Viscoelastic Earth Response Theory: From Tromp & Mitrovica (1999) to Sea Level Applications

Following the Normal-Mode Formalism for SLcode Implementation

1 Introduction: The Theoretical Foundation

This document presents the complete theoretical framework underlying viscoelastic Earth response calculations in SLcode, following the rigorous normal-mode formalism developed by Tromp & Mitrovica (1999) and applied to sea level problems by Kendall et al. (2005).

Key Insight: The exponential time dependence in viscoelastic Love numbers arises naturally from the normal-mode expansion of the Earth's response to surface loading, where each mode has its own characteristic decay time determined by the Earth's rheological structure.

2 Physical Picture: Why Normal Modes?

When a surface load is applied to a viscoelastic Earth:

1. The Earth responds with both **instantaneous elastic deformation** and **time-dependent viscous flow**
2. The viscous response can be decomposed into **normal modes** - fundamental oscillatory solutions of the viscoelastic Earth
3. Each normal mode has a characteristic **decay rate** s_k that depends on the Earth's rheological structure
4. The total response is a **superposition** of all these modes, each decaying exponentially as $e^{-s_k t}$

This is analogous to how a damped mechanical system responds to forcing - the response is a sum of damped oscillatory modes.

3 The Viscoelastic Normal-Mode Problem (Tromp & Mitrovica 1999)

3.1 Governing Equations

For a self-gravitating, viscoelastic Earth, the fundamental equations are:

1. Momentum balance (quasi-static):

$$\nabla \cdot \mathbf{T} - \nabla(\rho \mathbf{u} \cdot \nabla \Phi) + \rho \nabla \phi + \rho_1 \nabla \Phi = 0 \tag{1}$$

where $\mathbf{T}$ is the incremental Cauchy stress, $\mathbf{u}$ is displacement, ϕ is perturbed gravitational potential, and Φ is the background potential.

2. Maxwell rheology (in Laplace domain):

$$\mathbf{T} = \kappa(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu(s)\mathbf{D} \tag{2}$$

where $\mathbf{D}$ is the strain deviator and

$$\mu(s) = \frac{\mu s}{s + \mu/\eta} \tag{3}$$

is the Laplace-transformed shear modulus for Maxwell rheology.

3. Gravitational field equation:

$$\nabla^2 \phi = 4\pi G \rho_1 \tag{4}$$

where ρ_1 is the perturbed density.

3.2 Normal-Mode Eigenvalue Problem

The viscoelastic normal modes $\{\mathbf{u}_k, \phi_k\}$ with decay rates s_k satisfy:

$$\mathcal{L}(s_k)\mathbf{u}_k = 0, \quad \mathcal{L}_\phi \phi_k = 0 \tag{5}$$

where $\mathcal{L}(s)$ is the viscoelastic linear operator that depends on the Laplace variable s.

Key Point: The eigenvalues s_k are the **characteristic decay rates** of the Earth's viscoelastic response. The corresponding relaxation times are:

$$\tau_k = \frac{1}{s_k} \tag{6}$$

3.3 Biorthogonality Relations

A key feature of viscoelastic normal modes is that they satisfy **biorthogonality relations** (Tromp & Mitrovica 1999, Section 5):

For distinct modes $\{\mathbf{u}_k, \phi_k\}$ and $\{\mathbf{u}_{k'}, \phi_{k'}\}$:

$$[\mathbf{u}_k, \phi_k; \{\mathcal{L}(s_k) - \mathcal{L}(s_{k'})\}\mathbf{u}_{k'}, 0] = 0 \tag{7}$$

where $[\cdot;\cdot]$ denotes the duality product. These relations are essential for constructing the Green's tensor.

4 Surface-Load Response via Green's Tensor (Tromp & Mitrovica 1999, Section 7)

4.1 Green's Tensor Construction

The displacement response to a point force is expressed as a normal-mode expansion:

$$\mathbf{G}(\mathbf{r},\mathbf{r}';t) = -\sum_k \frac{1}{2s_k}\mathbf{u}_k(\mathbf{r})\mathbf{u}_k(\mathbf{r}')e^{s_k t}H(t) \tag{8}$$

where $H(t)$ is the Heaviside function.

