e-01
```

15. Log-Normal distribution with location μ and scale σ

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\,\pi}\,x\,\sigma} \exp\left[-\frac{(\log x-\mu)^2}{\sigma^2}\right], \quad x > 0.$$

• Bias expressions (Stočsić and Cordeiro, 2009):

$$\mathcal{B}\left(\widehat{\mu}\right) = 0 \text{ and } \mathcal{B}\left(\widehat{\sigma}\right) = -\frac{3\,\sigma}{4\,n}. \tag{23}$$

Using the data set from Kumagai et al. (1989), we have $n = 30$, $\widehat{\mu} = 2.164$, $\widehat{\sigma} = 1.1765$, $\widehat{se}\left(\widehat{\mu}\right) = 0.2148$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.1519$. Evaluating the analytical expressions (23) and the coxsnell.bc function, we have, respectively,

```
R> lognormal.bc(n = 30, mle = c(2.1643, 1.1765))

      mu    sigma
 0.00000 -0.02941

R> pdf <- quote(1 / (sqrt(2 * pi) * x * sigma) *
+   exp(-0.5 * (log(x) - mu)^2 / sigma^2))
R> lpdf <- quote(-log(sigma) - 0.5 * (log(x) - mu)^2 / sigma^2)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 30,
+   parms = c("mu", "sigma"), mle = c(2.1643, 1.1765), lower = 0)$bias

        mu      sigma
-5.952e-09 -2.941e-02
```

16. Log-Logistic distribution with shape β and scale α

$$f(x \mid \alpha, \beta) = \frac{(\beta/\alpha)\,(x/\alpha)^{\beta-1}}{\left[1+(x/\alpha)^{\beta}\right]^2}, \quad x > 0.$$

• For bias expressions, see Reath (2016).

From Reath (2016) we have $n = 19$, $\widehat{\alpha} = 6.2542$, $\widehat{\beta} = 1.1732$, $\widehat{se}\,(\widehat{\alpha}) = 2.1352$, $\widehat{se}\left(\widehat{\beta}\right) = 0.2239$, $\widehat{\mathcal{B}}\,(\widehat{\alpha}) = 0.3585$ and $\widehat{\mathcal{B}}\left(\widehat{\beta}\right) = 0.0789$. Evaluating the `coxsnell.bc` function, we have:

```
R> pdf <- quote((beta / alpha) * (x / alpha)^(beta - 1) /
+   (1 + (x / alpha)^beta)^2)
R> lpdf <- quote(log(beta) - log(alpha) + (beta - 1) * log(x / alpha) -
+   2 * log(1 + (x / alpha)^beta))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 19,
+   parms =  c("alpha", "beta"), mle = c(6.2537, 1.1734),
+   lower = 0)$bias

  alpha    beta
0.35854 0.07883
```

17. Gamma distribution with shape α and rate λ

$$f(x \mid \alpha, \lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)}\, x^{\alpha-1}\, \exp(-\lambda\, x), \quad x > 0.$$

• Bias expressions (Giles and Feng, 2009):

$$\mathcal{B}\,(\widehat{\alpha}) = \frac{\alpha\,\left[\Psi'(\alpha) - \alpha\Psi''(\alpha)\right] - 2}{2\,n\,\left[\alpha\Psi'(\alpha) - 1\right]^2} \tag{24}$$

and

$$\mathcal{B}\left(\widehat{\lambda}\right) = \frac{\lambda\,\left[2\,\alpha\,\left(\Psi'(\alpha)\right)^2 - 3\,\Psi'(\alpha) - \alpha\,\Psi''(\alpha)\right]}{2\,n\,\left[\alpha\Psi'(\alpha) - 1\right]^2}. \tag{25}$$

Using the data set from Delignette-Muller et al. (2008), we have $n = 254$, $\widehat{\alpha} = 4.0083$, $\widehat{\lambda} = 0.0544$, $\widehat{se}\,(\widehat{\alpha}) = 0.3413$ and $\widehat{se}\left(\widehat{\lambda}\right) = 0.0049$. Evaluating the analytical expressions (24), (25) and the `coxsnell.bc` function, we have, respectively,

```
R> gamma.bc(n = 254, mle = c(4.0082, 0.0544))

    alpha    lambda
0.0448278 0.0006618

R> pdf <- quote((lambda^alpha) / gamma(alpha) * x^(alpha - 1) *
+   exp(-lambda *x))
R> lpdf <- quote(alpha * log(lambda) - lgamma(alpha) + alpha * log(x) -
+   lambda * x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 254,
+   parms = c("alpha", "lambda"), mle = c(4.0082, 0.0544),
+   lower = 0)$bias
```

