P. Saavedra-Nieves
E-mail: paula.saavedra@usc.es
Department of Statistics, Mathematical Analysis and Optimization
Universidade de Santiago de Compostela, Spain

Catherine Hurley
Maynooth University, Ireland
Editor-in-Chief
E-mail: catherine.hurley@mu.ie
The R Journal

Dear Prof. Hurley,

First of all, thank you for your e-mail of June 7, 2022, including the referee report on our paper "HDiR: An R Package for Nonparametric Plug-in Estimation of Directional Highest Density Regions and Level Sets" with ID 2021-135 (coauthored by R. M. Crujeiras). We really appreciate this opportunity to send a revised version of the manuscript, following the reviewer's comments, which have improved considerably the initial version of the paper. A detailed point by point reply to the reviewer's report is given in the following pages. All proposed changes have been incorporated or, at least, discussed and justified in the updated versions of the manuscript and in the implementation of the new version of the package. Since CRAN submissions are offline from July 22 to August 5, updated package `HDiR` will be reuploaded on CRAN as soon as possible. If necessary, new version of `HDiR` could be installed from the local archive HDiR_1.1.2.tar.gz contained in the zip folder sent for revision.

We would like to thank the reviewer for the constructive comments which have considerably improved the article in several aspects, both for the presentation of the theoretical framework, the organization of the manuscript and the package. Mainly, we refer to the suggestion where the reviewer points out the possibility of estimating Highest Density Regions (HDRs) with `HDiR` from any user-defined density estimators (not necessarily the one implemented by default). The updated versions of the

paper and the package manual show an example where this functionality is illustrated. We have also envisioned how `HDiR` could be also used for exact computation and plug-in estimation of directional general level sets generalising the approach in (1). Although the main objective of package `HDiR` remains the computation and plug-in estimation of HDRs and density level sets, the paper has been modified conveniently for accommodating this broader perspective. In fact, the new title of the article tries to reflect this issue: "HDiR: An R Package for Computation and Nonparametric Plug-in Estimation of Directional Highest Density Regions and General Level Sets". So, the focus of this updated version, following the reviewer's suggestion, is more general than the initial submission.

Finally, we would like to take this opportunity to thank again the referee for his/her help with this paper.

Yours faithfully.

Best regards.

P. Saavedra-Nieves.

References

[1] Cuevas, A., González-Manteiga, W., Rodríguez-Casal, A. (2006). Plug-in estimation of general level sets. Australian New Zealand Journal of Statistics, 48(1), 7-19.

Reply to Referee (*RJournal 2021-135*)

We would like to thank the reviewer for the helpful comments provided. Following all the recommendations in the report, we have modified the original versions of the paper and also of the package. Since CRAN submissions are offline from July 22 to August 5, updated package `HDiR` will be reuploaded on CRAN as soon as possible. If necessary, new version of `HDiR` could be installed from the local archive HDiR_1.1.2.tar.gz contained in the zip folder sent for revision.

We are specially grateful for the suggestion where the reviewer points out the possibility of estimating Highest Density Regions (HDRs) with `HDiR` from any user-defined density estimators (not necessarily the one implemented by default). The updated versions of the paper and the package manual show an example where this functionality is illustrated. We have also envisioned how `HDiR` could be also used for exact computation and plug-in estimation of directional general level sets generalising the approach in (1). Although the main objective of package `HDiR` remains the computation and plug-in estimation of HDRs and density level sets, the paper has been modified conveniently for accommodating this broader perspective. In fact, the new title of the article tries to reflect this issue: "HDiR: An R Package for Computation and Nonparametric Plug-in Estimation of Directional Highest Density Regions and General Level Sets". So, the focus of this updated version, following the reviewer's suggestion, is more general than the initial submission.

In what follows, we provide a detailed answer to the issues raised by the reviewer, indicating the changes in the main text. Comments provided by the reviewer are transcribed in italics whereas our answers are written in normal font.

