In the Appendix useful facts about Wronskians are collected.

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2. Master functions and critical points

2.1. **Kac-Moody algebras.** Let $A = (a_{ij})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{ii} = 2$, $a_{ij} = 0$ if and only $a_{ji} = 0$, $a_{ij} \in \mathbb{Z}_{\le 0}$ if $i \ne j$. We assume that A is symmetrizable, there is a diagonal matrix $D = \mathrm{diag}\{d_1, \dots, d_r\}$ with positive integers d_i such that $B = DA$ is symmetric.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see [K], §1.2), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where d is the dimension of the kernel of the Cartan matrix A.

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \dots, r$, be the sets of simple roots, coroots, respectively. We have

$$\begin{aligned} (\alpha_i, \alpha_j) &= d_i \, a_{ij}, \\ \langle \lambda, \alpha_i^\vee \rangle &= 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i), \qquad \lambda \in \mathfrak{h}^*. \end{aligned}$$

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}\}$. A weight $\Lambda \in \mathfrak{h}^*$ is dominant integral if $\langle \Lambda, \alpha_i^\vee \rangle$ are non-negative integers for all i.

Fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \dots, r$. We have $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$.

The Weyl group $\mathcal{W} \in \mathrm{End}(\mathfrak{h}^*)$ is generated by reflections s_i, $i = 1, \dots, r$,

$$s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \qquad \lambda \in \mathfrak{h}^*.$$

We use the notation

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \qquad w \in \mathcal{W}, \ \lambda \in \mathfrak{h}^*,$$

for the shifted action of the Weyl group.

The Kac-Moody algebra $\mathfrak{g}^t = g(A^t)$ corresponding to the transposed Cartan matrix A^t is called Langlands dual to $\mathfrak{g}$.

2.2. **The definition of master functions and critical points.** Let $\boldsymbol{\Lambda} = (\Lambda_i)_{i=1}^n$, $\Lambda_i \in \mathcal{P}$; $\boldsymbol{z} = (z_i)_{i=1}^n \in \mathbb{C}^n$, $\boldsymbol{l} = (l_1, \dots, l_r) \in \mathbb{Z}_{\ge 0}^r$, $\boldsymbol{t} = (t_j^{(i)}, j = 1, \dots, l_i)_{i=1}^r$. We call Λ_i the *weight* at a point z_i; $t_j^{(i)}$ a *variable of color* i. Define

$$\Lambda_\infty \;=\; \sum_{i=1}^{n} \Lambda_i \;-\; \sum_{i=1}^{r} l_i\alpha_i \;\in\; \mathcal{P}$$

and $\bar{\mathbf{\Lambda}} = (\Lambda_1, \dots, \Lambda_n, \Lambda_\infty)$.

The *master function* $\Phi_{\mathfrak{g}(A)}(\boldsymbol{t}; \boldsymbol{z}; \bar{\mathbf{\Lambda}})$ is defined by

$$\Phi_{\mathfrak{g}(A)}(\boldsymbol{t}; \boldsymbol{z}; \bar{\mathbf{\Lambda}}) = \tag{2.1}$$
$$\prod_{i=1}^{r}\prod_{j=1}^{l_i}\prod_{s=1}^{n}(t_j^{(i)} - z_s)^{-(\Lambda_s,\alpha_i)} \prod_{i=1}^{r} \prod_{1\le j<s\le l_i} (t_j^{(i)} - t_s^{(i)})^{(\alpha_i,\alpha_i)} \prod_{1\le i<j\le r} \prod_{s=1}^{l_i}\prod_{k=1}^{l_j}(t_s^{(i)} - t_k^{(j)})^{(\alpha_i,\alpha_j)} ,$$

see [SV] . The function Φ is a rational function of variables $\boldsymbol{t}$ depending on parameters $\boldsymbol{z}$, $\mathbf{\Lambda}$. It is symmetric with respect to permutations of variables of the same color.

A point $\boldsymbol{t}$ with complex coordinates is called a *critical point* of the master function Φ if the following system of algebraic equations is satisfied

$$\sum_{s=1}^{n} \frac{-(\Lambda_s, \alpha_i)}{t_j^{(i)} - z_s} \;+\; \sum_{s,\ s\neq i}\sum_{k=1}^{l_s} \frac{(\alpha_s, \alpha_i)}{t_j^{(i)} - t_k^{(s)}} \;+\; \sum_{s,\ s\neq j} \frac{(\alpha_i, \alpha_i)}{t_j^{(i)} - t_s^{(i)}} \;=\; 0 \tag{2.2}$$

where $i = 1, \dots, r$, $j = 1, \dots, l_i$. In other words, a point $\boldsymbol{t}$ is a critical point if

$$\left(\Phi^{-1}\frac{\partial\Phi}{\partial t_j^{(i)}}\right)(\boldsymbol{t}) = 0, \qquad i = 1, \dots, r,\ j = 1, \dots l_i.$$

The product of symmetric groups $S_{\boldsymbol{l}} = S_{l_1} \times \cdots \times S_{l_r}$ acts on the critical set of the master function permuting the coordinates of the same color. All orbits have the same cardinality $l_1! \cdots l_r!$.

