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A robust varying coefficient approach to fuzzy multiple regression model

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ABSTRACT

The varying coefficient models are powerful tools for exploring the dynamic pattern between a response and a group of predictors in multiple regression models. In addition, robust regression is another solid approach in the regression analyses for cases whose data are contaminated with outliers or influential observations. This paper proposed a novel varying coefficient model with exact predictors and fuzzy responses which can be used in cases where outliers occur in the data set. For this purpose, a locally weighted approximation idea and a popular M-estimator were combined to estimate unknown fuzzy (nonparametric) varying coefficients. Some common goodness-of-fit criteria including an outlier detection criterion were also applied to examine the performance of the proposed method. The effectiveness of the presented method was then illustrated through two numerical examples including a simulation study. It was also compared with several common fuzzy multiple regression models. The numerical results clearly indicate that the proposed method is not sensitive to the outliers. Moreover, compared to the available fuzzy multiple regressions with constant coefficients, the proposed fuzzy varying coefficient model managed to provide more accurate results.
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1. Introduction

As the most basic statistical technique with extensive application, multiple regression analysis estimates the relationships between independent predictor variables and a dependent response variable. Although their properties have been well discussed in the literature, the parametric regression models are often unrealistic in applications [1]. In this context, varying coefficient models were introduced by Hastie and Tibshirani [2] which were more flexible than the traditional multiple linear regression model since they allow the regression coefficients (with another covariate) to smoothly vary in more than one dimension. Such a structural nonparametric regression can effectively avoid the curse of the model parameter heterogeneity (see, for instance [3–6]). On the contrary, as a solution to control the influential observations on over-fitting [7,8], the robust regression has gained a considerable deal of attention in the regression analysis. In this regard, M-estimation was introduced by Huber and has become the most common method of robust regression [8]. It generally offers higher accuracies compared to the least-square methods as it uses a weighting mechanism to weigh down the influential observations. However, traditional regression analyses often rely on exact information such as data or coefficients. Since introduced by Tanaka et al. [9] the fuzzy regression methods have been widely used in real-life applications. Such methods can be categorized in two classes: 1- the observations of the predictors can be either fuzzy

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numbers [10–28], or 2- real value quantities [9,29–44]. Moreover, the robust fuzzy linear regression techniques have been successfully applied at the presence of outliers over the past decades [45–50].

As a compromise between the fully nonparametric and the parametric models, it is worth noting that the varying coefficient models are capable of offering desirable flexibility to explore the hidden structure of the standard parametric regression models without running into the serious dimensionality issues [51]. In this paper, a common varying coefficient multiple regression model, introduced by Fan and Zhang [52], was extended in the fuzzy environment. For this purpose, a common and popular robust estimation method, called M-estimation [8] was employed in cases with exact predictors and fuzzy responses. The performance of the proposed method was also compared with several existing fuzzy regression models in terms of some common goodness-of-fit criteria. For practical reasons, the proposed method was further evaluated through a simulated study as well as an applied example. The numerical investigation clearly indicated that the proposed method provides sufficiently accurate results compared to the other methods at the presence of outliers in the data set.

The rest of this paper is organized as follows: Section 2 reviews some relevant concepts of the fuzzy numbers which will be used in the next sections. In Section 3, a methodology is proposed to estimate the fuzzy varying coefficients of a fuzzy multiple regression model with exact predictors and fuzzy responses. Section 4 illustrates two numerical examples to evaluate the effectiveness and performance of the proposed method relative to the other fuzzy multiple regression methods in terms of some common performance measures. Finally, the main contributions of this paper are summarized in Section 5.

2. Fuzzy numbers

This section reviews some basic definitions of fuzzy numbers based on [16].

A fuzzy set $\widetilde{A}$ of $\mathbb{R}$ (the real line) is defined by its membership function $\mu_{\widetilde{A}} : \mathbb{R} \to [0, 1]$ [53]. In addition, a fuzzy set $\widetilde{A}$ of $\mathbb{R}$ is called a fuzzy number if it is normal, i.e. there is a unique $x^*_{\widetilde{A}} \in \mathbb{R}$ so that $\mu_{\widetilde{A}}(x^*_{\widetilde{A}}) = 1$; and for every $\alpha \in [0, 1]$, the set of $\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\}$ is a nonempty compact interval in $\mathbb{R}$. This interval is denoted by $\widetilde{A}[\alpha] = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$, where $\widetilde{A}^L_\alpha = \inf\{x : x \in \widetilde{A}[\alpha]\}$ and $\widetilde{A}^U_\alpha = \sup\{x : x \in \widetilde{A}[\alpha]\}$. It is worth noting that fuzzy numbers are approximate assessments, given by experts and accepted by decision-makers when access to more accurate values is either impossible or unnecessary. To simplify the fuzzy numbers representation and handling, several authors have captured the information contained in a (unimodal) fuzzy number by the help of a functional parametric form known as *LR*-fuzzy number $\widetilde{A} = (a; l_a, r_a)_{LR}$ where $l_a, r_a > 0$. The membership function of an *LR*-fuzzy number of $\widetilde{A}$ is defined by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\dfrac{a-x}{l_a}), & x \leq a, \\ R(\dfrac{x-a}{r_a}), & x > a, \end{cases} \tag{1}$$

where L and R are continuous and strictly decreasing functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. An *LR*-number has been applied in various problems as a general function to model imprecision. In this paper, we employed the most commonly used *LR*-fuzzy numbers so-called triangular fuzzy numbers (**TFN**s), to handle the imprecision in data set during numerical evaluations. The membership function of a triangular fuzzy number, denoted by $\widetilde{A} = (a; l_a, r_a)_T$, is given by:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-(a-l_a)}{l_a} & a - l_a \leq x \leq a, \\ \frac{a+r_a-x}{r_a} & a \leq x \leq a + r_a, \\ 0 & x \in \mathbb{R} - [a - l_a, a + r_a]. \end{cases} \tag{2}$$

In addition, an L^p distance measure between the two fuzzy numbers of $\widetilde{A}$ and $\widetilde{B}$ was employed which can be defined as [33]:

$$D_p(\widetilde{A}, \widetilde{B}) = (\int_0^1 f(\alpha)(\frac{|\widetilde{A}^L_\alpha - \widetilde{B}^L_\alpha|^p + |\widetilde{A}^U_\alpha - \widetilde{B}^U_\alpha|^p}{2})d\alpha)^{1/p}, \ p \geq 1, \tag{3}$$

where $f(\alpha) = 2\alpha$. Application of $f(.)$ considers less significance to such heavy spreads compared to the values near the center. For any fuzzy numbers of $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$, D_p satisfies the following conditions:

- $D_p(\widetilde{A}, \widetilde{B}) = 0$ if and only if $\widetilde{A} = \widetilde{B}$,
- $D_p(\widetilde{A}, \widetilde{B}) = D_p(\widetilde{B}, \widetilde{A})$,
- $D_p(\widetilde{A}, \widetilde{C}) \leq \big(D_p(\widetilde{A}, \widetilde{B}) + D_p(\widetilde{B}, \widetilde{C})\big)$.

Both D_1 and D_2 were employed in optimization algorithm and performance evaluations in the next section.

