

## Lec 24. Warm up

$$\text{Ex. } \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

$$(1) \vec{v}_1 \perp \vec{v}_2 ?$$

$$\text{True. } \vec{v}_1 \cdot \vec{v}_2 = 0$$

$$(2) \vec{v}_1 \perp \vec{v}_3 ?$$

$$\vec{v}_1 \cdot \vec{v}_3 = (-1) + 2 \cdot 1 + (-1) \cdot 1 = 0. \text{ True.}$$

$$\text{Ex. } \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

(1) orthogonal set ?

$$\vec{v}_1 \cdot \vec{v}_2 = 2 - 2 + 0 = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = -2 + 2 + 0 = 0$$

True.

$$\vec{v}_2 \cdot \vec{v}_3 = -1 - 4 + 5 = 0$$

(2) orthonormal set ?

$$\|\vec{v}_1\| = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5} \neq 1$$

False.

Use diagonalization to  
save the world



Def .  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is called an orthogonal basis of subspace  $W \subseteq \mathbb{K}^n$  if

1) Orthogonal set

2) basis.

not too small :  $\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = W$

not too large :  $\{\vec{v}_1, \dots, \vec{v}_k\}$  lin. indep.

  
automatically satisfied by orthogonality

Thm.  $\{\vec{v}_1, \dots, \vec{v}_k\}$  is an orthonormal set, then this set is lin. indep.

Pf: Assume

$$c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$$

$$\vec{v}_1 \cdot (c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) = \vec{v}_1 \cdot \vec{0} = 0$$

$$c_1 \cdot 1 + 0 + \dots + 0 = 0 \Rightarrow c_1 = 0.$$

get rid  
of  $\vec{0}$

Apply to all  $\vec{v}_i$

$\Rightarrow c_1 = \dots = c_k = 0 \Rightarrow \text{lin. indep.} \quad \square$

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Why orthogonal basis is useful?

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$  is a basis of  $\mathbb{R}^n$ .

$\vec{v} \in \mathbb{R}^n$ .  $[\vec{v}]_B$

In Chap 4. form a lin. sys.

$$[\vec{b}_1 \dots \vec{b}_n \mid \vec{v}] \rightarrow \text{sol. } [\vec{v}]_{\mathcal{B}}.$$

Now if  $\mathcal{B}$  is an orthogonal basis.

$$\vec{c} \equiv [\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\vec{v} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$$

$$\vec{b}_1 \cdot \vec{v} = \vec{b}_1 \cdot (c_1 \vec{b}_1 + \dots + c_n \vec{b}_n)$$

$$= c_1 \vec{b}_1 \cdot \vec{b}_1 + 0 + \dots + 0$$

$$= c_1 \vec{b}_1 \cdot \vec{b}_1$$

$$c_i = \frac{\vec{b}_i \cdot \vec{v}}{\vec{b}_i \cdot \vec{b}_i}, \quad i=1, \dots, n.$$

No lin. sys. sol. Only 2 inner products.

$\mathcal{B}$  is orthonormal basis



$$c_i = \vec{b}_i \cdot \vec{v}$$

Ex.  $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ .

$$\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$$

Find  $[\vec{v}]_{\mathcal{B}}$ .  $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(exer) Find  $[\vec{v}]_{\mathcal{B}}$  by  $[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \mid \vec{v}]$ .

we have checked (in warm-up)  
that  $B$  is an orthogonal set.

