Math 190 — Homework 10/22

Classify All Theories on $\mathbf{S = \{a^0, b^0\}, V = \{x\}}$

First we observe that the set of terms is simply $T(S,V) = \{a, b, x\}$, since there are no functions of arity greater than 0. The set of all propositions is thus easily enumerable:

$$Prop(S,V) = \{a = a, b = b, x = x, a = b, a = x, b = x\}$$

(We omit the symmetric propositions, since any theory must include the symmetric proposition to any proposition it contains.) A theory on $T(S,V)$ is simply a subset of this list which is closed under the congruence properties and also substitution. As reflexivity is required, any theory must include $a = a, b = b$, and $x = x$. These three identities together certainly comprise one theory. Another theory may be obtained by also including $a = b$.

If, in addition to the reflexive identities, a theory also contains $a = x$, then by substitution it must also contain $a = b$, and then by transitivity it must contain $b = x$. Likewise, if a theory contains the reflexive identities and $b = x$, it must contain both $a = b$ and $a = x$ as well. So we have only three distinct theories:

$$\{a = a, b = b, x = x\}$$
$$\{a = a, b = b, x = x, a = b\}$$
$$\{a = a, b = b, x = x, a = b, a = x, b = x\}$$

(These theories are understood to contain the symmetric propositions to their members as well.)

Each theory may actually be explicitly exhibited as $T(S,V)/\sim$, where $\sim$ is defined by the congruence where two elements are related if and only if they are declared equal in the corresponding theory. The S-algebras with the above theories are, respectively,

$$\{[a], [b], [x]\}$$
$$\{[a] = [b], [x]\}$$
$$\{[a] = [b] = [x]\}$$

Classify All Theories on $\mathbf{S = \{f, 0\}, V = \{x\}}$

We observe that the set of terms is

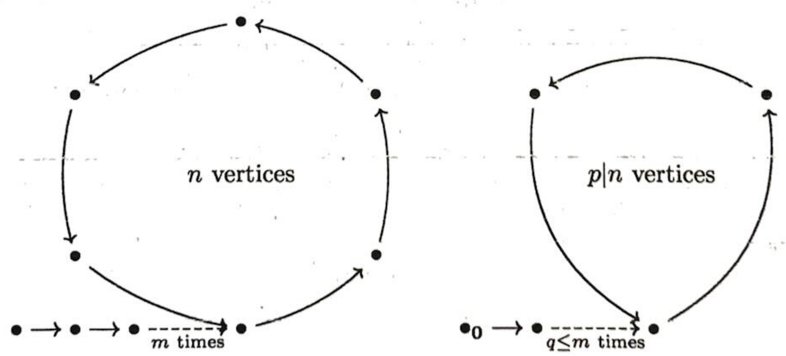
$$T(S,V) = \{x, fx, f^2x, ...\} \cup \{0, f0, f^20, ...\}$$

Notice that any theory restricted to only the terms containing x's must be a congruence which is closed under substitution — a redundant condition. We already know all such congruences are described as the set of all identities $f^kx = f^\ell x$ such that $k \equiv \ell \mod n$ and $k, \ell \geq m$ for some choices of $n > 0$ and $0 \leq m \leq \infty$ (with $m = \infty$ meaning no distinct terms are related). Likewise, any theory restricted to only the terms containing 0's is a congruence closed under substitution — a trivial condition in the absence of variables. So this relation is likewise described as all $f^k 0 = f^\ell 0$ such that $k \equiv \ell \mod p$ and $k, \ell \geq q$ for some choice of $p > 0$ and $0 \leq q \leq \infty$.