In the Bethe Ansatz method applied to the Gaudin models [G] the system of equations (2.2) is called the Bethe Ansatz equations. On relations of critical points to the Bethe Ansatz method of the Gaudin models see [RV]. See also [Fa, Sk1, Sk2, FFR, Fr, MV].

2.3. **The case of isolated critical points.** In this section we give a sufficient condition for the set of critical points to be finite and state a conjecture about its cardinality.

We say that the set of weights $\bar{\mathbf{\Lambda}}$ is *separating* if

$$(2\Lambda_\infty \;+\; 2\rho \;+\; \sum_{i=1}^{r} c_i\alpha_i,\ \sum_{i=1}^{r} c_i\alpha_i) \;\neq 0,$$

for all sets of integers $(c_i)_{i=1}^r$ such that $0 \le c_i \le l_i$, $\sum_i c_i \neq 0$.

For example, if the scalar product is non-negative on the root lattice and Λ_∞ is dominant integral, then $\bar{\mathbf{\Lambda}}$ is separating.

The following lemma is a generalization of Theorem 6 in [ScV].

Lemma 2.1. *If $\bar{\mathbf{\Lambda}}$ is separating, then the set of critical points is finite.*

Proof. (Cf. proof of Theorem 6 in [ScV].) If the algebraic set of critical points is infinite, then it is unbounded. Suppose we have a sequence of critical points which is unbounded. Without loss of generality, we assume that $t_j^{(i)}$ tends to infinity for $i = 1, \dots, r,\ j = 1, \dots, c_i$, and remains bounded for all other values of i, j.

Take the equation (2.2) corresponding to a variable $t_j^{(i)}$ and multiply it by $t_j^{(i)}$. Then add the resulting equations corresponding to $i = 1, \dots, r$, $j = 1, \dots, c_i$, and pass to the limit along our sequence of critical points. Then the resulting equation is

$$(2\Lambda_\infty \ + \ 2\rho \ + \ \sum_{i=1}^{r} c_i\alpha_i,\ \sum_{i=1}^{r} c_i\alpha_i) \ = \ 0\ .$$

This equation contradicts to our assumption. □

Conjecture 2.2. *If all components of $\bar{\mathbf{\Lambda}}$ are dominant integral weights, then for generic $z_1, \dots, z_n$ the number of $S_{\boldsymbol{l}}$-orbits of critical points of the master function $\Phi(\boldsymbol{t}; \boldsymbol{z}; \bar{\mathbf{\Lambda}})$ is equal to the multiplicity of the irreducible $\mathfrak{g}(A)$-module with highest weight Λ_∞ in the tensor product of irreducible $\mathfrak{g}(A)$-modules with highest weights Λ_i, $i = 1, \dots, n$. Moreover, all critical points are non-degenerate.*

Conjecture 2.2 is proved in [ScV] for $\mathfrak{g} = sl_2$. In this paper we prove for $\mathfrak{g} = sl_n$ that the number of $S_{\boldsymbol{l}}$-orbits of critical points is not greater than the above multiplicity, thus relating the number of critical orbits and multiplicities of irreducible representations in tensor products.

Conjecture 2.2 is related to the conjecture on completeness of the Bethe Ansatz for Gaudin models, see [RV]. In a Gaudin model to every orbit of critical points one assigns an eigenvector in the space of states of a family of commuting linear operators called Hamiltonians. The Bethe Ansatz conjecture predicts that the constructed eigenvectors span a basis in the space of states. The dimension of the space of states is equal to the above multiplicity. Therefore, if the Bethe Ansatz conjecture were true, then the number of orbits of critical points would be not less than the above multiplicity. The Bethe Ansatz conjecture is proved in [RV] for $\mathfrak{g} = sl_2$.

2.4. **On limits of critical points.** In this section we formulate auxiliary results which we use later.

It is useful to consider functions more general than master functions,

$$\Phi(\boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\mu}; \boldsymbol{\nu}) \ = \ \prod_{s=1}^{n}\prod_{i=1}^{l}(t_i - z_s)^{\mu_{i,s}} \prod_{1\le i<j\le l} (t_i - t_j)^{\nu_{i,j}}\ .$$

For $j > i$, set $\nu_{j,i} = \nu_{i,j}$. Say that $\boldsymbol{t} = (t_1, \dots, t_l)$ is a critical point of Φ, if

$$\sum_{s=1}^{n} \frac{\mu_{i,s}}{t_i - z_s} \ + \ \sum_{j,\ j\ne i} \frac{\nu_{i,j}}{t_i - t_j} \ = \ 0$$

for $i = 1, \dots, l$.

