

zoib: An R package for Bayesian Inference for Beta Regression and Zero/One Inflated Beta Regression

by Fang Liu and Yunchuan Kong

Abstract The beta distribution is a versatile function that accommodates a broad range of probability distribution shapes. Beta regression based on the beta distribution can be used to model a response variable y that takes values in open unit interval $(0, 1)$. Zero/one inflated beta (ZOIB) regression models can be applied when y takes values from closed unit interval $[0, 1]$. The ZOIB model is based a piecewise distribution that accounts for the probability mass at 0 and 1, in addition to the probability density within $(0, 1)$. This paper introduces an R package – **zoib** that provides Bayesian inferences for a class of ZOIB models. The statistical methodology underlying the **zoib** package is discussed, the functions covered by the package are outlined, and the usage of the package is illustrated with three examples of different data and model types. The package is comprehensive and versatile in that it can model data with or without inflation at 0 or 1, accommodate clustered and correlated data via latent variables, perform penalized regression as needed, and allow for model comparison via the computation of the DIC criterion.

Introduction

The beta distribution has two shape parameters α_1 and α_2 : $\text{Beta}(\alpha_1, \alpha_2)$. The mean and variance of a variable y that follows the beta distribution are $E(y) = \mu = \alpha_1(\alpha_1 + \alpha_2)^{-1}$ and $V(y) = \mu(1 - \mu)(\alpha_1 + \alpha_2 + 1)^{-1}$, respectively. A broad spectrum of distribution shapes can be generated by varying the two shapes values of α_1 and α_2 , as demonstrated in Figure 1. The beta regression has become more popular in recent years in modeling data bounded within open interval $(0, 1)$ such as rates and proportions, and more generally, data bounded within (a, b) as long as a and b are fixed and known and it is sensible to transform the raw data onto the scale of $(0, 1)$ by shifting and scaling, that is, $y' = (y - a)(b - a)^{-1}$.

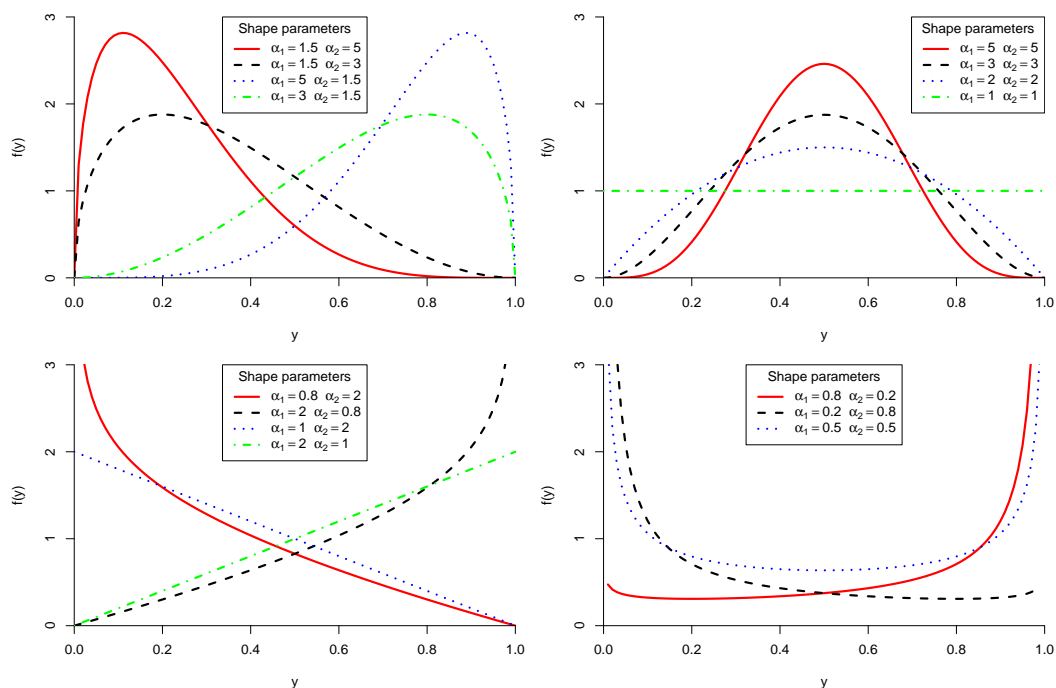


Figure 1: Beta distribution with various values of the two shape parameters.

Given the flexibility and increasing popularity of the beta regression, significant development has been made in the theory, methodology, and practical applications of the beta regression (Cepeda-Cuervo, 2001; Paolino, 2001; Williams, 1982; Prentice, 1986; Ferrari and Cribari-Neto, 2004; Smithson and Verkuilen, 2006; Simas et al., 2010; Smithson and Verkuilen, 2006; Hatfield et al., 2012;

Ospina and Ferrari, 2012; Cepeda-Cuervo, 2015). Mostly recently, Grün et al. (2012) apply the techniques of the model-based recursive partitioning (Zeileis et al., 2008) and the finite mixture model (Dalrymple et al., 2003) in the framework of beta regression to account for heterogeneity between groups/clusters of observations. They also propose bias-corrected or bias-reduced estimation in the beta regression by applying the unifying iteration technique (Kosmidis and Firth, 2010).

In many cases of real life data, exact 0's and 1's occur in addition to y values between 0 and 1, producing zero-inflated, one-inflated, or zero/one-inflated outcomes. Though the beta distribution covers a variety of the distribution shape, it does not accommodate excessive values at 0 and 1. Smithson and Verkuilen (2006) propose transformation $n^{-1}(y(n-1) + 0.5)$, where n is the sample size, so all data points after transformation are bounded within 0 and 1 and the regular beta regression can be applied. This approach, while offering a simple way to circumvent the complexity from modeling the boundary values, only shifts the excessiveness in point mass from one location to another. Hatfield et al. (2012) model the zero/one inflated VAS responses by relocating all 1 to 0.9995 and keep 0 as is, and apply the zero-inflated beta (ZIB) regression. The approach of only shifting 1 but not 0 when there is inflation at both is ad-hoc especially if there is no justification for treating 0 differently from 1. From a practical perspective, the observed 0's and 1's might carry practical meanings that would be otherwise lost if being replacing with other values, regardless how close the raw and substitutes values are. Ospina and Ferrari (2012) propose the zero-or-one inflated beta regression model (inflation at either 0 or 1, but not both) and obtain inferences via the maximum likelihood estimation (MLE). When there is inflation at both 0 and 1, it is sensible to model the excessiveness explicitly with the zero/one inflated beta (ZOIB) regression, especially when population 0's and 1's are real. For example, if the response variable is the death proportion of mice on different doses of a chemical entity; the death rate caused by administration of the chemical entity theoretically can be 0 when its dosage is 0, and 1 when the dosage increases to a 100% lethal level. The ZOIB regression technique has been previously discussed in the literature (Swearingen et al., 2012). Most beta regression and zoib models focus on fixed effects models only, and thus cannot handle clustered or repeated measurements. Liu and Li (2014) apply a joint model with latent variables to model the dependency structure among multiple $[0, 1]$ -bounded responses with repeated measures in the Bayesian framework.

From a software perspective, beta regression can be implemented in a software suite or package that accommodate nonlinear regression models, such as SPSS (NLR and CNLR) and SAS (PROC NLIN, PROC NLMIXED). There are also contributed packages or macros devoted specifically to beta regression, such as the SAS macro developed by Swearingen et al. (2011), which implements the beta regression directly and provides residuals plots for model fit diagnostics. In R, there are a couple of packages targeted specifically at beta regression. **betareg** (Zeileis et al., 2014) models a single response variable bounded within $(0, 1)$, with fixed-effects linear predictors in the link functions for the mean and precision parameter of the beta distribution (Cribari-Neto and Zeileis, 2010). The package is later updated by Grün et al. (2012) to perform bias correction/reduction, model-based recursive partitioning, and finite mixture models with added functions `betatree()` and `betamix()` in package **betareg**. In **betareg**, the coefficients of the regression are estimated by the MLE and inferences are based on large sample assumptions. **Bayesianbetareg** (Marin et al., 2014) allows the joint modelling of mean and precision of a single response in the Bayesian framework, as is proposed in Cepeda-Cuervo (2001), with logit link for the mean and logarithmic for the precision. Neither **betareg** nor **Bayesianbetareg** accommodate inflation at 0 or 1 (**betareg** transforms y with inflation at 0 and 1 using $(y(n-1) + 0.5)n^{-1}$); neither can model multiple response variables, repeated measures, or clustered/correlated response variables. In other words, the linear predictors in the link functions of the mean and precision parameters of the beta distribution in both **betareg** and **Bayesianbetareg** contain fixed effects only.

In this discussion, we introduce a new R package **zoib** (Liu and Kong, 2014) that models responses bounded within $[0, 1]$ – without inflation at 0 nor 1, with inflation at 0 only, at 1 only, or at both 0 and 1. The package can model a single response with or without repeated measures, or multiple or clustered $[0, 1]$ -bounded response variables, taking into account the dependency among them. Compared to the existing packages on beta regression in R, **zoib** is more comprehensive and flexible from the modeling perspective and can accommodate more data types. The inferences of the model parameters in package **zoib** are obtained in the Bayesian framework via the Markov Chain Monte Carlo (MCMC) approach as implemented in JAGS (Plummer, 2014a).

The rest of the paper is organized as follows. Section **Zero/one inflated beta regression** describes the methodology underlying the ZOIB regression. Section **Implementation in R** introduces the package **zoib**, including its functionality and outputs. Section **Examples** illustrates the usage of the package with 3 real-life data sets and 1 simulated data of different types. The paper ends in Section **Summary** with summaries and discussions.