orthogonal basis of  $\mathbb{R}^3$

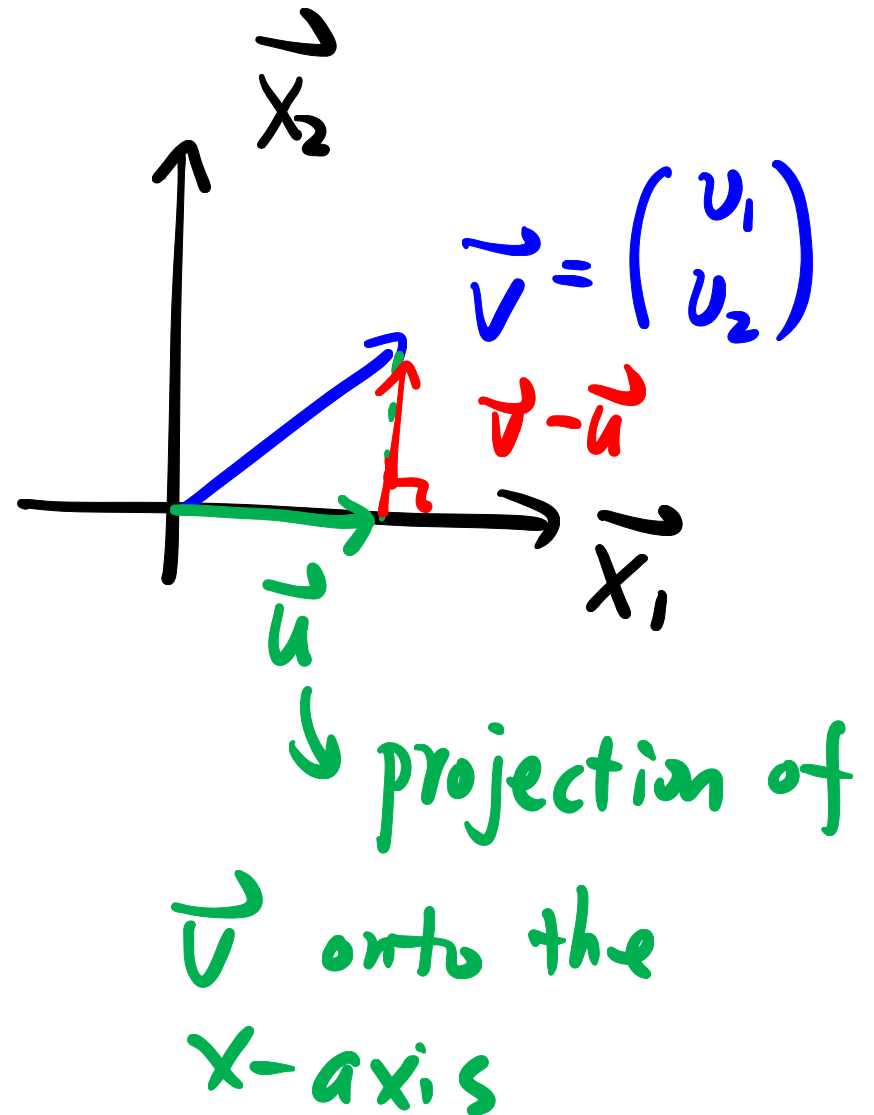
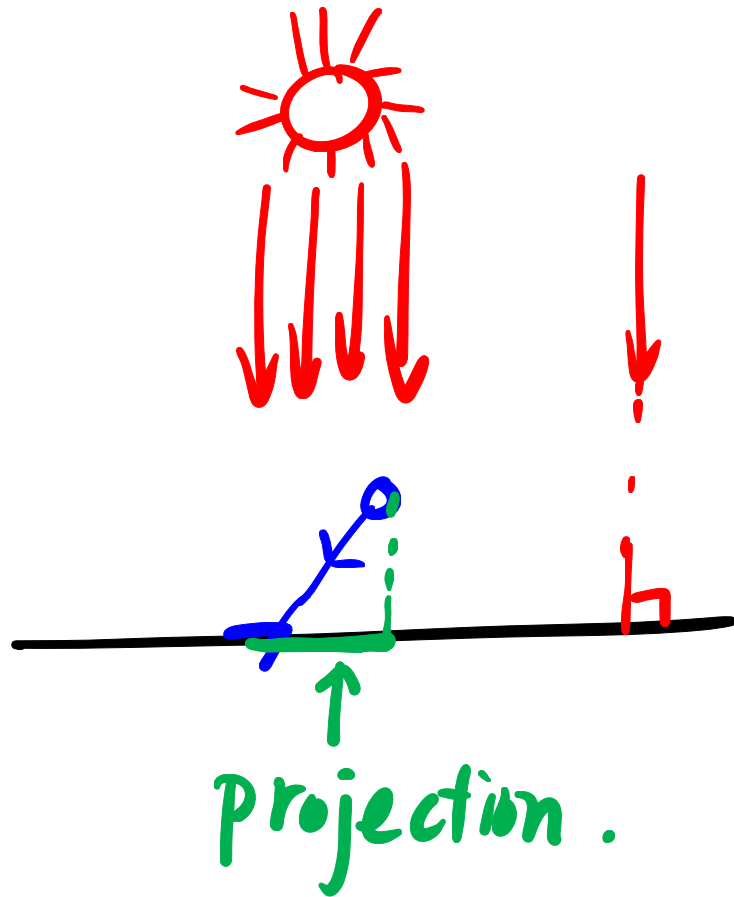
$$C_1 = \frac{\vec{b}_1 \cdot \vec{v}}{\vec{b}_1 \cdot \vec{b}_1} = \frac{2}{5}$$

