Deep Learning for Image Analysis 2024: Assignment 1

Adapted from Niklas Wahlström

March 21, 2024

Due: April 8, 2024

In this first assignment, you will implement a linear regression model by using gradient descent.[1] In the next assignment, we will extend this code to an implementation of a full neural network.

Tips

- **Clean code**: Keep your code structured and tidy, suitably divided into functions. The same code will provide a good start for the second hand-in assignment.
- **Vectorized code**: Do use NumPy's features such as vectorization and broadcasting `https://numpy.org/doc/stable/user/basics.broadcasting.html` for *clean* **and** *fast* code.
- **Debugging**: If something doesn't work, try to isolate the problem. Comment out code that possibly have bugs, do plenty of printouts and plots. Simplify the task until it works and then work from there. Try with a tiny dataset which you can check manually. Work in small iterations.
- **One-sample training**: Can the network converge if you only have one sample in the trainset?

Linear regression with gradient descent

Consider a dataset $\{\mathbf{x}_i, y_i\}_{i=1}^n$. Each input is a vector $\mathbf{x}_i = [x_{i1}, \dots, x_{ip}]^\mathsf{T}$ and each output $y_i \in \mathbb{R}$ is a scalar. We want to find a model for the output using linear regression. For one data point $i \in \{1, \dots, n\}$, the linear regression model can be expressed as

$$z_i = \sum_{j=1}^{p} w_j x_{ij} + b = \mathbf{w}^\mathsf{T}\mathbf{x}_i + b, \tag{1a}$$

where the weight vector $\mathbf{w} = [w_1, \dots, w_p]^\mathsf{T}$ and the offset b are the parameters of the model. The cost J is computed by summing the following loss L_i (mean squared error, MSE) over all training data points

$$L_i = (y_i - z_i)^2, \tag{1b}$$

$$J = \frac{1}{n}\sum_{i=1}^{n} L_i. \tag{1c}$$

To train this model with gradient descent, we need access to the gradient of the cost function with respect to the model parameters, i.e. the partial derivatives $\frac{\partial J}{\partial w_1}, \dots, \frac{\partial J}{\partial w_p}$, and $\frac{\partial J}{\partial b}$, which you will derive below.

[1]A linear regression problem can be also solved analytically using the normal equations. However, for the purpose of it being an initial code that you can extend in following assignments, you should solve it here using gradient descent.

Exercise 1. Partial derivatives:

Given a fixed dataset $\{\mathbf{x}_i, y_i\}_{i=1}^n$, based on the model in (1)*, derive expressions for*

$$\frac{\partial J}{\partial b} \quad \text{and} \quad \frac{\partial J}{\partial w_j} \quad \text{expressed in terms of} \quad \frac{\partial J}{\partial z_i}, \quad \frac{\partial z_i}{\partial b}, \quad \text{and} \quad \frac{\partial z_i}{\partial w_j}. \tag{2}$$

Note: The "In terms of" here means that the answer should only include these three terms, no other variables.

Exercise 2. Derivations:

Based on the model in (1)*, derive expressions for (the above used variables)*

$$\frac{\partial J}{\partial z_i}, \quad \frac{\partial z_i}{\partial b}, \quad \text{and} \quad \frac{\partial z_i}{\partial w_j}. \tag{3}$$

Exercise 3. Implement linear regression using gradient descent:

Implement in python, using `numpy`*, a gradient descent algorithm (Lindholm et al. 2021, Chp 5.4) that minimizes the cost J, with respect to the parameters $\mathbf{w}, b$, for the training data set $\{\mathbf{x}_i, y_i\}_{i=1}^n$. It should involve the following functionalities:*

- **Initialize.** *Write a function* `initialize_parameters` *that initializes the parameters $\mathbf{w}, b$. For least squares linear regression (being a convex problem) it is sufficient to initialize all parameters with zeros.*
- **Forward propagation.** *Write a function* `model_forward` *that evaluates the model shown in Equation 1a.*
- **Compute cost.** *Write a function* `compute_cost` *that computes the cost J according to the the mathematical expression in Equation 1c.*
- **Compute gradients.** *Write a function* `model_backward` *that computes the gradient of the cost function with respect to all parameters $\frac{\partial J}{\partial b}$ and $\frac{\partial J}{\partial \mathbf{w}}$. For this, you need the mathematical expression in Equations 2 and 3.*
- **Take a step.** *Write a function* `update_parameters` *that updates the parameters based on provided gradients. Scale the update step by the* learning rate *γ, which will determine the size of the steps you take in each iteration.*
- **Predict.** *Write a function* `predict` *which, using the trained model, predicts z_i based on the corresponding input $\mathbf{x}_i$. This function will in this task more or less only be a wrapper for* `model_forward`.

Your final function `train_linear_model` *should minimize the loss on the provided data using gradient descent. This function will iteratively call the above functions that you have defined. It should have as inputs the training data x_i, their labels y_i, the number of iterations and the learning rate γ. It should return the optimized parameters of the model. It is also recommended that you save and return the cost at each iteration in a vector.*

Note: The above function names are suggestions, you are allowed to structure your code in another way if you feel that it suits you more.

Exercise 4. Evaluate your linear regression model

Evaluate the model on the provided `Auto.csv` *dataset. You may use the supplied Python code* `load_auto.py` *for loading the data if you wish. Treat* `mpg` *(miles per gallon) as your output y_i and consider two different choices for your input $\mathbf{x}_i$: (i) only the input variable* `horsepower`*, and (ii) all input variables except* `name`.

