Assignment-3

Q1. Use Taylor's theorem to expand $f(x,y) = x^2 + xy + y^2$ in powers of $(x-1)$ and $(y-2)$

→ Here $x-1=0$ $\boxed{x=1}$ $y-2=0$ $\boxed{y=2}$

Now

	At (1,2)
$f(x,y) = x^2+xy+y^2$	$f(1,2) = 7$
$f_x = 2x+y$	$f_x = 4$
$f_y = x+2y$	$f_y = 5$
$f_{xx} = 2$	$f_{xx} = 2$
$f_{xy} = 1$	$f_{xy} = 1$
$f_{yy} = 2$	$f_{yy} = 2$

By Taylor's series

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}\Big)_{(a,b)} + (y-b)\frac{\partial f}{\partial y}\Big)_{(a,b)} + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2}\Big)_{(a,b)} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\Big)_{(a,b)} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x\partial y}\Big)_{(a,b)}\right] + \cdots$$

$$= 7 + (x-1)4 + (y-2)5 + \frac{1}{2!}\left[(x-1)^2 2 + (y-2)^2 2 + 2(x-1)(y-2)\right]$$

$$x^2+xy+y^2 = 7 + 4(x-1) + 5(y-2) + (x-1)^2 + (y-2)^2 + (x-1)(y-2)$$

Q2. Expand $f(x,y) = e^y \ln(1+x)$ in powers of x and y At $(0,0)$

$f(x,y) = e^y \ln(1+x)$

$f_x = \dfrac{e^y}{1+x}$

$f_y = e^y \ln(1+x)$

$f_{xx} = \dfrac{-e^y}{(1+x)^2}$

$f_{xy} = \dfrac{e^y}{1+x}$

$f_{yy} = e^y \ln(1+x)$

$f(0,0) = 0$

$f_x = 1$

$f_y = 0$

$f_{xx} = 1$

$f_{xy} = 1$

$f_{yy} = 0$

By Taylor's series

$$e^y \ln(1+x) = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_y(0,0)\right] + \cdots$$

$$= 0 + x(1) + y(0) + \frac{1}{2!}\left[x^2(1) + 2xy(1) + y^2(0)\right] + \cdots$$

$$= x + \frac{x^2}{2} + xy + \cdots$$

Q3. Expand $f(x,y) = \cos x \cos y$ at $(0,0)$ in powers of x and y

At $(0,0)$

$f(x,y) = \cos x \cos y$ | $f(0,0) = 1$

$f_x = -\sin x \cos y$ | $f_x = 0$

$f_y = -\sin y \cos x$ | $f_y = 0$

$f_{xx} = -\cos x \cos y$ | $f_{xx} = -1$

$f_{xy} = \sin x \sin y$ | $f_{xy} = 0$

$f_{yy} = -\cos y \cos x$ | $f_{yy} = -1$

By Taylor series

$$\cos x \cos y = f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \cdots$$

$$= 1 + x(0) + y(0) + \frac{1}{2!}\left[x^2(-1) + 2xy(0) + y^2(-1)\right] + \cdots$$

$$= 1 - \frac{1}{2}\left[x^2 + y^2\right] + \cdots$$

$$\cos x \cos y = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \cdots$$

Q4. Expand $\tan^{-1}\left(\frac{y}{x}\right)$ by taylor's series about $(1,1)$ and hence find the value of $\tan^{-1}\left(\frac{0.9}{1.1}\right)$ approximately.

At $(1,1)$

$f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ | $f(1,1) = \frac{\pi}{4}$

$f_x = \frac{-y}{x^2+y^2}$ | $f_x = -1/2$

$f_y = \frac{x}{x^2+y^2}$ | $f_y = 1/2$

$f_{xx} = \frac{2xy}{(x^2+y^2)^2}$ | $f_{xx} = 2/2$

$$f_{xy} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \qquad f_{xy} = 0$$

$$f_{yy} = \frac{-2xy}{(x^2+y^2)^2} \qquad \qquad f_{yy} = -1/2$$

By Taylor's series

$$\tan^{-1}\left(\frac{y}{x}\right) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1) + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)\right] + \dots$$

$$= \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\left(\frac{1}{2}\right) + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-1/2\right)\right] + \dots$$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{x-1}{2} + \frac{y-1}{2} + \frac{(x-1)^2}{4} - \frac{(y-1)^2}{4} + \dots$$

