

# Two-Tier Latent Class IRT Models in R

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**Abstract** In analyzing data deriving from the administration of a questionnaire to a group of individuals, Item Response Theory (IRT) models provide a flexible framework to account for several aspects involved in the response process, such as the existence of multiple latent traits. In this paper, we focus on a class of semi-parametric multidimensional IRT models, in which these traits are represented through one or more discrete latent variables; these models allow us to cluster individuals into homogeneous latent classes and, at the same time, to properly study item characteristics. In particular, we follow a within-item multidimensional formulation similar to that adopted in the two-tier models, with each item measuring one or two latent traits. The proposed class of models may be estimated through the package **MLCIRTwithin**, whose functioning is illustrated in this paper with examples based on data about quality-of-life measurement and about the propensity to commit a crime.

## Introduction

Several fields of human knowledge require the measurement of unobservable constructs (or latent traits) through ad hoc methods based on questionnaires consisting of multiple items having dichotomously or ordered polytomously scored response categories. This is the case of measurement of customer satisfaction, quality-of-life, level of physical and/or psychological disabilities, ability in certain subjects, and so on.

Item Response Theory (IRT) models (Hambleton and Swaminathan, 1985; Van der Linden and Hambleton, 1997; Bartolucci et al., 2015) are well-known statistical models to deal with these data. In their original formulation, these models are characterized by: (i) unidimensionality (i.e., only one latent trait is assumed to be measured by all items); (ii) a parametric (usually normal) distribution for the latent variables used to represent the trait of interest; and (iii) no effect of individual covariates on this latent trait. These elements often turn out to be restrictive in modern applications and, therefore, several extensions of IRT models have been proposed in the literature. Among the possible extensions, in this paper we consider the class of *multidimensional Latent Class (LC) IRT models* proposed by Bartolucci (2007) and von Davier (2008); see also Bacci et al. (2014). Models of this type are characterized by: (i) *multidimensionality*, in the sense that more latent traits may be measured by the set of items (Reckase, 2009); (ii) discreteness of the latent variables, so that homogeneous subpopulations (or latent classes; Lazarsfeld and Henry, 1968; Goodman, 1974) of individuals are detected with respect to the constructs measured by the questionnaire; and (iii) possible presence of individual covariates affecting the probabilities to belong to each latent class.

In particular, we focus on a specific extension of IRT models based on *within-item multidimensionality* (Adams et al., 1997), which is characterized by items affected by more than one latent variable. This is opposed to the more common *between-item multidimensionality*, where each item may measure only one latent variable as in the original approach of Bartolucci (2007). More in detail, the model here proposed represents a discrete version of the *item bifactor model* and of the more general *two-tier IRT model* (Bock et al., 1988; Gibbons and Hedeker, 1992; Gibbons et al., 2007; Cai, 2010; Cai et al., 2011; Reise, 2012; Bonifay, 2015), based on a particular within-item multidimensional formulation with each item loading on at most two latent variables that are mutually uncorrelated. With respect to traditional item bifactor and two-tier models, which assume the normality of the latent variables, the discreteness assumption increases the flexibility of the approach and allows us to cluster individuals in homogeneous latent classes. Formann and Kohlmann (2002) propose a general approach based on latent classes that includes the discrete two-tier model here proposed as special case. However, different from the proposal of these authors, we let the class membership probability depend on individual covariates and we also allow for more flexibility in terms of specification of model link function. Limited to binary items, a recent example of application of the proposed two-tier LC-IRT model is provided in Bacci and Bartolucci (2015) to jointly study certain students' abilities and the propensity to skipping item responses.

The procedures to estimate the proposed class of two-tier LC-IRT models are implemented in the R package **MLCIRTwithin** (Bartolucci and Bacci, 2016), downloadable from <http://CRAN.R-project.org/package=MLCIRTwithin>, whose illustration is the primary focus of the present paper. In particular, we are interested in providing a detailed description of the main functions of this package, named `est_multi_poly_within` and `search.model_within`, also through some applications.

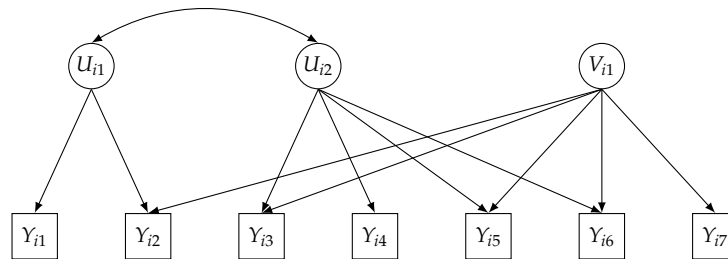
The remainder of the paper is organized as follows. In the next section we provide the formulation of the proposed class of two-tier LC-IRT models and, then, some details about likelihood inference for these models. Furthermore, we describe the main functions implemented in the R package **MLCIRTwithin** for model estimation. In the following, the functioning of the package is illustrated

through two applications: the first one concerns the measurement of Health-related Quality Of Life (HQOL) on cancer patients and the second one is about the measurement of propensity to commit crimes. Some final remarks conclude the work.

## The class of models

The proposed class of models is formulated on the basis of two independent vectors of latent variables representing the unobservable individual characteristics measured by the test items. For each unit  $i = 1, \dots, n$ , these vectors are denoted by  $\mathbf{U}_i = (U_{i1}, \dots, U_{iD_1})'$  and  $\mathbf{V}_i = (V_{i1}, \dots, V_{iD_2})'$  and are of dimension  $D_1$  and  $D_2$ , respectively. Similarly to the item bifactor model, we assume that each item response  $Y_{ij}$ , with  $i = 1, \dots, n$  and  $j = 1, \dots, r$ , where  $r$  is the number of items, may depend on (and then measures) at most two latent variables, under the constraint that these two variables do not belong to the same vector. This is formalized by introducing the disjoint subsets  $\mathcal{U}_1, \dots, \mathcal{U}_{D_1}$  and  $\mathcal{V}_1, \dots, \mathcal{V}_{D_2}$  of  $\mathcal{J} = \{1, \dots, r\}$ , where  $\mathcal{U}_{d_1}$  contains the indices of the items depending on latent variable  $U_{id_1}$  and  $\mathcal{V}_{d_2}$  is the set of those depending on latent variable  $V_{id_2}$ . Equivalently,  $Y_{ij}$  depends on  $U_{id_1}$  if and only if  $j \in \mathcal{U}_{d_1}$  and on  $V_{id_2}$  if and only if  $j \in \mathcal{V}_{d_2}$ . Note that, even if the subsets  $\mathcal{U}_1, \dots, \mathcal{U}_{D_1}$  cannot overlap, and the same is assumed for  $\mathcal{V}_1, \dots, \mathcal{V}_{D_2}$ , the same item  $j$  may belong both to a set of the first type and to a set of the second type (within-item multidimensionality); more formally, there may exist  $d_1$  and  $d_2$  such that  $j \in \mathcal{U}_{d_1}$  and  $j \in \mathcal{V}_{d_2}$ . In practice, some items belonging to  $\mathcal{U}_{d_1}$ ,  $d_1 = 1, \dots, D_1$ , will be present also in  $\mathcal{V}_{d_2}$ ,  $d_2 = 1, \dots, D_2$ . With respect to the specification commonly encountered in the literature on item bifactor and two-tier models, our proposal is more general, as any value of  $D_1$  and  $D_2$  is allowed, whereas  $D_1 = 1$  (or, alternatively,  $D_2 = 1$ ) in the item bifactor model and  $D_1 = 2$  (or, alternatively,  $D_2 = 2$ ) in the two-tier model. Moreover, components of  $\mathbf{U}_i$  are allowed to be correlated; the same holds for components of  $\mathbf{V}_i$ .

An illustrative example of the above assumptions is provided in Figure 1, where  $D_1 = 2$ ,  $D_2 = 1$ , and four items out of  $r = 7$  measure two latent traits (item 2 measures dimensions  $U_{i1}$  and  $V_{i1}$ ; items 3, 5, and 6 measure dimensions  $U_{i2}$  and  $V_{i1}$ ); the two dimensions  $U_{i1}$  and  $U_{i2}$  do not share any item.



**Figure 1:** Path diagram of the proposed two-tier model for two latent vectors with two and one dimension, respectively, and seven items ( $\mathcal{U}_1 = \{1, 2\}$ ,  $\mathcal{U}_2 = \{3, 4, 5, 6\}$ ,  $\mathcal{V}_1 = \{2, 3, 5, 6, 7\}$ ).

Adopting a semi-parametric approach for the latent distribution, the first latent vector  $\mathbf{U}_i$  is assumed to have a discrete distribution based on  $k_1$  support points  $\mathbf{u}_1, \dots, \mathbf{u}_{k_1}$  and, in absence of individual covariates, common mass probabilities  $\lambda_1, \dots, \lambda_{k_1}$ . Similarly, the distribution of the second latent vector  $\mathbf{V}_i$  has  $k_2$  support points  $\mathbf{v}_1, \dots, \mathbf{v}_{k_2}$  and, again in absence of individual covariates, common mass probabilities  $\pi_1, \dots, \pi_{k_2}$ . In both cases, the support points identify classes of individuals that are homogeneous with respect to the latent traits represented by  $\mathbf{U}_i$  and  $\mathbf{V}_i$ . Note that cases with  $k_1 = 1$  or  $k_2 = 1$  detect a special situation in which vector of latent variables  $\mathbf{U}_i$  or  $\mathbf{V}_i$ , respectively, has no role in explaining the observed item responses.

For binary response variables, the *measurement model* assumes that, for  $i = 1, \dots, n$ ,  $j = 1, \dots, r$ ,  $h_1 = 1, \dots, k_1$ , and  $h_2 = 1, \dots, k_2$ ,

$$\text{logit } p(Y_{ij} = 1 | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2}) = \gamma_{1j} \sum_{d_1=1}^{D_1} 1\{j \in \mathcal{U}_{d_1}\} u_{h_1 d_1} + \gamma_{2j} \sum_{d_2=1}^{D_2} 1\{j \in \mathcal{V}_{d_2}\} v_{h_2 d_2} - \beta_j, \quad (1)$$

where  $1\{\cdot\}$  is the indicator function and  $\gamma_{1j}$ ,  $\gamma_{2j}$ , and  $\beta_j$  are suitable item parameters. As usual for IRT models,  $\gamma_{1j}$  and  $\gamma_{2j}$  represent the discrimination power of item  $j$  with respect to the latent variables in  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively, whereas  $\beta_j$  denotes the difficulty level of item  $j$ . In the previous expression,  $u_{h_1 d_1}$  denotes the  $d_1$ -th element of  $\mathbf{u}_{h_1}$ , whereas  $v_{h_2 d_2}$  denotes the  $d_2$ -th element of  $\mathbf{v}_{h_2}$ .

