Suppose λ is an eigenvalue of T and v a corresponding eigenvector. Then

$$
\begin{aligned}
0 &= (T-2I)(T-3I)(T-4I)v \\
&= (T^3 - 9T^2 + 26T - 24I)v \\
&= T^3v - 9T^2v + 26T - 24v \\
&= \lambda^3 v - 9\lambda^2 v + 26\lambda v - 24v \\
&= (\lambda^3 - 9\lambda^2 + 26\lambda - 24)v
\end{aligned}
$$

If $\operatorname{null} T = \{0\}$ (because it implies surjectivity) or $\operatorname{range} T = \{0\}$ the result is obvious. Assume both contain non-zero vectors.

Since $\operatorname{null} T \neq \{0\}$, we have that 0 is an eigenvalue of T and that $E(0,T) = \operatorname{null} T$. Let $\lambda_1, \ldots, \lambda_m$ denote the other distinct eigenvalues of T. Now 5.41 implies that $V = \operatorname{null} T \oplus E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. We will prove that $\operatorname{range} T = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

Suppose $v \in E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Then $v = v_1 + \cdots + v_m$ for some $v_1, \ldots, v_m$, where each $v_j \in E(\lambda_j, T)$. Moreover, $v = T(\frac{1}{\lambda_1} v_1) + \cdots + T(\frac{1}{\lambda_m} v_m)$, which implies that $v \in \operatorname{range}(T)$. Hence

$$E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T) \subset \operatorname{range} T.$$

For the inclusion in the other direction, suppose $v \in \operatorname{range} T$. Note that $\operatorname{range} T$ stays the same when we restrict T to $E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Then $v = T(v_1 + \cdots + v_m)$ for some $v_1, \ldots, v_m$, where each $v_j \in E(\lambda_j, T)$. Therefore $v = \lambda_1 v_1 + \cdots + \lambda_m v_m$ and, because $\lambda_j v_j \in E(\lambda_j, T)$ for each j, this proves that $v \in E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$. Thus $\operatorname{range} T \subset E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, completing the proof.

$$
\begin{aligned}
\dim \operatorname{null} T + \dim \operatorname{range} T &= \dim V \\
&= \dim(\operatorname{null} T + \operatorname{range} T) \\
&= \dim \operatorname{null} T + \dim \operatorname{range} T - \dim(\operatorname{null} T \cap \operatorname{range} T)
\end{aligned}
$$

Therefore $\dim(\operatorname{null} T \cap \operatorname{range} T) = 0$, implying (c) is true.

Suppose (c) holds. By 1.45, $\operatorname{null} T + \operatorname{range} T$ is a direct sum. Then, by 2.43 and 3.22, we have

$$\dim(\operatorname{null} T \oplus \operatorname{range} T) = \dim \operatorname{null} T + \dim \operatorname{range} T = \dim V$$

Since $\operatorname{null} T \oplus \operatorname{range} T$ is a subspace of V, it follows that $\operatorname{null} T \oplus \operatorname{range} T = V$, implying (a) and completing the proof.

$$
\begin{aligned}
STv_j &= S(\lambda_j v_j) \\
&= \alpha_j \lambda_j v_j \\
&= \alpha_j T v_j \\
&= T(\alpha_j T v_j) \\
&= TSv_j
\end{aligned}
$$

For the converse, we will prove the contrapositive, that is, if $\langle u, v\rangle \neq 0$, then $||u|| > ||u+av||$ for some $a \in \mathbb{F}$.

Suppose $\langle u, v\rangle \neq 0$. Note that neither u nor v can equal 0. We have

$$\begin{aligned}
||u+av||^2 &= \langle u+av, u+av\rangle \\
&= \langle u, u\rangle + \langle u, av\rangle + \langle av, u\rangle + \langle av, av\rangle \\
&= ||u||^2 + \langle u, av\rangle + \langle av, u\rangle + |a|^2||v||^2 \\
&< ||u||^2
\end{aligned}$$

where the last line follows provided that $\langle u, av\rangle + \langle av, u\rangle + |a|^2||v||^2 < 0$. By 6.14, we can write $u = cv + w$ for some $c \in \mathbb{F}$ and $w \in V$ such that $\langle v, w\rangle = 0$. Note that $c \neq 0$, because $\langle v, v\rangle \neq 0$ and

$$0 \neq \langle u, v\rangle = \langle cv+w, v\rangle = \langle cv, v\rangle + \langle w, v\rangle = c\langle v, v\rangle$$

Choose $a = -c$, then

$$\begin{aligned}
\langle u, av\rangle + \langle av, u\rangle + |a|^2||v||^2 &= \langle cv+w, av\rangle + \langle av, cv+w\rangle + |a|^2||v||^2 \\
&= \langle cv, av\rangle + \langle w, av\rangle + \langle av, cv\rangle + \langle av, w\rangle + |a|^2||v||^2 \\
&= c\bar{a}\langle v, v\rangle + a\bar{c}\langle v, v\rangle + |a|^2||v||^2 \\
&= (c\bar{a} + a\bar{c} + ||c||^2)||v||^2 \\
&= (-c\bar{c} - c\bar{c} + ||c||^2)||v||^2 \\
&= -|c|^2||v||^2 \\
&< 0
\end{aligned}$$

$$\begin{aligned}\dim \operatorname{null} T^* &= \dim(\operatorname{range} T)^{\perp} \\ &= \dim W - \dim \operatorname{range} T \\ &= \dim W + \dim \operatorname{null} T - \dim V\end{aligned}$$

where the first line follows from 7.7 (a), the second from 6.50 and the third from 3.22. We alse have

$$\begin{aligned}\dim \operatorname{range} T^* &= \dim(\operatorname{null} T)^{\perp} \\ &= \dim V - \dim \operatorname{null} T \\ &= \dim \operatorname{range} T\end{aligned}$$

where the first line follows from 7.7 (b), the second from 6.50 and the third from 3.22.