Reviewer's comments on the article.

The general organization of the paper is somewhat weak, and does not help understanding the package potential. I suggest a global revision of the paper accounting for the following guidelines:

- *Methodological framework (Section 1-2).*

    1. *Section 'Overview...' is a little confusing: it mixes literature about level*

set estimation on the Euclidean space, the main contributions of the paper by Saavedra-Nieves and Crujeiras in the directional setting, R implementations in the two settings as well as R routines for directional data. I would separate the framework overview and literature from the related packages. Also, I would choose a focus for the explanation either on the euclidean case or on the directional one, since the continuous moving from one setting to the other makes not clear what the author is referring to.

Our answer: Thanks for your comment. Indeed, the initial distribution of the manuscript may not facilitate the introduction to the topic, being in some parts difficult to follow. We have made an effort in reorganizing the contents (and their presentation) and following your advice, we have modified Section 'Overview...' conveniently. In particular, literature on R packages was moved to Section 'Using `HDiR`'. After evaluation, we have decided not to remove the contributions on Euclidean level set estimation mainly for two reasons: (1) in the reviewer's report, there are also some literature suggestions for Euclidean data that should be included (see next item), and (2) there are very few contributions on this topic in the directional context. As far as we know, this problem is only formally addressed in (3), (2) and (7) mainly because directional level set estimation is an emerging research topic. Specifically, the last two cited articles extend Euclidean methodologies for directional data. Therefore, we think that it is relevant to include the Euclidean contributions on this topic for an easier reading. In any case, we have made an effort in presenting the contributions in a clearer way.

2. *The right credit should be given to some of the mentioned literature:*
   - *the association between t and τ dates back at least to Azzalini and Torelli (2007, Stat. Comp).*

     Our answer: Thanks for noting this. Of course, this reference has been included in the new version.
   - *package* ***`pdfCluster`*** *does not only implements kernel density estimation; while it is true that it focuses on clustering via kernel density estimation, it is in fact much more related to the idea of density level set estimation than it appears, since clusters are identified as the connected components of high density level sets.*

Our answer: Thanks also for noticing this issue. Of course, we were aware of the identification of clusters as connected components of density level sets but we did not mention it in the first version. We completely agree with you and we have incorporated this information in the updated version of the paper.

- *several papers*[1] *and R routines (package* `denpro`*) by Jussi Klemelä are closely related to the topic.*

  Our answer: Note that the previous link does not work. Even so, we have included a brief description on package `denpro` in the new version of the paper that contains a total of six references with contributions related to this library. Hopefully, they coincide with the references contained in the provided link. Please, let us know if more references should be cited, in order to provide right credit to contributors.

- *literature an R routines about modal clustering should be also mentioned, since several ideas are linked with the ones presented in the paper/package. The mean-shift algorithm, for instance, has been developed in the directional setting as well.*

  Our answer: Some literature on modal clustering (including directional mean-shift algorithm) has been incorporated to the Section 'Overview...'. Additionally, packages `LPCM` and `Modalclust` for modal clustering were also cited in Section 'Using `HDiR`' with a brief description.

- *in the Euclidean setting, along with Baíllo and Cuevas (2006) or Samworth and Wand (2010), also the work by Casa et al (2020, El. J. Stat.) is worth to be mentioned for bandwidth selection.*

  Our answer: This reference was also incorporated in the updated version of the manuscript. The inclusion of this work as of the previous references (pointed by the reviewer) justifies that we keep introducing both contributions on Euclidean and directional settings for HDRs and level sets.

3. *Theory underlying the R functions does not need to be explained in detail but the main underlying ideas need to be enlightened to the extent of*

[1] `http://www.jklm.fi/publications.html`

making the user understand if the methods implemented in the package are useful for her/his aims.