Consider for now all theories which only consist of the above "homogenous" identities, i.e., identities for which both sides contain an x or both sides contain a 0. For any choices of m, n, p, q, the above formulation satisfies the equivalence relation and replacement properties. Since substitution cannot be applied to the terms with only 0's, the choice of p, q does not restrict the choices of m, n, but the choice of m, n may certainly restrict the choice of p, q. Notice that if $f^k x = f^\ell x$, then by substitution, $f^{k+j} 0 = f^{\ell+j} 0$ for any $j \geq 0$. In particular, since the former identity holds whenever $k \equiv \ell \mod n$ for all $k, \ell \geq m$, we must have $f^k 0 \equiv f^\ell 0$ for all $k \equiv \ell \mod n$ and $k, \ell \geq m$. That is, p must divide n and q must be at most m. So if we define

$$\mathrm{Th}_{m,n,p,q} = \{f^k x = f^\ell x | k \equiv \ell \bmod n; k, \ell \geq m\} \cup \{f^k 0 = f^\ell 0 | k \equiv \ell \bmod p; k, \ell \geq q\}$$

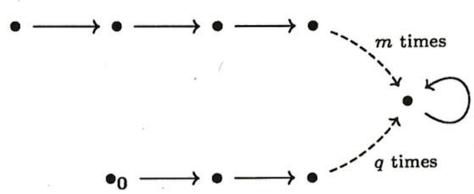
then we can describe all "homogeneous" theories as all $\mathrm{Th}_{m,n,p,q}$ such that $0 < p \leq n < \infty$, $0 \leq q \leq m \leq \infty$, and $p | n$. (If $m = \infty$ but q is finite, we may freely choose n, so this imposes no restriction on p.) Indeed, the below diagram consisting of two disjoint components and one marked vertex gives an S-algebra which exhibits precisely the theory $\mathrm{Th}_{m,n,p,q}$:



Now we consider theories more generally. Any theory must contain at least one of the theories $\mathrm{Th}_{m,n,p,q}$ above, possibly with some additional mixed terms. Since we already have all theories with no mixed terms, suppose our theory has one such term, i.e., there exist $u, v \geq 0$ such that $f^u x = f^v 0$ is in our theory. Take u to be minimal such that a v as above exists. Then by substitution of $fx, f^2 x$, etc. into x, it is clear that we also get $f^{u'} x = f^v 0$ for all $u' \geq u$, and in fact by transitivity $f^{u'} x = f^u x$. This fact imposes the immediate restriction that $n = 1$, so also $p = 1$. It also requires $m = u$, since we said u was minimal and m is by definition the smallest integer after which the terms $f^k x$ start having multiple elements in their equivalence classes.

We also have $f^{u'} 0 = f^u 0$ for all $u' \geq u$, but we have no guarantee that u is minimal such that this identity holds. The smallest such exponent after which all $f^k 0$ are related is, by definition, q, and we know $q \leq m = u$. We can certainly have q be strictly less than m. But since $f^q 0 = f^k 0$ for all $k \geq q$ and by replacement $f^{u+\ell}x = f^{v+\ell}0$ for all $\ell \geq 0$, by choosing k, ℓ large enough that the right-hand sides may be equal, we get $f^q 0 = f^{u+\ell}x = f^u x = f^v 0$, which by the minimality of q implies $q \leq v$, so indeed q is the smallest exponent for which $f^k 0$ may be related to any x-term.

Taking the above information together, the theories containing at least one mixed term can be described as all $\mathrm{Th}_{m,1,1,q}$, where $0 \leq q \leq m < \infty$, together with the additional identities $f^k x = f^\ell 0$ for all $k \geq m$ and $\ell \geq q$. Indeed, we can explicitly exhibit such an S-algebra with the below diagram.



In summary, any theory on $T(S, V)$ is one of the following, and each of the following is possible to construct:

$$\mathrm{Th}_{m,n,p,q} \quad s.t. \quad 0 < p \leq n < \infty, \quad 0 \leq q \leq m \leq \infty, \quad p | n$$

OR

$$\mathrm{Th}_{m,1,1,q} \cup \{f^k x = f^\ell 0 | k \geq m, \ell \geq q\} \quad s.t. \quad 0 \leq q \leq m < \infty$$