rnn: a Recurrent Neural Network in R*

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Abstract

The rnn package implements a Recurrent Neural Network (RNN). RNN algorithms have the ability to train neural networks to deal with greater levels of complexity . This package is purposely designed to demonstrate the self learning ability using the classic example of binary summation on a bit-by-bit (right to left) basis, which requires the model to develop the understanding that if a 1 and a 1 are added, the outcome is 0, but in the next iteration, it has to that it was carrying a 1 from the previous iteration.

 $^{{\}rm *https://cran.r-project.org/package=rnn\ |\ https://github.com/bquast/rnn}$

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1 Introduction

This package implements a Recurrent Neural Network which is trained to sum 8-bit binary numbers, teaching itself the complex task of carrying a 1 over to the next iteration if the sum of a column takes two bits of space.

to convert numbers in range of 0-127 to binary representation.

Of course, numbers < 128 can be represent in a 7-bit binary form, but since we are adding two numbers in the range 0-127, the total can reach and achieve 128, which requires 8 bits, it cannot be more than 254, the limit of 8 bit binary representation is 255, thereby preventing overflows.

At this point it is useful to clarrify the nomenclature in this article. I use the term RNN (capitalised) for the general concept of a Recurrent Neural Network and I use rnn (in miniscules and using a monospace font) to refer to the R package.

Table 1: Package

```
# load the package
library(rnn)

# list functions
ls('package:rnn')

## [1] "predictr" "sigmoid"
## [3] "sigmoid_output_to_derivative" "trainr"
```

2 Data

The main trainr() function takes three integer vectors as inputs: Y, X1, and X2. The vectors X1 and X2 are independent variables, the Y vector is the sum of X1 and X2 and is the response variable (for more info see help('trainr')).

Table 2: Training Data

```
# use the same random numbers
set.seed(123)

# create training inputs
X1 = sample(0:127, 7000, replace=TRUE)
X2 = sample(0:127, 7000, replace=TRUE)

# create training output
Y <- X1 + X2

# check that all vectors are integer
typeof(c(X1,X2,Y))

## [1] "integer"</pre>
```

Internally the trainr() function converts these characters into binary format using the intToBits() function and afterward converts it back into decimal format for printing using the packBits() function, both functions are included in the base package.

We can for instance take the first value of X1 and convert it to a binary representation, whereby the binary_dim argument to the trainr() function determines the length of the binary representation, throughout this paper we will use 8 bit representations (which limits numbers to the range 0-255), but the theoretical limit is 32 bits.

Table 3: Binary Representation

```
# manually define binary_dim
binary_dim = 8

X1[1]
## [1] 36

rev(as.numeric(intToBits( X1[1] ))[1:binary_dim])
## [1] 0 0 1 0 0 1 0 0
```

Lets check look at the first sum in decimal representation.

Table 4: Decimal Summation

```
X1[1]

## [1] 36

X2[1]

## [1] 119

X1[1] + X2[1]

## [1] 155

Y[1]

## [1] 155
```

and now in binary representation.

Table 5: Binary Summation

```
rev(as.numeric(intToBits( X1[1] ))[1:binary_dim])
rev(as.numeric(intToBits( X2[1] ))[1:binary_dim])
print('-----')
rev(as.numeric(intToBits( Y[1] ))[1:binary_dim])

## [1] 0 0 1 0 0 1 0 0
## [1] 0 1 1 1 0 1 1 1
## [1] "------"
## [1] 1 0 0 1 1 0 1 1
```

As can be seen from the above output, the first values of X1 and X2, 36 and 119 respectively, are both in the range 0-127, which can be represented with only 7 bits. Yet the sum of the two - 155 - is outside of the range 0-127, which is why an 8th bit is required (i.e. the 8th value from right to left in the bottom row is 1). If we sampled numbers great than 127 for X1 and X2 then the sum of the two could be greater than 255, which requires a ninth bit (or length=9)

The rnn() function will run until it has evaluated all values in the vector that it is fed. Since the training of the network, particularly the carrying part, takes many iterations to learn (the exact number of iterations varies but depends on the hyperparameters, more on this in the next section), it is therefore advisable to sample several thousand values (I use 7000).

