MATH 190 FINAL EXAMINATION
FALL 2024

1 (Math 145 in a single leap).
(a) Let $\phi : G \to H$ be a homomorphism of groups.
(b) Prove that the image of ϕ is a subgroup of H.
(c) Prove that the kernel of ϕ is a normal subroup of G.
(d) Prove that $\mathrm{im}\,(\phi) \simeq G/\ker(\phi)$.

2 (The Art of Abstraction).
(a) Let $\phi : G \to H$ be a homomorphism of S-algebras.
(b) Prove that the image of ϕ is a subalgebra of H.
(c) Prove that the kernel of ϕ is a congruence on G.
(d) Prove that $\mathrm{im}\,(\phi) \simeq G/\ker(\phi)$.

3 (Wait, what is a congruence again?). Let G be a group. In this problem, we will understand the relationship between congruences and normal subroups of G.
(a) Prove that if $\sim$ is a congruence on G, then $N := \{a \in G \,|\, a \sim e\}$ is a normal subroup of G.
(b) Prove that if N is a normal subroup of G, then the relation on G defined by $a \sim b := ab^{-1} \in N$ is a congruence on G.
(c) Conclude that there is a 1-1 correspondence between congruences on G and normal subgroups of G.

4 (The Art of Rereading).
(a) Choose anyone of the readings from this semester and reread it.
(b) Discuss.

5 (One formal system to rule them all...). Give formal deductions in equational logic corresponding to the following statements:
(a) $2 \times 2 = 4$
(b) A group in which every element has order 2 is abelian.
(c) If A, B, C are subsets of X and $A \subset B$ and $B \subset C$, then $A \subset C$

In each case, you should clearly specify what signature you are using, what equational axioms you are using, and then write out the deduction.

6 (The Most Significant Masters Thesis of the 20th Century.).
(a) Read the first 7 pages of Claude Shannon's Masters thesis - a symbolic analysis of switching circuits (available on canvas).
(b) Discuss. For the record, I believe this thesis was a turning point in switching conventional discussions of logic from being about sets to being about $\{0, 1\}$-functions.

7 (Terry Tao needs YOU!).
(a) Read this blog post: https://terrytao.wordpress.com/2024/09/25/a-pilot-project-in-universal-algebra-to-explore-new-ways-to-collaborate-and-use-machine-assistance/

(b) Try to verify for yourself some of the claims made in the post.
(c) Read this follow-up post: https://terrytao.wordpress.com/2024/10/12/the-equational-theories-project-a-brief-tour/

8 (Categorical Interpretation of Logical Connectives). Recall that we set up a category whose objects are propositional formulae in some fixed set of variables, and there is a single arrow from F to G if and only if $F \models G$. Verify carefully that this category has products, coproducts, an initial and terminal object, and exponentials. Put differently, the poset of formulas under logical implication has a global upper and lower bound, has least upper bounds, and greatest lower bounds, and the exponential.

9 (Linear Logic). Let V be a finite-dimensional vector space, e.g. $V = \mathbb{R}^n$. Consider the collection of all subspaces of V as the objects of a category and a put a single arrow from W_1 to W_2 if there is an inclusion from W_1 to W_2.

(a) What is the product of W_1 and W_2 in this category?
(b) What is the coproduct of W_1 and W_2 in this category?
(c) What is the initial and terminal object in this category?
(d) Does this category of exponentials?

10 (Functors are translations between languages). Let $S = \{\times\}$ be the signature consisting of a single binary function symbol, and let $T = \{f\}$ be the signature consisting of a single unary function symbol.

(a) If $(X, \times^X)$ is any S-algebra, then we can turn X a T-algebra by setting $f^X := x \times x$. Use this idea to construct a functor from the category of S-algebras to the category of T-algebras.
(b) Formulate a massive generalization of part (i), i.e. describe a general way of constructing functors from the category of S-algebras to the category of T-algebras for any pair of signatures S and T.

11 (Wait, what is a congruence again, pt. 2).

(a) Let R be a commutative ring with identity. Prove that there is a one-to-one correspondence between congruences on R and ideals of R.
(b) Try to formulate a condition on signatures S such that congruences on an S-algebra X can be identified with certain subsets of X.

12 (I need Yoneda's Lemma, and I need it now.). Suppose that $\mathbb{C}$ is a category with binary products, and let A, B, C be objects of $\mathbb{C}$. Prove that $A \times (B \times C)$ and $(A \times B) \times C$ are isomorphic objects of $\mathbb{C}$. (Annoying; there is ultimately a better way using Yoneda's lemma.)

13 (I was all good until part (d)). Recall that there is a category of posets, whose objects are posets $(X, \leq)$ and whose arrows are order-preserving maps.

(a) Prove that the category of posets has products. Hint: If $(X, \leq_X)$, $(Y, \leq_Y)$ are posets, then there is a natural partial order defined on $X \times Y$ such that $(X \times Y, \leq)$ is a product.
(b) Prove that the category of posets has coproducts.
(c) Prove that the category of posets has an initial and terminal object.
(d) Prove that that the category of posets has exponentials, $(X, \leq_X)$, $(Y, \leq_Y)$ are posets, then let Y^X denote the set of all poset homomorphisms from X to Y. Show that there is a natural partial order on Y^X such that if T is any poset, then homomorpisms from T to Y^X are in bijective correspondence with poset homorphisms $T \times X$ to Y. (Hard!)