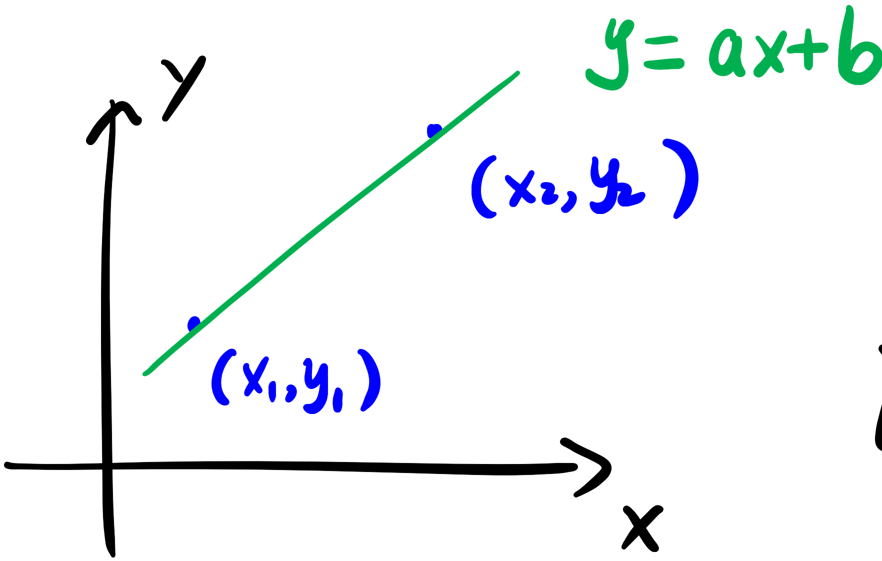
Lec 25.

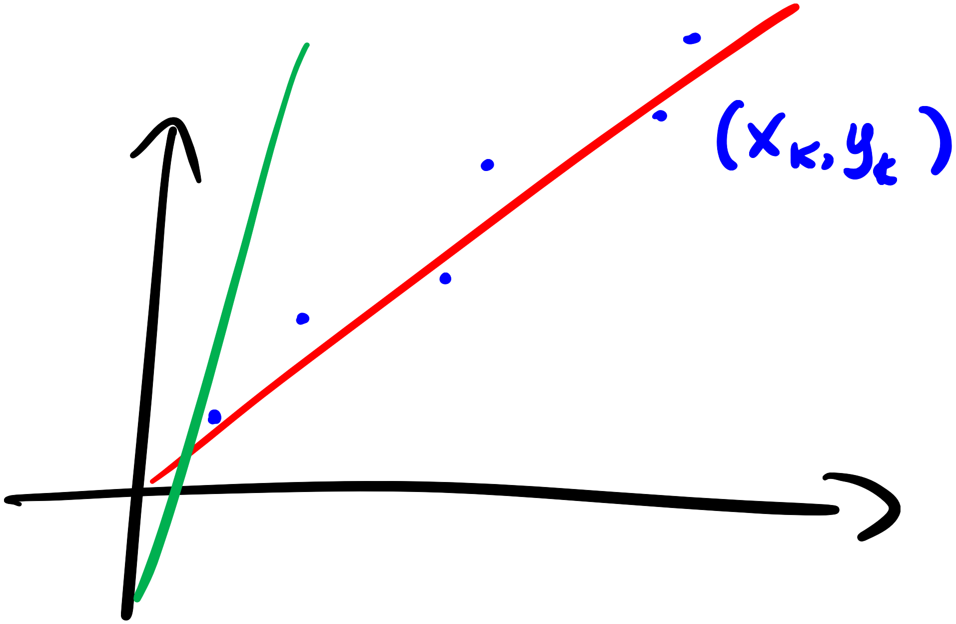
2 points in $\mathbb{R}^2$ $(x_1, y_1), (x_2, y_2)$



$$\begin{cases} a x_1 + b = y_1 \\ a x_2 + b = y_2 \end{cases}$$

$\Rightarrow$ solve (a, b).

$$\left[\begin{array}{cc|c} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \end{array}\right]$$