Our answer: You are totally right. Following your suggestion, we have reduced theoretical aspects in the updated version of the paper. Specifically, section 'Plug-in estimation methods' is the part of the manuscript that has undergone most changes. In this new version, we have tried to link the theory with the options offered by package `HDiR`. For instance, indicating which kernel density estimator is implemented by default in `HDiR` but noting that other density estimators (defined by the user) could be also used for plug-in methods. Smoothing parameter selection or distances between sets were also related to package `HDiR` in this section. Although more details are not given here, we have made the effort of linking theory and package throughout all the document. We hope that these modifications will facilitate the reading of the paper.

4. *Related to the previous point, section 'Plug-in ...' could be better designed. In my opinion, the issues which arise in the considered framework are: for a given threshold, and even if the true density was known, how to determine the connected components of the density level set? When the density is not know, how to provide a suitable estimate for the specific purpose? How to select the optimal threshold (and optimal for what) and to make inference on it?*

   Our answer: Thanks again for your suggestion. From our view-point, section 'Plug-in ...' must not contain the case when f is known. Under this assumption, the computation of both sets $G_f(t)$ or $L(f_\tau)$ does not involve statistical estimation. However, we have considered the following modifications in the manuscript to make clear the important issue that you mention. First, the abstract contains now the following information: *Package `HDiR` for R is designed for exact computation of directional (circular and spherical) highest density regions and density level sets when the density is fully known. Otherwise, `HDiR` implements nonparametric plug-in methods based on different kernel density estimates for reconstructing this kind of sets.*

   Secondly, we have added the next two paragraphs in the Section 'An overview on directional...' (just after defining the sets $G_f(t)$ and $L(f_\tau)$): *To sum up, given a value of t, the computation of the level set established*

in (1) (and of its connected components) is a quite simple mathematical task when f is known. Under this assumption and taking a fixed $\tau \in (0,1)$, determining the HDR introduced in (2) presents a similar complexity but, in this case, it is additionally necessary to determine the threshold f_τ. In particular, numerical integration methods can be applied to solve that problem. However, when the density f is assumed to be unknown and a random sample $\mathcal{X}_n \in S^{d-1}$ generated from f is the only available information to reconstruct the set, nonparametric set estimation techniques such as plug-in methods must be considered in order to reconstruct the connected components of the set. Perhaps due to its practical importance, Euclidean HDRs plug-in algorithms based on kernel smoothing have been widely studied even solving the problem of selecting an appropriate smoothing parameters specifically devised for the HDR reconstruction (see Baíllo and Cuevas, 2006, Samworth and Wand, 2010 or Casa et al., 2020). In the directional setting, given that a proper definition of the HDR $L(f_\tau)$ was not available, no work on this area had been carried out until the recent contribution by Saavedra-Nieves and Crujeiras (2021b).

The contents of this paper, describing the contributions in HDiR, mainly focus on computation and plug-in estimation of highest density regions (HDRs) and density level sets in the circle and the sphere. Although general level sets can be also analysed using HDiR, we will not formally detail aspects on their computation and on their plug-in reconstruction given that they can be seen as a direct generalisation of those introduced for density level sets by replacing the density by the general function under study. Therefore, with the objective of showing the capabilities of the HDiR package for exact computation of directional HDRs and density level sets when f is known and for plug-in estimation otherwise, this paper is organized as follows. [...].

Moreover, Section 'Threshold estimation of HDRs' is now entitled by 'Threshold estimation and confidence regions for HDRs'. Now, it also presents the contents on confidence regions that were disconnected in the previous version of the paper. To clarify again the difference between the exact computation and the estimation of a density level set, this new section contains the following sentence at the beginning:

For a given $\tau \in (0,1)$, determining the set $L(f_\tau)$ in (2) and its plug-in estimator $\hat{L}(\hat{f}_\tau)$ in (3) involve the exact computation and the estimation of the threshold f_τ, respectively. As in the Euclidean setting, both tasks require the use of numerical integration methods.[...].