Zero/one inflated beta regression

The ZOIB model

Suppose y_j is the j^{th} variable out of a total p response variables measured on n independent units, that is, $\mathbf{y}_j = (y_{1j}, \dots, y_{nj})^t$. The zoib model assumes y_{ij} follows a piecewise distribution when y_{ij} has inflation at both 0 and 1.

$$f(y_{ij}|\eta_{ij}) = \begin{cases} p_{ij} & \text{if } y_{ij} = 0 \\ (1 - p_{ij})q_{ij} & \text{if } y_{ij} = 1 \\ (1 - p_{ij})(1 - q_{ij})\text{Beta}(\alpha_{ij1}, \alpha_{ij2}) & \text{if } y_{ij} \in (0, 1). \end{cases} \quad (1)$$

p_{ij} is the probability of $y_{ij} = 0$, and q_{ij} is the conditional probability $\Pr(y_{ij} = 1|y_{ij} \neq 0)$, and α_{ij1} and α_{ij2} are the shape parameters of the beta distribution when $y_{ij} \in (0, 1)$. The probability parameters from the binomial distributions and the two shape parameters from the beta distributions are linked to observed explanatory variables \mathbf{x}_{ij} or unobserved latent variable \mathbf{z}_{ij} via link functions. Some natural choices for the link functions for p_{ij} , q_{ij} , and the mean of the beta distribution $\mu_{ij}^{(0,1)} = E(y_{ij}|y_{ij} \in (0, 1)) = \alpha_{ij1}(\alpha_{ij1} + \alpha_{ij2})^{-1}$, which are all parameters within $(0, 1)$, include the logit function, the probit function, or the complementary log-log (cloglog) function. While the binomial distribution is described by a single probability parameter, the beta distribution is characterized by two parameters. The variance of the beta distribution is not only a function of its mean but also the sum of two shape parameters $v_{ij} = \alpha_{ij1} + \alpha_{ij2}$; that is, $V(y_{ij}|y_{ij} \in (0, 1)) = \mu_{ij}^{(0,1)}(1 - \mu_{ij}^{(0,1)})(\alpha_{ij1} + \alpha_{ij2} + 1)^{-1} = \mu_{ij}^{(0,1)}(1 - \mu_{ij}^{(0,1)})(v_{ij} + 1)^{-1}$. v_{ij} is often referred to as the precision (dispersion) parameter and can also be affected by external explanatory variables or latent variables (Simas et al., 2010; Cribari-Neto and Zeileis, 2010). An example of the formulation of the zoib model, if the logit function is applied to p_{ij} , q_{ij} , and $\mu_{ij}^{(0,1)}$, and the log link function is applied to v_{ij} , is

$$\text{logit}(\mu_{ij}^{(0,1)}) = \mathbf{x}_{1,ij}\beta_{1j} + I_1(\mathbf{z}_{1,ij}\gamma_{1,i}) \quad (2)$$

$$\log(v_{ij}) = \mathbf{x}_{2,ij}\beta_{2j} + I_2(\mathbf{z}_{2,ij}\gamma_{2,i}) \quad (3)$$

$$\text{logit}(p_{ij}) = \mathbf{x}_{3,ij}\beta_{3j} + I_3(\mathbf{z}_{3,ij}\gamma_{3,i}) \quad (4)$$

$$\text{logit}(q_{ij}) = \mathbf{x}_{4,ij}\beta_{4j} + I_4(\mathbf{z}_{4,ij}\gamma_{4,i}), \quad (5)$$

where $\beta_{m,j}$ represents the linear fixed effects in link function m ($m = 1, 2, 3, 4$) for response j ($j = 1, \dots, p$); $\mathbf{x}_{m,ij}$ is the design matrix for the fixed effects; $I_m(\mathbf{z}_{m,ij}\gamma_{m,i})$ is an indicator function on whether link function m has a random component or not, that is, $I_m(\mathbf{z}_{m,ij}\gamma_{m,i}) = \mathbf{z}_{m,ij}\gamma_{m,i}$ if link function has a random component, $I_m(\mathbf{z}_{m,ij}\gamma_{m,i}) = 0$ otherwise. $\mathbf{z}_{m,i}$ represents the design matrix associated with the random components; Unless stated otherwise, we assume $\gamma_{m,i} \stackrel{\text{ind}}{\sim} N(\mathbf{0}, \Sigma_m)$ for $i = 1, \dots, n$ in link function m throughout this discussion. When $p \geq 2$, dependency among the p response variables are modeled through their sharing of $\gamma_{m,i}$ for each m .

When there are no random/latent components in the linear predictors, the mean of the beta distribution for $y_{ij} \in (0, 1)$ is given by $\exp(\mathbf{x}_{1,ij}\beta_{1j})(1 + \exp(\mathbf{x}_{1,ij}\beta_{1j}))^{-1}$ while $\exp(\mathbf{x}_{2,ij}\beta_{2j})$ is the sum of the two shape parameters. $\exp(\mathbf{x}_{3,ij}\beta_{3j})(1 + \exp(\mathbf{x}_{3,ij}\beta_{3j}))^{-1}$ is $\Pr(y_{ij} = 0)$, and $\exp(\mathbf{x}_{4,ij}\beta_{4j})(1 + \exp(\mathbf{x}_{4,ij}\beta_{4j}))^{-1}$ is $\Pr(y_{ij} = 1|y_{ij} > 0)$. The overall mean of y_{ij} is thus given by

$$\begin{aligned} E(y_{ij}) &= (1 - p_{ij}) \left(q_{ij} + (1 - q_{ij})\mu_{ij}^{(0,1)} \right) \\ &= \frac{\exp(\mathbf{x}_{1,ij}\beta_{1j})(1 + \exp(\mathbf{x}_{1,ij}\beta_{1j}))^{-1} + \exp(\mathbf{x}_{4,ij}\beta_{4j})}{(1 + \exp(\mathbf{x}_{3,ij}\beta_{3j}))(1 + \exp(\mathbf{x}_{4,ij}\beta_{4j}))} \end{aligned}$$

When there are random/latent components, then the conditional mean of y_{ij} given $\mathbf{z}_{m,i}$ is

$$\begin{aligned} E(y_{ij}|\gamma_{1,i}, \gamma_{2,i}, \gamma_{3,i}, \gamma_{4,i}) &= (1 - p_{ij}) \left(q_{ij} + (1 - q_{ij})\mu_{ij}^{(0,1)} \right) \\ &= \frac{\exp\{\mathbf{x}_{1,ij}\beta_{1j} + I_1(\mathbf{z}_{1,ij}\gamma_{1,i})\}(1 + \exp\{\mathbf{x}_{1,ij}\beta_{1j} + I_1(\mathbf{z}_{1,ij}\gamma_{1,i})\})^{-1} + \exp\{\mathbf{x}_{4,ij}\beta_{4j} + I_4(\mathbf{z}_{4,ij}\gamma_{4,i})\}}{(1 + \exp\{\mathbf{x}_{3,ij}\beta_{3j} + I_3(\mathbf{z}_{3,ij}\gamma_{3,i})\})(1 + \exp\{\mathbf{x}_{4,ij}\beta_{4j} + I_2(\mathbf{z}_{4,ij}\gamma_{4,i})\})} \end{aligned} \quad (6)$$

Calculation of the marginal mean of y_{ij} involves integrating out $\gamma_{m,i}$ over its distribution; that is, $E(y_{ij}) = \int E(y_{ij}|\gamma_{1,i}, \gamma_{2,i}, \gamma_{3,i}, \gamma_{4,i})f(\gamma_{1,i}|\Sigma_1)f(\gamma_{2,i}|\Sigma_2)f(\gamma_{3,i}|\Sigma_3)f(\gamma_{4,i}|\Sigma_4)d\gamma_{1,i}d\gamma_{2,i}d\gamma_{3,i}d\gamma_{4,i}$, which can become computationally and analytically tractable if the MLE approach is taken. In contrast, $E(y_{ij})$

is easier to obtain by the Monte Carlo approach in the Bayesian computational framework.

Equations (2) to (5) give a full parameterization of the ZOIB model. Various reduced forms of the fully parameterized model as given in equations (2) to (5) are available. For example, if a constant dispersion parameter is assumed, then equation (3) can be simplified $\log(v_{ij}) = c_j$ that differs only by response variable. In practice, it might also be reasonable to assume $\mathbf{z}_{m,ij}\gamma_{m,i}$ is the same across all links functions, that is, $\Sigma_m = \Sigma$, since information to distinguish among Σ_m 's is unlikely available in many real life applications.

Bayesian inference

Though the inferences of the parameters in the proposed ZOIB model can be obtained via the MLE approach, the task can be analytically and computationally challenging, considering the nonlinear nature of the model and existence of possible random effects. We adopt the Bayesian inferential approach in package **zoib**. Let $\Theta = \{\beta_1, \beta_2, \beta_3, \beta_4, \Sigma\}$ denote the set of the parameters from the ZOIB model (**zoib** sets $\gamma_{m,i} = \gamma_i$ and $\Sigma_m = \Sigma \forall m$). The joint posterior distribution of Θ and the random effects γ given data \mathbf{y} is $p(\Theta, \gamma | \mathbf{y}) \propto p(\mathbf{y} | \Theta, \gamma) p(\gamma | \Theta) p(\Theta)$. The likelihood $p(\mathbf{y} | \Theta, \gamma)$ is constructed from the ZOIB model in equation (1)

$$p(\mathbf{y} | \Theta, \gamma) \propto \prod_i \prod_j \left\{ p_{ij}^{I(y_{ij}=0)} (1 - p_{ij})^{I(y_{ij}=1)} q_{ij}^{I(y_{ij}=1)} \right\} \times \left\{ (1 - q_{ij}) \frac{\Gamma(v_{ij})}{\Gamma(v_{ij}\mu_{ij}^{(0,1)})\Gamma(v_{ij}(1 - \mu_{ij}^{(0,1)}))} (y_{ij})^{v_{ij}\mu_{ij}^{(0,1)} - 1} (1 - y_{ij})^{v_{ij}(1 - \mu_{ij}^{(0,1)}) - 1} \right\}^{I(y_{ij} \in (0,1))},$$

noting p_{ij}, q_{ij}, v_{ij} and $\mu_{ij}^{(0,1)}$ are functions of Θ , and $p(\gamma | \Theta) \sim N(0, \Sigma)$. **zoib** assumes all the parameters in Θ are *a priori* independent, thus $f(\Theta) = f(\Sigma) \prod_{j=1}^p \prod_{m=1}^4 f(\beta_{m,j})$. **zoib** offers the following prior choices on $\beta_{m,j}$:

- Diffuse normal (DN) on all intercept terms $\beta_{m,j0} \sim N(0, C)$, where C is the precision of the normal distribution that can be specified by users. The smaller C is, the more “diffuse” the normal distribution is (the less *a priori* information there is about $\beta_{m,j0}$). The default $C = 10^{-3}$.
- For the rest of elements in $\beta_{m,j}$ (minus the intercept term), there are 4 options:
 - diffuse normal (DN, default): $\beta_{m,jk} \stackrel{\text{ind}}{\sim} N(0, C)$ across $k = 1, \dots, p_m$ for a given j ($j = 1, \dots, q$) and m ($m = 1, \dots, 4$). C is the precision of the normal distribution that can be specified by users; the default $C = 10^{-3}$.
 - L2-prior (L2): The L2 prior shrinks the regression coefficients in the same link function m on the same variable y_j in a L_2 manner as in ridge regression (Lindley and Smith, 1972). The L2 prior helps when there is non-orthogonality among the covariates. $\beta_{m,jk} | \lambda_{m,jk} \stackrel{\text{ind}}{\sim} N(0, \lambda_{m,jk})$ for $k = 1, \dots, p_m$ and the precision parameter $\lambda_{m,j} \stackrel{\text{ind}}{\sim} \text{gamma}(\alpha, \beta)$ given j and m . α and β , the shape and scale parameters of the gamma distribution, are small constants that can be specified by the user. The default is $\alpha = \beta = 10^{-3}$ for all m and j .
 - L1-prior (L1): The L1 prior shrinks the regression coefficients in the same link function m on the same variable y_j in a L_1 manner (Lindley and Smith, 1972) as in Lasso regression (Park and Casella, 2008). As such, the L1-prior helps there is a large of covariates and sparsity in the regression coefficients is desirable. $\beta_{m,jk} | \lambda_{m,jk} \stackrel{\text{ind}}{\sim} N(0, \lambda_{m,jk})$ and $\lambda_{m,jk} \stackrel{\text{ind}}{\sim} \exp(\epsilon_{m,j})$ for $k = 1, \dots, p_m$ given j and m . $\epsilon_{m,j}$ is a small constant that can be specified by users. The default $\epsilon_{m,j} = 10^{-3}$ for all m and j .
 - automatic relevance determination (ARD): ARD, as the L2 and L1 priors, regularizes the regression coefficients toward sparsity. Different from the L2 prior, where every coefficient has the same precision parameter $\lambda_{m,j}$, the precision is coefficient-specific in the ARD prior (MacKay, 1996; Neal, 1994): $\beta_{m,jk} | \lambda_{m,jk} \stackrel{\text{ind}}{\sim} N(0, \lambda_{m,jk})$ and $\lambda_{m,jk} \stackrel{\text{ind}}{\sim} \text{gamma}(\alpha_{m,j}, \beta_{m,j})$ for $k = 1, \dots, p_m$ given j and m . $\alpha_{m,j}, \beta_{m,j}$ are small constants that can be specified by users. The default $\alpha_{m,j} = \beta_{m,j} = 10^{-3}$ for all m and j .

