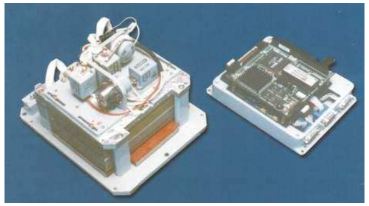
Inertial Navigation Systems

Integrated INS/GPS miniature navigation system, ELEKTROPRIBOR, Russia



Miniature integrated inertial/satellite navigation and reference system
MININAVIGATION-1

Intended for small vessels, airplanes and ground vehicles. Designed on the basis of the inertial measuring unit with fiber-optic gyros and miniature accelerometers, a GPS/GLONASS receiver and computer.

Specifications:

- Ultimate errors

- position	75 m
- velocity	0.25 m/s
- rocking and yawing angles	0.2°
- angular velocity components	0.1 °/s

IXBLUE
DEEP INSIGHT. SHARPER SENSES.

MARINS

Sigma 95

Sagem
Groupe SAFRAN

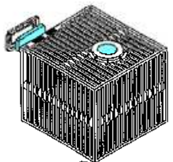
Sigma 40DX

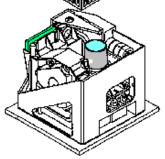
EPSILON 10

QUASAR 3000

THALES

ADIRU





Centrale
SEXTANT SIRAL

Kearfott

KN-4072A

LN120G

NORTHROP GRUMMAN

LASEREF VI

Honeywell
Solutions

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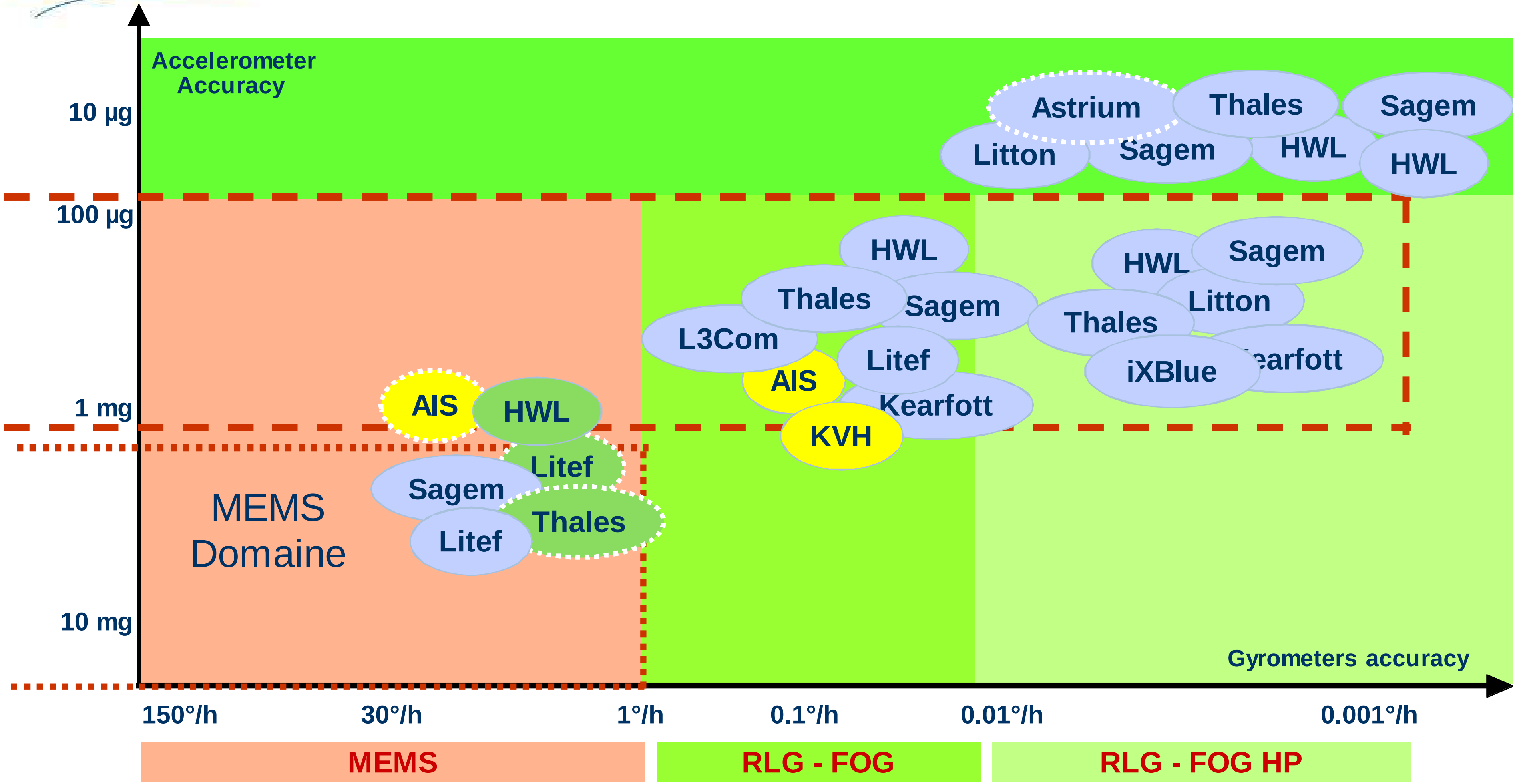
MINISTÈRE DE LA DÉFENSE
ET DES ANCIENS COMBATTANTS

DGA

Inertial Technology Manufacturers



MINISTÈRE DE LA DÉFENSE ET DES ANCIENS COMBATTANTS

Inertial navigation systems

Principles

- Basic ideas
- Compensation of $\vec{g}$
- Attitude reference

Architectures : strap-down and gimbaled

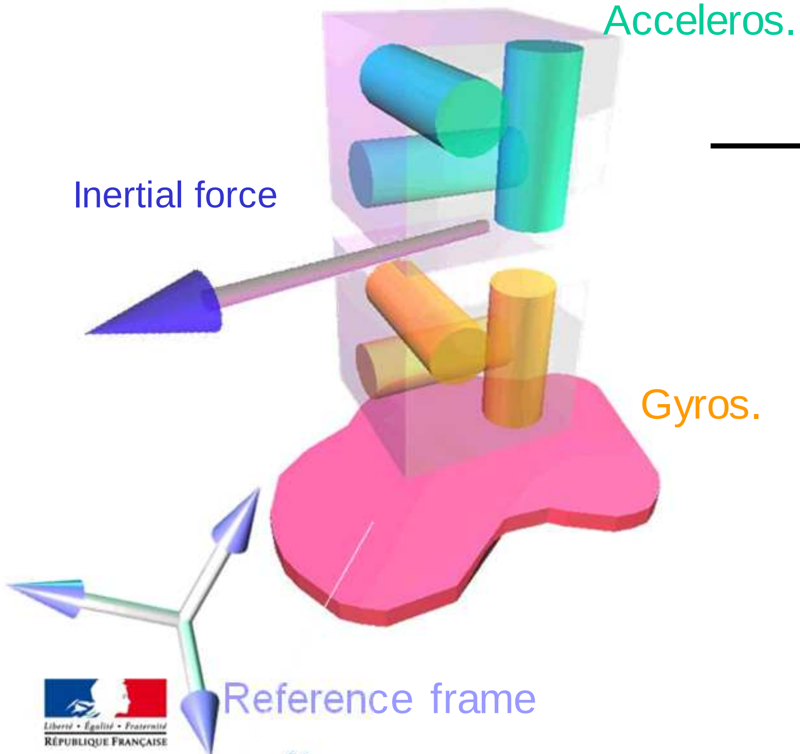
Gimbaled systems

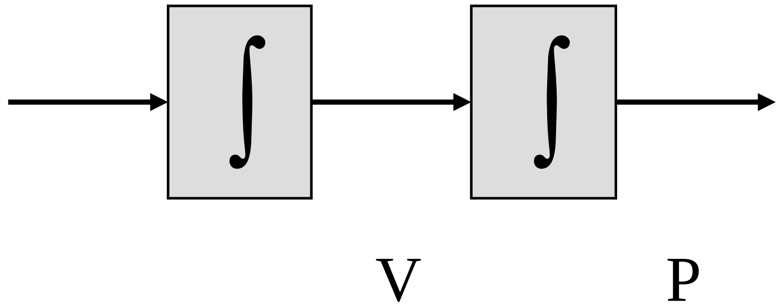
Strap-down systems

Key issues

Global design

- Basic idea : double integrate inertial forces in order to compute velocity and position





And then, how would the attitude be measured ?

(to be continued…)

What an accelerometer does measure

Reactive force to gravity mg

$R_1 = -mg$

Resulting force exerted by the seat to the pilot

$m\gamma - mg$

$R_2 = m\gamma$

Thrust transmission

Thrust

= true vehicle's acceleration $m\gamma$

Aboard the vehicle, the pilot is subject to all the forces exerted by the seat.

These forces are such that the pilot remains fixed in the cockpit.

$$\boxed{F = m(\gamma - g)}$$

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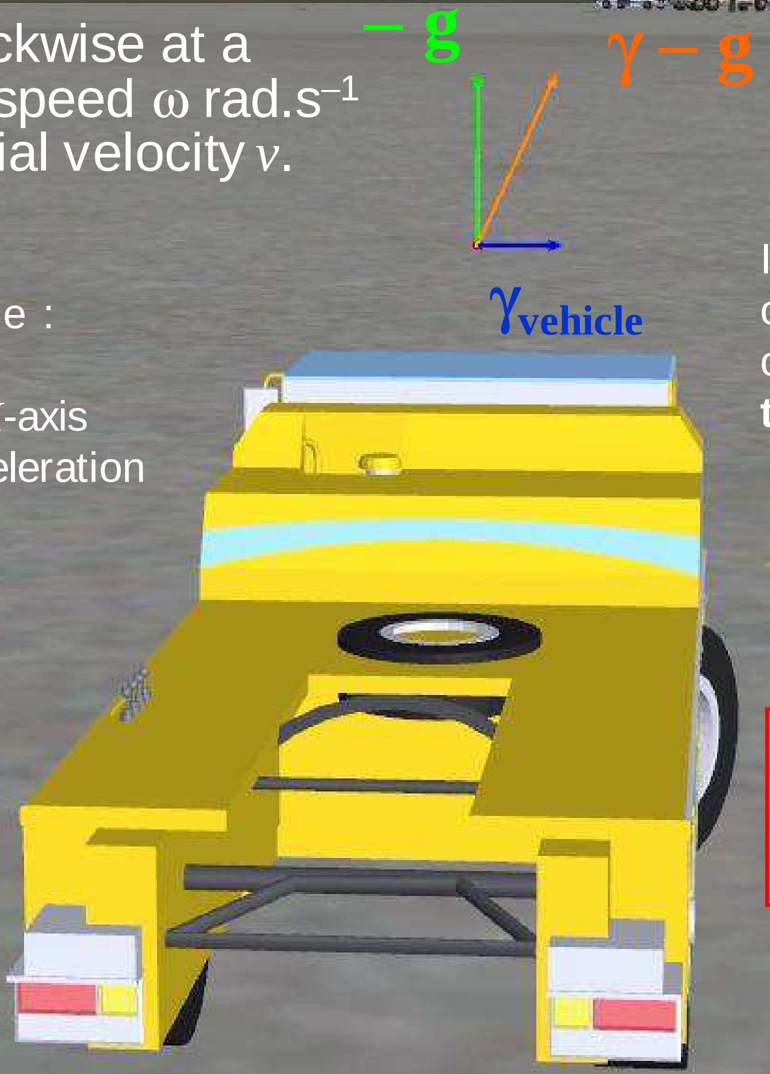
DGA

An angular reference frame is necessary !

Let a vehicle turn clockwise at a constant e rotational speed ω rad.s^{-1} and constant tangential velocity v.

The accelerometers provide :

- Nul acceleration on the X-axis
- Constant centripetal acceleration $R\omega^2$ on the Y-axis



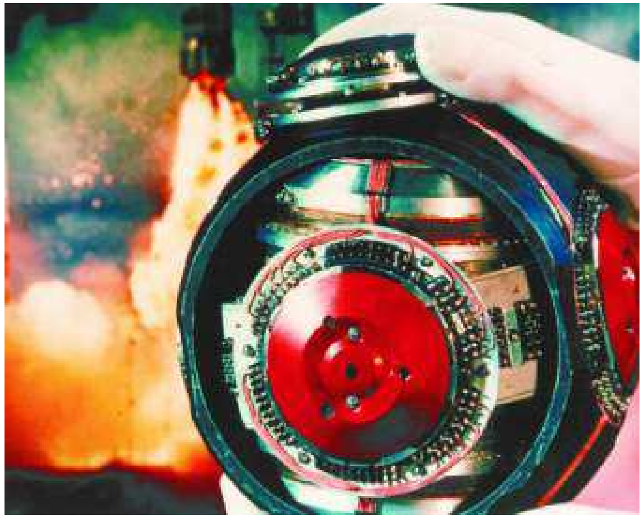
If the INS system assumed constant orientation, it would compute **a parabolic trajectory along the Y-axis !**

The orientation of the measured acceleration is needed.

Inertial navigation systems : two architectures

Strap-down INS

Gimbaled INS



Credits : Inertial Navigation – Forty Years of Evolution, by A. D. KING, B.Sc., F.R.I.N., Marconi Electronic Systems Ltd.

Design of inertial navigation systems : two architectures

The angular reference is used to compensate for $\vec{g}$. Then

- The linear acceleration may be integrated.
- The position is the result of numerical integration of the speed.

The angular reference is also the reference frame, with respect to which the attitude is defined: it may then be measured or computed.

Gimbaled	Strap-down
The attitude is <u>measured</u> on the gimbals : the reference frame is indeed a piece of hardware.	**The attitude of accelerometers is constantly integrated : the attitude of the vehicle is <u>computed</u>.**

Strap-down inertial navigation systems

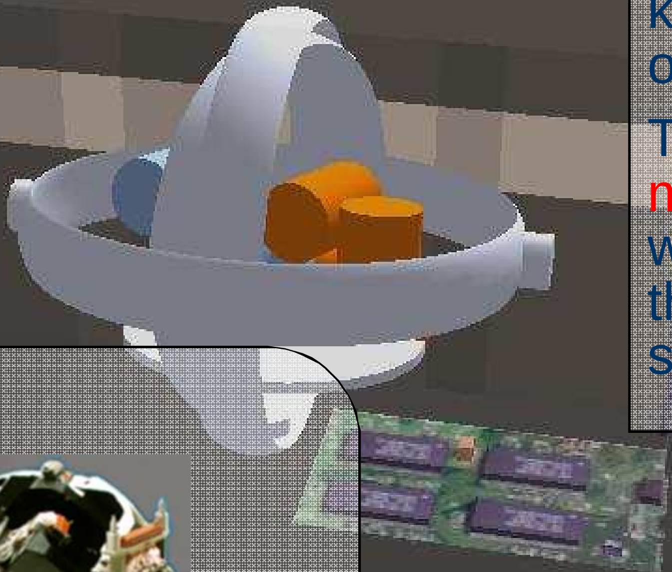
The inertial attitude of the accelerometer cluster is constantly integrated, so that the inertial force may be properly oriented in inertial space.

Other definitions of attitude are obtained from computations.

Elektropribor

THALES TOTEM 3k

Gimbaled inertial navigation systems



The angular position of the vehicle is constantly *mechanically compensated*, so that the accelerometer platform keeps steady. Implies the use of gimbals.

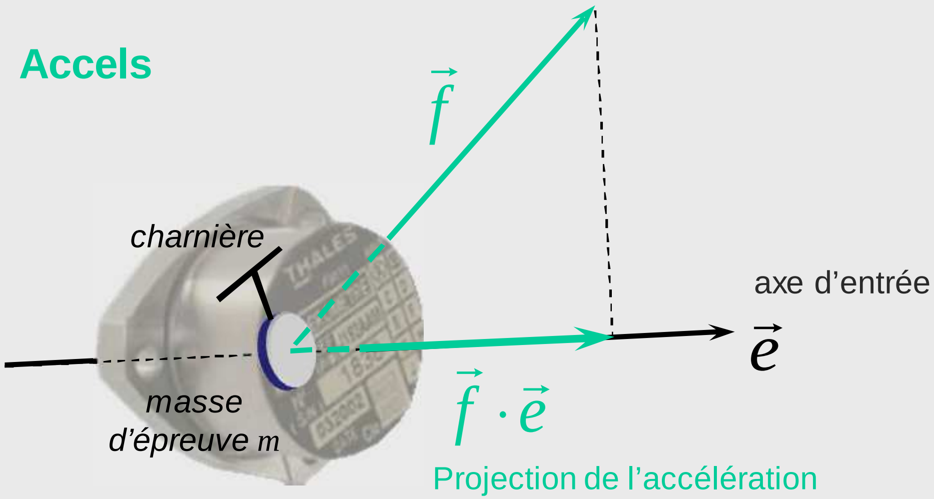
The vehicle attitude is measured on gimbal pickoffs, which provide the movement of the outer body around the steady platform.

Elektropribor

Strap-down inertial navigation systems

Accéléromètres et gyromètres

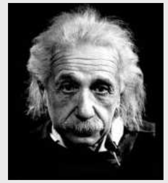
Accels



Projection de l'accélération apparente sur l'axe d'entrée → mesure

Principe de mesure : étudier et/ou rendre immobile une masse d'épreuve interne m par rapport au boîtier

$\oplus$



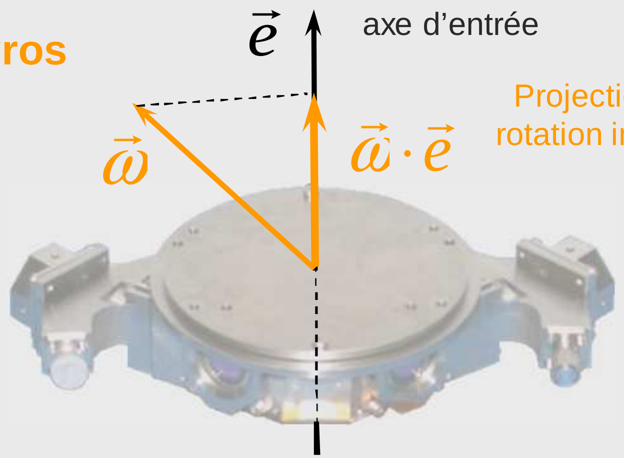
Impossibilité de séparer les effets de la gravité locale et des mouvements propres du véhicule

⬇

Mesure d'une « accélération apparente » dite force spécifique

$$\vec{f} = \vec{\gamma} - \vec{g}$$

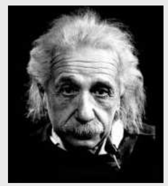
Gyros



Projection de la vitesse de rotation instantanée sur l'axe d'entrée → mesure

Principe de mesure : effets Coriolis sur une masse interne m ou étude d'effets relativistes

$\oplus$



Impossibilité de mesurer l'orientation absolue du véhicule

⬇

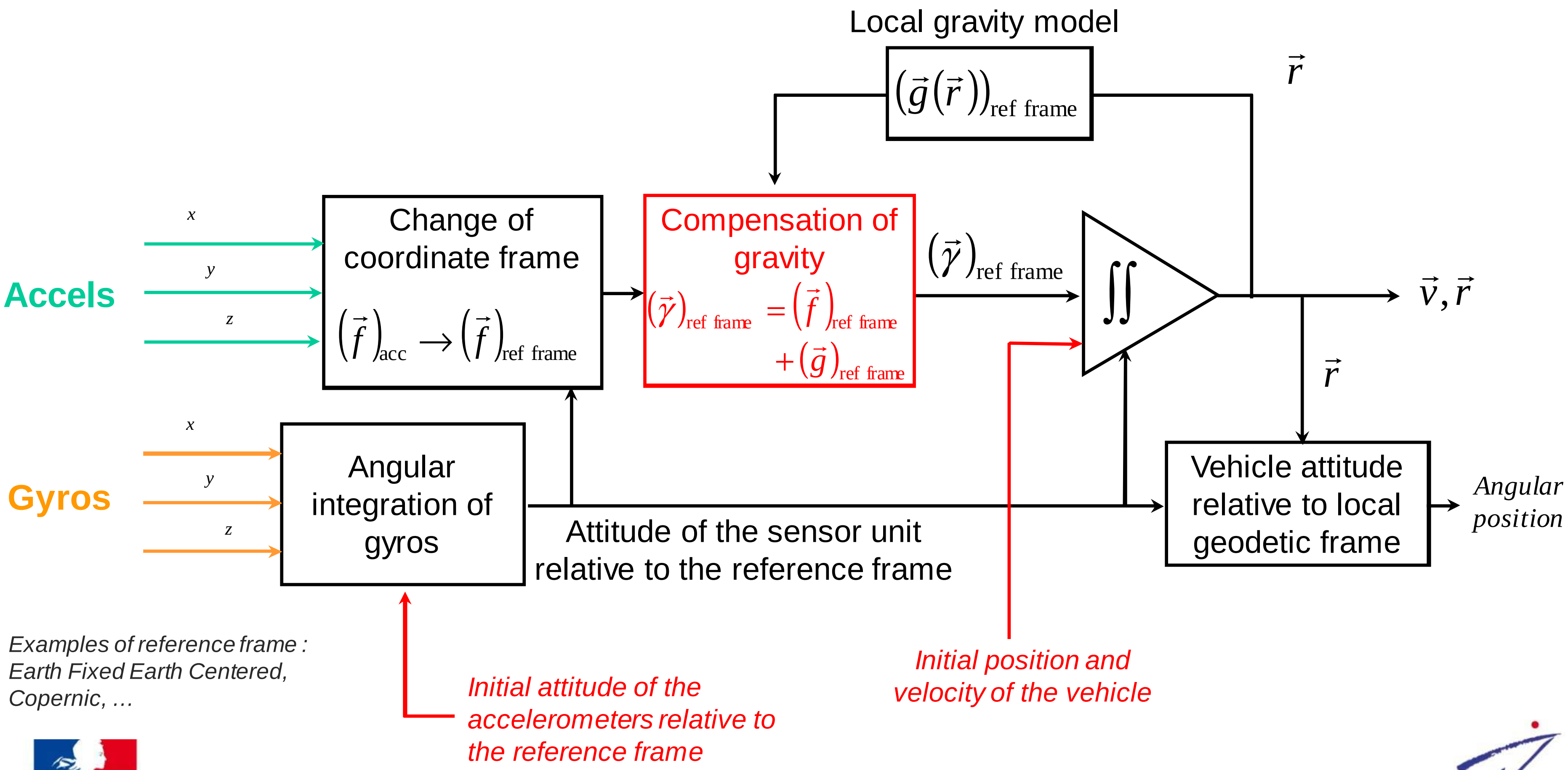
Mesure d'une vitesse de rotation instantanée (ou débattement angulaire)

