T5

$\xi \sim p(x) = \frac{1}{\theta} \{[\theta, 2\theta]\}$

$\overline{X_n}$ – выборка

1) ОММ

$\alpha_1 = M\xi = \int\limits_{\theta}^{2\theta} \frac{1}{\theta} \cdot x\,dx = \frac{3}{2}\theta$

$\alpha_1 = \widetilde{\alpha_1} = \overline{x}$

$\frac{3}{2}\theta = \overline{x} \to \widetilde{\theta} = \frac{2}{3}\overline{x}$

$M\xi^2 = \frac{1}{\theta}\int\limits_{\theta}^{2\theta} x^2 dx = \frac{7}{3}\theta^2$

$D\xi = M\xi^2 - M^2\xi = \frac{1}{12}\theta^2$

несмещ:

$M\widetilde{\theta} = M[\frac{2}{3}\overline{x}] = \frac{2}{3}M\xi = \theta$ → несмещ.

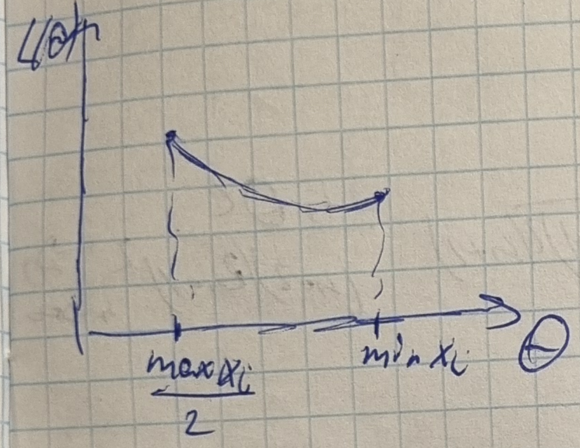
сост:

$D[\widetilde{\theta}] = \frac{4}{9}D[\overline{x}] = \frac{4}{9n}D\xi = \frac{1}{27n}\theta^2 \xrightarrow[n\to\infty]{} 0$

=> сост. по достат. усл.

2) ОМП

$L(\theta) = \frac{1}{\theta^n} \cdot \{\theta < x_i < 2\theta\} = \frac{1}{\theta^n} \cdot \{\frac{\max(x_i)}{2} < \theta < \min x_i\}$



$L(\theta) \to \max$

$\Rightarrow \hat{\theta} = \frac{x_{max}}{2}$

$M[\hat{\theta}] = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_{\theta}^{2\theta} \quad \left\{ \Psi = (F(x))^n = \left(\frac{x-\theta}{\theta}\right)^n \Rightarrow \psi = \Psi' = \frac{n}{\theta} \cdot \left(\frac{x-\theta}{\theta}\right)^{n-1} \right\} =$

$= \frac{1}{2}\int_{\theta}^{2\theta} \frac{nx}{\theta} \cdot \left(\frac{x-\theta}{\theta}\right)^{n-1} dx = \{\frac{x}{\theta} = t\} = \frac{\theta}{2}\int_1^2 nt(t-1)^{n-1} dt =$

$= \frac{\theta}{2}\{t(t-1)^n|_1^2 - \int_1^2 (t-1)^n dt\} = \frac{\theta}{2}[2 - \frac{1}{n+1}] =$

$= \frac{(2n+1)}{2(n+1)}\theta \to$ смещ.

$\Rightarrow \hat{\theta}_2' = \frac{2(n+1)}{2n+1} \cdot \frac{x_{max}}{2} = \frac{n+1}{2n+1} x_{max}$ – несмещ.