3 Methodology

The workhorse of the rnn package is the trainr() function.

For example, if we add the binary numbers 0 0 1 (decimal system: 1) and 1 0 1 (decimal system: 5), we start by adding the right column, 1 and 1 make 1 0 (similar to when 5 and 5 make 1 0 in the decimal system), the 0 is stored in the right column, the 1 is carried over to the middle column and added with the two existing bits 0 and 0, to form 1, which is stored in the middle column. This time nothing is carried over and the left column sums 0 and 1 to make 1, which gives the outcome 1 1 0 (decimal system: 6).

If we go back to the output of the int2binary() function for X1, X2, and Y, we see that in the 4th column (from right to left), a 0 and a 0 are added, resulting in an output of 1. This is because in the previous iteration 3rd column (from right to left) a 1 and a 1 are added, which becomes 1 0, so the 0 goes in column 3 and the 1 is carried over to colum 4. Since the summation is done bit by bit (or column by column), the neural network need to remember from the 3rd iteration until the 4th interation that it is carrying a 1 over. It is this remembering that a feed-forward neural network cannot teach itself.

The rnn() function internally makes use of the sigmoid() function, which is a very simple implementation of a sigmoid which takes the range (-Infinity, Infinity) and maps it to the range (0, 1).

Table 6: Sigmoid Source Code

```
# print source code of the sigmoid function
sigmoid

## function(x) {
## output = 1 / (1+exp(-x))
## return(output) }
## <environment: namespace:rnn>
```

For instance:

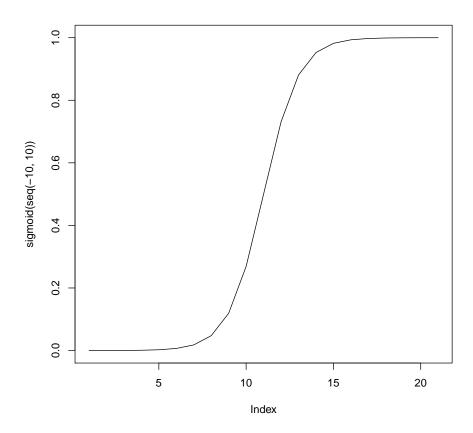
Table 7: Sigmoid Examples

```
sigmoid(-137)
sigmoid(5.3)
## [1] 3.174359e-60
## [1] 0.9950332
```

The shape of the sigmoid function is as follows.

Figure 1: Sigmoid Shape

plot(sigmoid(seq(-10,10)), type='l')



Additionally the rnn() function uses the sigmoid_output_to_derivative() function.

Table 8: Sigmoid Derivative Source Code

```
# print source code of the sigmoid_output_to_derivate function
sigmoid_output_to_derivative
## function(output) {
##
     return( output*(1-output) )
                                                       }
## <environment: namespace:rnn>
```

As the purpose of the package is to illustrate the working of a Recurrent Neural Network, the trainr() function is quite verbose (this can be controlled using the print argument).

Table 9: trainr() Output

```
## [1] "Summation number: 1000"
## [1] "x1: 1"
## [1] "x2: 1 +"
## [1] "y: 0"
## [1] "y: 0"
## [1] "x1: 0"
## [1] "x2: 0 +"
## [1] "y: 1"
## [1] "x2: 0 +"
## [1] "x2: 0 +"
## [1] "y: 0"
## [1] "y: 1"
## [1] "y: 0"
## [1] "y: 1"
## [1] "y: 0"
## [1] "y: 1000 ]: 0 1 0 1 0 0 0 1 ( 33 )"
## [1] "y: 1000 ]: 0 1 1 0 0 0 1 ( 49 )"
## [1] "y: 1000 ]: 0 1 1 0 0 0 1 1 ( 49 )"
## [1] "predict Y: 0 1 1 0 0 0 1 1 ( 99 )"
## [1] "predict Y: 0 1 1 0 0 0 1 1 ( 99 )"
## [1] "predict Y: 0 1 1 0 0 0 1 1 ( 99 )"
## [1] "predict Y: 0 1 1 0 0 0 1 1 ( 99 )"
```