```
    alpha    lambda
0.0448278 0.0006618
```

18. Inverse Gamma distribution with shape α and scale β

$$f(x \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right), \quad x > 0.$$

- Bias expressions (Stočsić and Cordeiro, 2009):

$$\mathcal{B}\left(\widehat{\alpha}\right) = \frac{-0.5\,\alpha^2\,\Psi''\left(\alpha\right) + 0.5\,\Psi'\left(\alpha\right)\alpha - 1}{n\,\alpha\,\left(\Psi'\left(\alpha\right) - 1\right)^2} \tag{26}$$

and

$$\mathcal{B}\left(\widehat{\beta}\right) = \frac{\beta\left(-0.5\,\alpha\,\Psi''\left(\alpha\right) - 1.5\,\Psi'\left(\alpha\right) + \left(\Psi'\left(\alpha\right)\right)^2\alpha\right)}{n\left(\Psi'\left(\alpha\right)\alpha - 1.0\right)^2}. \tag{27}$$

Using the data set from Kumagai and Matsunaga (1995), we have $n = 31$, $\widehat{\alpha} = 1.0479$, $\widehat{\beta} = 5.491$, $\widehat{se}\left(\widehat{\alpha}\right) = 0.2353$ and $\widehat{se}\left(\widehat{\beta}\right) = 1.5648$. Evaluating the analytical expressions (26), (27) and the `coxsnell.bc` function, we have, respectively,

```
R> invgamma.bc(n = 31, mle = c(5.4901, 1.0479))

   beta   alpha
0.60849 0.08388

R> pdf <- quote(beta^alpha / gamma(alpha) * x^(-alpha - 1) *
+   exp(-beta / x))
R> lpdf <- quote(alpha * log(beta) - lgamma(alpha) -
+   alpha * log(x) - beta / x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 31,
+   parms = c("beta", "alpha"), mle = c(5.4901, 1.0479), lower = 0)$bias

   beta   alpha
0.60847 0.08388
```

19. Lomax distribution with shape α and scale β

$$f(x \mid \alpha, \beta) = \alpha\,\beta\,(1 + \beta\,x)^{-(\alpha+1)}, \quad x > 0.$$

- Bias expressions (Giles et al., 2013):

$$\mathcal{B}\left(\widehat{\alpha}\right) = \frac{2\,\alpha\,\left(\alpha+1\right)\left(\alpha^2+\alpha-2\right)}{\left(\alpha+3\right)n} \tag{28}$$

and

$$\mathcal{B}\left(\widehat{\beta}\right) = -\frac{2\,\beta\,\left(\alpha+1.6485\right)\left(\alpha+0.3934\right)\left(\alpha-1.5419\right)}{n\,\alpha\,\left(\alpha+3\right)}. \tag{29}$$

Using the data set from Tahir et al. (2016), we have $n = 179$, $\widehat{\alpha} = 4.9103$, $\widehat{\beta} = 0.0028$, $\widehat{se}\left(\widehat{\alpha}\right) = 0.6208$ and $\widehat{se}\left(\widehat{\beta}\right) = 3.4803 \times 10^{-4}$. Evaluating the analytical expressions (28), (29) and the `coxsnell.bc` function, we have, respectively,

```
R> lomax.bc(n = 179, mle = c(4.9103, 0.0028))

     alpha       beta
 1.281e+00 -9.438e-05

R> pdf <- quote(alpha * beta / (1 + beta * x)^(alpha + 1))
R> lpdf <- quote(log(alpha) + log(beta) - (alpha + 1) *
+   log(1 + beta * x))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 179,
+   parms = c("alpha", "beta"), mle = c(4.9103, 0.0028),
+   lower = 0)$bias
```

```
    alpha       beta
1.281e+00 -9.439e-05
```

20. Weighted Lindley distribution with shape α and scale θ

$$f(x \mid \alpha, \theta) = \frac{\theta^{\alpha+1}}{(\theta+\alpha)\,\Gamma(\alpha)}\, x^{\alpha-1}\,(1+x)\,\exp(-\theta x), \quad x > 0.$$

• For bias expressions, see (Wang and Wang, 2017):
Using the data set from Ghitany et al. (2013), we have $n = 69$, $\widehat{\alpha} = 22.8889$, $\widehat{\theta} = 9.6246$, $\widehat{se}\,(\widehat{\alpha}) = 3.9507$ and $\widehat{se}\left(\widehat{\theta}\right) = 1.6295$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> wlindley.bc(n = 69, mle = c(22.8889, 9.6246))
```

```
 alpha  theta
1.0070 0.4167
```

```
R> pdf <- quote(theta^(alpha + 1) / ((theta + alpha) * gamma(alpha)) *
+   x^(alpha - 1) * (1 + x) * exp(-theta * x))
R> lpdf <- quote((alpha + 1) * log(theta) + alpha * log(x) -
+   log(theta + alpha) - lgamma(alpha) - theta * x)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 69,
+   parms = c("alpha", "theta"), mle = c(22.8889, 9.6246),
+   lower = 0)$bias
```

```
 alpha  theta
1.0068 0.4166
```