3. Robust varying fuzzy coefficient model

Inspired by Fan and Zhang [52], a robust-based varying coefficient model is introduced with non-fuzzy predictors, fuzzy responses and fuzzy varying coefficients. Here, their method is first briefly reviewed. Let $(\boldsymbol{x}_i, y_i, z_i)$ be the observation collected from the ith subject ($1 \le i \le n$), where y_i is the response of interest, $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{id})^\top$ shows the d-dimensional predictor, and $z_i \in [0, 1]$ denotes the so-called univariate index variable. A typical varying coefficient model assumes that $y_i = \sum_{j=0}^{d} \beta_j(z_i)x_{ij} + \epsilon_i$. Coefficient vector of $\boldsymbol{\beta}(z) = (\beta_0(z), \ldots, \beta_d(z))^\top$ is an unknown but smooth function in z. For an arbitrary index value of $z_i \in [a, b]$, $\boldsymbol{\beta}(z)$ can be estimated by minimizing the following locally weighted least squares function:

$$Q(\boldsymbol{B}) = \sum_{t=1}^{n}\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}_t)^2 K_h(z_i - z_t), \tag{4}$$

where $\boldsymbol{B} = (\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_n)^\top \in \mathbb{R}^{n\times d}$, $\boldsymbol{\beta}_t = (\beta_0(z_t), \beta_1(z_t), \ldots, \beta_d(z_t))^\top$, and K is a kernel. Now, assume that the observed data on n statistical units are denoted by $(\widetilde{y}_i, \boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{id})^\top, z_i)$ where $\widetilde{y}_i$ are **TFN**s. Notably, triangular fuzzy numbers can serve as an alternative tool for the same purpose to easily describe the fuzziness for most of the experts (as well as non-experts). Another advantage of using triangular fuzzy numbers is their feasible arithmetic operations. Based on the aforementioned data set, we will consider the following fuzzy varying coefficient multiple regression model:

$$\widetilde{y}_i = \widetilde{y}_i^*(z_i) \oplus \widetilde{\epsilon}_i, \ \ i = 1, 2, \ldots, n, \tag{5}$$

where

1. $\widetilde{y}_i = (y_i; l_{\widetilde{y}_i}, r_{\widetilde{y}_i})_T$ denote the fuzzy responses,
2. $\widetilde{y}_i^*(z_i) = \oplus_{j=0}^{d}(\widetilde{\beta}_j(z_i) \otimes x_{ij})$ where $x_{i0} = 1$
3. x_{ij}'s ($i = 1, 2, \ldots, n, j = 0, 1, 2, \ldots, d$) are non-fuzzy predictors for fuzzy linear regression,
4. $\widetilde{\beta}_j(z_i) = (\beta_j(z_i); l_{\beta_j(z_i)}, r_{\beta_j(z_i)})_T$ present unknown fuzzy varying coefficients to be estimated,
5. z_i are the observations of a distribution function on $[a, b]$ and
6. $\widetilde{\epsilon}_i$ indicates a fuzzy error term.

Remark 3.1. Applying the arithmetic operations of **TFN**s, the fuzzy responses can be shown by $\widetilde{y}_i^* = (y_i^*; l_{y_i^*}, r_{y_i^*})_T$ where

$$\widetilde{y}_i^*(z_i) = \sum_{j=0}^{n} x_{ij}\beta_j(z_i),$$

$$l_{y_i^*(z_i)} = \sum_{j=0}^{n}(s_{ij}x_{ij}l_{\beta_j(z_i)} - (1 - s_{ij})x_{ij}r_{\beta_j(z_i)}),$$

$$r_{y_i^*(z_i)} = \sum_{j=0}^{n}(s_{ij}x_{ij}r_{\beta_j(z_i)} - (1 - s_{ij})x_{ij}l_{\beta_j(z_i)}),$$

in which $s_{ij} = I(x_{ij} \ge 0)$ for $i = 1, 2, \ldots, n, j = 0, 1, 2, \ldots, d$. In addition, unknown fuzzy varying coefficients of model (5) can be presented as $\boldsymbol{B} = (B; \mathbf{L}_B, \mathbf{R}_B)_T$ where

$$B = \begin{pmatrix} \beta_{10} & \beta_{11} & \cdots & \beta_{1d} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_{n0} & \beta_{n1} & \cdots & \beta_{nd} \end{pmatrix}, \tag{6}$$

$$\mathbf{L}_B = \begin{pmatrix} l_{\beta_{10}} & l_{\beta_{11}} & \cdots & l_{\beta_{1d}} \\ \vdots & \vdots & \cdots & \vdots \\ l_{\beta_{n0}} & l_{\beta_{n1}} & \cdots & l_{\beta_{nd}} \end{pmatrix}, \tag{7}$$

$$\mathbf{R}_B = \begin{pmatrix} r_{\beta_{10}} & r_{\beta_{11}} & \cdots & r_{\beta_{1d}} \\ \vdots & \vdots & \cdots & \vdots \\ r_{\beta_{n0}} & r_{\beta_{n1}} & \cdots & r_{\beta_{nd}} \end{pmatrix}, \tag{8}$$

in which $\beta_{ij} = \beta_j(z_i)$, $l_{\beta_{ij}} = l_{\beta_j(z_i)}$ and $r_{\beta_{ij}} = r_{\beta_j(z_i)}$. It should be mentioned that Wang and Xia [54] showed the limited effect of the index variables on the model performance. Therefore, we only focused on uniform distribution for index distribution in this paper.

To be robust against the heavy-tailed errors or outliers in the fuzzy responses, instead of the least square part of the target function Q (Eq. (4)), we applied a weighted Huber's criterion [55] $(\rho_k(y_i, \widehat{y}_i) = (0.5|y_i - \widehat{y}_i|^2)I(|y_i - \widehat{y}_i| < k) + (k/|y_i - \widehat{y}_i|)I(|y_i - \widehat{y}_i| \ge k))$ based on fuzzy data. To this end, the following optimization criterion is suggested for evaluating $\boldsymbol{B} = (B; \mathbf{L}_B, \mathbf{R}_B)_T$:

$$\widehat{\boldsymbol{B}} = \arg\min_{I} \sum_{t=1}^{n} \sum_{i=1}^{n} w_k(\widetilde{y}_i, \widetilde{y}_i^*)\rho_k(\widetilde{y}_i, \widetilde{y}_i^*(z_t))K_h(z_i - z_t), \tag{9}$$

in which

-
$$I = \{B \in \mathbb{R}^{n\times(d+1)}, \mathbf{L}_B \in (\mathbb{R}^+)^{n\times(d+1)}, \mathbf{R}_B \in (\mathbb{R}^+)^{n\times(d+1)}\}, \tag{10}$$
- D_1 and D_2 denote the absolute and square error distance between two fuzzy numbers, respectively,
- ρ_k represents the extended Huber's criterion for the fuzzy data:

$$\rho_k(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) = \begin{cases} 0.5D_1^2(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) & D_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) < k, \\ kD_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) - 0.5k^2 & D_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) \geq k, \end{cases} \tag{11}$$

-
$$w_k(\widetilde{y}_i, \widetilde{y}_i^*) = \begin{cases} 1 & D_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) < k, \\ k/D_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) & D_1(\widetilde{y}_i, \widetilde{y}_i^*(z_t)) \geq k, \end{cases} \tag{12}$$
- $K(.)$ is a kernel satisfying the Mercer's conditions [56],
- k is a constant tuning and
- $h > 0$ controls the amount of smoothing called the bandwidth of K.

Remark 3.2. To examine the performance of the proposed fuzzy semi-parametric elastic net regression model, the following commonly used performance measures were employed to estimate and compare the prediction accuracy of different models:

1. Root mean square error:
$$RMSE = \sqrt{\frac{\sum_{i=1}^{n} D_2^2(\widetilde{y}_i, \widetilde{\widehat{y}}_i)}{n}}. \tag{13}$$
2. Mean absolute relative error:
$$MARE = \frac{1}{n}\sum_{i=1}^{n} \frac{\int_0^1 |\widetilde{y}_i(x) - \widetilde{\widehat{y}}_i(x)|dx}{\int_0^1 \widetilde{y}_i(x)dx}. \tag{14}$$
3. Similarity measure [16]:
$$MSM = \frac{1}{n}\sum_{j=1}^{n} S_{UI}(\widetilde{\widehat{y}}_j, \tilde{y}_j), \tag{15}$$
where
$$S_{UI}(\widetilde{\widehat{y}}_j, \tilde{y}_j) = \frac{Card(\widetilde{\widehat{y}}_j \cap \tilde{y}_j)}{Card(\widetilde{\widehat{y}}_j \cup \widehat{y}_j)},$$
in which $\cap, \cup$ denote the intersection and union operators on the space of fuzzy numbers, respectively; and $Card(\widetilde{A})$ shows the cardinal number of $\widetilde{A}$. Note that $MSM \in [0, 1]$. Moreover, $MSM = 0$ if and only if $\widetilde{\widehat{y}}_j \cap \tilde{y}_j = \emptyset$ (the worst prediction result) and $MSM = 1$ if and only if $\widetilde{\widehat{y}}_j = \tilde{y}_j$ (the best prediction result). Therefore, an MSM value near 1 can indicate a perfect fit, and it is thus a reliable model for future prediction; while an MSM value near 0 is indicative of the model failure inaccurate modeling of the available data set.
4. Coefficient of determination (COD):
$$COD = \frac{\sum_{i=1}^{n} D_2^2(\widetilde{y}_i, \widetilde{\widehat{y}}_i)}{\sum_{i=1}^{n} D_2^2(\widetilde{y}_i, \widetilde{\bar{y}})}. \tag{16}$$
The coefficient of determination was used to explain the extent of predictors variability caused by its relationship to responses. It should be noted that $0 \leq COD \leq 1$. COD values near to 0 indicate that the regression model does

not fit the set of data points while a closer value to 1 shows that the regression model perfectly fits the set of data points.