$M[\tilde{\theta}_2^2] = \frac{1}{4}\int_{\theta}^{2\theta} \frac{nx^2}{\theta} \cdot \left(\frac{x-\theta}{\theta}\right)^{n-1} dx = \{\frac{x}{\theta} = t\} =$

$= \frac{\theta^2}{4}\int_1^2 nt^2(t-1)^{n-1} dt = \frac{\theta^2}{4}[t^2(t-1)^n|_1^2 - 2\int_1^2 t(t-1)^n dt] =$

$= \dots = \frac{\theta^2}{4} \cdot \frac{(4n^2+8n+2)}{(n+1)(n+2)}$

$$D[\tilde{\theta}_2] = \frac{\theta^2}{4}\cdot\frac{(4n^2+8n+2)}{(n+1)(n+2)} - \frac{\theta^2(4n^2+4n+1)}{4(n+1)^2} =$$

$$= \frac{n}{4(n+1)^2(n+2)}\,\theta^2$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2 \cdot \frac{n}{4(n+1)^2(n+2)} = \frac{n\theta^2}{(n+2)(2n+1)^2} \xrightarrow[n\to\infty]{} 0$$

=> сост по дост. усл.

c)

$$D[\tilde{\theta}_1] = \frac{1}{27n}\theta^2, \quad D[\tilde{\theta}_2'] = \frac{n}{(n+2)(2n+1)^2}\theta^2$$

$$\frac{1}{27n}\theta^2 \geq \frac{n}{(n+2)(2n+1)^2}\theta^2 \quad \text{при } n \geq 3$$

=> $\tilde{\theta}_2'$ эффективнее

d) Точный ДИ

$\overline{X_n}$ – выборка $\quad \xi \sim R[\theta, 2\theta]$

$$f(\overline{X_n}, \theta) = \frac{X_{max}}{\theta} - 1$$

$$P(f < t) = P(X_{max} < \theta t + \theta) = (F(\theta t + \theta))^n$$

$= \left\{ F(x) = \begin{cases} 0, & x \le \theta \\ \frac{x}{\theta} - 1, & \theta \le x \le 2\theta \\ 1, & x > 2\theta \end{cases} \right\} = \begin{cases} 0, & t \le 0 \\ t^n, & 0 < t \le 1 \\ 1, & t \ge 1 \end{cases}$

$\lambda + \beta = 1$

$$t_1 = q_{\alpha/2} = \sqrt[n]{\frac{\alpha}{2}} = \sqrt[n]{\frac{1-\beta}{2}}$$

$$t_2 = q_{1-\frac{\alpha}{2}} = \sqrt[n]{1 - \frac{\alpha}{2}} = \sqrt[n]{\frac{1+\beta}{2}}$$

$$P\left(t_1 < \frac{x_{max}}{\theta} - 1 < t_2\right) = \beta$$

$$t_1 + 1 < \frac{x_{max}}{\theta} < t_2 + 1$$

$$\frac{1}{1+t_2} < \frac{\theta}{x_{max}} < \frac{1}{1+t_1}$$

$$\frac{x_{max}}{1 + \sqrt[n]{\frac{1+\beta}{2}}} < \theta < \frac{x_{max}}{1 + \sqrt[n]{\frac{1-\beta}{2}}}$$

е) Асимпт. доверительный интервал

ОММ: $\tilde{\theta}_1 = \frac{2}{3} \bar{x} = \frac{2}{3} \tilde{\alpha}_1 = g(\tilde{\alpha}_1)$

$g(\alpha_1) = \frac{2}{3} \alpha_1 = \theta$, $Dg = \frac{2}{3}$

$K_{11} = \alpha_2 - \alpha_1^2$

$$\widetilde{\mu_2} = \widetilde{\alpha_2} - \widetilde{\alpha_1}^2 = \frac{S^2(n-1)}{n}$$

$$\frac{\sqrt{n}\,(\widetilde{\theta_1} - \theta)}{\frac{2}{3}\cdot S\cdot\sqrt{\frac{n-1}{n}}} = \frac{3n(\widetilde{\theta_1} - \theta)}{2S\sqrt{n-1}} \sim N(0,1)$$

$$P\left(t_1 < \frac{3n(\widetilde{\theta_1} - \theta)}{2S\sqrt{n-1}} < t_2\right) = \beta$$

$$\frac{2St_1\sqrt{n-1}}{3n} < \widetilde{\theta_1} - \theta < \frac{2St_2\sqrt{n-1}}{3n}$$

$$P\left(\widetilde{\theta_1} - \frac{2St_2\sqrt{n-1}}{3n} < \theta < \widetilde{\theta_1} - \frac{2St_1\sqrt{n-1}}{3n}\right) = \beta$$