Physical Interpretation: Each normal mode k contributes to the Green's tensor with:

- **Spatial pattern**: $\mathbf{u}_k(\mathbf{r})\mathbf{u}_k(\mathbf{r}')$ (eigenfunction product)
- **Temporal evolution**: $e^{s_k t}$ (exponential decay)
- **Amplitude**: $1/2s_k$ (inversely proportional to decay rate)

4.2 Surface-Load Green's Vector

For surface loading, the relevant response function is:

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';t) = \mathrm{Re}\sum_k \frac{1}{2s_k}\mathbf{u}_k(\mathbf{r})[\mathbf{u}_k(\mathbf{r}')\cdot\nabla'\Phi(\mathbf{r}') + \phi_k(\mathbf{r}')]e^{s_k t}H(t) \tag{9}$$

The displacement field due to an arbitrary surface load $\sigma(\mathbf{r}',t')$ is:

$$\mathbf{u}(\mathbf{r},t) = \int_{-\infty}^{t}\int_{\partial V}\sigma(\mathbf{r}',t')\mathbf{\Gamma}(\mathbf{r},\mathbf{r}';t-t')d^2\mathbf{r}'dt' \tag{10}$$

5 Spherically Symmetric Earth: Love Numbers from Normal Modes

5.1 Spherical Harmonic Decomposition

For a spherically symmetric Earth, the normal modes separate by spherical harmonic degree ℓ:

$$\mathbf{u}_k(\mathbf{r}) = \mathbf{u}_{\ell,n}(r)Y_\ell^m(\theta,\phi) \tag{11}$$

$$\phi_k(\mathbf{r}) = \phi_{\ell,n}(r)Y_\ell^m(\theta,\phi) \tag{12}$$

where n indexes the radial modes for each degree ℓ.

The decay rates become $s_{\ell,n}$, giving degree-dependent relaxation times:

$$\tau_{\ell,n} = \frac{1}{s_{\ell,n}} \tag{13}$$

5.2 Love Number Construction

The **viscoelastic Love numbers** are constructed from the surface values of the radial eigenfunctions:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{n=1}^{\infty} r_{\ell,n}^h e^{-t/\tau_{\ell,n}} \tag{14}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{n=1}^{\infty} r_{\ell,n}^k e^{-t/\tau_{\ell,n}} \tag{15}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,n}^{h,k}$ are **residues** (amplitudes) determined from the normal-mode eigenfunctions
- $\tau_{\ell,n} = 1/s_{\ell,n}$ are **relaxation times** from the normal-mode eigenvalues

Key Result: The exponential terms $e^{-t/\tau_{\ell,n}}$ arise directly from the eigenvalue spectrum $s_{\ell,n}$ of the viscoelastic normal-mode problem.

6 Physical Interpretation of Normal Modes

6.1 Spherical Harmonic Degrees

- $\ell = 0$: Radial expansion/contraction (breathing mode)
- $\ell = 1$: Translational motion (center of mass)
- $\ell = 2$: Flattening/elongation (tidal deformation)
- **Higher** ℓ: Shorter wavelength surface deformation

6.2 Radial Mode Structure

Each degree ℓ has multiple relaxation times $\tau_{\ell,n}$ corresponding to different **radial modes**:

- **Fast modes (**$n = 1, 2, ...$**)**: Surface/shallow deformation (years to decades)
- **Slow modes (**n **large)**: Deep mantle flow (thousands of years)

The radial mode index n reflects the number of nodes in the radial eigenfunction - higher n corresponds to more complex radial structure and typically longer relaxation times.