Finally, we have also tried to clarify these aspects in the Section 'Using `HDiR`'. Several comments have been added in different parts of this section such as the following one:

Functions `circ.hdr` and `sphere.hdr` must be considered when the objective is to compute theoretical density level sets or HDRs from a fully known circular and spherical density f, respectively.

5. *Subsection 'Exploratory tools' has no much reason to be introduced in the methodological framework and could be better moved in the package description.*

   Our answer: Thanks for your proposal. Subsection 'Exploratory tools' has been moved to Section 'Using `HDiR`' under the title 'Exploring data with `HDiR`'. Additionally, the title of the section 'Plug-in estimation methods and exploratory tools' has been replaced by 'Plug-in estimation methods'.

- *Package description (Section 3)*

  1. *Instead of just listing the functions of the package, the authors could reorganize their presentation/illustration depending on their usage: data, accessory functions to generate data and compute densities on the sphere, exploratory tools, tools for high density regions estimations.*

     Our answer: Thanks for your comment. In the updated version of the article, Section 'Using `HDiR`' has been modified conveniently taking into account that the focus should be the usage of functions. Apart from information on related R packages, it contains the following subsections: 'Data description', 'Spherical density models', 'Computation of HDRs and general level sets with `HDiR`', 'Plug-in estimation of HDRs and general level sets with `HDiR`', 'Exploring data with `HDiR`' and 'Real data analysis with `HDiR`'. As you can check in the part of the letter corresponding to the package, `HDiR` allows now the computation and plug-in estimation of general level sets. Therefore, this task must appear in this section.

  2. *For better understanding how to use the package, and to provide infor-*

mation not easily derived from the help, not only the input, but especially the output of the functions should be (at least partially) printed in the paper, illustrated and discussed. Further arguments may also be illustrated, in order to better understand such output (I refer, for instance, to the `plot.hdr` *argument, which is mentioned for* `circ.plugin.hdr` *but not for the functions earlier described).*

Our answer: Thanks for your suggestion. Outputs of some function executions are shown and briefly explained in the updated version of the manuscript. As for argument `plot.hdr`, it is described when function `circ.hdr` is introduced as follows: *If no graphical representation is required, it is enough to consider plot.hdr=FALSE [...].* Then, the next sentence (or similar) is repeated in several parts of the manuscript: *The other basic arguments [...] and plot.hdr coincide with the usage description for function circ.hdr..* We believe that this argument has now a complete description in all functions that it is involved.

3. *About function* `sphere.hdr`*: when argument* `tau` *is set, the function is much slower than when argument* `level` *is set. I understand that some additional operation are required to be run in the former case, yet It would be helpful to highlight this aspect for the benefit of the user.*

   Our answer: Thanks for your suggestion. We have included the following clarification in the new version of the paper: *Moreover, it is worth to mention that execution time of* `sphere.hdr` *is considerably higher when* `tau` *is set instead level because, in this case, threshold estimation via numerical integration methods is required.*

4. *In general package illustration is rather long (and in same cases confused) with respect to the limited number of functions provided by the package. A more focused and compact illustration could be useful. Real data and simulated examples are mixed and the reader cannot fully appreciate what additional information is provided at each new example. In my opinion, the paper would benefit from a first illustration of all the functions with simple (simulated?) data, and a more comprehensive analysis with one of the real dataset afterwards.*

   Our answer: Thanks for your proposal. In the updated version of the article, we have illustrated first the usage of all functions only with simulated data. Moreover, we have added a new subsection a the end of

Section 'Using HDiR' where functions are also illustrated through the real datasets contained in HDiR. We think that it is important to include both datasets because one of them is distributed on the circle and the other, on the sphere.

Referee's comments on the package.