5. Area under coverage [57]:

$$AUCR_m = \frac{1}{2}\sum_{i=m}^{n}(RMSE_i + RMSE_{i-1})(coverage_i - coverage_{i-1}), \tag{17}$$

where

$$RMSE_i = \frac{\sum_{j=1}^{i} D_2^2(\widetilde{y}_j, \widetilde{\widetilde{y}}_j)}{n}, \ coverage_i = \frac{i}{n}.$$

Low AUCR between two regression models of A and B means that A acts better in setting the predicting accuracy than B.

In addition, to examine the relationship between $\widetilde{y}$s and $\widetilde{\widehat{y}}$s, we also applied Sugeno center of gravity [58] to convert $\widetilde{y}$s and $\widetilde{\widetilde{y}}$s to the exact values of $M_{\widetilde{y}}$s and $M_{\widetilde{\widehat{y}}}$s. Then, according to the conventional regression models, the relationship between the defuzzified values of $M_{\widetilde{y}}$s ($M_{\widetilde{\widehat{y}}}$s) was investigated according to their plots. It is worth noting that the center of gravity of $\widetilde{A} = (a; l_a, r_a)_T$ can be evaluated as:

$$M_{\widetilde{A}} = a + (r_a - l_a)/3. \tag{18}$$

Furthermore, an extended criterion was employed for fuzzy data to detect the outliers called Cook's distance based on the fuzzy data [44]. To identify the potential outliers, $\overline{d}$-chart with respective lower and upper control limits of $LCL = \overline{D} - 3\frac{S_D}{\sqrt{n}}$ and $UCL = \overline{D} + 3\frac{S_D}{\sqrt{n}}$ were employed.

3.1. Algorithm for estimating model's components

To estimate the unknown components of model (5) including $\boldsymbol{B}$, tuning constant (k) and bandwidth (h) should be simultaneously estimated based on a set of observed values of $(\widetilde{y}_1, \mathbf{x}_1^\top), \ldots, (\widetilde{y}_n, \mathbf{x}_n^\top)$ and a specified kernel function of K. Since all these parameters are connected to each other, a hybrid optimization algorithm is required. However, regarding the strong influence of the tuning parameter k and bandwidth h on the degree of estimation of $\widehat{\boldsymbol{B}}$, by focusing on a popular triweight kernel [59], the optimal value of tuning constant and bandwidth were evaluated using the mean absolute error and cross-validation criteria, i.e. $k_{opt} = \arg\min_{k>0}(1/n)\sum_{i=1}^{n} D_1(\widetilde{y}_i, \widetilde{\widetilde{y}}_i)$ and $h_{opt} = arg\ \min_{h>0} CV$ where

$$CV = \frac{1}{n}\sum_{j=1}^{n} D_2^2(\widetilde{y}_j, \widetilde{\widehat{y}}^{(j)}), \tag{19}$$

in which $\widetilde{\widehat{y}}^{(j)}, j = 1, 2, \ldots, n$ denotes the estimated value of fuzzy response related to the smoothing parameter of h based on the data $(\widetilde{y}_i, \boldsymbol{x}_i^T)$ in which $i \neq j$ [60]. For this purpose, an Algorithm 1 was suggested to find $\widehat{\boldsymbol{B}}$, optimal bandwidth and tuning constant of k. According to this algorithm, $\widehat{\boldsymbol{B}}^{(i+1)}$ and $\widehat{k}^{(i+1)}$ are the optimal estimators. The least local approximation idea was employed to optimize the target functions. For this purpose, an iterative procedure was implemented in Mathematica software [61].

4. Application examples

The feasibility and effectiveness of the proposed fuzzy robust-based varying coefficient multiple regression model were examined and compared with some common fuzzy multiple regression models. Moreover, the proposed regression model was also assessed in terms of several common fuzzy multiple regression models. In this regard, fuzzy multiple regression models including Taheri and Kekinnama [28], Choi and Buckley [32], Zeng et al. [43], Kula and Apaydin [48] and Choi and Yoon [62] were considered. Each method claimed to perform better than the other fuzzy multiple regression models. In order to conduct a competitive study, the measures explained in Remark 3.2 were applied to calculate the goodness-of-fit criteria.

Example 4.1 (*A Simulation Study*)**.** Here, a set of $m = 10$ simulated data set with the size of $n = 200$ is generated according to the following fuzzy varying coefficient regression model:

$$\widetilde{y}_i = \oplus_{j=0}^{3}(\widetilde{\beta}_j(z_i) \otimes x_{ij}) \oplus \widetilde{\epsilon}_i, \quad i = 1, 2, \ldots, 200. \tag{20}$$

Algorithm 1

1: $i \leftarrow 0$
2: $\widehat{h}^{(i)} \leftarrow n^{-0.2}$ [56], $\widehat{k}^{(i)} \leftarrow (1.345/0.6745)MAR$ [55], where

$$MAR \leftarrow Median_{i=1}^{n} D_1(\widetilde{y}_i, \oplus_{j=0}^{d}(\widetilde{\widetilde{\beta}}_j \otimes x_{ij})),$$

$$(\widetilde{\widetilde{\beta}}_0, \ldots, \widetilde{\widetilde{\beta}}_d) \leftarrow \arg\min_{\widetilde{\beta}_j}(\frac{1}{n})\sum_{i=1}^{n} D_1(\widetilde{y}_i, \oplus_{j=0}^{d}(\widetilde{\beta}_j \otimes x_{ij})).$$

3: **if** $k \leftarrow \widehat{k}^{(i)}$ and $h \leftarrow \widehat{h}^{(i)}$ **then**,

$$\widehat{\boldsymbol{B}}^{(i)} \leftarrow \arg\min_{I} \sum_{t=1}^{n}\sum_{i=1}^{n} w_k(\widetilde{y}_i, \widetilde{y}_i^*(z_t))\rho_{\widehat{k}^{(i)}}(\widetilde{y}_i, \widetilde{y}_i^*(z_t))K_{h^{(i)}}(z_i - z_t).$$

4: **if** $\boldsymbol{B} \leftarrow \widehat{\boldsymbol{B}}^{(i)}$ and $k \leftarrow \widehat{k}^{(i)}$ **then** $\widehat{h}^{(i+1)} \leftarrow \arg\min_h CV$.
5: **if** $h \leftarrow \widehat{h}^{(i+1)}$, $\boldsymbol{B} \leftarrow \widehat{\boldsymbol{B}}^{(i)}$ **then** $\widehat{k}^{(i+1)} \leftarrow \arg\min_{k>0}(1/n)\sum_{i=1}^{n} D_1(\widetilde{y}_i, \widehat{\widetilde{y}}_i)$.
6: **if** $h \leftarrow \widehat{h}^{(i+1)}$, $k \leftarrow \widehat{k}^{(i+1)}$ **then**

$$\widehat{\boldsymbol{B}}^{(i+1)} \leftarrow \arg\min_{I} \sum_{t=1}^{n}\sum_{i=1}^{n} w_k(\widetilde{y}_i, \widetilde{y}_i^*(z_t))\rho_{\widehat{k}^{(i+1)}}(\widetilde{y}_i, \widetilde{y}_i^*(z_t))K_{\widehat{h}^{(1)}}(z_i - z_t).$$

7: **if**

$$\max_{l=1}^{d+1} \max\{\| \widehat{B}^{(i+1)}(l) - \widehat{B}^{(i)}(l) \|, \| \mathbf{L}_{\widehat{B}^{(i+1)}}(l) - \mathbf{L}_{\widehat{B}^{(i)}}(l) \|, \| \mathbf{R}_{\widehat{B}^{(i+1)}}(l) - \mathbf{R}_{\widehat{B}^{(i)}}(l) \|\} < \varepsilon,$$

then $\widehat{\boldsymbol{B}} \leftarrow \widehat{\boldsymbol{B}}^{(i+1)}$ ($C(l)$ denotes the l^{th} column of the matrix C).
else $\{i + 1 \leftarrow i\}$ and **Repeat loop**.