Different from traditional LC models characterized by constant mass probabilities, a more general approach is based on assuming that the probabilities to belong to every latent class defined by the

distribution of  $\mathbf{U}_i$  and  $\mathbf{V}_i$  depend on individual covariates, when such covariates are observed. For this aim, we denote the vector of covariates for individual  $i = 1, \dots, n$  by  $\mathbf{X}_i$  and we assume that  $\mathbf{U}_i$  and  $\mathbf{V}_i$  are conditionally independent given  $\mathbf{X}_i$ . Moreover, we adopt the following multinomial logit parametrization (Formann, 2007) for each latent vector:

$$\log \frac{\lambda_{h_1}(\mathbf{x}_i)}{\lambda_1(\mathbf{x}_i)} = \mathbf{x}_i' \delta_{1h_1}, \quad h_1 = 2, \dots, k_1, \quad (2)$$

$$\log \frac{\pi_{h_2}(\mathbf{x}_i)}{\pi_1(\mathbf{x}_i)} = \mathbf{x}_i' \delta_{2h_2}, \quad h_2 = 2, \dots, k_2, \quad (3)$$

with  $\lambda_{h_1}(\mathbf{x}_i) = p(\mathbf{U}_i = \mathbf{u}_{h_1} | \mathbf{X}_i = \mathbf{x}_i)$  and  $\pi_{h_2}(\mathbf{x}_i) = p(\mathbf{V}_i = \mathbf{v}_{h_2} | \mathbf{X}_i = \mathbf{x}_i)$ , where  $\mathbf{x}_i$  contains the constant term. The vectors of coefficients  $\delta_{1h_1}$  and  $\delta_{2h_2}$  measure the effect of the covariates on the logit to belong to class  $h_1 = 2, \dots, k_1$  and  $h_2 = 2, \dots, k_2$ , with respect to class  $h_1 = 1$  and  $h_2 = 1$ , respectively. Alternatively, a global logit formulation may be adopted. This is related to a cumulative logit formulation (Agresti, 2013), where the logits in equations (2) and (3) are substituted with

$$\log \frac{p(\mathbf{U}_i \geq \mathbf{u}_{h_1} | \mathbf{X}_i = \mathbf{x}_i)}{p(\mathbf{U}_i < \mathbf{u}_{h_1} | \mathbf{X}_i = \mathbf{x}_i)} = \log \frac{\lambda_{h_1}(\mathbf{x}_i) + \dots + \lambda_{k_1}(\mathbf{x}_i)}{\lambda_1(\mathbf{x}_i) + \dots + \lambda_{h_1-1}(\mathbf{x}_i)}, \quad h_1 = 2, \dots, k_1$$

and

$$\log \frac{p(\mathbf{V}_i \geq \mathbf{v}_{h_2} | \mathbf{X}_i = \mathbf{x}_i)}{p(\mathbf{V}_i < \mathbf{v}_{h_2} | \mathbf{X}_i = \mathbf{x}_i)} = \log \frac{\pi_{h_2}(\mathbf{x}_i) + \dots + \pi_{k_2}(\mathbf{x}_i)}{\pi_1(\mathbf{x}_i) + \dots + \pi_{h_2-1}(\mathbf{x}_i)}, \quad h_2 = 2, \dots, k_2,$$

respectively. The main advantage of the global logit parametrization is the easier interpretation of the regression coefficients that now refer to the effect of the covariates on the logit to belong to a specific class (or higher) with respect to a lower class. However, this parameterization requires the latent classes to be ordered according to a specific criterion (e.g., requiring an increasing trend of the support points for a given dimension).

In the case of polytomously scored items with ordered categories indexed from 0 to  $l_j - 1$ , the model based on Equation (1) may be extended according to a global logit link function, so that a graded response model (Samejima, 1969) results in:

$$\log \frac{p(Y_{ij} \geq y | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2})}{p(Y_{ij} < y | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2})} = \gamma_{1j} \sum_{d_1=1}^{D_1} 1\{j \in \mathcal{U}_{d_1}\} u_{h_1 d_1} + \gamma_{2j} \sum_{d_2=1}^{D_2} 1\{j \in \mathcal{V}_{d_2}\} v_{h_2 d_2} - \beta_{jy}. \quad (4)$$

Alternatively, using a local logit link function, we may assume a partial credit model (Masters, 1982):

$$\log \frac{p(Y_{ij} = y | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2})}{p(Y_{ij} = y - 1 | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2})} = \gamma_{1j} \sum_{d_1=1}^{D_1} 1\{j \in \mathcal{U}_{d_1}\} u_{h_1 d_1} + \gamma_{2j} \sum_{d_2=1}^{D_2} 1\{j \in \mathcal{V}_{d_2}\} v_{h_2 d_2} - \beta_{jy}. \quad (5)$$

In the above expressions,  $y = 1, \dots, l_j - 1$  and the difficulty parameter  $\beta_{jy}$  is now specific of item  $j$  and response category  $y$ . A more parsimonious model is obtained by expressing  $\beta_{jy}$  as the sum of two components (rating scale parametrization; Andrich, 1978), that is,

$$\beta_{jy} = \beta_j + \tau_y, \quad j = 1, \dots, r; y = 1, \dots, l_j - 1, \quad (6)$$

so that the distance in terms of difficulty from category to category (i.e.,  $\tau_y$ ) is the same for all items. Note that the rating scale parametrization is allowed only when items have the same number of response categories (i.e.,  $l_j = l$ ,  $j = 1, \dots, r$ ). For more details about the possible item parametrizations in the presence of ordinal items see Bacci et al. (2014) and Bartolucci et al. (2015).

In order to ensure the identification of the proposed class of models, two necessary conditions must hold. First, as usual in the IRT modeling, we must constrain one discriminant index to be equal to 1 and one difficulty parameter to be equal to 0 for each dimension. More in detail, let  $j_{d_1}$  be a specific element of  $\mathcal{U}_{d_1}$  and  $j_{d_2}$  a specific element of  $\mathcal{V}_{d_2}$  for  $d_1 = 1, \dots, D_1$  and  $d_2 = 1, \dots, D_2$ . Then we assume  $\gamma_{1j_{d_1}} = 1$ ,  $\gamma_{2j_{d_2}} = 1$ , and, when item difficulties are free,  $\beta_{j_{d_1}1} = 0$  and  $\beta_{j_{d_2}1} = 0$ , whereas in the presence of a rating scale parametrization we assume  $\beta_{j_{d_1}} = 0$ ,  $\beta_{j_{d_2}} = 0$ , and  $\tau_1 = 0$ . In the case of binary items, constraints on difficulties simplify to  $\beta_{j_{d_1}} = 0$  and  $\beta_{j_{d_2}} = 0$ . Generally speaking,  $j_{d_1}$  and  $j_{d_2}$  may be chosen in an arbitrary way, paying attention to select a different item for each dimension. So, in the example illustrated in Figure 1, if we constrain item  $j = 1$  for dimension  $\mathcal{U}_{11}$  and item  $j = 3$  for dimension  $\mathcal{U}_{12}$ , then for dimension  $\mathcal{V}_{11}$  we may constrain any one of the items in the subset  $\mathcal{V}_1$  with the only exception of item  $j = 3$ . As an alternative to constraining item parameters, we may fix the support points, as in the general diagnostic model of von Davier (2008).

A further identification condition requires that at least one item belongs to one of the subsets  $\mathcal{U}_{d_1}$  or

to one of the subsets  $\mathcal{V}_{d_2}$ ; more formally, the union of  $\mathcal{U}_1, \dots, \mathcal{U}_{D_1}$  must be different from the union of  $\mathcal{V}_1, \dots, \mathcal{V}_{D_2}$ . In other words, we restrict  $\gamma_{1j} = 0$  or  $\gamma_{2j} = 0$  for at least one  $j$  and the maximum number of items shared by  $\mathbf{U}_i$  and  $\mathbf{V}_i$  is equal to  $r - 1$ . Alternatively, we may skip this restrictive condition by specifying in a suitable way linear constraints (e.g., equality restrictions) on some discriminant parameters (for some examples see Cai, 2010; Cai et al., 2011).

To specify in a flexible way constraints on the support points and item parameters, we denote the complete vectors of support points by  $\mathbf{u} = (u_{11}, u_{12}, \dots, u_{k_1 D_1})'$  for latent variable  $\mathbf{U}_i$  and  $\mathbf{v} = (v_{11}, v_{12}, \dots, v_{k_2 D_2})'$  for latent variable  $\mathbf{V}_i$ , the complete vectors of item discriminating indices as  $\gamma_1 = (\gamma_{11}, \dots, \gamma_{1r})'$  for items affected by  $\mathbf{U}_i$  (then  $\gamma_{1j}$  is missing if item  $j$  does not belong to  $\mathcal{U}_1, \dots, \mathcal{U}_{D_1}$ ) and  $\gamma_2 = (\gamma_{21}, \dots, \gamma_{2r})'$  for items affected by  $\mathbf{V}_i$  (then  $\gamma_{2j}$  is missing if item  $j$  does not belong to  $\mathcal{V}_1, \dots, \mathcal{V}_{D_2}$ ), and the complete vector of item difficulties as  $\beta = (\beta_{11}, \dots, \beta_{r, l_r-1})'$  (or  $\beta = (\beta_1, \dots, \beta_r)'$  in the binary case). The corresponding vectors of free support points and free item parameters are denoted by  $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \tilde{\gamma}_1, \tilde{\gamma}_2$ , and  $\tilde{\beta}$ , respectively. A wide range of linear constraints and fixed values of the parameters are specified through a suitable definition of matrices  $\mathbf{Z}_u, \mathbf{Z}_v, \mathbf{Z}_{\gamma_1}, \mathbf{Z}_{\gamma_2}$ , and  $\mathbf{Z}_\beta$  and vectors  $\mathbf{z}_u, \mathbf{z}_v, \mathbf{z}_{\gamma_1}, \mathbf{z}_{\gamma_2}$ , and  $\mathbf{z}_\beta$ , as follows:

$$\mathbf{u} = \mathbf{Z}_u \tilde{\mathbf{u}} + \mathbf{z}_u, \quad (7)$$

$$\mathbf{v} = \mathbf{Z}_v \tilde{\mathbf{v}} + \mathbf{z}_v, \quad (8)$$

$$\gamma_1 = \mathbf{Z}_{\gamma_1} \tilde{\gamma}_1 + \mathbf{z}_{\gamma_1}, \quad (9)$$

$$\gamma_2 = \mathbf{Z}_{\gamma_2} \tilde{\gamma}_2 + \mathbf{z}_{\gamma_2}, \quad (10)$$

$$\beta = \mathbf{Z}_\beta \tilde{\beta} + \mathbf{z}_\beta. \quad (11)$$

For instance, according to the usual IRT parametrization with free support points and constraints on the item parameters (i.e., one discriminant index equal to 1 and one difficulty parameter equal to 0 for each dimension),  $\mathbf{Z}_u$  and  $\mathbf{Z}_v$  are identity matrices of dimensions  $k_1 D_1 \times k_1 D_1$  and  $k_2 D_2 \times k_2 D_2$ , respectively, and  $\mathbf{z}_u$  and  $\mathbf{z}_v$  are null vectors. Moreover, matrices  $\mathbf{Z}_{\gamma_1}, \mathbf{Z}_{\gamma_2}$ , and  $\mathbf{Z}_\beta$  are defined as identity matrices without those columns corresponding to the constrained item parameters, whereas  $\mathbf{z}_{\gamma_1}$  and  $\mathbf{z}_{\gamma_2}$  are vectors with ones in correspondence of constrained item discrimination parameters and zeros otherwise;  $\mathbf{z}_\beta$  is a vector of zeros. Further examples of specification of constraints on model parameters are provided in the sequel, when the functioning of the estimation functions of the proposed R package and an example on criminal data (Example 2) are illustrated.

## Likelihood inference

The proposed two-tier LC-IRT model can be estimated by maximizing the *marginal log-likelihood*

$$\ell(\boldsymbol{\eta}) = \sum_{i=1}^n \log L_i(\mathbf{y}_i | \mathbf{x}_i), \quad (12)$$

where  $\boldsymbol{\eta}$  is the vector of free model parameters, that is, support points of  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , item difficulty and discrimination parameters, and regression coefficients for the covariates; in the previous expression,  $\mathbf{y}_i = (y_{i1}, \dots, y_{ir})'$  is the vector of observed item responses for subject  $i$ . Due to the *local independence* assumption, the marginal likelihood  $L_i(\mathbf{y}_i | \mathbf{x}_i)$  for subject  $i$  (or manifest probability of  $\mathbf{y}_i$ ) used in equation (12) is given by:

$$L_i(\mathbf{y}_i | \mathbf{x}_i) = \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \lambda_{h_1}(\mathbf{x}_i) \pi_{h_2}(\mathbf{x}_i) \prod_{j=1}^r p_{h_1 h_2}(y_{ij}),$$

where,  $p_{h_1 h_2}(y) = p(Y_{ij} = y | \mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2})$ , which depends on (1) in the case of binary items and on (4) or (5) in the case of ordinal items, if a global logit or a local logit parametrization is adopted.

