3-1-2019: Energy Density and RC Circuits

Friday, March 1, 2019 9:06 AM

Let's review from last time…

$q = CV$ Capacitors in parallel $C_{eq} = \sum_{i=1}^{n} C_i$

series $\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$

energy of capacitor…

$$U = \frac{q^2}{2C} = \frac{1}{2} qV = \frac{1}{2} CV^2$$

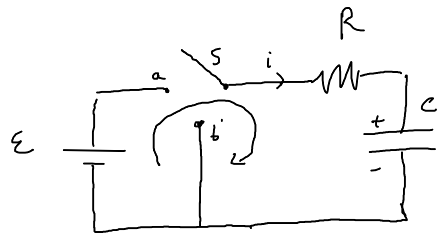
Parallel-plate capacitor: $C = \frac{\varepsilon_0 A}{d}$

what about energy density?

$$\boxed{u = \frac{U}{\text{Volume}}}$$

$$u = \frac{\frac{1}{2}\left(\frac{\varepsilon_0 A}{d}\right)V^2}{Ad} = \frac{1}{2}\varepsilon_0 \left(\frac{V}{d}\right)^2 = \frac{1}{2}\varepsilon_0 E^2$$

Now let's talk about the RC circuit. This circuit involves a capacitor and a resistor, and will be a dynamic system.



initially, no charge on C.

$S \to$ POSITION A

$q(t)$ $V = \frac{q}{C}$.

KLR

$$+\varepsilon - iR - V = 0$$

$$\varepsilon - iR - \frac{q}{C} = 0$$

REMEMBER, $i = \frac{dq}{dt}$.

$$\varepsilon - \frac{dq}{dt}R - \frac{q}{C} = 0$$

$$\Rightarrow \frac{dq}{dt}R = \varepsilon - \frac{q}{C} \Rightarrow \frac{dq}{dt} = \frac{\varepsilon}{R} - \frac{q}{RC}$$

$$\Rightarrow \frac{dq}{dt}\left(\frac{1}{\frac{\varepsilon}{R} - \frac{q}{RC}}\right) = 1$$

Now let's integrate both sides by dt…

$$\int_0^t \left(\frac{1}{\frac{\varepsilon}{R} - \frac{q}{RC}}\right)\frac{dq}{dt}\,dt = \int_0^t dt \Rightarrow \int_0^q \frac{dq}{\frac{\varepsilon}{R} - \frac{q}{RC}} = t \Rightarrow -RC\int_{\frac{\varepsilon}{R}}^{\frac{\varepsilon}{R} - \frac{q}{RC}} \frac{du}{u} = t \Rightarrow -RC\Big[\ln|u|\Big]_{\frac{\varepsilon}{R}}^{\frac{\varepsilon}{R} - \frac{q}{RC}} = t$$

$$u = \frac{\varepsilon}{R} - \frac{q}{RC}$$

$$u = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

$$du = dq\left(\frac{-1}{RC}\right)$$

$$\Rightarrow \ln\left|\frac{\frac{\mathcal{E}}{R} - \frac{q}{RC}}{\frac{\mathcal{E}}{R}}\right| = -\frac{t}{RC}$$

$$\Rightarrow \ln\left|1 - \frac{q}{\mathcal{E}C}\right| = -\frac{t}{RC} \Rightarrow 1 - \frac{q}{\mathcal{E}C} = e^{-\frac{t}{RC}} \Rightarrow C\mathcal{E} - q = C\mathcal{E}e^{-\frac{t}{RC}} \Rightarrow -q = C\mathcal{E}\left(e^{-\frac{t}{RC}} - 1\right)$$

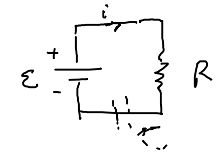
$$\Rightarrow \boxed{q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right) \quad \text{charging a capacitor}}$$

$$\boxed{i(t) = \frac{dq}{dt} = \frac{\mathcal{E}}{R}e^{-t/RC}} \qquad \boxed{V_C = \frac{q}{C} = \mathcal{E}\left(1 - e^{-\frac{t}{RC}}\right)}$$

@ $t = 0$ . . .

$i = \frac{\mathcal{E}}{R}$, $q = 0$, $V_C = 0$.

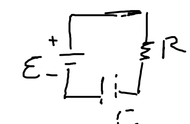
($C \sim$ WIRE)



@ $t = \infty$ . . .

$i = 0$, $q = C\mathcal{E}$, $V_C = \mathcal{E}$.

($C \sim$ BROKEN WIRE)



Let's think about the units here... note that the exponent of the e^() needs to be unitless! So,

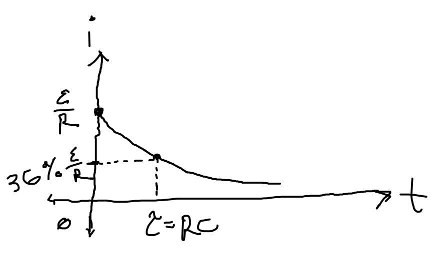
$$[t] = [R][C] \Rightarrow [s] = [R][C].$$

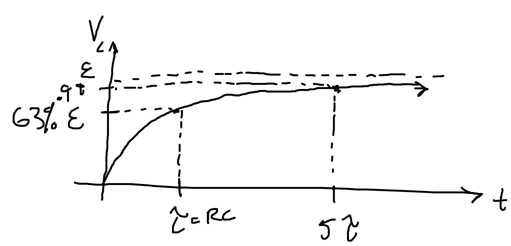
$\tau \equiv RC$ (UNITS, SECONDS)

TIME CONSTANT

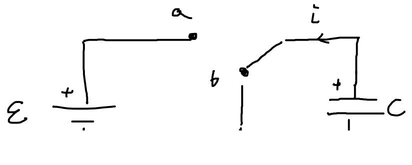
$$q = C\mathcal{E}\left(1 - e^{-t/\tau}\right) \qquad t \to \tau \Rightarrow q_\tau = C\mathcal{E}\left(1 - e^{-1}\right) = 0.63\, C\mathcal{E}.$$

$$t \to 5\tau \Rightarrow q \sim 99\% \text{ of } C\mathcal{E}.$$



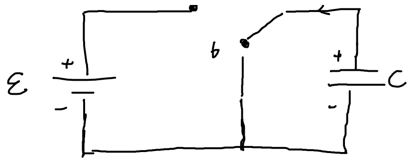


Now let's look at what happens in the circuit when we switch to point b instead!



Let's look at our KLR again... notice that our battery is no longer a part of the circuit.

$$\cancel{\mathcal{E}} - iR - \frac{q}{C} = 0$$



$$\cancel{\mathcal{E}} - iR - \frac{q}{C} = 0$$

$$\frac{dq}{dt} R = -\frac{q}{C}$$

$$\int_0^t \frac{1}{q} \frac{dq}{dt} dt = \int_0^t -\frac{1}{RC} dt$$

$$\int_0^q \frac{dq}{q} = -\frac{t}{RC} \Rightarrow q = q_0 e^{-\frac{t}{RC}} \qquad q_0 = C\mathcal{E}$$

$$i = \frac{dq}{dt} = -\frac{q}{RC} e^{-t/RC}$$