For this purpose, in each simulation process, a random sample of $(\widetilde{y}_i, (x_{i1}, x_{i2}, x_{i3})^T)$ is generated by the following steps:

Step 1:

1. Generate a random sample with $n = 200$ of fuzzy predictors $x_{ij}, j = 1, 2, 3$ where $x_{i1} \sim \exp(1)$, $(x_{i2}, x_{i3}) \sim N_2(3, \Sigma)$ in which $Cov(x_{i2}, x_{i3}) = 0.5$ and $Var(x_{i2}) = Var(x_{i3}) = 1$,
2. Let $\widetilde{t}_i = \widetilde{x}_{i1} \oplus \widetilde{x}_{i2} \oplus \widetilde{x}_{i3}$.

Step 2:

1. For a randomly selected $I_1 = \{i_1^1, i_2^1, \ldots, i_{10}^1\} \subseteq \{1, 2, 3, \ldots, 200\}$, let $\widetilde{y}_i = \widetilde{t}_i \oplus (20; u_1^i, u_2^i)_T$, $i \in I_1$.
2. For a randomly selected $I_2 = \{i_1^2, i_2^2, \ldots, i_{10}^2\} \subseteq \{1, 2, 3, \ldots, 200\} - I_1$, let $\widetilde{y}_i = \widetilde{t}_i \oplus (30; u_1^i, u_2^i)_T$, $i \in I_2$, where u_1^i and u_2^i are random variables observed from $U(1, 3)$ and $U(2, 4)$, respectively.

Step 3: Let $\widetilde{y}_i = \widetilde{t}_i$ for $i \in \{1, 2, 3, \ldots, 200\} - (I_1 \cup I_2)$.

Step 4: $\widetilde{\beta}_0(z_i) = (z_i + 1; 1, 0.5)$, $\widetilde{\beta}_1(z_i) = (\exp(2z_i - 1); \exp(2z_i - 1) - \exp(2z_i - 2), \exp(2z_i) - \exp(2z_i - 1))_T$, $\widetilde{\beta}_2(z_i) = (5z_i(1 - z_i); 2z_i(1 - z_i), 3z_i(1 - z_i))_T$ and $\widetilde{\beta}_3(z_i) = (\frac{3-z_i}{3+z_i}; \frac{3-z_i}{3+z_i} - \frac{2-z_i}{2+z_i}, \frac{3-z_i}{3+z_i} - \frac{1-z_i}{1+z_i})_T$.

Step 4: z_i are observed values from $U(0, 1)$.

With the help of the above simulation process, one could imagine a set of potential outliers in each simulation data set. In addition, we wish to compare the proposed method with the conventional fuzzy multiple regression models:

$$\widetilde{y}_i = \oplus_{j=0}^{3}(\widetilde{\beta}_j \otimes x_{ij}) \oplus \widetilde{\epsilon}_i. \tag{21}$$

To this end, the mean values of the performance measures along with other methods are reported in Table 1 in terms of $MSM = 0.86$ and $MARE = 7.39$, $RMSE = 6.22$, $AUCR_{160} = 10.73$ and $COD = 0.94$. Compared to the other methods, it can be seen that the proposed method presented more accurate performances, in terms of all goodness-of-fit criteria. Specifically, the high $MSM = 0.86$ and $COD = 0.94$ clearly indicate that the proposed fuzzy robust varying multiple regression model strongly fits the available fuzzy data set. In a specific case, consider the performance of the proposed fuzzy robust varying regression model for the 5th simulated data set. First, $\overline{D}$-chart was plotted in Fig. 1 to detect potential outliers. As can be readily seen, the data set involved some outliers. To compute the fuzzy varying coefficients, we need to determine a matrix of the estimated varying coefficients of $\widehat{\boldsymbol{B}}$ with dimension of 200×4. Applying the proposed optimization algorithm, the performance measures (for the 5th simulated data set) can be evaluated as $MSM = 0.90$ and

Table 1
Mean values of the performance measures corresponding to the proposed method and some fuzzy multiple regression techniques in Example 4.1.

Method	$\overline{MSM}$	$\overline{MARE}$	$\overline{RMSE}$	$\overline{COD}$	$\overline{AUCR}_{160}$
Choi and Yoon	0.62	18.73	15.53	0.77	24.01
Zeng et al.	0.64	17.52	16.55	0.76	24.81
Taheri and Kelkinnama	0.48	34.66	22.99	0.66	42.53
Choi and Buckley	0.52	26.88	15.53	0.68	35.76
Kula and Apaydin	0.63	18.52	17.01	0.75	25.05
Proposed	**0.86**	**7.39**	**6.22**	**0.94**	**10.73**

Table 2
Some fuzzy varying coefficients in Example 4.1 based on the 5th simulated data set.

Fuzzy coefficients
$\overset{\approx}{\widehat{\beta}}_0(z_{20}) = (1.18; 1.002, 0.511,)_T$, $\overset{\approx}{\widehat{\beta}}_1(z_{20}) = (0.43; 0.279, 0.759)_T$,
$\widehat{\beta}_2(z_{20}) = (0.41; 0.166, 0.25)_T$, $\overset{\approx}{\widehat{\beta}}_3(z_{20}) = (0.001; 0.0004, 0.0001)_T$
$\overset{\approx}{\widehat{\beta}}_1(z_{60}) = (2.56; 1.024, 0.586)_T$, $\overset{\approx}{\widehat{\beta}}_1 60 = (0.685; 0.434, 1.182)_T$,
$\widehat{\beta}_2(z_{60}) = (1.076; 0.430, 0.645)_T$, $\overset{\approx}{\widehat{\beta}}_3(z_{60}) = (0.0002; 0.00004, 0.0009)_T$
$\overset{\approx}{\widehat{\beta}}_0(z_{100}) = (2.56; 1.024, 0.586)_T$, $\overset{\approx}{\widehat{\beta}}_1(z_{100}) = (0.685; 0.434, 1.182)_T$,
$\widehat{\beta}_2(z_{100}) = (1.07; 0.430, 0.645)_T$, $\overset{\approx}{\widehat{\beta}}_3(z_{100}) = (0.0002; 0.00001, 0.0009)_T$
$\overset{\approx}{\widehat{\beta}}_0(z_{140}) = (1.918; 1.004, 0.640)_T$, $\overset{\approx}{\widehat{\beta}}_1(z_{140}) = (1.548; 0.928, 2.670,)_T$,
$\widehat{\beta}_2(z_{140}) = (1.008; 0.402, 0.604)_T$, $\overset{\approx}{\widehat{\beta}}_3(z_{140}) = (-0.003; 0, 1.534)_T$
$\overset{\approx}{\widehat{\beta}}_0(z_{180}) = (3.406; 1.147, 0.965,)_T$, $\overset{\approx}{\widehat{\beta}}_1(z_{180}) = (2.37; 1.497, 4.070,)_T$,
$\widehat{\beta}_2(z_{180}) = (0.322; 0.128, 0.193)_T$, $\overset{\approx}{\widehat{\beta}}_3(z_{180}) = (0.002; 0.0008, 0.002)_T$

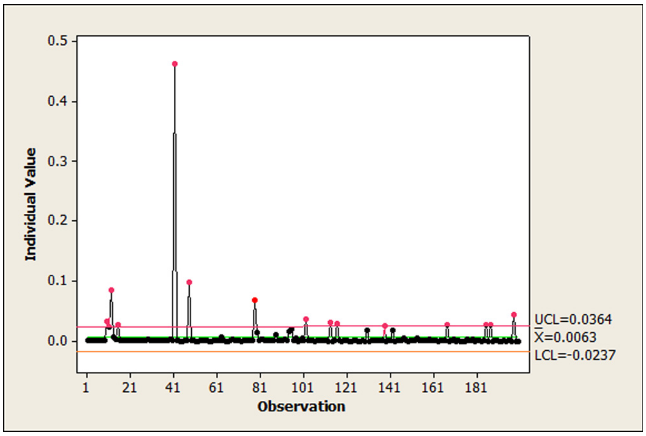

Fig. 1. $\overline{D}$-chart for detecting outliers based on the 5th simulated data set in Example 4.1.

