Lec 24. Warm up

Ex. $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$

(1) $\vec{v}_1 \perp \vec{v}_2$?

True. $\vec{v}_1 \cdot \vec{v}_2 = 0$

(2) $\vec{v}_1 \perp \vec{v}_3$?

$\vec{v}_1 \cdot \vec{v}_3 = (-1) + 2 \cdot 1 + (-1) \cdot 1 = 0$. True.

Ex. $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

(1) orthogonal set ?

$\vec{u}_1 \cdot \vec{u}_2 = 2 - 2 + 0 = 0$

$\vec{u}_1 \cdot \vec{u}_3 = -2 + 2 + 0 = 0$ True.

$\vec{u}_2 \cdot \vec{u}_3 = -1 - 4 + 5 = 0$

(2) orthornormal set ?

$\|\vec{u}_1\| = \sqrt{2^2 + 1^2 + 6^2} = \sqrt{5} \neq 1$

False.

Use diagonalization to save the world

Def. $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is called an orthogonal basis of subspace $W \subseteq \mathbb{R}^n$ if

1) Orthogonal set

2) basis.

not too small : $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = W$

not too large: $\underbrace{\{\vec{u}_1, \cdots, \vec{u}_k\} \text{ lin. indep}}$.

automatically satisfied by orthogonality

Thm. $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is an orthonormal set, then this set is lin. indep.

get rid of $\vec{0}$

pf: Assume

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$$

$$\vec{v}_1 \cdot \left(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k\right) = \vec{v}_1 \cdot \vec{0} = 0$$

$$c_1 \cdot 1 + 0 + \cdots + 0 = 0 \Rightarrow c_1 = 0.$$

Apply to all $\vec{v}_i$

$\Rightarrow c_1 = \cdots = c_k = 0 \Rightarrow$ lin. indep. $\square$

Why orthogonal basis is useful?

$\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ is a basis of $\mathbb{R}^n$.

$\vec{v} \in \mathbb{R}^n$. $[\vec{v}]_{\mathcal{B}}$

In Chap 4. form a lin. sys.

$$\left[\vec{b}_1 \cdots \vec{b}_n \;\middle|\; \vec{v}\right] \longrightarrow \text{sol. } [\vec{v}]_B.$$

Now if B is an orthogonal basis.

$$\vec{c} \equiv [\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

$$\vec{b_1} \cdot \vec{v} = \vec{b_1} \cdot \left(c_1 \vec{b_1} + \cdots + c_n \vec{b_n} \right)$$

$$= c_1 \vec{b_1} \cdot \vec{b_1} + 0 + \cdots + 0$$

$$= c_1 \vec{b_1} \cdot \vec{b_1}$$

$$c_i = \frac{\vec{b_i} \cdot \vec{v}}{\vec{b_i} \cdot \vec{b_i}}, \quad i = 1, \cdots, n.$$

No lin. sys. sol. Only 2 inner products.

$\mathcal{B}$ is orthonormal basis

$$\boxed{c_i = \vec{b_i} \cdot \vec{v}}$$

Ex. $\mathcal{B} = \{\vec{b_1}, \vec{b_2}, \vec{b_3}\}$.

$\vec{b_1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ $\vec{b_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{b_3} = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$

Find $[\vec{v}]_{\mathcal{B}}$. $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(exer) Find $[\vec{v}]_{\mathcal{B}}$ by $[\vec{b_1}\ \vec{b_2}\ \vec{b_3} \,|\, \vec{v}]$.

we have checked (in warm-up) that $\mathcal{B}$ is an orthogonal set.

