Lec 5. Linear transformation.

Map / Mapping / Transformation :

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\vec{x} \in \mathbb{R}^n \mapsto T(\vec{x}) \in \mathbb{R}^m$$

$m=1$. $T : \mathbb{R}^n \to \mathbb{R}$. function

$T(\vec{x})$: image of $\vec{x}$ under T.

$D \subseteq \mathbb{R}^n$: $T(D) = \{ T(\vec{x}) \mid x \in D \}$

image of a set D under T.

$$\text{Image}(T) = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n \}, \quad \text{i.e., } D = \mathbb{R}^n$$

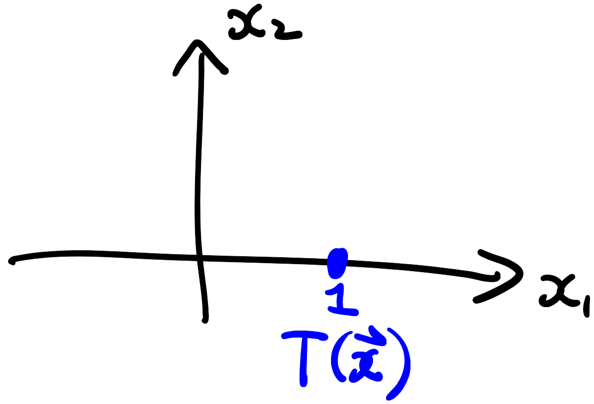
Matrix transformation. $A \in \mathbb{R}^{m \times n}$

$$T(\vec{x}) = A\vec{x} := x_1 \vec{a}_1 + \cdots + x_n \vec{a}_n$$

Ex. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2 \times 1}$ $\quad T : \mathbb{R} \to \mathbb{R}^2$

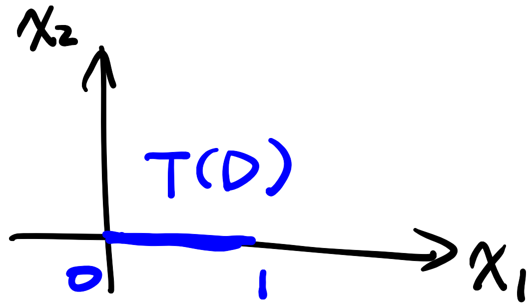
$x \in \mathbb{R}$, $T(x) = \begin{bmatrix} x \\ 0 \end{bmatrix}$

$x = 1$

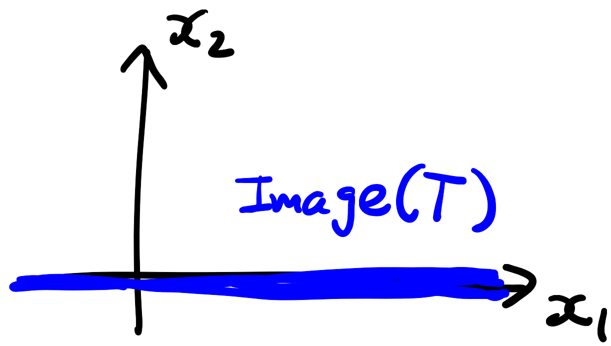


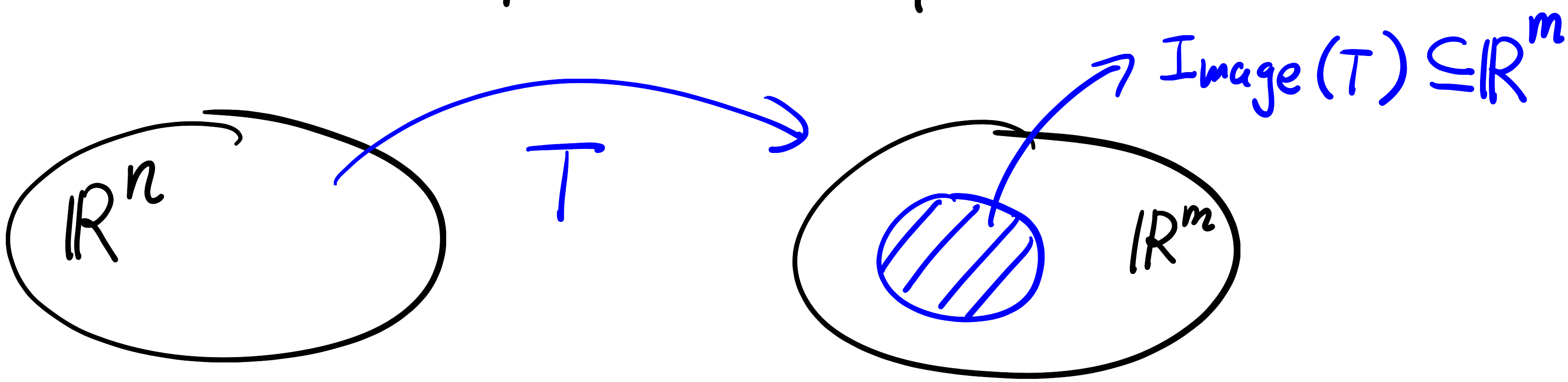
$D = [0, 1]$

$= \{x \mid 0 \le x \le 1\}$



Image(T)



