Derivation of the Viscoelastic Linear Operator

From Basic Equations to Eigenvalue Problem

1 Complete System of Equations

We start with the four fundamental equations for viscoelastic deformation:

1. Momentum balance (quasi-static):

$$\nabla \cdot \sigma = -f \tag{1}$$

where f is body force (gravitational loading).

2. Strain-displacement relation:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2}$$

3. Maxwell rheology:

$$\dot{\epsilon}_{ij} = \frac{1}{2\eta}\sigma'_{ij} + \frac{1}{2\mu}\dot{\sigma}'_{ij} \tag{3}$$

where $\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$ is deviatoric stress.

4. Elastic stress-strain relation:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij} \tag{4}$$

where λ, μ are Lamé parameters.

2 Elimination Process

2.1 Step 1: Express stress in terms of displacement

From equations (2) and (4):

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{5}$$

2.2 Step 2: Deviatoric stress

The deviatoric part of (5):

$$\sigma'_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2\mu}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{6}$$

2.3 Step 3: Time derivatives

From (2):

$$\dot{\epsilon}_{ij} = \frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) \tag{7}$$

From (6):

$$\dot{\sigma}'_{ij} = \mu\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) - \frac{2\mu}{3}\frac{\partial \dot{u}_k}{\partial x_k}\delta_{ij} \tag{8}$$

3 Substitution into Maxwell Equation

Substituting (7) and (8) into the Maxwell rheology (3):

$$\frac{1}{2}\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) = \frac{1}{2\eta}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2\mu}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right] + \frac{1}{2\mu}\left[\mu\left(\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i}\right) - \frac{2\mu}{3}\frac{\partial \dot{u}_k}{\partial x_k}\delta_{ij}\right] \tag{9}$$

Simplifying:

$$\frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} = \frac{\mu}{\eta}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2\mu}{3\eta}\frac{\partial u_k}{\partial x_k}\delta_{ij} + \frac{\partial \dot{u}_i}{\partial x_j} + \frac{\partial \dot{u}_j}{\partial x_i} - \frac{2}{3}\frac{\partial \dot{u}_k}{\partial x_k}\delta_{ij} \tag{10}$$

This gives us:

$$0 = \frac{\mu}{\eta}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2\mu}{3\eta}\frac{\partial u_k}{\partial x_k}\delta_{ij} - \frac{2}{3}\frac{\partial \dot{u}_k}{\partial x_k}\delta_{ij} \tag{11}$$

4 The Linear Operator

From momentum balance (1) with stress (5):

$$\mu\nabla^2 u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = -f_i \tag{12}$$

Combined with the Maxwell constraint (11), we get the **viscoelastic wave equation**:

$$\boxed{\mu\nabla^2 \dot{u}_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial \dot{u}_k}{\partial x_k}\right) + \frac{\mu^2}{\eta}\nabla^2 u_i + \frac{\mu(\lambda+\mu)}{\eta}\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = -\dot{f}_i} \tag{13}$$

5 Eigenvalue Problem

For time-harmonic solutions $u_i = \hat{u}_i e^{-\lambda t}$, equation (13) becomes:

$$\left[-\lambda\mu\nabla^2 + \frac{\mu^2}{\eta}\nabla^2\right]\hat{u}_i + \left[-\lambda(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k} + \frac{\mu(\lambda+\mu)}{\eta}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\right]\hat{u}_k = \lambda\hat{f}_i \tag{14}$$

This gives us the **linear operator**:

$$\boxed{L_{ij}[\hat{u}] = \left[\mu\left(\frac{\mu}{\eta} - \lambda\right)\nabla^2\delta_{ij} + (\lambda+\mu)\left(\frac{\mu}{\eta} - \lambda\right)\frac{\partial^2}{\partial x_i \partial x_j}\right]\hat{u}_j} \tag{15}$$

6 Eigenvalue Interpretation

The eigenvalue equation is:

$$L_{ij}[\hat{u}_j] = \lambda\hat{f}_i \tag{16}$$

The eigenvalues λ satisfy:

$$\lambda = \frac{\mu}{\eta} \pm \text{corrections from elastic terms} \tag{17}$$

These eigenvalues determine the **relaxation times** $\tau = 1/\lambda$ that appear in the viscoelastic Love numbers:

$$h_\ell(t) = h_\ell^E\delta(t) + \sum_k r_\ell^k e^{-t/\tau_k} \tag{18}$$

7 Spherical Harmonic Decomposition

For a **spherically symmetric Earth**, we transform to spherical coordinates (r, θ, ϕ) and expand in spherical harmonics.