For any subset $I \subset \{1, \dots, l\}$, denote $\bar{I} = \{1, \dots, l\} - I$ its complement. Say that the pair $\boldsymbol{\mu}, \boldsymbol{\nu}$ is *separating*, if

$$\sum_{i \in I} \sum_{j \in \bar{I}} \nu_{i,j} \;+\; \sum_{i,j \in I,\ i<j} \nu_{i,j} \;+\; \sum_{i \in I} \sum_{s=1}^{n} \mu_{i,s} \;\neq\; 0$$

for any non-empty subset $I \subset \{1, \dots, l\}$.

Lemma 2.3. *If the pair $\boldsymbol{\mu}, \boldsymbol{\nu}$ is separating, then the set of critical points of Φ is finite.* $\square$

Assume that $\boldsymbol{t}_k = (t_{1,k}, \dots, t_{l,k})$, $k = 1, 2, \dots$, is a sequence of critical points of Φ. Assume that for every i, the sequence $\{t_{i,k}\}$ has a limit, finite or infinite, as k tends to infinity. We show that, if we ignore the coordinates whose limit belong to $\{z_1, \dots, z_n, \infty\}$, then the limits of the remaining coordinates form a critical point of a suitable function $\tilde{\Phi}$ defined below.

For $w \in \mathbb{C} \cup \infty$, denote $I(w)$ the subset of $\{1, \dots, l\}$ consisting of all i such that $\lim_{k\to\infty} t_{i,k} = w$. Let $t_{1,\infty}, \dots t_{\tilde{l},\infty}$ be all pairwise distinct limiting points lying in $\mathbb{C} - \{z_1, \dots, z_n\}$.

Define numbers $\tilde{\mu}_{\tilde{\imath},s}$, $\tilde{\imath} = 1, \dots \tilde{l}$, $s = 1, \dots, n$, by

$$\tilde{\mu}_{\tilde{\imath},s} \;=\; \sum_{i \in I(w_{\tilde{\imath},\infty})} \mu_{i,s} \;+\; \sum_{i \in I(w_{\tilde{\imath},\infty})} \sum_{j \in I(z_s)} \nu_{i,j} \;.$$

Define numbers $\tilde{\nu}_{\tilde{\imath},\tilde{\jmath}}$, $\tilde{\imath}, \tilde{\jmath} \in \{1, \dots, \tilde{l}\}, \tilde{\imath} \neq \tilde{\jmath}$, by

$$\tilde{\nu}_{\tilde{\imath},\tilde{\jmath}} \;=\; \sum_{i \in I(w_{\tilde{\imath},\infty})} \sum_{j \in I(w_{\tilde{\jmath},\infty})} \nu_{i,j} \;.$$

Define a function

$$\tilde{\Phi}(\tilde{\boldsymbol{t}}; \boldsymbol{z}; \tilde{\boldsymbol{\mu}}; \tilde{\boldsymbol{\nu}}) \;=\; \prod_{s=1}^{n} \prod_{\tilde{\imath}=1}^{\tilde{l}} (\tilde{t}_{\tilde{\imath}} - z_s)^{\tilde{\mu}_{\tilde{\imath},s}} \prod_{1 \le \tilde{\imath} < \tilde{\jmath} \le \tilde{l}} (\tilde{t}_{\tilde{\imath}} - \tilde{t}_{\tilde{\jmath}})^{\tilde{\nu}_{\tilde{\imath},\tilde{\jmath}}} \;.$$

Lemma 2.4. *The point $(t_{1,\infty}, \dots, t_{\tilde{l},\infty})$ is a critical point of the function $\tilde{\Phi}$.* $\square$

Lemma 2.5. *Assume that the pair $\boldsymbol{\mu}^0 = \{\mu_{i,s},\ i \in \{1, \dots, l\} - I(\infty), s = 1, \dots, n\}$ and $\boldsymbol{\nu}^0 = \{\nu_{i,j},\ i, j \in \{1, \dots, l\} - I(\infty), i \neq j\}$ is separating. Then the pair $\tilde{\boldsymbol{\mu}} = \{\tilde{\mu}_{\tilde{\imath},s},\ \tilde{\imath} = 1, \dots, \tilde{l}, s = 1, \dots, n\}$ and $\tilde{\boldsymbol{\nu}} = \{\tilde{\nu}_{\tilde{\imath},\tilde{\jmath}},\ \tilde{\imath}, \tilde{\jmath} \in \{1, \dots, \tilde{l}\}, \tilde{\imath} \neq \tilde{\jmath}\}$ is separating.* $\square$

3. Populations of critical points

3.1. **Remarks on Fuchsian equations.** Consider a differential equation for a function $u(x)$

$$u^{(k)} \;+\; p_1 u^{(n-1)} \;+\; \dots \;+\; p_k u \;=\; 0 \;, \tag{3.1}$$

where $p_i = p_i(x)$ are rational functions. A point $z \in \mathbb{C}$ is called an *ordinary point* if all $p_i(x)$ are holomorphic at z. A non-ordinary point is called *singular.*

A singular point $z \in \mathbb{C}$ is called *regular* if the order of the pole of p_i at z is at most i.