The text printed here is of the 8 steps of the summation of the 1000th value of X1 and X2, or iteration 7993-8000.

Each iteration is printed individually, with the two input bits, the prediction for the response value and the actual response value.

After each iternation the difference between the predicted value and the actual value is fed back into the neural network using a method called back-propagation (an application the chain rule of differential calculus).

At the end of the 8 iterations that it here takes to add two values of X1 and X2, the results are printed in a more human legible form. It should be clear from the results that after 1000 numbers, which 8 iterations each, the model is still performing very poorly.

However, progress can be observed:

Table 10: trainr() Output

```
# use the same random numbers
set.seed(1)
# train the network
m1 <- trainr(Y,</pre>
                             X1,

X2,

binary_dim = 8,

alpha = 0.1,

input_dim = 2,

hidden_dim = 10,

output_dim = 1,

print = 'minimal')
## [1] "----"
## [1] "Y[ 1000 ]: 0 0 1 1 1 1 0 0 ( 60 )"
## [1] "predict Y^: 0 1 1 1 1 1 1 1 ( 127 )"
## [1] "summation number: 2000"
## [1] "Summation number: 2000"
## [1] "Error: 4.03678792609062"
## [1] "XI[ 2000 ]: 0 0 1 1 1 0 0 1 ( 57 )"
## [1] "XZ[ 2000 ]: 0 1 1 1 0 0 1 0 0 + ( 100 )"
## [1] "Y[ 2000 ]: 1 0 0 1 1 1 0 1 ( 157 )"
## [1] "Y[ 2000 ]: 1 0 0 1 1 1 0 1 ( 157 )"
## [1] "predict Y: 1 1 1 1 1 1 1 1 ( 255 )"
## [1] "Summation number: 3000"
## [1] "Summation number: 3000"
 ## [1] "Error: 4.0117145610462"
## [1] "XI[ 3000 ]: 0 1 0 1 0 1 1 1 (87 )"
## [1] "X2[ 3000 ]: 0 1 1 1 0 0 1 1 + (115 )"
## [1] "X2[ 3000 ]: 0 1 1 1 0 0 1 1 + (115 )"
 ## [1] "----"
## [1] "Y[ 3000 ]: 11 0 0 1 0 1 0 0 ( 202 )"
## [1] "predict Y^: 1 0 0 0 0 1 0 0 ( 132 )"
## [1] "-------"
## [1] "Summation number: 4000"
 ## [1] "Error: 3.43636651888"

## [1] "XI[ 4000 ]: 0 0 1 1 0 1 1 0 ( 54 )"

## [1] "X2[ 4000 ]: 0 0 1 1 1 0 0 0 + ( 56 )"

## [1] ".................."
## [1]
"Y[ 7000 ]: 1 0 0 1 0 0 1 0 (146 )"
"predict Y^: 1 0 0 1 0 0 1 0 (146 )"
```

In fact, from the 6000th summation on, all the printed estimates are in fact correct.

4 Results

The eventual purpose is to use the model generated by the trainr() function as an input to the predictr() function, in order to predict the values of new inputs.

