Viscoelastic Earth Response Theory: From Tromp & Mitrovica (1999) to Sea Level Applications

Following the Normal-Mode Formalism for SLcode Implementation

1 Introduction: The Theoretical Foundation

This document presents the complete theoretical framework underlying viscoelastic Earth response calculations in SLcode, following the rigorous normal-mode formalism developed by Tromp & Mitrovica (1999) and applied to sea level problems by Kendall et al. (2005).

Key Insight: The exponential time dependence in viscoelastic Love numbers arises naturally from the normal-mode expansion of the Earth's response to surface loading, where each mode has its own characteristic decay time determined by the Earth's rheological structure.

2 Physical Picture: Why Normal Modes?

When a surface load is applied to a viscoelastic Earth:

1. The Earth responds with both **instantaneous elastic deformation** and **time-dependent viscous flow**
2. The viscous response can be decomposed into **normal modes** - fundamental oscillatory solutions of the viscoelastic Earth
3. Each normal mode has a characteristic **decay rate** s_k that depends on the Earth's rheological structure
4. The total response is a **superposition** of all these modes, each decaying exponentially as $e^{-s_k t}$

This is analogous to how a damped mechanical system responds to forcing - the response is a sum of damped oscillatory modes.

3 The Viscoelastic Normal-Mode Problem (Tromp & Mitrovica 1999)

3.1 Governing Equations

For a self-gravitating, viscoelastic Earth, the fundamental equations are:

1. Momentum balance (quasi-static):

$$\nabla \cdot \mathbf{T} - \nabla(\rho \mathbf{u} \cdot \nabla \Phi) + \rho \nabla \phi + \rho_1 \nabla \Phi = 0 \tag{1}$$

where $\mathbf{T}$ is the incremental Cauchy stress, $\mathbf{u}$ is displacement, ϕ is perturbed gravitational potential, and Φ is the background potential.

2. Maxwell rheology (in Laplace domain):

$$\mathbf{T} = \kappa(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu(s)\mathbf{D} \tag{2}$$

where $\mathbf{D}$ is the strain deviator and

$$\mu(s) = \frac{\mu s}{s + \mu/\eta} \tag{3}$$

is the Laplace-transformed shear modulus for Maxwell rheology.

3. Gravitational field equation:

$$\nabla^2 \phi = 4\pi G \rho_1 \tag{4}$$

where ρ_1 is the perturbed density.

3.2 Normal-Mode Eigenvalue Problem

The viscoelastic normal modes $\{\mathbf{u}_k, \phi_k\}$ with decay rates s_k satisfy:

$$\mathcal{L}(s_k)\mathbf{u}_k = 0, \quad \mathcal{L}_\phi \phi_k = 0 \tag{5}$$

where $\mathcal{L}(s)$ is the viscoelastic linear operator that depends on the Laplace variable s.

Key Point: The eigenvalues s_k are the **characteristic decay rates** of the Earth's viscoelastic response. The corresponding relaxation times are:

$$\tau_k = \frac{1}{s_k} \tag{6}$$

4 Elimination Process

4.1 Step 1: Express stress in terms of displacement

From equations (2) and (4):

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{7}$$

4.2 Step 2: Deviatoric stress

The deviatoric part of (5):

$$\sigma'_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2\mu}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \tag{8}$$

4.3 Step 3: Time derivatives

From (2):

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) \tag{9}$$

From (6):

$$\dot{\sigma}'_{ij} = \mu \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) - \frac{2\mu}{3} \frac{\partial \dot{u}_k}{\partial x_k} \delta_{ij} \tag{10}$$

5 Substitution into Maxwell Equation

Substituting (7) and (8) into the Maxwell rheology (3):

$$\frac{1}{2} \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) = \frac{1}{2\eta} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2\mu}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right] + \frac{1}{2\mu} \left[\mu \left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} \right) - \frac{2\mu}{3} \frac{\partial \dot{u}_k}{\partial x_k} \delta_{ij} \right] \tag{11}$$

Simplifying:

$$\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} = \frac{\mu}{\eta} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2\mu}{3\eta} \frac{\partial u_k}{\partial x_k} \delta_{ij} + \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} - \frac{2}{3} \frac{\partial \dot{u}_k}{\partial x_k} \delta_{ij} \tag{12}$$

This gives us:

$$0 = \frac{\mu}{\eta} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2\mu}{3\eta} \frac{\partial u_k}{\partial x_k} \delta_{ij} - \frac{2}{3} \frac{\partial \dot{u}_k}{\partial x_k} \delta_{ij} \tag{13}$$

6 The Linear Operator

From momentum balance (1) with stress (5):

$$\mu \nabla^2 u_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) = -f_i \tag{14}$$

Combined with the Maxwell constraint (11), we get the **viscoelastic wave equation**:

$$\boxed{\mu \nabla^2 \dot{u}_i + (\lambda + \mu) \frac{\partial}{\partial x_i} \left(\frac{\partial \dot{u}_k}{\partial x_k} \right) + \frac{\mu^2}{\eta} \nabla^2 u_i + \frac{\mu(\lambda + \mu)}{\eta} \frac{\partial}{\partial x_i} \left(\frac{\partial u_k}{\partial x_k} \right) = -\dot{f}_i} \tag{15}$$