Strap-down inertial navigation systems

Inertial navigation system

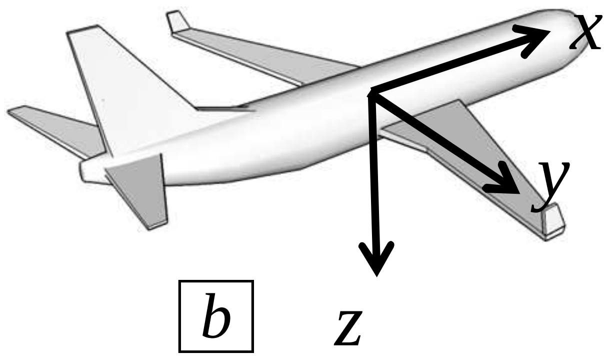
Double integration of an apparent, measured acceleration

- reconstructed angular position of the accel cluster, through gyros
- compensated by local gravity, modeled as a function of the estimated position

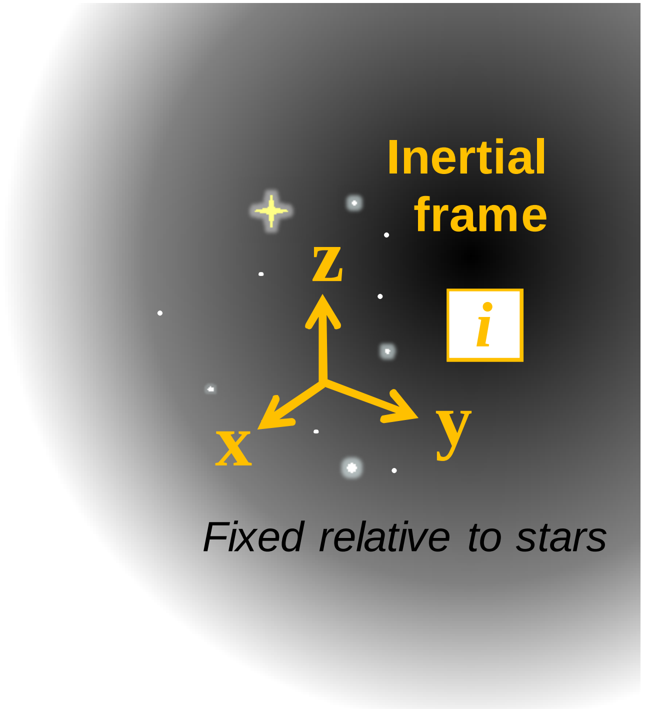


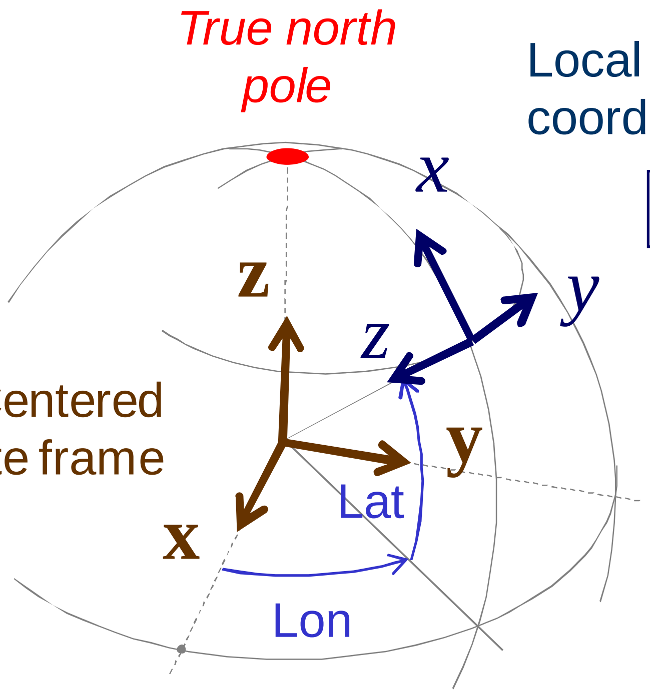
Examples of reference frame : Earth Fixed Earth Centered, Copernic, …

Common coordinate frames



Body coordinate frame





Local geodetic / navigation coordinate frame

n

Earth Fixed Earth Centered coordinate frame

e t

Computation of position and velocity (1)

Velocities

Time derivatives wrt rotary frames

$$\vec{v} \overset{\Delta}{=} \overset{t}{\vec{r}}$$ **for terrestrial applications**

t **: EFEC or North-East-Down**

$$\vec{v} \overset{\Delta}{=} \overset{i}{\vec{r}}$$ **for spatial vessels**

So, for terrestrial applications, need to integrate :

$$\overset{ii}{\vec{r}} = \vec{f} + \vec{g}(\vec{r})$$

$$= \overset{t}{\vec{v}} + \overset{i}{\vec{\omega}}_{t/i} \times \vec{r} + \vec{\omega}_{t/i} \times \left(2\,\vec{v} + \vec{\omega}_{t/i} \times \vec{r}\right)$$

Hence for $\boldsymbol{t} = \boldsymbol{EFEC}$:

$$\boxed{\overset{t}{\vec{v}} = \vec{f} + \vec{g}(\vec{r}) - 2\vec{\Omega}_E \times \vec{v} + \Omega_E^2\ \vec{r}_\perp}$$

Plumb bob vs Newton gravity

Earth is rotating in the inertial frame, thus inducing an inertial force that adds to the gravity field.

By the definition, we denote

$$\boxed{\vec{g}_{\text{Plumb bob}}(\vec{r}) = \vec{g}(\vec{r}) - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)}$$

And will omit the « Plumb bob » subscript in the following. Therefore :

$$\boxed{\overset{e}{\vec{v}} = \vec{f} + \vec{g}(\vec{r}) - 2\vec{\Omega}_E \times \vec{v}}$$

Computation of position and velocity (2)

Actual computation

- should be able to add $\vec{f}$ measured in accelerometer axes to $\vec{g}(\vec{r})$ usually available in some global frame (eg. ECEF)

Therefore : *a coordinate transform between accelerometer axes and global axes is absolutly needed !*

↳ **gyros**

- some data is
  - measured : $\text{data}^{\text{meas}}$
  - modeled : data^{mod}
  - computed : data^{c}
  - perfectly known : data

Computation of position and velocity (3)

How do gyros provide proper coordinate transform between global and accelerometer axes (body) ?

They measure : $\vec{\omega}_{b/i}$ in body axes.

and the transition matrix from $\boldsymbol{i}$ to $\boldsymbol{b}$ obeys

$$(\vec{\omega}_{b/i})_b = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\frac{d}{dt} T_{b/i} = -\left[(\vec{\omega}_{b/i})_b \times\right] \cdot T_{b/i}$$

$$= \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \cdot T_{b/i}$$

Computation of position and velocity (4)

(accelero compensation)

Henceforth :

$$f_{ECEF} = \left(\vec{f}\right)_{ECEF} = T_{e/b} \cdot \left(\vec{f}\right)_b$$

$$= T_{e/i} \cdot T_{i/b} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Then

$$\left(\vec{f} + \vec{g}(\vec{r})\right)_{ECEF} = T_{e/i} \cdot T_{i/b} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} + \left(\vec{g}\right)_e$$

Similarly :

$$f_n = \left(\vec{f}\right)_n = T_{n/b} \cdot \left(\vec{f}\right)_b$$

$$= T_{n/e} \cdot T_{e/i} \cdot T_{i/b} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Computation of position and velocity (5)

Actual computation ; let's summarize :

- $\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$ is measured in accelerometer axes

- $(\vec{g})_{ECEF}$ is a model of the true gravitational field, computed at $\vec{r}^{\,c}$, which is the best estimate of the position we have so far.

- $T_{i/ECEF}$ is a mathematical function of time

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Computation of position and velocity (6)

So the actual computation of position and velocity will involve the following in the ***ECEF*** axes :

$$
\begin{cases}
\dfrac{d}{dt} r^c = v^c \\[2ex]
\dfrac{d}{dt} v^c = T^c_{b/i} \cdot T_{i/ECEF}(t) \cdot \begin{pmatrix} \boldsymbol{f_x} \\ \boldsymbol{f_y} \\ \boldsymbol{f_z} \end{pmatrix}^{\mathbf{meas}} + \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}^{\mathrm{mod}} \left(r^c\right) - 2 \begin{pmatrix} 0 \\ 0 \\ \Omega_E \end{pmatrix} \times v^c + \Omega_E^2 \, r_\perp^c \\[2ex]
\dfrac{d}{dt} T^c_{b/i} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\omega_z} & \boldsymbol{\omega_y} \\ \boldsymbol{\omega_z} & \mathbf{0} & -\boldsymbol{\omega_x} \\ -\boldsymbol{\omega_y} & \boldsymbol{\omega_x} & \mathbf{0} \end{pmatrix}^{\mathbf{meas}} \cdot T^c_{b/i}
\end{cases}
$$

Nota bene : $r^c, v^c, T^c_{b/i}$ need initial estimates $\boldsymbol{r^c(0), v^c(0), T^c_{b/i}(0)}$!

Computation of position and velocity in n (7)

Le cas de n = Trièdre Géographique Local (TGL) ou North-East-Down

$$\boxed{\overset{n}{\dot{\vec{v}}} = \vec{f} + \vec{g}(\vec{r}) - \left(\vec{\rho} + 2\vec{\Omega}_E\right) \times \vec{v}}$$

avec

$$\vec{\rho} \overset{\Delta}{=} \vec{\omega}_{n/e}$$

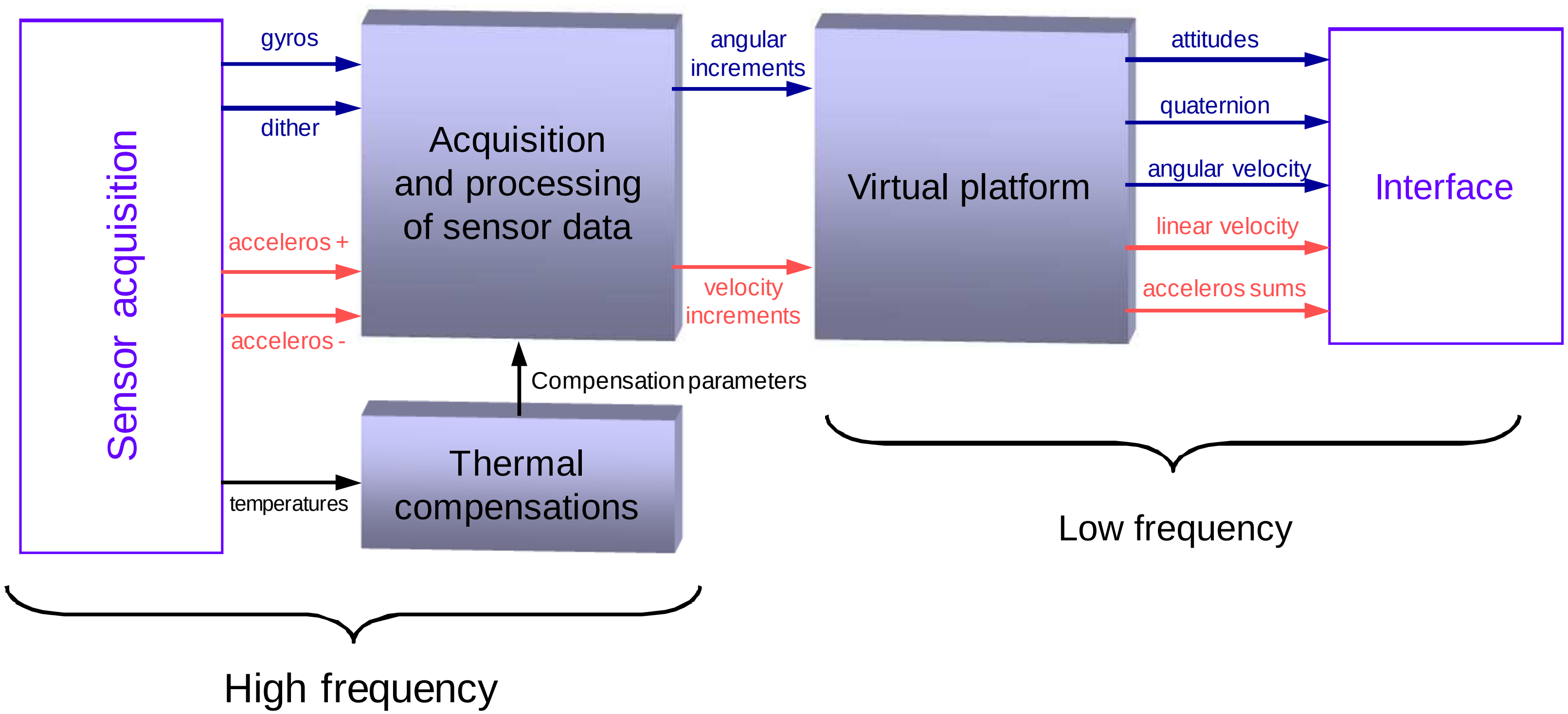
Computation of position and velocity in n (8)

$$
\begin{cases}
\dfrac{d}{dt} r^c = v^c - \rho\left(v^c\right) \times r^c \\[2ex]
\dfrac{d}{dt} v^c = T_{n/e}\left(r^c\right) \cdot T_{e/i}(t) \cdot T_{i/b}^c \cdot \begin{pmatrix} \boldsymbol{f}_{\mathbf{x}} \\ \boldsymbol{f}_{\mathbf{y}} \\ \boldsymbol{f}_{\mathbf{z}} \end{pmatrix}^{\mathbf{meas}} + \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}^{\mathrm{mod}} \left(r^c\right) - \left[\rho\left(v^c\right) + 2 \cdot T_{n/e}\left(r^c\right) \cdot \begin{pmatrix} 0 \\ 0 \\ \Omega_E \end{pmatrix} \right] \times v^c \\[2ex]
\dfrac{d}{dt} T_{b/i}^c = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\omega}_{\mathbf{z}} & \boldsymbol{\omega}_{\boldsymbol{y}} \\ \boldsymbol{\omega}_{\mathbf{z}} & \mathbf{0} & -\boldsymbol{\omega}_{\boldsymbol{x}} \\ -\boldsymbol{\omega}_{\boldsymbol{y}} & \boldsymbol{\omega}_{\boldsymbol{x}} & \mathbf{0} \end{pmatrix}^{\mathbf{meas}} \cdot T_{b/i}^c
\end{cases}
$$

Hardware design of computation

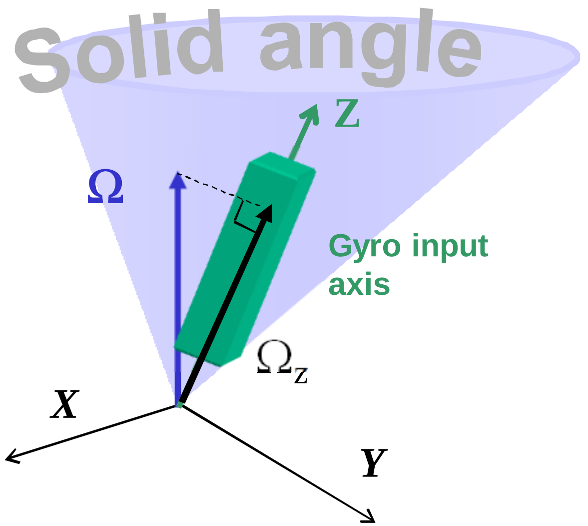
Strap-down INS : two time scales

Schematic of strap-down calculus



Strap-down INS : two time scales, why ?

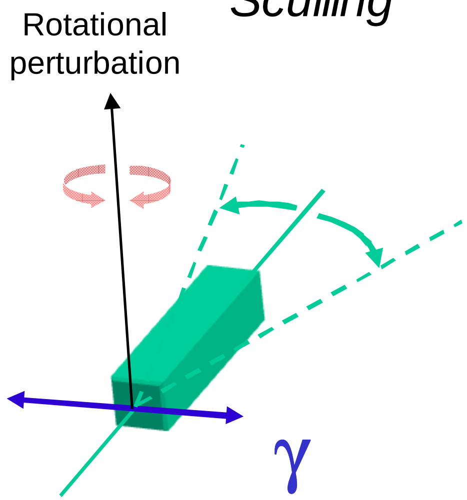
Coning



$$\boxed{\int \Omega_z \mathrm{d}t = 2k\pi - a}$$

(Goodman theorem)