Regarding the random effects specification in **zoib**, it is assumed $z \sim N(0, \sigma^2)$ in the case of a single random variable z ; when there are multiple random variables z_1, \dots, z_L , it is assumed $\mathbf{z} \sim N(0, \Sigma)$. **zoib** offers two structures on Σ : variance components (VC) and unstructured (UN).

- In the VC case, Σ is diagonal, indicating all the random variables are independent. **zoib** offers two priors on σ_l , the standard deviation of z_l ($l = 1, \dots, L$): 1) $\sigma_l \sim \text{unif}(0, C)$, where C is a large constant that can be specified by users (default $C = 20$); 2) $\sigma_l \sim \text{half-Cauchy}(C)$, the half- t distribution with degree freedom equal to 1. Symbolically, $f(\sigma_j) \propto \left(1 + \sigma_j^2 C^{-2}\right)^{-1}$, where C is the scale parameter (Gelman, 2006) (default $C = 20$). The half-Cauchy distribution is the default in **zoib**.
- When Σ is fully parameterized with $L(L+1)/2$ parameters (the UN structure), we write $\Sigma = \text{Diag}(\sigma_l) \cdot \mathbf{R} \cdot \text{Diag}(\sigma_j)$, where \mathbf{R} is the correlation matrix. The priors for σ_l for $l = 1, \dots, L$ are the same as in the VC case. **zoib** supports L up to 3 in the UN structure. When $L = 2$, there is a single correlation parameter and a uniform prior is imposed $\rho \sim \text{unif}(0, 1)$. When $L = 3$, the uniform prior is imposed on two out of three correlation coefficients, say $\rho_{12} \sim \text{unif}(0, 1)$ and $\rho_{13} \sim \text{unif}(0, 1)$. In order to ensure positive definitiveness of \mathbf{R} , ρ_{23} has to be bounded within (L, U) , where $L = \rho_{12}\rho_{13} - \sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}$ and $U = \rho_{12}\rho_{13} + \sqrt{(1 - \rho_{12}^2)(1 - \rho_{13}^2)}$. The prior on ρ_{13} is thus specified as $\text{unif}(L, U)$.

All taken together, **zoib** offers 4 options on the prior for the covariance matrix Σ in the case of more than one random variables: VC.unif, VC.half, UN.unif, and UN.half.

Implementation in R

The joint distribution $f(\Theta, \gamma|y)$ in the zoib model is not available in closed form. We apply slice sampling (Neal, 2003), a Markov chain Monte Carlo (MCMC) method, to draw posterior samples on the parameters leveraging on the available software JAGS (Plummer, 2014a). Before using **zoib**, users need to download JAGS and the R package **rjags** that offers a connection between R and JAGS. The main function in **zoib** generates a JAGS model object, and the posterior samples on the model parameters, the observed y and their posterior predictive values, and the design matrices x_1, x_2, x_3 and x_4 , as applicable. Convergence diagnostics, mixing of the MCMC chains, summary of the posterior draws, and the deviance information criterion (DIC) (Spiegelhalter et al., 2002) of the model can be calculated using the functions already available in packages **coda** (Plummer et al., 2006) and **rjags** (Plummer, 2014b). Trace plots and auto-correlation plots can be generated, the Gelman-Rubin's potential scale reduction factor (psrf) (Gelman and Rubin, 1992) and multivariate psrf (Brooks and Gelman, 1998) can be computed. To check on the mixing and convergence of the Markov chains, multiple independent Markov chains should be run. More details on the output and functions of **zoib** are provided in Section Implementation in R below.

The package **zoib** contains 23 functions (Table 1). Users can call the main function `zoib()`, which produces a MCMC (JAGS) model object and posterior samples of model parameters as an MCMC object, among others. Convergence of the MCMC chains can be checked using the `traceplot(MCMC.object)`, `autocorr.plot(MCMC.object)` and `gelman.diag(MCMC.object)` functions provided by package **coda**. Posterior summary of the parameters can be obtained by function `summary()` if the posterior draws are in a format of a `mcmc.list`. The DIC of the proposed model can be calculated using function `dic.samples(JAGS.object)` available in **rjags** for model comparison purposes. Besides these existing functions, **zoib** provides an additional function `check.psr()` that checks whether multivariate psrf value can be calculated for multi-dimensional model parameters, provides box plots and summary statistics on multiple univariate psrf values, and the `paraplot()` function which provides a visual display on the posterior inferences on the model parameters. The remaining 20 functions are called internally by function `zoib()`. The main function `zoib()` is used as follows:

```
zoib(model, data, zero.inflation = TRUE, one.inflation = TRUE, joint = TRUE,
      random = 0, EUID, link.mu = "logit", link.x0 = "logit", link.x1 = "logit",
      prior.beta = rep("DN", 4), prec.int = 0.001, prec.DN = 0.001,
      lambda.L2 = 0.001, lambda.L1 = 0.001, lambda.ARD = 0.001,
      prior.Sigma = "VC.half", scale.unif = 20, scale.half = 20,
      n.chain = 2, n.iter = 5000, n.burn=200, n.thin = 2)
```

`data` represents the data set to be modeled. `model` presents a symbolic description of the zoib model in the format of formula responses $y \sim \text{covariates } x$. `zero.inflation` and `one.inflation` contain the information on whether the data has inflation at zero or one. `joint` specifies whether to model multiple response variables jointly or separately. `random = 0` indicates the ZOIB model has no random effects; `random = m` (for $m = 1, 2, 3, 4$) instructs **zoib** which linear predictor(s) out of the four (as given in equations (2) to (5)) have a random component. For example, if `random = 13`, then the linear predictors associated with the link function of the mean of the beta regression (1) and the probability of zero inflation (3) have a random component, while the linear predictors associated with the link function of the precision parameters of (2) and the probability one inflation (4) do not have a random

Function	Description
Functions called by users	
<code>zoib()</code>	main function; produces a MCMC (JAGS) model object and posterior samples of model parameters
<code>check.psr()</code>	checks whether the multivariate psrf value can be calculated for multi-dimensional parameters; provides a box plot and summary statistics for multiple univariate psrf values
<code>paraplot()</code>	plots the posterior mode, mean, or median with Bayesian credible intervals for the parameters from a zoib model.
Internal functions called by <code>zoib()</code>	
	<i>fixed-effect model</i>
<code>fixed()</code>	without y inflation at 0 or 1
<code>fixed0()</code>	with y inflation at 0 only
<code>fixed1()</code>	with y inflation at 1 only
<code>fixed01()</code>	with y inflation at both 0 and 1
	<i>joint modeling of ≥ 2 response variables when there is a single random variable</i>
<code>join.1z()</code>	without y inflation at 0 or 1
<code>join.1z0()</code>	with y inflation at 0 only
<code>join.1z1()</code>	with y inflation at 1 only
<code>join.1z01()</code>	at both 0 and 1
	<i>joint modeling of ≥ 2 response variables when there are ≥ 2 random variables</i>
<code>join.2z()</code>	without y inflation at 0 or 1
<code>join.2z0()</code>	with y inflation at 0 only
<code>join.2z1()</code>	with y inflation at 1 only
<code>join.2z01()</code>	with y inflation at both 0 and 1
	<i>separate modeling of ≥ 2 response variables when there is a single random variable</i>
<code>sep.1z()</code>	without y inflation at 0 or 1
<code>sep.1z0()</code>	with y inflation at 0 only
<code>sep.1z1()</code>	with y inflation at 1 only
<code>sep.1z01()</code>	at both 0 and 1. called by function <code>zoib()</code> .
	<i>separate modeling of ≥ 2 response variables when there are ≥ 2 random variables</i>
<code>sep.2z()</code>	without y inflation at 0 or 1
<code>sep.2z0()</code>	with y inflation at 0
<code>sep.2z1()</code>	with y inflation at 1 only
<code>sep.2z01()</code>	with y inflation at both 0 and 1

Table 1: Functions developed in package **zoib**.

component. Similarly, if `random = 124`, then the linear predictors associated with the mean (1) and precision parameters (2) of the beta distribution, and the probability of one inflation (4) have a random component, but the link function associated with the probability of zero inflation (4) does not. `random = 1234` would suggest all 4 linear predictors have random components. There are total $2^4 - 1 = 15$ possibilities to specify the random components and **zoib** supports all 15 possibilities.

The remaining arguments in function `zoib()` are necessary for Bayesian model formulation and computation, including the hyper-parameter specification in the prior distributions of the parameters in the ZOIB model (`prior.beta`, `prec.int`, `prec.DN`, `lambda.L2`, `lambda.L1`, `lamdda.ARD`, `scale.unif`, `scale.halft`, `prior.Sigma`), the number of Markov chains to run (`n.chain`), the number of MCMC iterations per chain (`n.iter`), and the burn-in period (`n.burn`) and thinning period (`n.thin`). In addition, the link functions that relate linear predictors to $\Pr(y = 0)$, $\Pr(y = 1)$, and $\mu^{(0,1)}$ can be chosen from `logit` (the default), `probit`, and `cloglog`. The link function that links a linear predictor to the sum of the two shape parameters of the beta distribution is the log function.

Table 2 lists the functions offered by packages **coda** and **rjags** that can be used to check the convergence of the MCMC chains from the ZOIB models, to compute the posterior summaries of the model parameters, and to calculate the penalized deviance of the converged models.