Equation (3.1) has an ordinary (resp., singular, regular singular) point at infinity if after the change of variable $x = 1/\xi$ the point $\xi = 0$ is ordinary (resp., singular, regular singular).

A differential equation with only regular singular points is called *Fuchsian.*

Let $f_1, \dots, f_k$ be linearly independent polynomials. There is a unique (up to multiplication by a function) linear differential equation of order k with solutions $f_1, \dots, f_k$,

$$W(u, f_1, \dots, f_k) \ / \ W(f_1, \dots, f_k) \ = \ 0 \ ,$$

where

$$W(g_1, \dots, g_s) \ = \ \det(g_i^{(j-1)})_{i,j=1}^{s}$$

is the *Wronskian* of functions $g_1, \dots, g_s$. This equation is Fuchsian.

Consider a Fuchsian equation (3.1) and write in a neighborhood of a point $z \in \mathbb{C}$

$$p_i \ = \ \sum_{s=0}^{\infty} p_{is} \ (x-z)^{s-i}, \qquad i = 1, \dots, k.$$

If a function

$$u \ = \ (x-z)^{\lambda} \ (1 \ + \ \sum_{s=1}^{\infty} a_s \ (x-z)^s)$$

is a solution of equation (3.1), then λ is a root of the *indicial equation* at the point z

$$\lambda(\lambda-1)\dots(\lambda-k+1) \ + \ p_{10}\lambda(\lambda-1)\dots(\lambda-k+2) \ + \ \dots \ + \ p_{k0} \ = \ 0.$$

The roots of the indicial equation at a point z are called *exponents* at z of the Fuchsian equation. If V is the space of solutions of the Fuchsian equation, then the roots of the indicial equation are called the exponents of V at z.

3.2. **Polynomials representing critical points.** Let $\boldsymbol{t} = (t_j^{(i)})$ be a critical point of a master function $\Phi = \Phi(\boldsymbol{t}; \boldsymbol{z}; \bar{\boldsymbol{\Lambda}})$. Introduce polynomials $\boldsymbol{y} = (y_1(x), \dots, y_r(x))$,

$$y_i(x) \ = \ \prod_{j=1}^{l_i} (x - t_j^{(i)}).$$

The r-tuple $\boldsymbol{y}$ determines the $S_{\boldsymbol{l}}$-orbit of the critical point $\boldsymbol{t}$. We say that the r-tuple of polynomials $\boldsymbol{y}$ *represents a critical point* of the master function Φ. Usually we do not distinguish between the critical points of the same $S_{\boldsymbol{l}}$-orbit and identify a critical point with the r-tuple of polynomials $\boldsymbol{y}$ representing the point.

We consider the r-tuple $\boldsymbol{y}$ up to multiplication of each coordinate by a non-zero number, since we are interested only in the roots of polynomials $y_1, \dots, y_r$. Thus the r-tuple defines a point in the direct product $\boldsymbol{P}(\mathbb{C}[x])^r$ of r copies of the projective space associated with the vector space of polynomials of x.

Introduce polynomials

$$T_i(x) = \prod_{s=1}^{n} (x - z_s)^{\langle \Lambda_s, \alpha_i^\vee \rangle}, \qquad i = 1, \ldots, r. \tag{3.2}$$

We say that a given r-tuple of polynomials $\boldsymbol{y} \in \boldsymbol{P}(\mathbb{C}[x])^r$ is *generic with respect to integral dominant weights* $\Lambda_1, \ldots, \Lambda_n$ *of the Kac-Moody algebra* $\mathfrak{g}(A)$ *and points* $z_1, \ldots z_n$ if

- each polynomial $y_i(x)$ has no multiple roots;
- all roots of $y_i(x)$ are different from roots of the polynomial T_i;
- any two polynomials $y_i(x)$, $y_j(x)$ have no common roots if $i \neq j$ and $a_{ij} \neq 0$.

If $\boldsymbol{y}$ represents a critical point of Φ, then $\boldsymbol{y}$ is generic.

Now we reformulate the property of $\boldsymbol{y}$ to represent a critical point.

Write $f' = \partial f / \partial x$ and $\ln'(f) = f'/f$. Let polynomials F_i, G_i, $i = 1, \ldots, r$, be given by

$$F_i \;=\; \prod_{s=1}^{n} (x - z_s) \prod_{j,\ a_{ij} \neq 0} y_j, \qquad G_i \;=\; F_i \; \ln' \left(T_i \prod_{j,\ j \neq i} y_j^{-\langle \alpha_j, \alpha_i^\vee \rangle} \right).$$

Lemma 3.1. *A generic r-tuple $\boldsymbol{y}$ represents a critical point if and only if for every $i = 1, \ldots, r$ the polynomial $F_i\ y_i'' \ -\ G_i\ y_i'$ is divisible by the polynomial y_i. In other words, a generic r-tuple $\boldsymbol{y}$ represents a critical point if and only if for every $i = 1, \ldots, r$ there exists a polynomial H_i, such that deg $H_i \ \leq$ deg $F_i \ -\ 2$, and the polynomial y_i is a solution of the differential equation*

$$F_i\ u'' \ -\ G_i\ u' \ +\ H_i\ u \ =\ 0\ . \tag{3.3}$$

Proof. The lemma is a direct corollary of a classical result of Heine-Stieltjes, see [S], Section 6.8. We sketch the proof.