We maximize  $\ell(\boldsymbol{\eta})$  through the Expectation Maximization (EM) algorithm (Dempster et al., 1977), which is based on alternating two steps until convergence:

**E-step:** the expected value of the *complete data log-likelihood* (i.e., the log-likelihood for the observed and latent variables) is computed, given the current parameter vector  $\boldsymbol{\eta}$ . In practice, this consists in computing the posterior probability  $q_{h_1 h_2 i}$  for  $h_1 = 1, \dots, k_1, h_2 = 1, \dots, k_2$ , and  $i = 1, \dots, n$ ; this is the probability that unit  $i$  belongs to latent class  $h_1$ , according to the first vector of latent variables, and to latent class  $h_2$ , according to the second vector of latent variables, given the observed data, that is,  $p(\mathbf{U}_i = \mathbf{u}_{h_1}, \mathbf{V}_i = \mathbf{v}_{h_2} | \mathbf{x}_i, \mathbf{y}_i)$ . By the Bayes' theorem we have that

$$q_{h_1 h_2 i} = \frac{\lambda_{h_1}(\mathbf{x}_i) \pi_{h_2}(\mathbf{x}_i) \prod_{j=1}^r p_{h_1 h_2}(y_{ij})}{L_i(\mathbf{y}_i | \mathbf{x}_i)}.$$

The resulting complete data log-likelihood is, in expected value, equal to

$$\ell^*(\eta) = \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{i=1}^n q_{h_1 h_2 i} \log \left[ \lambda_{h_1}(x_i) \pi_{h_2}(x_i) \prod_{j=1}^r p_{h_1 h_2}(y_{ij}) \right]. \quad (13)$$

**M-step:** the parameter vector  $\eta$  is updated by maximizing function (13) obtained at the previous step. Note that single parameter subvectors of  $\eta$  may be updated separately, as this function factorizes in three components involving the mass probabilities  $\lambda_{h_1}(x_i)$ , the mass probabilities  $\pi_{h_2}(x_i)$ , and the conditional response probabilities  $p_{h_1 h_2}(y_{ij})$ , respectively. Iterative algorithms of Newton-Raphson type are necessary to maximize all components (for details see [Bacci and Bartolucci, 2015](#), and references therein) with the exception of the first two in the case of absence of individual covariates. In fact, in this case we have the following explicit expressions to update the class weights:

$$\lambda_{h_1} = \frac{1}{n} \sum_{h_2=1}^{k_2} \sum_{i=1}^n q_{h_1 h_2 i}, \quad h_1 = 1, \dots, k_1,$$

$$\pi_{h_2} = \frac{1}{n} \sum_{h_1=1}^{k_1} \sum_{i=1}^n q_{h_1 h_2 i}, \quad h_2 = 1, \dots, k_2.$$

Similar to the other iterative algorithms, the first iteration of the EM algorithm needs to be initialized through suitable values for the model parameters that can be chosen according to certain deterministic or random rules. A common problem with finite mixture models, and then with the proposed model, is due to the presence of several local maximum points of the log-likelihood function. Therefore, in order to avoid a solution that does not correspond to the global maximum, a good practice consists in repeating the estimation process for a specific model a certain number of times using random starting values and, in the presence of different values of the log-likelihood at convergence, the solution corresponding to the highest log-likelihood value is selected.

A crucial point is that of model selection, mainly as concerns the choice of the number of support points (or latent classes) for both latent vectors (i.e.,  $k_1$  and  $k_2$ ). For this aim, a likelihood-ratio test cannot be directly used, as the regularity conditions for having an asymptotic null distribution of  $\chi^2$ -type are not satisfied for this type of test when it is applied to compare two models with different values of  $k_1$  and  $k_2$ . We then suggest to rely on suitable forms of penalization of the maximum log-likelihood, such as the Akaike Information Criterion (AIC; [Akaike, 1973](#)), which is related to Kullback-Leibler distance between the true density and the estimated density of a model. This criterion is based on the following index:

$$AIC = -2\hat{\ell}(\eta) + 2\#\text{par},$$

with  $\hat{\ell}(\cdot)$  denoting the estimated maximum log-likelihood and  $\#\text{par}$  the number of free parameters. Alternatively, we suggest the use of the Bayesian Information Criterion (BIC; [Schwarz, 1978](#)) based on the index

$$BIC = -2\hat{\ell}(\eta) + \log(n)\#\text{par}.$$

According to both these criteria, one should select the model with the minimum value of AIC or BIC.

Other selection criteria may be based on the entropy, whose computation involves the individual posterior probabilities. Entropy is a measure of the capability of the model to provide a neat partition of the sample units, which is computed as

$$E = - \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{i=1}^n q_{h_1 h_2 i} \log q_{h_1 h_2 i}$$

based on the posterior probabilities  $q_{h_1 h_2 i}$ . If the components are well separated, the posterior probabilities tend to define a clear partition of the units, assuming values close to one, and, as a consequence, the entropy will be close to zero. Usually, the entropy is not directly used to assess the number of support points and, to also account for the goodness of fit of the model, a normalized version of entropy is used by [Celeux and Soromenho \(1996\)](#). This is defined as

$$NEC = \frac{E}{\hat{\ell}_{k_1 k_2} - \hat{\ell}_{11}}, \quad k_1 > 1, k_2 > 1,$$

where  $\hat{\ell}_{k_1 k_2}$  is the maximum log-likelihood of the model with  $k_1$  and  $k_2$  support points and  $\hat{\ell}_{11}$  is the maximum log-likelihood value of the model with just one component for both latent variables. According to this criterion, the optimal number of components is the one that minimizes the NEC index. Note that NEC is not defined when  $k_1 = k_2 = 1$ , in which case  $NEC = 1$  by convention.



In practice, we propose to fit a series of models with similar specifications that distinguish one other for values assigned to  $k_1$  and  $k_2$  and, then, to make comparisons through one or more of the mentioned criteria. In more detail, given  $k_1$ , we consider increasing values of  $k_2$  and, similarly, given  $k_2$  we consider increasing values of  $k_1$  until *AIC*, *BIC*, or *NEC* do not start to increase and, then, the previous value of support points is taken as the optimal one.

More in general, for certain values of  $k_1$  and  $k_2$  we suggest that the choice between two competing models is driven by the likelihood-ratio test in the presence of nested models (i.e., when one model is obtained by the other one through constraints on the parameters), whereas the selection criteria above mentioned are suitable in the presence of non-nested models. The likelihood-ratio test is also used to evaluate the global fit of a model, when this is compared with the saturated model, that is, the largest model one can fit. Note that in the context at issue the saturated model is defined only for model specifications without covariates. Other proposals, coming from the literature on logistic regression models and on IRT models, consist of parametric and non-parametric tests that allow us to verify specific hypotheses concerning, among others, the unidimensionality of the questionnaire, the validity of the Rasch paradigm, the validity of the local independence assumption. In addition to the global fit of a model, also item-specific fit statistics, which are usually based on the comparison between observed and expected item responses, are useful to evaluate the goodness of each item and the need of removing it from the questionnaire. For a wide review of the mentioned methods see [Bartolucci et al. \(2015\)](#), Chap. 5.7, and the references therein.

Finally, in order to facilitate the interpretation of the results, we suggest to standardize the estimated support points  $\hat{u}_{d_1 h_1}$  and  $\hat{v}_{d_2 h_2}$ , so as to obtain latent variables that have mean 0 and variance equal to 1, and coherently transform the estimated item parameters  $\hat{\gamma}_{1j}$ ,  $\hat{\gamma}_{2j}$ , and  $\hat{\beta}_{jy}$ .

Let  $\hat{\mu}_{U_{d_1}}$  and  $\hat{\sigma}_{U_{d_1}}$  denote the mean and the standard deviation of  $\hat{u}_{d_1 1}, \dots, \hat{u}_{d_1 k_1}$  and let  $\hat{\mu}_{V_{d_2}}$  and  $\hat{\sigma}_{V_{d_2}}$  denote the mean and the standard deviation of  $\hat{v}_{d_2 1}, \dots, \hat{v}_{d_2 k_2}$ . Then,  $\hat{u}_{d_1 h_1}$  and  $\hat{v}_{d_2 h_2}$  are standardized as follows:

$$\hat{u}_{d_1 h_1}^* = \frac{\hat{u}_{d_1 h_1} - \hat{\mu}_{U_{d_1}}}{\hat{\sigma}_{U_{d_1}}}, \quad d_1 = 1, \dots, D_1, \quad (14)$$

$$\hat{v}_{d_2 h_2}^* = \frac{\hat{v}_{d_2 h_2} - \hat{\mu}_{V_{d_2}}}{\hat{\sigma}_{V_{d_2}}}, \quad d_2 = 1, \dots, D_2. \quad (15)$$

Moreover,  $\hat{\gamma}_{1j}$ ,  $\hat{\gamma}_{2j}$ , and  $\hat{\beta}_{jy}$  are transformed as

$$\hat{\gamma}_{1j}^* = \hat{\gamma}_{1j} \sum_{d_1=1}^{D_1} 1\{j \in U_{d_1}\} \hat{\sigma}_{U_{d_1}}, \quad (16)$$

$$\hat{\gamma}_{2j}^* = \hat{\gamma}_{2j} \sum_{d_2=1}^{D_2} 1\{j \in V_{d_2}\} \hat{\sigma}_{V_{d_2}}, \quad (17)$$

$$\hat{\beta}_{jy}^* = \hat{\beta}_{jy} - \hat{\gamma}_{1j} \sum_{d_1=1}^{D_1} 1\{j \in U_{d_1}\} \hat{\mu}_{U_{d_1}} - \hat{\gamma}_{2j} \sum_{d_2=1}^{D_2} 1\{j \in V_{d_2}\} \hat{\mu}_{V_{d_2}}, \quad (18)$$

for  $j = 1, \dots, r$  and  $y = 0, \dots, l_j - 1$ , with  $l_j = 2$  for a dichotomously scored item.

## The R package MLCIRTwithin

The class of two-tier LC-IRT models previously described may be estimated through the R package **MLCIRTwithin**; for technical details see the official documentation provided in CRAN ([Bartolucci and Bacci, 2016](#)).

Before illustrating the main functions in the package at issue, it is worth mentioning some alternative R packages, which estimate models with a formulation resembling the one proposed. A first example is provided by the R package **MultiLCIRT** ([Bartolucci et al., 2014, 2016](#)), whose functions are similar, in terms of input and output, to those of **MLCIRTwithin**; however, **MultiLCIRT** is limited to the estimation of LC-IRT models under between-item multidimensionality, in the sense that items loading on more than one latent trait are not allowed and then a single vector of latent variables  $U_i$  is used. Moreover, constraints on the item parameters or fixed values for the support points cannot be specified. Package **CDM** ([Robitzsch et al., 2016](#)) performs the estimation of the class of cognitive diagnostic models ([Tatsuoka, 1983](#); [Jang, 2008](#); [Rupp and Templin, 2008](#)), in which the proposed, discrete, two-tier model may be included. The class of models estimated through **CDM** may be characterized, among the main options, by normally distributed latent traits or, alternatively, discrete

latent traits whose support points may be freely estimated or may be specified as fixed values. As concerns item parameters, item-by-category specific slopes as well as linear constraints are allowed. However, different from our proposal, individual covariates affecting the class membership as well as the specification of a global logit link are not allowed. Moreover, attention must be paid to the interpretation of the latent classes, which are defined in a quite different and general way with respect to our proposal. The estimation of within-item multidimensional IRT models and item bifactor models is also performed through packages **mirt** (Chalmers, 2012; Chalmers et al., 2016) and **flirt** (Jeon et al., 2014), under the assumption of normally distributed latent variables. Package **mirt** also allows for discrete latent variables; however, in such a case just the multidimensional LC model without a classical IRT parametrization (mainly, without item difficulties) and without covariates is estimated. A major flexibility with respect to **mirt** is provided by package **covLCA** (Bertrand and Hafner, 2013), which is focused on multidimensional LC models with covariates affecting both the class membership and the manifest variables. Other two packages to mention are **lavaan** (Rosseel et al., 2015) and **OpenMx** (Neale et al., 2016) that perform the estimation of the wide class of structural equation models, in which unidimensional and multidimensional IRT models are included, under the assumption of normality of the latent variables. Finally, we mention two general and flexible softwares that may accommodate the estimation of the model here proposed, that is, Mplus (Muthén and Muthén, 2012) and LatentGold (Vermunt and Magidson, 2005): the former is tailored to the estimation of latent variable models under the assumption of normal or discrete latent variables, whereas the latter is focused on LC models. In both cases, the user may formulate IRT models with a variety of features, among which multidimensionality and presence of covariates.

### Functions `est_multi_poly_within` and `est_multi_poly_between`

The main function of **MLCIRTwithin** is `est_multi_poly_within`, which performs the maximum likelihood estimation of the model specified through equations (1) to (5), allowing for several options.

