Group Meeting

March 2024

1 Introduction

Let $f \in K[x,y,z]$ a degree 2 homogeneus polynomial, $I = \langle f \rangle$ and $M_d = \mathrm{Mac}_d(I)$ its associated Macaulay matrix in degree d that is size $\binom{d}{2} \times \binom{d+2}{2}$.

Proposition 1 *Let $k \geq 2$. If B is a minor of M_d such that the set of columns of B that are indexed by $m \cdot \mathrm{Mon}_k$ satisfies $|B \cap m \cdot \mathrm{Mon}_k| < \dim I_k = \binom{k}{2}$ for some monomial $m \in \mathrm{Mon}_{d-k}$, then* $\det B = 0$.

With the notations above, fix $m \in \mathrm{Mon}_{d-k}$. We can write $Mac_d(I)$ in the following block form:

$$\begin{array}{c|c|c} & m \cdot \mathrm{Mon}_k & \mathrm{Mon}_d - m \cdot \mathrm{Mon}_k \\ n \in \mathrm{Mon}_k : m|n & * & 0 \\ \mathrm{Mon}_k - \{n \in \mathrm{Mon}_k : m|n\} & * & * \end{array}$$

Note that the top right square is zero because the rows are indexed by monomials divisible by m and the columns are precisely the monomials that are not divisible by m.

As the cardinality of $\{n \in \mathrm{Mon}_k : m|n\}$ is $\binom{d-2-(d-k)+2}{2} = \binom{k}{2}$ and cardinality of $m \cdot \mathrm{Mon}_k$ is $\binom{k+2}{2}$, we deduce that the bottom left square has size $\binom{d}{2} - \binom{k}{2} \times \binom{d+2}{2} - \binom{k+2}{2}$.

Therefore, if $|B \cap \mathrm{Mon}_d - m \cdot \mathrm{Mon}_k| > \binom{d}{2} - \binom{k}{2}$, these columns would be linearly dependent because they span a $(\binom{d}{2} - \binom{k}{2})$-dimensional subspace and we conclude.