$$C_3 = \frac{\vec{b}_3 \cdot \vec{v}}{\vec{b}_3 \cdot \vec{b}_3} = \frac{4}{30}$$

$$C_2 = \frac{\vec{b}_2 \cdot \vec{v}}{\vec{b}_2 \cdot \vec{b}_2} = \frac{2}{6} = \frac{1}{3}$$

$$= \frac{2}{15}$$

# Projection



$$\vec{u} = v_1 \vec{e}_1 = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \quad \vec{z} = \vec{v} - \vec{u} = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$$

$$\vec{u} \cdot \vec{z} = 0 \Rightarrow \vec{u} \perp \vec{z}$$

generalize to a more algebraic setting?

$$\text{Ex. } \vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find  $\alpha \in \mathbb{R}$  s.t.

$$\vec{v} = \alpha \vec{u} + \vec{z} \quad \text{and} \quad \vec{u} \cdot \vec{z} = 0$$

$$\vec{u} \cdot \vec{v} = \vec{u} \cdot (\alpha \vec{u} + \vec{z})$$

$$\Rightarrow \alpha = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} = \frac{3}{5}$$

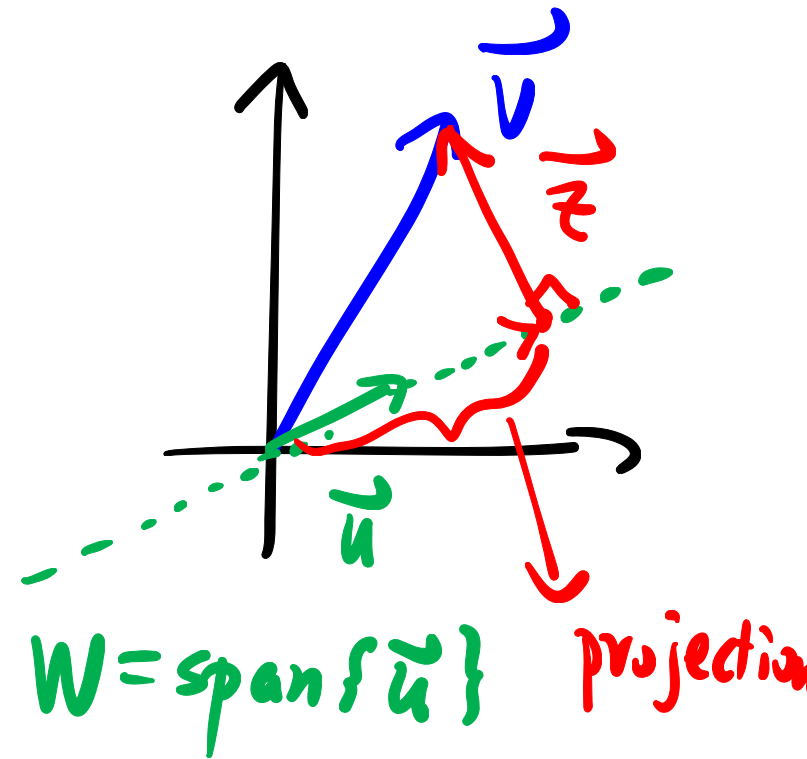
$$\vec{z} = \vec{v} - \alpha \vec{u} = \begin{bmatrix} 1 \\ \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$$

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$$\vec{v} \in \mathbb{R}^n, \quad \vec{0} \neq \vec{u} \in \mathbb{R}^n$$

$$W = \text{span}\{\vec{u}\} \rightarrow 1 \text{ dim.}$$

$$\vec{v} = \underbrace{\left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right)}_{\text{projection}} \vec{u} + \vec{z}$$



