T6

$$p(x) = \begin{cases} \frac{\theta-1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases} \qquad \theta > 1$$

$\overline{x_n}$ – выборка

a) ОМП

$$L(\theta) = \prod p(x_i, \theta) = \cancel{\left(\frac{\theta-1}{x^{\theta}}\right)^n} \cdot \frac{(\theta-1)^n}{\prod_{i=1}^{n} x_i^{\theta}} \cdot \{x_{\ldots} \geq 1\}$$

$$\ln L = n \ln(\theta-1) - \theta \left(\sum_{i=1}^{n} \ln x_i\right)$$

$$(\ln L)'_{\theta} = \frac{n}{\theta-1} - \sum \ln x_i = 0$$

$$\Rightarrow \theta = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$(\ln L)''_{\theta} = \frac{-n}{(\theta-1)^2} < 0 \quad \text{в т. экстр}$$

$\Rightarrow$ max.

$$\hat{\theta} = 1 + \frac{1}{\overline{\ln x}}$$

b) Проверим модель на сильную регулярность

1) p – непр дифф

2) $\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta} p\right)dx = 0$?

$$p'_\theta = \frac{\ln x - \theta\ln x + 1}{x^\theta}$$

$$\int_1^{\infty} x^{-\theta}(\ln x(1-\theta)+1)\,dx = \frac{1}{\theta-1} - \int_1^{\infty} x^{-\theta}dx =$$

$$= \frac{1}{\theta-1} - \frac{1}{\theta-1} = 0$$

$$p''_\theta = \ln x \left(\frac{\theta\ln x - \ln x - 2}{x^\theta}\right)$$ — непр. дифф

$$\int_1^{+\infty} p''\,dx = \int_1^{+\infty}\left((\theta-1)x^{-\theta}\ln^2 x - 2x^{-\theta}\ln x\right)dx =$$

$$=0$$

3) $$I(\theta) = \int_1^{+\infty}\left(\frac{1}{(\theta-1)^2} + (-2)\frac{\ln x}{\theta-1} + \ln^2 x\right)\frac{\theta-1}{x^\theta}dx$$

$$= \int_1^{+\infty}\left(\frac{x^{-\theta}}{\theta-1} - \frac{2\ln x}{x^\theta} + (\theta-1)\left(\frac{\ln^2 x}{x^\theta}\right)\right)dx =$$

$$= \frac{1}{(\theta-1)^2} - 2\left(\ln x\,\frac{x^{1-\theta}}{1-\theta}\Big|_1^{+\infty} - \int_1^{+\infty}\frac{x^{-\theta}}{1-\theta}dx\right) +$$

$$+ \int_1^{+\infty}\frac{(\theta-1)\ln^2 x}{x^\theta}dx = \frac{-1}{(\theta-1)^2} - \left(\ln^2 x\cdot x^{1-\theta}\right|_1^{+\infty}$$

$$= 2\int_1^{+\infty} \frac{\ln x}{x} x^{1-\theta} = \frac{-1}{(1-\theta)^2} \cdot 2\left(\frac{x^{1-\theta}\ln x}{1-\theta}\Big|_1^{+\infty} - \int_1^{+\infty}\frac{x^{-\theta}}{1-\theta}\,dx\right) = \frac{1}{(1-\theta)^2}$$ — непр., > 0 на $(1, +\infty)$

$\Rightarrow$ сильно регулярна

$\Rightarrow$ ОМП (сост., асимп. несмещ., асимп. эффективна и асимп. норм.)

медиана: $F(x_{\frac12}) = \frac12$

$$\int_1^{x_{\frac12}} \frac{\theta-1}{x^\theta}\,dx = -\frac{1}{x_{\frac12}^{\theta-1}} + 1 = \frac12$$

$$\Rightarrow x_{\frac12} = 2^{\frac{1}{\theta-1}}$$

$$g(\hat\theta) = \sqrt[\theta-1]{2}$$

$$\nabla g = 2^{\frac{1}{\theta-1}} \cdot \ln(2) \cdot \frac{-1}{(\theta-1)^2}$$

$$\sqrt{n}\,\frac{g(\hat\theta) - g(\theta)}{\sqrt{\nabla^T g(\hat\theta)\, I^{-1}(\hat\theta)\, \nabla g(\hat\theta)}} \rightsquigarrow N(0,1)$$

$$\sqrt{n}\cdot\frac{(g(\hat\theta) - g(\theta))\cdot(\hat\theta - 1)}{2^{\frac{1}{\hat\theta-1}}\cdot\ln 2} \rightsquigarrow N(0,1)$$

$$P\left(\tilde{g}(\tilde{\theta}) - \frac{1{,}96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}\,(\tilde{\theta}-1)} < x_{\frac{1}{2}} < \tilde{g}(\tilde{\theta}) + \frac{1{,}96 \cdot \ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}}{\sqrt{n}\,(\tilde{\theta}-1)}\right) \approx \beta$$

c) Для θ

ОМП:

$$\frac{\sqrt{n}(\tilde{\theta}-\theta)}{\tilde{\theta}-1} \rightsquigarrow N(0;1)$$

$$P\left(1 - \frac{1{,}96 - \sqrt{n}}{\sqrt{n} \cdot \overline{\ln x}} < \theta < 1 + \frac{1{,}96 - \sqrt{n}}{\sqrt{n}\,\overline{\ln x}}\right) \approx \beta$$