Sculling



Sinusoidal acceleration input $\gamma = \gamma_0 \cos \omega t$

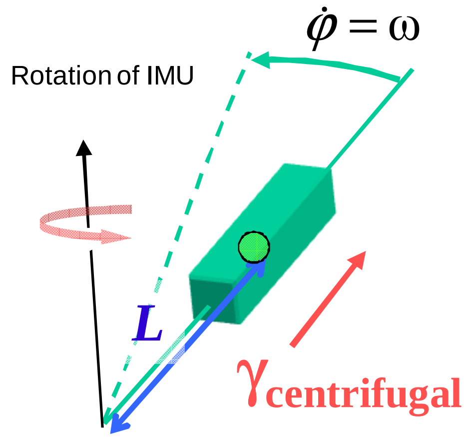
Simultaneous rotation of input axis at ω rad/s

$$\psi = \psi_0 \cos(\omega t + \phi)$$

then the accelerometer X will be biased by

$$\boxed{B_{\text{sculling}} = \frac{1}{2}\gamma_0 \psi_0 \cos\phi}$$

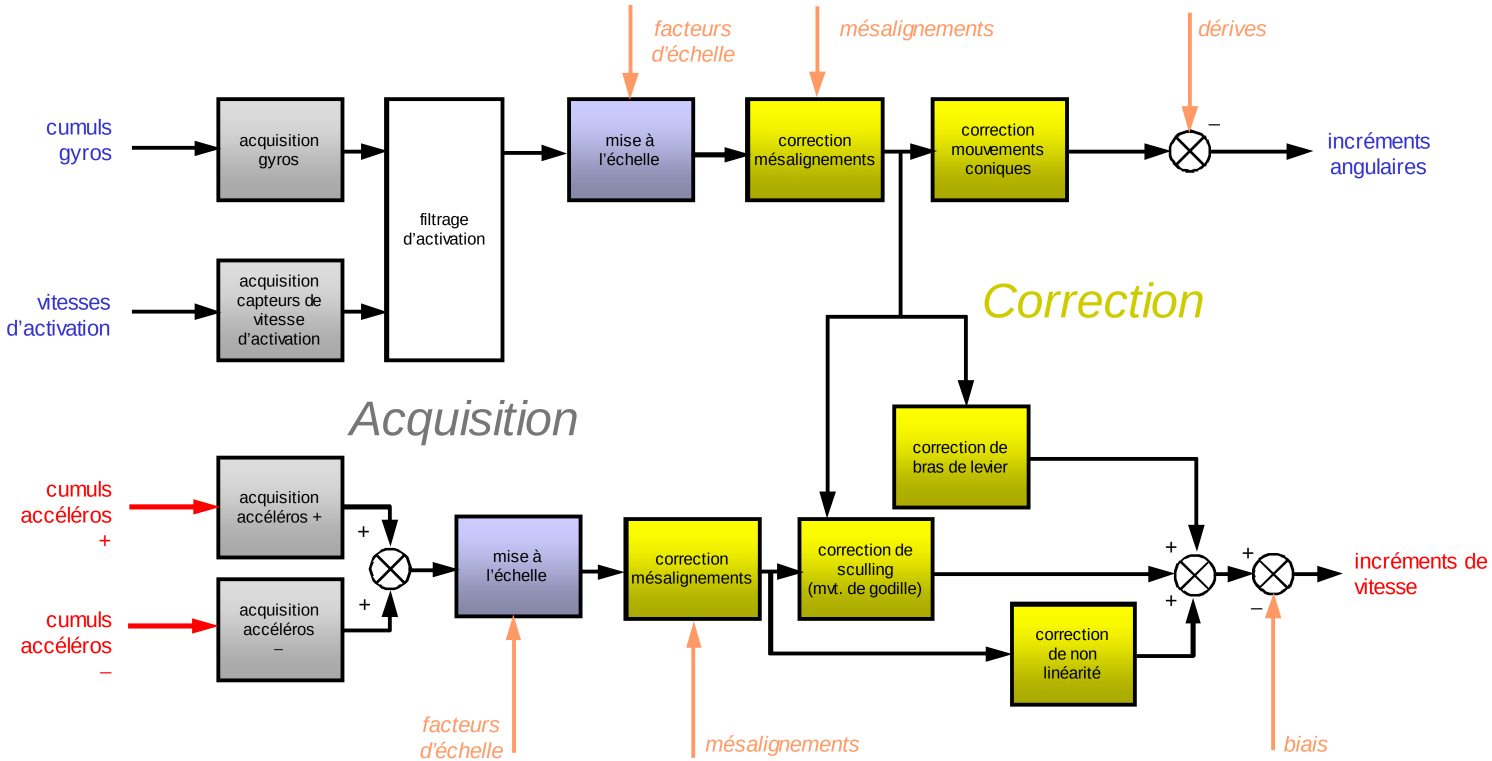
Centrifugal accelerations



$$\boxed{\gamma_{\text{centrifuge}} = L\,\omega^2}$$

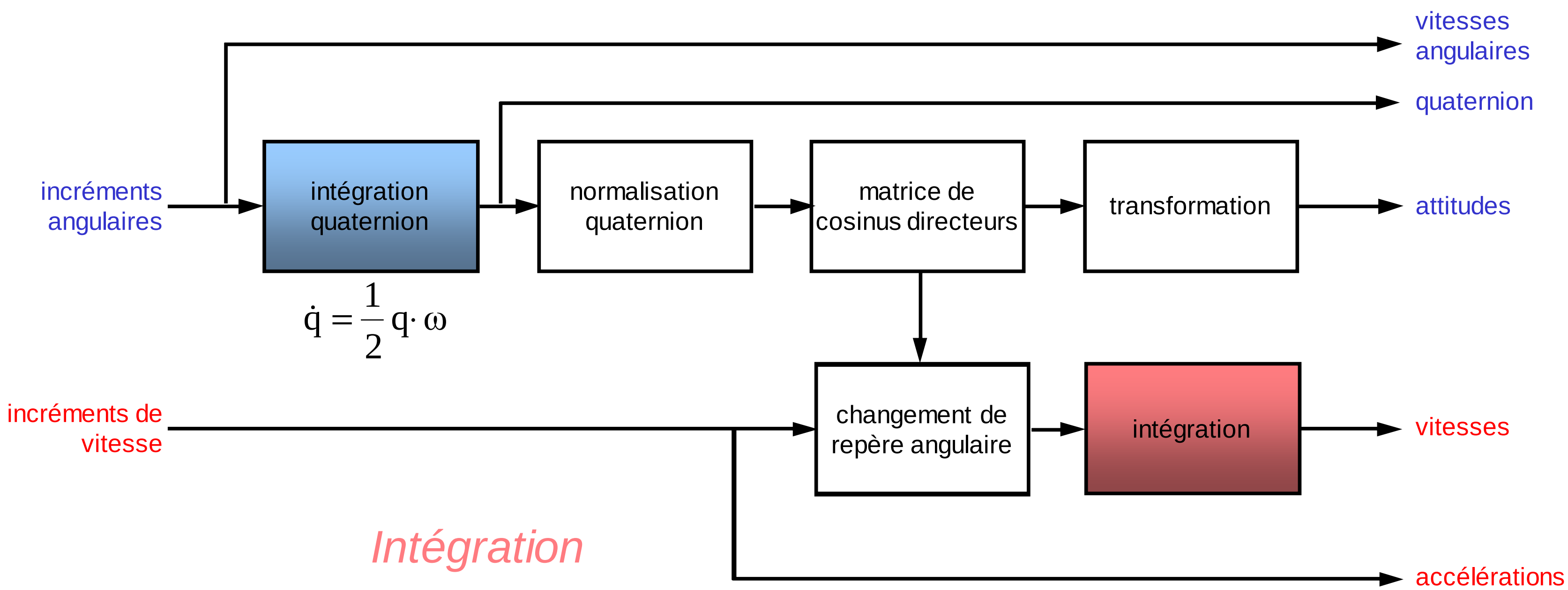
Strap-down INS : two time scales, why ?

High frequency process



Strap-down INS : two time scales, why ?

Low frequency process



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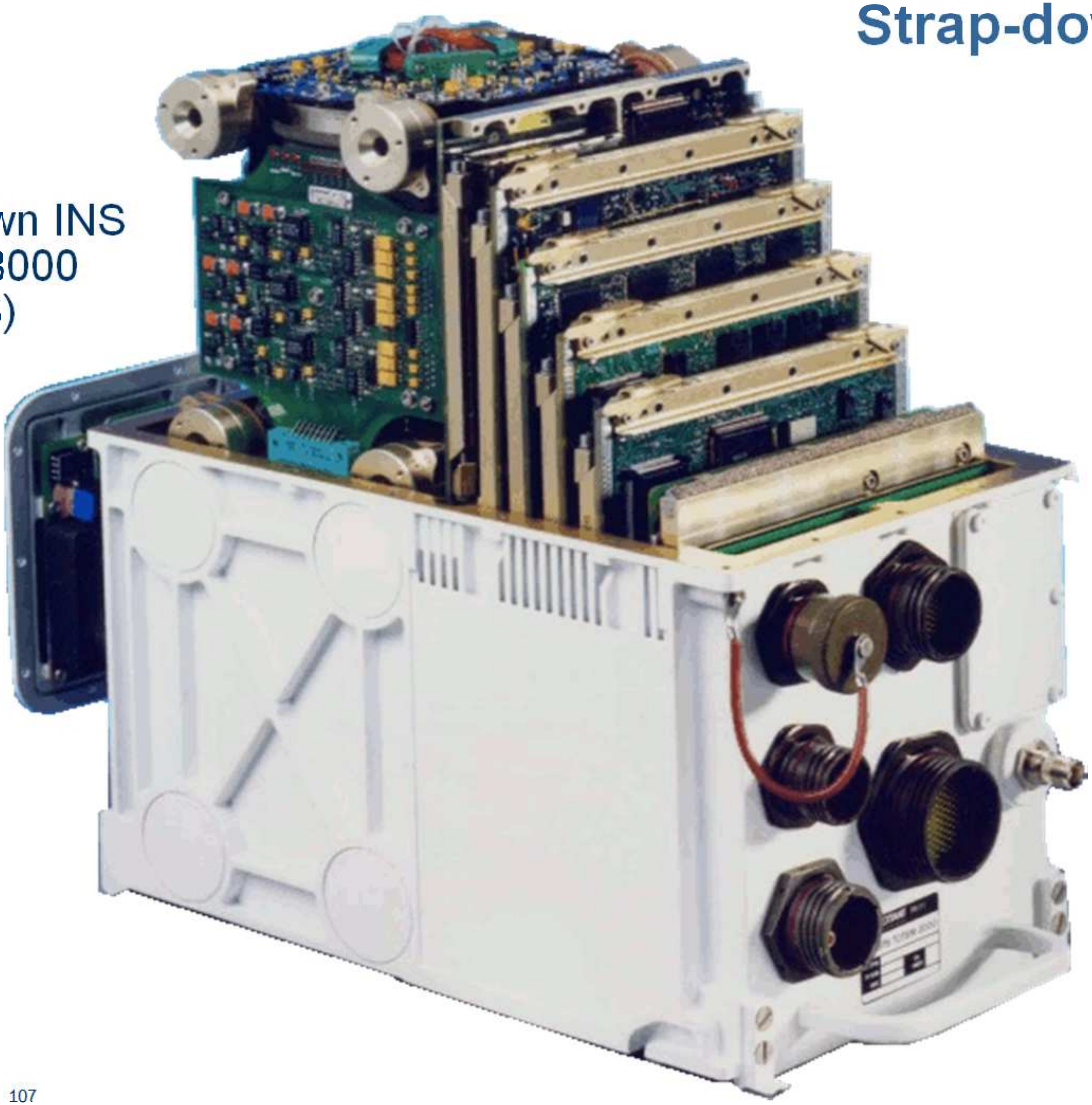
Hardware design of system

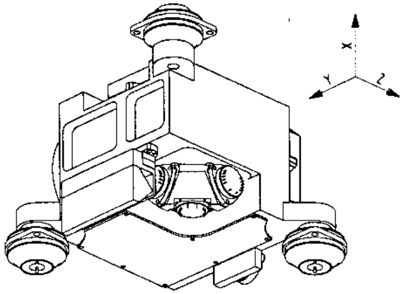
Strap-down INS

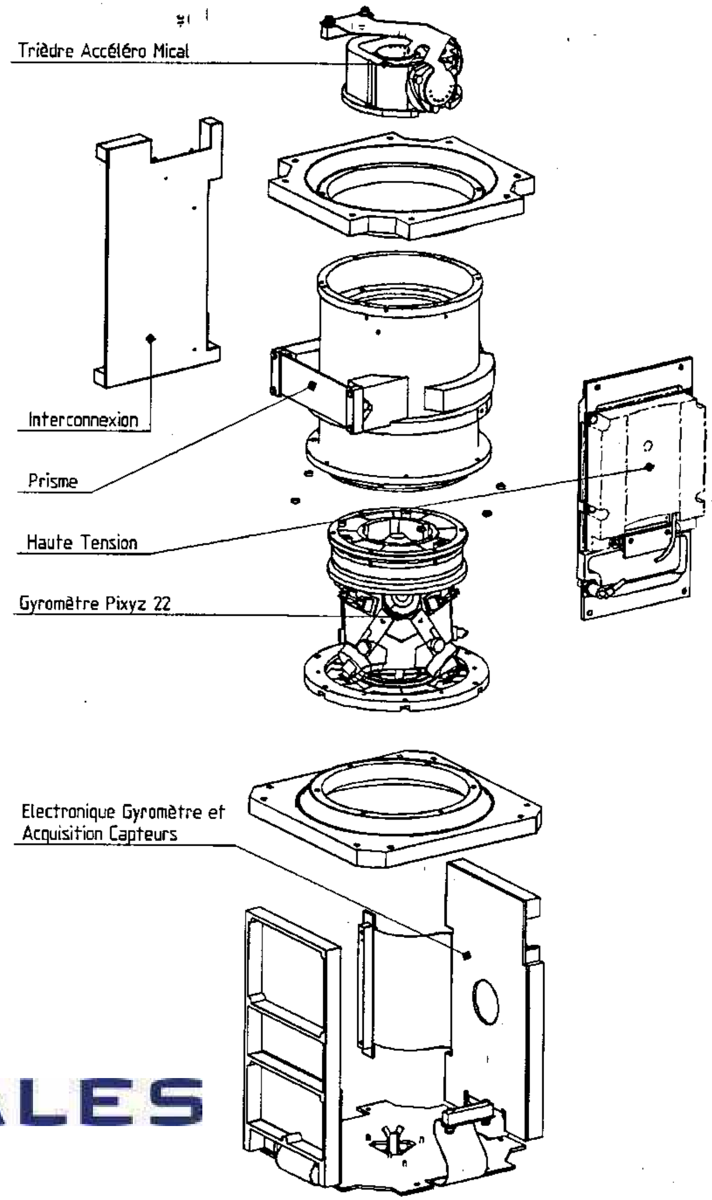
Strap-down INS
TOTEM 3000
(THALES)



Strap-down INS

Inertial measurement unit





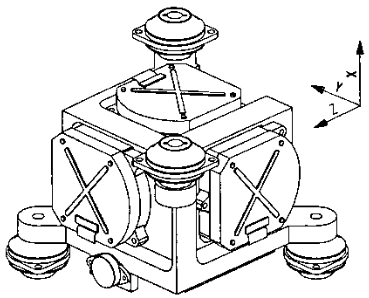


Figure 4.12 - Vue de l'UMI

THALES

MINISTÈRE DE LA DÉFENSE
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DGA

Strap-down INS

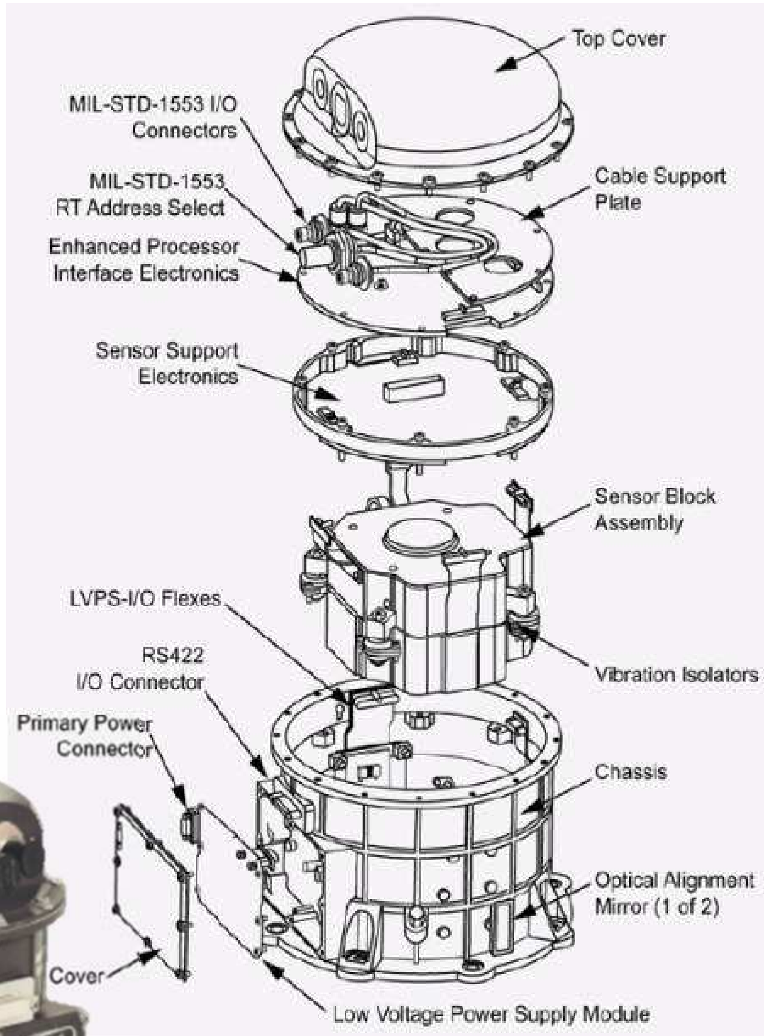
FIN3110 instrument cluster

Marconi FIN3110 ring laser gyro inertial navigation unit

Credits : Inertial Navigation – Forty Years of Evolution, by A. D. KING, B.Sc., F.R.I.N., Marconi Electronic Systems Ltd.

Strap-down INS

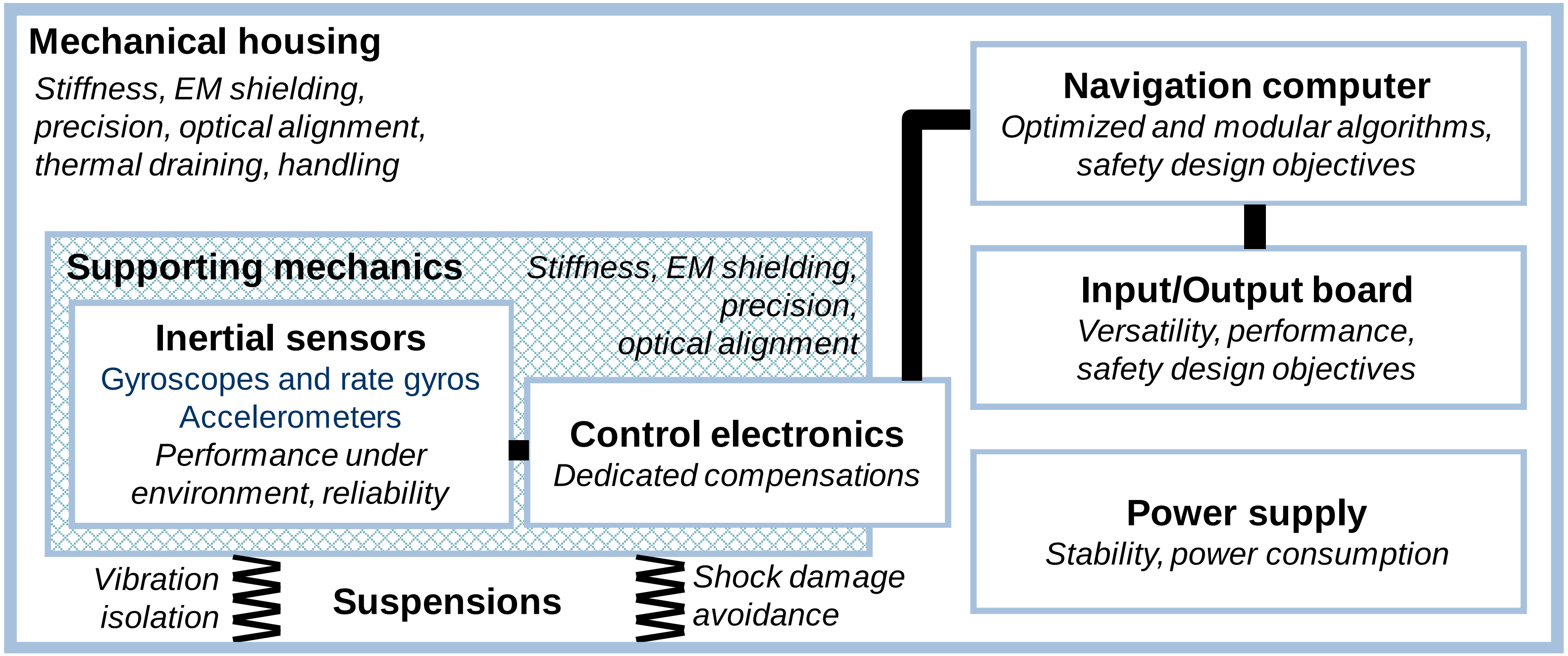
HG9900 IMU (from datasheet online)



MIMU (Space applications)

Inertial Navigation Systems
Key issues

Key elements of inertial systems



INS : thermal issues

Thermal stabilization

The performances of inertial sensors may significantly depend on temperature. Therefore it might be necessary to control the effects of thermal drifts :

- control the temperature of the sensors and/or environment : needs thermal insulation materials and thermostating devices

  Gimbaled INS are more concerned.

- correct thermal drifts using models : needs calibration

  Models of order 2, 3 or 4 are used on strap-down systems, whose computation power is appropriate.

INS : vibratory environment issues

Suspension

- The main purpose is to reject vibratory perturbances due to the environment or dithers. It might be crucial for
  - the inertial sensors to survive the mechanical environment !
  - performances (coning, sculling, …)
- The design of the vibration dampers is a compromise between
  - the passband
  - the efficiency of correction algorithms (coning, sculling)
  - the need to activate laser gyros (dithers).
- The design of a strap-down system usually requires the cancellation of assymetric effects and angular disturbances.