orthogonal basis of $\mathbb{R}^3$

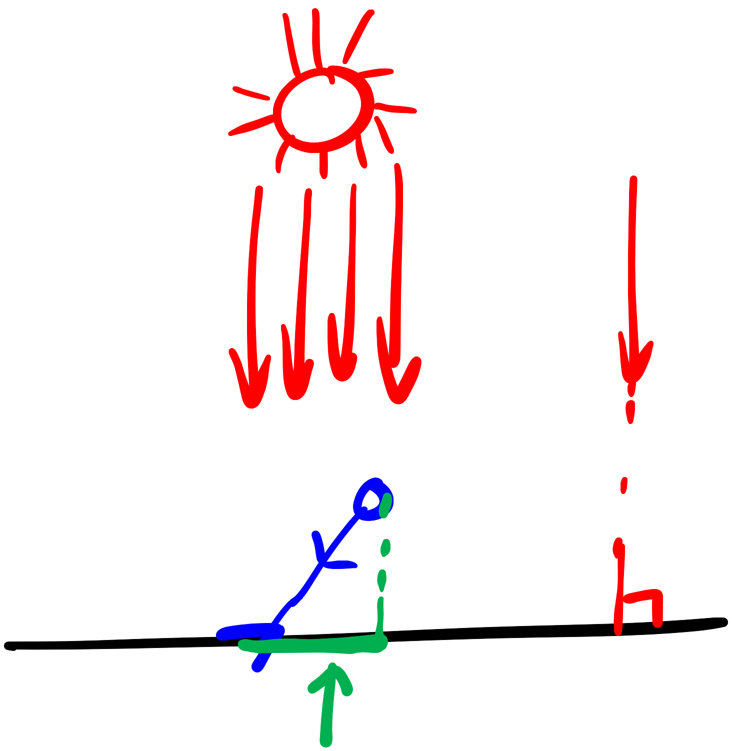
$$c_1 = \frac{\vec{b_1} \cdot \vec{v}}{\vec{b_1} \cdot \vec{b_1}} = \frac{2}{5}$$

$$c_3 = \frac{\vec{b_3} \cdot \vec{v}}{\vec{b_3} \cdot \vec{b_3}} = \frac{4}{30}$$

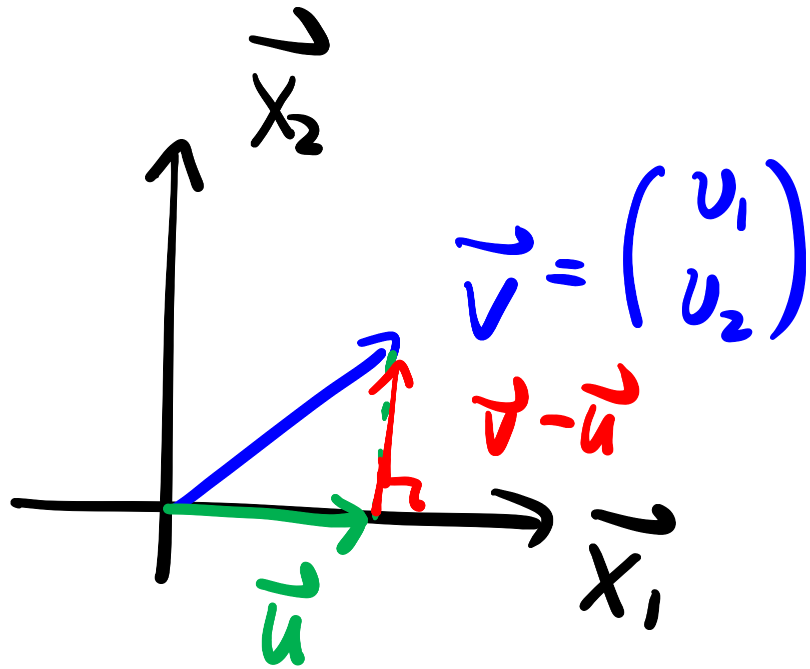
$$c_2 = \frac{\vec{b_2} \cdot \vec{v}}{\vec{b_2} \cdot \vec{b_2}} = \frac{2}{6} = \frac{1}{3}$$

$$= \frac{2}{15}$$

Projection



projection.



projection of $\vec{v}$ onto the x-axis

$$\vec{u} = v_1 \vec{e_1} = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \qquad \vec{z} = \vec{v} - \vec{u} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$$

$$\vec{u} \cdot \vec{z} = 0 . \Rightarrow \vec{u} \perp \vec{z}$$

generalize to a more algebraic setting?