When $y = 0.9$ $x = 1.1$

$$\tan^{-1}\frac{y}{x} = \frac{\pi}{4} - \frac{(1.1-1)}{2} + \frac{(0.9-1)}{2} + \frac{(1.1-1)^2}{4} - \frac{(0.9-1)^2}{4}$$

$$= \frac{\pi}{4} - \frac{0.1}{2} - \frac{0.1}{2} + \cancel{\frac{0.01}{4}} - \cancel{\frac{0.01}{4}}$$

$$= \frac{\pi}{4} - \frac{0.2}{2} = \frac{\pi}{4} - 0.1$$

$$\tan^{-1}\left(\frac{0.9}{1.1}\right) = 0.6855$$

Q5. Using differential calculus, calculate the approximate value of $f(1.997)$ where $y(x) = x^4 - 2x^3 + 9x + 7$

$$f(1.997) = f(x+\Delta x)$$
$$= f(x) + \Delta f$$
$$= f(x) + f'(x)\Delta x$$
$$= f(2) + f'(2)(-0.003)$$

$f(x) = x^4 - 2x^3 + 9x + 7$ $\qquad$ $f(2) = 25$

$f'(x) = 4x^3 - 6x^2 + 9$ $\qquad$ $f'(2) = 17$

$$\therefore f(1.997) = 25 + 17(0.003)$$
$$= 25 - 0.051$$
$$f(1.997) = 24.949$$

Q6. The time T of a complete oscillation of a simple pendulum of length L is governed by the eq^n $T = 2\pi\sqrt{\frac{L}{g}}$, where g is constant. Find error in T, when error in L is 2%.

$$T = 2\pi\sqrt{\frac{L}{g}}$$

Given: Error in $L = 2\%$

Find: Error in T

Taking log on both sides

$$\log T = \log 2\pi + \frac{1}{2}\log L - \frac{1}{2}\log g$$

Differentiating

$$\frac{dT}{T} = \frac{1}{2L} \cdot dL$$

$$\frac{dT}{T}\times 100 = \frac{dL}{2L}\times 100$$

$$= \frac{1}{2}\times 2$$

$$\frac{dT}{T}\times 100 = 1\%$$

$\therefore$ Error in $T = 1\%$

Q7 The diameter ~~of~~ and height of a right circular cylinder are measured to be 5 and 8 inches respectively. If each of these dimensions may be in error of ± 0.1 inch. find relative precentage error in volume of cylinder

Let diameter of cylinder be x

Height of cylinder be h

$$V = \pi r^2 h$$

As ~~dia~~ $x = 2r$

$$r^2 = \frac{x^2}{4}$$

$$V = \frac{1}{4}\pi x^2 h$$

Taking log on both sides

$$\log V = \log\frac{\pi}{4} + 2\log x + \log h$$

Differentiating

$$\frac{dV}{V} = 2\frac{dx}{x} + \frac{dh}{h}$$

Given $x = 5$ inches, $h = 8$ inches, $dx = dh = \pm 0.1$

$$\frac{dV}{V} = \pm\left(2\times\frac{0.1}{5} + \frac{0.1}{8}\right) = \pm 0.0525$$

$$\frac{dV}{V}\times 100 = \pm 0.0525\times 100 = \pm 5.25\%$$

$\therefore$ error in volume $= \pm 5.25\%$

Q8 What are the advantages of Lagrange's method over ordinary method of calculating maxima and minima?

→ It can be used for more than two variables. Also this method can be extended to a function of several 'n' variables $x_1, x_2, x_3, ---, x_n$ and subject to many (more than one) 'm' constraints by forming auxiliary equation.

$$F(x_1, x_2, ---, x_n) = f(x_1, x_2, ---x_n) + \sum_{i=1}^{m} \lambda_i \phi_i (x_1, x_2, ---, x_n).$$

The stationary values are obtained by solving $n+m$ equations consisting of n equations $\frac{\partial F}{\partial x_i} = 0$, for $i=1,2,3,---,n$ and m constraint, $\phi_i = 0$, for $i=1,2,3,---,m$.

Q9. Find minimum and maximum value of $f(x,y) = x^3 + 3xy^2 \cancel{+} -15x^2 - 15y^2 + 72x$.