1. *The package imports a chain of other packages which are of very slow installation. Check if all the imports are needed or maybe some functions may be simply borrowed from other packages with proper citation (I guess with* `importFrom`*) – I'm not sure this is allowed in fact, please check it. This comment holds in general, but to install* `HDiR`*, I had especially troubles with installation of package* `Rfast2` *required to install* `Directional`*. There is a code error in the routine to load package* `Rfast2`*, hence I had to download and correct the source file, then build the package again and install it. I know that this is not a fault of the authors of the paper under review, yet due the dependency of* `HDiR` *on package* `Directional`*, the usability of* `HDiR` *itself may be compromised.*

   Our answer: Thanks for your comments. Following your advice, we have reduced the installation time of HDiR package by using the option importFrom instead of import. As for the trouble with package Rfast2, we had not detected it maybe because we use the R version 4.2.1. We have written to the authors and maintainers of the Directional and Rfast2 packages to analyse the situation. They have indicated that if the latest versions of these two packages (and their dependencies) are installed, there should be no problem. We have checked it and it works correctly. These kind of issues are frequent but they should not compromise at all the usability of HDiR. Please, let us know if you can not solve it when latest versions are installed.

2. *A main concern about the package is that it looks as unseparable from the paper of Saavedra-Nieves and Crujeiras. While the ideas proposed in that papers deserve some tools for implementation, I think that with just a slight effort of abstraction the package could be largely wider: it appears to me that high density regions can be possibly estimated with other, possibly user-defined density estimators than the one considered in the package.*

   Our answer: Thanks for this critical and completely right comment. Imple-

mentation of functions `circ.hdr` and `sphere.hdr` has been modified in the new version of the package `HDiR`. Now, they allow to compute level sets of general real-valued functions (not necessarily a density) defined on the unit circle or on the unit sphere, respectively. Following (3), the plug-in estimation of level sets for general functions (such as regression ones) is also an interesting issue that, as far as we know, has not yet been addressed in the directional literature. A new example for function `circ.hdr` has been added to the reference manual of `HDiR` in order to illustrate the level set plug-in reconstruction of Nadaraya-Watson regression estimator with a circular explanatory variable and linear response. Of course, as the referee suggest, argument `f` could be also a different density estimator than the one implemented in `HDiR` that only considers von Mises Misher kernel. Another example for function `sphere.hdr` has been included in the manual of our package in order to show this possibility. Specifically, `sphere.hdr` is used in this example for plug-in reconstruction of a density level set from a kernel density estimator with uniform kernel implemented in `DirStats`. Therefore, the combination of `HDiR` with functions available in other R packages increases considerably the functionality of our package. Besides performing plug-in estimation of level sets and HDRs for alternative density estimators already implemented in R code, `circ.hdr` and `sphere.hdr` also allow to solve this task from density estimators defined directly by the user (for instance, (6) that is not implemented yet in R). We have remarked in the reference manual of `HDiR` that argument `tau` must be specified only when `f` is a density. Otherwise, establishing the probability content of the level set of an arbitrary function make no sense. The new version of the paper has been also updated in order to reflect this more general perspective.

3. *Beside focusing on estimation of directional highest density regions,* ***`HDiR`*** *extends to the sphere some features of package* ***`NPCirc`****, such as functions* ***`dspheremix`*** *and* ***`rspheremix`****. In fact, I do not see very much the point of these functions, aimed at generating data and computing density of some specific models. I see they can be useful for the package illustration, but in my opinion do not enhance the package usability. If the authors see specific reasons to keep such routines in the package, at least they should be described a part from the illustration of the main features of the package (perhaps beforehand, similarly to the data description? See also comment 1 in 'Package Description').*

Our answer: Thanks for your comment. The next sentence was added to the new version of the paper "*A complete description of the* `HDiR` *package capabilities is provided in this section. The complete list of functions, illustrative density models (density functions and random sample generation) and the two novel datasets available in HDiR, with a brief description, can be seen in Table 1.*". Moreover, subsection 'Spherical density models' in Section 'Using `HDiR`' contains a brief description on their usage. Their main purpose is illustration due to they are densities representing complex structures of multimodality and/or asymetry. Therefore, we believe that it is interesting that they are also available (for simulation or even for illustration) for any package user.