Ex. $\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Find $\alpha \in \mathbb{R}$ s.t.

$$\vec{v} = \alpha \vec{u} + \vec{z} . \text{ and } \vec{u} \cdot \vec{z} = 0$$

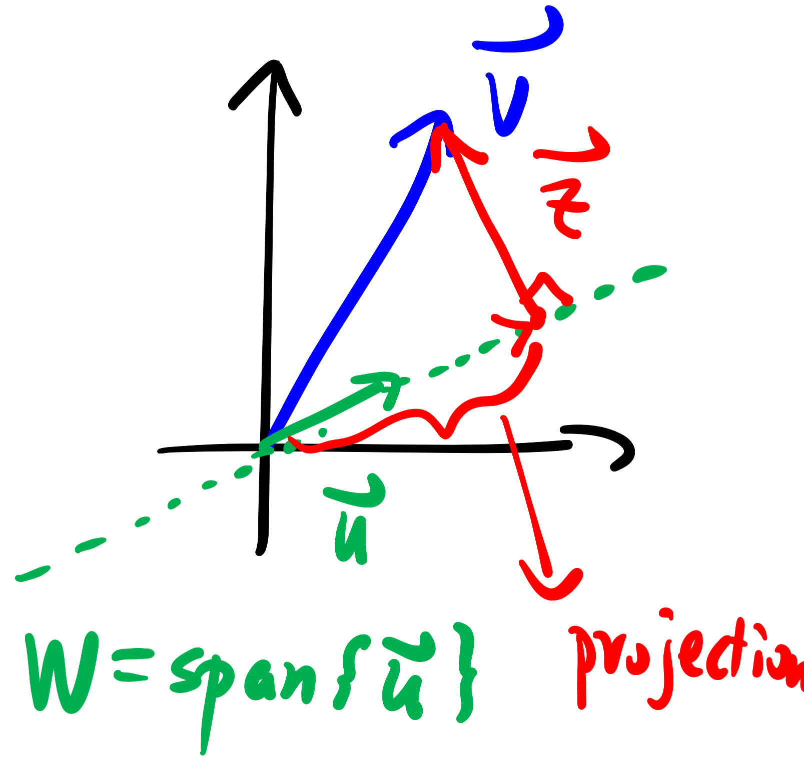
$$\vec{u}\cdot\vec{v} = \vec{u}\cdot\left(\alpha\vec{u} + \vec{z}\ \right)$$

$$\Rightarrow \quad \alpha = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} = \frac{3}{5}$$

$$\vec{z} = \vec{v} - \alpha\vec{u} = \begin{bmatrix} 1 \\ \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

$$\vec{v} \in R^n, \quad \vec{0} \neq \vec{u} \in \mathbb{R}^n$$

$W = \text{span}\{\vec{u}\} \rightarrow 1 \text{ dim.}$



$$\vec{v} = \underbrace{\left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)}\vec{u} + \vec{z}$$