7 Connection to Kendall et al. (2005): Time-Dependent Sea Level Equations

7.1 From Normal Modes to Sea Level Response

Kendall et al. (2005) applied the Tromp & Mitrovica (1999) framework to sea level problems. The time-dependent sea level response in spherical harmonic space is:

$$SL_{\ell m}(t_j) = T_\ell E_\ell[\rho_I I^*_{\ell m}(t_j)+\rho_w S_{\ell m}(t_j)]+T_\ell \sum_{i=1}^{j-1} \Delta t \sum_k A^k_\ell[\rho_I \Delta I^*_{\ell m}(t_i)+\rho_w \Delta S_{\ell m}(t_i)]e^{-(t_j-t_i)/\tau_k} \tag{16}$$

where:

- $E_\ell = 1 + k^{el}_\ell - h^{el}_\ell$ (elastic response coefficient)
- $T_\ell = \frac{4\pi a^3}{M_e(2\ell+1)}$ (normalization factor)
- A^k_ℓ are the normal-mode amplitudes from Tromp (1999)
- $\tau_k = 1/s_k$ are the normal-mode relaxation times
- The exponential terms $e^{-(t_j-t_i)/\tau_k}$ encode the **viscoelastic memory**

7.2 Implementation in SLcode: MATLAB vs Python

Key Implementation Note: The complete Tromp (1999) → Kendall (2005) framework is implemented differently across SLcode versions:

7.2.1 MATLAB Implementation (Complete Viscoelastic Theory)

The MATLAB files (`SL_equation_viscoelastic_*.m`) implement the **full Tromp (1999) normal-mode theory**:

$$\boxed{\beta_\ell(t) = \sum_{k=1}^{N_{modes}} \frac{(k_{amp,\ell,k} - h_{amp,\ell,k})}{s_{\ell,k}} \left(1 - e^{-s_{\ell,k}\cdot t}\right)} \tag{17}$$

where:

- `spoles(ℓ,k)` $= s_{\ell,k}$ (normal-mode decay rates)
- `k_amp(ℓ,k)`, `h_amp(ℓ,k)` = amplitude coefficients from normal-mode eigenfunctions
- `mode_found(ℓ)` = number of computed modes for degree ℓ

The MATLAB code loads these parameters from precomputed normal-mode solutions:

```
load SavedLN/prem.l90C.umVM2.lmVM2.mat
% Contains: spoles, k_amp, h_amp, k_amp_tide, h_amp_tide, mode_found
```

7.2.2 Python Implementation (Elastic Only)

The Python files (`SL_equation_elastic.py`) implement **only the elastic limit**:

$$E_{ml} = 1 + k_{lm} - h_{lm} \tag{18}$$

using static Love numbers without time dependence. The Python version:

- Loads elastic Love numbers: `love['h_el']`, `love['k_el']`
- Implements Kendall (2005) elastic sea level equation
- **Does not include** `spoles`, `k_amp`, `h_amp` parameters
- **No viscoelastic time dependence** or normal-mode theory

Status: According to the repository documentation, implementing the full viscoelastic Python version is a "Long Term" development goal.

8 Mathematical Derivation: From Continuum Mechanics to Implementation

8.1 The Complete Theoretical Chain

The complete progression from fundamental physics to numerical implementation follows:

Maxwell Rheology
↓
Viscoelastic Normal-Mode Problem
↓
Eigenvalue Spectrum $\{s_{\ell,k}\}$
↓
Green's Tensor Construction
↓
Love Number Time Dependence
↓
Kendall (2005) Sea Level Equations
↓
SLcode MATLAB Implementation

8.2 Physical Significance of Normal-Mode Parameters

The parameters in SLcode MATLAB files have direct physical meaning:

- `spoles(`ℓ`,k)` $= s_{\ell,k}$: Characteristic decay rates of Earth's viscoelastic response
  - Determined by mantle viscosity structure and elastic moduli
  - Fast modes: $s_{\ell,k} \sim 10^{-1}$ yr^{-1} (surface/lithosphere)
  - Slow modes: $s_{\ell,k} \sim 10^{-4}$ yr^{-1} (deep mantle flow)
- `k_amp(`ℓ`,k), h_amp(`ℓ`,k)`: Amplitude coefficients from surface values of normal-mode eigenfunctions
  - Related to gravitational potential and radial displacement eigenfunctions
  - Encode the coupling between loading and deformation for each mode
- `mode_found(`ℓ`)`: Number of computed normal modes for each spherical harmonic degree
  - Higher ℓ typically requires more modes for convergence
  - Reflects the complexity of radial eigenfunction structure

