θ $n \to \infty$

Задача №1 (T1)

$\xi \sim R(0, \theta)$ $\theta > 0$

$\overline{X_n}$ – выборка объёма n

$$\tilde{\theta} = 2\overline{x} = 2\frac{1}{n}\sum_{i=1}^{n} x_i$$

$\tilde{\Theta}_2 = \min x_i$

$\tilde{\Theta}_3 = \max x_i$

$\tilde{\Theta}_4 = x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i$

Решение:

$M[\xi] = \int_{-\infty}^{\infty} x p(x) dx = \int_0^{\Theta} \frac{x}{\Theta} dx = \frac{\Theta}{2}$

$M \xi^2 = \int_0^{\Theta} \frac{x^2}{\Theta} dx = \frac{\Theta^2}{3}$

$D[\xi] = \frac{\Theta^2}{3} - \frac{\Theta^2}{4} = \frac{\Theta^2}{12}$

а)

1) $\tilde{\Theta}_1 = 2\overline{X}$

$\forall \Theta > 0 \quad M[\tilde{\Theta}_1] = \Theta$

$M[\frac{2}{n} \sum_1^n x_i] = \frac{2}{n} \sum_{i=1}^{n} M[x_i] = 2M[\xi] = \Theta$

несмещённая

$D[\tilde{\Theta}_1] = D[\frac{2}{n} \sum x_i] = \frac{4}{n^2} \sum D x_i =$

$= \frac{4}{n} D\xi = \frac{\Theta^2}{3n} \xrightarrow[n \to \infty]{} 0$

по дост. усл-ю оценка состоятельная

2) $\tilde{\theta}_2 = \min(x_i)$

$M[\tilde{\theta}_2] = \int_{-\infty}^{\infty} y\, \varphi(y)\, dy$

$\Phi(y) = 1-(1-F(y))^n$

$\varphi(y) = \Phi' = n\,(\underbrace{1-F(y)}_{\frac{y}{\theta}})^{n-1} \cdot \underbrace{p(y)}_{\frac{1}{\theta}\{(0,\theta)\}}$

$M[\tilde{\theta}_2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\, y\, dy =$

$t = 1-\frac{y}{\theta}$

$= -\int_1^0 n t^{n-1}(1-t)\,\theta\, dt = \int_0^1 n\theta t^{n-1}\, dt - \int n\theta t^n\, dt$

$= \theta\left[1-\frac{n}{n+1}\right] = \frac{\theta}{n+1}$ смещённая

$\tilde{\theta}_2' = (n+1)\, x_{min}$ — несмещ.

$M[\tilde{\theta}_2'] = \theta$

$M[\tilde{\theta}_2^2] = \int_0^{\theta} n\left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\, y^2\, dy =$

$= -\int_1^0 n t^{n-1}\, \theta^2 (1-t)^2\, dt = \ldots = \theta^2 \cdot \frac{2}{(n+1)(n+2)}$

$D[\tilde{\theta}_2] = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \theta^2\left[\frac{2(n+1)-(n+2)}{(n+1)^2(n+2)}\right]$

$= \theta^2\left\{\frac{n}{(n+1)^2(n+2)}\right\} \to 0$

$D[\tilde{\theta}_2] = \frac{\theta^2 n}{n+2}$

$\tilde{\theta}_2$ по бер

$\forall \theta > 0 \quad \forall \varepsilon > 0 \qquad P(|\tilde{\theta}_2 - \theta| \geq \varepsilon) \longrightarrow 0$

$P(| \; | \geq \varepsilon) \geq P(\tilde{\theta}_2 \geq \theta + \varepsilon) = P((n+1)x_{min} \geq \theta + \varepsilon)$

$= P(x_{min} \geq \frac{\theta+\varepsilon}{n+1}) = 1 - \varphi(\frac{\theta+\varepsilon}{n+1}) =$

$= (1 - (\frac{\theta+\varepsilon}{\theta(n+1)}))^n \xrightarrow[n \to \infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$

$\tilde{\theta}_2$

$P(\tilde{\theta}_2 < \theta - \varepsilon) + \underbrace{P(\tilde{\theta}_2 > \theta + \varepsilon)}_{\to 0}$

$P(x_{min} < \theta - \varepsilon) = \varphi(\theta - \varepsilon) = 1 - (1 - \frac{\theta - \varepsilon}{\theta})^n =$

$\varepsilon < \theta$

$= 1 - (\frac{\varepsilon}{\theta})^n \longrightarrow 1$

3) $\tilde{\theta}_3 = x_{max}$

$M\tilde{\theta}_3 = \int_{-\infty}^{\infty} z \psi(z) dz$

$\psi(z) = (F(z))^n$

$$\psi(z) = \Psi'(z) = n(F(z))^{n-1} \cdot p(z) =$$

$\frac{1}{\theta}\{(0,\theta)$

$$= n\left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta} \{(0;\theta)\}$$

$$M\{\bar{\theta}\} = \int_0^{\theta} n\frac{z^n}{\theta^n}\,dz = \frac{n}{n+1}\theta \quad \text{смещ.}$$

$$\tilde{\theta}_3' = \frac{n+1}{n} x_{max} \quad \text{несмещ.}$$

$$D[\bar{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2 \cdot \frac{n}{(n+2)(n+1)^2}$$

$$D[\tilde{\theta}_3'] = \frac{(n+1)^2}{n^2} D\bar{\theta}_3 = \frac{\theta^2}{n(n+2)} \to 0$$

$\tilde{\theta}_3$ сост.