$$\vec{z}\cdot\vec{u} = 0$$

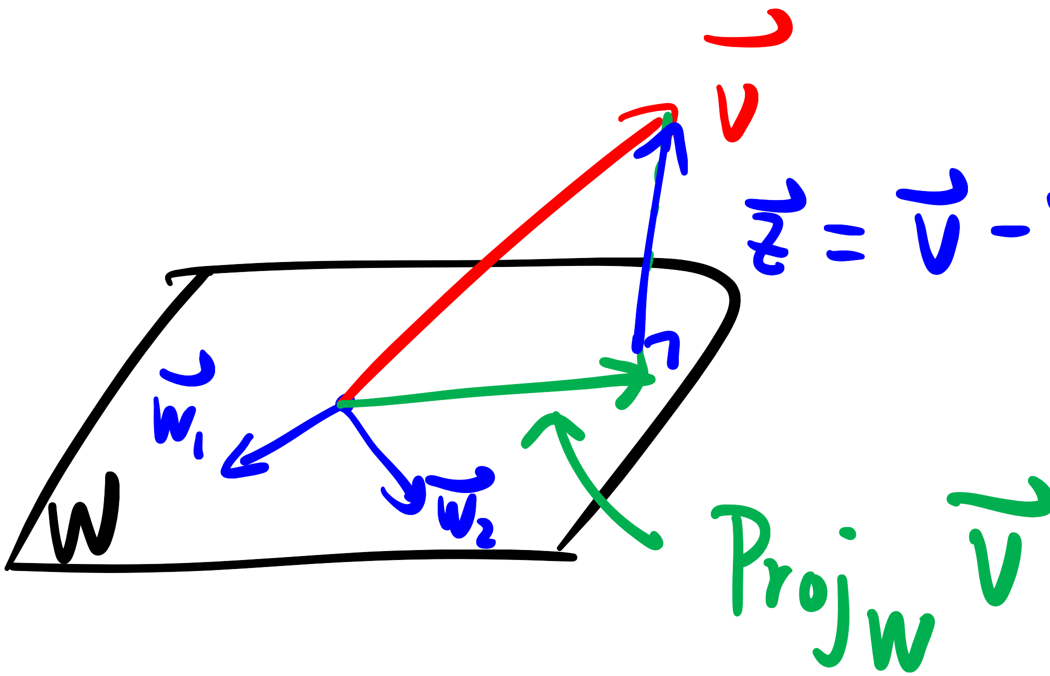
$$\text{Proj}_W \vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u} \in W$$

$$\Rightarrow \vec{v} = \text{Proj}_W \vec{v} + \vec{z}$$

$W = \text{span}\{\vec{w}_1, \cdots, \vec{w}_k\} \rightarrow$ k-dim subspace of $\mathbb{R}^n$

orthogonal basis set

$k=2$



$\vec{z} = \vec{v} - \text{Proj}_W \vec{v}$

$\vec{z} \cdot \vec{w}_1 = 0, \quad \vec{z} \cdot \vec{w}_2 = 0$

Take any $\vec{w} \in W$

$$\vec{w} = c_1 \vec{w_1} + c_2 \vec{w_2}, \quad c_1, c_2 \in \mathbb{R}.$$

$$\vec{z} \cdot \vec{w} = c_1 \vec{z} \cdot \vec{w_1} + c_2 \vec{z} \cdot \vec{w_2} = 0$$

$$\vec{z} \perp \vec{w}, \quad \vec{w} \in W$$

$$\mathrm{Proj}_W \vec{v} = \left(\frac{\vec{v} \cdot \vec{w_1}}{\vec{w_1} \cdot \vec{w_1}}\right) \vec{w_1} + \cdots + \left(\frac{\vec{v} \cdot \vec{w_k}}{\vec{w_k} \cdot \vec{w_k}}\right) \vec{w_k}$$

You can check $\vec{z} \perp (\vec{w_1}, \cdots, \vec{w_k})$

Ex. $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$W = \text{span}\{\vec{w}_1, \vec{w}_2\}$.

Compute $\text{Proj}_W \vec{v}$

Check first whether $\{\vec{w}_1, \vec{w}_2\}$ is an orthogonal basis set.

$$\vec{w}_1 \cdot \vec{w}_2 = 1+1 = 2 \neq 0.$$

So construct an orthogonal basis first

$$\vec{w}_2 = \mathrm{Proj}_{\mathrm{span}\{\vec{w}_1\}} \vec{w}_2 + \vec{\tilde{w}}_2$$

$\vec{\tilde{w}}_2 \perp \vec{w}_1 \Rightarrow \{\vec{w}_1, \vec{\tilde{w}}_2\}$ is an orthogonal basis.

$$\tilde{\vec{w}}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now apply formula (exer)

$$\mathrm{Proj}_{\vec{w}} \vec{v} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$