Viscoelastic Earth Response Theory: From Tromp & Mitrovica (1999) to Sea Level Applications

Following the Normal-Mode Formalism for SLcode Implementation

1 Introduction: The Theoretical Foundation

This document presents the complete theoretical framework underlying viscoelastic Earth response calculations in SLcode, following the rigorous normal-mode formalism developed by Tromp & Mitrovica (1999) and applied to sea level problems by Kendall et al. (2005).

Key Insight: The exponential time dependence in viscoelastic Love numbers arises naturally from the normal-mode expansion of the Earth's response to surface loading, where each mode has its own characteristic decay time determined by the Earth's rheological structure.

2 Physical Picture: Why Normal Modes?

When a surface load is applied to a viscoelastic Earth:

1. The Earth responds with both **instantaneous elastic deformation** and **time-dependent viscous flow**
2. The viscous response can be decomposed into **normal modes** - fundamental oscillatory solutions of the viscoelastic Earth
3. Each normal mode has a characteristic **decay rate** s_k that depends on the Earth's rheological structure
4. The total response is a **superposition** of all these modes, each decaying exponentially as $e^{-s_k t}$

This is analogous to how a damped mechanical system responds to forcing - the response is a sum of damped oscillatory modes.

3 The Viscoelastic Normal-Mode Problem (Tromp & Mitrovica 1999)

3.1 Governing Equations

For a self-gravitating, viscoelastic Earth, the fundamental equations are:

1. Momentum balance (quasi-static):

$$\nabla \cdot \mathbf{T} - \nabla(\rho \mathbf{u} \cdot \nabla\Phi) + \rho\nabla\phi + \rho_1 \nabla\Phi = 0 \tag{1}$$

where $\mathbf{T}$ is the incremental Cauchy stress, $\mathbf{u}$ is displacement, ϕ is perturbed gravitational potential, and Φ is the background potential.

2. Maxwell rheology (in Laplace domain):

$$\mathbf{T} = \kappa(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu(s)\mathbf{D} \tag{2}$$

where $\mathbf{D}$ is the strain deviator and

$$\mu(s) = \frac{\mu s}{s + \mu/\eta} \tag{3}$$

is the Laplace-transformed shear modulus for Maxwell rheology.

3. Gravitational field equation:

$$\nabla^2\phi = 4\pi G\rho_1 \tag{4}$$

where ρ_1 is the perturbed density.

3.2 Normal-Mode Eigenvalue Problem

The viscoelastic normal modes $\{\mathbf{u}_k, \phi_k\}$ with decay rates s_k satisfy:

$$\mathcal{L}(s_k)\mathbf{u}_k = 0, \quad \mathcal{L}_\phi \phi_k = 0 \tag{5}$$

where $\mathcal{L}(s)$ is the viscoelastic linear operator that depends on the Laplace variable s.

Key Point: The eigenvalues s_k are the **characteristic decay rates** of the Earth's viscoelastic response. The corresponding relaxation times are:

$$\tau_k = \frac{1}{s_k} \tag{6}$$

3.3 Biorthogonality Relations

A key feature of viscoelastic normal modes is that they satisfy **biorthogonality relations** (Tromp & Mitrovica 1999, Section 5):

For distinct modes $\{\mathbf{u}_k, \phi_k\}$ and $\{\mathbf{u}_{k'}, \phi_{k'}\}$:

$$[\mathbf{u}_k, \phi_k; \{\mathcal{L}(s_k) - \mathcal{L}(s_{k'})\}\mathbf{u}_{k'}, 0] = 0 \tag{7}$$

where $[\cdot;\cdot]$ denotes the duality product. These relations are essential for constructing the Green's tensor.

4 Surface-Load Response via Green's Tensor (Tromp & Mitrovica 1999, Section 7)

4.1 Green's Tensor Construction

The displacement response to a point force is expressed as a normal-mode expansion:

$$\mathbf{G}(\mathbf{r}, \mathbf{r}'; t) = -\sum_k \frac{1}{2s_k} \mathbf{u}_k(\mathbf{r})\mathbf{u}_k(\mathbf{r}') e^{s_k t} H(t) \tag{8}$$

where $H(t)$ is the Heaviside function.

Physical Interpretation: Each normal mode k contributes to the Green's tensor with:

- **Spatial pattern**: $\mathbf{u}_k(\mathbf{r})\mathbf{u}_k(\mathbf{r}')$ (eigenfunction product)
- **Temporal evolution**: $e^{s_k t}$ (exponential decay)
- **Amplitude**: $1/2s_k$ (inversely proportional to decay rate)

4.2 Surface-Load Green's Vector

For surface loading, the relevant response function is:

$$\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; t) = \mathrm{Re} \sum_k \frac{1}{2s_k} \mathbf{u}_k(\mathbf{r}) [\mathbf{u}_k(\mathbf{r}') \cdot \nabla' \Phi(\mathbf{r}') + \phi_k(\mathbf{r}')] e^{s_k t} H(t) \tag{9}$$

The displacement field due to an arbitrary surface load $\sigma(\mathbf{r}', t')$ is:

$$\mathbf{u}(\mathbf{r}, t) = \int_{-\infty}^{t} \int_{\partial V} \sigma(\mathbf{r}', t') \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; t - t') d^2\mathbf{r}' dt' \tag{10}$$

5 Spherically Symmetric Earth: Love Numbers from Normal Modes

5.1 Spherical Harmonic Decomposition

For a spherically symmetric Earth, the normal modes separate by spherical harmonic degree ℓ:

$$\mathbf{u}_k(\mathbf{r}) = \mathbf{u}_{\ell,n}(r) Y_\ell^m(\theta, \phi) \tag{11}$$

$$\phi_k(\mathbf{r}) = \phi_{\ell,n}(r) Y_\ell^m(\theta, \phi) \tag{12}$$

where n indexes the radial modes for each degree ℓ.