8.3 Displacement Field Expansion

The displacement field separates as:

$$u_r(r, \theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_r^{\ell m}(r, t) Y_\ell^m(\theta, \phi) \tag{19}$$

$$u_\theta(r, \theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\theta^{\ell m}(r, t) \frac{\partial Y_\ell^m}{\partial \theta} \tag{20}$$

$$u_\phi(r, \theta, \phi, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} u_\phi^{\ell m}(r, t) \frac{1}{\sin\theta} \frac{\partial Y_\ell^m}{\partial \phi} \tag{21}$$

8.4 Spherical Laplacian Eigenvalue Property

The key insight is that spherical harmonics are eigenfunctions of the angular part of ∇^2:

$$\nabla^2 Y_\ell^m = \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) - \frac{\ell(\ell+1)}{r^2} \right] Y_\ell^m \tag{22}$$

8.5 Separation by Degree

When we substitute the expansion (18)-(20) into the linear operator L_{ij}, **the problem separates by spherical harmonic degree ℓ**.

For each ℓ, we get a **radial ODE system**:

$$\mathcal{L}_\ell^{(r)} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} = \lambda_{\ell,k} \begin{pmatrix} u_r^{\ell m}(r) \\ u_\theta^{\ell m}(r) \end{pmatrix} \tag{23}$$

where $\mathcal{L}_\ell^{(r)}$ is the **radial differential operator** that depends on degree ℓ.

8.6 Radial Operator Structure

The radial operator has the form:

$$\mathcal{L}_\ell^{(r)} = \begin{pmatrix} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} & \text{coupling terms} \\ \text{coupling terms} & \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu}{\eta} \end{pmatrix} \tag{24}$$

9 Connection to Love Numbers

9.1 Eigenvalue Spectrum

For each degree ℓ, the radial operator $\mathcal{L}_\ell^{(r)}$ has eigenvalues:

$$\lambda_{\ell,1}, \lambda_{\ell,2}, \lambda_{\ell,3}, \ldots \tag{25}$$

These give the **relaxation times** for degree ℓ:

$$\tau_{\ell,k} = \frac{1}{\lambda_{\ell,k}} \tag{26}$$

9.2 Love Number Time Dependence

The **viscoelastic Love numbers** are constructed from these eigenvalues:

$$h_\ell(t) = h_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^h e^{-t/\tau_{\ell,k}} \tag{27}$$

$$k_\ell(t) = k_\ell^E \delta(t) + \sum_{k=1}^{\infty} r_{\ell,k}^k e^{-t/\tau_{\ell,k}} \tag{28}$$

where:

- h_ℓ^E, k_ℓ^E are **elastic Love numbers** (instantaneous response)
- $r_{\ell,k}^{h,k}$ are **residues** (amplitudes of each mode)
- $\tau_{\ell,k} = 1/\lambda_{\ell,k}$ are **relaxation times** from eigenvalues

9.3 Physical Interpretation

- **$\ell = 0$**: Radial expansion/contraction (breathing mode)
- **$\ell = 1$**: Translational motion (center of mass)
- **$\ell = 2$**: Flattening/elongation (tidal deformation)
- **Higher ℓ**: Shorter wavelength surface deformation

Each degree ℓ has multiple relaxation times $\tau_{\ell,k}$ corresponding to different **radial modes**:

- Fast modes: Surface/shallow deformation (years to decades)
- Slow modes: Deep mantle flow (thousands of years)

10 Summary

The complete chain is:

Maxwell rheology $\rightarrow$ Linear operator L $\rightarrow$ Spherical harmonic separation $\rightarrow$ Radial eigenvalue problems $\rightarrow$ Relaxation times $\tau_{\ell,k}$ $\rightarrow$ Love number exponentials

The exponential terms $e^{-t/\tau_{\ell,k}}$ in viscoelastic Love numbers arise directly from the **eigenvalue spectrum** of the radial viscoelastic operator for each spherical harmonic degree ℓ.