Popular vibration damping :

- pods
- rings

INS : design issues

Challenges \ Technology	Gimbaled	Strap-down
Design	**Hard & Complex** **Servoing of gyros** **Transmitting signals through gimbals requires slip rings**	**Easy** **…but HF computations and balancing of IMU**
Manufacturing & servicing/maintenance	**Complex** **Many pieces** **Resolvers**	**Easy** **Very few pieces**
Calibration	**Easy**	**Hard**
Computation time	**Small**	**Large**
Range of vehicle orientations	**Restricted** **Gimbals may lock when reaching some positions**	**Full** **If gyro integration algorithm appropriate (quaternion)**
Integrity	**Intrinsic property** **of the « inertial » platform**	**Hard** **fault in electronics may induce loss of all computations !**

Inertial navigation systems : take home ideas

Discrete, independent
Non-jammable
Available, reliable
Universal coverage

Very expensive
Accuracy is also a function of volume and mass

Divergent accuracy
Operational constraints

Structure of INS errors

Computation of position and velocity (recall)

The actual computation of position and velocity will involve the following in the ***ECEF*** axes :

$$
\begin{cases}
\dfrac{d}{dt} r^c = v^c \\[2ex]
\dfrac{d}{dt} v^c = T^c_{b/i} \cdot T_{i/ECEF}(t) \cdot \begin{pmatrix} \boldsymbol{f}_\mathbf{x} \\ \boldsymbol{f}_\mathbf{y} \\ \boldsymbol{f}_\mathbf{z} \end{pmatrix}^{\mathbf{meas}} + \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}^{\mathrm{mod}} \left(r^c\right) - 2 \begin{pmatrix} 0 \\ 0 \\ \Omega_E \end{pmatrix} \times v^c + \Omega_E^2 \, r^c_\perp \\[2ex]
\dfrac{d}{dt} T^c_{b/i} = \begin{pmatrix} \mathbf{0} & -\boldsymbol{\omega}_\mathbf{z} & \boldsymbol{\omega}_\mathbf{y} \\ \boldsymbol{\omega}_\mathbf{z} & \mathbf{0} & -\boldsymbol{\omega}_\mathbf{x} \\ -\boldsymbol{\omega}_\mathbf{y} & \boldsymbol{\omega}_\mathbf{x} & \mathbf{0} \end{pmatrix}^{\mathbf{meas}} \cdot T^c_{b/i}
\end{cases}
$$

Nota bene : $r^c, v^c, T^c_{b/i}$ need initial estimates $\boldsymbol{r^c(0), v^c(0), T^c_{b/i}(0)}$!

Error analysis : coordinate transform error (1)

Coordinate transform errors

Let's denote $\vec{\psi}$ the (small) vector such that

$$T_{b/i}^{c} \cong \left(\mathrm{Id} - \left(\vec{\psi}\right)_i \times\right) \cdot T_{b/i}$$

We could show that $\vec{\psi}$ satisfies the following :

$$\boxed{\overset{i}{\dot{\vec{\psi}}} = -\vec{\varepsilon}_{\mathrm{gyro}}}$$

« The speed of coordinate transform error as seen from the stars is exactly the gyro error. »

where $\vec{\varepsilon}_{\mathrm{gyro}}$ is defined in the accelerometer axes as

$$\left(\vec{\varepsilon}_{\mathrm{gyro}}\right)_b = \left(\vec{\omega}_{b/i}\right)_b^{\mathrm{meas}} - \left(\vec{\omega}_{b/i}\right)_b$$

Position & Velocity errors

Error analysis : position and velocity errors (1)

Velocity error

Let's denote $\vec{\varepsilon}_{\text{acc}}$ the amount of specific force measurement error :

$$\left(\vec{\varepsilon}_{\text{acc}}\right)_b = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}^{\text{meas}} - \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

One can show that

$$\dot{v}^c - \dot{v} \quad = \quad T^c_{b/ECEF} \cdot \varepsilon_{\text{acc}} + \left[T_{b/ECEF} \cdot \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \right] \times \left(\vec{\psi}\right)_{ECEF}$$

$$+ \frac{\partial}{\partial \vec{r}} g^{\text{mod}}{}_{ECEF} \cdot \left(r^c - r\right) + g^{\text{mod}}\left(r\right)_{ECEF} - g\left(r\right)_{ECEF}$$

$$- \begin{pmatrix} 0 \\ 0 \\ \Omega_E \end{pmatrix} \times \left(v^c - v\right) + \Omega_E^2 \cdot \left(r^c - r\right)_{\perp}$$

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Error analysis : position and velocity errors (2)

(velocity error continued)

With the assumption that $g^{\mathrm{mod}}(r)_{ECEF} \approx g(r)_{ECEF}$, we end up with the following « grand model » :

$$\boxed{\delta \overset{t}{\dot{\vec{v}}} + 2\vec{\Omega}_E \times \delta \vec{v} = \vec{f} \times \vec{\psi} + \vec{\varepsilon}_{\mathrm{acc}} + \frac{\partial \vec{g}}{\partial \vec{r}} \cdot \delta\vec{r} + \Omega_E^2 \cdot \delta\vec{r}_\perp}$$

t **here :** $ECEF$

Error analysis : 3D mid-term behaviour (1)

For short to mid term applications (a few hours at most), the effects of $\vec{\Omega}_E$ are negligible if $\delta\vec{r}$ and $\delta\vec{v}$ are small enough.

The full model then reads :

$$\boxed{\begin{cases} \delta\overset{t}{\vec{v}} = \vec{f}\times\vec{\psi} + \vec{\varepsilon}_{\text{acc}} + \dfrac{\partial\vec{g}}{\partial\vec{r}}\cdot\delta\vec{r} \\ \overset{NED}{\vec{\psi}} = -\vec{\varepsilon}_{\text{gyro}} \end{cases}}$$

Mid-Term Error Model (MTERM)

In North-East-Down axes (NED),

$$\left(\frac{\partial\vec{g}}{\partial\vec{r}}\right)_{NED} \cong \begin{pmatrix} -\omega_s{}^2 & 0 & 0 \\ 0 & -\omega_s{}^2 & 0 \\ 0 & 0 & 2\omega_s{}^2 \end{pmatrix} \quad \textbf{with} \quad \omega_s{}^2 = \frac{g_0}{R}$$

(spherical Earth with radius R)

Error analysis : vertical mid-term behaviour (2)

Vertical divergence

On the local vertical axis (D in NED), MTERM may be further developed as :

$$\boxed{\delta \dot{v}_D = \left(\vec{f} \times \vec{\psi}\right)_D + \left(\vec{\varepsilon}_{\mathrm{acc}}\right)_D + 2\omega_s{}^2 \delta z}$$

This is an exponentially unstable dynamical system, with divergence time

$$\tau = \frac{1}{\omega_s \sqrt{2}} \cong 566\,\mathrm{s}$$

Consequently, when a stable navigation system is needed on the vertical axis, it may not be inertial purely !

Error analysis : horizontal mid-term behaviour (1)

Horizontal errors

On the local horizontal plane of ***NED***, equation MTERM now reads

$$\left\{\begin{array}{rcl}\left(\delta\dot{\vec{v}}\right)_{NE} & = & \vec{f}_{NE}\times\vec{\psi}_D+\vec{f}_D\times\vec{\psi}_{NE} \quad + \quad \left(\vec{\varepsilon}_{\mathrm{acc}}\right)_{NE} \quad - \quad \omega_s{}^2\left(\delta\vec{r}\right)_{NE} \\ \left(\dot{\vec{\psi}}\right)_{NE} & = & -\left(\vec{\varepsilon}_{\mathrm{gyro}}\right)_{NE} \\ \dot{\vec{\psi}}_D & = & -\left(\vec{\varepsilon}_{\mathrm{gyro}}\right)_D\end{array}\right.$$

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Error analysis : horizontal mid-term behaviour (2)

Horizontal errors : simplified model

When assuming horizontal accelerations only,

$$\begin{cases} \left(\delta\dot{\vec{v}}\right)_{NE} = \vec{\gamma}_{NE} \times \vec{\psi}_D - \vec{g} \times \vec{\psi}_{NE} + \left(\vec{\varepsilon}_{\text{acc}}\right)_{NE} - \omega_s^{\ 2} \left(\delta\vec{r}\right)_{NE} \\ \left(\dot{\vec{\psi}}\right)_{NE} = -\left(\vec{\varepsilon}_{\text{gyro}}\right)_{NE} \\ \dot{\vec{\psi}}_D = -\left(\vec{\varepsilon}_{\text{gyro}}\right)_D \end{cases}$$

So :

- The gravitational field is responsible for errors on the horizontal plane when gyros show significant horizontal reference error ($\vec{\psi}_{NE} \neq \vec{0}$)
- Significant errors due to azimut misalignment $\vec{\psi}_D$ occur only if significant horizontal accelerations $\vec{\gamma}_{NE}$ occur.

Error analysis : solving for horizontal errors in mid-term (1)

Now assume constant sensor errors, as seen from the ***EFEC*** coordinate frame

Then :

$$\boxed{\begin{aligned}\vec{\varepsilon}_{\text{gyro}} &= \textbf{stationary}\\ \vec{\varepsilon}_{\text{acc}} &= \textbf{stationary}\end{aligned}}$$

in *EFEC*

- $\vec{\psi}_{NE} = \vec{\varepsilon}_{\text{gyro},\,NE} \cdot t + \vec{\psi}_{NE}^{0}$ and $\vec{\psi}_{D} = \vec{\varepsilon}_{\text{gyro},\,D} \cdot t + \vec{\psi}_{D}^{0}$

Error analysis : solving for horizontal errors in mid-term (2)

(constant sensor errors continued)

- short term contributions of $-\vec{g}\times\vec{\psi}_{NE}$ simply follow

$$\vec{\varepsilon}_{\mathrm{gyro},\,NE}\times\vec{g}\cdot\frac{t^2}{2}+\vec{\psi}^0_{NE}\times\vec{g}\cdot t \quad =\vec{\varepsilon}_{\mathrm{gyro},\,NE}\times\text{position of Newton's apple}$$

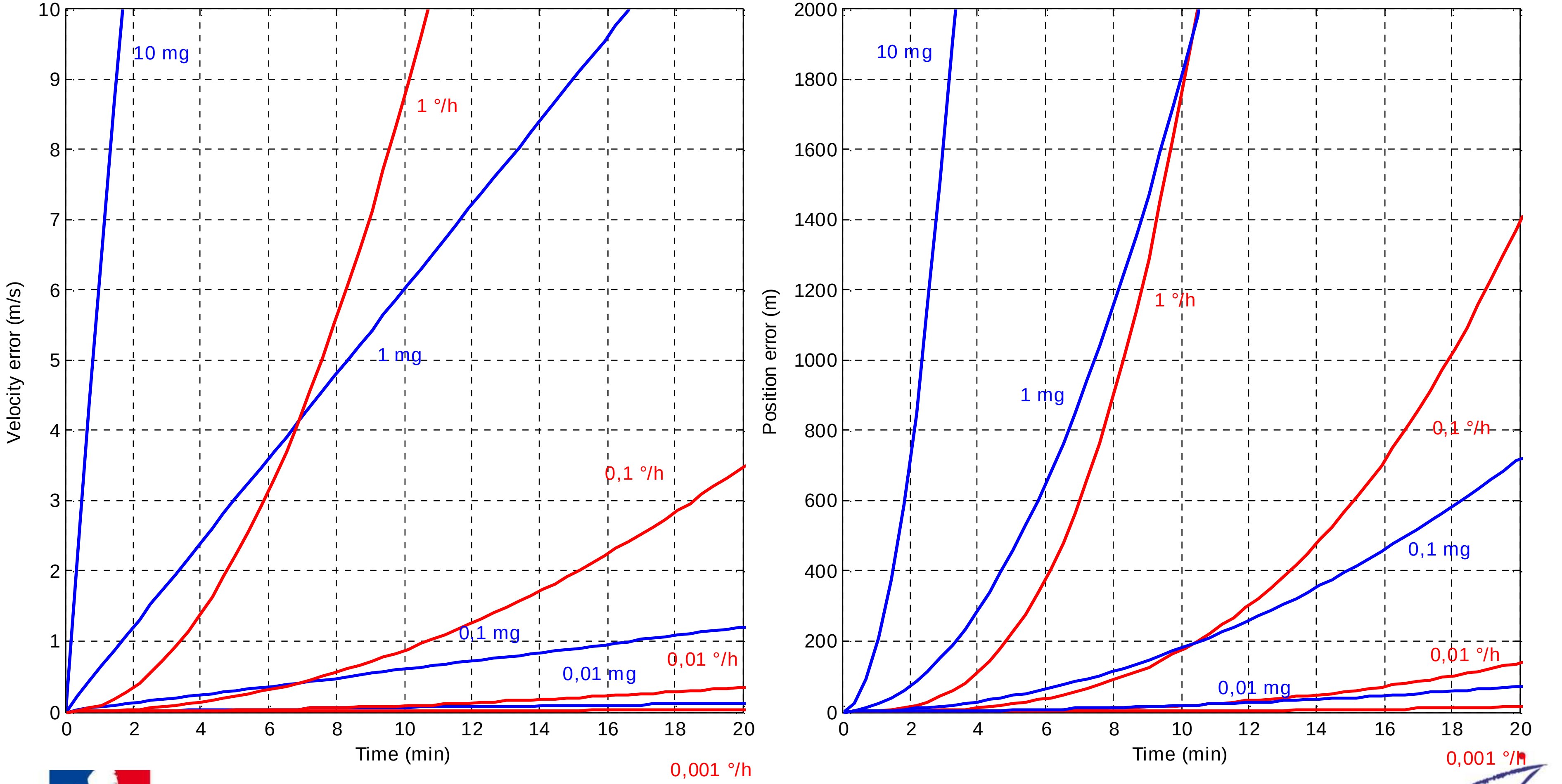
$$+\vec{\psi}^0_{NE}\times\text{speed of Newton's apple !}$$

- short term contributions of $\vec{\varepsilon}_{\mathrm{acc}}$ are linear (!) : $\vec{\varepsilon}_{\mathrm{acc}}\cdot t$

- global, mid-term oscillatory behaviour at fundamental Schüler’s frequency

Error analysis : solving for horizontal errors in short-term (more graphically !)

accelerometer bias or $g \times$ initial alignment error **rate gyro bias**



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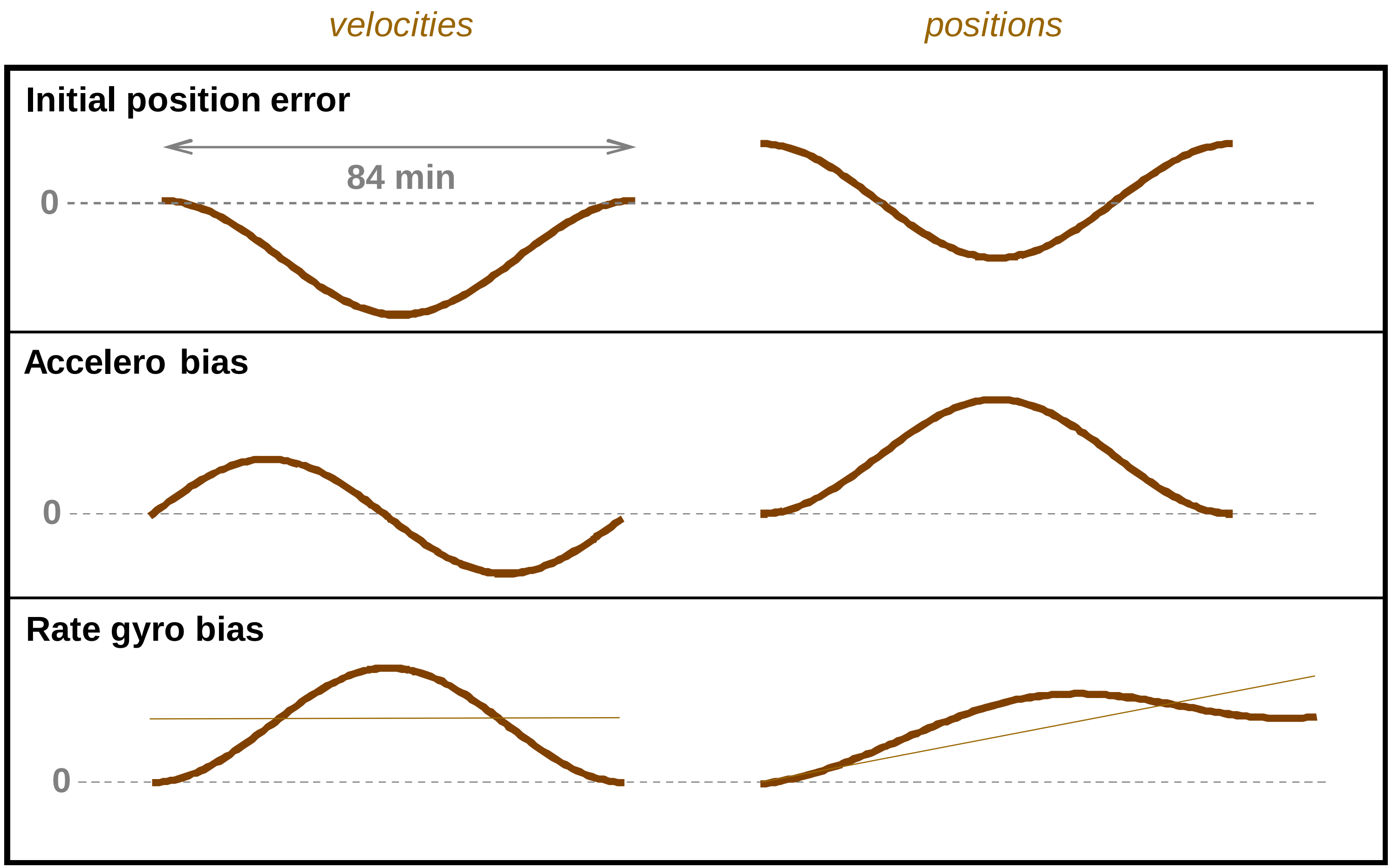
Error analysis : solving for horizontal errors in mid-term (more graphically !)

Mid-term Schüler damping effect



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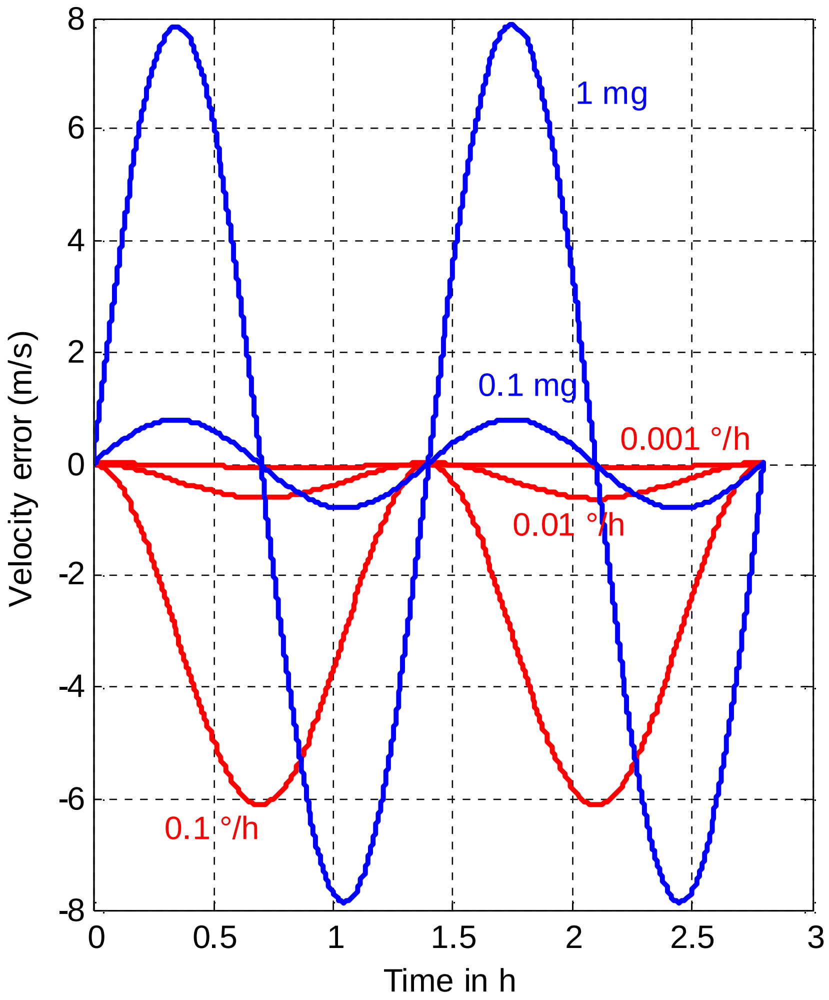
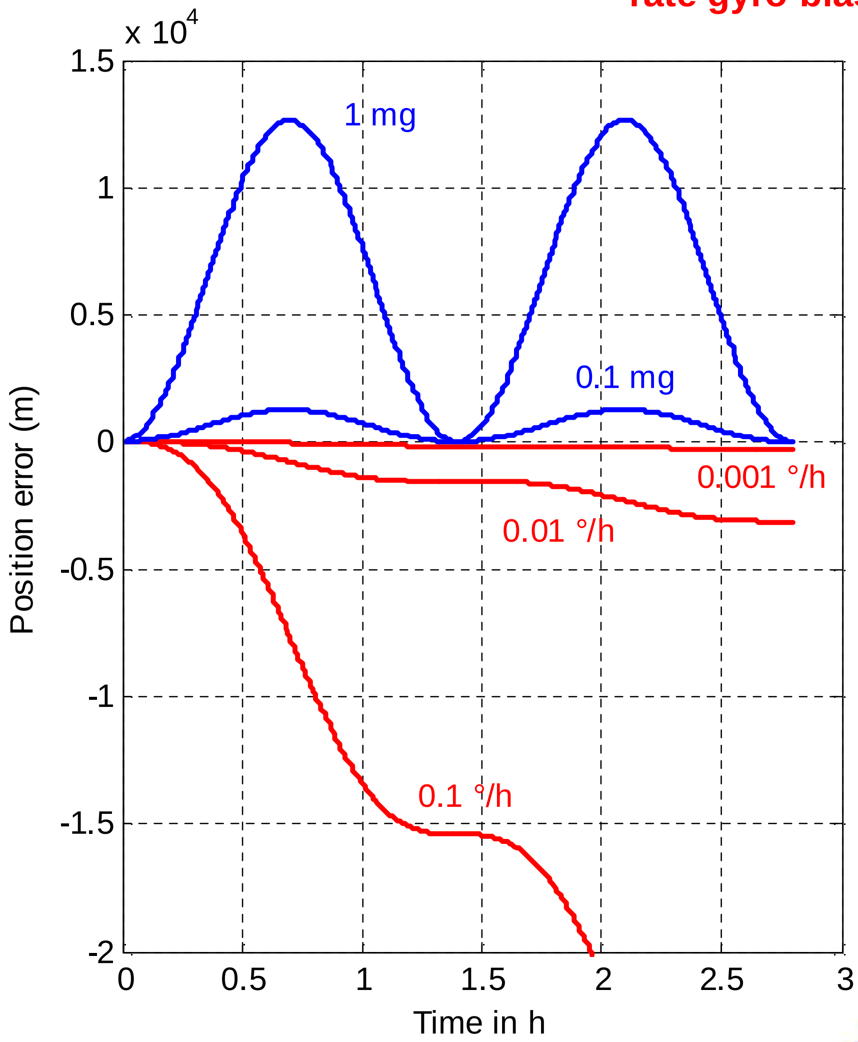
P & V mid-term errors : orders of magnitude

accelerometer bias or $g \times$ initial alignment error



rate gyro bias



Analytical expressions of mid-term velocity errors

It can be shown that velocity errors have a response to gyro drift as follows

$$\begin{aligned}\delta v_N &= -R_{\text{Terre}} \cdot D_W \cdot \left(1 - \cos \omega_S t\right)\\ \delta v_W &= \quad R_{\text{Terre}} \cdot D_N \cdot \left(1 - \cos \omega_S t\right)\end{aligned}$$