4. *Related to the previous comment, yet more in general, I have some concerns about the fact that for some aspects, the package looks like an extension of package* `NPCirc`*. I guess that this comment is beyond the scope of my reviewing, which should focus on the paper more than the package; yet if the authors prefer not to consider to merge the two packages (which, in my opinion, would be helpful for the final user), they should at least highlight in the paper that package* `HDiR` *represents a sort of appendix of* `NPCirc`*.*

   Our answer: Thanks for your comment. Unfortunately, we disagree with this suggestion. `NPCirc` is focused on nonparametric smoothing methods for density and regression estimation only involving circular data. Actually, the `NPCirc` package has been recently updated (and more updates are in progress) in the context of (nonparametric) circular regression (the co-author of this paper is also one of the authors of NPCirc). The plan is that this will be the focus of NPCirc. On the contrary, `HDiR` solves the specific problem of estimating general level sets and HDRs in the spherical setting, not only for circular variables. In fact, spherical kernel density estimation (not available in `NPCirc`) is imported from `Directional` package. Moreover, functionalities in `HDiR` could be extended in future for data lying on a torus (see (4)) or even in more general manifolds (see (6)). Density-based clustering tools in (1) and (5) for euclidean data could be also naturally adapted to the directional setting in `HDiR`. Therefore, we think that `HDiR` has weight/entity enough not to merge with either `NPCirc` or `Directional`. From our point of view, a merger could result in HDiR's functionalities being hidden for the final user.

5. *Function* `circ.hdr` *and* `sphere.hdr`*:*

- *(related to point 2) Description of functions is somewhat misleading; about* `circ.hdr` *it says that the "function must be considered when the objective is to compute theoretical level sets or HDRs from a fully known circular density f", and a similar explanation is left to be intended for function* `sphere.hdr`*. As far as I understand, argument f is not necessarily the true density function, and the output are not necessarily the theoretical level sets. Conversely, the user might provide here any function defined on the circle, possibly even estimated. I would suggest to highlight this potential, and focus more on the format of the wrapper function required in input.*

  Our answer: Thanks for your interesting comment. As we mention previously, implementation of functions `circ.hdr` and `sphere.hdr` has been modified in the new version of the package `HDiR`. Now, they allow to compute level sets of general real-valued functions (not necessarily a density) defined on the unit circle or on the unit sphere, respectively. Following (3), the plug-in estimation of level sets for general functions (such as regression ones) is also an interesting issue that, as far as we know, has not yet been addressed in the directional literature. A new example for function `circ.hdr` has been added to the reference manual of `HDiR` in order to illustrate the level set plug-in reconstruction of Nadaraya-Watson regression estimator with a circular explanatory variable and linear response. Of course, as the referee suggest, argument `f` could be also a different density estimator than the one implemented in `HDiR` that only considers von Mises-Fisher kernel. Another example for function `sphere.hdr` has been included in the manual of our package in order to show this possibility. Specifically, `sphere.hdr` is used in this example for plug-in reconstruction of a density level set from a kernel density estimator with uniform kernel implemented in `DirStats`. Therefore, the combination of `HDiR` with functions available in other R packages increases considerably the functionality of our package. Apart from performing plug-in estimation of level sets and HDRs for alternative density estimators already implemented in R code, `circ.hdr` and `sphere.hdr` also allow to consider density estimators defined directly by the user (for instance, the proposal presented in (6) that, as far as we know, is not implemented yet in R). We have remarked in the reference manual of `HDiR` that argument `tau`

must be specified only when `f` is a density. Otherwise, establishing the probability content of the level set of an arbitrary function make no sense. The new version of the paper has been also updated in order to reflect this more general perspective. Finally, it is worth to mention that the reference manual of the package has been also updated. More details on the format of the wrapper function `f` required as an input in `circ.hdr` and `sphere.hdr`.