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$

Differentiating f partially w.r.t. x and y

$$f_x = 3x^2 + 3y^2 - 30x + 72$$

$$f_y = 6xy - 30y$$

Now $f_x = 0$ and $f_y = 0$

$$6xy - 30y = 0$$

$$6y(x-5) = 0$$

$\boxed{y=0}$ or $\boxed{x=5}$

$$3x^2 + 3y^2 - 30x + 72 = 0$$

when $y=0$

$3x^2 - 30x + 72 = 0$ $\boxed{x = 6 \text{ or } 4}$

when $x=5$

$75 + 3y^2 - 150 + 72 = 0$ $\boxed{y = \pm 1}$

$\therefore$ Points:- $(6,0), (4,0), (5,1), (5,-1)$

Now

$r = f_{xx} = 6x - 30$ $\quad$ $t = f_{yy} = 6x - 30$

$s = f_{xy} = 6y$

At $(6,0)$

$r = 36 - 30 = 6 > 0$

$rt - s^2 = 36 > 0.$

$\therefore (6,0)$ is minimum point and minimum value is

$6^3 + 0 - 15.36 + 72.6 = 108$

At $(4,0)$

$r = 24 - 30 = -6 < 0$

$rt - s^2 = 36 > 0$

$(4,0)$ is maximum point and maximum value is 112

At $(5,1)$

$r = 0$

$rt - s^2 = -36 < 0$

$\therefore (5,1)$ is neither maximum nor minimum

At $(5,-1)$

$r = 0$

$rt - s^2 = -36 < 0$

$(5,-1)$ is neither maximum nor minimum.

Q10. Find the shortest distance from origin to surface $xyz^2 = 2$

→ Let d be distance from origin $(0,0,0)$ to any pt. (x,y,z)

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$

$d = x^2 + y^2 + z^2$

$$xyz^2 = 2$$
$$z^2 = \frac{2}{xy}$$

$$f = d^2 = x^2 + y^2 + z^2$$
$$= x^2 + y^2 + \frac{2}{xy}$$

$$f = x^2 + y^2 + \frac{2}{xy}$$

$$\therefore f_x = 2x - \frac{2}{x^2 y}, \quad f_y = 2y - \frac{2}{xy^2}$$

Now $f_y = 0$ and $f_x = 0$

$$\frac{x^3 y - 1}{x^2 y} = 0 \quad \text{and} \quad \frac{xy^3 - 1}{xy^2} = 0$$

$$x^3 y = 1 = xy^3 \text{ or } xy(x^2 - y^2) = 0$$

Since $x \neq 0, y \neq 0$ or $x = \pm y = 1$

$\therefore$ Points $= (1,1), (-1,-1)$

$r = f_{xx} = 2 + \frac{4}{x^3 y}$ $\quad s = f_{xy} = \frac{2}{x^2 y^2}$ $\quad t = f_{yy} = 2 + \frac{4}{xy^3}$

At $(1,1)$

$r = 6 > 0$

$rt - s^2 = 32 > 0$

At $(-1,-1)$

$r = 6 > 0$

$rt - s^2 = 32 > 0$

$\therefore$ Minimum ~~distance~~ occurs at $(1, 1, \sqrt{2})$ and $(-1, -1, \sqrt{2})$

Shortest distance,

$$d = \sqrt{1 + 1 + 2} = \sqrt{4}$$
$$d = 2 \text{ units}$$

Q11. Find shortest distance from origin to plane $x-2y-2z=3$

Let d be distance from origin $(0,0,0)$ to any pt (x,y,z) in plane

$$d=\sqrt{x^2+y^2+z^2}$$

$$f=d^2=x^2+y^2+z^2$$

$$f=x^2+y^2+\frac{(x-2y-3)^2}{4}$$

$$\left[\begin{array}{c} x-2y-3=2z \\ z=\frac{x-2y-3}{2} \\ z^2=\frac{(x-2y-3)^2}{4} \end{array}\right]$$

$$f_x=2x+2\frac{(x-2y-3)}{4} \qquad f_y=2y+2\frac{(x-2y-3)}{4}(-2)$$

$$=\frac{5x}{2}-y-\frac{3}{2} \qquad\qquad =4y-x+3$$

Now $\quad f_x=0 \qquad f_y=0$

$5x-2y=3 \quad -(1) \qquad x-4y=3$

$x=4y+3 \quad -(2)$

Solving (1) and (2)