Function `est_multi_poly_within` requires the following main input arguments:

- `S`: matrix of item response configurations listed row-by-row; items with a different number of categories and missing responses are allowed.
- `yv`: vector of the frequencies of every row in `S`; by default, `yv` is a vector of ones.
- `k1`: number of latent classes for latent variable  $U_i$ .
- `k2`: number of latent classes for latent variable  $V_i$ .
- `X`: matrix of covariates affecting the class weights; by default, `X` is `NULL`.
- `start`: method of initialization of the algorithm: "deterministic" (default value) for values chosen according to a deterministic rule, "random" for values randomly drawn from suitable distributions (continuous uniform between 0 and 1 for the class weights and standard normal for the other parameters), and "external" for values provided by the researcher through inputs `Phi`, `ga1t`, `ga2t`, `De1`, and `De2`.
- `link`: type of link function: "global" for global logits as in Equation (4) and "local" for local logits as in Equation (5). With binary items, any type of link function may be specified, resulting in a Rasch (Rasch, 1960) or a two-parameter logit (2PL; Birnbaum, 1968) type model depending on the value assigned to input `disc`.
- `disc`: constraints on the discriminating item parameters: `FALSE` (default value) for parameters  $\gamma_{1j}$  and  $\gamma_{2j}$  all equal to one and `TRUE` for free values. With binary items, option `disc = FALSE` results in a Rasch model, whereas a 2PL model is obtained when option `disc = TRUE`.
- `dif1`: constraints on the difficulty item parameters, in the case of ordinal polytomously scored items: `FALSE` (default value) for unconstrained parameters  $\beta_{jy}$  and `TRUE` for a rating scale parametrization as in (6). This option is not allowed in the presence of items with a different number of response categories.
- `multi1`: matrix with one row for each component of  $U_i$  and elements in each cell indicating the indices of the items measuring the dimension corresponding to that row; the number of rows is  $D_1$  and that of columns equals the number of items in the largest dimension. If dimensions differ in the number of items, zeros are inserted in the empty cells. Each item corresponding to the first column of each row has discriminating index constrained to 1 and difficulty parameter constrained to 0 to ensure model identifiability. For instance, in the presence of 6 items, with items 1 and 2 measuring the first dimension of latent variable  $U_i$  and the remaining items 3 to 6 measuring another dimension of  $U_i$ , as in Figure 1, matrix `multi1` is specified as

```
(multi1 <- rbind(c(1,2,0,0), c(3,4,5,6)))
```

```

      [,1] [,2] [,3] [,4]
[1,]    1    2    0    0
[2,]    3    4    5    6

```

- `multi2`: same as `multi1` for latent variable  $V_i$ . For model identifiability, attention must be paid on the item indices in the first column of `multi2` that cannot be the same as the indices in the first column of `multi1`. For instance, in the situation described in Figure 1, the matrix `multi2` specified as

```
(multi2 <- c(7, 2:3, 5:6))
```

```
[1] 7 2 3 5 6
```

implies  $\gamma_{27} = 1$  and  $\beta_7 = 0$ . A particular case is when the intersection between matrices `multi1` and `multi2` is empty: in such a case a between-item multidimensional LC-IRT model is specified, based on two completely independent latent vectors  $U_i$  and  $V_i$ .

- `fort`: if TRUE, Fortran routines are used whenever possible to speed up computation.
- `tol`: level of tolerance of the algorithm in terms of relative difference between the log-likelihood corresponding to two consecutive algorithm iterations (default value is  $10^{-10}$ ).
- `disp`: if TRUE, the log-likelihood evolution is displayed step-by-step.
- `output`: if TRUE, additional output arguments are returned.
- `out_se`: if TRUE, standard errors and variance-covariance matrix for the parameter estimates are returned.
- `glob`: type of parametrization for the sub-model assumed on the individual-specific latent class weights: FALSE (default value) for a multinomial logit model as in (2)-(3) and TRUE for a global logit model.
- `Zth1`, `Zth2`: matrices for the specification of linear constraints on the support points, according to (7) and (8), respectively; by default these are identity matrices with a number of rows (and columns) equal to the total number of support points, that is,  $k_1 D_1$  and  $k_2 D_2$ , respectively.
- `zth1`, `zth2`: vectors of length  $k_1 D_1$  and  $k_2 D_2$ , respectively, for the specification of linear constraints and fixed support points, according to (7) and (8), respectively; by default they are null vectors.

Under the default specifications of `Zth1`, `Zth2`, `zth1`, and `zth2`, the support points are freely estimated and, for the model identification, certain constraints are assumed on the item parameters. On the contrary, to fix the values of the support points, `Zth1`, `Zth2`, `zth1`, and `zth2` must be supplied by the user. For instance, in the situation described in Figure 1 under the assumption  $k_1 = k_2 = 2$ , we define  $u = (-1, -1, 1, 1)'$  and  $v = (-0.5, 0.5)'$  as follows:

```

Zth1 <- matrix(0, 2*2, 0)
zth1 <- c(rep(-1, times=2), rep(1, times = 2))
Zth2 <- matrix(0, 2, 0)
zth2 <- c(-0.5, 0.5)

```

- `Zga1`, `Zga2`, `Zbe`: matrices for the specification of linear constraints on the vectors of item parameters  $\gamma_1$ ,  $\gamma_2$ , and  $\beta$ , as in (9), (10) and (11), respectively. In more detail, the number of rows of `Zga1` and `Zga2` is equal to the number of non-null entries in `multi1` and `multi2`, respectively, and coincides with the length of vectors  $\gamma_1$  and  $\gamma_2$ ; whereas the number of rows of `Zbe` corresponds to the total number of item difficulties and coincides with the length of vector  $\beta$ . The number of columns of `Zga1`, `Zga2`, `Zbe` is equal to the total number of free parameters, corresponding to the length of vectors  $\tilde{\gamma}_1$ ,  $\tilde{\gamma}_2$ , and  $\tilde{\beta}$ , respectively. By default these are identity matrices without those columns corresponding to the constrained parameters. For instance, in the situation described in Figure 1 with the usual IRT constraints  $\gamma_{11} = \gamma_{13} = 1$  and  $\gamma_{27} = 1$  resulting from matrices `multi1` and `multi2` defined above, and  $\beta_1 = \beta_3 = \beta_7 = 0$  in the case of binary items, the following matrices are used by default in function `est_multi_poly_within`:

```

(Zga1 <- diag(6)[, -c(1,3)])
      [,1] [,2] [,3] [,4] [,5]
[1,]    0    0    0    0    0
[2,]    1    0    0    0    0
[3,]    0    0    0    0    0
[4,]    0    1    0    0    0
[5,]    0    0    1    0    0
[6,]    0    0    0    0    1

```



```
(Zga2 <- diag(5)[, -5])
```

```
      [,1] [,2] [,3] [,4]
[1,]    1    0    0    0
[2,]    0    1    0    0
[3,]    0    0    1    0
[4,]    0    0    0    1
[5,]    0    0    0    0
```

```
(Zbe <- diag(7)[, -c(1,3,7)])
```

```
      [,1] [,2] [,3] [,4]
[1,]    0    0    0    0
[2,]    1    0    0    0
[3,]    0    0    0    0
[4,]    0    1    0    0
[5,]    0    0    1    0
[6,]    0    0    0    1
[7,]    0    0    0    0
```

Whenever we are interested in introducing further constraints, then the matrices at issue must be supplied by the user. For instance, to restrict  $\gamma_{14} = \gamma_{15}$ , then matrix Zga1 must be defined as

```
Zga1 <- diag(6)[, -c(1, 3, 5)]; Zga1[5, 2] <- 1
Zga1
```

```
      [,1] [,2] [,3]
[1,]    0    0    0
[2,]    1    0    0
[3,]    0    0    0
[4,]    0    1    0
[5,]    0    1    0
[6,]    0    0    1
```

- zga1, zga2, and zbe: vectors whose length is equal to the number of rows of Zga1, Zga2, and Zbe, respectively. In other words, length of zga1, zga2, and zbe is given by the number of elements in  $\gamma_1$ ,  $\gamma_2$ , and  $\beta$ . The suitable specification of these vectors, combined with that of matrices Zga1, Zga2, and Zbe, allows for linear constraints and fixed values of the item parameters, as in (9), (10), and (11). By default, zga1 and zga2 are vectors with elements 1 for each constrained item and 0 otherwise; by default zbe is a null vector. For instance, in the situation depicted in Figure 1 and matrices multi1 and multi2, default values assumed for vectors at issue are

```
zga1 <- c(1, 0, 1, 0, 0, 0)
zga2 <- c(0, 0, 0, 0, 1)
zbe <- rep(0, times = 7)
```

Any other constraint may be defined by modifying in a suitable way these three vectors and matrices Zga1, Zga2, Zbe. For instance, if we are interested in fixing the difficulty of (binary) item 4 to be equal to 2 (i.e.,  $\beta_4 = 2$ ), then we define Zbe and zbe as follows:

```
Zbe <- diag(7)[, -c(1,3,4,7)]
Zbe
```

```
      [,1] [,2] [,3]
[1,]    0    0    0
[2,]    1    0    0
[3,]    0    0    0
[4,]    0    0    0
[5,]    0    1    0
[6,]    0    0    1
[7,]    0    0    0
```

```
zbe <- c(0, 0, 0, 2, 0, 0, 0)
```

Function `est_multi_poly_within` supplies the following output:

- piv1 and piv2: vectors of the estimated weights of latent classes for  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively; in the presence of individual covariates, these are averages of the individual-specific mass probabilities.

- Th1 and Th2: matrices of estimated and constrained support points for each dimension (by row) and each latent class (by column) for  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively.
- Bec: matrix of estimated and constrained item difficulty parameters; exact zeros correspond to identifiability constraints.
- ga1c and ga2c: vectors of estimated and constrained item discriminating parameters for  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively; exact ones correspond to identifiability constraints and NA to items that do not load on the latent variable.
- th1t, th2t, bet, ga1t, and ga2t: estimated parameters (i.e., parameters without constraints) related to Th1, Th2, Bec, ga1c, ga2c, respectively.
- Th1s, Th2s, Becs, ga1cs, and ga2cs: standardized values of Th1, Th2, Bec, ga1c, and ga2c, respectively; in Th1s and Th2s classes are re-ordered according to the increasing values of the support points for the first dimension.
- piv1s and piv2s: the same as piv1 and piv2, but re-ordered according to Th1s and Th2s.
- fv1 and fv2: vectors indicating the reference items for each dimension of  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively.
- Phi: conditional response probabilities for every item and each pair of latent classes of  $\mathbf{U}_i$  and  $\mathbf{V}_i$ .
- De1 and De2: matrices of estimated regression coefficients for the model on the class weights for  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , respectively, in the presence of individual covariates; for each covariate and the constant term, the number of estimated coefficients is equal to the number of latent classes minus one.
- Piv1, Piv2, Pp1, Pp2, and lkv: optional output (obtained if `output = TRUE`) referred to the matrices of weights for every covariate configuration for latent variables  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , the matrices of the posterior probabilities for each response configuration and latent class for latent variables  $\mathbf{U}_i$  and  $\mathbf{V}_i$ , and the values of the log-likelihood during the estimation process, respectively.
- XX1dis and XX2dis: design matrices for the covariates affecting the first and the second vector of latent variables, respectively (optional output obtained if `output = TRUE`).
- lk: value of the log-likelihood at convergence.
- np: number of estimated model parameters.
- aic, bic, and ent: AIC, BIC, and entropy indices, respectively.
- seDe1, seDe2, seTh1, seTh2, seBec, sega1, sega2, seth1t, seth2t, sebet, sega1t, sega2t, and Vn: standard errors of the corresponding estimated parameters and estimated variance-covariance matrix (if `out_se = TRUE`).

Some relevant commands to display output from function `est_multi_poly_within` are based on the S3 methods summary for the main estimates; `coef` and `confint` for the point estimates and confidence intervals (at a specified level of confidence) of support points, item parameters, and regression coefficients; `logLik` for the value of log-likelihood at convergence; and `vcov` for the estimated variance-and-covariance matrix.

Another relevant function of package **MLCIRTwithin** is `est_multi_poly_between`, which performs the maximum likelihood estimation of an LC-IRT model under between-item multidimensionality. The main differences with respect to function `est_multi_poly` of the R package **MultiLCIRT** are that the latter does not allow for items with a different number of response categories and refers to a slightly different specification of item difficulties (for details see [Bacci et al., 2014](#)).