21. Generalized Rayleigh with shape α and scale θ

$$f(x \mid \beta, \mu) = \frac{2\,\theta^{\alpha+1}}{\Gamma(\alpha+1)}\, x^{2\,\alpha+1}\, \exp\left(-\theta\, x^2\right), \quad x > 0.$$

• For bias expressions, see (Xiao and Giles, 2014):
Using the data set from Gomes et al. (2014), we have $n = 384$, $\widehat{\theta} = 0.5195$, $\widehat{\alpha} = 0.0104$, $\widehat{se}\left(\widehat{\theta}\right) = 0.2184$ and $\widehat{se}\,(\widehat{\alpha}) = 0.0014$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> generalizedrayleigh.bc(n =  384, mle = c(0.5195, 0.0104))
```

```
    alpha     theta
1.035e-02 8.865e-05
```

```
R> pdf <- quote(2 * theta^(alpha + 1) / gamma(alpha + 1) *
+   x^(2 * alpha + 1) * exp(-theta * x^2 ))
R> lpdf <- quote((alpha + 1) * log(theta) - lgamma(alpha + 1) +
+   2 * alpha * log(x) - theta * x^2)
R> coxsnell.bc(density = pdf, logdensity = lpdf,  n = 384,
+   parms = c("alpha", "theta"), mle = c(0.5195, 0.0104),
+   lower = 0)$bias
```

```
    alpha     theta
1.035e-02 8.865e-05
```

22. Weibull distribution with shape β and scale μ

$$f(x \mid \beta, \mu) = \frac{\beta}{\mu^{\beta}}\, x^{\beta-1}\, \exp\left(-\frac{x}{\mu}\right)^{\beta}, \quad x > 0.$$

• Bias expressions (the expressions below differs from Stočsić and Cordeiro (2009)):

$$\mathcal{B}\,(\widehat{\mu}) = \frac{\mu\,(0.5543324495 - 0.3698145397\,\beta)}{n\,\beta^2} \tag{30}$$

and

$$\mathcal{B}\left(\widehat{\beta}\right) = \frac{1.379530692\,\beta}{n}. \tag{31}$$

From Datta and Datta (2013), we have $n = 50$, $\widehat{\mu} = 2.5752$, $\widehat{\beta} = 38.0866$, $\widehat{se}\,(\widehat{\mu}) = 0.2299$ and $\widehat{se}\left(\widehat{\beta}\right) = 2.2299$. Evaluating the analytical expression (30), (31) and the `coxsnell.bc` function, we have, respectively,

```
R> weibull.bc(n = 50, mle = c(38.0866, 2.5751))

      mu     beta
-0.04572  0.07105

R> pdf <- quote(beta / mu^beta * x^(beta - 1) *
+   exp(-(x / mu)^beta))
R> lpdf <- quote(log(beta) - beta * log(mu) + beta * log(x) -
+   (x / mu)^beta)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 50,
+   parms = c("mu", "beta"), mle = c(38.0866, 2.5751), lower = 0)$bias

      mu     beta
-0.04572  0.07105
```

23. Inverse Weibull distribution with shape β and scale μ

$$f(x \mid \beta, \alpha) = \beta\, \mu^{\beta}\, x^{-(\beta+1)} \exp\left[-\left(\frac{\mu}{x}\right)^{\beta}\right], \quad x > 0.$$

• Bias expressions (not previously reported in the literature):

$$\mathcal{B}\left(\widehat{\beta}\right) = \frac{1.379530690\,\beta}{n} \tag{32}$$

and

$$\mathcal{B}\,(\widehat{\mu}) = \frac{\mu\,\left(0.3698145391\,\beta + 0.5543324494\right)}{n\beta^2}. \tag{33}$$

Using the data set from Nichols and Padgett (2006), we have $n = 100$, $\widehat{\beta} = 1.769$, $\widehat{\mu} = 1.8917$, $\widehat{se}\left(\widehat{\beta}\right) = 0.1119$ and $\widehat{se}\,(\widehat{\mu}) = 0.1138$. Evaluating the analytical expressions (32), (33) and the `coxsnell.bc` function, we have, respectively,