N points

$a x_k + b = y_k$

$k = 1, \cdots, N.$

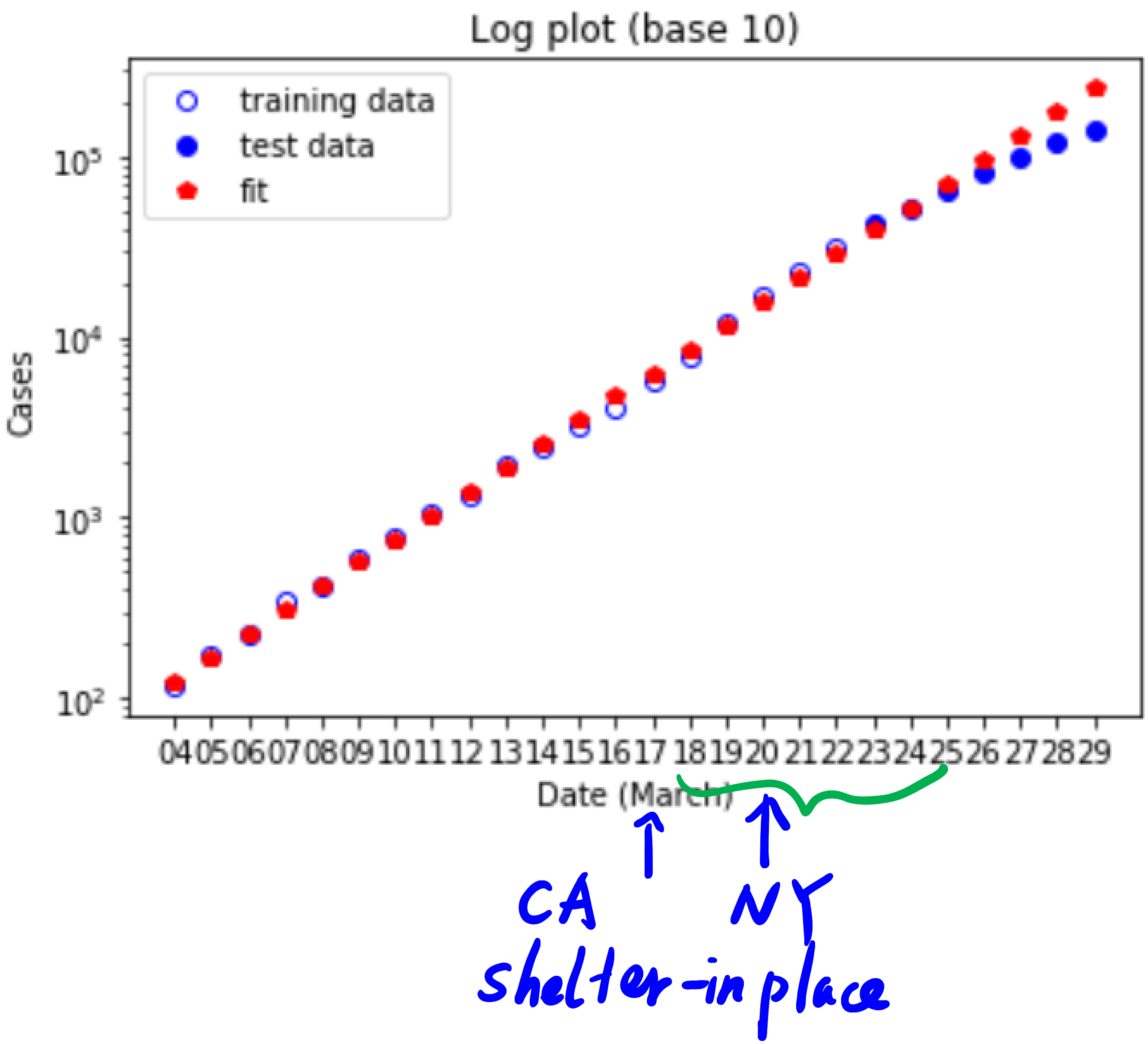
$$\left[\begin{array}{cc|c} x_1 & 1 & y_1 \\ x_2 & 1 & \vdots \\ \vdots & & \vdots \\ x_N \cdots & 1 & y_N \end{array}\right]$$

→ most likely no sol.