Examples

We apply the package **zoib** to three examples. In the first example **zoib** is applied to analyze the `GasolineYield` data available in R package **betareg**, to provide a comparison between the results obtained from the two packages. There is no inflation in either 0 or 1 in data `GasolineYield`. In

Function	Description
<code>traceplot()</code>	plots number of iterations vs. drawn values for each parameter in per Markov chain (from package coda)
<code>autocorr.plot()</code>	plots the autocorrelation for each parameter in each Markov chain (from package coda)
<code>gelman.diag()</code>	calculates the potential scale reduction factor (psrf) value for each variable drawn from at least two Markov chains, together with the upper and lower 95% confidence limits. When there are multiple variables, a multivariate psrf value is calculated (from package coda)
<code>dic.samples()</code>	extracts random samples of the penalized deviance from a jags model (from package rjags)
<code>summary()</code>	calculates posterior mean, standard deviation, 50%, 2.5% and 97.5% for each parameters using the posterior draws from Markov chains

Table 2: Existing functions for checking the convergence and mixing of the Markov chain of the ZOIB model and summarizing the posterior samples.

Example 2, **zoib** is applied to a simulated data with two correlated beta variables, where joint modeling of the variable is used with a single random variable. In Example 3, **zoib** is applied to a real life data on alcohol use in California teenagers. Example 3 is used to demonstrate how to model clustered beta variables via **zoib**. The data set in Example 3 can be downloaded from website <http://www.kidsdata.org>. In all three example, the ZOIB model is specified using the generic function `formula` in R. When there are multiple response variables, each variable should be separated by `|`, such as `y1|y2|y3` on the left hand side of the formula. On the right side of the formula, it can take up to 5 parts in the following order:

1. fixed-effect variables x_1 in the link function of the mean of the beta distribution;
2. fixed-effect variables x_2 in the link function of the precision parameter of the beta distribution;
3. fixed-effect variables x_3 in the link function of $\Pr(y = 0)$;
4. fixed-effect variables x_4 in the link function of $\Pr(y = 1)$; and
5. random-effects variables z .

x_1 and x_2 should always be specified, even if x_2 contains only an intercept (represented by 1). If there is no zero inflation in any of the y 's, then the x_3 part can be omitted, similarly with x_2 and the random component z . For example, if there are 3 response variables y_1, y_2, y_3 and 2 independent variables ($xx1, xx2$), and none of the y 's has zero inflation, then model $y_1 | y_2 | y_3 \sim xx1 + xx2 | 1 | xx1 | xx2$ implies $x_1 = (1, xx1, xx2)$, $x_2 = 1$ (intercept), $x_3 = \text{NULL}$, $x_4 = (1, xx1)$, $z = (1, xx2)$. If y_1 has inflation at zero, y_3 has inflation at one, and there is no random effect, model $y_1 | y_2 | y_3 \sim xx1 + xx2 | xx1 | xx1$ implies $x_1 = (1, xx1, xx2)$, $x_2 = (1, xx1)$, $x_3 = c(1, xx1)$ for y_1 , $x_4 = (1, xx1)$ for y_3 . The details on how to specify the model using `formula` can be found in the user manual of package **zoib**.

Example 1: univariate fixed-effect beta regression

According to the description in **betareg**, the GasolineYield data was collected by Prater (1956) and analyzed by Atkinson (1985). The data set contains 32 observations and 6 variables. The dependent variable is the proportion of crude oil after distillation and fractionation. There is no 0 or 1 inflation in y . **betareg** fits a beta regression model with all 32 observations and 2 independent variables: batch ID (1, ..., 10) corresponding to 10 different crudes that were subjected to experimentally controlled distillation conditions, and temp (quantitative, Fahrenheit temperature at which all gasoline has vaporized). Both batch and temp are treated as fixed effects.

$$\text{logit} \left(\frac{\alpha_1}{\alpha_1 + \alpha_2} \right) = \beta_0 + \beta_1 \cdot \text{temp} + \beta_2 \cdot \text{batch1} + \dots + \beta_{10} \cdot \text{batch9},$$

$$\log(\alpha_1 + \alpha_2) = \eta.$$

The R command for fitting the model using **betareg** is

```
library(zoib)
library(betareg)
data("GasolineYield", package = "zoib")
GasolineYield$batch <- as.factor(GasolineYield$batch)
```

```
#### betareg
gy <- betareg(yield ~ temp + batch, data = GasolineYield)
summary(gy)$coeff
```

Fitting the same model in **zoib**, we have

```
#### zoib: fixed effect on batch.
d <- GasolineYield
eg1.fixed <-
  zoib(yield ~ temp + as.factor(batch) | 1, data = GasolineYield, joint = FALSE,
        random = 0, EUID = 1:nrow(d), zero.inflation = FALSE,
        one.inflation = FALSE, n.iter = 1050, n.thin = 5, n.burn = 50)
sample1 <- eg1.fixed$coeff
# check convergence of the MCMC chains
traceplot(sample1); autocorr.plot(sample2); gelman.diag(sample1)
```

The procedure took about 25 seconds on a PC with Intel Core-i5 2520M CPU 2.5GHz (2 chains, 1050 iterations, burn-in periods = 5, and thinning period = 5 per chain). The trace plots, the autocorrelation plots, and the values of the potential scale reduction factors (psrf) (Gelman and Rubin, 1992; Brooks and Gelman, 1998) suggest the Markov chains mixed well and reached satisfactory convergence (Appendix Figure A1 and Table A1).

If the 10 batches constitute a random sample of many possible batches and the mean of each batch is of little interest, we can treat batch as a random variable rather than dummyming code batch. The following model is a mixed-effects model with a random component in the link function of the mean of the beta distribution.

```
#### zoib: random effect on batch
eg1.random <- zoib(yield ~ temp | 1 | 1, data = GasolineYield, joint = FALSE,
                  random = 1, EUID = GasolineYield$batch, zero.inflation = FALSE,
                  one.inflation = FALSE, n.iter = 10200, n.thin = 50, n.burn = 200)
sample2 <- eg1.random$oripara
summary(sample2); traceplot(sample2); autocorr.plot(sample2); gelman.diag(sample2)
```

The above procedure took about 42 seconds on a PC with Intel Core-i5 2520M CPU 2.5GHz (2 chains, 10200 iterations, burn-in periods = 50, and thinning period = 50 per chain). The trace plots, the auto-correlation plots, and the psrf values suggest that the Markov chains mixed well and converged (Appendix Figure A2 and Table A1).

The inferential results on the parameters from all three analyses (betareg, zoib-fixed, zoib-random) are depicted in Figure 2, which is generated using the `paraplot` function.

```
### posterior inferences from zoib: fixed
summ1 <- summary(sample1); summ1 <- cbind(summ1$stat[, 1], summ1$quant[, c(1, 5)])
### posterior inferences from zoib: random
summ2 <- summary(sample2); summ2 <- cbind(summ2$stat[, 1], summ2$quant[, c(1, 5)])
summ2 <- summ2[-4, ]
### inferences from betareg
summ3 <- cbind(c(summ3$mean[, 1], summ3$precision[, 1]), confint(gy))
summ3[12,] <- log(summ3[12, ]) # log-precision
### plot
names <- rownames(summ3); names[1] <- "intercept"; names[12] <- "log(precision)"
rownames(summ1) <- names; rownames(summ3) <- names
rownames(summ2) <- names[c(1, 2, 12)]
paraplot(summ1, summ2, summ3, legpos = c(2, 10), annotate = TRUE,
         legtext = c("zoib: fixed", "zoib: random", "betareg"))
```

Figure 2 suggests minimal difference between the Bayesian and the frequentist approaches in the inferences of the intercept and regression coefficients in the fixed-effects model. The posterior mean of η (log sum of the two shape parameters in the beta distribution) is numerically smaller from the Bayesian approach compared to the MLE of the frequentist approach. The mixed-effects approach yields similar estimates on η and the regression coefficients as in the fixed effect approaches, but the point estimate on the intercept is smaller. Appendix Figure A3 also presents the posterior mean of y plotted against the observed y for the two zoib models, and suggests both zoib models provide satisfactory goodness-of-fit.

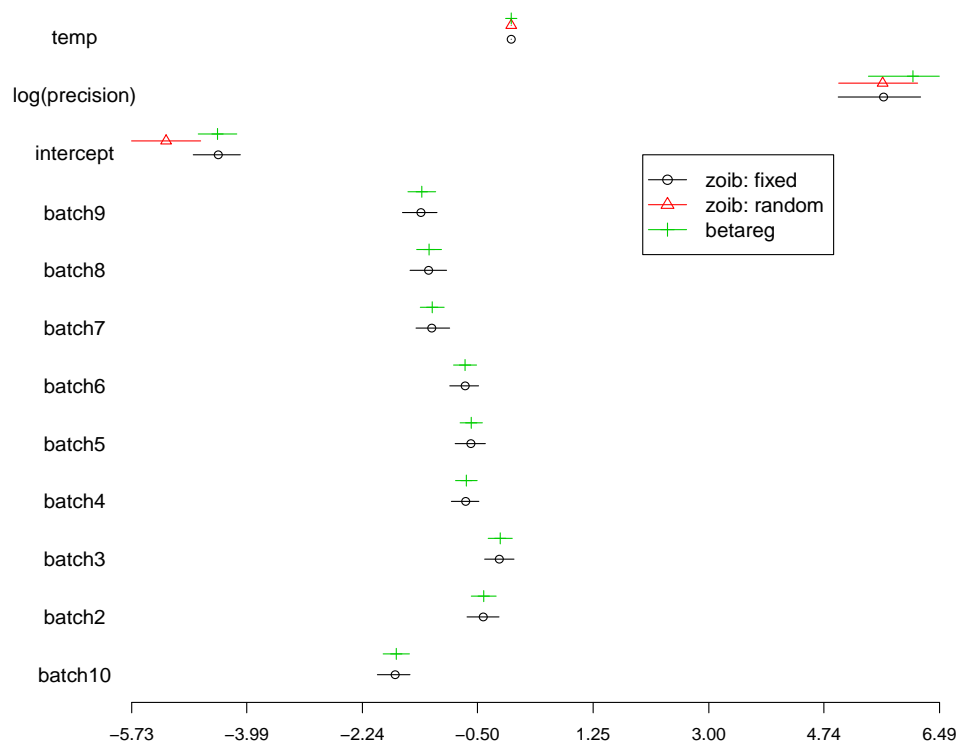


Figure 2: Inferences of model parameters in Example 1 (posterior mean and 95% posterior interval from **zoib**; MLE and 95% CI from **betareg**).