```
R> inverseweibull.bc(n = 100, mle = c(1.7690, 1.8916))

    beta       mu
0.024404 0.007305

R> pdf <- quote(beta * mu^beta * x^(-beta - 1) *
+   exp(-(mu / x)^beta))
R> lpdf <- quote(log(beta) + beta * log(mu) - beta * log(x) -
+   (mu / x)^beta)
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 100,
+   parms = c("beta", "mu"), mle = c(1.7690, 1.8916), lower = 0)$bias

    beta       mu
0.024404 0.007305
```

24. Generalized Half-Normal distribution with shape α and scale θ

$$f(x \mid \alpha, \theta) = \sqrt{\frac{2}{\pi}}\, \frac{\alpha}{\theta^{\alpha}}\, x^{\alpha-1} \exp\left[-\frac{1}{2}\left(\frac{x}{\theta}\right)^{2\alpha}\right].$$

• Bias expressions (Mazucheli and Dey, 2017):

$$\mathcal{B}\,(\widehat{\alpha}) = 1.483794456\,\frac{\alpha}{n} \tag{34}$$

and

$$\mathcal{B}\left(\widehat{\theta}\right) = \left(0.2953497661 - 0.3665611957\,\alpha\right) \frac{\theta}{n\,\alpha^2}. \tag{35}$$

Using the data set from Nadarajah (2008a), we have $n = 119$, $\widehat{\alpha} = 3.8096$, $\widehat{\theta} = 4.9053$, $\widehat{se}\,(\widehat{\alpha}) = 0.2758$ and $\widehat{se}\left(\widehat{\theta}\right) = 0.0913$. Evaluating the analytical expressions (34), (35) and the `coxsnell.bc` function, we have, respectively,

```
R> genhalfnormal.bc(n = 119, mle = c(3.8095, 4.9053))

    alpha     theta
 0.047500 -0.003127

R> pdf <- quote(sqrt(2 / pi) * alpha / theta^alpha * x^(alpha - 1)*
+   exp(- 0.5 * (x / theta)^(2 * alpha) ))
R> lpdf <- quote(log(alpha) - alpha * log(theta) + alpha * log(x) -
+   0.5 * (x / theta)^(2 * alpha))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 119,
+   parms = c("alpha", "theta"), mle = c(3.8095, 4.9053),
+   lower = 0)$bias

    alpha     theta
 0.047500 -0.003127
```

25. Inverse Generalized Half-Normal distribution with shape α and scale θ

$$f(x \mid \alpha, \theta) = \sqrt{\frac{2}{\pi}} \left(\frac{\alpha}{x}\right) \left(\frac{1}{\theta\, x}\right)^{\alpha} \exp\left[-\frac{1}{2}\left(\frac{1}{\theta\, x}\right)^{2\,\alpha}\right], \quad x > 0.$$

• For bias expressions (not previously reported in the literature, see the “analyticalBC.R” file. Using the data set from Nadarajah et al. (2011), we have $n = 20$, $\widehat{\alpha} = 3.0869$, $\widehat{\theta} = 0.6731$, $\widehat{se}\,(\widehat{\alpha}) = 0.5534$ and $\widehat{se}\left(\widehat{\theta}\right) = 0.0379$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> invgenhalfnormal.bc(n = 20, mle = c(3.0869, 0.6731))

    alpha     theta
 0.229016 -0.002953

R> pdf <- quote(sqrt(2) * pi^(-0.5) * alpha * x^(-alpha - 1) *
+   exp(-0.5 * x^(-2 * alpha) * (1 / theta)^(2 * alpha))
+   * theta^(-alpha))
R> lpdf <- quote(log(alpha) - alpha  * log(x) - 0.5e0 / (x^alpha)^2*
+   theta^(-2 * alpha) - alpha * log(theta))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 20,
+   parms = c("alpha", "theta"), mle = c(3.0869, 0.6731),
+   lower = 0)$bias

    alpha     theta
 0.229016 -0.002953
```

26. Marshall-Olkin Extended Exponential distribution with shape α and rate λ

$$f(x \mid \alpha, \lambda) = \frac{\lambda\, \alpha\, \exp\left(-\lambda\, x\right)}{\left[1 - (1-\alpha)\, \exp\left(-\lambda\, x\right)\right]^2}, \quad x > 0.$$

