К #5

Задача 1)

$f = x^4 + 2x^3 - x^2 - 4x - 2$

$g = x^4 + x^3 - x^2 - 2x - 2$

$$\begin{array}{r|l} x^4 + 2x^3 - x^2 - 4x - 2 & x^4 + x^3 - x^2 - 2x - 2 \\ -\ x^4 + x^3 - x^2 - 2x - 2 & 1 \\ \hline x^3 \qquad - 2x & \end{array} = q_1(x)$$

$r_1(x) = x^3 - 2x$

$$\begin{array}{r|l} x^4 + x^3 - x^2 - 2x - 2 & x^3 - 2x \\ -\ x^4 - 2x^2 & x + 1 \\ \hline x^3 + x^2 - 2x - 2 & \\ -\ x^3 - 2x & \\ \hline x^2 - 2 & \end{array} = q_2(x)$$

$r_2(x) = x^2 - 2$

$$\begin{array}{r|l} x^3 - 2x & x^2 - 2 \\ -\ x^3 - \frac{2}{3}x & x \\ \hline -\frac{4}{3}x & \end{array} = q_3(x)$$

$r_3(x) = -\frac{4}{3}x$

$$\begin{array}{r|l} 3x^2 - 2 & -\frac{4}{3}x \\ -\ 3x^2 & -\frac{9}{4}x \\ \hline -2 & \end{array} = q_4(x)$$

$r_4(x) =$

$r_1(x)$	$r_2(x)$
$x^3 - 2x$	$x^2 - 2$
$-\; x^3 - 2x$	$x \quad = q_3(x)$
$r_3(x) = 0$	

$$\Rightarrow (f(x), g(x)) = x^2 - 2 = d$$

$$r_2(x) = g(x) - r_1(x)\, q_2(x) =$$

$$= g(x) - (f(x) - g(x)\, q_1(x))\, q_2(x) =$$

$$= g(x) - f(x)\, q_2(x) + g(x)\, q_1(x)\, q_2(x)$$

$$r_2(x) = -q_2(x)\, f(x) + (q_1(x)\, q_2(x) + 1)\, g(x)$$

$$d = u f + v g$$

$$\Rightarrow u = -q_2(x) = -x - 1$$

$$v = (x+1)(1) + 1 = x + 2$$

$$\underset{d}{x^2 - 2} = \underset{u}{(-x-1)} f + \underset{v}{(x+2)} g$$

задача 2) $I = (13+5\sqrt{6}) \triangleleft \mathbb{Z}[\sqrt{6}]$

Д(Д) $I = \{c+d\sqrt{6} \mid c,d \in \mathbb{Z},\ 19 \mid (5c-13d)\} = J$: $I = J$

и $\mathbb{Z}[\sqrt{6}]/I \cong \mathbb{Z}_{19}$

$\mathbb{Z}[\sqrt{6}] = \{a+b\sqrt{6} \mid a,b \in \mathbb{Z}\}$

$I = \{(13+5\sqrt{6})(a+b\sqrt{6}) \mid a,b \in \mathbb{Z}\} = \{13a+13\sqrt{6}b+5\sqrt{6}a+30ab \mid a,b\in\mathbb{Z}\}$

$\cancel{= \{(13+5\sqrt{6})a + (13\sqrt{6}+30)b \mid a,b\in\mathbb{Z}\}}$

$= \{(13a+30b)+(5a+13b)\sqrt{6} \mid a,b \in \mathbb{Z}\}$

$(I \subseteq J)$ Нека $\underbrace{(13a+30b)}_{=c}+\underbrace{(5a+13b)}_{=d}\sqrt{6} \in I$

$5.(13a+30b) - 13(5a+30b) =$

$= 65a+150b-65a-169b = -19b \Rightarrow 19 \mid d \in I$

$\Rightarrow I \subseteq J$

$(J \subseteq I)$ Нека $B \in J$ и $B = c+d\sqrt{6}$

$\Rightarrow 19 \mid (5c-13d)$

Откриваме $a,b \in \mathbb{Z}$: $\begin{cases} 13a+30b = c \\ 5a+13b = d \end{cases}$

$\Rightarrow 19b = 13d-5c$

$19 \mid 13d-5c$

$\Rightarrow b \in \mathbb{Z}$

$\bullet\ 5a + 13(13d - 5c) = d$

$5a = -168d + 65c$

$\bullet\ 13a + 30(13d - 5c) = c$

$13a = 151c - 390d$

$\Leftrightarrow 3a + 2\cdot(-168d + 65c) = 151c - 390d$

$3a = 21c - 54d$

$a = 7c - 18d$

$\Rightarrow a \in \mathbb{Z}$

$\Rightarrow J \subseteq I \Rightarrow I = J$

Д.С.Д. $\mathbb{Z}[\sqrt{6}]/I \cong \mathbb{Z}_{19}$

~~Ker~~

Погледни $\varphi : \mathbb{Z}[\sqrt{6}] \to \mathbb{Z}_{19}$

$\operatorname{Ker}\varphi = I = \{a + b\sqrt{6} \mid \varphi(a + b\sqrt{6}) = \bar{0}\}$

$\operatorname{Im}\varphi = \mathbb{Z}_{19} = \{\varphi(z) \mid z \in \mathbb{Z}[\sqrt{6}]\}$