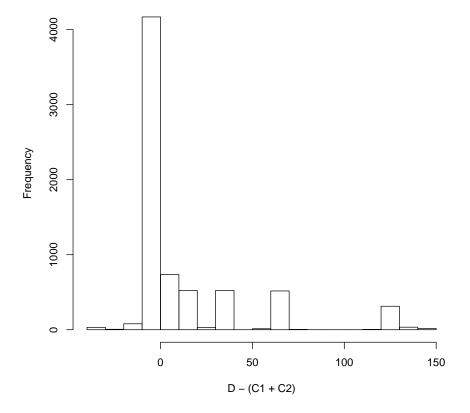
Table 11: Test Data

```
# create test inputs
C1 = sample(0:127, 7000, replace=TRUE)
C2 = sample(0:127, 7000, replace=TRUE)
```

Now predict using the predictr() function.

Table 12: predictr()

Histogram of D - (C1 + C2)



5 Conclusion

CRAN and the rest of the R ecosystem show that there is a strong interest in using the R language for neural network analysis. Existing package such as the built in nnet package and the caret package make available very powerful neural network tools to R users. The RSNNS package acts as an R wrapper for the Stutgard Neural Network Simulator library, which is written in C, and thereby makes available to partial RNNs such as Elman and Jordan networks.

The enormous popularity of full Recurrent Neural Networks in other languages, primarily Python and C, show that there is a great amount of interest for using this methodology, including interest from Economist, Data Scientists, and other non-professional programmers. Although Python is a relatively accessible programming language for laymen, it has a smaller user base in terms of data analists. The rnn package attempts to address this need by showing that Recurrent Neural Networks can be made available and perhaps more importantly, made available in native R, which allows user to delve into the code and understand the method and developer a more thorough understanding of how to use it.