One possibility: least-squares

$$\min_{a,b} \sum_{i=1}^{N} \left(a x_i + b - y_i\right)^2 := f(a,b)$$

Least-squares study of COVID-19 in US



Optimization problem

$f(a,b) \geq 0$, if $f(a,b)=0$

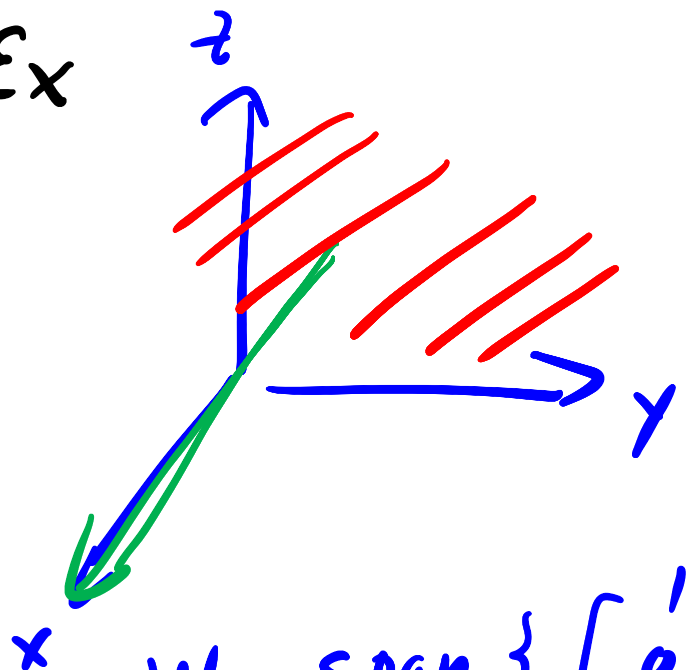
$\Rightarrow$ $a x_i + b = y_i$, $i=1,\cdots,N$

$\rightarrow$ linear sys. is solvable.

<u>Def</u>. $W \subseteq \mathbb{R}^N$ subspace. The subset in R^N orthogonal to W

$$W^{\perp} = \{ \vec{v} \in \mathbb{R}^n \text{ s.t } \vec{v} \perp \vec{w} \text{ for all } \vec{w} \in W \}.$$

Ex

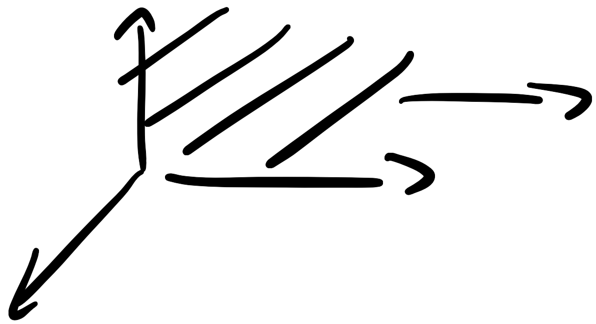


$W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$W^{\perp} := \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

y-z plane.

$W^{\perp}$ is a subspace of $\mathbb{R}^3$.

Ex.  W y-z plane

$W^\perp$: x-axis.

$$(W^\perp)^\perp = W$$

Properties:

1) $W^\perp$ is a subspace

2) $W = \text{span}\{w_1, \dots, w_k\} \subseteq \mathbb{R}^n$.

Then $W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w_i} = 0,\ i = 1, \dots, k\}$.

Ex. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ Find $Col(A)^{\perp}$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. $Col(A) = span\{\vec{a}_1, \vec{a}_2\}$

$= span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Find $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t. $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

One sol: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow col(A)^{\perp} = span\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$.

More generally.

$$A = [\vec{a}_1, \cdots, \vec{a}_n]$$

$$\begin{cases} \vec{a}_1 \cdot \vec{x} = 0 \\ \vdots \\ \vec{a}_n \cdot \vec{x} = 0 \end{cases} \Longrightarrow \begin{array}{l} \vec{a}_1^T \vec{x} = 0 \\ \vdots \\ \vec{a}_n^T \vec{x} = 0 \end{array} \Rightarrow A^T \vec{x} = \vec{0}$$