Example 2: bivariate repeated measures

Example 2 demonstrates how to jointly model multiple $[0, 1]$ -bounded response variables using **zoib**. The data set contains two beta variables (y_{i1}, y_{i2}) from 200 independent cases ($i = 1, \dots, 200$). Both y_{i1} and y_{i2} are repeatedly measured at a set of covariate values $\mathbf{x} = (0.1, 0.2, 0.3, 0.4, 0.5, 0.6)$. That is, $\mathbf{y}_{i1} = (y_{i11}, y_{i12}, \dots, y_{i16})$, and $\mathbf{y}_{i2} = (y_{i21}, y_{i22}, \dots, y_{i26})$. BiRepeated is a simulated data set from the following model,

$$\begin{aligned} \text{logit} \left(\frac{\alpha_{ijk,1}}{\alpha_{ijk,1} + \alpha_{ijk,2}} \right) &= (\beta_{0j} + u_{i1}) + (\beta_{1j} + u_{i2})x_{ijk} \\ \log(\alpha_{ijk,1} + \alpha_{ijk,2}) &= \eta_j, \text{ and} \\ \mathbf{u}_i = (u_{i1}, u_{i2}) &\sim N_{(2)}(0, \Sigma), \text{ where } \Sigma = \text{Diag}(\sigma) \mathbf{R} \text{Diag}(\sigma) \end{aligned} \quad (7)$$

where $i = 1, \dots, 200$ and $k = 1, \dots, 6$, $\beta_{01} = -1, \beta_{11} = 1, \beta_{02} = -2, \beta_{12} = 2, \rho = 0$, and $\sigma^2 = (\sigma_1^2, \sigma_2^2) = (0.2, 0.2)$. σ_1 and σ_2 are the marginal standard deviation of the two random variables u_{i1} and u_{i2} , and $\mathbf{R} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ is the correlation matrix. The data is available in R by name BiRepeated in package **zoib**. The joint model as given in equation (7) is applied to the data. The priors for the model parameters are $\beta_{0j} \sim N(0, 10^{-3}), \beta_{1j} \sim N(0, 10^{-3}), \eta_j \sim N(0, 10^{-3}), \sigma_j \sim \text{unif}(0, 20)$, and $\rho \sim \text{unif}(-1, 1)$. The R codes for realizing the above model are

```
library(zoib)
data("BiRepeated", package = "zoib")
eg2 <- zoib(y1|y2 ~ x|1|x, data = BiRepeated, random = 1, EUID = BiRepeated$id,
           zero.inflation = FALSE, one.inflation = FALSE, prior.Sigma = "UN.unif",
           n.iter = 7000, n.thin = 25, n.burn = 2000)
post.sample <- eg2$oripara; summary(post.sample)
```

The above procedure took about 3 hours 55 minutes on a Linux server with 2.4 GHz AMD Opteron processors (2 chains, 7000 iterations, burn-in periods is 2000, and thinning period is 25 per chain). The trace plots, the auto-correlation plots, and the psrf values suggest that the Markov chains mixed well

and converged (Appendix Figures A4 and Table A2).

The Bayesian inferences on β_{0j} , β_{1j} and η_j for $j = 1, 2$ are given in Table 3. Note the purpose of Example 2 is to demonstrate how **zoib** can model the correlated data; so only one simulated data set is used. The posterior means of β and η_j are nevertheless close to the true parameter values used to the simulate the data, even with the finite not-so-large sample size ($n = 200$).

Parameter	posterior mean	posterior median	2.5% quantile	97.5% quantile
β_{01}	-0.957	-0.958	-1.059	-0.866
β_{11}	0.906	0.903	0.720	1.092
β_{02}	-2.022	-2.020	-2.119	-1.935
β_{12}	2.040	2.040	1.865	2.234
η_1	2.505	2.506	2.433	2.582
η_2	3.014	3.017	2.926	3.091
σ_1^2	0.180	0.179	0.135	0.235
σ_2^2	0.334	0.331	0.152	0.530
ρ	-0.70	-0.70	-0.84	-0.62

Table 3: Bayesian inferences of the joint ZOIB model parameters in Example 2.

Example 3: clustered zero-inflated beta regression

In this example, **zoib** is applied to the county-level monthly alcohol use data collected from students in California from year 2008 to 2010. The data is available in **zoib** by name `AlcoholUse`. The data can be downloaded at <http://www.kidsdata.org>. `AlcoholUse` contains the percentage of public school students in grades 7, 9, and 11 reporting the number of days in which they drank alcohol in the past 30 days by gender (students at the “Non-Traditional” grade level refer to those enrolled in Community Day Schools or Continuation Education and are not included in this analysis). The following model is fitted to the data

$$\begin{aligned}\text{logit}\left(\frac{\alpha_{ij,1}}{\alpha_{ij,1} + \alpha_{ij,2}}\right) &= (\beta_{1,0} + u_i) + \beta_1 \mathbf{x}_{ij} \\ \log(\alpha_{ij,1} + \alpha_{ij,2}) &= \eta \\ \text{logit}(p_{ij}) &= \beta_{2,0} + \beta_2 \mathbf{x}_{ij} \\ u_i &\sim N(0, \sigma^{-2})\end{aligned}$$

where u_i is the cluster-level (county-level) random variable ($i = 1, \dots, 56$) and j indexes the j^{th} case in cluster i . $\beta_{1,1}$ contains the regression coefficients associated with the main effects associated with gender, grade, and the mid-point of each days bucket on which teenagers drank alcohol, and the interaction between gender and grade; so does $\beta_{2,1}$. σ^{-2} is the precision of the distribution of random effect u_i . The prior specification of the model parameters are: $\beta_{1,0} \sim N(0, 10^{-3})$, $\beta_{2,0} \sim N(0, 10^{-3})$, $\beta_{1,k} \stackrel{\text{ind}}{\sim} N(0, 10^{-3})$ and $\beta_{2,k} \stackrel{\text{ind}}{\sim} N(0, 10^{-3})$ for $k = 1, \dots, 6$, $\eta \sim N(0, 10^{-3})$, and $\sigma \sim \text{unif}(0, 20)$. The R codes for realizing the model in **zoib** are

```
data("AlcoholUse", package = "zoib")
AlcoholUse$Grade <- as.factor(AlcoholUse$Grade)
eg3 <- zoib(Percentage ~ Grade * Gender + MedDays|1|Grade * Gender + MedDays|1,
  data = AlcoholUse, random = 1, EUID = AlcoholUse$County,
  zero.inflation = TRUE, one.inflation = FALSE, joint = FALSE,
  n.iter = 5000, n.thin = 20, n.burn = 1000)
sample1 <- eg3$coeff
summary(sample1)
```

The above procedure took about 10 hours 56 minutes on a Linux server with 2.4 GHz AMD Opteron processors (2 chains, 5000 iterations, burn-in periods is 1000, and thinning period is 20 per chain). The trace plots, the auto-correlation plots, and the psrf values suggest that the Markov chains mixed well and reached satisfactory convergence (Appendix Figures A5 and Table A4).

The results from Example 3 are presented in Table 4. The posterior mean difference in the $\text{logit}(\text{mean})$ of the beta distribution between a 9-th grader and a 7-th grader is 0.702 ($\beta_{1,1}$), assuming they are of the same gender, and fall in the same Days Bucket. Similarly, the posterior mean difference in $\text{logit}(\text{mean})$ of the beta distribution between a male and a female students is -0.0503 ($\beta_{1,3}$), assuming

they are equal with regard to other covariates. The posterior mean difference $\text{logit}(\Pr(y = 0))$ between a 9-th grader and a 7-th grader is $\beta_{2,1} = -0.563$; in other words, the ratio in the odds of not drinking alcohol between the a 7-th grader and a 9-th grader is $\exp(0.563) = 1.75$. The other parameters in β_1 and β_2 can be interpreted in a similar manner. The posterior mean of the log(sum of the two shape parameters) η in the beta distribution is 4.389, and the posterior mean of the variance of the random effect u_i is 0.021.

Effect	Parameter	mean	median	2.5% quantile	97.5% quantile
Intercept	$\beta_{1,0}$	-2.392	-2.392	-2.484	-2.299
Grade 9	$\beta_{1,1}$	0.702	0.702	0.609	0.791
Grade 11	$\beta_{1,2}$	0.955	0.956	0.869	1.036
Gender M	$\beta_{1,3}$	-0.053	-0.052	-0.156	0.054
MedDays	$\beta_{1,4}$	-0.092	-0.092	-0.096	-0.088
Grade 9*Gender M	$\beta_{1,5}$	-0.123	-0.118	-0.255	0.003
Grade 11*Gender M	$\beta_{1,6}$	0.053	0.055	-0.087	0.193
intercept	$\beta_{2,0}$	-3.365	-3.332	-4.158	-2.635
Grade 9	$\beta_{2,1}$	-0.563	-0.572	-1.648	0.427
Grade 11	$\beta_{2,2}$	-0.874	-0.884	-2.027	0.181
Gender M	$\beta_{2,3}$	0.465	0.469	-0.382	1.329
MedDays	$\beta_{2,4}$	0.028	0.028	-0.003	0.062
Grade 9*Gender M	$\beta_{2,5}$	-0.246	-0.213	-1.628	0.999
Grade 11*Gender M	$\beta_{2,6}$	-0.664	-0.695	-2.117	1.015
	η	4.384	4.385	4.302	4.463
	σ^2	0.021	0.020	0.011	0.034

Table 4: Posterior inferences (Example 3).

Discussion

We have introduced an R package for obtaining the Bayesian inferences from the beta regression and zero/one inflated beta regression. We have provided the methodological background behind the package and demonstrated how to apply the package using both real-life and simulated data. **zoib** is more versatile and comprehensive from a modeling perspective compared to other R packages **betareg** and **Bayesainbetareg** on beta regression. First, **zoib** accommodates boundary inflation at 0 or 1. Second, it models clustered and correlated beta variables by introducing random components into the linear predictors of the link functions, and users can specify which linear predictors have a random component. Last but not least, the Bayesian inferential approach provides a convenient way for obtaining inferences for parameters that can be computationally expensive in the frequentist approach, such as the marginal means of response variables when there are random effects. For the regression coefficients in a linear predictor, 4 different priors are offered with options for penalized regression if needed. DIC criteria can be calculated using existing function from package **rjags** for model comparison purposes. Future updates to the **zoib** package include the development of more efficient computational algorithms to shorten the computational time in running the MCMC chains, especially when a zoib model contains a relatively large number of parameters.

Acknowledgements

The authors would like to thank two anonymous reviewers for their valuable comments and suggestions that have greatly improve the quality of the manuscript.