Input arguments required by `est_multi_poly_between` are very similar to those of function `est_multi_poly_within`. The main difference is that only one vector of latent variables is involved in the model specification. Consequently, the number of latent classes (input `k`) is common to all the dimensions and the multidimensional structure of the items is specified through one matrix (input `multi`), having one row for each dimension. Constraints on model parameters are also possible through a suitable definition of arguments `Zth`, `zth`, `Zbe`, `zbe`, `Zga`, and `zga`, whose functioning is the same as the corresponding arguments in function `est_multi_poly_within`. The function provides as its main output argument a vector of estimated average weights of the latent classes (output `piv`) and a matrix of estimated support points for each dimension and each latent class of the latent trait before (output `Th`) and after the standardization (output `Ths`). Besides, a matrix of difficulty item parameters (outputs `Bec` and, in the case of standardization, `Becs`), a vector of discriminating indices (output `gac` and, in the case of standardization, `gacs`), and a matrix of regression coefficients (output `De`) are provided, other than other output arguments similar to those above described for `est_multi_poly_within`, included the S3 methods.

Finally, we clarify that a model specification of type

```
out1 <- est_multi_poly_between(S, k = k0, link = "global", multi = rbind(1:3, 4:6)),
```

with  $k_0$  latent classes, is substantially different from a model specification of type

```
out2 <- est_multi_poly_within(S, k1 = k0, k2 = k0, link = "global", multi1 = c(1:3),
                             multi2 = c(4:6)).
```

In fact, the model corresponding to `out2` involves two completely independent latent variables, having incidentally the same number of latent classes: thus, an individual belonging to a specific class (say, class 1) according to the first latent variable may belong to any latent class under the second latent variable (say, class 2). On the contrary, model `out1` involves only one latent variable decomposed in two dimensions: thus, belonging to a given latent class under one dimension implies belonging to the same class under the other dimension. Overall, model `out2` has  $k_0 - 1$  free parameters more than model `out1`.

### Functions `search.model_within` and `search.model_between`

As outlined in Section “Likelihood inference,” the selection of a two-tier LC-IRT model may be a quite demanding procedure, requiring the choice of the number of support points for the latent variables and a check for the possible presence of local maxima. Function `search.model_within` allows us to search for the global maximum of the log-likelihood of a model with a specific formulation (in terms of multidimensional structure, link function, and constraints on the item parameters) given a vector of possible number of latent classes to try for.

In practice, function `search.model_within` applies function `est_multi_poly_within` a given number of times for each pair of values for  $k_1$  and  $k_2$ , initializing the estimation algorithm with deterministic and random values of the model parameters and holding, for each pair of  $k_1$  and  $k_2$ , that model with the highest value of the log-likelihood at convergence. To make the entire process computationally less demanding, the search of the global maximum may be performed with a relatively large tolerance level for checking convergence of the estimation algorithm. Then, in order to improve the precision of parameters estimates, the estimates provided by the model with the best value of the log-likelihood are used as starting values in the last step of the model selection process, using an augmented tolerance level. Note that when  $k_1 = 1$  or  $k_2 = 1$  the model estimation is actually performed by the function `est_multi_poly_between`, which is automatically retrieved by `search.model_within`.

The function at issue requires the following main input arguments:

- `S`, `yv`, `X`, `link`, `disc`, `dif1`, `multi1`, `multi2`, `fort`, `disp`, `output`, `out_se`, `Zth1`, `zth1`, `Zth2`, `zth2`, `Zbe`, `zbe`, `Zga1`, `zga1`, `Zga2`, and `zga2`: are the same as in function `est_multi_poly_within`.
- `kv1` and `kv2`: vectors of number of latent classes to try for latent variable  $U_i$  and  $V_i$ , respectively; single values are also allowed for.
- `tol1` and `tol2`: tolerance levels (default value are  $10^{-6}$  and  $10^{-10}$ , respectively) for checking convergence of the algorithm as relative difference between consecutive log-likelihoods. The value of `tol1` is used for checks based on deterministic and random starting values, whereas the value of `tol2` is used for improving the precision of estimates for the model with the best log-likelihood level.
- `nrep`: constant value that drives the number of estimations of each model with random starting values, given by  $nrep(k_1 k_2 - 1)$ ; the default value for `nrep` is 2. In the case of `nrep` equal to 0, only the estimation with deterministic starting values is performed.

Note that if single values for `kv1` and `kv2` are specified and `nrep` equals 0, function `search.model_within` performs just one call of function `est_multi_poly_within` (or function `est_multi_poly_between` if `kv1` or `kv2` equal 1) with option `start = "deterministic"`.

Function `search.model_within` supplies the following output:

- `aicv`, `bicv`, and `necv`: vectors of AIC, BIC, and NEC indices, respectively, for each of the estimated models.
- `errv`: trace of any error occurred during the estimation process.
- `lkv`: values of log-likelihood at convergence for each of the estimated models.
- `out.single`: output of each single model, similar to the output of `est_multi_poly_within`, with the addition of values of  $k_1$  (output `k1`);  $k_2$  (output `k2`); and the sequence of log-likelihoods (output `lktrace`) for the deterministic start, for each random start, and for the final estimation provided by a tolerance level equal to `tol2` (if `tol2 > tol1`).

Finally, we outline that a function with input and output arguments similar to those of the function `search.model_within` is available to perform the model selection in the case of between-item multidimensional LC-IRT models. This function is named `search.model_between` and it relies on `est_multi_poly_between`.

## Examples in R

In the following we illustrate package **MLCIRTwithin** through two data analysis examples. In Example 1 we describe the model selection procedure, as well as the interpretation of the output, considering a set of ordered items measuring two latent variables. In Example 2, the specification of constraints on the support points and on the item parameters is illustrated through the analysis of data concerning repeated item responses along two time occasions. The detailed software scripts to implement the two examples, named Example1.R and Example2.R, are available in the Supplementary Online Material at [https://sites.google.com/site/bartstatistics/sm\\_mlcirtwithin.zip](https://sites.google.com/site/bartstatistics/sm_mlcirtwithin.zip).

### Example 1: analysis of multidimensionality

Data set SF12\_nomiss, already provided in the R package **MLCIRTwithin**, refer to a sample of 493 oncological Italian patients who were asked to fill in the Italian validated Short Form 12 version 2 questionnaire (SF-12; [Stewart and Ware, 1992](#); [Ware et al., 2002](#)) concerning the assessment of HQOL. The questionnaire is comprised by 12 items having five ordered response modalities, except items 2 and 3 having only three modalities; a high score means a worse level of HQOL and vice-versa (note that, in the original scoring system, modalities of items 9 and 10 are reversed). Also the age is available for each patient. In the following we show the first few records of the data set and the related summaries.

```
library(MLCIRTwithin)

data(SF12_nomiss)
head(SF12_nomiss)

  Y1 Y2 Y3 Y4 Y5 Y6 Y7 Y8 Y9 Y10 Y11 Y12    age
1  1  0  1  1  0  2  2  2  1  1  1  0 74.94593
2  0  0  1  1  2  1  2  1  0  2  1  1 84.49829
3  1  1  1  2  1  0  0  2  1  1  1  1 77.44285
4  3  2  2  4  4  4  4  3  4  4  4  4 80.55305
6  1  1  2  2  2  2  2  2  2  2  2  2 81.68104
7  1  0  1  3  2  2  1  2  1  2  1  3 78.55168

str(SF12_nomiss)

'data.frame':      493 obs. of  13 variables:
 $ Y1 : num  1 0 1 3 1 1 1 1 2 2 ...
 $ Y2 : num  0 0 1 2 1 0 0 2 1 1 ...
 $ Y3 : num  1 1 1 2 2 1 1 2 1 2 ...
 $ Y4 : num  1 1 2 4 2 3 1 3 2 3 ...
 $ Y5 : num  0 2 1 4 2 2 2 2 2 3 ...
 $ Y6 : num  2 1 0 4 2 2 2 2 3 3 ...
 $ Y7 : num  2 2 0 4 2 1 1 2 3 3 ...
 $ Y8 : num  2 1 2 3 2 2 2 1 3 3 ...
 $ Y9 : num  1 0 1 4 2 1 2 3 3 3 ...
 $ Y10: num  1 2 1 4 2 2 1 2 2 2 ...
 $ Y11: num  1 1 1 4 2 1 2 1 4 3 ...
 $ Y12: num  0 1 1 4 2 3 1 2 3 3 ...
 $ age: num  74.9 84.5 77.4 80.6 81.7 ...

# For the description of each item see the online documentation
?SF_nomiss
```

According to the main current literature (see, mainly, [Ware et al., 2002](#)), the SF-12 questionnaire may be used to properly evaluate two main aspects of HQOL: physical and emotional. The standard scoring algorithm for summarizing these two latent dimensions is based on an orthogonal factor analysis, on the basis of which positive and negative weights are assigned to each item. More in detail, items 1 to 5 and item 8 have positive weights for physical HQOL and negative weights for emotional HQOL, whereas items 6, 7, 9, 11, and 12 have negative weights for physical HQOL and positive weights for emotional HQOL; item 10 has positive weights for both components. According to this scoring system, the scores of physical and emotional HQOL result by a suitable weighted average of the item responses.

A low score on physical HQOL has the following meaning (Ware and Gandek, 1998): substantial limitations in self-care, physical, social and role activities; severe bodily pain; frequent tiredness; health rated as poor. On the contrary, a high level of physical component corresponds to: no physical limitations, disabilities or decrements in well-being; high energy level; health rated as excellent. As regards the emotional component, a low score implies: frequent psychological distress, social and role disability due to emotional problems; health rated poor. On the other hand, a high level of emotional component corresponds to: frequent positive affect; absence of psychological distress and limitations in usual social activities due to emotional problems; health rated excellent.

The main drawback of the above algorithm based on the orthogonal factor analysis is that the summary score may be inconsistent due to weights of opposite sign for the same items, as higher emotional health scores drive physical health scores down and, similarly, higher physical health scores drive emotional health scores down (Farivar et al., 2007). An alternative approach for clustering patients according to their physical and emotional health status is based on IRT analysis (see, among others, Hays et al., 1993). In such a context, we analyze the multidimensional structure of SF-12 questionnaire through a two-dimensional model allowing items measuring both latent variables. In more detail, we compare several plausible multidimensional structures, defined through the following matrices, with the first one referred to the physical HQOL and the second one referred to the emotional HQOL:

**Type 1:** within-item multidimensional model with two independent latent variables and no shared item; items are allocated according to the sign of weights resulting by the factor analysis mentioned above:

```
(multi1_dim1 <- c(1:5, 8))

[1] 1 2 3 4 5 8

(multi1_dim2 <- c(6:7, 9:12))
[1] 6 7 9 10 11 12
```

**Type 2:** model with two latent variables sharing items that do not explicitly affect a specific dimension

```
(multi2_dim1 <- c(1:5, 8:12))

[1] 1 2 3 4 5 8 9 10 11 12

(multi2_dim2 <- c(6:12, 1))

[1] 6 7 8 9 10 11 12 1
```

**Type 3:** multidimensional structure similar to the previous one, but with three items (9, 10, and 11) assigned only to the emotional HQOL

```
(multi3_dim1 <- c(1:5, 8, 12))

[1] 1 2 3 4 5 8 12

(multi3_dim2 <- c(6:12, 1))

[1] 6 7 8 9 10 11 12 1
```

**Type 4:** multidimensional structure similar to that defined through multi21 and multi22, but one more item (number 8), concerning the presence of pain, is assigned only to physical HQOL, since pain is usually intended in terms of physical health (i.e., bodily pain)

```
(multi4_dim1 <- c(1:5, 8, 12))

[1] 1 2 3 4 5 8 12

(multi4_dim2 <- c(6:7, 9:12, 1))

[1] 6 7 9 10 11 12 1
```

The allocation of every item according to one of the above proposed structures is suggested by the more or less explicit reference of the item text to physical or emotional component of HQOL (or to both of them).