Maximum errors are thus

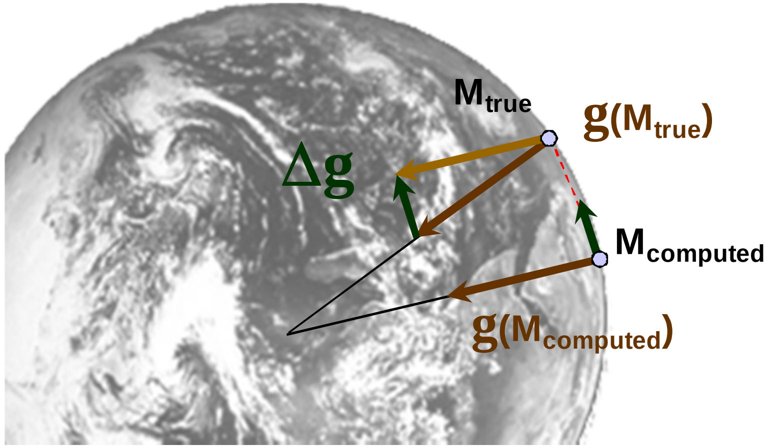
$$\left|\delta v_{Max}\right| = 2 \cdot R_{\text{Terre}} \cdot D_{W,N}$$

Application: consider 0,01°/h on each input axis N, W, U
$\Rightarrow$ 0,01×60' /h = 0,60 '/h = 0,60 Nm/h = 0,60 kts
since 1°= 60' = 60 Nm on the Earth (otherwise stated: R_{Terre} x 1° = 60 Nm)

Schüler stabilizing effect

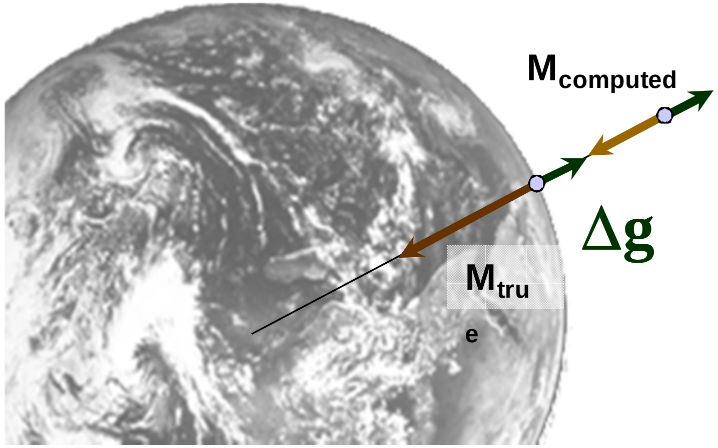
Medium term error propagation :

- Schüler's term $\frac{\partial \vec{g}}{\partial \vec{r}} \Delta \vec{r}$ starts beeing more and more sensitive



The difference between the computed gravitational horizontal component and the true component has an opposite sign to the horizontal position error.

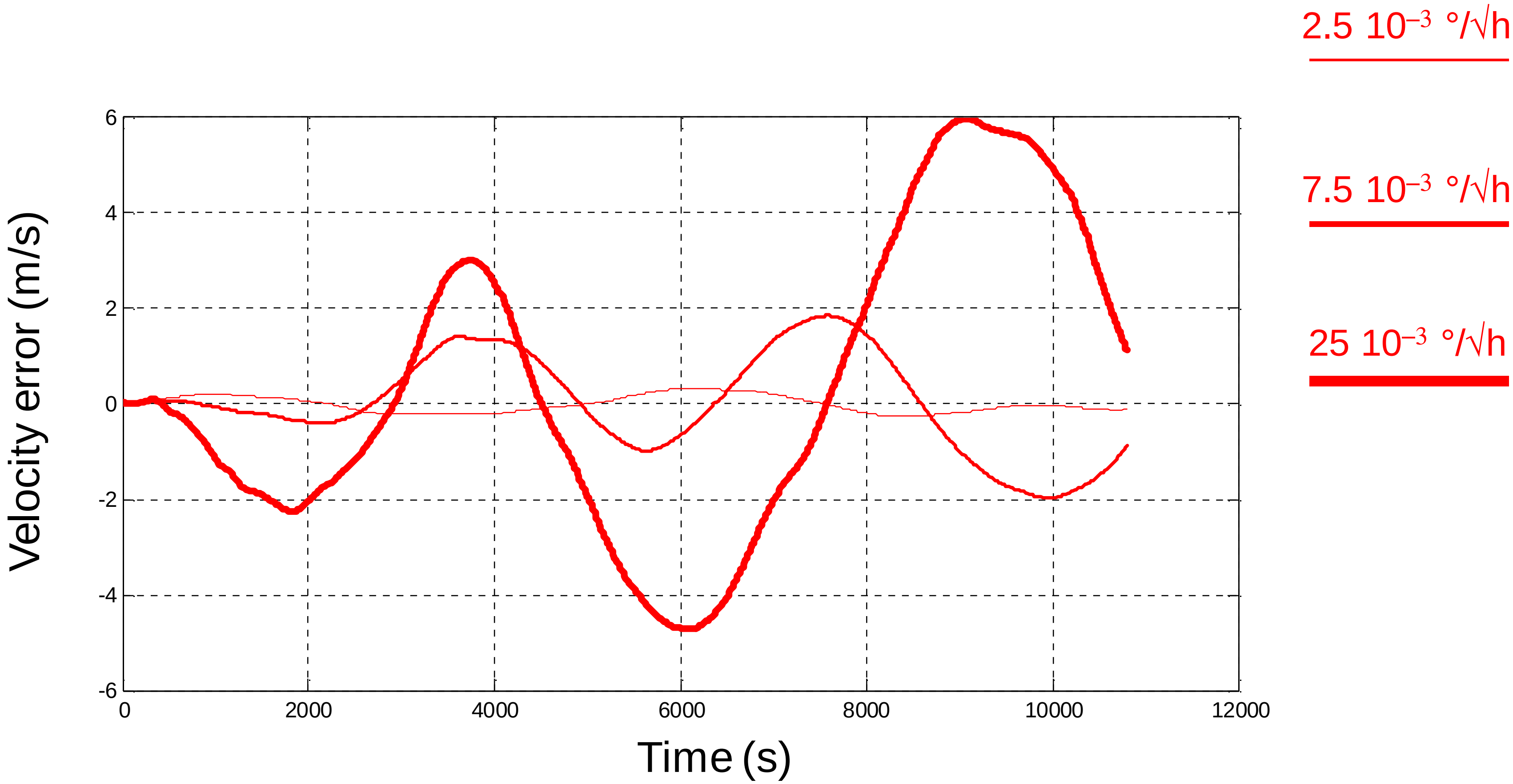
Henceforth the horizontal Schüler component has a stabilizing effect on horizontal error.



The difference between the computed gravitational vertical component and the true component has the same sign as the vertical position error.

Henceforth the vertical Schüler component has a destabilizing effect on vertical error.

Mid-term response of velocity to laser gyro Random Walk



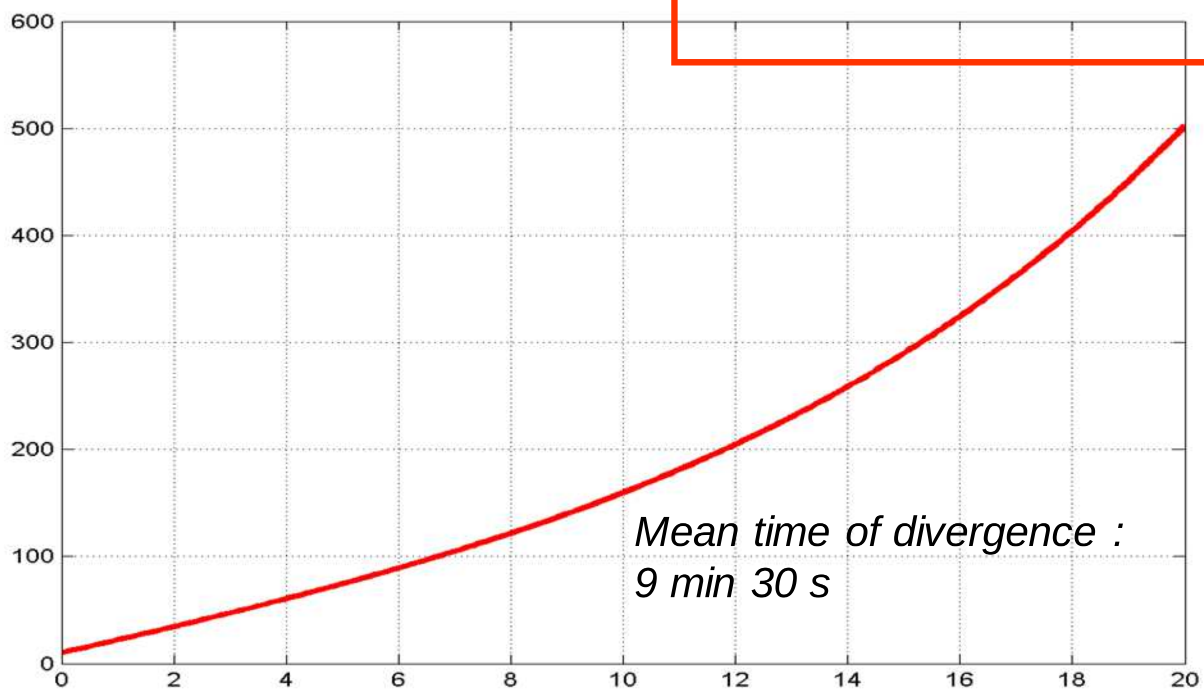
Altitude error

Exponentially divergent altitude errors

Such a divergent nature

- **precludes using of INS as a sole reference for altitude**
- **calls upon vertical hybridization, with GPS and/or pressure sensor**

Altitude error in m



Time in min

Initial errors :
altitude = 10 m
speed = 0,2 $m.s^{-1}$

No sensor error !

Attitude & Heading errors

Error analysis : attitude error (1)

The attitude is defined as the transition matrix from the angular reference frame to the body frame :

$$T_{b/\mathrm{reference}}$$

So for instance, when reference = NED (which is a local frame):

$$\begin{aligned} T_{b/\mathrm{reference}} &= T_{b/NED} \\ &= T_{b/i} \cdot T_{i/NED} \end{aligned}$$

This leads to the actual computation of attitude :

$$T_{b/\mathrm{reference}}{}^{c} = T_{b^c/NED^c}$$

function of computed position

Error analysis : attitude error (2)

With the following conventional notations

$$T^c_{b/i} \cong (\mathrm{Id} - \psi\times)\cdot T_{b/i} \Rightarrow T_{b^c/b} = \mathrm{Id} - \psi\times$$

$$T^c_{NED/i} \cong (\mathrm{Id} - \theta\times)\cdot T_{NED/i} \Rightarrow T_{NED^c/NED} = \mathrm{Id} - \theta\times$$

one can show that

$$T_{b^c/NED^c} \cong (\mathrm{Id} - \varphi\times)\cdot T_{b/NED}$$

Therefore the computed attitude error reads :

$$\boxed{\vec{\psi} = \vec{\theta} + \vec{\varphi}}$$

$$\vec{\varphi} = \vec{\theta} - \vec{\psi}$$

Error analysis : attitude error (3)

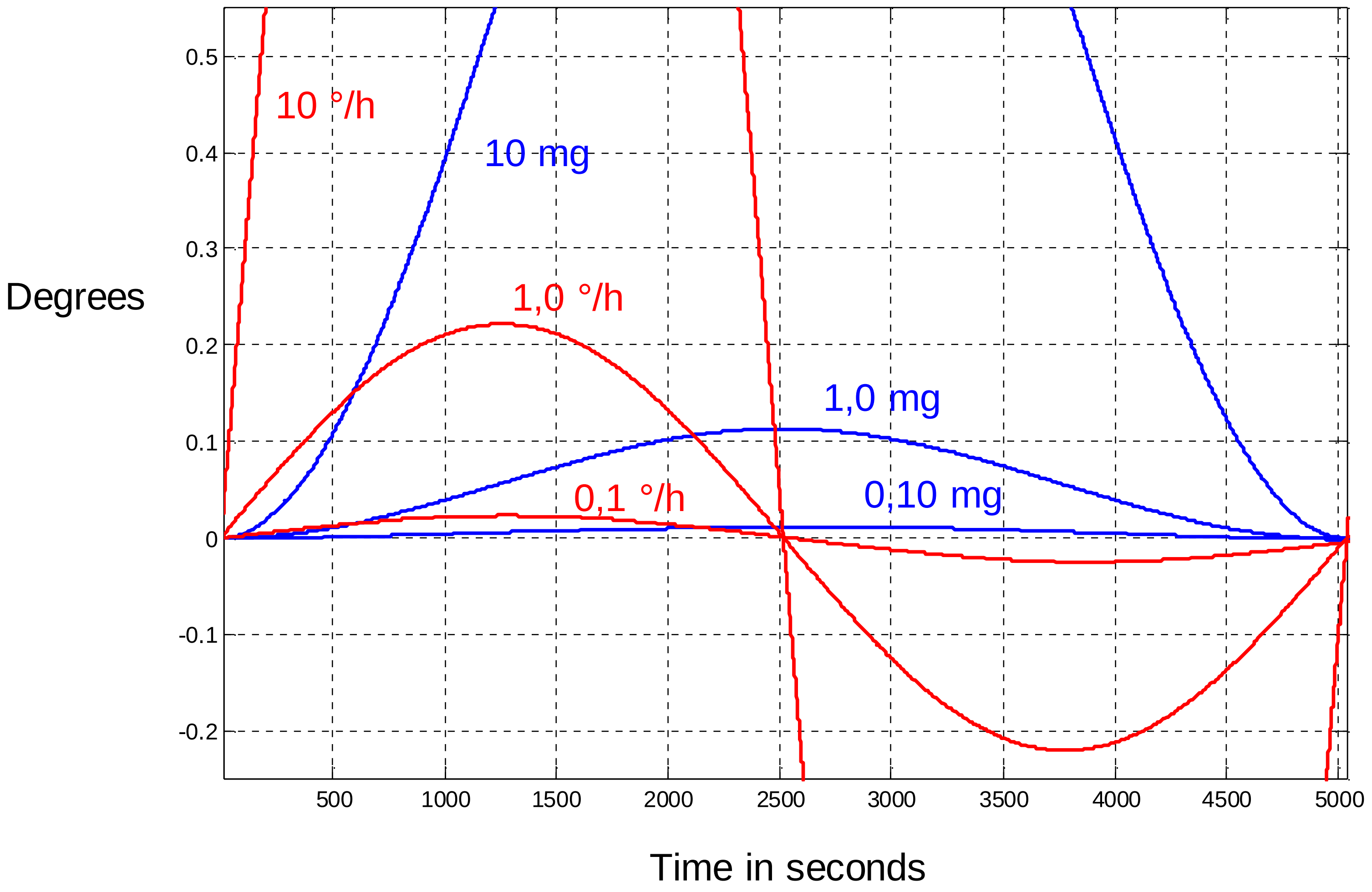
The main result to keep in mind about attitude errors wrt the local geodetic frame (NED) is that horizontal components are pretty much bounded !

- The divergent horizontal position errors $\vec{\theta}$ are compensating for the inertial reference error $\vec{\psi}$: horizontal errors are reflecting inertial reference errors.

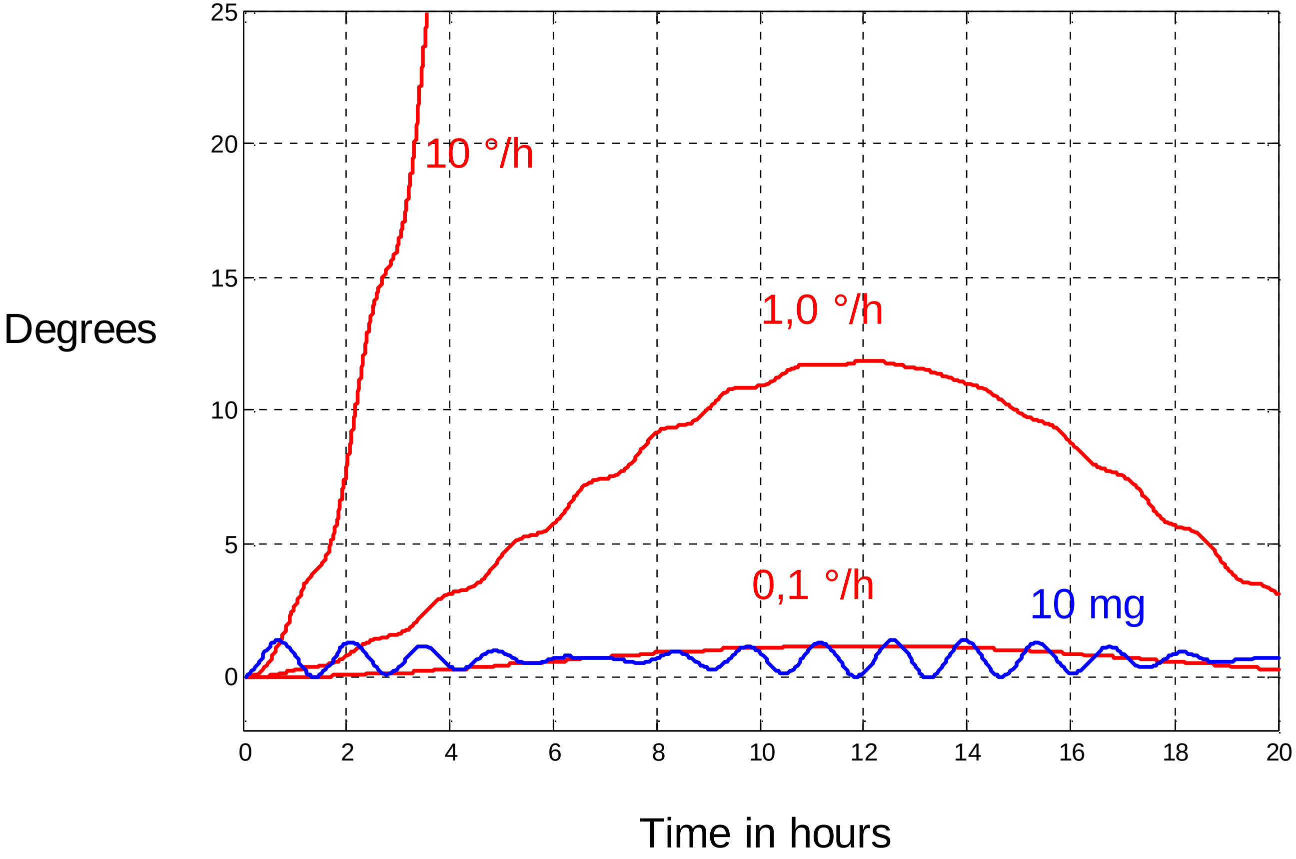
- Horizontal components of attitude errors are damped by gravity through Schüler loops (accelerometers) : since the gravitation field is essentially vertical, the local horizontal plane is the orthogonal to accelerometers' signals. This information is biased when the vehicle is accelerating along the horizontal plane – but it shan't be long term.