Bibliography

- A. Atkinson. *Plots, Transformations and Regression: An Introduction to Graphical Methods of Diagnostic Regression Analysis*. New York: Oxford University Press, 1985. [p40]
- S. Brooks and A. Gelman. General methods for monitoring convergence of iterative simulations. *Journal of Computational and Graphical Statistics*, 7:434–455, 1998. [p38, 41]
- E. Cepeda-Cuervo. *Modelagem da variabilidade em modelos lineares generalizados*. PhD thesis, Instituto de Matemáticas, Universidade Federal de Rio de Janeiro, 2001. [p34, 35]

- E. Cepeda-Cuervo. Beta regression models: Joint mean and variance modeling. *Journal of Statistical Theory and Practice*, 9(1):134–145, 2015. [p35]
- F. Cribari-Neto and A. Zeileis. Beta regression in R. *Journal of Statistical Software*, 34(2):1–24, 2010. [p35, 36]
- M. Dalrymplea, I. Hudsona, and R. Ford. Finite mixture, zero-inflated poisson and hurdle models with application to sids. *Computational Statistics & Data Analysis*, 41:491–504, 2003. [p35]
- S. Ferrari and F. Cribari-Neto. Beta regression for modelling rates and proportions. *Journal of the Royal Statistical Society, Series B*, 31:799–815, 2004. [p34]
- A. Gelman. Prior distributions for variance parameters in hierarchical models. *Bayesian Analysis*, 1: 515–533, 2006. [p38]
- A. Gelman and D. Rubin. Inference from iterative simulation using multiple sequences. *Statistical Science*, 7(4):457–511, 1992. [p38, 41]
- B. Grün, I. Kosmidis, and A. Zeileis. Extended beta regression in R: Shaken, stirred, mixed, and partitioned. *Journal of Statistical Software*, 48(11):1–25, 2012. [p35]
- L. Hatfield, M. Boye, M. Hackshaw, and B. Carlin. Models for survival times and longitudinal patient-reported outcomes with many zeros. *Journal of the American Statistical Association*, 107:875–885, 2012. [p34, 35]
- I. Kosmidis and D. Firth. A generic algorithm for reducing bias in parametric estimation. *Electronic Journal of Statistics*, 4:1097–1112, 2010. [p35]
- D. V. Lindley and A. F. M. Smith. Bayes estimates for the linear model. *Journal of the Royal Statistical Society: Series B*, 34(1):1–41, 1972. [p37]
- F. Liu and Y. Kong. *zoib: Bayesian Inference for Beta Regression and Zero/One Inflated Beta Regression*, 2014. URL <https://CRAN.R-project.org/package=zoib>. R package version 1.0. [p35]
- F. Liu and Q. Li. A Bayesian model for joint analysis of multivariate repeated measures and time to event data in crossover trials. *Statistics Methods in Medical Research*, 2014. doi: 10.1177/0962280213519594. [p35]
- D. MacKay. Bayesian methods for back-propagation networks. In E. Domany, J. van Hemmen, and K. Schulten, editors, *Models of Neural Networks III: Association, Generalization, and Representation*, pages 211–254. Springer-Verlag, New York, 1996. [p37]
- M. Marin, J. Rojas, D. Jaimes, H. A. G. Rojas, and E. Cepeda-Cuervo. *Bayesianbetareg: Bayesian Beta Regression: Joint Mean and Precision Modeling*, 2014. URL <https://CRAN.R-project.org/package=Bayesianbetareg>. R package version 1.2. [p35]
- R. Neal. *Bayesian Learnings for Neural Networks*. University of Toronto, Canada, 1994. [p37]
- R. M. Neal. Slice sampling. *Annals of Statistics*, 31(3):705–767, 2003. [p38]
- R. Ospina and S. L. Ferrari. A general class of zero-or-one inflated beta regression models. *Computational Statistics & Data Analysis*, 56(6):1609–1623, 2012. [p35]
- P. Paolino. Maximum likelihood estimation of models with beta-distributed dependent variables. *Political Analysis*, 9(4):325–346, 2001. [p34]
- T. Park and G. Casella. Bayesian lasso. *Journal of the American Statistical Association*, 103(482):681–686, 2008. [p37]
- M. Plummer. JAGS. <http://mcmc-jags.sourceforge.net/>, 2014a. [p35, 38]
- M. Plummer. *rjags: Bayesian Graphical Models using MCMC*, 2014b. URL <https://CRAN.R-project.org/package=rjags>. R package version 3-14. [p38]
- M. Plummer, N. Best, K. Cowles, and K. Vines. CODA: Convergence diagnosis and output analysis for MCMC. *R News*, 6(1):7–11, 2006. URL <https://CRAN.R-project.org/doc/Rnews/>. [p38]
- N. Prater. Estimate gasoline yields from crudes. *Petroleum Refiner*, 35:236–238, 1956. [p40]
- R. Prentice. Binary regression using an extended beta-binomial distribution, with discussion of correlation induced by covariate measurement. *Journal of the American Statistical Association*, 81: 321–327, 1986. [p34]

- A. Simas, W. Barreto-Souza, and A. Rocha. Improved estimators for a general class of beta regression models. *Computational Statistics & Data Analysis*, 54:348–366, 2010. [p34, 36]
- M. Smithson and J. Verkuilen. A better lemon squeezer? Maximum-likelihood regression with beta-distributed dependent variables. *Psychological Methods*, 11:54–71, 2006. [p34, 35]
- D. J. Spiegelhalter, N. G. Best, B. P. Carlin, and A. van der Linde. Bayesian measures of model complexity and fit (with discussion). *Journal of the Royal Statistical Society, Series B*, 64(4):583–639, 2002. [p38]
- C. Swearingen, M. Melguizo Castro, and Z. Bursac. Modeling percentage outcomes: The %beta_regression macro. *SAS Global Forum 2011: Statistics and Data Analysis*, paper 335, 2011. [p35]
- C. Swearingen, M. Melguizo Castro, and Z. Bursac. Inflated beta regression: Zero, one, and everything in between. *SAS Global Forum 2012: Statistics and Data Analysis*, paper 325, 2012. [p35]
- D. Williams. Extra binomial variation in logistic linear models. *Applied Statistics*, 31:144–148, 1982. [p34]
- A. Zeileis, T. Hothorn, and K. Hornik. Model-based recursive partitioning. *Journal of Computational and Graphical Statistics*, 17:492–514, 2008. [p35]
- A. Zeileis, F. Cribari-Neto, B. Grün, I. Kosmidis, A. B. Simas, and A. V. Rocha. *Beta Regression for Rates and Proportions*, 2014. URL <https://CRAN.R-project.org/package=betareg>. R package version 3.0-5. [p35]

Fang Liu
University of Notre Dame
Notre Dame, IN 46530
USA
fang.liu.131@nd.edu

Yunchuan Kong
Chinese University of Hong Kong
Shatin, N.T.
Hong Kong
ykong@cuhk.edu.hk

Appendices

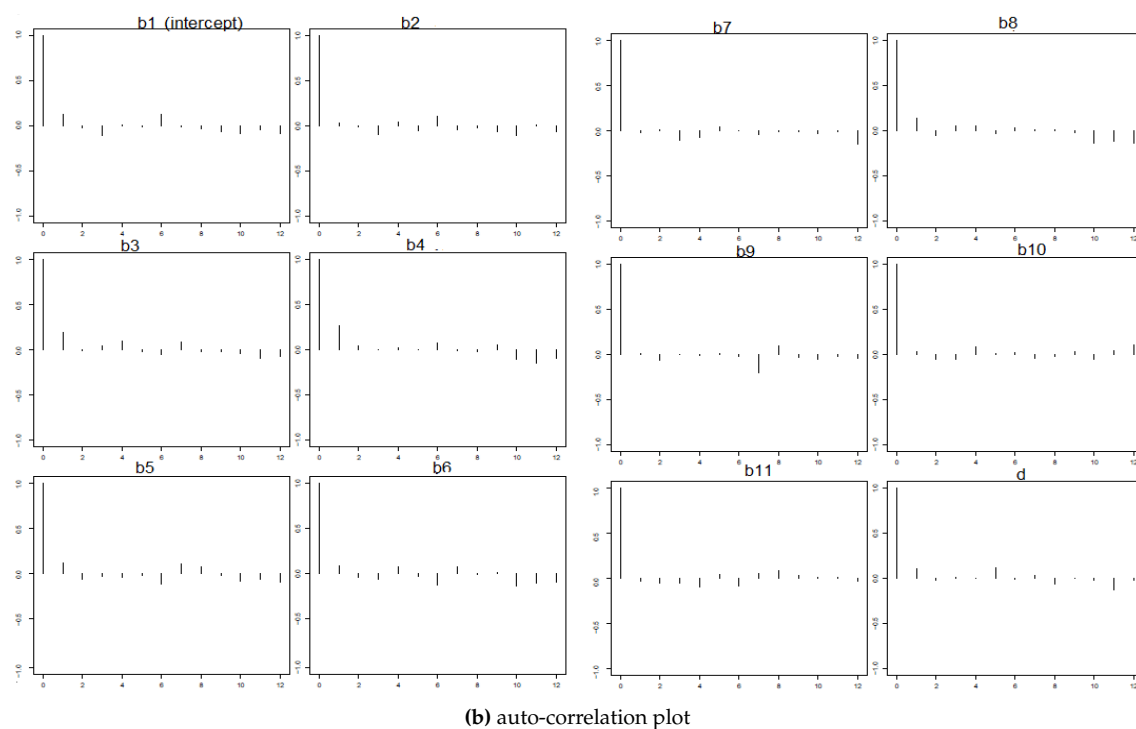
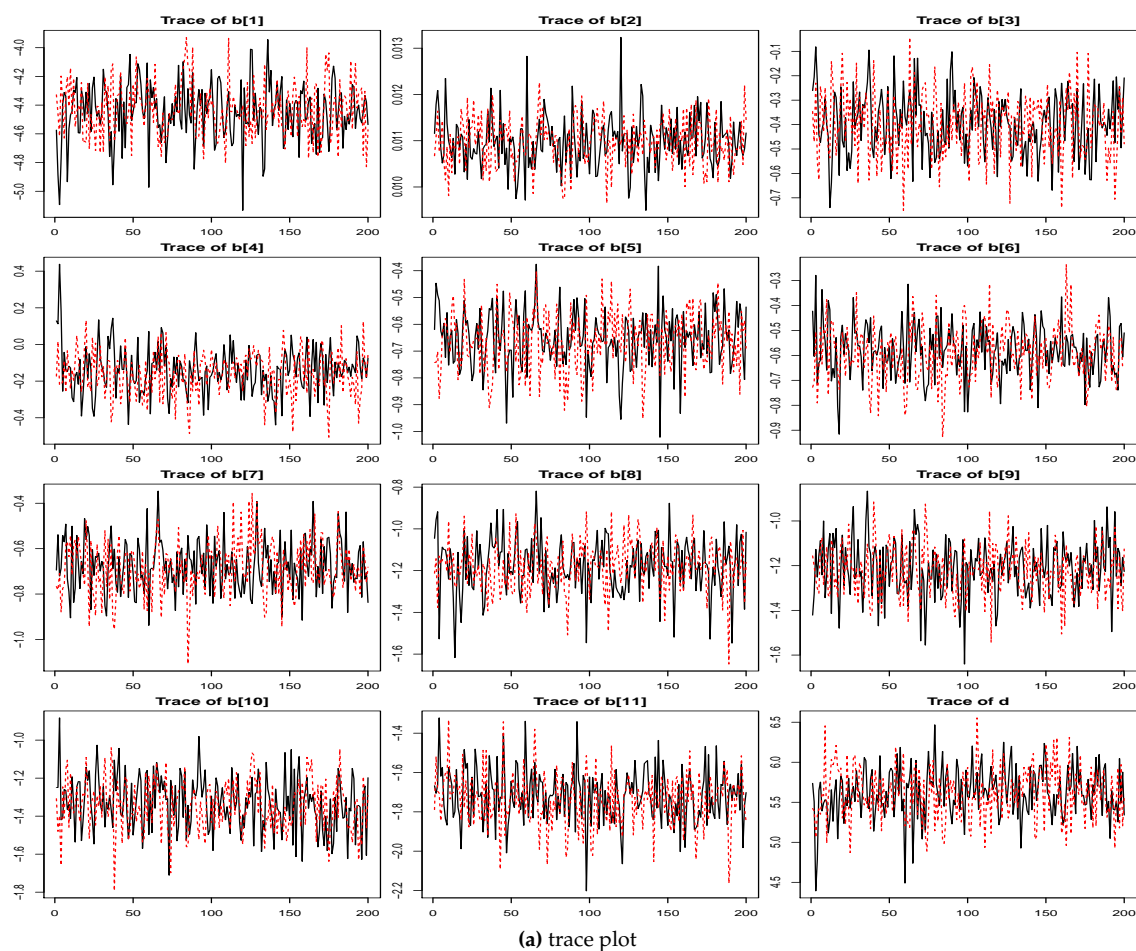