$$A^T = \begin{bmatrix} \vec{a}_1^T \\ \vdots \\ \vec{a}_n^T \end{bmatrix}$$

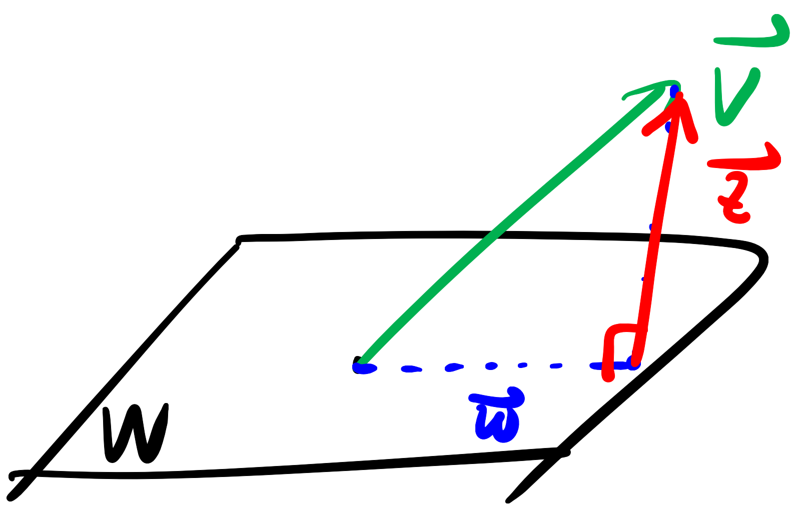
$\vec{x} \in \text{Col}(A)^{\perp}$.

$$\Longleftrightarrow \vec{x} \in \text{Null}(A^T) \Rightarrow \boxed{\text{Col}(A)^{\perp} = \text{Null}(A^T)}$$

Thm. (Uniqueness of Projection)

$W \subseteq \mathbb{R}^n$ is a subspace. any vector $\vec{v} \in \mathbb{R}^n$ has a unique decomposition

$$\vec{v} = \vec{w} + \vec{z}, \quad \vec{w} \in W, \quad \vec{z} \in W^{\perp}.$$



$\vec{w} = \text{Proj}_W \vec{v}$

Pf: Only need to show uniqueness.

Suppose this is not true

$$\vec{v} = \vec{w} + \vec{z} = \vec{w}' + \vec{z}', \quad \vec{w}, \vec{w}' \in W, \quad \vec{z}, \vec{z}' \in W^{\perp}$$

$$\Rightarrow \underbrace{(\vec{w} - \vec{w}')}_{\in W} + \underbrace{(\vec{z} - \vec{z}')}_{\in W^{\perp}} = \vec{0}$$

If $\vec{w} - \vec{w}' \neq 0$, $\vec{z} - \vec{z}' \neq 0$, then they are lin. indep. $\Rightarrow \vec{w} = \vec{w}', \vec{z} = \vec{z}'$

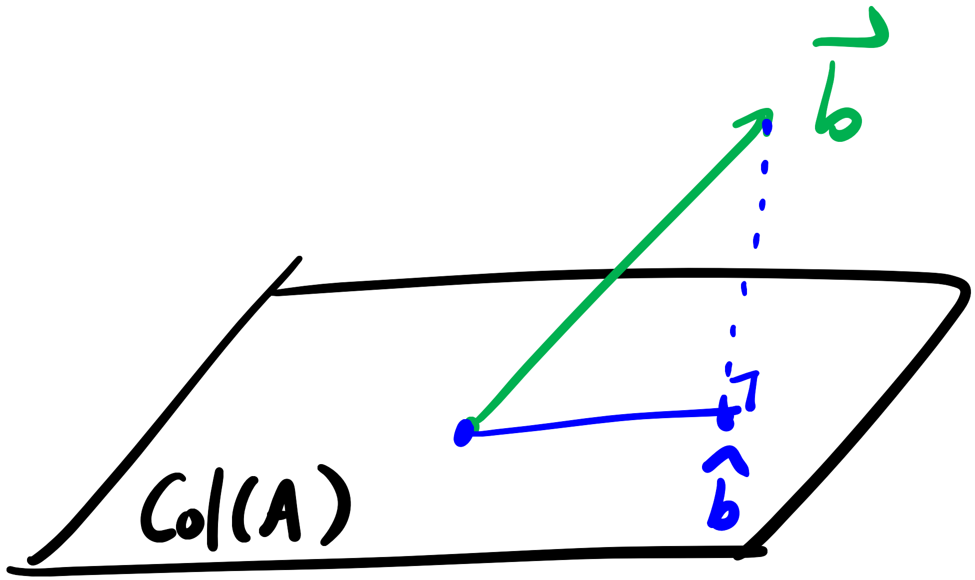
$\Rightarrow$ uniqueness.