The decay rates become $s_{\ell,n}$, giving degree-dependent relaxation times:

$$\tau_{\ell,n} = \frac{1}{s_{\ell,n}} \tag{13}$$

5.2 Love Number Construction

The **viscoelastic Love numbers** are constructed from the surface values of the radial eigenfunctions:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{n=1}^{\infty} r_{\ell,n}^h e^{-t/\tau_{\ell,n}} \tag{14}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{n=1}^{\infty} r_{\ell,n}^k e^{-t/\tau_{\ell,n}} \tag{15}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,n}^{h,k}$ are **residues** (amplitudes) determined from the normal-mode eigenfunctions
- $\tau_{\ell,n} = 1/s_{\ell,n}$ are **relaxation times** from the normal-mode eigenvalues

Key Result: The exponential terms $e^{-t/\tau_{\ell,n}}$ arise directly from the eigenvalue spectrum $s_{\ell,n}$ of the viscoelastic normal-mode problem.

6 Physical Interpretation of Normal Modes

6.1 Spherical Harmonic Degrees

- $\ell = 0$: Radial expansion/contraction (breathing mode)
- $\ell = 1$: Translational motion (center of mass)
- $\ell = 2$: Flattening/elongation (tidal deformation)
- **Higher** ℓ: Shorter wavelength surface deformation

6.2 Radial Mode Structure

Each degree ℓ has multiple relaxation times $\tau_{\ell,n}$ corresponding to different **radial modes**:

- **Fast modes (**$n = 1, 2, ...$**)**: Surface/shallow deformation (years to decades)
- **Slow modes (**n **large)**: Deep mantle flow (thousands of years)

The radial mode index n reflects the number of nodes in the radial eigenfunction - higher n corresponds to more complex radial structure and typically longer relaxation times.

7 Spherical Harmonic Decomposition

For a **spherically symmetric Earth**, we transform to spherical coordinates (r, θ, ϕ) and expand in spherical harmonics.

7.1 Displacement Field Expansion

The displacement field separates as:

$$u_r(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_r^{\ell m}(r,t) Y_\ell^m(\theta,\phi) \tag{16}$$

$$u_\theta(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\theta^{\ell m}(r,t) \frac{\partial Y_\ell^m}{\partial \theta} \tag{17}$$

$$u_\phi(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\phi^{\ell m}(r,t) \frac{1}{\sin\theta} \frac{\partial Y_\ell^m}{\partial \phi} \tag{18}$$

7.2 Spherical Laplacian Eigenvalue Property

The key insight is that spherical harmonics are eigenfunctions of the angular part of ∇^2:

$$\nabla^2 Y_\ell^m = \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\ell(\ell+1)}{r^2} \right] Y_\ell^m \tag{19}$$

7.3 Separation by Degree

When we substitute the expansion (18)-(20) into the linear operator L_{ij}, **the problem separates by spherical harmonic degree ℓ**.

For each ℓ, we get a **radial ODE system**:

$$\mathcal{L}_\ell^{(r)} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} = \lambda_{\ell,k} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} \tag{20}$$

where $\mathcal{L}_\ell^{(r)}$ is the **radial differential operator** that depends on degree ℓ.

7.4 Radial Operator Structure

The radial operator has the form:

$$\mathcal{L}_\ell^{(r)} = \begin{pmatrix} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} & \text{coupling terms} \\ \text{coupling terms} & \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} \end{pmatrix} \tag{21}$$

8 Connection to Love Numbers

8.1 Eigenvalue Spectrum

For each degree ℓ, the radial operator $\mathcal{L}_\ell^{(r)}$ has eigenvalues:

$$\lambda_{\ell,1}, \lambda_{\ell,2}, \lambda_{\ell,3}, \ldots \tag{22}$$

These give the **relaxation times** for degree ℓ:

$$\tau_{\ell,k} = \frac{1}{\lambda_{\ell,k}} \tag{23}$$

8.2 Love Number Time Dependence

The **viscoelastic Love numbers** are constructed from these eigenvalues:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^h e^{-t/\tau_{\ell,k}} \tag{24}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^k e^{-t/\tau_{\ell,k}} \tag{25}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,k}^{h,k}$ are **residues** (amplitudes of each mode)
- $\tau_{\ell,k} = 1/\lambda_{\ell,k}$ are **relaxation times** from eigenvalues

8.3 Physical Interpretation

- **$\ell = 0$**: Radial expansion/contraction (breathing mode)
- **$\ell = 1$**: Translational motion (center of mass)
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- **Higher ℓ**: Shorter wavelength surface deformation

Each degree ℓ has multiple relaxation times $\tau_{\ell,k}$ corresponding to different **radial modes**:

- Fast modes: Surface/shallow deformation (years to decades)
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9 Summary

The complete chain is:

Maxwell rheology → Linear operator L → Spherical harmonic separation → Radial eigenvalue problems → Relaxation times $\tau_{\ell,k}$ → Love number exponentials

The exponential terms $e^{-t/\tau_{\ell,k}}$ in viscoelastic Love numbers arise directly from the **eigenvalue spectrum** of the radial viscoelastic operator for each spherical harmonic degree ℓ.