Figure A1: Trace and auto-correlation plots in the zoib-fixed model in Example 1.

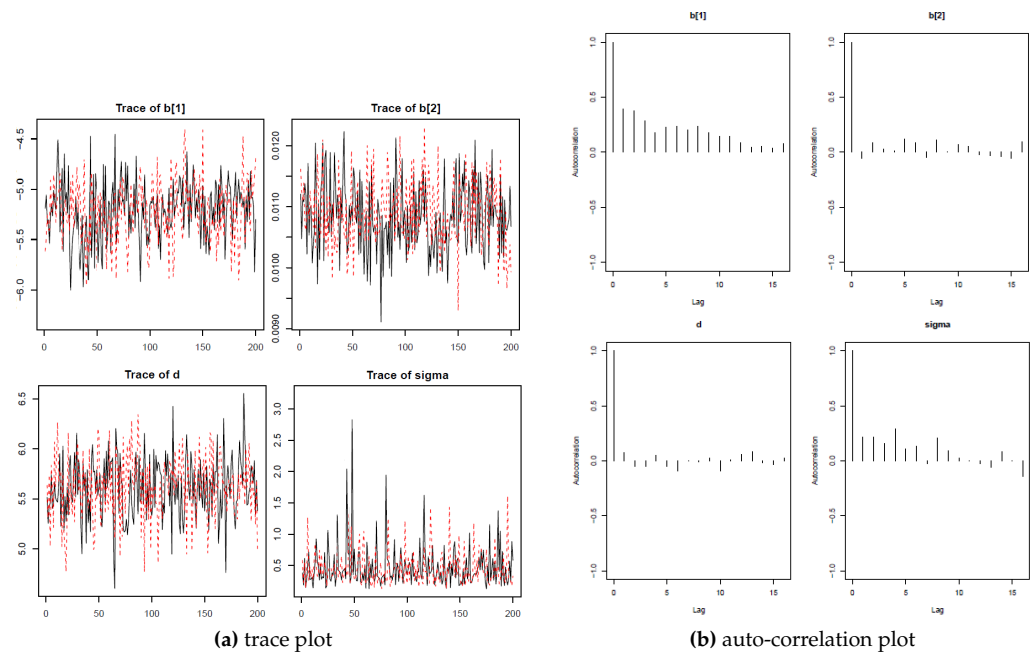


Figure A2: Trace and auto-correlation plots in the zoib-random model in Example 1.

Parameter	zoib-fixed		zoib-random	
	point	upper bound (95%)	point	upper bound (95%)
$\beta_{1,0}$ (intercept)	0.997	0.998	1.036	1.037
$\beta_{1,1}$	1.001	1.002		
$\beta_{1,2}$	1.008	1.008		
$\beta_{1,3}$	1.013	1.040		
$\beta_{1,4}$	0.998	1.006		
$\beta_{1,5}$	1.005	1.007		
$\beta_{1,6}$	1.000	1.014		
$\beta_{1,7}$	0.999	1.004		
$\beta_{1,8}$	1.003	1.033		
$\beta_{1,9}$	1.001	1.015		
$\beta_{1,10}$	1.007	1.048	1.003	1.017
β_{20} (temperature)	1.018	1.032	1.002	1.006
σ^2			0.997	0.997

Table A1: Potential scale reduction factors of the zoib model parameters in Example 1.

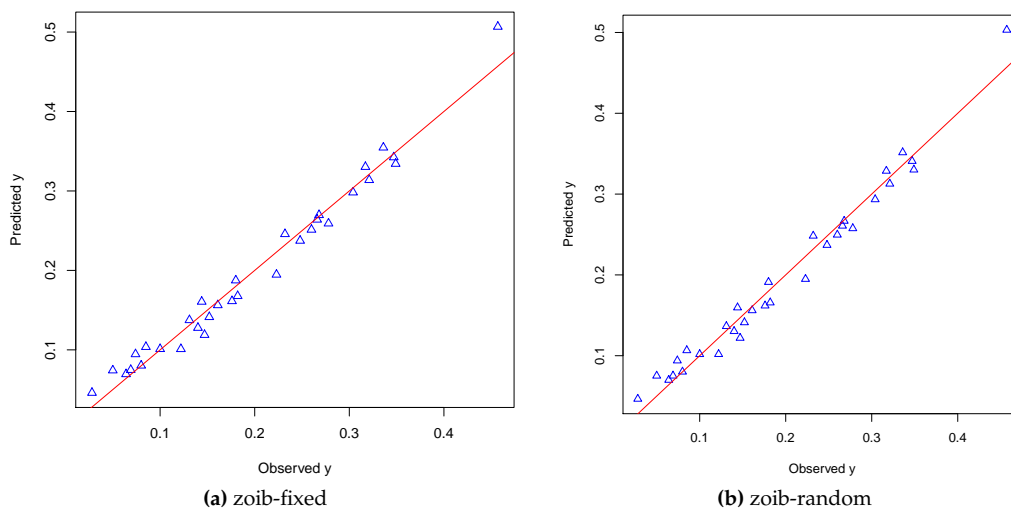


Figure A3: Posterior mean of Y vs. observed Y in Example 1.

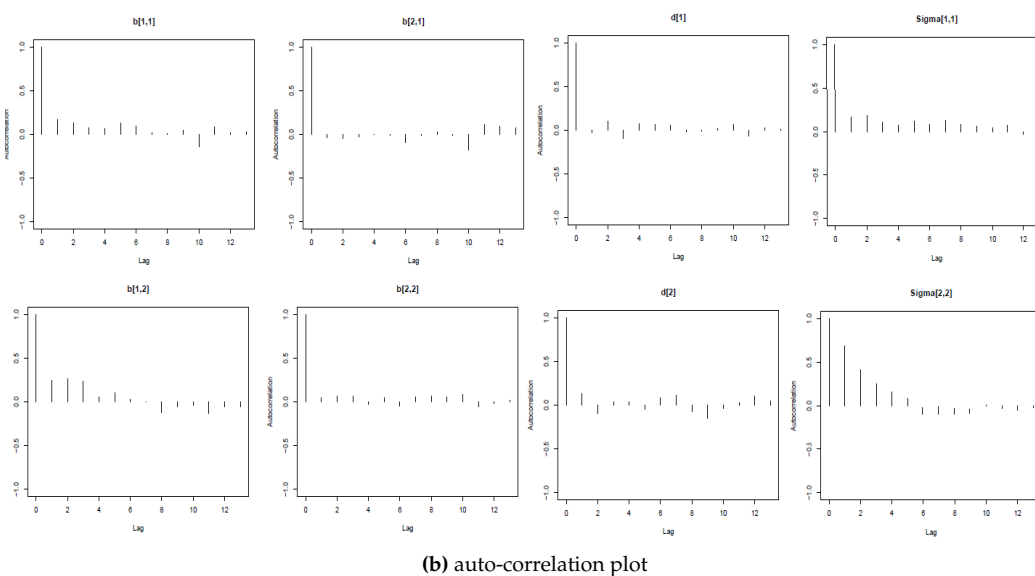
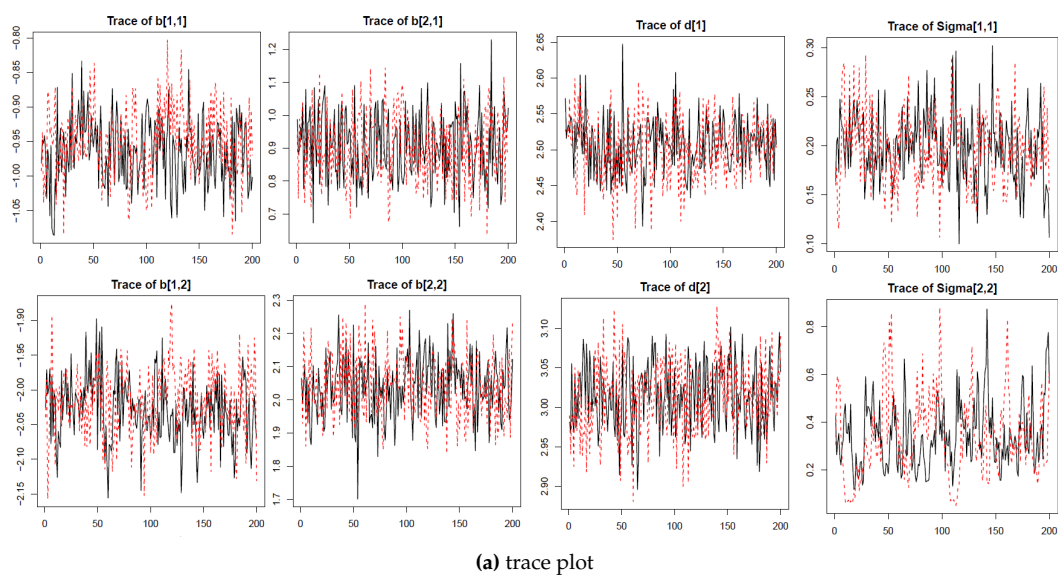


Figure A4: Example 1, zoib-fixed.