• For bias expressions (not previously reported in the literature, see the “analyticalBC.R” file. Using the data set from Linhart and Zucchini (1986), we have $n = 20$, $\widehat{\alpha} = 0.2782$, $\widehat{\lambda} = 0.0078$, $\widehat{se}\,(\widehat{\alpha}) = 0.2321$ and $\widehat{se}\left(\widehat{\lambda}\right) = 0.0049$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> moeexp.bc(n = 20, mle = c(0.2781, 0.0078))

   alpha   lambda
0.210919 0.003741

R> pdf <- quote(alpha * lambda * exp(-x * lambda) /
+   ((1- (1 - alpha) * exp(- x * lambda)))^2)
R> lpdf <- quote(log(alpha) + log(lambda) - x * lambda -
+   2 * log((1 - (1-alpha) * exp(- x * lambda))))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 20,
+   parms = c("alpha", "lambda"), mle = c(0.2781, 0.0078),
+   lower = 0)$bias
```

```
  alpha  lambda
0.21086 0.00374
```

27. Beta distribution with shapes α and β

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \quad 0 < x < 1.$$

• For bias expressions, see (Cordeiro et al., 1997).
Using the data set from Javanshiri et al. (2015), we have $n = 48$, $\widehat{\alpha} = 5.941$, $\widehat{\beta} = 21.2024$, $\widehat{se}\,(\widehat{\alpha}) = 1.1812$ and $\widehat{se}\left(\widehat{\beta}\right) = 4.3462$. Evaluating the analytical expressions in Cordeiro et al. (1997), our analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> beta.gauss.bc(n = 48, mle = c(5.941, 21.2024))

 alpha   beta
-4.784 -4.125

R> beta.bc(n = 48, mle = c(5.941, 21.2024))

 alpha   beta
0.3582 1.3315

R> pdf <- quote(gamma(alpha + beta) / (gamma(alpha) * gamma(beta)) *
+   x^(alpha - 1) * (1 - x)^(beta - 1))
R> lpdf <- quote(lgamma(alpha + beta) - lgamma(alpha) -
+   lgamma(beta) + alpha * log(x) + beta * log(1 - x))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 48,
+   parms = c("alpha", "beta"), mle = c(5.941, 21.2024),
+   lower = 0, upper =  1)$bias

 alpha   beta
0.3582 1.3315
```

28. Kumaraswamy distribution with shapes α and β

$$f(x \mid \alpha, \beta) = \alpha\,\beta\, x^{\alpha-1}\,(1-x^{\alpha})^{\beta-1}, \quad 0 < x < 1.$$

• For bias expressions, see (Lemonte, 2011).
Using the data set from Wang et al. (2017), we have $n = 20$, $\widehat{\alpha} = 6.3478$, $\widehat{\beta} = 4.4898$, $\widehat{se}\,(\widehat{\alpha}) = 1.5576$ and $\widehat{se}\left(\widehat{\beta}\right) = 2.0414$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> kum.bc(n = 20, mle = c(6.3478, 4.4898))

  alpha    beta
 -6.573 -13.323

R> pdf <- quote(alpha * beta * x^(alpha - 1) *
+   (1 - x^alpha)^(beta - 1))
R> lpdf <- quote(log(alpha) + log(beta) + alpha * log(x) + (beta - 1) *
+   log(1 - x^alpha))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 20,
+   parms = c("alpha", "beta"), mle = c(6.3478, 4.4898),
+   lower = 0, upper = 1)$bias

alpha  beta
0.514 1.013
```

29. Inverse Beta distribution with shapes α and β

$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\, x^{\alpha-1}\,(1+x)^{-(\alpha+\beta)}, \quad x > 0.$$

• For bias expressions, see (Stočsić and Cordeiro, 2009).
Using the data set from Nadarajah (2008b), we have $n = 116$, $\widehat{\alpha} = 28.5719$, $\widehat{\beta} = 1.3783$, $\widehat{se}\,(\widehat{\alpha}) = 4.0367$ and $\widehat{se}\left(\widehat{\beta}\right) = 0.1637$. Evaluating the analytical expressions and the coxsnell.bc function, we have, respectively,

```
R> invbeta.bc(n = 116, mle = c(28.5719, 1.3782))

 alpha   beta
534.26  17.73

R> pdf <- quote(gamma(alpha + beta) * x^(alpha - 1) *
+   (1 + x)^(- alpha - beta) / gamma(alpha)/gamma(beta))
R> lpdf <- quote(lgamma(alpha + beta) + alpha * log(x)
+   - (alpha + beta) * log(1 + x) - lgamma(alpha) - lgamma(beta))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 116,
+   parms = c("alpha", "beta"), mle = c(28.5719, 1.3782),
+   lower = 0)$bias

