Measure Theory

Measure: Intuition

Roughly: Measure = Integral as a function of its region

$$\mu(A) = \int_A dx \qquad \text{or} \qquad \mu(A) = \int_A p(x)dx$$

Interpretation

$\mu(A)$ is mass of A, eg:

- Geometric case: Volume of A, or physical mass of a body.
- Probability case: Probability mass of event "random variable X takes value in A"

Integration: Abstract properties

Integrals: Decomposition properties

Write $\mu(A)$ for integral $\int_A dx$.

- $\mu(\emptyset) = 0$ (integral over empty set is zero)
- Pairwise disjoint sets A_n:

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) \quad \text{and} \quad \mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n)$$

- If $B \subset A$:

$$\mu(B) \leq \mu(A) \qquad \text{and} \qquad \mu(A \setminus B) = \mu(A) - \mu(B)$$

Henri Lebesgue's Approach

Call any set function an integral (a measure) if it decomposes like an integral.

σ-algebras (1)

Motivation

P here indicates the Powerset

- Defining measure: Often difficult/impossible on $\mathcal{P}(\Omega)$
- Idea: Restrict μ to subset $\mathcal{A}$ ("measurable sets") of $\mathcal{P}(\Omega)$
- Measurable sets = sets over which we can integrate

Intuition: σ-algebra

- Always assume we can integrate over Ω
- If integrals on $A_1, A_2, \dots$ given: Write $\mathcal{A} = \sigma(\{A_1, A_2, \dots\})$ for system of all sets with deducable integrals.
- Compeleted set system $\mathcal{A}$ is called *σ-algebra*.

σ-algebras (2)

Def: σ-algebra

A system of sets $\mathcal{A} \subset \mathcal{P}(\Omega)$ is called a σ-algebra if:

1. $\emptyset, \Omega \in \mathcal{A}$
2. If $A \in \mathcal{A}$, then $\complement A \in \mathcal{A}$
3. If $A_n \in \mathcal{A}$ (for $n \in \mathbb{N}$), then $\cup_n^{\infty} A_n \in \mathcal{A}$

Constructing σ-algebras

Most important method:

- Start with: $\mathcal{T}$ = all open sets in Ω.
- σ-algebra: $\mathcal{B}(\Omega) := \sigma(\mathcal{T})$
  Read: $\sigma(\mathcal{T})$ = smallest σ-algebra that includes $\mathcal{T}$
- $\mathcal{B}(\Omega)$ is called the *Borel σ-algebra* of Ω
- Contains all open and closed sets