Bayesian Statistics, Problem Set 11 (Nov. 5) SOLUTION

The Mug Breakage Problem

Step 1 — Computing the Likelihood Function

Act as if a were known. (Remember the likelihood is $P(\text{data} \mid a)$. That means we are **pretending** that a is given.

(a) Write down the probability of having 1 mug broken in the first week. You may have to remind yourself the formula for the Poisson distribution. It was on p. 59 of Young. You are just being asked to plug $n = 1$ into this formula.

$$\frac{a^1}{1!} e^{-a} = a\, e^{-a}$$

(b) Repeat, but with $n = 6$ for the second week.

$$\frac{a^6}{6!} e^{-a} = \frac{a^6}{720} e^{-a}$$

(c) Repeat, but with $n = 4$ for the third week.

$$\frac{a^4}{4!} e^{-a} = \frac{a^4}{24} e^{-a}$$

(d) Take the product of what you got in (a), (b), and (c). This is the likelihood of getting this data.

$$a\, e^{-a} \times \frac{a^6}{720} e^{-a} \times \frac{a^4}{24} e^{-a} = \frac{a^{11}}{720 \cdot 24} e^{-3a}$$

What a crazy function!

Step 2 — Tabulate the Likelihood

In[70]:= `poisson[n_, a_] :=` $\dfrac{a^n}{\text{Factorial}[n]}$ `Exp[-a];`

`likelihood[{n1_, n2_, n3_}, a_] := poisson[n1, a] * poisson[n2, a] * poisson[n3, a];`

`TableForm[Table[{a, likelihood[{1, 6, 4}, a]}, {a, 0, 8, 0.5}],`
` TableHeadings → {None, {"a", "P(data|a)"}}]`

Out[71]//TableForm=

a	P(data|a)
0.	0.
0.5	6.30499×10^{-9}
1.	2.8812×10^{-6}
1.5	0.0000556077
2.	0.000293778
2.5	0.000763111
3.	0.00126514
3.5	0.00153855
4.	0.00149136
4.5	0.00121568
5.	0.000864389
5.5	0.000550284
6.	0.000319756
6.5	0.000172092
7.	0.0000867662
7.5	0.0000413527
8.	0.0000187663

Step 3 — Graph the Likelihood

In[72]:= `ListPlot[Table[{a, likelihood[{1, 6, 4}, a]}, {a, 0, 8, 0.5}],`
`  PlotRange → {{0, 8}, {0, 0.002}}]`

Out[72]=
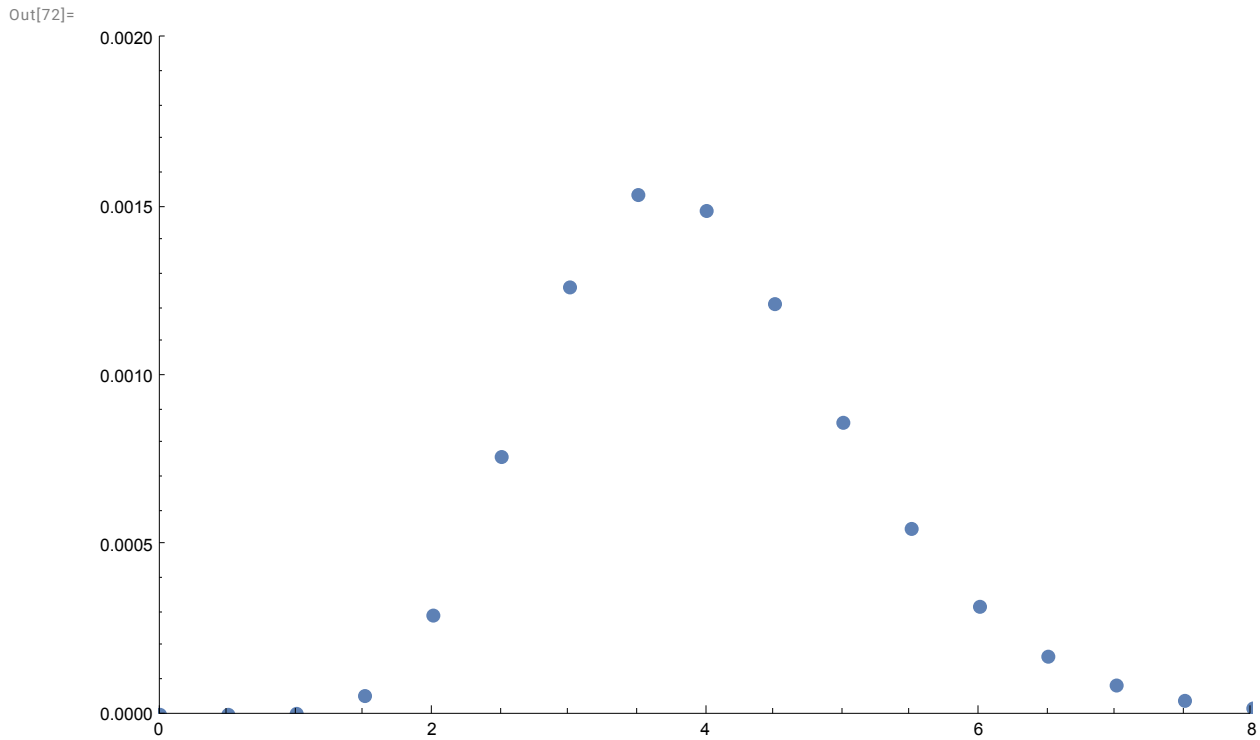
Once you have all the points plotted, freehand a nice smooth curve through them.