$MARE = 6.34$, $RMSE = 5.22$, $AUCR_{160} = 6.86$ and $COD = 0.96$. For this case, time series plots of the centers as well as the left and right spread of fuzzy varying coefficients $\widetilde{\beta}_j(z) = (\widehat{\beta}_j(z); l_{\widehat{\beta}_j(z)}, r_{\widehat{\beta}_j(z)})_T, j = 0, 1, 2, 3$ were plotted in Fig. 2. Some of the estimated fuzzy varying coefficients corresponding to some index variables are also summarized in Table 2.

Example 4.2. In order to illustrate the findings of this research, consider a study that predicts housing prices in Boston. Here the Boston Housing Data () with the size of $n = 507$ was considered. Twelve predictors were used for subjective evaluation of the housing price (as fuzzy quantity) $\widetilde{y}$: median value of owner-occupied homes in 1000 United States dollar (USD) as x_1: crim (per capita crime rate by town), x_2: zn (proportion of residential land zones for lots over 25,000 sq.ft), x_3: indus (proportion of non-retail business acres per town), x_4: chas (Charles River dummy variable = 1 if tract bounds river; otherwise, 0), x_5: nox (nitric oxides concentration (parts per 10 million)), x_6: rm (average number of rooms

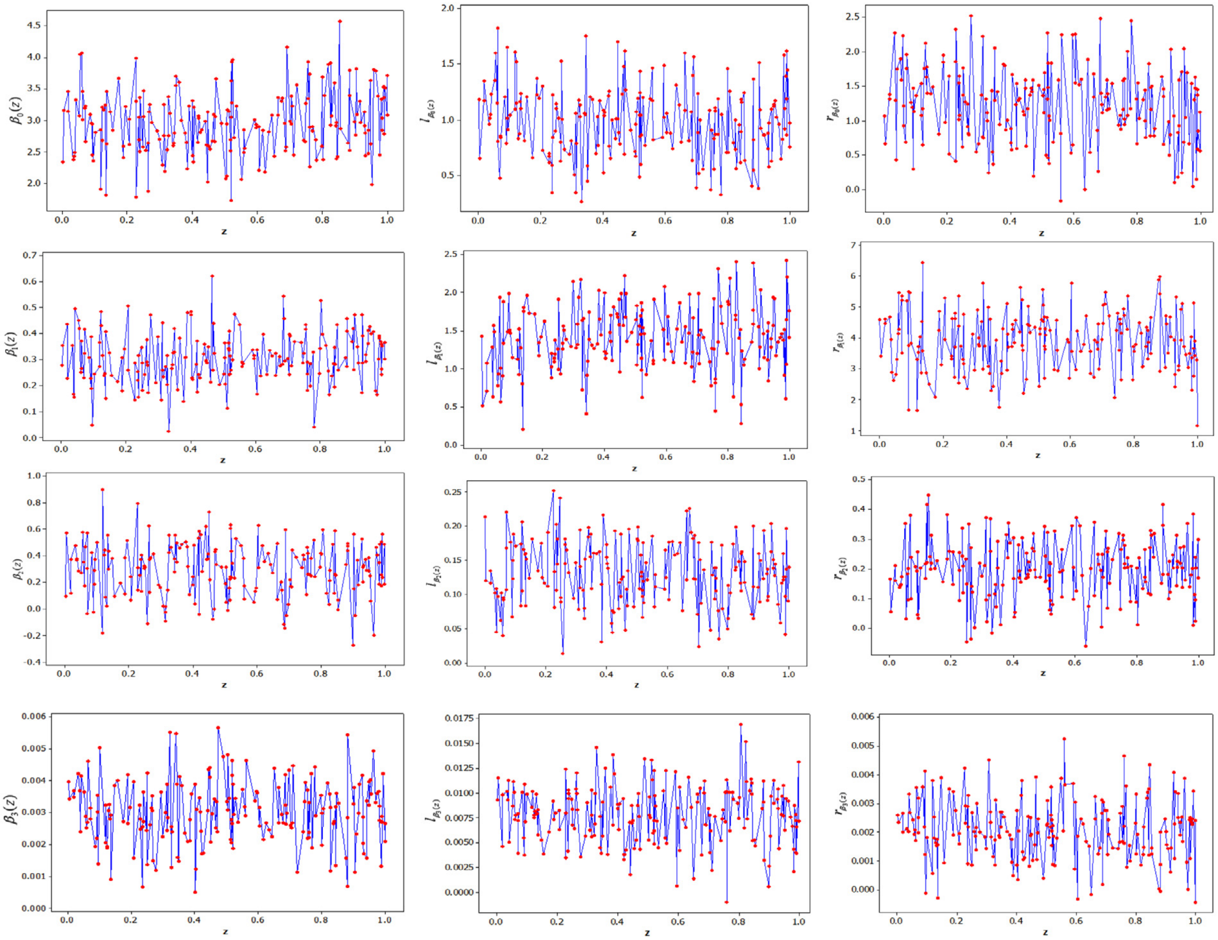

Fig. 2. Plots of $\widetilde{\beta}_j(z) = (\beta_j(z); l_{\beta_j(z)}, r_{\beta_j(z)})_T$ for $j = 0, 1, 2, 3$ in Example 4.1 based on 5th simulated data set.

per dwelling), x_7: age (proportion of owner-occupied units built before 1940), x_8: dis (weighted distances to 5 Boston employment centers), x_9: rad (index of accessibility to radial highways), x_{10}: tax (full-value property-tax rate per USD 10,000), x_{11}: ptratio (pupil-teacher ratio by town), x_{12}: b (the proportion of blacks by town $(1000(b-0.63)^2)$) and z: LSTAT (the percentage of lower status of the population) as the index variable. The observed fuzzy responses were assumed to be symmetric **TFN**s as:

$$\widetilde{y} = \begin{cases} (y; 2, 2)_T & y < 20, \\ (y; 3, 3)_T & 20 \le y < 30, \\ (y; 4, 4)_T & 30 \le y < 40. \end{cases} \tag{22}$$

First, let us investigate the data set to check the presence of outliers. To this end, the $\overline{D}$-chart is plotted in Fig. 3. Clearly, the data set involved numerous potential outliers. For this data set, the proposed method was then employed to estimate unknown fuzzy varying coefficients. The performance results are summarized in terms of the following goodness-of-fit criteria: *MSM* and *MARE*, *RMSE*, $AUCR_{460}$, and *COD* as listed in Table 3. Furthermore, the proposed fuzzy method was compared with several well-known fuzzy multiple regression models whose performance measures are also listed in Table 3. Comparing the methods in terms of the mentioned goodness-of-fit criteria, it can be seen that the proposed method exhibited a better performance compared to the other methods ($MSM = \mathbf{0.64}$ and $MARE = \mathbf{65.10}$, $RMSE = \mathbf{81.67}$, $AUCR_{460} = \mathbf{89.60}$, and $COD = \mathbf{0.82}$). Performance of the proposed method is also compared with the other methods in Fig. 7. As can be seen, the values of $M_{\widehat{\widetilde{y}}}$ in our method are closer to $M_{\widetilde{y}}$ compared to other methods. Therefore, it can be concluded that the proposed fuzzy robust-based varying coefficient gave more accurate results compared to the others.