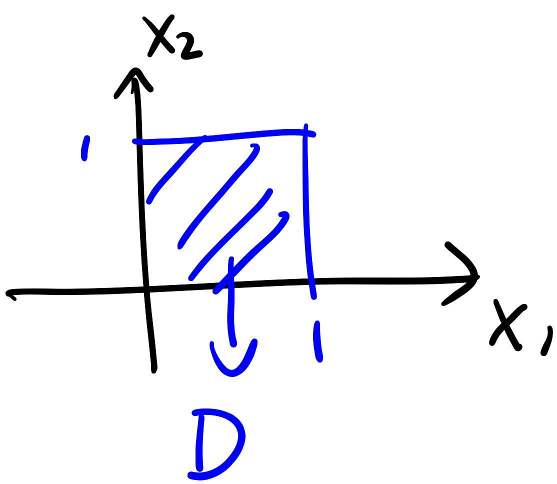


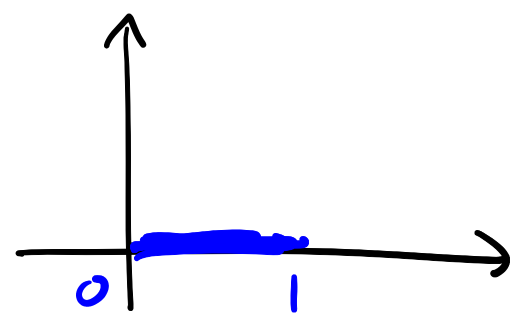
Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(\vec{x}) = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$D = [0,1] \times [0,1] := \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle|\ 0 \le x_1 \le 1,\ 0 \le x_2 \le 1 \right\}.$$





Projection

$$T(D) = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \middle|\ 0 \le x_1 \le 1 \right\}.$$

Def A transformation T is linear trans. if $\forall \vec{u}, \vec{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$

(1) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(2) $T(c\vec{u}) = c\,T(\vec{u})$

matrix trans. is a special case of lin. trans.

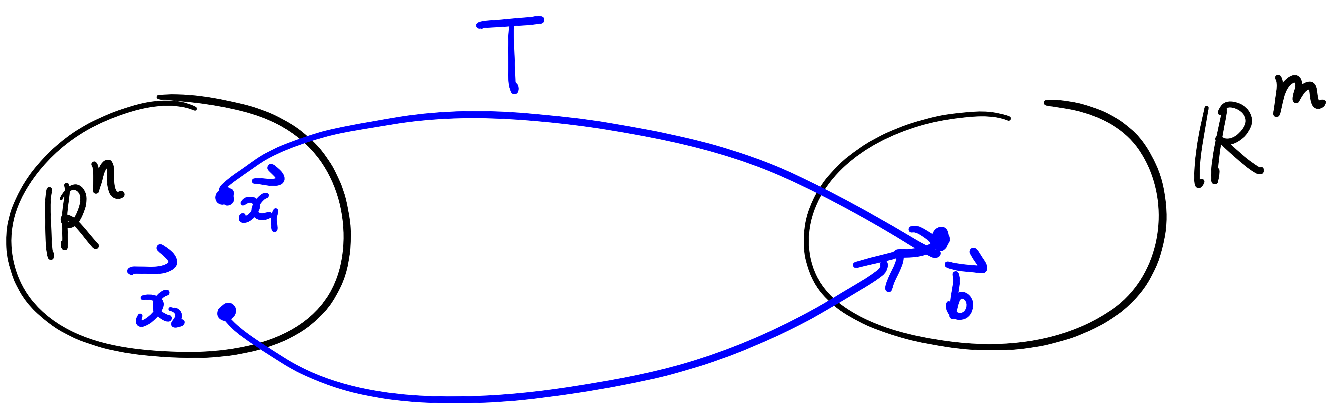
Def $T: \mathbb{R}^n \to \mathbb{R}^m$ lin. trans.

is onto (a.k.a. surjective)

if for each $\vec{b} \in \mathbb{R}^m$ there is

at least one $\vec{x} \in \mathbb{R}^n$ s.t.