Assume that there exist such polynomials $H_1, \ldots, H_r$. Substitute $x = t_j^{(i)}$ into $F_i\ y_i'' \ -\ G_i\ y_i' \ +\ H_i\ y_i \ =\ 0$. We get $y''(t_j^{(i)})/y_i'(t_j^{(i)}) \ =\ G_i(t_j^{(i)})/F_i(t_j^{(i)})$. This is exactly equation (2.2) multiplied by $2/(\alpha_i, \alpha_i)$, since for $f = \prod_s (x - a_s)$ we have

$$\frac{f''}{f'}(a_s) \ =\ \sum_{k,\ k \neq s} \frac{2}{a_s - a_k}\ .$$

This means that the roots of polynomials $y_1, \ldots, y_r$ form a critical point. This argument is reversible. □

Let $\boldsymbol{y}$ represent a critical point. Then equation (3.3) is Fuchsian. The singular points and exponents of that equation are

$$\begin{array}{ll} x = z_s : & \{0, \langle \Lambda_s, \alpha_i^\vee \rangle + 1\}, \\ x = t_j^{(k)},\ k \neq i : & \{0, -\langle \alpha_k, \alpha_i^\vee \rangle + 1\}, \\ x = \infty : & \{-l_i, -\sum_s \langle \Lambda_s, \alpha_i^\vee \rangle + \sum_{k,\ k \neq i} l_k \langle \alpha_k, \alpha_i^\vee \rangle - 1\}. \end{array}$$

Lemma 3.2. *Let $\boldsymbol{y}$ be generic and let $\langle \Lambda_s, \alpha_i^\vee \rangle$ be non-negative integers for some i and all $s = 1, \dots, n$. Then $F_i\, y_i'' \,-\, G_i\, y_i'$ is divisible by y_i if and only if there exists a polynomial $\tilde y_i(x)$ such that the Wronskian $W(y_i, \tilde y_i)$ is given by*

$$W(y_i, \tilde y_i) \;=\; T_i \prod_{j,\ j\neq i} y_j^{-\langle \alpha_j, \alpha_i^\vee \rangle} \;. \tag{3.4}$$

Proof. If $H_i \;=\; (F_i\, y_i'' \,-\, G_i\, y_i')/y_i$ is a polynomial, then by [ScV], Lemma 7, all solutions of $F_i\, u'' - G_i\, u' \,+ H_i\, u \;=\; 0$ are polynomials. Then $\tilde y_i$ is any second linearly independent solution multiplied by a suitable constant. This proves the "only if" part of the lemma.

Let a polynomial $\tilde y_i$ exist. The polynomial y_i is a solution of the equation $W(u, y_i, \tilde y_i) = 0$. After multiplying this equation by $F_i / W(y_i, \tilde y_i)$ we get $F_i u'' - G_i u' + H_i u = 0$, where

$$H_i(x) \;=\; F_i\, \frac{y_i'\, \tilde y_i'' \,-\, y_i''\, \tilde y_i'}{W(y_i, \tilde y_i)} \;=\; F_i\, \frac{\ln'(W(y_i, \tilde y_i))\; y_i' - y_i''}{y_i} \;.$$

It is clear that $F_i \ln'(W(y_i, \tilde y_i))$ is a polynomial. Therefore, poles of H_i are common zeros of y_i and $W(y_i, \tilde y_i)$. Equation (3.4) implies that the polynomials y_i and $W(y_i, \tilde y_i)$ do not have common zeros since $\boldsymbol{y}$ is assumed to be generic. Therefore H_i is a polynomial. □

Corollary 3.3. *Let the weights $\Lambda_1, \dots, \Lambda_n$ be dominant integral. Then a generic r-tuple $\boldsymbol{y}$ represents a critical point if and only if for every $i = 1, \dots, r$ there is a polynomial $\tilde y_i$ satisfying (3.4).*