Table 3
Coefficients of model and performance measures corresponding to the proposed method and some fuzzy multiple regression techniques in Example 4.2.

Method	Fuzzy coefficients	*MSM*	*MARE*	*RMSE*	$AUCR_{460}$	*COD*
Choi and Yoon	$\widetilde{\widehat{\beta}}_0 = (0.312; 0.621, 1.18)_T$, $\widetilde{\widehat{\beta}}_1 = (1.655; 0.744, 1.059)_T$, $\widehat{\beta}_2 = (0.083; 0.020, 0.632)_T$, $\widetilde{\widehat{\beta}}_3 = (0.098; 0, 0.921)_T$, $\widetilde{\widehat{\beta}}_4 = (0.156; 0.916, 0.594)_T$, $\widetilde{\widehat{\beta}}_5 = (-0.112; 0.815, 1.314)_T$, $\widetilde{\widehat{\beta}}_6 = (2.914; 0, 1.537)_T$, $\widetilde{\widehat{\beta}}_7 = (0.010; 0, 1.268)_T$. $\widetilde{\widehat{\beta}}_8 = (-0.590; 0.047, 0.539)_T$, $\widetilde{\widehat{\beta}}_9 = (0.760; 0, 0.808)_T$, $\widetilde{\widehat{\beta}}_{10} = (-0.005; 0.0003, 0.741)_T$, $\widetilde{\widehat{\beta}}_{11} = (-0.103; 0.027, 1.146)_T$, $\widetilde{\widehat{\beta}}_{12} = (0.012; 0, 0.706)_T$.	0.45	120.55	86.68	134.93	0.63
Zeng et al.	$\widetilde{\widehat{\beta}}_0 = (0.441; 0.598, 1.182)_T$, $\widetilde{\widehat{\beta}}_1 = (3.006; 0, 1.057)_T$, $\widehat{\beta}_2 = (0.063; 0.029, 0.621)_T$, $\widetilde{\widehat{\beta}}_3 = (0.147; 0.011, 0.951)_T$, $\widetilde{\widehat{\beta}}_4 = (0.190; 0.746, 0.594)_T$, $\widetilde{\widehat{\beta}}_5 = (-0.311; 0.934, 1.314)_T$, $\widetilde{\widehat{\beta}}_6 = (6.338; 0.001, 1.532)_T$, $\widetilde{\widehat{\beta}}_7 = (-0.046; 0, 1.223)_T$. $\widetilde{\widehat{\beta}}_8 = (-0.737; 0, 0.534)_T$, $\widetilde{\widehat{\beta}}_9 = (0.338; 2.770, 0.804)_T$, $\widetilde{\widehat{\beta}}_{10} = (-0.013; 0.001, 0.463)_T$, $\widetilde{\widehat{\beta}}_{11} = (0.058; 0, 1.130)_T$, $\widetilde{\widehat{\beta}}_{12} = (-0.030; 0.001, 0.352)_T$.	0.52	98.73	75.90	107.01	0.71
Taheri and Kelkinnama	$\widetilde{\widehat{\beta}}_0 = (0.002; 0.595, 1.197)_T$, $\widetilde{\widehat{\beta}}_1 = (-0.500; 1.018, 0.599)_T$, $\widehat{\beta}_2 = (0.628; 1.417, 1.181)_T$, $\widetilde{\widehat{\beta}}_3 = (-0.513; 4.718, 1.216)_T$, $\widetilde{\widehat{\beta}}_4 = (-0.530; 1.073, 0.993)_T$, $\widetilde{\widehat{\beta}}_5 = (0.233; 0.397, 1.357)_T$, $\widetilde{\widehat{\beta}}_6 = (0.220; 0, 3.038)_T$, $\widetilde{\widehat{\beta}}_7 = (-0.444; 0.00006, 0.00005)_T$. $\widetilde{\widehat{\beta}}_8 = (0.514; 0.126, 0.1333)_T$, $\widetilde{\widehat{\beta}}_9 = (0.406; 0.0001, 3.629)_T$, $\widetilde{\widehat{\beta}}_{10} = (0.296; 0.0001, 0.00009)_T$, $\widetilde{\widehat{\beta}}_{11} = (0.064; 0, 0)_T$, $\widetilde{\widehat{\beta}}_{12} = (-0.077; 0.0001; 0.00007)_T$.	0.34	142.65	111.307	160.01	0.56
Kula and Apaydin	$\widetilde{\widehat{\beta}}_0 = (0.098; 0.603; 1.311)_T$, $\widetilde{\widehat{\beta}}_1 = (1.730; 0, 0.685)_T$, $\widehat{\beta}_2 = (0.095; 0.025, 0)_T$, $\widetilde{\widehat{\beta}}_3 = (0.033; 0.0006; 0)_T$, $\widetilde{\widehat{\beta}}_4 = (-0.073; 0.925, 0.802)_T$, $\widetilde{\widehat{\beta}}_5 = (-0.570; 0.826, 1.554)_T$, $\widetilde{\widehat{\beta}}_6 = (1.243; 0.001, 0.118)_T$, $\widetilde{\widehat{\beta}}_7 = (-0.012; 0.00001, 0)_T$. $\widetilde{\widehat{\beta}}_8 = (-1.260; 0.010, 4.203)_T$, $\widetilde{\widehat{\beta}}_9 = (0.652; 0, 4.227)_T$, $\widetilde{\widehat{\beta}}_{10} = (-0.004; 0, 0)_T$, $\widetilde{\widehat{\beta}}_{11} = (-0.459; 8.117, 0)_T$, $\widetilde{\widehat{\beta}}_{12} = (0.069; 0.001, 0)_T$.	0.50	103.46	70.58	117.65	0.74
Choi and Buckley	$\widetilde{\widehat{\beta}}_0 = (-0.182; 0.823, 0.626)_T$, $\widetilde{\widehat{\beta}}_1 = (0.050; 0.616, 0.707)_T$, $\widehat{\beta}_2 = (0.247; 0.025, 0.036)_T$, $\widetilde{\widehat{\beta}}_3 = (0.081; 0, 0.0005)_T$, $\widetilde{\widehat{\beta}}_4 = (-0.616; 0.562, 0.873)_T$, $\widetilde{\widehat{\beta}}_5 = (0.652; 0.011, 1.250)_T$. $\widetilde{\widehat{\beta}}_6 = (0.742; 0, 0.00009)_T$, $\widetilde{\widehat{\beta}}_7 = (0.100; 0, 0.006)_T$, $\widetilde{\widehat{\beta}}_8 = (0.078; 0.430, 0.0001)_T$, $\widetilde{\widehat{\beta}}_9 = (0.368; 0, 0)_T$, $\widetilde{\widehat{\beta}}_{10} = (0.018; 0, 0.003)_T$, $\widetilde{\widehat{\beta}}_{11} = (-0.267; 0, 0.0004)_T$. $\widetilde{\widehat{\beta}}_{12} = (0.018; 0, 0)_T$	0.43	125.34	85.77	136.74	0.59
Proposed	(See Figs. 4–6)	**0.64**	**65.10**	**81.67**	**89.60**	**0.82**

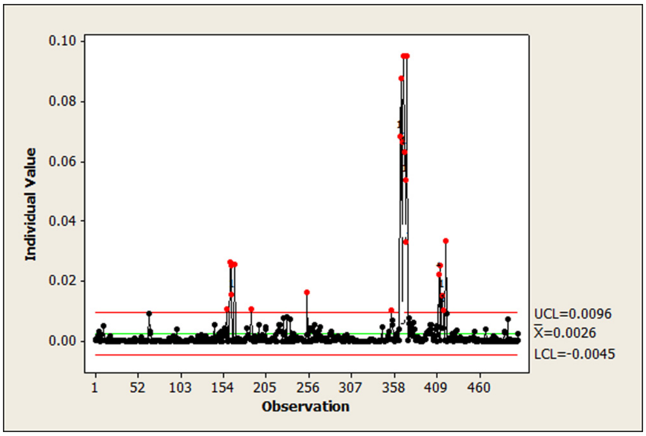

Fig. 3. $\overline{D}$-chart for detecting the outliers in Example 4.2.