- ***Pre-processing.*** *As pre-processing steps to construct your dataset $\{\mathbf{x}_i, y_i\}_{i=1}^n$ you need to*
  1. *Remove all rows with* `NA` *values*[2]*.*

[2]Already done if you use the provided loading code.

2. *Normalize the input variables, for example by subtracting the mean and dividing with the standard deviation.*

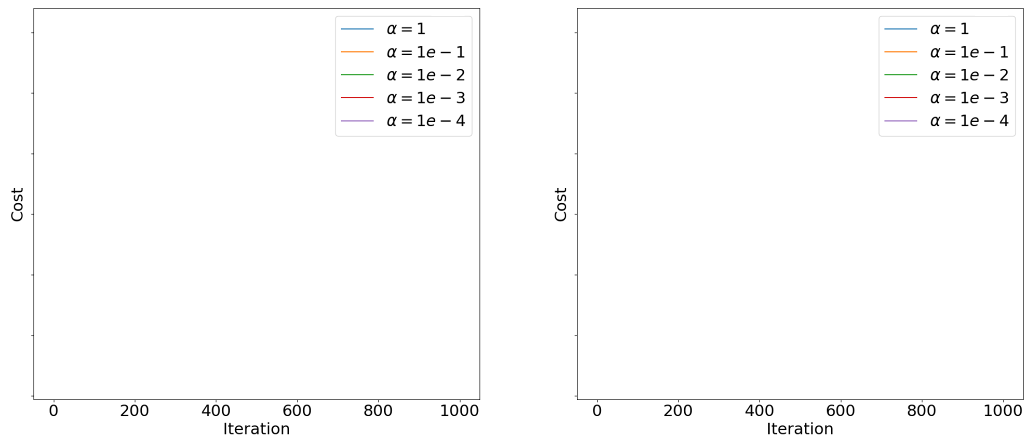
$$\tilde{x}_{ij} = \frac{x_{ij} - \bar{x}_j}{s_j}$$

- ***Evaluate.*** *Now use the code developed in Exercise 1.3 (and* `matplotlib` *for visualization) to*

  (a) Fit two linear regression models for the two choices of inputs ((i) only `horsepower` *and (ii) all except* `name`*). With a correct implementation, the final cost for the two models should be less than 25 and less than 12, respectively. State the final loss for each model.*

  (b) Write out the mathematical expressions of your two models (with sensible number of decimals). The first being a scalar function of a single variable $z(x)$, and the second a scalar function of a vector variable $z(\mathbf{x})$. For the second model, which of the input variables has the highest weight in the expression? Does this seem reasonable?

  (c) To study the relationship between learning rate *and* number of iterations*, produce a plot of the cost J versus 1,000 iterations for learning rates* `[1, 1e-1, 1e-2, 1e-3, 1e-4]` *for the two models. Present the results as two plots, as shown in Figure 1. Provide a high level interpretation of what you observe in the plots.*



(a) Model 1 (horsepower only) (b) Model 2 (all features)

Figure 1: Cost vs iterations for different learning rates γ. The actual lines and the y-axis values have been omitted.

  (d) Try repeating any of the previous experiments without normalizing the input. What happens then?

  (e) For the `horsepower`*-only model, produce a scatter plot with* `horsepower` *vs* `mpg` *together with a drawn line representing the linear regression model which you computed. Is it a good fit?*

Exercise 5. Classification (not manditory)

Voluntary extra task (for your own learning pleasure): Add a softmax function to the output to handle this as a classification problem. Try to predict the `origin` *of the car based on the other variables.*

Submission instructions: Submit a pdf with answers to Exercise 1.1, Exercise 1.2 and Exercise 1.4. For Exercise 1.3, submit a zip file with the code. If you don't want to use LaTeX for the whole report you can generate equations here `https://latex.codecogs.com/eqneditor/editor.php` and take a screenshot.

Checklist

Please check these things before handing in:

- ☐ **No miss**: Make sure you didn't miss answering a task: 1, 2, 3, 4a, 4b, 4c, 4d and 4e.
- ☐ **Repeat the problem**: Start the answer to each task with the abbreviated problem formulation (a short sentence where you write with your own words what you did). For example "Exercise 3: In exercise 3 we implement ... using ... since ..." instead of just "Exercise 3: Answer:...".
- ☐ **Readable plots**: Make sure all plots have caption, legend and axis labels; a background grid often improves readability. Also, make sure the plots are large enough to see the variations and all labels, x- and y-ticks etc.
- ☐ **Reasonable costs**: The cost of the two models should be less than 25 and less than 12 respectively, if you did everything correctly.
- ☐ **Partial derivatives**: Are the partial derivatives in Exercise 1 expressed in terms of the required terms?

Also see the 'Assignment template' (on the Course homepage).

The auto.csv data

Origin: The dataset was used in the 1983 American Statistical Association Exposition.

Description: Gas mileage, horsepower, and other information for 392 vehicles.

Format: A data frame with 392 observations on the following 9 variables.

`mpg`: miles per gallon
`cylinders`: Number of cylinders between 4 and 8
`displacement`: Engine displacement (cu. inches)
`horsepower`: Engine horsepower
`weight`: Vehicle weight (lbs.)
`acceleration`: Time to accelerate from 0 to 60 mph (sec.)
`year`: Model year (modulo 100)
`origin`: Origin of car (1. American, 2. European, 3. Japanese)
`name`: Vehicle name

References

Lindholm, Andreas, Niklas Wahlström, Fredrik Lindsten, and Thomas B. Schön (2021). *Supervised Machine Learning*. URL: `https://smlbook.org`.