θ_3 по опр

$$P(|\theta_3 - \theta| \geq \varepsilon) = P(x_{max} < \theta - \varepsilon) +$$

$$+ \underbrace{P(x_{max} > \theta + \varepsilon)}_{0} = (F(\theta-\varepsilon))^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \to$$

$$0 < \varepsilon < \theta$$

$$\xrightarrow[n\to\infty]{} 0 \quad \varepsilon \geq 0 \quad (0)^n \to 0$$

сост.

4) $\tilde{\Theta}_4 = x_1 + \frac{1}{n-1} \sum_{i=2}^{n} x_i$

$M[\tilde{\Theta}_4] = M[x_1 + \frac{1}{n-1} \sum x_i] = M[x_1] + \frac{1}{n-1} \cdot \sum_{2}^{n} M x_i$

$= \frac{\Theta}{2} + \frac{\Theta}{2} = \Theta$

не смещ.

$D[\tilde{\Theta}_4] = D[x_1 + \frac{1}{n-1} \sum x_i] = D\xi + \frac{1}{(n-1)^2} (n-1) D\xi$

$= \{ D\xi = \frac{\Theta^2}{12} \} = \frac{\Theta^2}{12} + \frac{\Theta^2}{12(n-1)} = \frac{\Theta^2}{12} \frac{n}{n-1} \not\to 0$

Дост. усл. не работает

По опр.

$\tilde{\Theta}_4 \xrightarrow{P} \Theta$

$x_1 \xrightarrow{P} x_1$

$\frac{1}{(n-1)} \sum_{i=2}^{n} x_i \longrightarrow M[\xi] = \frac{\Theta}{2}$

$\tilde{\Theta}_4 \xrightarrow{P} x_1 + \frac{\Theta}{2} \neq \Theta$

не явл. сост.

д)

$\tilde{\theta}_1 = 2\bar{x}$

$\tilde{\theta}_3' = \frac{n+1}{n} x_{\cancel{min}\,max}$

$D\{\tilde{\theta}_1\} = \frac{\theta^2}{3n}$

$D\{\tilde{\theta}_3'\} = \frac{\theta^2}{n(n+2)}$

$\forall \theta > 0 \quad \frac{\theta^2}{n(n+2)} < \frac{\theta^2}{3n}$

$3n < n^2 + 2n$

$n^2 > n$

$n > 1$

П1.

„Подарок"

$\hat{Q}_3' = \frac{n+1}{n} x_{max}$

не сост

$P(|\hat{Q}_3' - Q| \geq \varepsilon) = P(x_{max} \cdot \frac{n+1}{n} \leq \theta - \varepsilon) +$
$+ P(\frac{n+1}{n} x_{max} \geq \theta + \varepsilon) = P(x_{max} \leq \frac{n(\theta-\varepsilon)}{n+1}) +$
$+ 1 - P(x_{max} \leq \frac{n(\theta+\varepsilon)}{n+1}) = F^n(\frac{n(\theta-\varepsilon)}{n+1}) +$
$+ 1 - F^n(\frac{n(\theta+\varepsilon)}{n+1}) = \varphi_1 + 1 - \varphi_2$

$F = \begin{cases} 0, & x \leq 0 \\ \frac{x}{\theta}, & x > 0 \end{cases}$

φ_1:

при $\varepsilon \geq \theta$; $0^n \to 0$, $n \to \infty$

при $\varepsilon < \theta$: $\left(\frac{n(\theta-\varepsilon)}{\theta(n+1)}\right)^n = \left(\frac{\theta-\varepsilon}{\theta(1+\frac{1}{n})}\right)^n \to \left(\underbrace{\frac{\theta-\varepsilon}{\theta}}_{<1}\right)^n \to$

$\to 0$, $n \to \infty$

φ_2:

при $\frac{n(\theta+\varepsilon)}{\theta(n+1)} > \theta$: $1^n \to 1$, $n \to \infty$

при $\frac{n(\Theta+\varepsilon)}{n+1} \le \Theta$:

$n(\Theta+\varepsilon) \le n\Theta + \Theta$

$n \le \frac{\Theta}{\varepsilon} = N_\varepsilon$

Тогда $\forall \varepsilon \ \exists N_\varepsilon = \frac{\Theta}{\varepsilon}: \forall n \geq N_\varepsilon \hookrightarrow$

$\frac{n(\Theta+\varepsilon)}{n+1} > \Theta \ \wedge \ \varphi_2 \to 1$ при $n \to \infty$

Получаем:

$P(|\hat{\Theta}_3' - \Theta| \geq \varepsilon) = \varphi_1 + 1 - \varphi_2 \to 0 + 1 - 1 = 0$

$\Rightarrow \hat{\Theta}_3'$ сост.