 alpha   beta
0.8025 0.0306
```

30. Birnbaum-Saunders distribution with shape α and scale β

$$f(x \mid \alpha, \beta) = \frac{1}{2\,\alpha\,\beta\,\sqrt{2\,\pi}} \left[\left(\frac{\beta}{x}\right)^{1/2} + \left(\frac{\beta}{x}\right)^{3/2} \right] \exp\left[-\frac{1}{2,\alpha^2} \left(\frac{x}{\beta} + \frac{\beta}{x} - 2 \right) \right], \quad x > 0.$$

• Bias expressions (Lemonte et al., 2007):

$$\mathcal{B}\left(\widehat{\alpha}\right) = -\frac{\alpha}{4\,n} \left(1 + \frac{2 + \alpha^2}{\alpha\,(2\,\pi)^{-1/2}\,h(\alpha) + 1} \right) \tag{36}$$

and

$$\mathcal{B}\left(\widehat{\beta}\right) = \frac{\beta^2\,\alpha^2}{2\,n\,\left[\alpha\,(2\,\pi)^{-1/2}\,h(\alpha) + 1\right]}, \tag{37}$$

where

$$h(\alpha) = \alpha\,\sqrt{\frac{\pi}{2}} - \pi\,e^{2/\alpha^2} \left[1 - \Phi\left(\frac{2}{\alpha}\right) \right].$$

Using the data set from Gross and Clark (1976), we have $n = 20$, $\widehat{\alpha} = 0.3149$, $\widehat{\beta} = 1.8105$, $\widehat{se}\left(\widehat{\alpha}\right) = 0.0498$ and $\widehat{se}\left(\widehat{\beta}\right) = 0.1259$. Evaluating the analytical expressions (36), (37) and the `coxsnell.bc` function, we have, respectively,

```
R> birnbaumsaunders.bc(n = 20, mle = c(0.3148, 1.8104))

    alpha      beta
-0.011991  0.004374

R> pdf <- quote(1 / (2 * alpha * beta * sqrt(2 * pi)) *
+   ((beta / x)^0.5 + (beta / x)^1.5) *
+   exp(- 1/(2 * alpha^2) * (x / beta + beta/ x - 2)))
R> lpdf <- quote(-log(alpha) - log(beta) - 1 / (2 * alpha^2) *
+   (x / beta + beta/ x - 2) + log((beta / x)^0.5 + (beta / x)^1.5))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 20,
+   parms = c("alpha", "beta"), mle = c(0.3148, 1.8104), lower = 0)$bias

    alpha      beta
-0.011991  0.004374
```

31. Generalized Pareto distribution with shape ξ and scale σ

$$f(x \mid \xi, \sigma) = \frac{1}{\sigma} \left(1 + \frac{\xi\,x}{\sigma} \right)^{-(1/\xi + 1)}, \quad x > 0,\ \xi \neq 0.$$

• Bias expressions (Giles et al., 2016):

$$\mathcal{B}\left(\widehat{\xi}\right) = -\frac{(1+\xi)\,(3+\xi)}{n\,(1+3\,\xi)} \tag{38}$$

and

$$\mathcal{B}\left(\widehat{\sigma}\right) = -\frac{\sigma\,\left(3 + 5\,\xi + 4\,\xi^2\right)}{n\,(1+3\,\xi)}. \tag{39}$$

Using the data set from Ross and Lott (2003), we have $n = 58$, $\widehat{\xi} = 0.736$, $\widehat{\sigma} = 1.709$, $\widehat{se}\left(\widehat{\xi}\right) = 0.223$ and $\widehat{se}\left(\widehat{\sigma}\right) = 0.41$. Evaluating the analytical expressions (38), (39) and the `coxsnell.bc` function, we have, respectively,

```
R> genpareto.bc(n = 58, mle = c(0.736, 1.709))

      xi    sigma
-0.03486  0.08126

R> pdf <- quote(1 / sigma * (1 + xi * x / sigma )^(-(1 + 1 / xi)))
R> lpdf <- quote(-log(sigma) - (1 + 1 / xi) * log(1 + xi * x / sigma))
R> coxsnell.bc(density = pdf, logdensity = lpdf, n = 58,
+   parms = c("xi", "sigma"), mle = c(0.736, 1.709), lower = 0)$bias

