Lec 16. **Warm up**

$\mathbb{P}_2$ basis $\mathcal{B} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$

$C = \{1, 2t, t^2\}$

$\vec{v} \in \mathbb{P}_2$, $[\vec{v}]_C = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(1) Compute $[\vec{v}]_{\mathcal{B}}$ directly

(2) Compute $\underset{C \leftarrow B}{P}$ and use it to compute $[\vec{v}]_{\mathcal{B}}$.

(1) $\vec{v} = -1 + 2t$

In standard basis, $\{1, t, t^2\} := \mathcal{E}$

Solve $\left[\begin{array}{ccc|c} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{array}\right] \longrightarrow [\vec{v}]_B = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$

(2) $\underset{C \leftarrow B}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & -\frac{5}{2} & 1 \\ 1 & 4 & 3 \end{bmatrix}$

$$[\vec{v}]_C = \underset{C \leftarrow B}{P} [\vec{v}]_B$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & -1 \\ -1 & -\frac{5}{2} & 1 & 1 & 1 \\ 1 & 4 & 3 & 1 & 0 \end{array}\right] \longrightarrow [\vec{v}]_B$$

Matrix representation of general linear transformation.

Fact: $T : \mathbb{R}^n \to \mathbb{R}^m$. lin trans

$\iff$ matrix trans.

standard matrix A

$$T(\vec{x}) = A\vec{x} \quad , \quad \vec{x} \in \mathbb{R}^n , A \in \mathbb{R}^{m\times n}$$

$$A = \left[T(\vec{e}_1) \cdots T(\vec{e}_n) \right]$$

This lecture :

$$T : V \longrightarrow W .$$

V, W finite dimensional vec. space

$V: \mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$. n-dim

$W: C = \{\vec{c}_1, \cdots \vec{c}_m\}$. m-dim

How to define matrix transformation?

"We share a philosophy about linear algebra: we think basis-free, we write basis-free , but when the chips are down we close the office door and compute with matrices like fury."

Kaplansky, Irving

"Don't forget to call it a 'procedure'—makes it less scary."

$$\begin{tikzcd}
V \arrow[r, "T"] & W \arrow[d, "P_C^{-1}"] \\
\mathbb{R}^n \arrow[u, "P_B"] \arrow[r, "P_C^{-1} T P_B"] & \mathbb{R}^m
\end{tikzcd}$$

$T: V \longrightarrow W$

matrix transformation

Apply Chap 1.

"unpack notation". $\vec{x} \in \mathbb{R}^n$

$$\left(P_C^{-1} T P_B\right)(\vec{x}) = P_C^{-1}\left(T\left(P_B(\vec{x})\right)\right)$$

Find the (standard matrix)

$$(P_C^{-1} T P_B)(\vec{x}) = A\vec{x}, \quad \vec{x} \in \mathbb{R}^n$$

$$A = \left[(P_C^{-1} T P_B)(\vec{e}_1) \quad \cdots \quad (P_C^{-1} T P_B)(\vec{e}_n) \right]$$

$$P_B(\vec{e}_1) = \vec{b}_1, \quad T(P_B(\vec{e}_1)) = T(\vec{b}_1)$$

$$P_C^{-1}(T(\vec{b}_1)) = [T(\vec{b}_1)]_C$$

$$A = \left[[T(\vec{b}_1)]_C \quad \cdots \quad [T(\vec{b}_n)]_C \right]$$

$$\in \mathbb{R}^{m \times n}$$

matrix rep. of T w.r.t. basis sets $\mathcal{B}, C$.

Ex. $T: \mathbb{P}_3 \to \mathbb{P}_2$

$$f \mapsto x f''(x)$$

Find mat. rep. w.r.t. basis

$\mathcal{B} = \{1, x, x^2, x^3\}$, $C = \{1, x, x^2\}$.

Take any $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$[T(f)](x) = x \cdot (2a_2 + 6a_3 x)$$

$$= 2a_2 x + 6a_3 x^2 \in \mathbb{P}_2$$

linearity follows that of derivative.

$$A = \left[[T(\vec{b}_1)]_C \ \cdots \ [T(\vec{b}_4)]_C \right]$$

$$= \begin{array}{cc} & \begin{array}{cccc} & n & & \end{array} \\ m & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \end{array}$$

$\dim \mathrm{Null}(T) = 2.$

$\dim \mathrm{Image}(T) := \mathrm{rank}(T) = 2.$

<u>Thm</u>. (Rank theorem)

V, W finite dim vector spaces.

$$\underset{\substack{\mathbin{\rotatebox{90}{$\subset$}} \\ n}}{\dim V} = \mathrm{rank}(T) + \dim \mathrm{Null}(T)$$

Another look at change of coord. Vector space V.

Basis $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$

$C = \{\vec{c}_1, \cdots, \vec{c}_n\}$.

Identity map

$$
\begin{tikzcd}
V \arrow[r, "I"] & V \arrow[d, "P_C^{-1}"] \\
\mathbb{R}^n \arrow[u, "P_B"] \arrow[r, "P_C^{-1} P_B"] & \mathbb{R}^n
\end{tikzcd}
$$

$I : V \longrightarrow V$

$$\underset{C \leftarrow B}{P} := P_C^{-1} P_B \quad \text{matrix rep. of } I \text{ w.r.t. } B, C.$$