CS 357 - 06 Taylor Series

Boyang Li (boyangl3)

Degree n polynomial: Consider a polynomial respect to variable x,

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Using the summation notation, we can express the polynomial as $\sum_{k=0}^{n} a_k x^k$.

A polynomial can also be considered as the linear combination of monorails. i.e. ax^n where a is non-zero constant and n is a non-negative integer.

Taylor Series: Used to approximate function $f(x)$ where x is closed to x_0.

$$f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

There is a special case of Taylor Series, when $x_0 = 0$, the series is called Maclaurin Series:

$$f(x) = f(0) + \frac{f'(0)}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!}$$

The infinite Taylor series expansion of any polynomial is the polynomial itself. However, sometimes a function is not infinite differentiable, so we need to truncate the Taylor Series at some point to approximate the function.

Taylor Series at degree n: Just take the first $n+1$ terns of the Taylor Series:

$$T_n(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \cdots = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$$

Taylor Series Error:

- **Error bound when truncating:** When $h = |x - x_0| \to 0$, the error bound is

  $$|f(x) - T_n(x)| \leq C \cdot h^{n+1} = O(h^{n+1})$$

- **Taylor remainder theorem:** Let $R_n(x)$ denote the difference between $f(x)$ and $T_n(x)$ which centered at x_0, then

  $$R_n(x) = f(x) - T_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

  for some ξ between x and x_0. Thus the constant C mentioned above is $\max_{\xi} \frac{|f^{(n+1)}(\xi)|}{(n+1)!}$.

- **Asymptotic behavior of the error:** If $e_1 \propto h_1^n$ and $e_2 \propto h_2^n$, then $e_2 = \left(\frac{h_2}{h_1}\right)^n e_1$.