Lemma 3.4. *Let $y_1, \dots, y_r$, T_i be given and let $\tilde y_i$ satisfy equation (3.4). Then, up to multiplication by a non-zero number, the function $\tilde y_i$ has the form*

$$\tilde y_i(x) \;=\; c_1\, y_i(x) \int T_i(x) \prod_{j=1}^{r} y_j^{-\langle \alpha_j, \alpha_i^\vee \rangle}\, dx \;+\; c_2\, y_i(x) \;, \tag{3.5}$$

where c_1, c_2 are complex numbers. □

Notice that formula (3.5) gives all solutions of the differential equation

$$F_i\, u'' \;-\; G_i\, u' \;+\; H_i\, u \;=\; 0 \;.$$

Lemma 3.4 shows that the r-tuples

$$\boldsymbol{y}^{(i)} \;=\; (y_1, \dots, \tilde y_i, \dots, y_r) \qquad \in \qquad \boldsymbol{P}(\mathbb{C}[x])^r \;, \tag{3.6}$$

where $\tilde y_i$ is given by (3.5), form a one-parameter family. The parameter space of the family is identified with the projective line $\boldsymbol{P}^1$ with projective coordinates $(c_1 : c_2)$. We have a map

$$Y_{\boldsymbol{y},i} \;:\; \boldsymbol{P}^1 \;\to \boldsymbol{P}(\mathbb{C}[x])^r \;, \tag{3.7}$$

which sends a point $c = (c_1 : c_2)$ to the corresponding r-tuple $\boldsymbol{y}^{(i)}$.

Lemma 3.5. *If $\boldsymbol{y}$ is generic, then almost all r-tuples $\boldsymbol{y}^{(i)}$ are generic. The exceptions form a finite set in $\boldsymbol{P}^1$.* □

3.3. **Fertile r-tuples.** Let $\Lambda_1, \dots, \Lambda_n$ be dominant integral weights, $z_1, \dots, z_n$ complex numbers. Let $\boldsymbol{y} = (y_1, \dots, y_r) \in \boldsymbol{P}(\mathbb{C}[x])^r$ and let l_i be the degree of the polynomial y_i. The weight

$$\Lambda_\infty = \sum_{s=1}^{n} \Lambda_s - \sum_{i=1}^{r} l_i \, \alpha_i$$

is called *the weight at infinity* of the r-tuple $\boldsymbol{y}$ with respect to $\Lambda_1, \dots, \Lambda_n$ and $z_1, \dots, z_n$.

An r-tuple $\boldsymbol{y}$ is called *fertile* with respect to dominant integral weights $\Lambda_1, \dots, \Lambda_n$ and complex numbers $z_1, \dots, z_n$, if for every i there exists a polynomial $\tilde{y}_i$ satisfying equation (3.4). If $\boldsymbol{y}$ is fertile, then the r-tuples $\boldsymbol{y}^{(i)}$ given by (3.6) are called *immediate descendants* of $\boldsymbol{y}$ in the i-th direction.

A generic r-tuple $\boldsymbol{y}$ represents a critical point of a master function associated to dominant integral weights $\Lambda_1, \dots, \Lambda_n$ and points $z_1, \dots, z_n$ if and only if it is fertile, see Corollary 3.3.

Lemma 3.6. *Assume that a sequence $\boldsymbol{y}_k$, $k = 1, 2, \dots$, of fertile r-tuples of polynomials has a limit $\boldsymbol{y}_\infty$ in $\boldsymbol{P}(\mathbb{C}[x])^r$ as k tends to infinity.*

- *Then the limiting r-tuple $\boldsymbol{y}_\infty$ is fertile.*
- *Let $i \in \{1, \dots, r\}$. Let $\boldsymbol{y}_\infty^{(i)}$ be an immediate descendant of $\boldsymbol{y}_\infty$ in the i-th direction. Then for any k, there exists an immediate descendant $\boldsymbol{y}_k^{(i)}$ of $\boldsymbol{y}_k$ such that $\boldsymbol{y}_\infty^{(i)}$ is the limit of $\boldsymbol{y}_k^{(i)}$ as k tends to infinity.*

Proof. Let $\boldsymbol{y}_k = (y_{k,1}, \dots, y_{k,r})$. For every k, including $k = \infty$, consider the differential equation

$$F_{k,i} \, u'' - G_{k,i} \, u' + H_{k,i} \, u = 0, \tag{3.8}$$

where $F_{k,i}$, $G_{k,i}$ are as in (3.3) with y_j replaced by $y_{k,j}$ and $H_{j,i} = (G_{k,i} y'_{k,i} - F_{k,i} y''_{k,i})/y_{k,i}$. For every $k < \infty$, all solutions of that equation are polynomials, and the polynomial $y_{k,i}$ is one of solutions.

The lemma would follow if we proved that all solutions of equation (3.8) for $k = \infty$ were polynomials.

Since the sequence of $\boldsymbol{y}_k$ has a limit, there is a point $z \in \mathbb{C}$, such that $x = z$ is an ordinary point of equation (3.8) for all k. Fix $a, b \in \mathbb{C}$, and let $\tilde{y}_{k,i}$ be the solution of (3.8) with the initial condition $\tilde{y}_{k,i}(z) = a$, $\tilde{y}'_{k,i}(z) = b$.

By the standard theorem on continuous dependence of solutions on the coefficients of the equation, the function $\tilde{y}_{\infty,i}$ is the limit of functions $\tilde{y}_{k,i}$ as k tends to infinity.