7.1 Displacement Field Expansion

The displacement field separates as:

$$u_r(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_r^{\ell m}(r,t)Y_\ell^m(\theta,\phi) \tag{19}$$

$$u_\theta(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_\theta^{\ell m}(r,t)\frac{\partial Y_\ell^m}{\partial\theta} \tag{20}$$

$$u_\phi(r,\theta,\phi,t) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_\phi^{\ell m}(r,t)\frac{1}{\sin\theta}\frac{\partial Y_\ell^m}{\partial\phi} \tag{21}$$

7.2 Spherical Laplacian Eigenvalue Property

The key insight is that spherical harmonics are eigenfunctions of the angular part of ∇^2:

$$\nabla^2 Y_\ell^m = \left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell(\ell+1)}{r^2}\right] Y_\ell^m \tag{22}$$

7.3 Separation by Degree

When we substitute the expansion (18)-(20) into the linear operator L_{ij}, **the problem separates by spherical harmonic degree ℓ**.

For each ℓ, we get a **radial ODE system**:

$$\mathcal{L}_\ell^{(r)} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} = \lambda_{\ell,k} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} \tag{23}$$

where $\mathcal{L}_\ell^{(r)}$ is the **radial differential operator** that depends on degree ℓ.

7.4 Radial Operator Structure

The radial operator has the form:

$$\mathcal{L}_\ell^{(r)} = \begin{pmatrix} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} & \text{coupling terms} \\ \text{coupling terms} & \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} \end{pmatrix} \tag{24}$$

8 Connection to Love Numbers

8.1 Eigenvalue Spectrum

For each degree ℓ, the radial operator $\mathcal{L}_\ell^{(r)}$ has eigenvalues:

$$\lambda_{\ell,1}, \lambda_{\ell,2}, \lambda_{\ell,3}, \ldots \tag{25}$$

These give the **relaxation times** for degree ℓ:

$$\tau_{\ell,k} = \frac{1}{\lambda_{\ell,k}} \tag{26}$$

8.2 Love Number Time Dependence

The **viscoelastic Love numbers** are constructed from these eigenvalues:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^h e^{-t/\tau_{\ell,k}} \tag{27}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^k e^{-t/\tau_{\ell,k}} \tag{28}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,k}^{h,k}$ are **residues** (amplitudes of each mode)
- $\tau_{\ell,k} = 1/\lambda_{\ell,k}$ are **relaxation times** from eigenvalues

8.3 Physical Interpretation

- **$\ell = 0$**: Radial expansion/contraction (breathing mode)
- **$\ell = 1$**: Translational motion (center of mass)
- **$\ell = 2$**: Flattening/elongation (tidal deformation)
- **Higher ℓ**: Shorter wavelength surface deformation

Each degree ℓ has multiple relaxation times $\tau_{\ell,k}$ corresponding to different **radial modes**:

- Fast modes: Surface/shallow deformation (years to decades)
- Slow modes: Deep mantle flow (thousands of years)

9 Summary

The complete chain is:

Maxwell rheology $\to$ Linear operator L $\to$ Spherical harmonic separation $\to$ Radial eigenvalue problems $\to$ Relaxation times $\tau_{\ell,k}$ $\to$ Love number exponentials

The exponential terms $e^{-t/\tau_{\ell,k}}$ in viscoelastic Love numbers arise directly from the **eigenvalue spectrum** of the radial viscoelastic operator for each spherical harmonic degree ℓ.