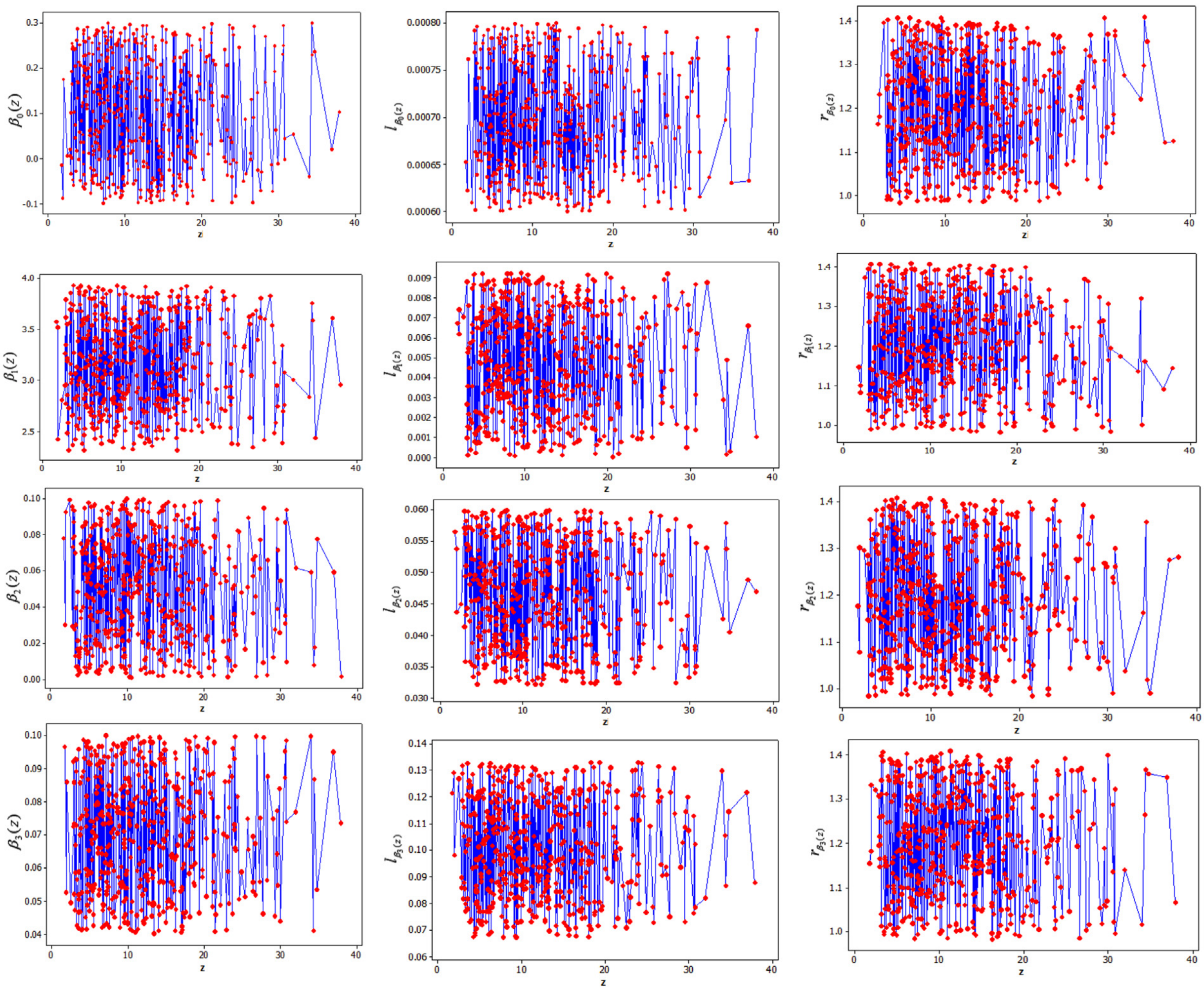

Fig. 4. Plots of $\widetilde{\beta}_j(z) = (\beta_j(z); l_{\beta_j(z)}, r_{\beta_j(z)})_T$ for $j = 0, 1, 2, 3$ in Example 4.2.

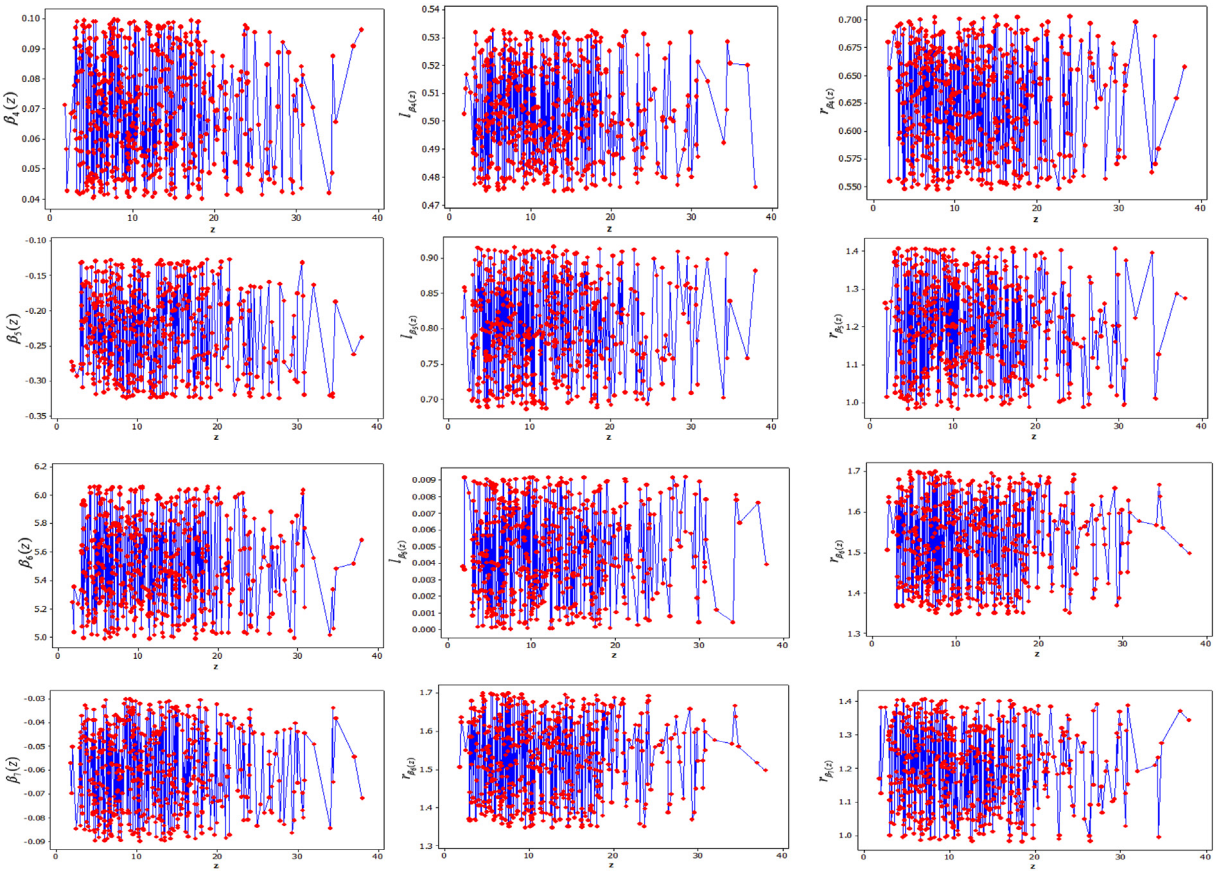

Fig. 5. Plots of $\widetilde{\beta}_j(z) = (\beta_j(z); l_{\beta_j(z)}, r_{\beta_j(z)})_T$ for $j = 4, 5, 6, 7$ in Example 4.2.