The function $\tilde{y}_{\infty,i}$ is univalued and regular, since $\tilde{y}_{k,i}$ is a polynomial for every finite k. Hence the function $\tilde{y}_{\infty,i}$ is a polynomial. Thus all solutions of equation (3.8) for $k = \infty$ are polynomials. This implies the lemma. $\square$

Let $\boldsymbol{y}$ represent a critical point of a master function $\Phi(\boldsymbol{t}; \boldsymbol{z}; \boldsymbol{\Lambda}, \Lambda_\infty)$. Let $\boldsymbol{y}^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_r)$, be an immediate descendant of $\boldsymbol{y}$ and let $\Lambda_\infty^{(i)}$ be its weight at infinity with respect to $\Lambda_1, \dots, \Lambda_n$ and $z_1, \dots, z_n$.

The following observation is crucial in this paper.

Theorem 3.7. *If $\boldsymbol{y}^{(i)}$ is generic, then $\boldsymbol{y}^{(i)}$ represents a critical point of the master function $\Phi(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda},\Lambda_\infty^{(i)})$.*

Proof. Denote $\tilde t_s^{(i)}$ the roots of $\tilde y_i$. For any j, such that $j \neq i$ and $a_{ij} \neq 0$, choose a root $t_k^{(j)}$ of the polynomial y_j. We have $W(y_i,\tilde y_i)(t_k^{(j)}) = 0$ by (3.4). Hence

$$\sum_s \frac{1}{t_k^{(j)} - t_s^{(i)}} \ = \ \sum_s \frac{1}{t_k^{(j)} - \tilde t_s^{(i)}} \ .$$

This implies that the roots of $\boldsymbol{y}^{(i)}$ satisfy the equation of system (2.2) corresponding to the coordinate $t_k^{(j)}$.

The roots of $\boldsymbol{y}^{(i)}$ satisfy the equations of system (2.2) corresponding to coordinates $\tilde t_s^{(i)}$ according to Lemma 3.2. □

Thus, starting with an r-tuple $\boldsymbol{y}$ representing a critical point of a master function $\Phi(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda},\Lambda_\infty)$ and an index $i \in \{1,\dots,r\}$, we construct in (3.7) a family $Y_{\boldsymbol{y},i} : \boldsymbol{P}^1 \to \boldsymbol{P}(\mathbb{C}[x])^r$ of fertile r-tuples. For almost all $c \in \boldsymbol{P}^1$ (with only finitely many exceptions), the r-tuple $Y_{\boldsymbol{y},i}(c)$ represents a critical point of a master function associated with integral dominant weights $\Lambda_1,\dots,\Lambda_n$ and points $z_1,\dots,z_n$.

We call this construction the *simple reproduction procedure in the i-th direction.*

3.4. **General reproduction procedure.** Assume that the weights $\Lambda_1,\dots,\Lambda_n$ are dominant integral and an r-tuple $\boldsymbol{y}^0 \in \boldsymbol{P}(\mathbb{C}[x])^r$ represents a critical point of a master function $\Phi(\boldsymbol{t};\boldsymbol{z};\boldsymbol{\Lambda},\Lambda_\infty)$.

Let $\boldsymbol{i} = (i_1, i_r, \dots, i_k)$, $1 \le i_j \le r$, be a sequence of natural numbers. We define a k-parameter family of fertile r-tuples

$$Y_{\boldsymbol{y}^0,\boldsymbol{i}} \ : \ (\boldsymbol{P}^1)^k \ \to \boldsymbol{P}(\mathbb{C}[x])^r$$

by induction on k, starting at $\boldsymbol{y}^0$ and successively applying the simple reproduction procedure in directions $i_1,\dots,i_k$.

More precisely, for $k=1$, it is the family $Y_{\boldsymbol{y}^0,i_1} : \boldsymbol{P}^1 \ \to \boldsymbol{P}(\mathbb{C}[x])^r$ defined by (3.7). If $k>1$, $\boldsymbol{i}' = (i_1, i_2, \dots, i_{k-1})$, $\boldsymbol{c} = (c^1,\dots,c^k) \in (\boldsymbol{P}^1)^k$, and $\boldsymbol{c}' = (c^1,\dots,c^{k-1}) \in (\boldsymbol{P}^1)^{k-1}$, then we set

$$Y_{\boldsymbol{y}^0,\boldsymbol{i}}(\boldsymbol{c}) \ = \ Y_{Y_{\boldsymbol{y}^0,\boldsymbol{i}'}(\boldsymbol{c}'),i_k}(c^k) \ .$$

The image $P_{\boldsymbol{y}^0,\boldsymbol{i}} \ \subset \ \boldsymbol{P}(\mathbb{C}[x])^r$ of the map $Y_{\boldsymbol{y}^0,\boldsymbol{i}}$ is called the *population in the direction of $\boldsymbol{i}$ originated at $\boldsymbol{y}^0$*. The set $P_{\boldsymbol{y}^0,\boldsymbol{i}}$ is an irreducible algebraic variety.