$\boxed{x=\frac{1}{3}} \qquad \boxed{y=\frac{-2}{3}}$

$r=f_{xx}=\frac{5}{2} \qquad s=f_{xy}=-1 \qquad t=f_{yy}=4$

At $\left(\frac{1}{3},-\frac{2}{3}\right)$

$r=5/2>0$

$rt-s^2=10-1=9>0$

$\therefore z=\left(\frac{1/3+4/3-3}{2}\right)=-2/3$

$\therefore$ Minimum occur at $(1/3,-2/3,-2/3)$

$\therefore$ Shortest distance, $d=\sqrt{\left(\frac{1}{3}\right)^2+(-2/3)^2+(-2/3)^2}$

$$=\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}}=1$$

$\boxed{d=1 \text{ unit}}$

Q12. Find volume of largest rectangular parallelopiped with edges parallel to the axes that can be inscribed in the

(i) sphere $x^2+y^2+z^2=a^2$

(ii) ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(i)

$$V = 2x.2y.2z$$

$$V = 8xyz$$

$$V^2 = 64x^2y^2z^2$$

$$f = V^2 = 64x^2y^2\left[a^2-(x^2+y^2)\right]$$

$$= 64\left[a^2x^2y^2 - x^4y^2 - x^2y^4\right]$$

$$f_x = 64\left[2a^2xy^2 - 4x^3y^2 - 2xy^4\right]$$

$$= 64(2xy^2)\left[a^2-2x^2-y^2\right]$$

$$f_y = 64\left[2a^2x^2y - 2x^4y - 4x^2y^3\right]$$

$$= 64(2x^2y)\left[a^2-x^2-2y^2\right]$$

Now $\quad f_x = 0 \qquad\qquad f_y = 0$

$128\,xy^2\left[a^2-2x^2-y^2\right]=0 \qquad\qquad 128\,x^2y\left[a^2-x^2-2y^2\right]=0$

$2x^2+y^2=a^2 \quad -(1) \qquad\qquad x^2+2y^2=a^2 \quad -(2)$

Solving (1) and (2), we get

$$x=\frac{a}{\sqrt{3}} \qquad y=\frac{a}{\sqrt{3}}$$

Now

$r = f_{xx} = 64\left[2a^2y^2 - 12x^2y^2 - 2y^4\right]$

$t = f_{yy} = 64\left[2a^2x^2 - 2x^4 - 12x^2y^2\right]$

$s = f_{xy} = 64\left[4a^2xy - 8x^3y - 8xy^3\right]$

At $(a/\sqrt{3}, a/\sqrt{3})$

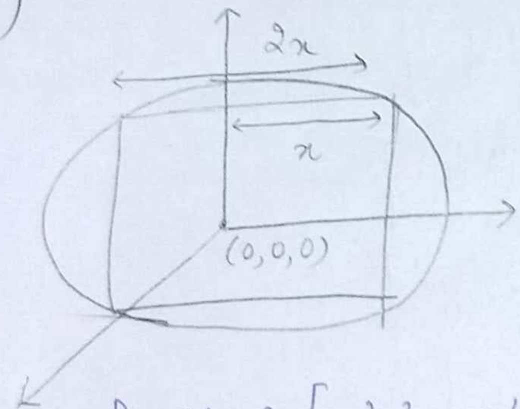
$r = -\frac{\overset{512}{\cancel{64}}}{9}a^4 < 0$

$rt - s^2 = \frac{196608}{81}a^8 > 0$

$\therefore$ Maximum occur at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

$\therefore$ Maximum volume $= \frac{8a^3}{3\sqrt{3}}$

(ii)



$V = 2x \cdot 2y \cdot 2z$

$= 8xyz$

$V^2 = 64x^2y^2z^2$

$f = 64x^2y^2c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

$f = 64c^2\left[x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right]$

$f_x = 64c^2\left[2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right]$

$f_y = 64c^2\left[2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right]$

Now $f_x = 0$ and $f_y = 0$

$128xy^2c^2\left[1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right] = 0$

$\frac{2x^2}{a^2} + \frac{y^2}{b^2} = 1$ — (1)

$128x^2yc^2\left[1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right] = 0$

$\frac{x^2}{a^2} + \frac{2y^2}{b^2} = 1$ — (2)

Solving ① and ②, we get

$\boxed{x = \frac{a}{\sqrt{3}}}$ $\boxed{y = \frac{b}{\sqrt{3}}}$

$r = f_{xx} = 64c^2\left[2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right]$

$s = f_{xy} = 64c^2\left[4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right]$

$t = f_{yy} = 64c^2\left[2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right]$

At $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$

$r = \frac{-512\,b\,a\,c^2}{9} < 0$

$rt - s^2 = \frac{196608\,c^4a^2b^2}{81} > 0$

$\therefore$ Maximum occur at $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

$\therefore$ Maximum volume $= \frac{8abc}{3\sqrt{3}}$

Q13. Find dimensions of a rectangular box with open top, so that total surface area of box is minimum, given that volume of box is constant say V.