Нека пробваме $\varphi(a + b\sqrt{6}) = \overline{5a - 13b} \pmod{19}$

$\bullet\ \varphi(a + b\sqrt{6}) +_{\mathbb{Z}_{19}} \varphi(c + d\sqrt{6}) = \overline{5a - 13b} +_{\mathbb{Z}_{19}} \overline{5c - 13d} =$

$= \overline{5(a + c) - 13(b + d)} = \varphi((a + b\sqrt{6}) + (c + d\sqrt{6}))$

• $\varphi(a+b\sqrt{6})\varphi(c+d\sqrt{6}) = \overline{(5a-13b)(5c-13d)} =$

$= \overline{25ac-65ad-65bc+169bd} =$

$= 5 \cdot \overline{5ac-13ad-13bc+30bd}$

$\varphi((a+b\sqrt{6})(c+d\sqrt{6})) = \varphi(ac+ad\sqrt{6}+bc\sqrt{6}+bd\sqrt{6}) =$

$= \varphi((ac+6bd)+(ad+bc)\sqrt{6}) =$

$= \overline{5ac+30bd-13ad-13cb}$

Нека $f(\alpha) = \underset{\in \mathbb{Z}_{19}}{k} \cdot \varphi(\alpha)$, като $\varphi(\alpha+\beta) = \varphi(\alpha) + \varphi(\beta)$, $\varphi(\alpha)\varphi(\beta) = 5\varphi(\alpha\beta)$, за $\alpha, \beta \in \mathbb{Z}[\sqrt{6}]$

$f(\alpha+\beta) = k \cdot \varphi(\alpha+\beta) = k\varphi(\alpha) + k\varphi(\beta) = f(\alpha) + f(\beta)$

$f(\alpha\beta) = k\,\varphi(\alpha\beta)$

$f(\alpha)f(\beta) = k\varphi(\alpha) \cdot k\varphi(\beta) = 5k^2\varphi(\alpha\beta)$

$\Rightarrow k = 5k^2$

$k(\bar{5}k-\bar{1}) = \bar{0}$ $(k \neq 0)$

$\bar{5}k - \bar{1} = \bar{0}$

$\bar{5}k = \bar{1}$

$k = \bar{4}$

$\Rightarrow f(\alpha) = \bar{4}\,\overline{(5a-13b)}$ е хомоморфизъм.

• $\mathrm{Im} f \subseteq \mathbb{Z}_{19}$

$f(a+0\sqrt{6}) = \bar{a} \Rightarrow \mathrm{Im} f \supseteq \mathbb{Z}_{19} \Rightarrow \mathrm{Im} f = \mathbb{Z}_{19}$

• $\mathrm{Ker} f = I$

$\Rightarrow \mathbb{Z}[\sqrt{6}]/I \cong \mathbb{Z}_{19}$

Задача 3) R и S са пръстени

$\therefore$ Д (Д) $R \times S$ не е поле

Понеже R има поне 2 ел $\Rightarrow \exists\, r \neq 0 \in R$

$(r, 0_S) \in R \times S$ и $(r, 0_S) \neq (0_R, 0_S)$

Аналогично за $(0_R, s) \in R \times S$

Сега да разгледаме:

(да приемем, че $(r, 0_S)^{-1}$ е обр. на $(r, 0_S)$)

$$(r, 0_S) . (0_R, s) = (0_R, 0_S) \quad / \circ (r, 0_S)^{-1}$$

~~$R \times S$ не е област~~ има дел. на 0

$$((r, 0_S)^{-1} (r, 0_S)) (0_R, s) = (0_R, 0_S)$$

$$(0_R, s) = (0_R, 0_S)$$

Противоречие с $s \neq 0 \Rightarrow (r, 0_S)$ няма обр. ел.

$\Rightarrow R \times S$ не е поле

- Нека $K \triangleleft R \times S$, Д(Д) $\exists\, I \triangleleft R, J \triangleleft S : K = I \times J$

Нека $I = \{ i \in R \mid \exists j : (i, j) \in K \}$, ugl gок, е $I \triangleleft R$

И аналогично се док. за J

- Нека $(a, b), (a', b') \in K$, тогава ~~а~~ $a, a' \in I$

$(a, b) - (a', b') = (a - a', b - b') \in K$ (защото е идеал), като $a - a' \in I$ (от gл)

- Нека $(a, b) \in K$ и $(c, d) \in R \times S$

$(a, b)(c, d) = (ac, bd) \in K \Rightarrow ac \in I$ за $\forall c \in R$

$(c, d)(a, b) = (ca, db) \in K \Rightarrow ca \in I$ за $\forall c \in R$

$\Rightarrow I \triangleleft R$

Аналогично за J, док. е $J \triangleleft S$

Сега да видим защо $I \times J = K$

$I \times J \subseteq K$ Нека $i \in I$ и $j \in J \Rightarrow \exists s : (i,s) \in K$ и
$\exists r : (r,j) \in K$

Тогава $(i,s)\cdot(1,0) + (r,j)\cdot(0,1) = (i,j) \in K$

$K \subseteq I \times J$ Нека $(i,j) \in K \Rightarrow i \in I$ и $j \in J$
$\Rightarrow K \subseteq I \times J$

$\Rightarrow I \times J = K$

~~([illegible] ук. за първото [illegible])~~

(Бел. Приемаме че пръстените имат 1.)
за първото включване