Considering the possible multidimensional structures above defined, we focus on models with global logit link function and free discriminating item parameters; also the effect of age on the mass probabilities is investigated.



```
# Item responses and covariates
S <- SF12_nomiss[ , 1:12]
X <- SF12_nomiss[ , 13]
```

For each type of multidimensional structure, we select the optimal number of latent classes on the basis of BIC index according to the procedure described in Section “Likelihood inference”. Also the check of local maxima solutions follows the same lines described therein. For these aims we use function `search.model_within`, as follows:

```
### Model selection
maxk1 <- 6
maxk2 <- 6

tol1 <- 10^-3
tol2 <- 10^-6

# Multidimensional structure of Type 1
set.seed(0)
out1 <- search.model_within(S, kv1 = 1:maxk1, kv2 = 1:maxk2, X = X, link = "global",
  disc = TRUE, multi1 = multi1_dim1, multi2 = multi1_dim2,
  fort = TRUE, tol1 = tol1, tol2 = tol2, nrep = 1)

# Multidimensional structure of Type 2
set.seed(0)
out2 <- search.model_within(S, kv1 = 1:maxk1, kv2 = 1:maxk2, X = X, link = "global",
  disc = TRUE, multi1 = multi2_dim1, multi2 = multi2_dim2,
  fort = TRUE, tol1 = tol1, tol2 = tol2, nrep = 1)

# Multidimensional structure of Type 3
set.seed(0)
out3 <- search.model_within(S, kv1 = 1:maxk1, kv2 = 1:maxk2, X = X, link = "global",
  disc = TRUE, multi1 = multi3_dim1, multi2 = multi3_dim2,
  fort = TRUE, tol1 = tol1, tol2 = tol2, nrep = 1)

# Multidimensional structure of Type 4
set.seed(0)
out4 <- search.model_within(S, kv1 = 1:maxk1, kv2 = 1:maxk2, X = X, link = "global",
  disc = TRUE, multi1 = multi4_dim1, multi2 = multi4_dim2,
  fort = TRUE, tol1 = tol1, tol2 = tol2, nrep = 1)
```

We advise that the entire estimation process may take a very long computational time; then, we suggest to reduce the tolerance level of the algorithm (we adopted  $10^{-3}$  instead of the default value  $10^{-6}$  for argument `tol1` and  $10^{-6}$  instead of the default value  $10^{-10}$  for argument `tol2`) and the number of repetitions with random initializations (we specified `nrep = 1` instead of the default value `nrep = 2`). Outputs `out1`, `out2`, `out3`, and `out4` are contained in the file ‘Example1.RData’, available in the supplementary online material at [https://sites.google.com/site/bartstatistics/sm\\_mlcirtwithin.zip](https://sites.google.com/site/bartstatistics/sm_mlcirtwithin.zip).

In order to select the optimal model, values of BIC index are displayed in the following 36-by-4 matrix, having one row for each model and one column for each multidimensional structure:

```
# BIC indices
BIC <- cbind(out1$bicv, out2$bicv, out3$bicv, out4$bicv)
colnames(BIC) <- c("Type 1", "Type 2", "Type 3", "Type 4")
k1 <- rep(1:6, times = 1, each = 6)
k2 <- rep(1:6, times = 6)
BIC <- cbind(k1, k2, BIC)
BIC
```

	k1	k2	Type 1	Type 2	Type 3	Type 4
[1,]	1	1	16041.95	16041.95	16041.95	16041.95
[2,]	1	2	14952.57	14758.90	14758.90	14857.40
[3,]	1	3	14637.43	14397.05	14397.05	14523.30
[4,]	1	4	14457.90	14186.90	14186.90	14323.75
[5,]	1	5	14457.16	14182.29	14182.29	14322.70
[6,]	1	6	14456.62	14181.76	14181.76	14323.05

```

[7,] 2 1 15245.74 14755.76 15106.47 15106.47
[8,] 2 2 14180.93 14010.29 14071.65 14097.89
[9,] 2 3 13865.85 13721.04 13750.05 13784.53
[10,] 2 4 13686.27 13543.49 13554.24 13595.26
[11,] 2 5 13685.48 13543.77 13553.30 13599.42
[12,] 2 6 13684.91 13559.40 13563.59 13604.91
[13,] 3 1 14945.39 14292.43 14748.77 14748.77
[14,] 3 2 13880.61 13674.91 13783.16 13791.13
[15,] 3 3 13565.50 13435.72 13482.17 13495.04
[16,] 3 4 13385.96 13291.34 13303.73 13319.80
[17,] 3 5 13385.83 13294.18 13306.15 13324.50
[18,] 3 6 13392.96 13308.42 13318.83 13334.18
[19,] 4 1 14897.97 14183.55 14678.01 14678.01
[20,] 4 2 13837.48 13606.69 13728.08 13732.33
[21,] 4 3 13522.40 13369.83 13441.26 13451.80
[22,] 4 4 13342.82 13245.49 13262.09 13271.06
[23,] 4 5 13342.45 13251.35 13269.85 13277.34
[24,] 4 6 13341.56 13265.77 13271.07 13293.80
[25,] 5 1 14858.93 14122.61 14610.54 14610.54
[26,] 5 2 13851.00 13549.28 13744.75 13674.28
[27,] 5 3 13538.57 13353.08 13383.57 13387.67
[28,] 5 4 13356.25 13196.32 13215.00 13219.66
[29,] 5 5 13298.70 13207.16 13216.58 13226.12
[30,] 5 6 13298.20 13212.85 13237.74 13242.02
[31,] 6 1 14932.45 14110.19 14613.70 14613.70
[32,] 6 2 13805.78 13557.76 13683.30 13683.43
[33,] 6 3 13495.08 13339.91 13393.67 13410.61
[34,] 6 4 13311.17 13205.40 13231.35 13230.88
[35,] 6 5 13312.05 13214.68 13237.86 13240.88
[36,] 6 6 13310.90 13222.19 13241.50 13243.07

```

On the basis of the above matrix we observe that the minimum value of the BIC index, that is, 13196.32, is displayed in column 2, denoting the multidimensional structure of type 2, and row 28, which refers to models with  $k_1 = 5$  and  $k_2 = 4$  latent classes. The output of the selected model is contained in the object `out2$out.single[[28]]`.

```

# Minimum BIC
min(BIC[ , 3:6])

[1] 13196.32

# Detect model with the minimum BIC
arrayInd(which.min(BIC[ , -c(1:2)]), .dim = dim(BIC))

      [,1] [,2]
[1,]    28     2

# Number of support points for the best model
out2$out.single[[28]]$k1

[1] 5

out2$out.single[[28]]$k2
[1] 4

# Selected model
outsel = out2$out.single[[28]]

We also observe that the same number of support points is selected for the other multidimensional
structures with shared items (i.e., structures of types 3 and 4).

# Minimum BIC and selected model for multidimensional structures of types 1, 3, 4
min(BIC[, "Type 1"])

[1] 13298.2

```

```

which.min(BIC[, "Type 1"])

[1] 30

min(BIC[, "Type 3"])

[1] 13215

which.min(BIC[, "Type 3"])

[1] 28

min(BIC[, "Type 4"])

[1] 13219.66

which.min(BIC[, "Type 4"])

[1] 28

```

After selecting the number of latent classes and the type of within-item multidimensional structure, we estimate again the selected model out with option `out_se = T` for the computation of standard errors. To reduce the computational time, we use function `est_multi_poly_within` with option `start = "external"` and specified as input for options `Phi`, `ga1c`, `ga2c`, `De1`, and `De2` the corresponding output from the selected model.

```

# Re-estimate model to compute standard errors
out <- est_multi_poly_within(S = S, k1 = outsel$k1, k2 = outsel$k2, X = X,
  start = "external",
  multi1 = multi2_dim1, multi2 = multi2_dim2,
  Phi = outsel$Phi, ga1t = outsel$ga1t, ga2t = outsel$ga2t,
  De1 = outsel$De1, De2 = outsel$De2,
  link = "global", disc = TRUE, fort = TRUE, output = TRUE,
  out_se = TRUE, disp = TRUE)

```

Details of the output of the estimated model may be displayed through the usual methods `summary`, `coef`, and `confint`.

```

### Display output of the estimated model
# summary(out)
# coef(out)
# confint(out)

# Estimates of support points and average mass probabilities for physical HQOL
lv1 <- rbind(out$Th1, t(out$piv1))
rownames(lv1) <- c("Physical HQOL", "Average prob.")
round(lv1, 3)

          1      2      3      4      5
Physical HQOL 0.823 1.684 2.854 0.093 -2.024
Average prob. 0.199 0.387 0.310 0.083 0.021

# Estimates of support points and average mass probabilities for emotional HQOL
lv2 <- rbind(out$Th2, t(out$piv2))
rownames(lv2) <- c("Emotional HQOL", "Average prob.")
round(lv2, 3)

          1      2      3      4
Emotional HQOL 0.471 11.526 7.585 4.151
Average prob. 0.149 0.141 0.390 0.321

```

According to the estimated model, patients are clustered in 5 latent classes denoting different levels of physical HQOL (output `out$Th1`) and in 4 latent classes denoting different levels of emotional HQOL (output `out$Th2`). To simplify the interpretation of the latent classes, it is useful to re-order and standardize the estimated support points. For this aim, function `est_multi_poly_within` provides the values of support points, which are standardized according to equations (14)-(18) and re-ordered

according to increasing values of the first dimension of the corresponding latent variable (outputs out\$Th1s and out\$Th2s with related weights out\$piv1s and out\$piv2s, respectively).

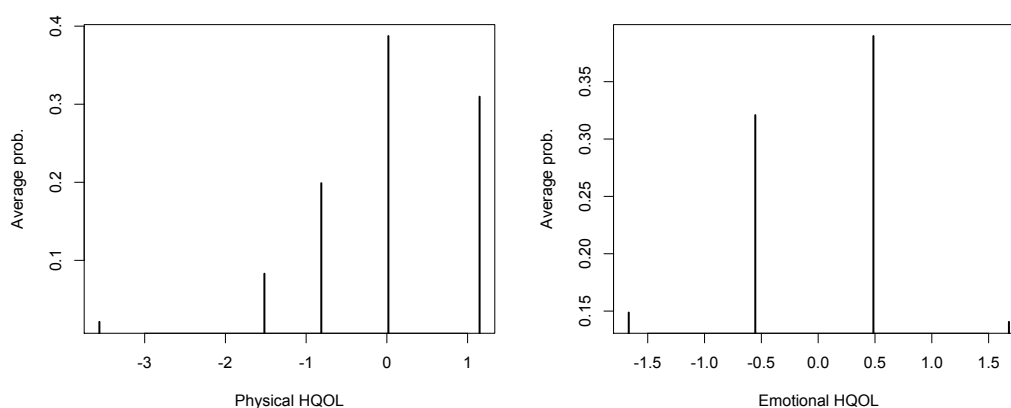
```
# Standardized support points of physical HQOL
lv1s <- rbind(out$Th1s, t(out$piv1s))
rownames(lv1s) <- c("Stand. Physical HQOL", "Average prob.")
round(lv1s, 3)
```

	5	4	1	2	3
Stand. Physical HQOL	-3.561	-1.517	-0.812	0.019	1.149
Average prob.	0.021	0.083	0.199	0.387	0.310

```
# Standardized support points of emotional HQOL
lv2s <- rbind(out$Th2s, t(out$piv2s))
rownames(lv2s) <- c("Stand. Emotional HQOL", "Average prob.")
round(lv2s, 3)
```

	1	4	3	2
Stand. Emotional HQOL	-1.667	-0.553	0.486	1.679
Average prob.	0.149	0.321	0.390	0.141

We observe that classes 5, 4, and 1 of physical HQOL and classes 1 and 4 of emotional HQOL collect patients with negative levels of the related latent trait, whereas patients with levels of HQOL above the mean belong to the remaining classes (i.e., classes 2 and 3 for physical HQOL and classes 3 and 2 for emotional HQOL). We also observe that the distribution of physical HQOL is strongly skewed to negative values, whereas that of emotional HQOL is symmetric (Figure 2).