Let x be length, y be breadth, z be height of box

$\therefore V = xyz$

$\therefore$ Total surface area, $S = xy + 2xz + 2yz$

$f = S = xy + \frac{2V}{y} + \frac{2V}{x}$

$f_x = y - \frac{2V}{x^2}$ $f_y = x - \frac{2V}{y^2}$

Now $f_x = 0$ and $f_y = 0$

$y - \frac{2V}{x^2} = 0$ — ① $x - \frac{2V}{y^2} = 0$ — ②

Solving ① and ②, we get
$x = (2V)^{1/3}$ and $y = (2V)^{1/3}$

$f_{xx} = r = \frac{4V}{x^3}$ $\quad s = f_{xy} = 1$ $\quad t = f_{yy} = \frac{4V}{y^3}$

At $[(2V)^{1/3}, (2V)^{1/3}]$

$r = 2 > 0$

$rt - s^2 = 3 > 0$

$\therefore$ Minimum occur at $\left[(2V)^{1/3}, (2V)^{1/3}, \frac{(2V)^{1/3}}{2}\right]$

$\therefore x = (2V)^{1/3}$ $\quad y = (2V)^{1/3}$ $\quad 2z = (2V)^{1/3}$

Q14. Find minimum and maximum distance from origin to curve $3x^2 + 4xy + 6y^2 = 140$

Let d be distance from $(0,0)$ to any pt (x,y)

$$d = \sqrt{(x-0)^2 + (y-0)^2}$$

$$f = d^2 = x^2 + y^2$$

$$\phi = 3x^2 + 4xy + 6y^2 - 140$$

$$F = f + \lambda\phi$$
$$= x^2 + y^2 + \lambda(3x^2 + 4xy + 6y^2 - 140)$$

$F_x = 2x + 6\lambda x + 4y$ $\quad F_y = 2y + 12\lambda y + 4x$

~~Now $F_x = 0$ and $F_y = 0$~~

Solving for $\lambda = \frac{-x}{(3x+2y)} = \frac{-y}{6y+2x}$

$$-\lambda = \frac{x^2}{3x^2+2xy} = \frac{y^2}{6y^2+2xy} = \frac{x^2+y^2}{3x^2+6y^2+4xy}$$

$$-\lambda = \frac{f}{140}$$

Substituting λ in $F_x = 0$ and $F_y = 0$

$$(140-3f)x + 2fy = 0$$
$$-2fx + (140-6f)y = 0$$

This system has non-trival solution if

$$\begin{vmatrix} 140-3f & -2f \\ -2f & 140-6f \end{vmatrix} = 0$$

$$(140-3f)(140-6f) - 4f^2 = 0$$
$$14f^2 - 1260f + 140^2 = 0$$
$$f^2 - 90f - 1400 = 0$$
$$(f-70)(f-20) = 0$$
$$\therefore f = 70, 20$$

$\therefore$ Maximum and minimum distances are $\sqrt{70}, \sqrt{20}$

Q15. Find minimum value of $x^2+y^2+z^2$ subject to condition $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$f = x^2+y^2+z^2$$

$$\phi = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1$$

$$F = f + \lambda\phi$$
$$= x^2+y^2+z^2+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

$F_x = 2x - \frac{\lambda}{x^2}$ $\quad$ $F_y = 2y - \frac{\lambda}{y^2}$ $\quad$ $F_z = 2z - \frac{\lambda}{z^2}$

Now $F_x = 0$, $F_y = 0$, $F_z = 0$

$2x - \frac{\lambda}{x^2} = 0$ $\quad$ $2y - \frac{\lambda}{y^2} = 0$ $\quad$ $2z - \frac{\lambda}{z^2} = 0$

$x = \left(\frac{\lambda}{2}\right)^{1/3}$ $\quad$ $y = \left(\frac{\lambda}{2}\right)^{1/3}$ $\quad$ $z = \left(\frac{\lambda}{2}\right)^{1/3}$