Note : **due to Earth's rotation around the North-South axis (so-called « World axis »), $\vec{\psi}$ is roughly spiraling around this axis : hence the equatorial components are bounded, the world component is linearly diverging in time. This property leads to longitude divergence of horizontal errors, that is to say North component of $\vec{\theta}$.**

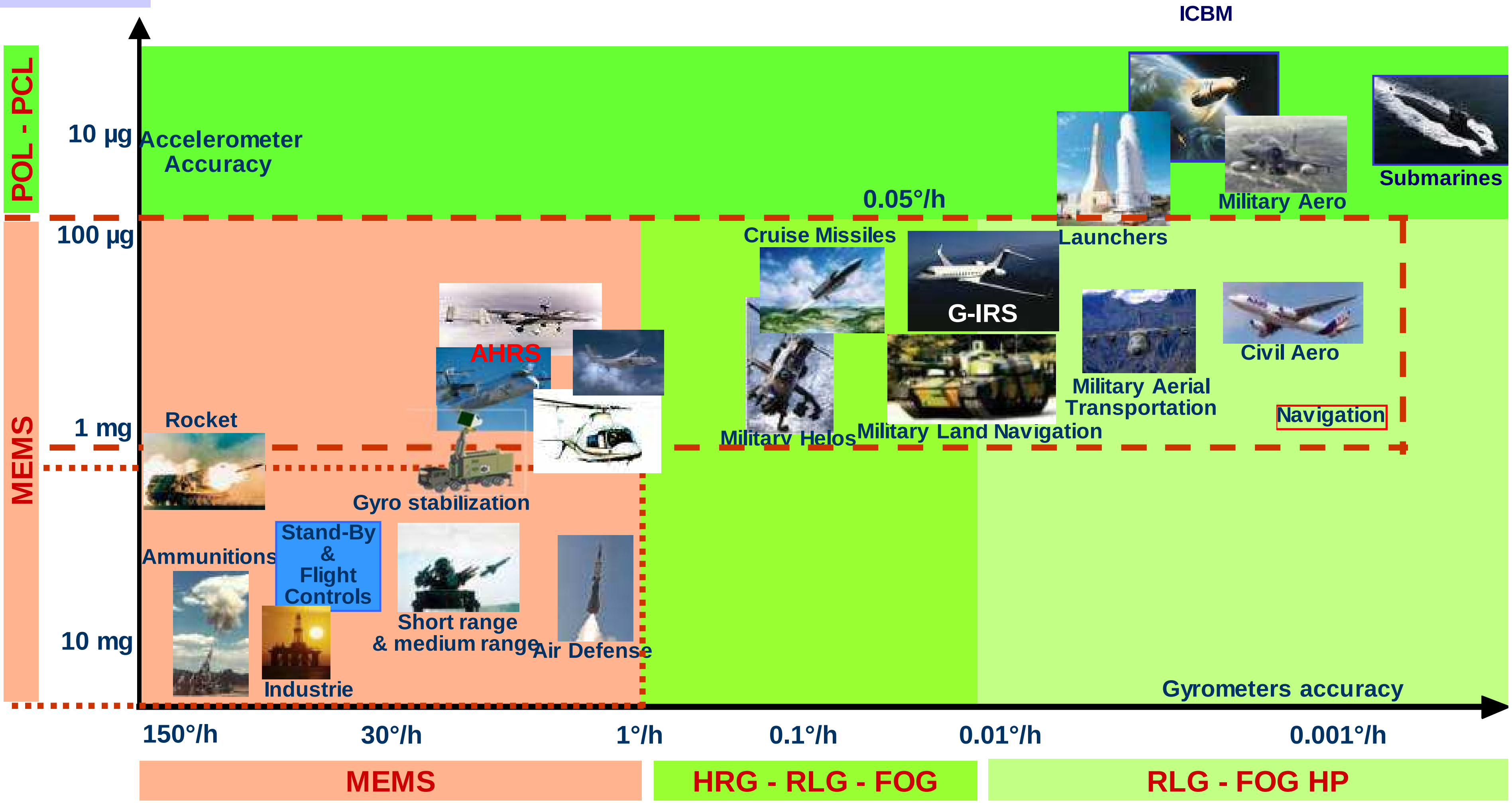
Mid-term vertical errors (roll & pitch)



Mid and long term true heading errors



Inertia Market Segmentation (Precision)



Hybridization

Hybrid technologies (inertial system/*x*)

The hybridization of an inertial navigation system with a non-inertial is necessary to prevent from the vertical divergence (when mid-term navigation is a requirement).

- altitude sensor : GNSS, baro-altimeters, vision, …
- altitude blocking

The hybridization is also a mean to preclude horizontal (velocity) oscillations, using :

- odometers
- GNSS
- zero velocity updates (the user or some sensor raises a flag as soon as the vehicle stops in the environment)

The most popular technique of hybridization is the Kalman filter (and brother filters)

Stochastic hybridization techniques (1)

Kalman filter

- Optimal choice when the dynamics and measurement are linear with gaussian noise, and **perfectly known**
- If the linear dynamics and noises are not perfectly known, a robust tuning of the filter might fit : nevertheless a robust set of parameters may provide ultra-conservative results !

Extended Kalman filter

- Used when dynamics and/or measurement are analytically non-linear
- Poor performance if non-linearities are too strong
- No theoretical guarantee of optimality

Stochastic hybridization techniques (2)

Particle filtering

- Used when dynamics and/or measurement are strongly non-linear
- Particle = (roughly speaking) one simulation of the system at a given initial state. Hence particle filtering makes massively parallel simulations of the real system
  $\rightarrow$ the particle which best explains measurements is selected probabilistically
- No theoretical guarantee of optimality, unless the number of particles is infinite !

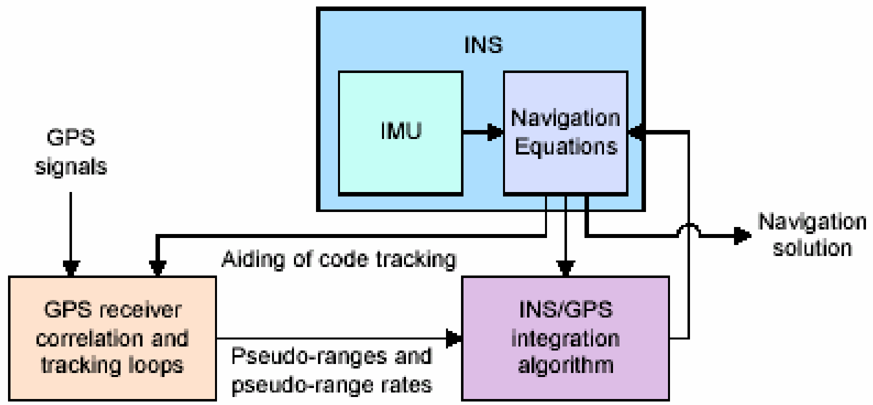
Unscented (or Sigma points) Kalman filter

- Used when dynamics and/or measurement are analytically non-linear
- Tries to inherit nice properties of particle filtering in presence of strong non-linearities : the idea is to extend the exploration space at most, without increasing the number of particles. Hence UKF chooses deterministic values in state space to explore (so-called « Sigma points »).

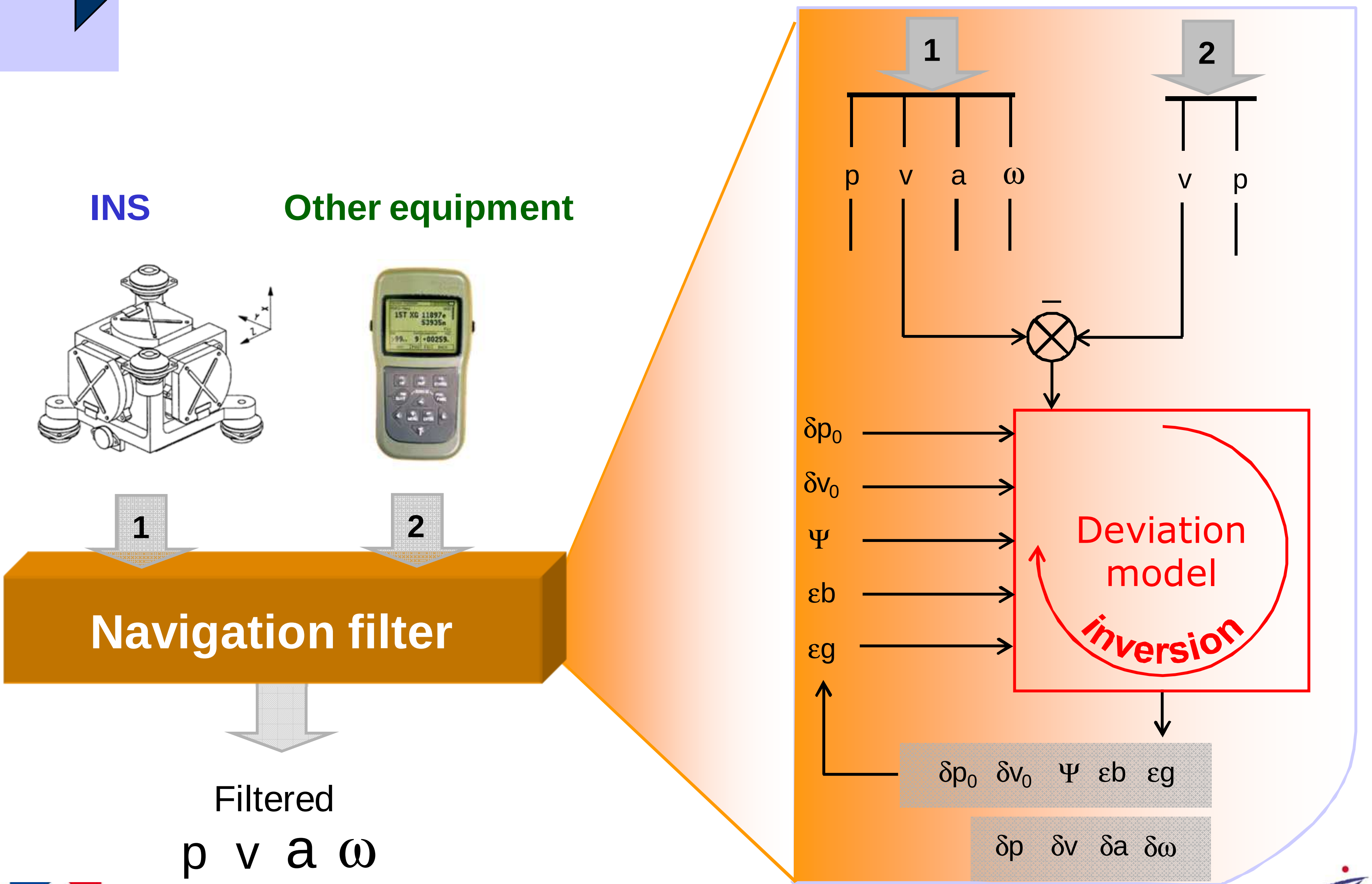
Hybrid technologies (inertial system/x)

Various architectures are practiced: from « loose » to « ultra-tight »

- « loose » = GPS position and/or velocity updates
- « ultra-tight » = « radio-inertial navigation system »



Principle of a navigation filter



Advantages and drawbacks… hybridized !

The hybridization often allows for better accuracy and/or cheaper sensors

- mutual calibration
- « in run calibration »
- fault detection and analysis
- sensor redundancy

Be always aware ! Appropriate modeling of causality (*e.g.* dynamical models) is the fundamental condition to these positive effects of hybridization.

An hybridized system also inherits drawbacks of subsystems

- less availability
- dependancy on American DoD, when hybridizing GPS

INS and GPS : two navigation means that complement each other

Discrete, independent

Non-jammable

Available, reliable, highpass

Universal coverage

Provides whole navigation data

Very accurate over the short term

Fully autonomous

Very expensive

Accuracy is a function of volume and mass

Poor, divergent accuracy over the long term

Operational constraints

Highly jammable

Dependent on the USA

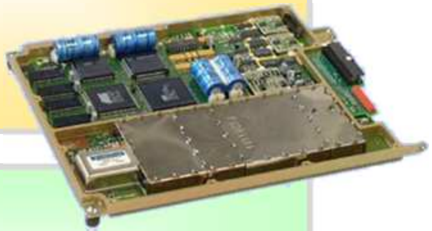
Provides position and velocity only (no angles)

Availability not compatible with high integrity reqs.

Low/no availability in certain zones

Operational constraints in military services (key management)

Noisy over the short term



Very accurate anytime

Free service → very cheap function (civil)

No operational constraint in civil services

Very compact form factor

Hybridization of INS and GPS provides better equilibrium in cost, accuracy, accuracy, dependency, vulnerability

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Hybridization example : INS/GPS (2)

Specific terminology

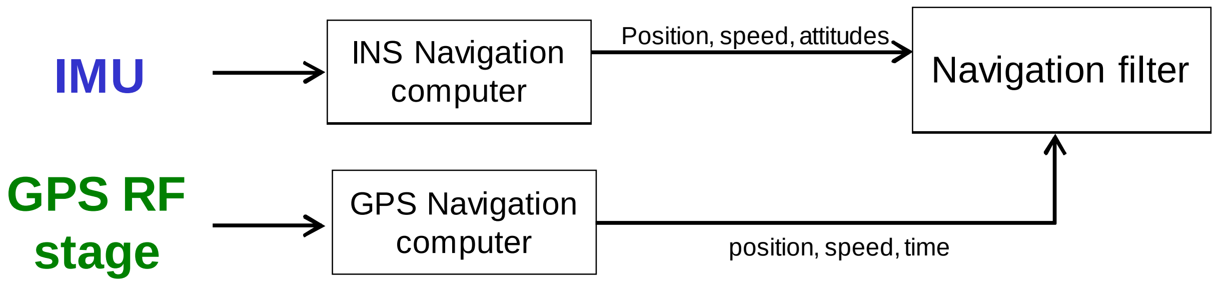
- closed loop : the filter estimates are backwarded to the INS loops (velocity for instance)
- tight coupling : GPS pseudo-ranges are provided by the receiver, as updates to the hybridization filter
- Ultra-tight coupling : updates are INS increments and GPS pseudo-ranges (no separate computation of velocity or position)
- Inertial aid : the GPS receiver stage is provided with inertial informations

Hybridization example : INS/GPS (3)

Loose coupling

- **Advantage :**
  - easy to integrate

- **Drawbacks :**
  - Limited precision when jammed or antenna hidden
  - Cascade of two locally optimal filters $\rightarrow$ suboptimal
  - No GPS navigation with less than 4 satellites
  - low frequency of GPS

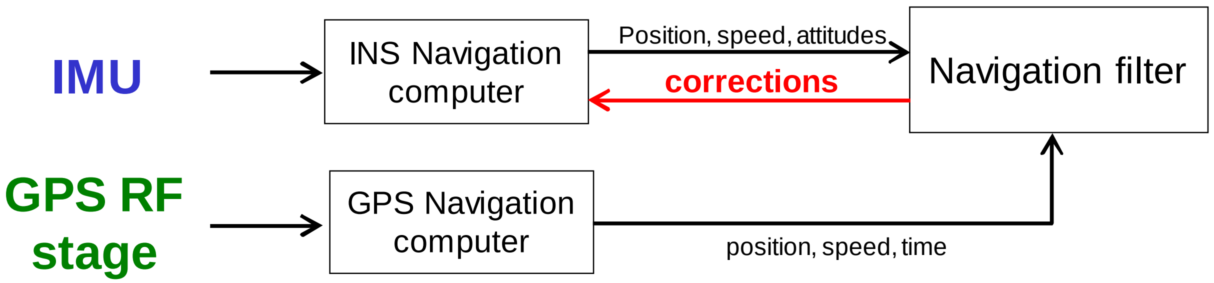


Hybridization example : INS/GPS (4)

Loose INS / GPS with filter loopback

- Advantage :
  - INS errors maintained small $\rightarrow$ linear error models

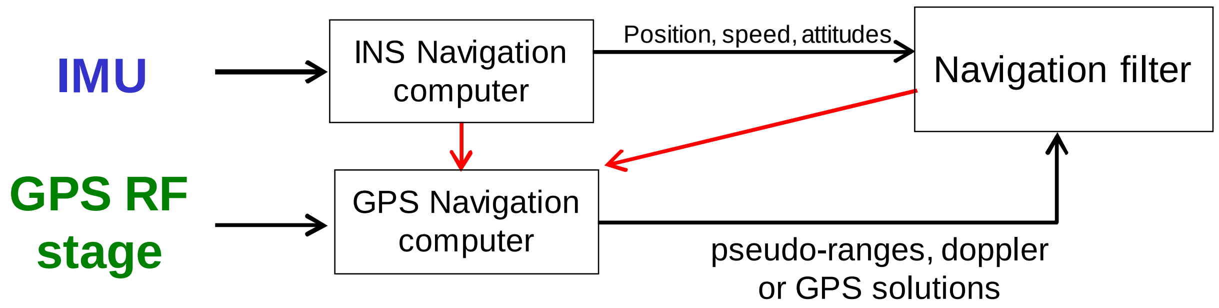
- Drawbacks :
  - possible biased GPS measurements loop back to INS
  - short term noise pollution of the INS



Hybridization example : INS/GPS (5)

Tight INS / GPS coupling with INS or filter aid of GPS

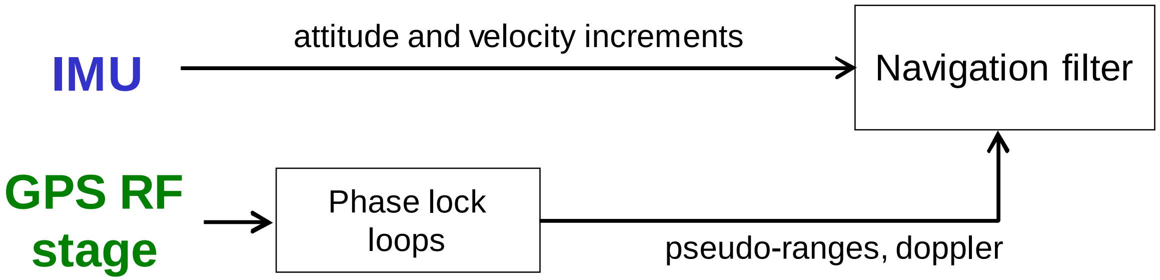
- Advantages :
  - reduced lock-on time of the GPS receiver stage
  - more robust to jamming
  - higher robustness against signal losses due to dynamics
  - higher GPS frequency
- Drawbacks :
  - IMU is necessary
  - robustness against jammers can be deceiptful



Hybridization example : INS/GPS (6)

Ultra-tight INS / GPS coupling

- Advantages :
  - very compact system
  - auto-adaptive GPS signal processing
  - navigation under limit of 4 satellites
  - integrity checks of satellite lines of sight
  - high anti-jamming capacity
- Drawbacks :
  - high technological skills required in the field of GNSS and INS design