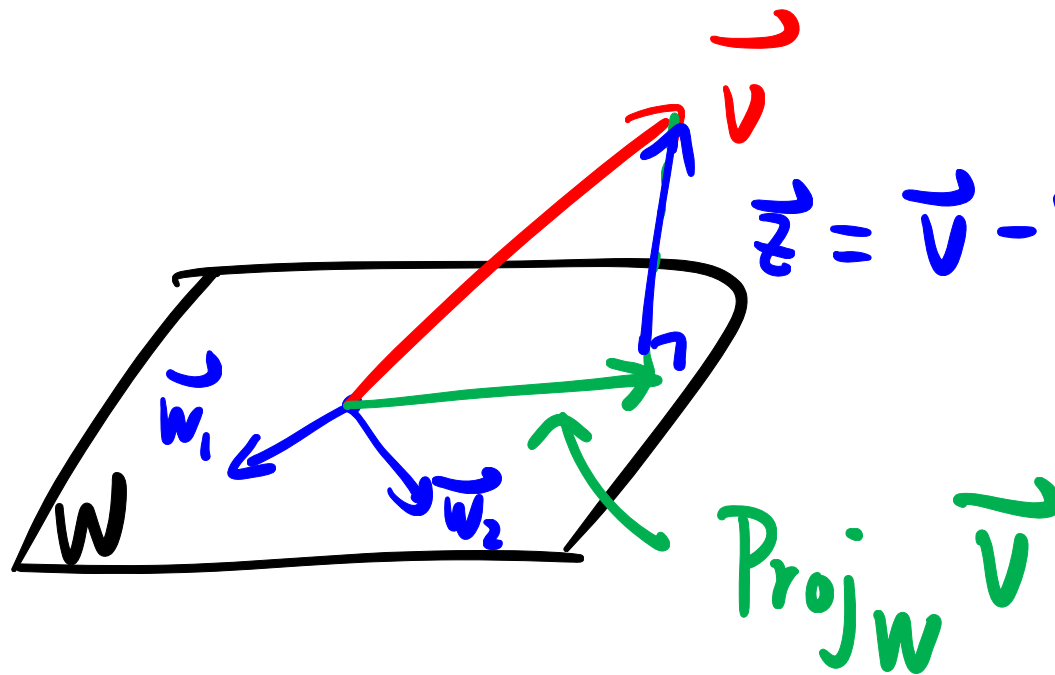
$$\vec{z} \cdot \vec{u} = 0$$

$$\text{Proj}_W \vec{v} = \left( \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u} \in W$$

$$\Rightarrow \vec{v} = \text{Proj}_W \vec{v} + \vec{z}$$

$W = \text{span}\{\vec{w}_1, \dots, \vec{w}_k\} \rightarrow k\text{-dim}$   
orthogonal basis set subspace of  $\mathbb{R}^n$

$k=2$



$$\vec{z} \cdot \vec{w}_1 = 0, \quad \vec{z} \cdot \vec{w}_2 = 0$$

Take any  $\vec{w} \in W$

$$\vec{w} = c_1 \vec{w}_1 + c_2 \vec{w}_2, \quad c_1, c_2 \in \mathbb{R}.$$

$$\vec{z} \cdot \vec{w} = c_1 \vec{z} \cdot \vec{w}_1 + c_2 \vec{z} \cdot \vec{w}_2 = 0$$

$$\vec{z} \perp \vec{w}, \quad \vec{w} \in W$$

$$\text{Proj}_W \vec{v} = \left( \frac{\vec{v} \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \right) \vec{w}_1 + \dots + \left( \frac{\vec{v} \cdot \vec{w}_k}{\vec{w}_k \cdot \vec{w}_k} \right) \vec{w}_k$$

You can check  $\vec{z} \perp (\vec{w}_1, \dots, \vec{w}_k)$



$$\text{Ex. } \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$W = \text{span} \{ \vec{w}_1, \vec{w}_2 \}.$$

Compute  $\text{Proj}_W \vec{v}$

Check first whether

$\{ \vec{w}_1, \vec{w}_2 \}$  is an orthogonal  
basis set.

$$\vec{w}_1 \cdot \vec{w}_2 = 1+1 = 2 \neq 0.$$

so Construct an orthogonal basis first

$$\vec{w}_2 = \text{Proj}_{\text{span}\{\vec{w}_1\}} \vec{w}_2 + \vec{w}_2^\perp$$

$$\vec{w}_2^\perp \perp \vec{w}_1 \Rightarrow \{\vec{w}_1, \vec{w}_2^\perp\} \text{ is}$$

an orthogonal basis.

$$\vec{\tilde{w}}_2 = \vec{w}_2 - \frac{\vec{w}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1} \vec{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Now apply formula (exer)

$$\text{Proj}_{\vec{w}} \vec{v} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$