Given: $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

$$\left(\frac{2}{\lambda}\right)^{1/3}+\left(\frac{2}{\lambda}\right)^{1/3}+\left(\frac{2}{\lambda}\right)^{1/3}=1$$

$$(\lambda)^{1/3}=3(2)^{1/3}$$

Taking cube on both sides

$$\lambda=3^3(2)$$

$$\therefore x=\left(\frac{3^3\cdot 2}{2}\right)^{1/3} \quad y=\left(\frac{3^3\cdot 2}{2}\right)^{1/3} \quad z=\left(\frac{3^3\cdot 2}{2}\right)^{1/3}$$

$\therefore x=y=z=27 \quad \therefore$ Minimum is 27

Q16. Find extreme value of $\sqrt{x^2+y^2}$ when $13x^2-10xy+13y^2=72$

$$f=\sqrt{x^2+y^2}$$

$$\phi=13x^2-10xy+13y^2-72$$

$$F=f+\lambda\phi$$
$$=\sqrt{x^2+y^2}+\lambda(13x^2-10xy+13y^2-72)$$

$$F_x=\frac{x}{\sqrt{x^2+y^2}}+\lambda(26x-10y) \qquad F_y=\frac{y}{\sqrt{x^2+y^2}}+\lambda(26y-10x)$$

$$\lambda=\frac{-x}{\sqrt{x^2+y^2}(26x-10y)}=\frac{-y}{\sqrt{x^2-y^2}(26y-10x)}$$

$$-\lambda=\frac{-x^2}{\sqrt{x^2+y^2}(26x^2-10xy)}=\frac{-y^2}{\sqrt{x^2-y^2}(26y^2-10xy)}=\frac{x^2+y^2}{\sqrt{x^2+y^2}(26x^2+26y^2-20xy)}$$

$$-\lambda=\frac{\sqrt{x^2+y^2}}{2[13x^2-10xy+13y^2]}=\frac{f}{144}$$

Substituting λ in $f_x = 0$ and $f_y = 0$, we get

$$(144 - 26f^2)x + 10f^2 y = 0$$

$$(144 - 26f^2)y + 10f^2 x = 0$$

This system has non-trivial solution if

$$\begin{vmatrix} 144 - 26f^2 & 10f^2 \\ \cancel{144}\ 10f^2 & 144 - 26f^2 \end{vmatrix} = 0$$

$$(144 - 26f^2)^2 - (10f^2)^2 = 0$$

$$(144 - 36f^2)(144 - 16f^2) = 0$$

$$\therefore f = 2, 3$$

Maximum and minimum value are 3, 2

Q17. Find maximum and minimum distance of point (3,14,12) from sphere $x^2+y^2+z^2=1$ using Lagrange's Method.

$$d = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

$$f = d^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$$\phi = x^2 + y^2 + z^2 - 1$$

$$F = f + \lambda\phi$$

$$F = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2+y^2+z^2-1)$$

$F_x = 2(x-3) + 2\lambda x$ $\quad F_y = 2(y-4) + 2\lambda y$ $\quad F_z = 2(z-12) + 2\lambda z$

Now $F_x = 0$, $F_y = 0$, $F_z = 0$, we get

$$x = \frac{3}{1+\lambda}, \quad y = \frac{4}{1+\lambda}, \quad z = \frac{12}{1+\lambda}$$

Given $x^2+y^2+z^2=1$

$$\left(\frac{3}{1+\lambda}\right)^2+\left(\frac{4}{1+\lambda}\right)^2+\left(\frac{12}{1+\lambda}\right)^2=1$$

$$169=(1+\lambda)^2$$

$$(1+\lambda)^2-(13)^2=0$$

$$(\lambda+1-13)(\lambda+1+13)=0$$

$$\lambda=12,-14$$

When $\lambda=12$

$x=\frac{3}{13}$ $y=\frac{4}{13}$ $z=\frac{12}{13}$

$F=\sqrt{\left(\frac{3}{13}-3\right)^2+\left(\frac{4}{13}-3\right)^2+\left(\frac{12}{13}-12\right)^2}$

$=11.9\simeq 12$

$\lambda=-14$

$x=\frac{-3}{13}$ $y=-\frac{4}{13}$ $z=\frac{-12}{13}$

$F=\sqrt{\left(\frac{-3}{13}-3\right)^2+\left(\frac{-4}{13}-4\right)^2+\left(\frac{-12}{13}-12\right)^2}$

$=14$

$\therefore$ Maximum and minimum distances are 14 and 12 respectively.