Parameter point upper bound (95%)		
$\beta_{0,1}$	1.083	1.332
$\beta_{1,1}$	1.004	1.035
$\beta_{0,2}$	1.163	1.582
$\beta_{1,2}$	1.008	1.100
σ_1^2	0.997	0.997
σ_2^2	1.036	1.165

Table A2: Potential scale reduction factors in Example 2.

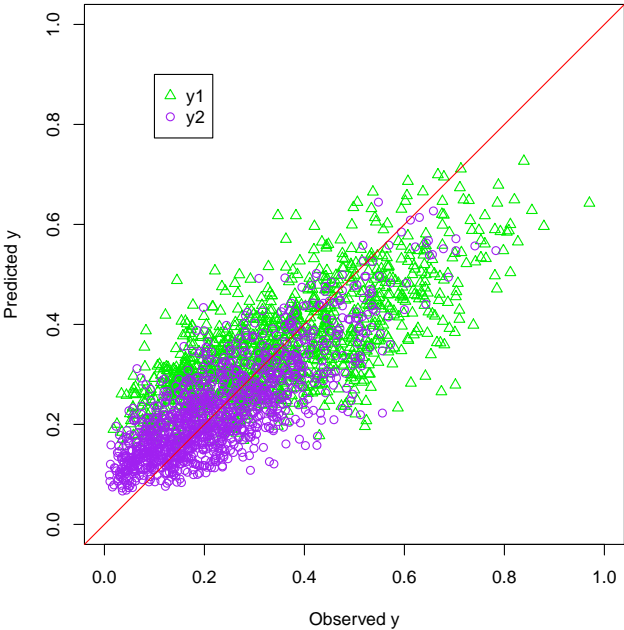


Table A3: Posterior mean of Y vs. observed Y (Example 2).

Parameter	$\beta_{1,0}$	$\beta_{1,1}$	$\beta_{1,2}$	$\beta_{1,3}$
point	1.018	1.005	1.000	0.997
upper limit (95% CI)	1.092	1.042	1.015	0.999
parameter	$\beta_{1,4}$	$\beta_{1,5}$	$\beta_{1,6}$	η
point	0.997	1.006	0.997	1.068
upper limit (95% CI)	1.000	1.047	0.998	1.279
parameter	$\beta_{2,0}$	$\beta_{2,1}$	$\beta_{2,2}$	$\beta_{2,3}$
point	0.998	1.001	0.998	0.998
upper limit (95% CI)	1.000	1.001	0.999	1.004
parameter	$\beta_{2,4}$	$\beta_{2,5}$	$\beta_{2,6}$	σ^2
point	0.996	1.012	0.996	1.001
upper limit (95% CI)	0.997	1.012	0.999	1.020

Table A4: Potential scale reduction factors in Example 3.

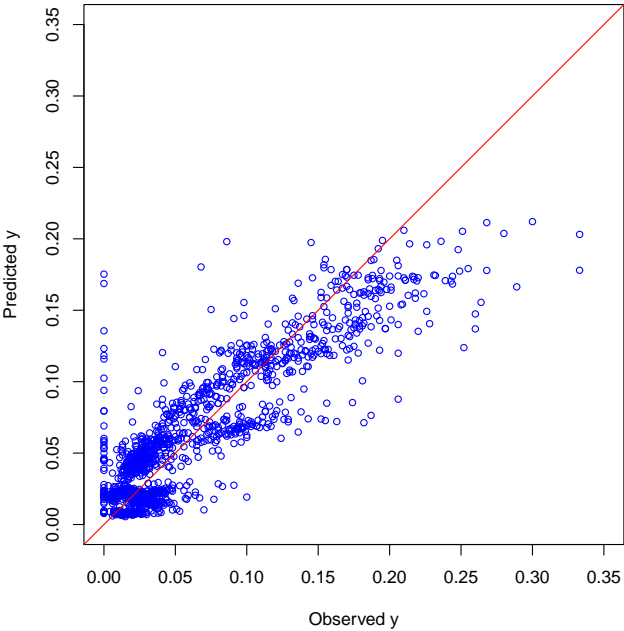


Table A5: Posterior mean of Y vs. observed Y (Example 2).

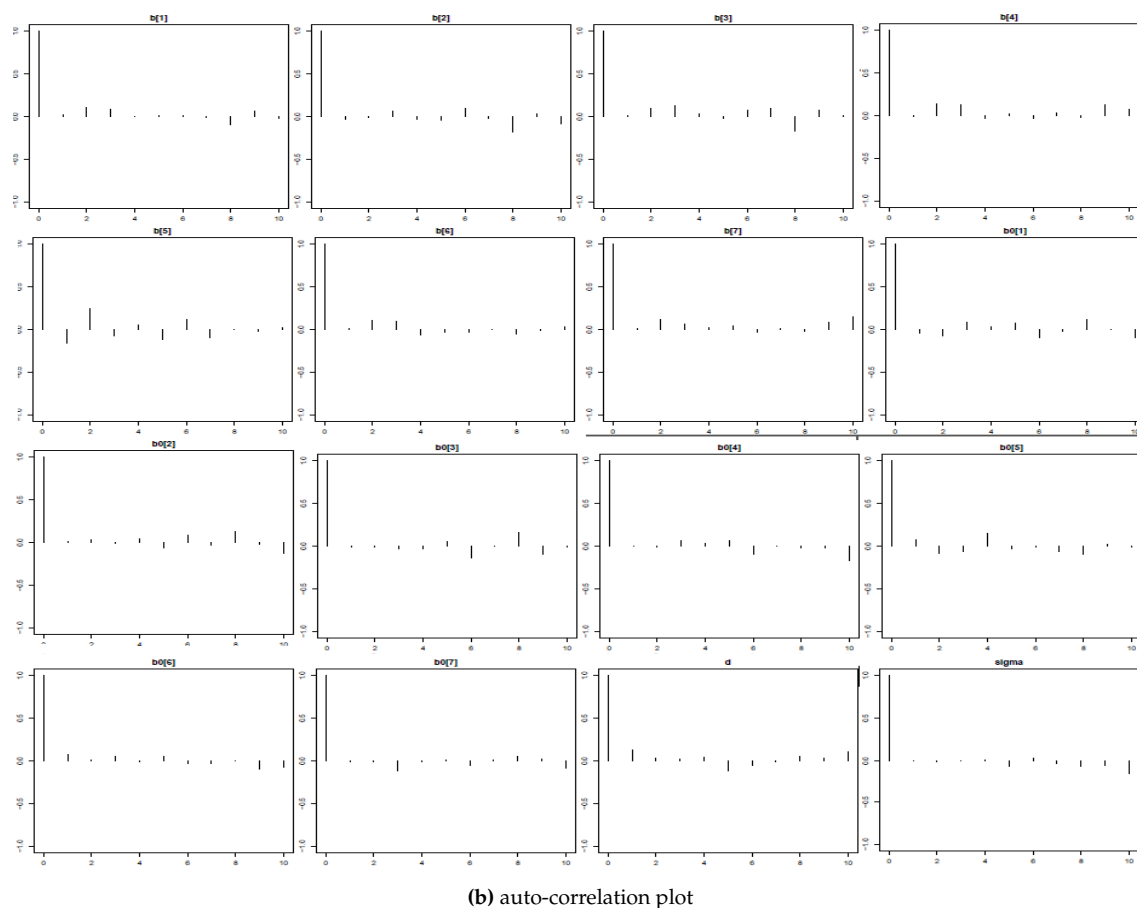
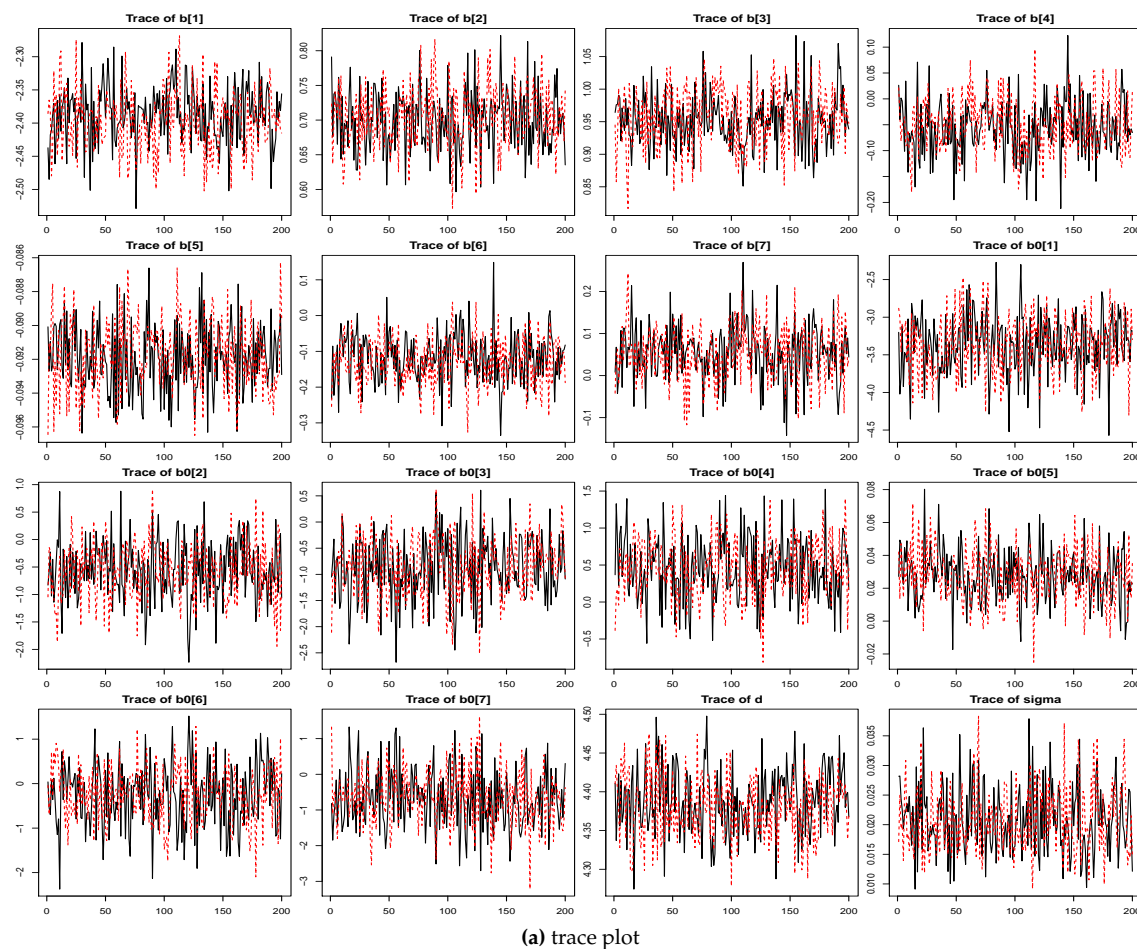


Figure A5: Trace and auto-correlation plots in the zoib-fixed model in Example 1.