Step 4 — Tabulate the Product of the Likelihood and the Prior

In Step 2, you tabulated the likelihood. Take a look at the graph of the prior. It is 0 unless a is between 2 and 6. If a is between 2 and 6, it is 1/4. Using these facts, you should be able to quickly tabulate the product of the likelihood and the prior.

```
In[77]:=  prior[a_] := If[2 ≤ a ≤ 6, 0.25, 0];
          product[a_] := likelihood[{1, 6, 4}, a] * prior[a];
          TableForm[Table[{a, product[a]}, {a, 0, 8, 0.5}],
           TableHeadings → {None, {"a", "P(data|a)*P(a)"}}]
```

Out[79]//TableForm=

a	P(data|a)*P(a)
0.	0.
0.5	0.
1.	0.
1.5	0.
2.	0.0000734445
2.5	0.000190778
3.	0.000316286
3.5	0.000384638
4.	0.00037284
4.5	0.000303919
5.	0.000216097
5.5	0.000137571
6.	0.0000799391
6.5	0.
7.	0.
7.5	0.
8.	0.

Step 5 — Graph the Product of the Likelihood and the Prior

In Step 4, you tabulated the product of the likelihood and the prior. Now you are going to graph it.

In[80]:= `Plot[{a, product[a]}, {a, 0, 8}, PlotRange → {{0, 8}, {0, 0.0004}}]`

Out[80]=
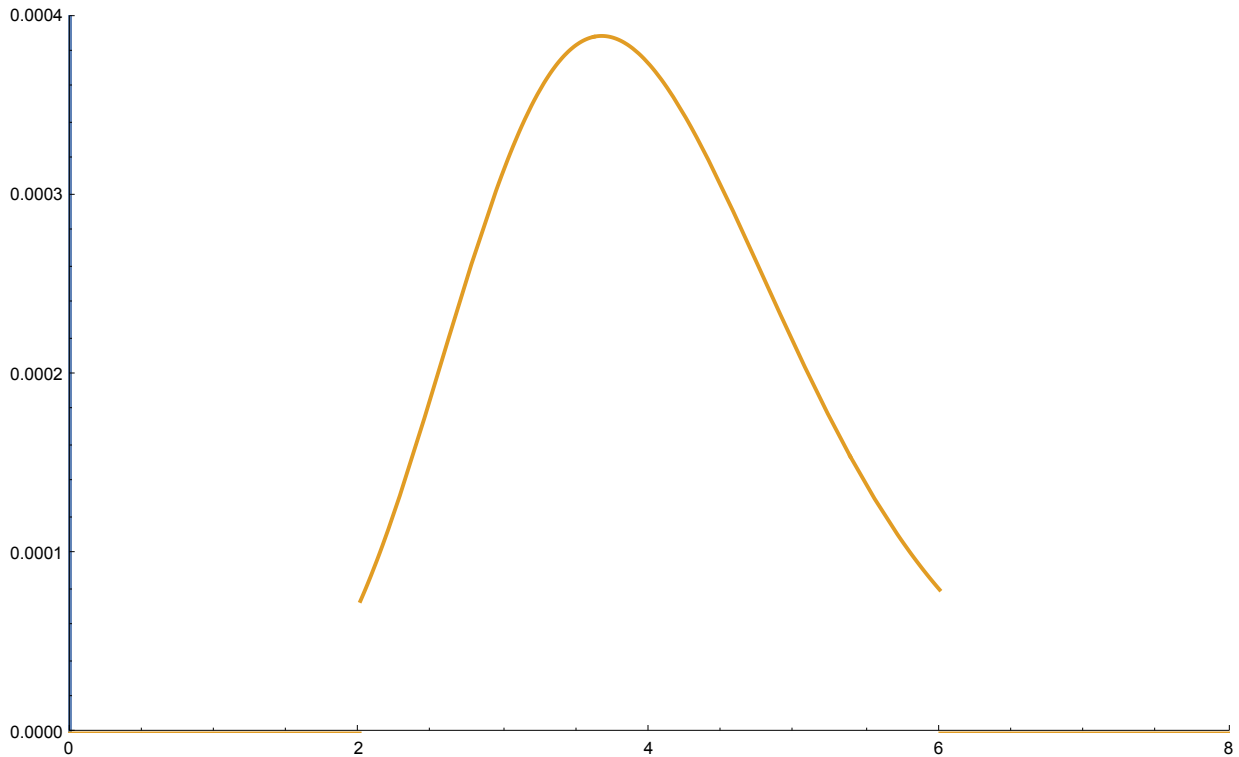
Remaining Steps

There are still two a few more steps to go!! Remember that we are trying to compute

$$P(a^* \mid \text{data})\, \Delta a^* = \frac{P(\text{data} \mid a^*)\, P(a^*)\, \Delta a^*}{\int_{a_{\min}}^{a_{\max}} P(\text{data} \mid a)\, P(a)\, da}$$

Step 6 is to do the integral in the denominator,
Step 7 is to make a table of the posterior, which is $P(\text{data} \mid a^*)\, P(a^*)$ divided by the denominator from Step 6.
Step 8 is to plot the posterior.
Step 9 is to interpret the posterior.

HOWEVER! We have to leave something for our presenters to do, so we will stop at Step 5.