A Source code of trainr() function

1_Users_quast_rnn_R_trainr.R

```
@name \ trainr
 1
       @export
 2
   #' @importFrom stats runif
3
   #' @title Recurrent Neural Network
   \#' @description Trains a Recurrent Neural Network.
   #' @param Y vector of output values
   \#' @param X1 vector of input values
   \#' @param X2 vector of input values
8
   \#' @param binary_dim dimension of binary representation
   \#' @param alpha \overline{s}ize of alpha
10
   \#' @param input dim dimension of input layer, i.e. how many numbers
11
         to sum
   \#' @param \ hidden\_dim \ dimension \ of \ hidden \ layer
12
13
   \#' @param output dim dimension of output layer
   #' @param print should train progress be printed
14
   \#' @return a model to be used by the predictr function
15
   \#' @examples
16
   #' # create training inputs
17
   \#'X1 = sample(0:127, 7000, replace=TRUE)
   \#' X2 = sample(0:127, 7000, replace=TRUE)
19
   \#\,'\,\,\#\,\,create\ training\ output
21
22
   \#' Y < - X1 + X2
   #'
23
   \#' \# train the model
24
   \#' trainr(Y,
   #'
               X1.
26
   #'
27
               X2,
   #'
               binary\_dim = 8,
28
   #'
               alpha
                        = 0.1,
29
   #'
               input dim = 2,
               hidden_dim = 10,
output_dim = 1,
   #'
31
   #'
   #'
               print = 'full'
33
34
35
36
    \begin{array}{lll} trainr <& -function(Y,\ X1,\ X2,\ binary\_dim,\ alpha\,,\ input\_dim,\ hidden\_dim,\ output\_dim,\ print = c('none', 'minimal', 'full'))\ \{ \end{array}
37
38
      \#\ check\ what\ largest\ possible\ number\ is
39
      largest number = 2^binary dim
40
41
      # initialize neural network weights
42
      synapse 0 = matrix(stats::runif(n = input dim*hidden dim, min=-1,
43
           max=1), nrow=input dim)
      synapse 1 = matrix(stats::runif(n = hidden dim*output dim, min
44
          =-1, max=1), nrow=hidden_dim)
      synapse h = matrix(stats::runif(n = hidden dim*hidden dim, min
45
          =-1, \max=1), nrow=hidden dim)
46
      synapse 0 update = matrix(0, nrow = input dim, ncol = hidden dim)
47
```

```
synapse 1 update = matrix(0, nrow = hidden dim, ncol = output dim
48
      synapse\_h\_update = matrix(0, nrow = hidden dim, ncol = hidden dim
49
50
      # training logic
51
52
      for (j in 1: length(Y)) {
53
        if(print != 'none' && j %% 1000 == 0) {
54
55
          print(paste('Summation_number:', j))
56
57
        \# generate a simple addition problem (a + b = c)
58
        a int = X1[j] \# int version
        a = rev(as.numeric(intToBits(a_int))[1:binary_dim])
60
61
62
        b int = X2[j] # int version
        b = rev(as.numeric(intToBits(b int))[1:binary dim])
63
        \# true answer
65
        \mathbf{c} int = Y[j]
66
        \mathbf{c} = \mathbf{rev}(\mathbf{as}.\mathbf{numeric}(intToBits(\mathbf{c}\ int))[1:binary\ \mathbf{dim}])
67
68
        # where we'll store our best guesss (binary encoded)
69
        d = matrix(0, nrow = 1, ncol = binary_dim)
70
71
        overallError = 0
72
73
        layer 2 deltas = \mathbf{matrix}(0)
74
        layer_1_values = matrix(0, nrow=1, ncol = hidden_dim)
75
        \# layer_1\_values = rbind(layer_1\_values, matrix(0, nrow=1, ncol))
76
            =hi\overline{d}d\overline{e}n _dim))
        \# moving along the positions in the binary encoding
78
        for (position in 0:(binary dim-1)) {
79
80
          # generate input and output
81
82
          X = cbind(a[binary_dim - position], b[binary_dim - position])
          y = c[binary dim - position]
83
84