It is easy to see that if $\boldsymbol{i}' = (i_1', i_r', \dots, i_{k'}')$, $1 \le i_j' \le r$, is a sequence of natural numbers, and the sequence $\boldsymbol{i}'$ is contained in the sequence $\boldsymbol{i}$ as an ordered subset, then $P_{\boldsymbol{y}^0,\boldsymbol{i}'}$ is a subset of $P_{\boldsymbol{y}^0,\boldsymbol{i}}$.

The union

$$P_{\boldsymbol{y}^0} \ = \ \cup_{\boldsymbol{i}} \, P_{\boldsymbol{y}^0,\boldsymbol{i}} \ \subset \boldsymbol{P}(\mathbb{C}[x])^r \ ,$$

where the summation is over all sequences $\boldsymbol{i}$, is called *the population of critical points associated* with the Kac-Moody algebra $\mathfrak{g}$, weights $\Lambda_1, \dots, \Lambda_n$, points $z_1, \dots, z_n$, and *originated* at y^0.

Lemma 3.8. *For a given $\boldsymbol{i} = (i_1, \dots, i_k)$, almost all r-tuples $Y_{\boldsymbol{y}^0, \boldsymbol{i}}(\boldsymbol{c})$ represent critical points of master functions associated to weights $\Lambda_1, \dots, \Lambda_n$, and points $z_1, \dots, z_n$. Exceptional values of $c \in (\boldsymbol{P}^1)^k$ are contained in a proper algebraic subset.* □

Lemma 3.9. *If two populations intersect, then they coincide.* □

3.5. **Populations and flag varieties.**

Example. Consider the population of critical points associated to $\mathfrak{g} = sl_3$ and $n = 0$ and originated at $\boldsymbol{y}^0 = (1, 1)$. The pair $(1, 1)$ represents the critical point of the function with no variables. This population consists of pairs of non-zero polynomials $\boldsymbol{y} = (y_1, y_2)$, where

$$y_i \; = \; a_{2,i}\, x^2 \; + \; a_{1,i}\, x \; + \; a_{0,i} \; , \qquad i = 1, 2 \; , \tag{3.9}$$

and

$$a_{1,1}\; a_{1,2} \; = \; 2\; a_{0,1}\; a_{2,2} \; + 2\; a_{2,1}\; a_{0,2} \; .$$

For any pair $\boldsymbol{y} = (y_1, y_2)$, if y_1, y_2 do not have multiple roots and do not have common roots, then the roots of the polynomials y_1, y_2 form a critical point of the function

$$\Phi \; = \prod_{1 \le i < j \le l_1} (t_i^{(1)} - t_j^{(1)})^2 \prod_{1 \le i < j \le l_2} (t_i^{(2)} - t_j^{(2)})^2 \prod_{i=1}^{l_1} \prod_{j=1}^{l_2} (t_i^{(1)} - t_j^{(2)})^{-1} \; ,$$

where $l_1 = \deg\, y_1$ and $l_2 = \deg\, y_2$.

In this case equations (3.4) take the form

$$W(y_1, \tilde{y}_1) \; = \; y_2 \; , \qquad W(y_2, \tilde{y}_2) \; = \; y_1 \; , \tag{3.10}$$

and the reproduction procedure works as follows. We start with $\boldsymbol{y}^0 = (1, 1)$. Equations (3.10) have the form $W(1,\ \tilde{y}_1) = 1,\ \ W(1,\ \tilde{y}_2) = 1$. Using, the first of them, we get pairs $\boldsymbol{y} = (x + a,\ 1)$ for all numbers a. Equations (3.10) now are $W(x + a,\ \tilde{y}_1) = 1,\ \ W(1,\ \tilde{y}_2) = x + a$. Using the second equation we get pairs $\boldsymbol{y} = (x + a,\ x^2/2 + ax + b)$ for all a, b. Equations (3.10) take the form $W(x + a,\ \tilde{y}_1) = x^2/2 + ax + b,\ \ W(x^2/2 + ax + b,\ \tilde{y}_2) = x + a$. Using the first of them we get $\boldsymbol{y} = (x^2/2 + cx + ac - b,\ x^2/2 + ax + b)$ for all a, b, c.

If we started the procedure using equation $W(1,\ \tilde{y}_2) = 1$, then the constructed pairs would have been of the form $\boldsymbol{y} = (1,\ x + a)$, $\boldsymbol{y} = (x^2/2 + ax + b,\ x + a)$, $\boldsymbol{y} = (x^2/2 + ax + b,\ x^2/2 + cx + ac - b)$.

It is easy to see that the union of all those pairs is our population, and nothing else can be constructed starting from $\boldsymbol{y}^0 = (1, 1)$.

It is easy to see that the family of pairs (3.9) (where each pair is considered up to multiplication of its coordinates by non-zero numbers) is isomorphic as an algebraic variety to the variety of all full flags in the three dimensional vector space V of the first coordinates