5. Conclusion

The varying coefficient multiple regression models are more flexible than the traditional multiple regression models due to the infinite dimensionality of their corresponding parameter spaces. In addition, the robust regression is an alternative to the least square regression methods for data set contaminated with outliers or influential observations. This paper developed a novel robust-based varying coefficient multiple regression model with exact predictors, fuzzy responses and fuzzy varying coefficients. To solve the problem, the ideas of the local kernel smoothing method and an improved M-estimation were combined to estimate the fuzzy varying coefficients and control the effect of outliers on regression performances. The cross-validation criterion was also utilized to evaluate the optimal value of tuning constant. The effectiveness and advantages of the proposed regression model were examined and compared with several existing methods via some experimental results including a simulation study. For this purpose, some common goodness-of-fit criteria were utilized in a fuzzy environment. The results clearly indicated that our prediction/estimation is more efficient than other methods, specifically, in cases where outliers occur in the data set. Therefore, the superior feasibility and effectiveness of the proposed method over the other fuzzy multiple regression model can be assigned to several reasons: 1- it results in a better performance measures, 2- it can restrict the effect of outliers on prediction, 3- it provides flexible fuzzy varying regression coefficients through some smooth functions. Future studies could be focused on extending the proposed model to the cases whose predictors are also reported by non-fuzzy numbers. Extension of the proposed method to nonlinear varying coefficient regression models could be another potential topic for further investigations. The problem of multicollinearity in the proposed method at high dimensional frameworks is another interesting issue that requires a deeper understanding.

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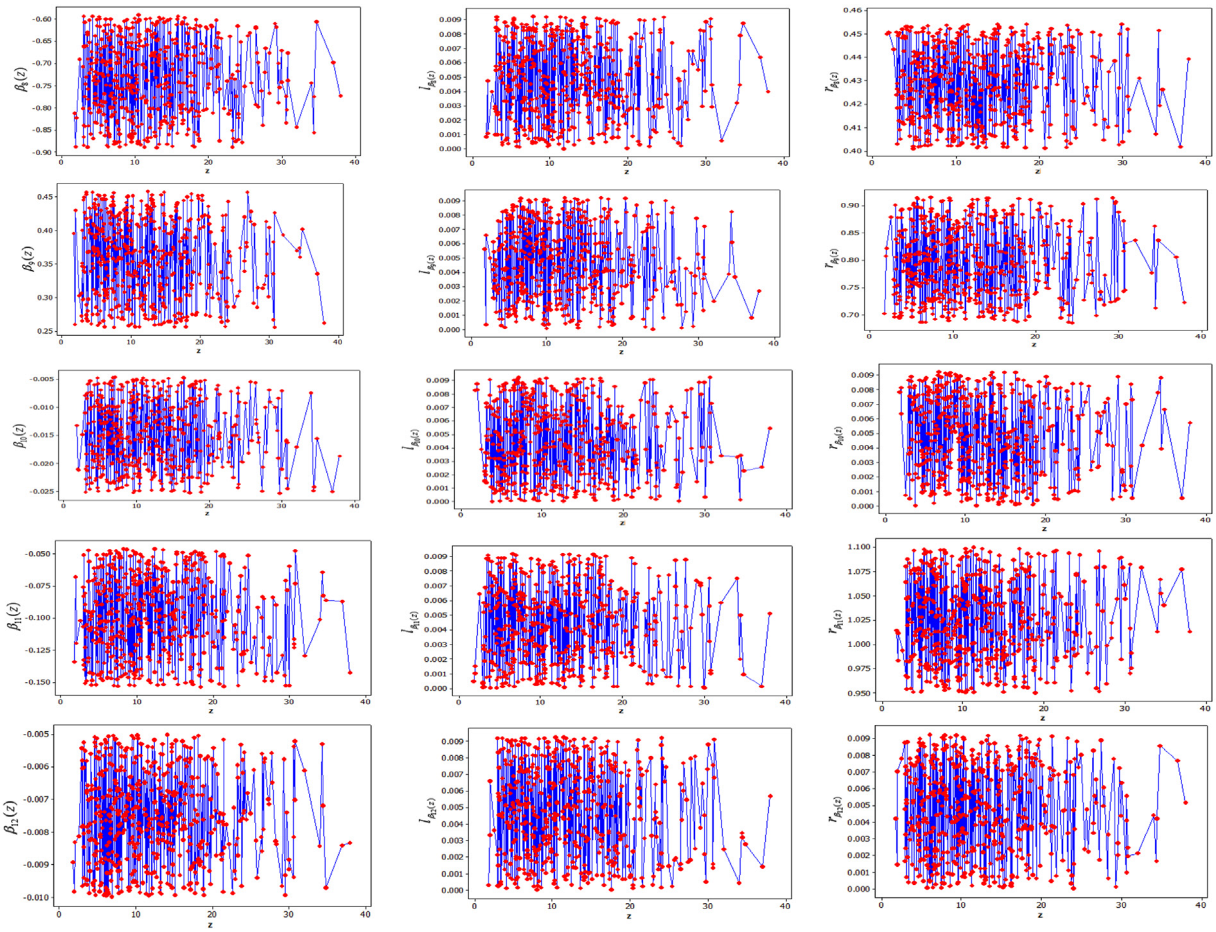

Fig. 6. Plots of $\widetilde{\beta}_j(z) = (\beta_j(z); l_{\beta_j(z)}, r_{\beta_j(z)})_T$ for $j = 8, 9, 10, 11, 12$ in Example 4.2.

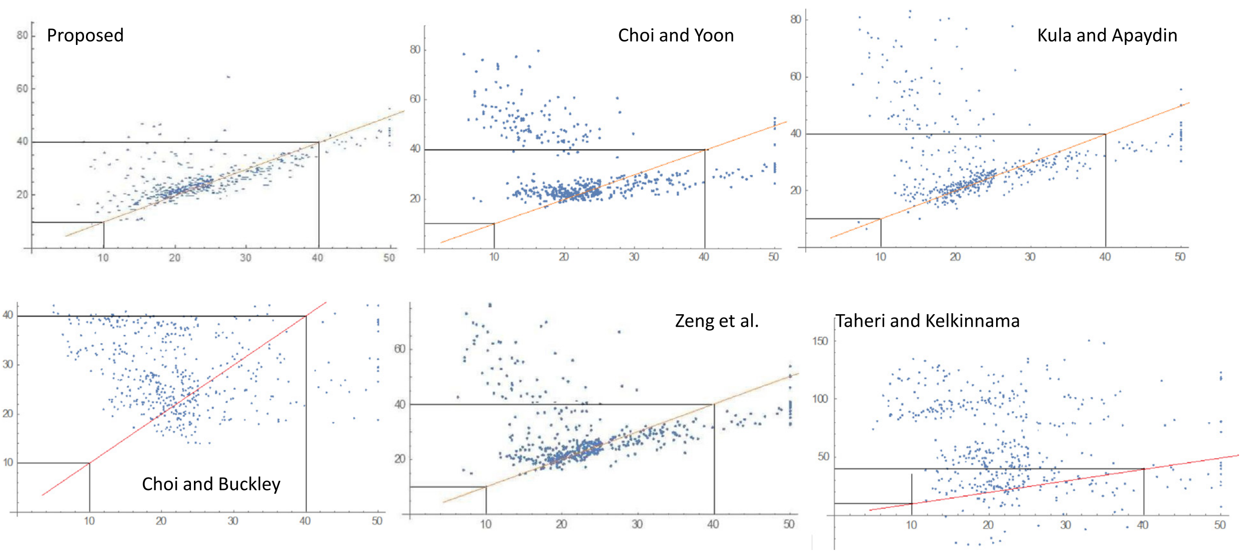

Fig. 7. Comparison of $M_{\widehat{\widetilde{y}}}$ with $M_{\widetilde{y}}$ values in Example 4.2.

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