**Figure 2:** Standardized mass probability distribution of physical (left panel) and emotional (right panel) HQOL.

As concerns item parameters, the discriminating indices and the corresponding standard errors and confidence intervals are displayed as follows:

```
# Item discriminating parameters and related standard errors
gamma1 <- cbind(out$galc, out$segalc)
colnames(gamma1) <- c("gamma1", "st.err.")
round(gamma1, 3)
```

	gamma1	st.err.
[1,]	1.000	0.000
[2,]	1.988	0.329
[3,]	1.193	0.199
[4,]	3.519	0.570
[5,]	3.394	0.582
[6,]	NA	NA
[7,]	NA	NA
[8,]	1.579	0.255

```
[9,] -0.006  0.117
[10,]  0.732  0.151
[11,]  0.145  0.125
[12,]  1.002  0.184
```

```
gamma2 <-cbind(out$ga2c, out$sega2c)
colnames(gamma2) <- c("gamma2", "st.err.")
round(gamma2, 3)
```

```
      gamma2 st.err.
[1,]  0.212  0.042
[2,]    NA    NA
[3,]    NA    NA
[4,]    NA    NA
[5,]    NA    NA
[6,]  1.000  0.000
[7,]  0.809  0.076
[8,]  0.128  0.037
[9,]  0.599  0.079
[10,] 0.429  0.059
[11,] 0.793  0.093
[12,] 0.606  0.074
```

```
# Confidence intervals at 95% of item discriminating parameters (columns 1-4)
confint(out)
```

```
[...] Output omitted
```

```
Confidence interval for the item parameters:
```

	gamma1_1	gamma1_2	gamma2_1	gamma2_2	beta1_1	beta1_2	beta2_1	beta2_2	beta3_1	beta3_2	beta4_1	beta4_2
1	1.0000	1.0000	0.1293	0.2939	0.0000	0.0000	2.2927	3.0763	5.4703	6.6560	7.0976	8.9132
2	1.3425	2.6326	NA	NA	-0.2428	1.8308	3.6945	6.1378	NA	NA	NA	NA
3	0.8036	1.5825	NA	NA	-1.4885	-0.0974	1.1539	2.5872	NA	NA	NA	NA
4	2.4010	4.6368	NA	NA	-1.3097	2.1941	1.2833	4.9066	5.2791	9.4358	8.5939	13.4278
5	2.2542	4.5348	NA	NA	-0.8427	2.5507	1.9437	5.5126	5.3510	9.4330	8.2434	12.8277
6	NA	NA	1.0000	1.0000	0.0000	0.0000	2.4866	3.9477	5.4657	7.6420	8.4771	11.2498
7	NA	NA	0.6612	0.9574	-0.4397	0.7189	1.7730	3.0777	4.1602	5.8143	7.1608	9.3353
8	1.0790	2.0796	0.0556	0.2010	-2.8325	-0.8067	0.3711	2.0693	3.0574	4.9355	4.5989	6.6225
9	-0.2348	0.2235	0.4451	0.7529	-1.9089	-0.6523	1.0767	2.2309	3.5534	5.0154	6.9565	9.1180
10	0.4366	1.0274	0.3135	0.5451	-0.7823	0.3940	1.9615	3.1901	4.6207	6.1860	7.3341	9.3617
11	-0.0999	0.3905	0.6100	0.9762	-1.1600	0.1005	1.6032	2.9576	4.8700	6.7528	7.4612	9.8398
12	0.6411	1.3637	0.4611	0.7503	-1.5644	0.0124	1.4481	2.8845	4.6337	6.4619	6.9167	9.0782

```
[...] Output omitted
```

In the previous output, the first two tables show the estimates of item discriminating parameters  $\hat{\gamma}_{1j}$  and  $\hat{\gamma}_{2j}$ , respectively, and the related standard errors, whereas the last table shows the corresponding inferior and superior limits of the confidence intervals at 95% level. Entries denoted by NA refer to those items that do not load on the corresponding latent variable.

We observe that discriminating parameters are generally highly significant, with two notable exceptions: discriminating indices for items 9 and 11 referred to physical HQOL (i.e., parameters  $\gamma_{19}$  and  $\gamma_{1,11}$ ). This result suggests that these two items do not contribute in a significant way to the measurement of physical HQOL.

The standardized estimates of parameters  $\hat{\gamma}_{1j}^*$  and  $\hat{\gamma}_{2j}^*$  are provided by

```
# Standardized discriminating parameters
gammas <- rbind(out$galcs, out$ga2cs)
rownames(gammas) <- c("Physical HQOL", "Emotional HQOL")
round(gammas, 3)
```

```
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12]
Physical HQOL 1.036 2.059 1.236 3.645 3.516    NA    NA 1.636 -0.006 0.758 0.151 1.038
Emotional HQOL 0.699    NA    NA    NA    NA 3.304 2.674 0.424 1.979 1.419 2.621 2.001
```

Note that the judgment about general health (item 1) is affected by both components of HQOL, but the physical dimension has a more relevant role ( $\hat{\gamma}_{11}^* = 1.036$  vs  $\hat{\gamma}_{21}^* = 0.699$ ). On the contrary, the self-evaluation of the consequences of physical or emotional health on social activities (item 12) is mainly affected by the emotional component of HQOL ( $\hat{\gamma}_{1,12}^* = 1.038$  vs  $\hat{\gamma}_{2,12}^* = 2.001$ ).



We also highlight relevant differences among items in terms of difficulty, as results by the related standardized item parameters:

```
# Standardized difficulty parameters
round(out$Becs, 3)
```

```
      cutoff
item      1      2      3      4
 1 -2.929 -0.245  3.134  5.076
 2 -2.514  1.608    NA    NA
 3 -2.778 -0.115    NA    NA
 4 -5.414 -2.761  1.501  5.155
 5 -4.795 -1.921  1.743  4.887
 6 -5.979 -2.762  0.575  3.884
 7 -4.700 -2.414  0.148  3.409
 8 -5.215 -2.175  0.601  2.215
 9 -4.853 -1.918  0.712  4.465
10 -3.979 -1.209  1.618  4.563
11 -5.514 -2.704  0.827  3.667
12 -6.066 -3.124  0.258  2.707
```

In particular, positive responses to items related only (items 4 and 5) or mainly (item 1) to physical HQOL require a level of the latent trait in average higher than items related only (items 6 and 7) or mainly (items 9, 11, and 12) to emotional HQOL. In other words, the emotional component of HQOL interferes less than physical component of HQOL on work, daily activities, and social life.

In the estimated model we assume an effect of patient's age on the HQOL, according to the multinomial logit parametrization, as in equation (3). Estimates of regression coefficients  $\delta_{1h_1}$  and  $\delta_{2h_2}$ , with  $h_1 = 2, 3, 4, 5$  and  $h_2 = 2, 3, 4$ , and related standard errors, are provided by:

```
# Effect of covariate age on physical HQOL
De1 <- cbind(out$De1, out$seDe1)
colnames(De1) <- c("delta12", "delta13", "delta14", "delta15", "se(delta12)",
                  "se(delta13)", "se(delta14)", "se(delta15)")
round(De1, 3)
```

```
      delta12 delta13 delta14 delta15 se(delta12) se(delta13) se(delta14) se(delta15)
intercept  2.960   3.021  -0.997  -4.773      1.035      0.937      1.862      2.453
X1         -0.038  -0.043   0.002   0.039      0.017      0.015      0.026      0.035
```

```
# Effect of covariate age on emotional HQOL
De2 <- cbind(out$De2, out$seDe2)
colnames(De2) <- c("delta22", "delta23", "delta24", "se(delta22)", "se(delta23)",
                  "se(delta24)")
round(De2, 3)
```

```
      delta22 delta23 delta24 se(delta22) se(delta23) se(delta24)
intercept  0.743   2.581   1.905      0.951      0.791      0.850
X1         -0.013  -0.027  -0.019      0.015      0.013      0.014
```

```
# Confidence intervals at 95% of regression coeffic. for physical and emotional HQOL
confint(out)
```

[...] Output omitted

Confidence interval for the regression coefficients for the 1st latent variable:

```
      logit
      2_1    2_2    3_1    3_2    4_1    4_2    5_1    5_2
intercept  0.9301  4.9890  1.1833  4.8581 -4.6451  2.6520 -9.5808  0.0355
X1         -0.0709 -0.0055 -0.0717 -0.0146 -0.0489  0.0528 -0.0295  0.1075
```

Confidence interval for the regression coefficients for the 2nd latent variable:

```
      logit
      2_1    2_2    3_1    3_2    4_1    4_2
intercept -1.1204  2.606  1.0306  4.1313  0.2383  3.5707
X1         -0.0434  0.017 -0.0521 -0.0021 -0.0458  0.0081
```

We observe a negative, although not particularly significant, effect of age on the probability to belong to a specific latent class with respect to the worst one (i.e., class 1), for both components of HQOL. A more parsimonious and more easily interpretable solution is provided by specifying a global logit parametrization through option `glob = TRUE` of the function `est_multi_poly_within`, as follows:

```
# Model with global logit parametrization for covariates
outgl <- est_multi_poly_within(S = S, k1 = 5, k2 = 4, X = X,
  multi1 = multi2_dim1, multi2 = multi2_dim2, link = "global",
  disc = TRUE, fort = TRUE, tol = 10^-8, disp = TRUE, output = TRUE,
  out_se = TRUE, glob = TRUE)

# Effect of covariate age on physical HQOL
De1glob <- cbind(outgl$De1, outgl$seDe1)
colnames(De1glob) <- c("coef", "se")
round(De1glob, 3)

      coef    se
cutoff1  5.093 0.544
cutoff2  2.827 0.467
cutoff3  1.050 0.443
cutoff4 -0.777 0.477
X1       -0.028 0.007

# Effect of covariate age on emotional HQOL
De2glob <- cbind(outgl$De2, outgl$seDe2)
colnames(De2glob) <- c("coef", "se")
round(De2glob, 3)

      coef    se
cutoff1  2.306 0.463
cutoff2  0.664 0.453
cutoff3 -1.312 0.461
X1       -0.010 0.007

# Confidence intervals at 95% of regression coeffic. for physical and emotional HQOL
confint(outgl)

[...] Output omitted

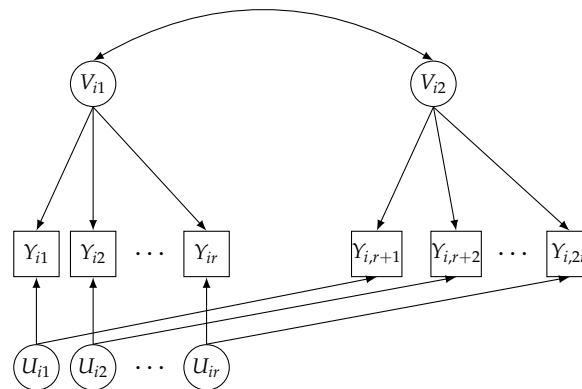
Confidence interval for the regression coefficients for the 1st latent variable:
      logit
      _1    _2
cutoff1  4.0265  6.1586
cutoff2  1.9112  3.7427
cutoff3  0.1816  1.9187
cutoff4 -1.7110  0.1575
X1       -0.0422 -0.0132

Confidence interval for the regression coefficients for the 2nd latent variable:
      logit
      _1    _2
cutoff1  1.3987  3.2124
cutoff2 -0.2229  1.5509
cutoff3 -2.2153 -0.4088
X1       -0.0240  0.0049
```

In such a case just one regression coefficient for each latent variable is estimated instead of  $k_1 - 1$  or  $k_2 - 1$ , denoting a negative effect of patient's age on the logit to belong to a specific class or higher with respect to a lower class.

### Example 2: analysis of repeated item responses

An interesting example of a real problem that can be solved through the discrete two-tier models is encountered when a questionnaire is used on multiple waves to measure the same latent variable at each time occasion. In such a situation, following Cai (2010) we distinguish a “specific” dimension  $U_{id_1}$  for each item within the questionnaire to capture the dependence between an individual’s responses to the same item  $d_1$  ( $d_1 = 1, \dots, D_1, D_1 = r$ ) at the different occasions, and we introduce one “primary” dimension  $V_{id_2}$  for each wave  $d_2$  ( $d_2 = 1, \dots, D_2$ ), representing the latent variable of interest at a given occasion. Through a suitable choice of identifiability and equality constraints on the item parameters, it is possible to estimate the support points of  $V_i$  and, then, obtain a class-specific time trajectory of the latent variable measured by the questionnaire. The resulting multidimensional structure is illustrated in Figure 3 for the case  $D_2 = 2$ .