      xi    sigma
-0.03486  0.08126
```

Additional Applications

In this section, we present additional numerical results returned by `cosnell.bc`, `observed.varc` and `expected.varcov`. For the data describing the times between successive electric pulses on the surface of isolated muscle fiber (Cox and Lewis, 1966; Jørgensen, 1982), we fitted the exponentiated Weibull, Marshall-Olkin extended Weibull, Weibull, Marshall-Olkin extended Exponential and Exponential distributions. These distributions were also fitted by Cordeiro and Lemonte (2013). There are 799 observations and for each distribution we report the MLEs, the bias corrected MLEs, the observed variance-covariance obtained from the numerical Hessian $\boldsymbol{H}_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$, the observed variance-covariance obtained from the analytical Hessian $\boldsymbol{H}_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$, the expected variance-covariance $\boldsymbol{I}^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$ and the expected variance-covariance evaluated at the bias corrected MLEs $\boldsymbol{I}^{-1}\left(\widetilde{\boldsymbol{\theta}}\right)$. The MLEs and the $\boldsymbol{H}_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right)$ matrix were obtained by the **fitdistrplus** package (Delignette-Muller et al., 2017). The R codes used to obtain the numerical results are available in the supplementary material.

It is important to emphasize that for the Marshall-Olkin extended Weibull and exponentiated Weibull distributions, it is not possible to obtain analytical expressions for bias corrections. The exponentiated-Weibull family was proposed by Mudholkar and Srivastava (1993). Its probability density function is:

$$f(x \mid \lambda, \beta, \alpha) = \alpha\, \beta\, \lambda\, x^{\beta-1}\, \mathrm{e}^{-\lambda\, x^{\beta}} \left(1 - \mathrm{e}^{-\lambda\, x^{\beta}}\right)^{\alpha-1},$$

where $\lambda > 0$ is the scale parameter and $\beta > 0$ and $\alpha > 0$ are the shape parameters. The Marshall-Olkin extended Weibull distribution was introduced by Marshall and Olkin (1997). Its probability density function is:

$$f(x \mid \lambda, \beta, \alpha) = \frac{\alpha\, \beta\, \lambda\, x^{\beta-1}\, \mathrm{e}^{-\lambda\, x^{\beta}}}{\left(1 - \bar{\alpha}\, \mathrm{e}^{-\lambda\, x^{\beta}}\right)^2},$$

where $\lambda > 0$ is the scale parameter, $\beta > 0$ is the shape parameter, $\alpha > 0$ is an additional shape parameter and $\bar{\alpha} = 1 - \alpha$.

The fitted parameter estimates and their bias corrected estimates are shown in Table 1. We see that the bias corrected MLEs for α and λ of the MOE-Weibull and Exp-Weibull distributions are quite different from the original MLEs.

Table 1: MLEs and bias corrected MLEs.

Distribution	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\lambda}$	$\widetilde{\alpha}$	$\widetilde{\beta}$	$\widetilde{\lambda}$
MOE-Weibull	0.3460	1.3247	0.0203	0.3283	1.3240	0.0188
Exp-Weibull	1.9396	0.7677	0.2527	1.8973	0.7625	0.2461
Weibull	–	1.0829	0.0723	–	1.0811	0.0723
MOE-Exponential	1.1966	–	0.0998	1.1820	–	0.0994
Exponential	–	–	0.0913	–	–	0.0912

It is important to assess the accuracy of MLEs. The two common ways for this are through the inverse observed Fisher information and the inverse expected Fisher information matrices. The results below show large differences between the observed H^{-1} and expected I^{-1} information matrices. As demonstrated by Cao (2013), the I^{-1} outperforms the H^{-1} under a mean squared error criterion, hence with **mle.tools** the researchers may choose one of them and not use the easier. Furthermore, in general, we observe that the bias corrected MLEs decrease the variance of estimates.

- Exponentiated Weibull distribution:

$$H_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00726 & -0.00717 & 0.03564 \\ -0.00717 & 0.00718 & -0.03493 \\ 0.03564 & -0.03493 & 0.18045 \end{bmatrix}, \quad H_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00729 & -0.00720 & 0.03579 \\ -0.00720 & 0.00721 & -0.03509 \\ 0.03579 & -0.03509 & 0.18120 \end{bmatrix},$$

$$I^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00532 & -0.00524 & 0.02609 \\ -0.00524 & 0.00527 & -0.02545 \\ 0.02609 & -0.02545 & 0.13333 \end{bmatrix}, \quad I^{-1}\left(\widetilde{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00510 & -0.00510 & 0.02482 \\ -0.00510 & 0.00519 & -0.02454 \\ 0.02482 & -0.02454 & 0.12590 \end{bmatrix}.$$

- Marshall-Olkin Extended Weibull distribution:

$$H_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & -0.00036 & 0.00052 \\ -0.00036 & 0.00361 & -0.00430 \\ 0.00052 & -0.00430 & 0.00748 \end{bmatrix}, \quad H_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00005 & -0.00047 & 0.00068 \\ -0.00047 & 0.00468 & -0.00582 \\ 0.00068 & -0.00582 & 0.00967 \end{bmatrix},$$