          # hidden layer (input ~+ prev hidden)
85
          layer 1 = sigmoid ((X%*%synapse 0) + (layer 1 values [dim(layer
86
               _1_values)[1],] %*% synapse h))
87
          # output layer (new binary representation)
88
          layer_2 = sigmoid(layer_1 %*% synapse 1)
89
90
          \# did we miss?... if so, by how much?
91
          layer_2_error = y - layer_2
layer_2_deltas = rbind(layer_2_deltas, layer_2_error *
92
93
               sigmoid_output_to_derivative(layer_2))
          overallError = overallError + abs(layer 2 error)
94
95
          \# decode estimate so we can print it out
96
97
          d[binary dim - position] = round(layer 2)
98
          # store hidden layer so we can print it out
99
```

```
layer 1 values = rbind(layer 1 values, layer 1)
100
101
           if(print = 'full' && j %% 1000 == 0) {
102
             print(paste('x1:', a[binary_dim - position]))
print(paste('x2:', b[binary_dim - position], '+'))
103
104
             print ( '--
                            - ' )
105
             print(paste('y: ', c[binary_dim - position]))
print(paste('y^:', d[binary_dim - position]))
106
107
             print('===')
108
109
          }
110
111
         future layer 1 delta = matrix(0, nrow = 1, ncol = hidden dim)
112
113
         for (position in 0:(binary dim-1)) {
114
115
116
          X = cbind(a[position + 1], b[position + 1])
           layer 1 = layer 1 \ values [dim(layer 1 \ values)[1] - position,]
117
           prev_layer_1 = \overline{lay}er_1_values[dim(\overline{lay}er_1_values)[1] - (
               \overline{position} + 1),
119
           \# error at output layer
120
           layer_2_delta = layer_2_deltas[dim(layer_2_deltas)[1] -
121
               position,
           # error at hidden layer
122
           layer 1 delta = (future layer 1 delta \% t (synapse h) +
               124
             sigmoid output to derivative (layer 1)
125
          # let's update all our weights so we can try again
126
           synapse 1 update = synapse 1 update + matrix(layer 1) %*%
127
               layer 2 delta
           synapse h update = synapse h update + matrix(prev layer 1) %*
128
               % layer 1 delta
           synapse 0 \overline{update} = synapse 0 \overline{update} + \overline{t}(X) %*% layer 1 delta
129
130
           future\_layer\_1\_delta = layer\_1\_delta
131
132
133
        134
135
         synapse h = synapse h + ( synapse h update * alpha )
136
137
         synapse_0_update = synapse_0_update * 0
138
         synapse 1 update = synapse 1 update * 0
139
140
         synapse h update = synapse h update * 0
141
        \# print out progress
142
         if(print != 'none' &  1000 == 0)  {
143
           print(paste('Error:', overallError))
144
           print(paste('X1[', j, ']:', paste(a, collapse = '_'), '_', '(
145
               ', a int, ')'))
           print(paste('X2[', j, ']:', paste(b, collapse = '_'), '+', '(
146
                  b_int , ')'))
           print ( '-
147
           print(paste('Y[', j, ']: ', paste(c, collapse = '.'), '.', '(
148
               ', c_int, ')'))
```

```
\# convert d to decimal
149
          out = packBits(as.raw(rev(c(rep(0, 32-binary dim), d))),
              integer')
          print(paste('predict_Y^:',
                                         paste(d, collapse = '__'), '__', '
151
              (', out, ')'))
          print ( '=
152
153
154
155
156
      \# output object with synapses
      return(list(synapse 0 = synapse 0, synapse 1 = synapse 1, synapse
157
          _{h} = synapse_{h})
158
```