**Figure 3:** Path diagram of the discrete two-tier model for longitudinal data for  $r$  items and  $D_2 = 2$  waves.

To illustrate specification and estimation of the discrete two-tier model for longitudinal data we refer to a simulated data set about 10 different types of crime committed by a cohort of 10,000 hypothetical subjects on 6 waves; each crime corresponds to a binary item, which assumes value 1 if the crime was committed and 0 otherwise. The latent variable of interest denotes the propensity to commit a crime at each time occasion. The data set is available from the R package **LMest** (Bartolucci and Pandolfi, 2016).

```
library(MLCIRTwithin)
```

```
### Load and prepare data
```

```
library(LMest)
```

```
data(data_criminal_sim)
```

```
data_criminal_sim[1:12,]
```

```

      id sex time y1 y2 y3 y4 y5 y6 y7 y8 y9 y10
[1,]  1  1   1   0  0  0  0  0  0  0  0  0  0
[2,]  1  1   2   0  0  0  0  0  0  0  0  0  0
[3,]  1  1   3   0  0  0  0  0  0  0  0  0  0
[4,]  1  1   4   0  0  0  0  0  0  0  0  0  0
[5,]  1  1   5   0  0  0  0  0  0  0  0  0  0
[6,]  1  1   6   0  0  0  0  0  0  0  0  0  0
[7,]  2  1   1   0  0  0  0  0  0  0  0  0  0
[8,]  2  1   2   0  0  0  0  0  0  0  0  0  0
[9,]  2  1   3   0  0  0  0  0  0  0  0  0  0
[10,] 2  1   4   0  0  0  0  0  0  0  0  0  0
[11,] 2  1   5   0  0  0  0  0  0  0  0  0  0
[12,] 2  1   6   0  0  0  0  0  0  0  0  0  0

```

In order to simplify the illustration of the application, we keep the first two waves and five types of crime, as follows:

```
# Keep items: y1,y3,y5,y7,y10; keep occasions: 1, 2
```

```
criminal_red <- data_criminal_sim[(data_criminal_sim[,3]==1 | data_criminal_sim[,3]==2),
                                   c(1:3,4,6,8,10,13)]
```

Moreover, in order to estimate the model at issue through the R package **MLCIRTwithin**, we need to reshape the data from “long” to “wide” format; we also aggregate records corresponding to the same patterns. Note that, after reshaping the data set, the total amount of items is 10, that is, 5 items observed at time 1 ( $j = 1, \dots, 5$ ) and the same 5 items observed again at time 2 ( $j = 6, \dots, 10$ ).

```
# Data reshape in wide format
criminal_red <- data.frame(criminal_red)
crim_wide <- reshape(criminal_red, v.names = c("y1", "y3", "y5", "y7", "y10"),
                      timevar = "time", idvar = "id", direction = "wide")

head(crim_wide)

   id sex y1.1 y3.1 y5.1 y7.1 y10.1 y1.2 y3.2 y5.2 y7.2 y10.2
1   1  1    0    0    0    0    0    0    0    0    0    0
3   2  1    0    0    0    0    0    0    0    0    0    0
5   3  1    0    0    0    0    0    0    0    0    0    0
7   4  1    0    0    0    0    0    0    0    0    0    0
9   5  1    0    0    0    0    0    0    0    0    0    0
11  6  1    0    0    0    0    0    0    0    0    0    0
dim(crim_wide)
[1] 10000    12

# Aggregate records with the same pattern
crim_wide <- as.matrix(crim_wide)
crim_wide2 <- aggr_data(crim_wide[, -1])

# Item responses, covariates, and vector of weights
S <- crim_wide2$data_dis[, -1]
X <- crim_wide2$data_dis[, 1]; X <- X - 1
yv <- crim_wide2$freq
```

The multidimensional structure is defined through the specification of matrices `multi1` and `multi2`, with `multi1` referring to specific dimensions capturing the dependence between responses to the same item at the two different waves, whereas `multi2` refers to the propensity to commit a crime at time 1 and at time 2. Then,  $\mathbf{U}_i = (U_{i1}, \dots, U_{i5})'$  and  $\mathbf{V}_i = (V_{i1}, V_{i2})'$ ; we also assume  $k_1 = k_2 = 2$ .

```
### Define the multidimensional structure
multi1 <- matrix(0, nrow=5, ncol=2)
multi2 <- matrix(0, nrow=2, ncol=5)
multi1[1,] <- c(6, 1)
multi1[2,] <- c(2, 7)
multi1[3,] <- c(3, 8)
multi1[4,] <- c(4, 9)
multi1[5,] <- c(5, 10)
multi2[1,] <- c(1:5)
multi2[2,] <- c(7, 6, 8:10)
multi1
      [,1] [,2]
[1,]    6    1
[2,]    2    7
[3,]    3    8
[4,]    4    9
[5,]    5   10
multi2
      [,1] [,2] [,3] [,4] [,5]
[1,]    1    2    3    4    5
[2,]    7    6    8    9   10

# Number of latent classes
k1 <- 2
k2 <- 2
```

The next step of the model specification consists in fixing some support points to identify the model and adding equality constraints on difficulties and discriminating indices to properly account for the longitudinal data structure, similarly to Cai (2010). In more detail, we fix the support points of the latent variable  $\mathbf{U}_i$ , that is,  $\mathbf{u} = (-1, -1, -1, -1, -1, 1, 1, 1, 1, 1)'$ , and on the first dimension of the latent variable  $\mathbf{V}_i$ , whereas the support points on the second dimension of  $\mathbf{V}_i$  are freely estimated, that is,

$v = (-1, v_{12}, 1, v_{22})'$ : in such a way, we estimate how the propensity to commit a crime changes over the time. Moreover, we constrain difficulty and discriminating parameters of each item at time 1 to be equal to the parameters of the same item at time 2, that is,  $\beta_j = \beta_{5+j}$ ,  $\gamma_{1j} = \gamma_{1,5+j}$ ,  $\gamma_{2j} = \gamma_{2,5+j}$ , with  $j = 1, \dots, 5$ .

```
### Specification of model constraints
# Fix support points on latent variable U
# Fix support points on the first dimension of latent variable V
# Free support points on the second dimension of latent variable V

(Zth1 <- matrix(0, nrow(multi1)*k1, 0))

[1,]
[2,]
[3,]
[4,]
[5,]
[6,]
[7,]
[8,]
[9,]
[10,]

(zth1 <- c(rep(-1, times = nrow(multi1)), rep(1, times = nrow(multi1))))
[1] -1 -1 -1 -1 -1 1 1 1 1 1

(Zth2 <- diag(nrow(multi2)*k2)[ , -c(1,3)])
      [,1] [,2]
[1,]    0    0
[2,]    1    0
[3,]    0    0
[4,]    0    1

(zth2 <- c(-1, 0, 1, 0))
[1] -1 0 1 0

# Equality constraints on difficulties and discriminating indices to account
# for the longitudinal data structure

(Zbe <- matrix(1, nrow(multi2), 1) %x% diag(nrow(multi1)))
      [,1] [,2] [,3] [,4] [,5]
[1,]    1    0    0    0    0
[2,]    0    1    0    0    0
[3,]    0    0    1    0    0
[4,]    0    0    0    1    0
[5,]    0    0    0    0    1
[6,]    1    0    0    0    0
[7,]    0    1    0    0    0
[8,]    0    0    1    0    0
[9,]    0    0    0    1    0
[10,]   0    0    0    0    1
Zga2 <- Zbe
Zga1 <- Zbe
```

The final step consists in estimating the model according to the structure specified above. The main difference with respect to the model presented in the previous section (Example 1) is that now all items are shared by both the latent variables  $U_i$  and  $V_i$  and suitable constraints on the model parameters are introduced through arguments  $Zth1$ ,  $zth1$ ,  $Zth2$ ,  $zth2$ ,  $Zbe$ ,  $Zga1$ , and  $Zga2$ , according to equations (7)-(11).

```
out <- est_multi_poly_within(S = S, yv = yv, k1 = k1, k2 = k2, X = X, link = "global",
  disc = TRUE, multi1 = multi1, multi2 = multi2, disp = TRUE,
  output = TRUE, out_se = TRUE, Zth1 = Zth1, zth1 = zth1, Zth2 = Zth2,
  zth2 = zth2, Zbe = Zbe, Zga1 = Zga1, Zga2 = Zga2)
```

Estimates of support points and item parameters may be displayed through the usual methods



```

summary, coef, and confint:

### Display output
# summary(out)
# coef(out)
# confint(out)

# Support points and average weights of latent variable V
out$Th2

      class
dimension 1      2
      1 -1.0000000 1.000000
      2 -0.6925198 1.679367

out$piv2

      [,1]
[1,] 0.91757919
[2,] 0.08242081

# Conditional response probabilities for latent variable V
# Note that the latent classes are numbered following an increasing order,
# starting from latent variable U
round(out$Phi[, , 3:4], 3)

, , class = 3

      item
category 1      2      3      4      5      6      7      8      9     10
      0 0.994 0.982 0.944 0.986 0.998 0.991 0.974 0.919 0.981 0.997
      1 0.006 0.018 0.056 0.014 0.002 0.009 0.026 0.081 0.019 0.003

, , class = 4

      item
category 1      2      3      4      5      6      7      8      9     10
      0 0.897 0.799 0.551 0.874 0.882 0.758 0.618 0.335 0.758 0.628
      1 0.103 0.201 0.449 0.126 0.118 0.242 0.382 0.665 0.242 0.372

```

We observe that 91.8% of the individuals belong to class 1, which is characterized by the smallest propensity to commit a crime, whereas the remaining 8.24% of individuals are allocated in class 2. Both classes present a tendency to increase the propensity to commit a crime from time 1 to time 2. Indeed, the estimated support points of the propensity to commit a crime at time 2 (i.e., estimates of  $v_{12}$  and  $v_{22}$ ) are higher than the corresponding values at time 1, for both the latent classes ( $-0.693$  vs  $-1$  for class 1 and  $1.679$  vs  $1$  for class 2). Moreover, the conditional probabilities of observing a given crime is higher for items observed at time 2, that is, items 6 to 10, with respect to the same items observed at time 1, that is, items 1 to 5, mainly in the case of latent class 2. For instance, for an individual belonging to class 2 the conditional probability of observing a crime of type 1 equals 10.3% at time 1 and it increases to 24.2% at time 2.

Finally, we outline that the constraints specified through matrices  $Z_{be}$ ,  $Z_{ga1}$ , and  $Z_{ga2}$  result in the following estimates, which are equal for every pair of items referring to the same type of crime (i.e., items 1-6, 2-7, 3-8, 4-9, 5-10):

```

# Estimates of item parameters
beta <- cbind(out$Bec[1:5], out$Bec[6:10])
colnames(beta) <- c("time 1", "time 2")
gamma1 <- cbind(out$ga1c[1:5], out$ga1c[6:10])
colnames(gamma1) <- c("time 1", "time 2")
gamma2 <- cbind(out$ga2c[1:5], out$ga2c[6:10])
colnames(gamma2) <- c("time 1", "time 2")

beta
      time 1  time 2
[1,] 4.994018 4.994018

```

```
[2,] 4.891019 4.891019
[3,] 2.700745 2.700745
[4,] 5.991825 5.991825
[5,] 5.544927 5.544927
```

```
gamma1
      time 1   time 2
[1,] 1.318531 1.318531
[2,] 2.195804 2.195804
[3,] 1.184268 1.184268
[4,] 2.883623 2.883623
[5,] 1.353772 1.353772
```

```
gamma2
      time 1   time 2
[1,] 1.510292 1.510292
[2,] 1.317424 1.317424
[3,] 1.311341 1.311341
[4,] 1.169848 1.169848
[5,] 2.183823 2.183823
```

## Summary

In this paper we illustrate the R package **MLCIRTwithin**, whose main function `est_multi_poly_within` implements an Expectation Maximization based approach to estimate the parameters of two-tier latent class item response theory (IRT) models. This class of models, which extends in a flexible way the class of basic IRT models, is based on two independent vectors of latent variables. Moreover, two main assumptions hold: (i) items are allowed to measure one or two latent variables (within-item multidimensionality) and (ii) latent variables are assumed to have a discrete distribution with a finite number of support points, which identify homogeneous latent classes of individuals, and related mass probabilities that may depend on individual covariates.

We illustrate the R package through two examples based on data about the measurement of health-related quality of life in cancer patients (Example 1) and about the measurement of the propensity to commit a crime in two time occasions (Example 2). The first example investigates the multidimensional structure of the questionnaire and is focused on the interpretation of the estimated model parameters. The second example illustrates how to treat longitudinal item responses through the specification of suitable constraints on support points and item parameters.

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