7 Eigenvalue Problem

For time-harmonic solutions $u_i = \hat{u}_i e^{-\lambda t}$, equation (13) becomes:

$$\left[-\lambda\mu\nabla^2 + \frac{\mu^2}{\eta}\nabla^2\right]\hat{u}_i + \left[-\lambda(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} + \frac{\mu(\lambda+\mu)}{\eta}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\right]\hat{u}_k = \lambda\hat{f}_i \tag{16}$$

This gives us the **linear operator**:

$$\boxed{L_{ij}[\hat{u}] = \left[\mu\left(\frac{\mu}{\eta}-\lambda\right)\nabla^2\delta_{ij} + (\lambda+\mu)\left(\frac{\mu}{\eta}-\lambda\right)\frac{\partial^2}{\partial x_i \partial x_j}\right]\hat{u}_j} \tag{17}$$

8 Eigenvalue Interpretation

The eigenvalue equation is:

$$L_{ij}[\hat{u}_j] = \lambda\hat{f}_i \tag{18}$$

The eigenvalues λ satisfy:

$$\lambda = \frac{\mu}{\eta} \pm \text{corrections from elastic terms} \tag{19}$$

These eigenvalues determine the **relaxation times** $\tau = 1/\lambda$ that appear in the viscoelastic Love numbers:

$$h_\ell(t) = h_\ell^E\delta(t) + \sum_k r_\ell^k e^{-t/\tau_k} \tag{20}$$

9 Spherical Harmonic Decomposition

For a **spherically symmetric Earth**, we transform to spherical coordinates (r,θ,ϕ) and expand in spherical harmonics.

9.1 Displacement Field Expansion

The displacement field separates as:

$$u_r(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_r^{\ell m}(r,t)Y_\ell^m(\theta,\phi) \tag{21}$$

$$u_\theta(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_\theta^{\ell m}(r,t)\frac{\partial Y_\ell^m}{\partial\theta} \tag{22}$$

$$u_\phi(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_\phi^{\ell m}(r,t)\frac{1}{\sin\theta}\frac{\partial Y_\ell^m}{\partial\phi} \tag{23}$$

9.2 Spherical Laplacian Eigenvalue Property

The key insight is that spherical harmonics are eigenfunctions of the angular part of ∇^2:

$$\nabla^2 Y_\ell^m = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}\right] Y_\ell^m \tag{24}$$

9.3 Separation by Degree

When we substitute the expansion (18)-(20) into the linear operator L_{ij}, **the problem separates by spherical harmonic degree ℓ**.

For each ℓ, we get a **radial ODE system**:

$$\mathcal{L}_\ell^{(r)} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} = \lambda_{\ell,k} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} \tag{25}$$

where $\mathcal{L}_\ell^{(r)}$ is the **radial differential operator** that depends on degree ℓ.

9.4 Radial Operator Structure

The radial operator has the form:

$$\mathcal{L}_\ell^{(r)} = \begin{pmatrix} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} & \text{coupling terms} \\ \text{coupling terms} & \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} \end{pmatrix} \tag{26}$$

10 Connection to Love Numbers

10.1 Eigenvalue Spectrum

For each degree ℓ, the radial operator $\mathcal{L}_\ell^{(r)}$ has eigenvalues:

$$\lambda_{\ell,1}, \lambda_{\ell,2}, \lambda_{\ell,3}, \ldots \tag{27}$$

These give the **relaxation times** for degree ℓ:

$$\tau_{\ell,k} = \frac{1}{\lambda_{\ell,k}} \tag{28}$$

10.2 Love Number Time Dependence

The **viscoelastic Love numbers** are constructed from these eigenvalues:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^h e^{-t/\tau_{\ell,k}} \tag{29}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^k e^{-t/\tau_{\ell,k}} \tag{30}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,k}^{h,k}$ are **residues** (amplitudes of each mode)
- $\tau_{\ell,k} = 1/\lambda_{\ell,k}$ are **relaxation times** from eigenvalues

10.3 Physical Interpretation

- **$\ell = 0$**: Radial expansion/contraction (breathing mode)
- **$\ell = 1$**: Translational motion (center of mass)
- **$\ell = 2$**: Flattening/elongation (tidal deformation)
- **Higher ℓ**: Shorter wavelength surface deformation

Each degree ℓ has multiple relaxation times $\tau_{\ell,k}$ corresponding to different **radial modes**:

- Fast modes: Surface/shallow deformation (years to decades)
- Slow modes: Deep mantle flow (thousands of years)

11 Summary

The complete chain is:

Maxwell rheology → Linear operator L → Spherical harmonic separation → Radial eigenvalue problems → Relaxation times $\tau_{\ell,k}$ → Love number exponentials

The exponential terms $e^{-t/\tau_{\ell,k}}$ in viscoelastic Love numbers arise directly from the **eigenvalue spectrum** of the radial viscoelastic operator for each spherical harmonic degree ℓ.