B Source code of predictr() function

2 Users quast rnn R predictr.R

```
@name \ predictr
   #' @export
2
   \#' @importFrom stats runif
   #' @title Recurrent Neural Network
   \#' @description Trains a Recurrent Neural Network.
   \#' @param model output of the trainr function
   \#' @param X1 vector of input values
   \#' @param X2 vector of input values
   \#' \ @param \ binary\_dim \ dimension \ of \ binary \ representation
   \#' @param alpha \overline{s}ize of alpha
   \#' @param input_dim dimension of input layer, i.e. how many numbers
         to sum
   \#' @param hidden dim dimension of hidden layer
12
   #' @param output dim dimension of output layer
13
   \#' @param print \overline{s} hould train progress be printed
   \#' @return vector of predicted values
15
   \#' @examples
   \#\,'\,\,\#\,\,create\ training\ inputs
   #' X1 = sample(0:127, 7000, replace=TRUE)
#' X2 = sample(0:127, 7000, replace=TRUE)
20
   \#' \# create training output
21
   #' Y <- X1 + X2
22
   #'
23
   \#' \# train the model
24
   \#' m1 < - trainr(Y,
25
   #'
                      X1,
   #'
                      X2,
27
   #'
                      binary\_dim = 8,
   #'
                              = 0.1,
29
                      alpha
                      input dim = 2,
30
   #'
                      hidde\overline{n} dim = 10,
31
                      output dim = 1
32
   #' # create test inputs
  \#'A1 = sample(0:127, 7000, replace=TRUE)
```

```
|\#'| A2 = sample(0:127, 7000, replace=TRUE)
36
   #'
37
   \#' \# predict
38
   \left| \stackrel{"}{\#} \right| \stackrel{"}{B} \stackrel{"}{<-} predictr(m1,
39
   #'
                        A1.
40
   #'
                        A2,
41
   #'
42
                        binary dim = 8,
   #'
                        alpha
                                = 0.1,
43
   #'
                        input dim = 2,
44
   #'
                        hidden_dim = 10,
output_dim = 1
45
46
47
   #' # inspect the differences
48
   \#' table (B-(A1+A2))
49
50
51
52
   predictr <- function (model, X1, X2, binary dim, alpha, input dim,
53
        hidden dim, output dim, print = c('none', 'minimal', 'full')) {
54
      # check what largest possible number is
55
      largest number = 2^binary dim
56
57
58
      # create output vector
     Y <- vector(mode = 'integer', length = length(X1))
59
60
61
     # load neural network weights
62
      synapse 0 = model\$synapse 0
63
      synapse_1 = model synapse_1
64
      synapse h = model$synapse h
65
66
     \# synapse 0 update = matrix(0, nrow = input dim, ncol = hidden
67
          dim.)
       synapse\_1\_update = matrix(0, nrow = hidden dim, ncol = output
68
          dim)
      \# \ synapse\_h\_update = matrix(0, nrow = hidden\_dim, ncol = hidden\_
69
          dim)
70
71
      # training logic
      for (j in 1: length(X1)) {
72
73
        if(print != 'none' && j %% 1000 == 0) {
74
          print(paste('Summation_number:', j))
75
76
77
        \# generate a simple addition problem (a + b = c)
78
79
        a int = X1[j] \# int version
        a = rev(as.numeric(intToBits(a int))[1:binary dim])
80
81
        b_{int} = X2[j] \# int version
82
        b = rev(as.numeric(intToBits(b int))[1:binary dim])
83
84
        # where we'll store our best guesss (binary encoded)
85
        d = matrix(0, nrow = 1, ncol = binary dim)
86
87
        overallError = 0
88
```

```
89
          layer_2_deltas = matrix(0)
layer_1 values = matrix(0, nrow=1, ncol = hidden dim)
90
91
          \# layer^{-1} values = rbind(layer 1 values, matrix(\overline{0}, nrow=1, ncol))
92
               =hi\overline{d}d\overline{e}n\ dim))
93
94
          \# moving along the positions in the binary encoding
          for (position in 0:(binary dim-1)) {
95
96
97
             \# generate input and output
             X = cbind(a[binary dim - position], b[binary dim - position])
98
99
             # hidden layer (input ~+ prev hidden)
100
             layer 1 = \operatorname{sigmoid}((X\% - y) + (\operatorname{layer} 1 \operatorname{values} \operatorname{\mathbf{dim}}(\operatorname{layer} 1))
                  \overline{1} values) [1], \%*% synapse h))
102
103
             \# output layer (new binary representation)
             layer 2 = sigmoid(layer 1 %*% synapse 1)
104
105
             \#\ decode\ estimate\ so\ we\ can\ print\ it\ out
106
             d[binary dim - position] = round(layer 2)
107
108
             # store hidden layer so we can print it out
109
             layer 1 values = rbind(layer 1 values, layer 1)
110
111
             if(print == 'full' & j %% 1000 == 0) {
               print(paste('x1:', a[binary_dim - position]))
print(paste('x2:', b[binary_dim - position], '+'))
113
114
                             ____')
                print ( '---
115
                print(paste('y^:', d[binary_dim - position]))
116
                print('===')
117
118
             }
119
120
          # output to decimal
121
          out = packBits(as.raw(rev(c(rep(0, 32-binary dim), d))),
122
               integer')
123
          \# synapse\_0 = synapse\_0 + ( synapse\_0\_update * alpha )
124
          \# synapse\_1 = synapse\_1 + (synapse\_1\_update * alpha) \\ \# synapse\_h = synapse\_h + (synapse\_h\_update * alpha)
125
126
127
          \# \ synapse\_0\_update = synapse\_0\_update * 0
          129
130
131
          \# print out progress
132
          if(print != 'none' && j %% 1000 == 0) {
             print(paste('Error:', overallError))
134
             print(paste('X1[', j, ']:', paste(a, collapse = 'o'), 'o', '(
    ', a_int, ')'))
135
              \mathbf{print} \left( \mathbf{paste} \left( \ 'X2[\ '\ ,\ j\ ,\ '\ ] \colon '\ ,\ \mathbf{paste} \left( b\ ,\ \mathbf{collapse} \ =\ '\ '\ ' \right) \right),\ '+'\ ,\ '(
136
                  ', b_int, ')'))
             print ('-
137
             \# convert d to decimal
139
```