- *Is there reason for providing the* `hdr` *output of function* `circ.hdr` *as a vector? I would find more natural to use a matrix with different rows for different boundaries of the connected components*

  Our answer: Thanks for your comment. Following your advice, we have modified the output of this function.

- *Since* `tau` *and* `level` *are in one-to-one correspondence, it would be useful that, as when* `tau` *is provided the associated* `level` *is returned, if* `level` *is provided, the associated* `tau` *could be also returned. In general, disregarding the input, I would provide the very same output. Also, i would rename the output* `threshold` *in* `level`*, to avoid misunderstanding.*

  Our answer: Thanks for your right indication. We have modified the functions `circ.plugin.hdr` and `sphere.plugin.hdr` following your comment because they also present the pointed issue. Now, if `tau` is provided, the associated `level` is returned and vice versa. However, this change was not incorporated for functions `circ.hdr` and `sphere.hdr`. After updates, level sets of general functions `f` (not necessarily densities) can be computed from `circ.hdr` and `sphere.hdr`. Then, we have remarked in the reference manual that argument `tau` must be specified only when `f` is a density. Otherwise, establishing the probability content of the level set makes no sense. Therefore, when argument `level` is specified by the user as an input, `f` it may not be a density function and only `level` is shown as an output. If `tau` is an input (therefore, `f` is a density), `level` is also returned. The new version of the paper has been also modified according to these changes. Finally, we have renamed the output `threshold` of the four functions (`circ.hdr`, `sphere.hdr`, `circ.plugin.hdr` and `sphere.plugin.hdr`) by `level` such as you suggest.

6. *Function* `circ.plugin.hdr` *and* `sphere.plugin.hdr`

- *The simulated example produces a warning, which occurs with all the seeds I have tried. Since this looks so common, with the considered density, a discussion about it would be helpful*

  Our answer: We have detected just a warning message that occurs for the simulated example with function `circ.plugin.hdr`:
  `Warning message:`
  `In bw.CV(circular(sample)): minimum/maximum occurred at one end of the range`
  This warning is produced by function `bw.CV()` in `NPCirc` that computes the smoothing parameter by default when `circ.plugin.hdr` is executed. This function obtains the smoothing parameter as the value that maximizes the logarithm of the likelihood cross-validation function. The message indicates that optimal value is reached at the lower or upper boundary of the interval considered for optimization. To avoid this message, we have modified directly this upper limit in `circ.plugin.hdr` and `circ.scatterplot`. Remark that we have increased the default value (50) of the argument `upper` in `bw.CV()` in order to increase the length of the interval for optimization and to eliminate the warning.

- *Similar to function* `circ.hdr`*: is there reason for providing the output of function* `circ.plugin.hdr` *as a vector? I would find more natural to use a matrix with different rows for different boundaries of the connected components*

  Our answer: Thanks for your comment. Following your advice, we have also modified the output of this function.

- *Similar to function* `circ.hdr`*: since* `tau` *and* `level` *are in one-to-one correspondence, it would be useful that, as when* `level` *is provided the associated level is returned, if* `level` *is provided, the associated* `level` *could be also returned. In general, disregarding the input, I would provide the very same output. Also, i would rename the output* `threshold` *in* `level`*, to avoid misunderstanding.*

  Our answer: Thanks for your right indication. As we mention before, we have modified the functions `circ.plugin.hdr` and `sphere.plugin.hdr`

following your comment because they also present the pointed issue. Now, if `tau` is provided the associated `level` is returned and vice versa. The new version of the paper has been also modified according to this changes. Finally, we have also renamed the output `threshold` of both functions by `level` such as you suggest.