$$I^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00006 & -0.00056 & 0.00082 \\ -0.00056 & 0.00542 & -0.00699 \\ 0.00082 & -0.00699 & 0.01146 \end{bmatrix}, \quad I^{-1}\left(\widetilde{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00005 & -0.00051 & 0.00072 \\ -0.00051 & 0.00526 & -0.00651 \\ 0.00072 & -0.00651 & 0.01030 \end{bmatrix}.$$

- Weibull distribution:

$$H_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & -0.00018 \\ -0.00018 & 0.00086 \end{bmatrix}, \qquad H_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & -0.00018 \\ -0.00018 & 0.00087 \end{bmatrix},$$

$$I^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & -0.00018 \\ -0.00018 & 0.00089 \end{bmatrix}, \qquad I^{-1}\left(\widetilde{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & -0.00018 \\ -0.00018 & 0.00089 \end{bmatrix}.$$

- Marshall-Olkin Extended Exponential distribution:

$$H_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & 0.00081 \\ 0.00081 & 0.02022 \end{bmatrix}, \qquad H_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & 0.00081 \\ 0.00081 & 0.02023 \end{bmatrix},$$

$$I^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & 0.00083 \\ 0.00083 & 0.02094 \end{bmatrix}, \qquad I^{-1}\left(\widetilde{\boldsymbol{\theta}}\right) = \begin{bmatrix} 0.00004 & 0.00082 \\ 0.00082 & 0.02047 \end{bmatrix}.$$

- Exponential distribution:

$$H_1^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = 0.000010433, \qquad H_2^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = 0.000010436,$$

$$I^{-1}\left(\widehat{\boldsymbol{\theta}}\right) = 0.000010436, \qquad I^{-1}\left(\widetilde{\boldsymbol{\theta}}\right) = 0.000010410.$$

Concluding Remarks

As pointed out by several works in the literature, the Cox-Snell methodology, in general, is efficient for reducing the bias of the MLEs. However, the analytical expressions are either notoriously cumbersome or even impossible to deduce. To the best of our knowledge, there are only two alternatives to obtain the analytical expressions automatically, those presented in Stočsić and Cordeiro (2009) and Johnson et al. (2012a). They use the commercial softwares *Maple* (Maple, 2017) and *Mathematica* (Wolfram Research, Inc., 2010).

In order to calculate the bias corrected estimates in a simple way, Mazucheli (2017) developed an R (R Core Team, 2016) package, uploaded to CRAN on 2 February 2017. Its main function, `coxsnell.bc`, evaluates the bias corrected estimates. The usefulness of this function has been tested for thirty one continuous probability distributions. Bias expressions, for most of them, are available in the literature.

It is well known that the Fisher information can be computed using the first or second order derivatives of the log-likelihood function. In our implementation, the functions `expected.varcov` and `coxsnell.bc` are using the second order derivatives, analytically returned by the `D` function. In a future work, we intend to check if there is any gain in calculating the Fisher information from the first order derivatives of the log-hazard rate function or from the first order derivatives of the log-reversed-hazard rate function. Efron and Johnstone (1990) showed that the Fisher information can be computed using the hazard rate function. Gupta et al. (2004) computed the Fisher information from the first order derivatives of the log-reversed-hazard rate function. In general, expressions of the first order derivatives of the log-hazard rate function (log-reversed-hazard rate function) are simpler than second order derivatives of the log-likelihood function. In this sense, the `integrate` function can work better. It is important to point out that the hazard rate function and the reversed hazard rate function are given, respectively, by $h\left(x \mid \boldsymbol{\theta}\right) = -\frac{d}{dx} \log\left[S(x \mid \boldsymbol{\theta})\right]$ and $\overline{h}\left(x \mid \boldsymbol{\theta}\right) = \frac{d}{dx} \log\left[F(x \mid \boldsymbol{\theta})\right]$, where $S\left(x \mid \boldsymbol{\theta}\right)$ and $F\left(x \mid \boldsymbol{\theta}\right)$ are, respectively, the survival function and the cumulative distribution function.

In the next version of **mle.tools**, we will include, using analytical first and second-order partial derivatives, the following:

- the MLEs of $g\left(\boldsymbol{\theta}\right)$ and $\text{Var}\left[g\left(\boldsymbol{\theta}\right)\right]$,
- the negative log likelihood value $-2\log(L)$,
- the Akaike's information criterion $-2\log(L) + 2p$,
- the corrected Akaike's information criterion $-2\log(L) + \frac{2np}{n-p-1}$,
- the Schwarz's Bayesian information criterion $-2\log(L) + p\log(n)$,
- the Hannan-Quinn information criterion $-2\log(L) + 2\log\log(n)p$,

where L is the value of the likelihood function evaluated at the MLEs, n is the number of observations, and p is the number of estimated parameters.

Also, the next version of the package will incorporate analytical expressions for the distributions studied in Section 2.4 implemented in the supplementary file "analyticalBC.R".

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