Measures

Def: Measure

Given σ-algebra $\mathcal{A}$, a *measure* is a function $\mu : \mathcal{A} \to \mathbb{R}_+$ with:

1. $\mu(\emptyset) = 0$
2. $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ if $A_n \in \mathcal{A}$ pairwise disjoint.

μ is *probability measure* if additionally

3. $\mu(\Omega) = 1$

Note: (1) and (2) imply all integral decomposition properties.

Most important measures

- *Lebesgue measure*: “Flat” measure on $\mathbb{R}^d$ (d-volume).
- *Counting measure*: $|A|$ if A finite set, $+\infty$ otherwise.

Densities

Intuition

Density = function that transforms measure μ_1 into
measure μ_2 by pointwise reweighting (on Ω)

Derivative Notation

Here, x is a set (eg. an event)

$$d\mu_2(x) = f(x) d\mu_1(x) \qquad \text{or} \qquad \frac{d\mu_2}{d\mu_1}(x) = f(x)$$

Is this a Jacobian (of..?)?

Motivation: f = a function that is integrated to obtain μ_2
$\to$ “derivative” of μ_2

Immediate Question:

Is there always a density for μ_1, μ_2 given?

Radon-Nikodym Theorem

Absolute Continuity

- “Reweighting” by density

$$\mu_2(A) = \int_A d\mu_2(x) = \int_A f(x) d\mu_1(x)$$

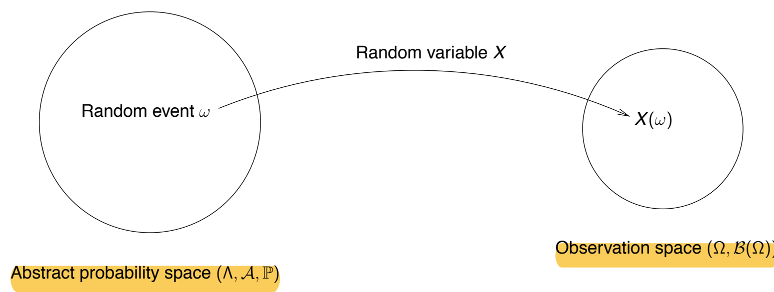
cannot work if $\mu_1(A) = 0$ and $\mu_2(A) \neq 0$.

- If that never happens for any $A \in \mathcal{A}$, then μ_2 is called “absolutely continuous wrt μ_1”, in symbols: $\mu_2 \ll \mu_1$

Theorem (Radon-Nikodym)

Let μ_1, μ_2 be two finite measures on $\mathcal{A}$. Then μ_2 has a density w.r.t. μ_1 if and only if $\mu_2 \ll \mu_1$.

Probability: Formal Framework



- ω: atomic random event, “state of the universe”
- X: Random variable (mapping $\Lambda \to \Omega$)
- $X(\omega)$: observed random value
- $\mathbb{P}$: probability measure (distribution of ω)
- For set $A \in \mathcal{B}(\Omega)$: Probability of “$X(\omega) \in A$” = $\mathbb{P}(X^{-1}(A))$