- *The opportunity, allowed by the package, of user-defined bandwidth selection should be highlighted.*

  Our answer: Thanks for your suggestion. This possibility is highlighted in the updated version of the paper and of the package.

Minor remarks

1. *The concept that large values of t are are associated with with the greatest modes and small values with the whole support is illustrated twice in section 'Overview'*

   Our answer: This error has been removed.

2. *p. 2 'Unfortunately, despite a practitioner may be interested in determining this type of regions, the specific value of the level t in (1) is usually ignored for most applications'. It is not clear what specific value is ignored. If that value is the one which identifies all the high-density regions, it does not necessarily exist; also if that is the case, I would also mention that tools such the cluster tree allows for identifying all such regions by moving t (or τ).*

   Our answer: Thanks for your comment. Cluster trees have been mentioned in the new version through the references (1) and (5). Furthermore, we have rewritten the paragraph you pointed out in order to emphasise that selecting τ (or the empirical probability content, $1-\tau$, of the HDR) instead t is simpler and useful for practical purposes. Note that the value of t is (in principle) unknown if the level set $\{f \geq t\}$ must satisfy a fixed probability content by the user.

3. *p.2 'this drawback is solved by Saavedra-Nieves and Crujeiras': which drawback? The one of previous point? How?*

   Our answer: Thanks for your comment. This part of the manuscript has been modified.

4. *A whole paragraph in Section Data Description (p.8) is repeated.*

   Our answer: Thanks for your annotation. This part of the manuscript has been also modified.

5. *p.4 formula (4) is just one example of kernel density estimator for directional data; other densities than the von Mises Misher make sense to be selected as possible kernels for directional data.*

   Our answer: Thanks for your suggestion. We have included this comment in the updated version of the paper. Furthermore, a new example for function `sphere.hdr` has been also added to the reference manual of `HDiR` in order to show this possibility. As we mentioned before, `sphere.hdr` is used in this example for plug-in reconstruction of a density level set from a kernel density estimator with uniform kernel implemented in `DirStats` package. Besides performing plug-in estimation of level sets and HDRs for alternative density estimators implemented in other R packages, `circ.hdr` and `sphere.hdr` also allow to consider density estimators defined directly by the user (for instance, the proposal presented in (6) that, as far as we know, is not implemented yet in R).

6. *p.10 What is argument* ***k*** *in* ***circ.plugin.hdr****?*

   Our answer: Thanks for your comment. Argument `k` is a graphical parameter that only takes positive integer values. It controls the distance between the unit circle and the HDR confidence region in the output plot. Confidence regions will be plotted nearer the unit circle for large values of `k` and farther away for small values of `k`.

7. *Tables and Figures should be placed either at the top or at the bottom of page.*

   Our answer: Thanks for your comment. We have incorporated these changes.

References

[1] Azzalini, A., Torelli, N. (2007). Clustering via nonparametric density estimation. Statistics and Computing, 17(1), 71-80.

[2] Cholaquidis, A., Fraiman, R., Moreno, L. (2022). Level set and density estimation on manifolds. Journal of Multivariate Analysis, 189, 104925.

[3] Cuevas, A., González-Manteiga, W., Rodríguez-Casal, A. (2006). Plug-in estimation of general level sets. Australian New Zealand Journal of Statistics, 48(1), 7-19.

[4] Di Marzio, M., Panzera, A., Taylor, C. C. (2011). Kernel density estimation on the torus. Journal of Statistical Planning and Inference, 141(6), 2156-2173.

[5] Menardi, G., Azzalini, A. (2014). An advancement in clustering via nonparametric density estimation. Statistics and Computing, 24(5), 753-767.

[6] Pelletier, B. (2005). Kernel density estimation on Riemannian manifolds. Statistics probability letters, 73(3), 297-304.

[7] Saavedra-Nieves, P., Crujeiras, R. M. (2021). Nonparametric estimation of directional highest density regions. Advances in Data Analysis and Classification, 1-36.