№3

$$p(x) = \begin{cases} \frac{e^{-x/\theta}}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases} \qquad \theta > 0$$

$n=3$

$\widehat{\theta}_1 = \overline{x}$, $\widehat{\theta}_2 = x_{(2)}$

$$M[\xi] = \int_{-\infty}^{\infty} e^{-\frac{x}{\theta}} \frac{x}{\theta} dx = \left\{ \begin{matrix} \frac{x}{\theta} = t \\ dt = \frac{dx}{\theta} \end{matrix} \right\} = \int_0^{+\infty} e^{-t} t \theta \cdot dt =$$

$$= \theta \cdot \left(-e^{-t} t \Big|_0^{\infty} + \underbrace{\int_0^{\infty} e^{-t} dt}_{1} \right) = \theta$$

$$M[\xi^2] = \int_{-\infty}^{\infty} e^{-\frac{x}{\theta}} \frac{x^2}{\theta} dx = \left\{ \begin{matrix} \frac{x}{\theta} = t \\ dt = \frac{dx}{\theta} \end{matrix} \right\} = \theta^2 \int_0^{\infty} e^{-t} t^2 dt =$$

$$= \theta^2 \left(-e^{-t} t^2 \Big|_0^{\infty} + 2 \underbrace{\int_0^{\infty} e^{-t} t\, dt}_{1} \right) = 2\theta^2$$

$$D[\xi] = M[\xi^2] - M^2[\xi] = \theta^2$$

a)

$\widetilde{\theta}_1 = \overline{x}$

$$M[\bar{x}] = \frac{1}{n} \cdot n M[\xi] = \theta$$

несмещ.

$$\tilde{\theta}_2 = x_{(2)}$$

$$\varphi(x) = n \cdot p(x) \cdot C_{n-1}^{k-1} \, (1 - F(x))^{n-k} \cdot (F(x))^{k-1}$$

$k=2$, $n=3$, $F(x) = \int_0^x p(t)\,dt = 1 - e^{-\frac{x}{\theta}}$

$$\varphi = 3 \cdot \left(\frac{e^{-\frac{x}{\theta}}}{\theta}\right) \cdot C_2^1 \cdot \left(e^{-\frac{x}{\theta}}\right)^1 \cdot \left(1 - e^{-\frac{x}{\theta}}\right) =$$

$$= \frac{6}{\theta} \cdot e^{-\frac{2x}{\theta}} \cdot \left(1 - e^{-\frac{x}{\theta}}\right) = \frac{6}{\theta} \left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right)$$

$$M[x_{(2)}] \int_{-\infty}^{\infty} \varphi(x) \cdot x \, dx = \frac{6}{\theta} \int_0^{\infty} x \cdot \left(e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}\right) dx =$$

$$= \left\{ \begin{matrix} \frac{x}{\theta} = t \\ dt = \frac{dx}{\theta} \end{matrix} \right\} = 6\theta \int_0^{\infty} t \cdot \left(e^{-2t} - e^{-3t}\right) dt =$$

$$= 6\theta \cdot \left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{6} \theta$$

смещ..

$$\tilde{\theta}_2' = \frac{6}{5} x_{(2)}$$

b) $D[\tilde{\theta}_1] = D[\bar{x}] = \frac{1}{4^2} \cdot 4 \cdot D[\xi] = \frac{\theta^2}{4} =$

$= \frac{\theta^2}{3}$

для $\tilde{\theta}_2'$:

$M[\tilde{\theta}_2'^2] = \frac{36}{25} \cdot 6 \cdot \int_0^\infty \frac{x^2}{\theta} \cdot (e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}) dx =$

$= \left\{ \begin{array}{l} \frac{x}{\theta} = t \\ dt = \frac{dx}{\theta} \end{array} \right\} = \frac{216}{25} \cdot \theta^2 \cdot \int_0^\infty t^2 \cdot (e^{-2t} - e^{-3t}) dt =$

$= \frac{216}{25} \theta^2 \cdot \left(\frac{1}{4} - \frac{2}{27} \right) = \frac{38}{25} \theta^2$

$D[\tilde{\theta}_2'] = M[\tilde{\theta}_2'^2] - M^2[\tilde{\theta}_2'] = \frac{38}{25}\theta^2 - \theta^2 =$

$= \frac{13}{25} \theta^2$

$\frac{\theta^2}{3} < \frac{13}{25} \theta^2 \Rightarrow \tilde{\theta}_1$ более эффективна

c)

регулярность $\rho(x)$:

1) $\rho(x)$ непр дифф по θ при $\theta > 0$

2) $\frac{\partial}{\partial\theta}\int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta}\,dx = 0$

$\int_0^\infty \left(\frac{x}{\theta^3}e^{-\frac{x}{\theta}} - \frac{1}{\theta^2}e^{-\frac{x}{\theta}}\right)dx = \frac{1}{\theta} - \frac{1}{\theta} = 0$

равенство есть

3) $I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \left\{\ln p = -\frac{x}{\theta} - \ln\theta\right\}$

$= M\left[\left(\frac{x}{\theta^2} - \frac{1}{\theta}\right)^2\right] = \int_0^\infty \left(\frac{x^2}{\theta^4} - \frac{2x}{\theta^3} + \frac{1}{\theta^2}\right)\cdot\frac{1}{\theta}\cdot$

$\cdot e^{-\frac{x}{\theta}}\,dx = \cancel{\frac{1}{\theta}}\left\{\begin{array}{l}\frac{x}{\theta} = t \\ dt = \frac{dx}{\theta}\end{array}\right\} = \frac{1}{\theta^2}\left(\int_0^\infty t^2 e^{-t}\,dt - \right.$

$\left. -2\int_0^\infty t e^{-t}\,dt + \int_0^\infty e^{-t}\,dt\right) = \frac{1}{\theta^2}$

непр и >0 при $\theta > 0$

$\Rightarrow \rho(x)$ регулярна

регулярность оценок:

по дост. усл.

$\tilde{\theta}_1$: модель регулярна несмещ.

$D\{\tilde{\theta}_1\} = \frac{\theta^2}{3}$ – огр. на любом компакте из $(0; +\infty)$ по θ.

=> ~~$\tilde{\theta}_1$~~ $\tilde{\theta}_1$ – регулярна

$\tilde{\theta}_2$: ~~рег~~ модель регулярна несмещ.

$D\{\tilde{\theta}_2\} = \frac{13}{25}\theta^2$ – огр. на $\forall$ компакте из $(0; +\infty)$ по θ

=> регулярна

$$D\{\tilde{g}\} \geq \frac{g'^2(\theta)}{n I(\theta)}$$

для $\tilde{\theta}_1$:

$$\frac{\theta^2}{3} \geq \frac{1}{3 \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{3}$$

для $\tilde{\theta}_2$:

$$\frac{13}{25}\theta^2 \geq \frac{\theta^2}{3}$$

по дост. усл. эффект:

$\tilde{\theta}_1$ – эффективная

$\tilde{\theta}_2$ – не эффект.