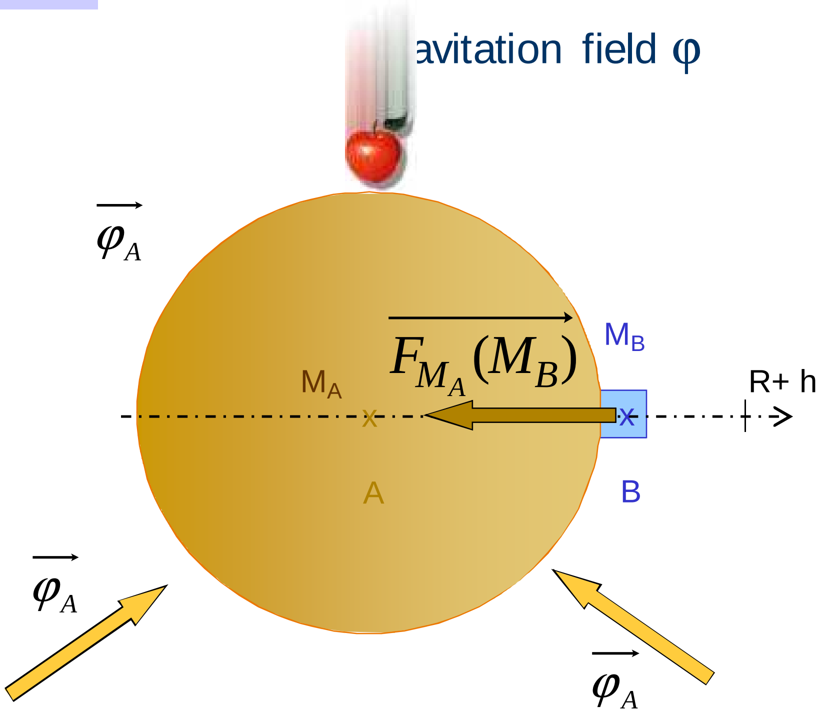
Appendices

French/English terminology

Centrale inertielle de navigation	Inertial navigation system
Cardan	Gimbal
Plate-forme à cardans	Gimbaled platform
Contacts tournants	Slip rings
Thermostat	Thermostat
Thermostatisation	Thermostating
Suspension	Suspension
Suspension antivibratoire	Vibration damping element
Mécanisation	Mechanization
Spire	Coil
Fibre optique	Optical fiber
Odomètre	Odometer
Centrifugeuse	Centrifuge
Calage angulaire	Setting angle
Panne	Fault
Intégrité	Fault tolerance, integrity

Guidage	Guidance
Pilotage	Control
Navigation	Navigation
Localisation	Localization
Missile balistique	Ballistic missile
Spécification	Requirement
Prédiction	Predication phase
Recalage	Update phase
Moindres carrés	Least squares
Hybridation, couplage	Coupling
Couplage lâche, serré, très serré (GPS)	Loose, tight, ultra-tight coupling (speaking of GPS)

Accelerometers - Gravity



$$\vec{\varphi}_A(R+h) = \vec{\varphi}_A(R) \cdot \frac{R^2}{(R+h)^2}$$

$$\vec{\varphi}_A(R+h) \approx \vec{\varphi}_A(R) \cdot (1 - \frac{2h}{R})$$

M_{earth} = 5,98 .10^{24} kg R_{earth} = 6 370 km

φ_{earth}(M at R) ≈ -9.83 N/kg (or -9.83 m/s²)

LRBA (h = 130 m) : 1 g ≈ 9.809370 m/s²
Vernon (h = 30 m) : 1g ≈ 9.809670 m/s²

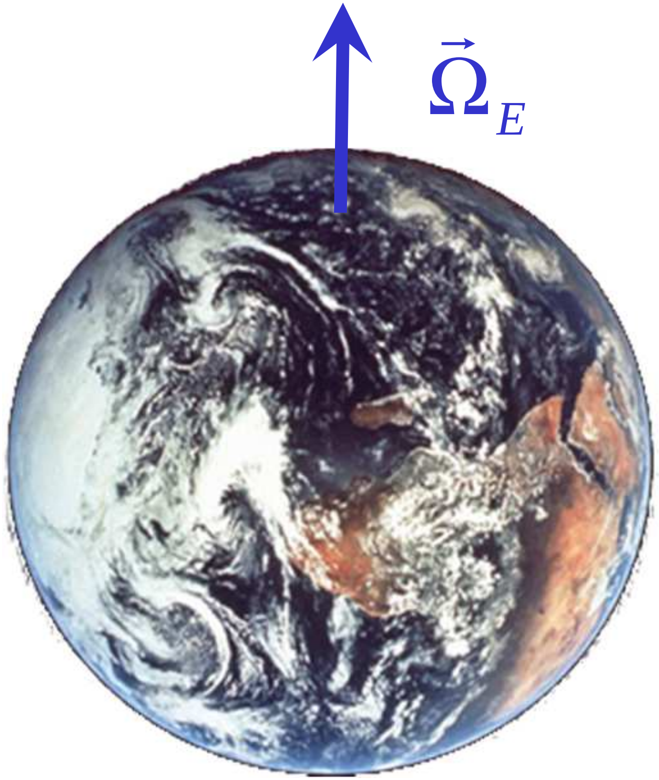
Variation LRBA - Vernon (down town) ≈ 30 µg

Gyros : what is deg.h^{-1} ?

Gyro's main performance expressed in deg.h^{-1}

Earth rotation	= 1 revolution / day
	= 360 deg / 24 hours
	~ 15 deg.h^{-1}

Think of 1 deg.h^{-1} as roughly 1/15 of the earth rotation during 1 hour



Earth rotation rate

15.041 deg.h^{-1}

Sensor performance modeling

Main concepts

Output repeatability

Property lying underneath the performance : ability of the sensor to provide identical output when submitted to identical input and environment

Identical input

Identical environment

Identical output

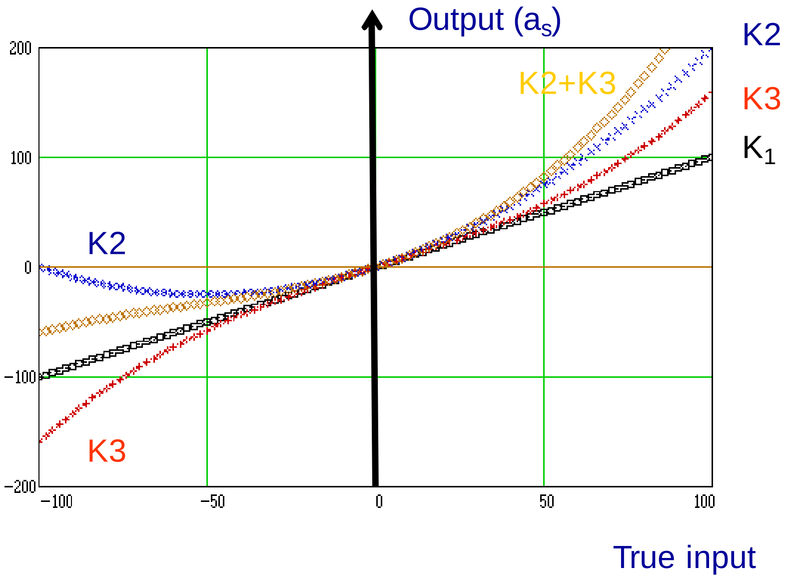
+ Should be a **very simple function of the input** and the environment

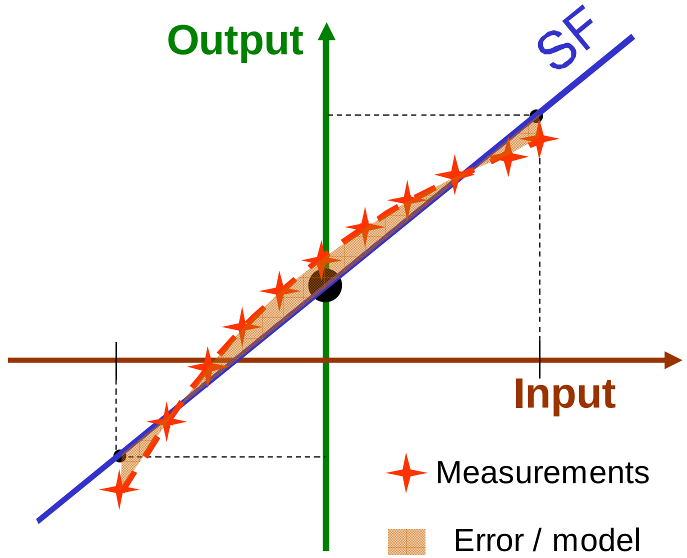
« Ideal sensor » =
• output is purely proportional to input, with permanent scale factor
• output is insensitive to environment

Unitary sensor model

$$S = K_1 \cdot \left\{ K_0 + \left(1 + \frac{\Delta K_1}{K_1}\right) \cdot E + K_2 \cdot E^2 + K_3 \cdot E^3 + \varepsilon \right\}$$

E	**input**
K_0	**offset**
K_1	**first order sensitivity (scale factor)**
K_2	**second order sensitivity**
K_3	**third order sensitivity**
ε	**residuals**





Thermal sensitivity

Repeatability of the parameters K0, K1, K2, K3, … as functions of temperature is modeled with polynomials

Offset/bias

$$K_0 = K_{00} + K_{01}\cdot(T - T_{\text{ref}}) + K_{02}\cdot(T - T_{\text{ref}})^2 + K_{03}\cdot(T - T_{\text{ref}})^3$$

Scale factor

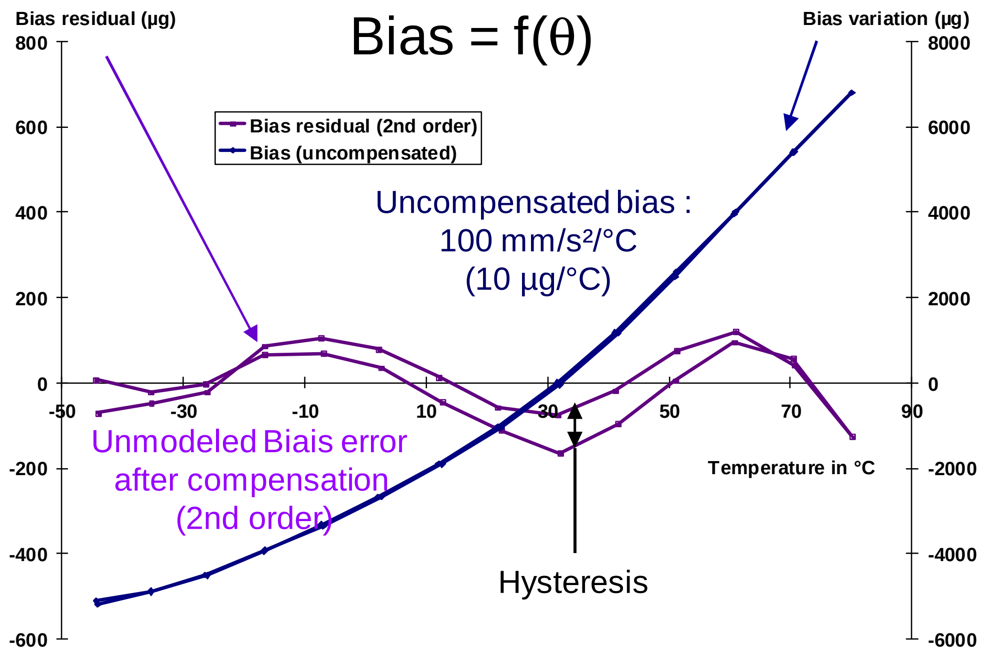
$$K_1 = K_{10} + K_{11}\cdot(T - T_{\text{ref}}) + K_{12}\cdot(T - T_{\text{ref}})^2 + K_{13}\cdot(T - T_{\text{ref}})^3$$

2nd order NL

$$K_2 = K_{20} + K_{21}\cdot(T - T_{\text{ref}})$$

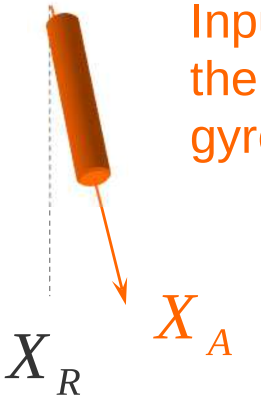
3rd order NL

$$K_3 = K_{30} + K_{31}\cdot(T - T_{\text{ref}})$$

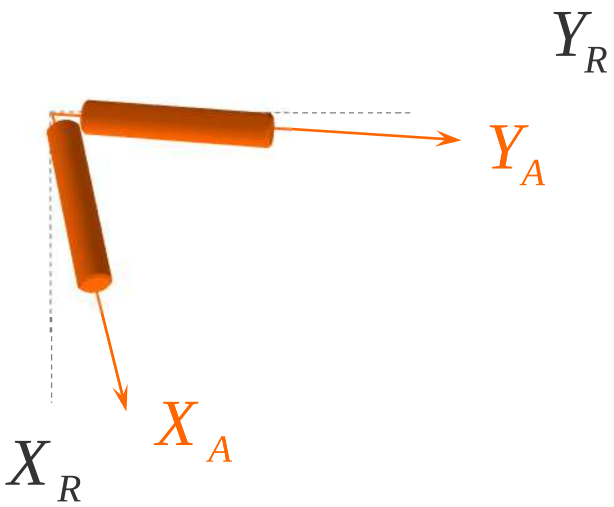


Sensor mounting errors (outside of the packaging)

Input X-axis defined on the packaging of the X-gyro/accelero-meter

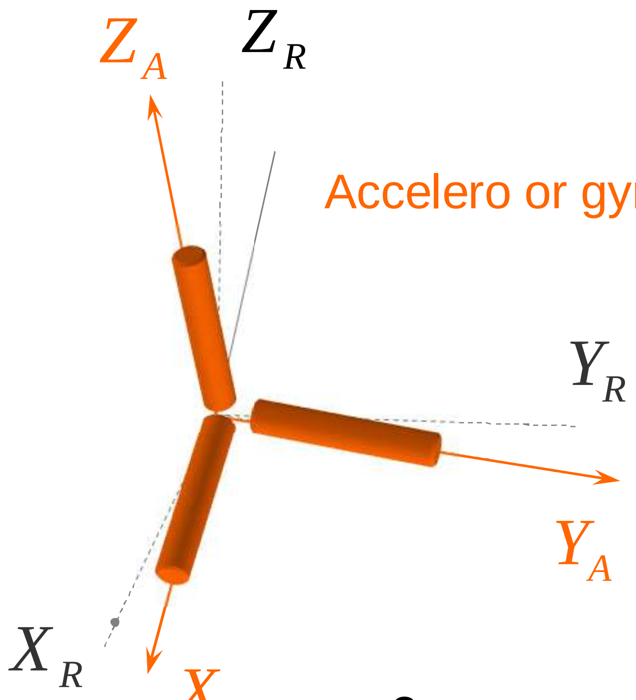


Reference X-axis of the sensor cluster



2 axes - seen from above

Accelero or gyro

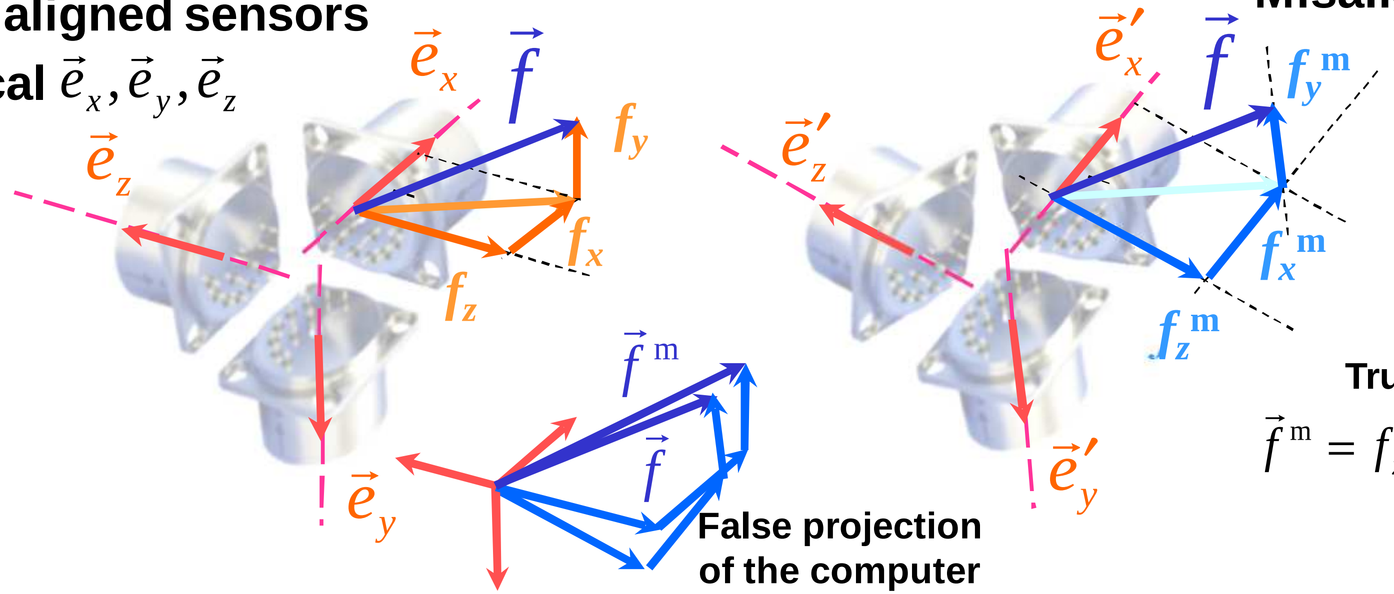


3 axes

3D sensor errors

Perfectly aligned sensors

Theoretical $\vec{e}_x, \vec{e}_y, \vec{e}_z$



Misaligned (=real) sensors

$\vec{e}\,'_x, \vec{e}\,'_y, \vec{e}\,'_z$

True measurement

$$\vec{f}^{\,\mathrm{m}} = f_x \vec{e}\,'_x + f_y \vec{e}\,'_y + f_z \vec{e}\,'_z$$

Measurement equation as seen from the computer, inducing false 3D measurement :

$$\vec{f} = f_x \vec{e}_x + f_y \vec{e}_y + f_z \vec{e}_z$$

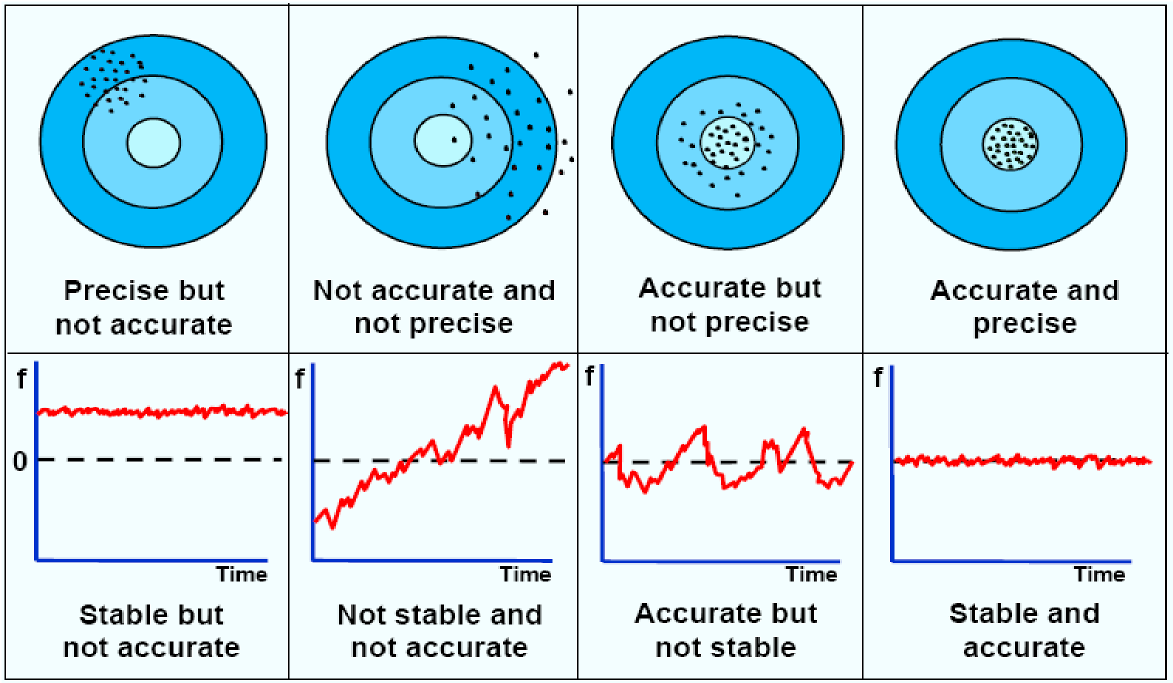
$$\vec{f}^{\,\mathrm{m}} = \overline{\overline{\mathrm{K}}} \cdot \vec{f}$$

$$\overline{\overline{\mathrm{K}}} = \begin{pmatrix} 1 & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & 1 & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & 1 \end{pmatrix}$$