Solve least-squares problem.

$\underbrace{A}_{\cap\ Col(A)} \vec{x} = \vec{b}$ has no sol.

$\vec{b}$: target vector.

$A \in \mathbb{R}^{m \times n}$. $\vec{x} \in \mathbb{R}^n$. $\vec{b} \in \mathbb{R}^m$



$\mathbb{R}^m$

$A\vec{x} = \hat{b}$, by definition solvable problem.

$\hat{b} = \mathrm{Proj}_{Col(A)} \vec{b}$

Least-squares.

$$\min_{\vec{x}} \| A\vec{x} - \vec{b} \|^2$$

<u>Thm</u>. (Best approximation).

$W \subseteq \mathbb{R}^n$ is a subspace. $\vec{v} \in \mathbb{R}^n$

$\vec{v} = \vec{w} + \vec{z}$, $\vec{w} \in W$, $\vec{z} \in W^\perp$.

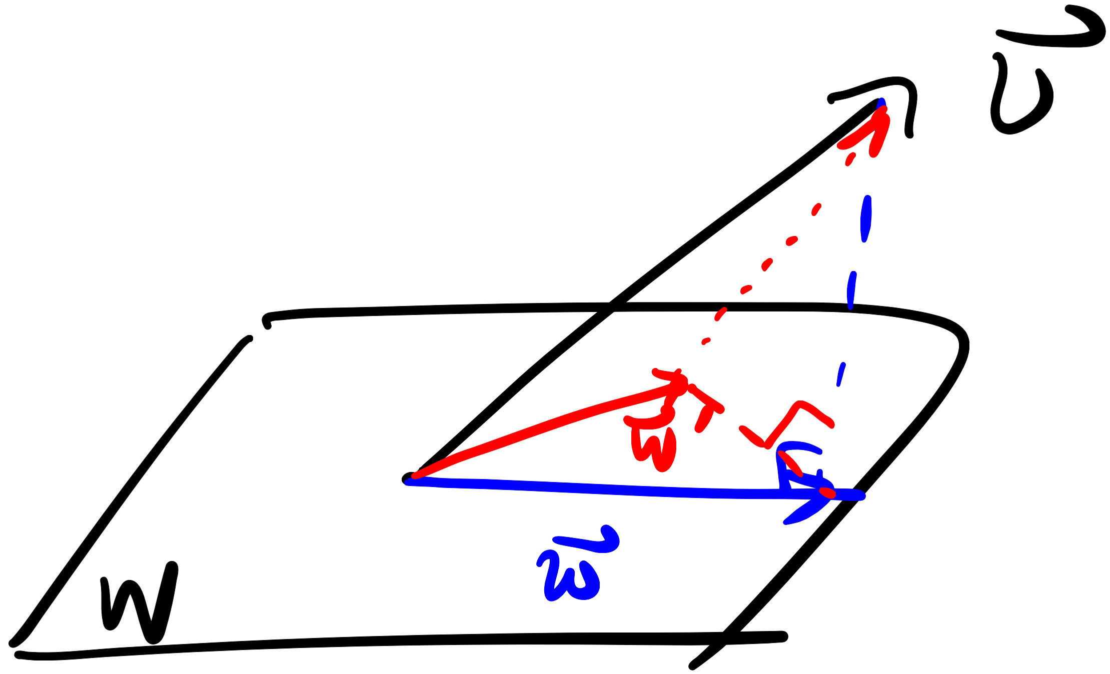
$\vec{w} = \mathrm{Proj}_W \vec{v}$. Then

$$\|\vec{z}\| = \|\vec{v} - \mathrm{Proj}_W \vec{v}\|$$

$\longrightarrow$ best approx.

$$\leq \|\vec{v} - \vec{w}'\|$$

For any $\vec{w}' \in W$.



$\vec{v}$
$\vec{w}$
$\vec{w}$
W