Question 1

Problem. *Classify all congruences on $S = \{f\}$, $V = \{x\}$, $T(S,V) = \{x, fx, f^2x, \ldots\}$ and describe $T(S,V)_{/\sim}$.*

Solution. Consider an arbitrary congruence $\sim$ on S, V, $T(S,V) = T$. In this congruence, any pair takes the form $f^k x \sim f^l x$, where k and l are non-negative integers. We claim that the relation is fully determined by the pair $f^{k_0}x \sim f^{l_0}x$, where k_0 is the least non-negative integer such that f^{k_0} is related to other elements than itself and l_0 is the least non-negative integer such that $l_0 - k_0 > 0$ and $f^{k_0}x \sim f^{l_0}x$. To prove this, we need to prove several sub-claims.

Claim 1. Setting $d_0 = l_0 - k_0$, we get for every $k \geq k_0$ the equivalence $f^k x \sim f^{k+d_0}x$. This can be shown by induction. First notice that

$$\begin{aligned} f^{k_0}x \sim f^{l_0}x &\implies f^T(f^{k_0}x) \sim f^T(f^{l_0}x) \\ &\implies ff^{k_0}x \sim ff^{l_0}x \\ &\implies f^{k_0+1}x \sim f^{l_0+1}x. \end{aligned}$$

By induction, if $f^{k_0+n}x \sim f^{l_0+n}x$, then

$$\begin{aligned} f^{k_0+n}x \sim f^{l_0+n}x &\implies f^T(f^{k_0+n}x) \sim f^T(f^{l_0+n}x) \\ &\implies ff^{k_0+n}x \sim ff^{l_0+n}x \\ &\implies f^{k_0+n+1}x \sim f^{l_0+n+1}x. \end{aligned}$$

Substitute $k_0 + n + 1 = k$ and $l_0 + n + 1 = k_0 + l_0 - k_0 + n + 1 = k_0 + d_0 + n + 1 = k + d_0$, and we obtain the property claimed.

Claim 2. There exist no k, l such that $l > k > k_0$, $d = l - k < d_0$, and $f^k x \sim f^l x$. By contradiction, assume that this exists. Then for any $k' \geq k$ such that $d_0 | k' - k$, we have $f^{k_0}x \sim f^{k'}x$ and $f^{k_0+d}x \sim f^{k'+d}x$ by Claim 1. Deducing $f^{k'}x \sim f^{k'+d}x$ from $f^k x \sim f^{k+d}x$ by Claim 1, we get by transitivity $f^{k_0}x \sim f^{k_0+d}x$, contradicting the assumption that l_0 is the least non-negative integer such that $l_0 - k_0 > 0$ and $f^{k_0}x \sim f^{l_0}x$.

Claim 3. There exist no k, l such that $l > k > k_0$, $d = l - k > d_0$, $d_0 \nmid d$, and $f^k x \sim f^l x$. By contradiction, assume this exists, and thus the remainder r of dividing d by d_0 is positive. Consider $f^k x \sim f^{k+d_0}x, \ldots f^{k+d_0(\lfloor \frac{d}{d_0} \rfloor - 1)}x \sim f^{k+d_0 \lfloor \frac{d}{d_0} \rfloor}x$, where $k + d_0 \cdot \lfloor \frac{d}{d_0} \rfloor = l - r$. Then we get by transitivity $f^{l-r}x \sim f^l x$. Since $l - r < d_0$, we get a contradiction of Claim 2.

Hence, we see from k_0 onwards a congruence relation analogous to congruence modulo. Specifically, for any $k_0 \leq k < l$, $f^k x \sim f^l x$ if and only if $k \equiv l \pmod{d_0}$. The congruence classes are $\{[x], [fx], \ldots, [f^{k_0-1}x], [f^{k_0}x], [f^{k_0+1}x], \ldots, [f^{k_0+(d_0-1)}x]\}$, where $[x], [fx], \ldots, [f^{k_0-1}x]$ are singleton sets.

$\square$

Question 2

Problem. *Let $S = \{f, g\}$, where f, g are unary, $V = \{x\}$, $\Sigma = \{fgx = gfx\}$. Then answer:*

1. *What is an S-algebra?*
2. *How is $T(S, V)$ an S-algebra?*
3. *Describe $D(\Sigma)$.*
4. *$p \sim q \Leftrightarrow p = q \in D(\Sigma)$, describe $T(S, V)_{/\sim}$.*

Solution. For Item 1, let S be an algebraic signature. Then an S-algebra is a set A together with a k-ary function $f^A : A^k \to A$ for each k-ary function symbol $f \in S$.

For Item 2, we require an interpretation of the unary functions f, g on the set $T(S, V)$. Note that each term $t \in T(S, V)$ is the variable x prepended by an arbitrary string of 'f's and 'g's. Setting $T = T(S, V)$, we have the natural interpretations

$$\begin{aligned} f^T :& T \to T \\ & t \to ft, \\ g^T :& T \to T \\ & t \to gt. \end{aligned}$$

For Item 3, there is an equality in $D(\Sigma)$ for each pair of permutations on any fixed number of 'f's and 'g's prepending x. We need to reason about both inclusions. There is no other kind of equality because the application of an equality rule, substitution, or replacement, starting with $fgx = gfx \in \Sigma$, preserves the number of 'f's and 'g's on each side. Additionally, every permutation of 'f's and 'g's prepending x is related to every other because repeated use of the replacement rule with $fgx = gfx \in \Sigma$ allows to move any 'g' that is left of an 'f' to its right, hence any permutation of 'f's and 'g's can be rearranged to have all 'f's first and 'g's second, by transitivity relating any two permutations.

For Item 4, we thus have $T(S, V)_{/\sim} = \{[f^i g^j x] \mid i, j \in \mathbb{N}_0^+\}$. $\square$