Stability, accuracy, precision

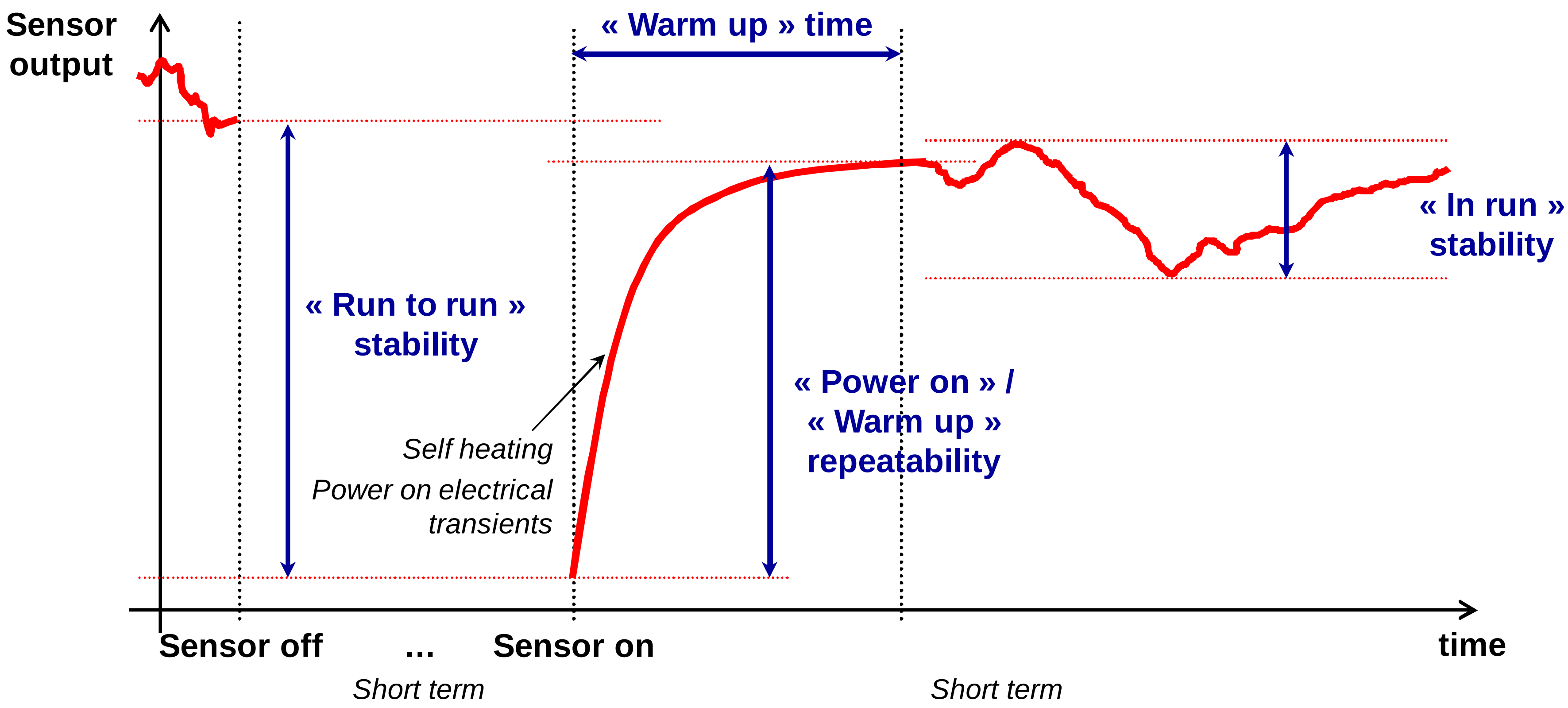
Accuracy, stability, temporal drift

Environment : temperature, in run, vibrations, radiations, pressure, acceleration,…



MINISTÈRE DE LA DÉFENSE ET DES ANCIENS COMBATTANTS

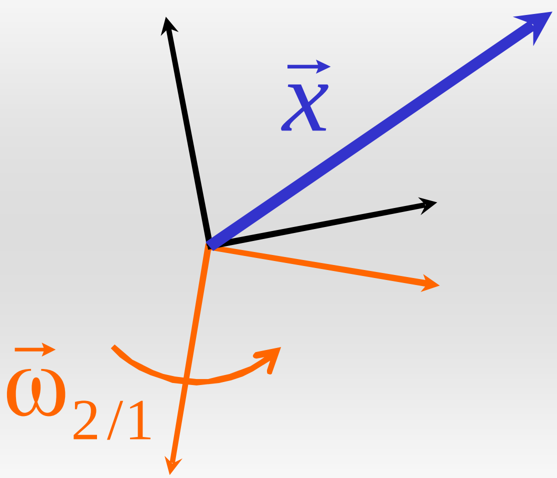
Vocabulary of sensor stability



Note : on IESI, run-to-run does <u>not</u> include power-on & warm-up.

Transform of apparent velocity

Reference frame T_1



Reference frame T_2

First derivative

$$\boxed{\overset{2}{\vec{x}} = \overset{1}{\vec{x}} - \vec{\omega}_{2/1} \times \vec{x}}$$

hence $\overset{2}{\vec{\omega}}_{2/1} = \overset{1}{\vec{\omega}}_{2/1} = \dot{\vec{\omega}}_{2/1}$

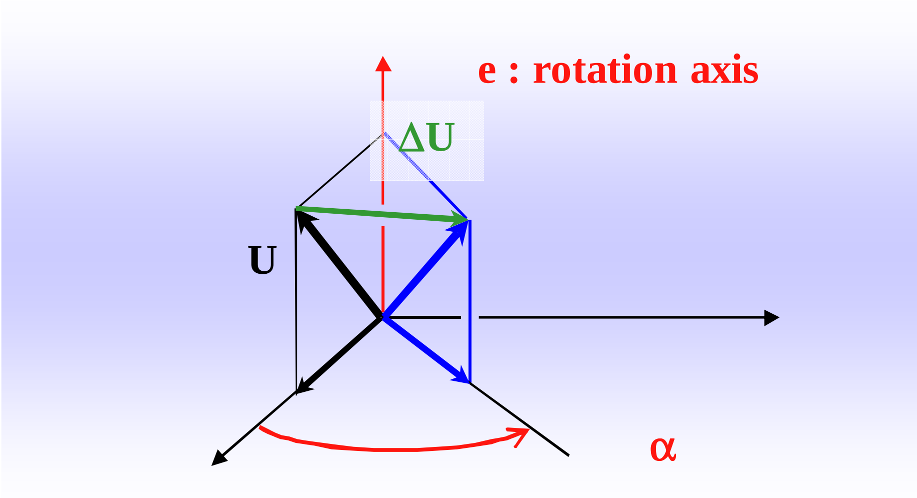
Second derivative

$$\overset{22}{\vec{x}} = \overset{2}{\overline{\left(\overset{1}{\vec{x}} - \vec{\omega}_{2/1} \times \vec{x} \right)}}$$

$$= \overset{1}{\overline{\left(\overset{1}{\vec{x}} - \vec{\omega}_{2/1} \times \vec{x} \right)}} - \vec{\omega}_{2/1} \times \left(\overset{1}{\vec{x}} - \vec{\omega}_{2/1} \times \vec{x} \right)$$

$$= \overset{11}{\vec{x}} - \dot{\vec{\omega}}_{2/1} \times \vec{x} - 2\vec{\omega}_{2/1} \times \overset{1}{\vec{x}} - \vec{\omega}_{2/1} \times \left(\vec{\omega}_{2/1} \times \vec{x} \right)$$

Spectral Representation of a Rotation



For any angle α,

$$\boxed{\mathrm{Rot}(e,\alpha) = I + \sin\alpha\,[e\times\cdot] + (1-\cos\alpha)[e\times\cdot]^2}$$

When α tends to 0, the exact development simplifies to the first order as:

$$\boxed{\mu\mathrm{Rot}(e,\alpha) = I + [\alpha\, e\times\cdot] + O(\alpha^2)}$$

Rotations *vs.* pass matrices

Let (a) and (b) be two orthonormal, clockwise bases.

The pass-matrix from (a) to (b) is denoted $T_{a/b}$. Then the pass-matrix from (a) to (b) is $T_{a/b}^{-1} = T_{b/a}$.

In this context, the rotation operator $R = Rot(e, \alpha)$ transforming (a) into (b), that is to say such that

$$R(a) = (b)$$

Is represented in the base (a) by the matrix:

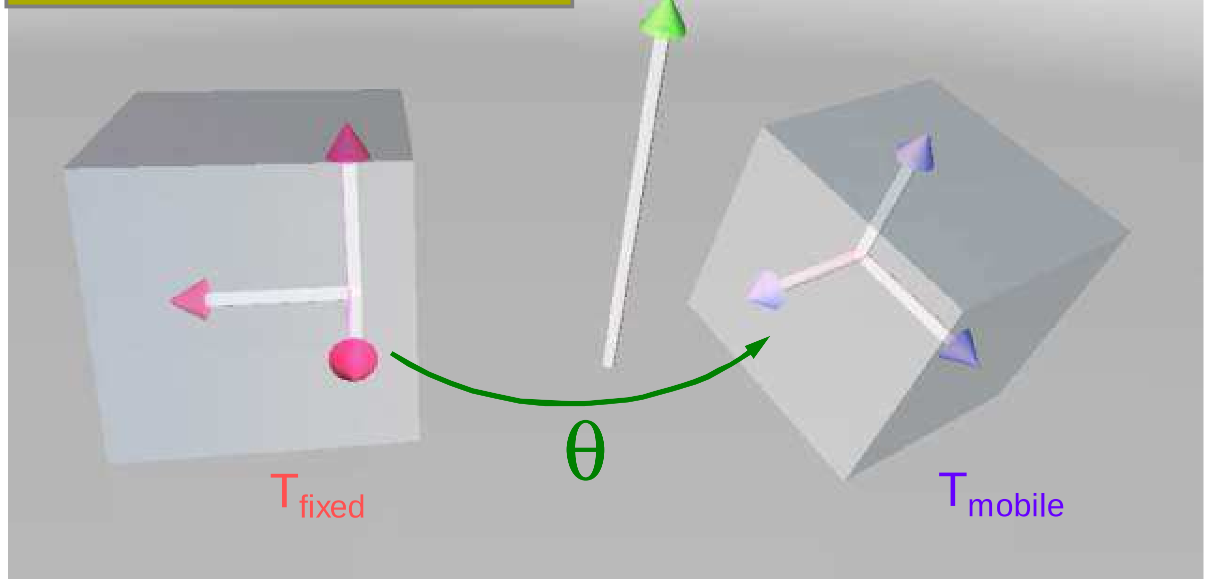
$$\boxed{R \overset{a}{\equiv} T_{a/b}}$$

MINISTÈRE DE LA DÉFENSE ET DES ANCIENS COMBATTANTS

How is the angular orientation computed from the rate gyros signals ?

- Using a mathematical tool known as « attitude quaternion » instead of the more classical rotation matrix.
- The quaternion associated with the rotation with angle θ and unit vector $\mathbf{u}$, that is such that T_{fixed} is transformed into T_{mobile} reads:

$$q = \cos(\theta/2) + \vec{u}\,\sin(\theta/2)$$



Then

- The attitude integration in terms of quaternion space reads :

$$\boxed{\dot{q} = \frac{1}{2} q \cdot \omega}$$

with ω : the (« quaternionized ») rotation vector from the rate gyros measurement.

- This is the theoretical ordinary differential equation, that needs to be numerically integrated, using specific techniques.

Quaternion representation

Using the following notations,

$$\begin{cases} \rho_1 = e_1 \sin\left(\dfrac{\alpha}{2}\right) \\ \rho_2 = e_2 \sin\left(\dfrac{\alpha}{2}\right) \\ \rho_3 = e_3 \sin\left(\dfrac{\alpha}{2}\right) \\ \rho_4 = \cos\left(\dfrac{\alpha}{2}\right) \end{cases}$$

We have :

$$\boxed{T_{a/b} = \begin{pmatrix} \rho_1^2 - \rho_2^2 - \rho_3^2 + \rho_4^2 & 2(\rho_1\rho_2 - \rho_3\rho_4) & 2(\rho_3\rho_1 + \rho_2\rho_4) \\ 2(\rho_1\rho_2 + \rho_3\rho_4) & \rho_1^2 + \rho_2^2 - \rho_3^2 + \rho_4^2 & 2(\rho_2\rho_3 - \rho_1\rho_4) \\ 2(\rho_3\rho_1 - \rho_2\rho_4) & 2(\rho_2\rho_3 + \rho_1\rho_4) & \rho_1^2 - \rho_2^2 + \rho_3^2 + \rho_4^2 \end{pmatrix}}$$

Quaternions' algebra H

Set of real linear combinations of 1, i, j and k, where

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k \quad jk = -kj = i \quad ki = -ik = j$$

$$i^* = -i \quad j^* = -j \quad k^* = -k$$

Isomorphism between H and the direct rotation group

With $\mathrm{T}_{a/b}$ written as the previous slide, we uniquely associate

$$\boxed{\begin{aligned} Q_{a/b} &= \rho_1 i + \rho_2 j + \rho_3 k + \rho_4 \\ &= \cos\frac{\alpha}{2} + u \sin\frac{\alpha}{2} \end{aligned}}$$

denoting $u = i\,\mathrm{e}_1 + j\,\mathrm{e}_2 + k\,\mathrm{e}_3$

Quaternions

Base (a) to base (b) change of coordinates, matrix and quaternion form

$$\boxed{\underset{a}{r} = \mathrm{T}_{a/b}\,\underset{b}{r} \quad \Leftrightarrow \quad \underset{a}{x} = Q_{a/b}\,\underset{b}{x}\,Q_{a/b}{}^{*}}$$

MINISTÈRE DE LA DÉFENSE
ET DES ANCIENS COMBATTANTS

Attitude integration

$$\boxed{\dot{\mathrm{T}}_{\mathrm{a/b}} = \left[-(\omega_{\mathrm{a/b}})_{\mathrm{a}} \times \cdot\right]\mathrm{T}_{\mathrm{a/b}} \quad \Leftrightarrow \quad \dot{Q}_{\mathrm{a/b}} = \frac{1}{2} Q_{\mathrm{a/b}} (\Omega_{\mathrm{a/b}})_{\mathrm{a}}}$$

With the « quaternionized » rotation vector:

$$(\omega_{\mathrm{a/b}})_{\mathrm{a}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \quad \Leftrightarrow \quad (\Omega_{\mathrm{a/b}})_{\mathrm{a}} = \underbrace{\omega_x\, i + \omega_y\, j + \omega_z\, k}_{\text{pure imaginary quaternion!}}$$

Since $Q_{\mathrm{a/b}}$ is a unit quaternion, and $\Omega_{\mathrm{a/b}}$ a pure imaginary quaternion, the integration formula is equivalent to its quaternion conjugate : $\dot{Q}_{\mathrm{b/a}} = -\frac{1}{2} (\Omega_{\mathrm{a/b}})_{\mathrm{a}} Q_{\mathrm{b/a}}$

Quaternion numerical integration (1)

We need to numerically integrate the following attitude quaternion differential equation :

$$\dot{Q}_{a/b} = \frac{1}{2} Q_{a/b} \cdot \left(\Omega_{a/b}\right)_a \qquad (1)$$

with, as previously stated :

A convenient way to see this equation is to consider that a quaternion is a 4-dimensional vector in IR^4 :

$$\left(\omega_{a/b}\right)_a = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \Leftrightarrow \left(\Omega_{a/b}\right)_a = \underbrace{\omega_x\, i + \omega_y\, j + \omega_z\, k}_{\text{pure imaginary quaternion!}}$$

Then (1) is equivalent to the vector integration :

with

$$Q_{a/b} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

q_0 real part

q_1, q_2, q_3 imaginary parts

$$\dot{Q}_{a/b} = \frac{1}{2} \cdot M \cdot Q_{a/b}$$

$$M = \begin{pmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{pmatrix}$$

Quaternion numerical integration (2)

We can solve this last equation between t_{k-1} and t_k as:

$$Q_{a/b}(t_k) = \exp\left(\frac{1}{2}\int_{t_{k-1}}^{t_k} M\right) \cdot Q_{a/b}(t_{k-1})$$

provided that the vector $\vec{\omega}_{a/b}$ is constant between $t_{k\text{-}1}$ and t_k

$$A = \int_{t_{k-1}}^{t_k} M = \begin{pmatrix} 0 & -\alpha_x & -\alpha_y & -\alpha_z \\ \alpha_x & 0 & \alpha_z & -\alpha_y \\ \alpha_y & -\alpha_z & 0 & \alpha_x \\ \alpha_z & \alpha_y & -\alpha_x & 0 \end{pmatrix}$$ where $\alpha_{x,y,z} = \int_{t_{k-1}}^{t_k} \omega_{x,y,z}$ are the (debiased) angular increments

When the time step is small enough, angular increments are very small, and so is the argument of the $\exp$. Then:

BTW, note that $\exp\left(\frac{1}{2}\int_{t_{k-1}}^{t_k} M\right) \approx \mathrm{Id}_4 + \frac{1}{2}\int_{t_{k-1}}^{t_k} M$

$$A = \begin{pmatrix} 0 & -\alpha_x & -\alpha_y & -\alpha_z \\ \alpha_x & 0 & \alpha_z & -\alpha_y \\ \alpha_y & -\alpha_z & 0 & \alpha_x \\ \alpha_z & \alpha_y & -\alpha_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha^T \\ \alpha & [\alpha\times.] \end{pmatrix}$$ where $\alpha = \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix}$

Quaternion numerical integration (3)

Typical numerical schemes

- in practice, more accurate approximations of the previous discrete equation may be derived
- a popular one :

$$Q_k = \underbrace{Q_{k-1} + \frac{1}{2} A \cdot Q_{k-1}}_{\text{propagation}} \underbrace{- \frac{1}{8} \cdot \|\alpha\|_2^{\,2} \cdot \left(Q_{k-1} + \frac{1}{6} A \cdot Q_{k-1} \right)}_{\text{corrections}}$$

where $Q_k = Q_{a/b}(t_k) \qquad Q_{k-1} = Q_{a/b}(t_{k-1})$

$\alpha_{x,y,z} = \int_{t_{k-1}}^{t_k} \omega_{x,y,z}$ are the (debiased) angular increments

$$A = \begin{pmatrix} 0 & -\alpha_x & -\alpha_y & -\alpha_z \\ \alpha_x & 0 & \alpha_z & -\alpha_y \\ \alpha_y & -\alpha_z & 0 & \alpha_x \\ \alpha_z & \alpha_y & -\alpha_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha^T \\ \alpha & [\alpha \times .] \end{pmatrix}$$

$$\|v\|_2^{\,2} = v_1^{\,2} + v_2^{\,2} + v_3^{\,2}$$

Check [Titterton] pages 319-324 for more details

Wilcox integration of quaternion

Second order

$$Q_k = Q_{k-1} + \frac{1}{2} A \cdot Q_{k-1} - \frac{1}{8} \cdot \|\alpha\|^2 \cdot Q_{k-1}$$

Third order

$$Q_k = Q_{k-1} + \frac{1}{2} A \cdot Q_{k-1} - \frac{1}{8} \cdot \|\alpha\|^2 \cdot \left(Q_{k-1} + \frac{1}{6} A \cdot Q_{k-1} \right)$$

Forth order

$$Q_k = Q_{k-1} + \frac{1}{2} A \cdot Q_{k-1} - \frac{1}{8} \cdot \|\alpha\|^2 \cdot \left(Q_{k-1} + \frac{1}{6} A \cdot Q_{k-1} \right) + \frac{1}{384} \cdot \|\alpha\|^4 \cdot \left(Q_{k-1} + \frac{1}{10} A \cdot Q_{k-1} \right)$$

Strategies to reduce computation load when under coning motion

Any of the previous numerical schemes can lead to apparent drift errors (when coning motion exists) :

- finite angular resolution of the RLG
- superactivation
- high frequency vibrations vs frequency of digitial computation
- digital signal processing stages inside the angular integration loop

When integrated at very high frequency.

Classes of numerical schemes (Miller, Bortz) are intended to allow (rather) low frequency computation while combining high frequency « correction » terms, to circumvent virtual drifts.

Elements of proof

Recall that for a square matrix X, $\exp(X) = \sum_{p=0}^{+\infty} \frac{1}{p!} X^p$

Observe that for $A = \begin{pmatrix} 0 & -\alpha^T \\ \alpha & [\alpha\times.] \end{pmatrix}$ we have $A^2 = \begin{pmatrix} -\|\alpha\|^2 & 0_{1\times3} \\ 0_{3\times1} & -\alpha\cdot\alpha^T + [\alpha\times.][\alpha\times.] \end{pmatrix}$ and $[\alpha\times.][\alpha\times.] = \alpha\cdot\alpha^T - \|\alpha\|^2\cdot I_3$
so A^2 simplifies to

$$A^2 = -\|\alpha\|^2 \cdot I_4$$

further leading to an exact expression of any power of A :

$$\forall\ p \geq 0 \qquad A^{2p} = (-1)^p \|\alpha\|^{2p} \cdot I_4 \qquad A^{2p+1} = (-1)^p \|\alpha\|^{2p} \cdot A$$

The remarkable consequence for the exp is that it has the following closed form:

$$\exp\left(\frac{A}{2}\right) = \cos\frac{\|\alpha\|}{2} \cdot I_4 + \frac{\sin\frac{\|\alpha\|}{2}}{\|\alpha\|} \cdot A$$

Taylor expansions of cos and sine to the 3rd order give the numerical scheme of previous slide.

$$\boxed{Q_k = \cos\frac{\|\alpha\|}{2} \cdot Q_{k-1} + \sin\frac{\|\alpha\|}{2} \cdot \frac{A}{\|\alpha\|} \cdot Q_{k-1}}$$