$$T(\vec{x}) = \vec{b}.$$

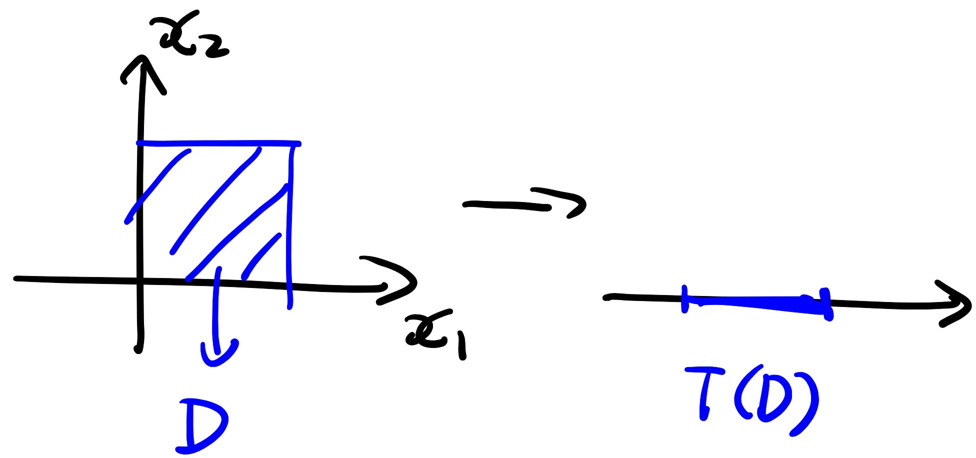


$\text{Image}(T) = \mathbb{R}^m$

Ex. $T: \mathbb{R}^2 \to \mathbb{R}$

$A = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$T(\vec{x}) = x_1$



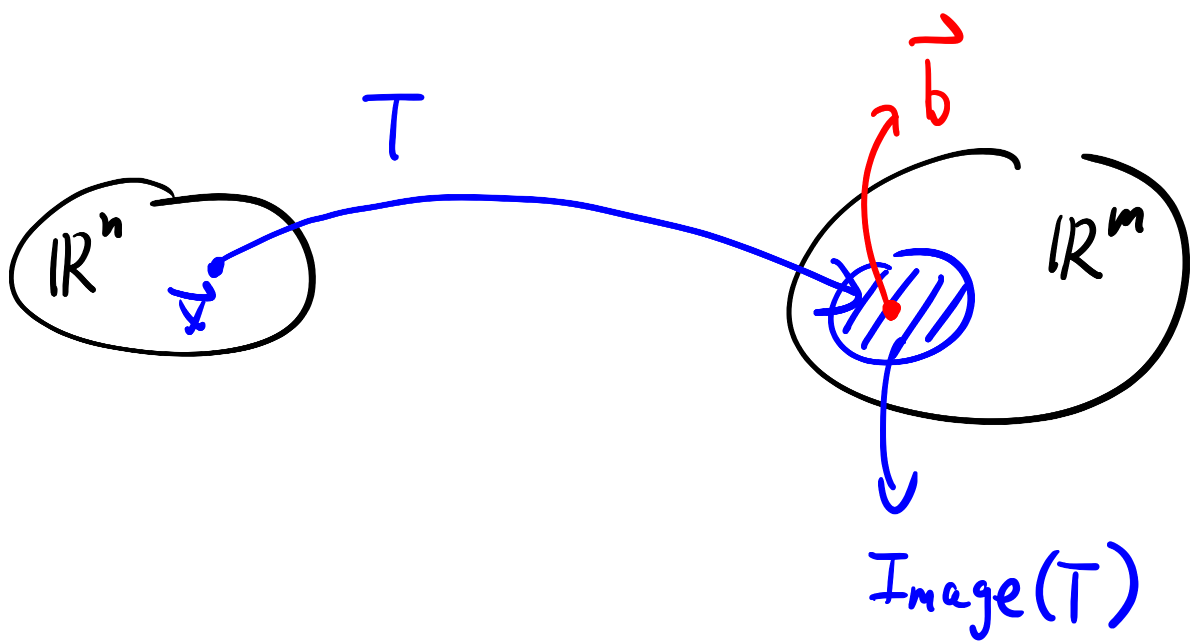
onto.

Def $T: \mathbb{R}^n \to \mathbb{R}^m$. lin. trans. is

one-to-one (a.k.a. injective)

if for each $\vec{b} \in \mathbb{R}^m$, there is at most one $\vec{x} \in \mathbb{R}^n$. s.t.

$$T(\vec{x}) = \vec{b}.$$



Both injective and surjective : bijective.

For any $\vec{b} \in \mathbb{R}^n$, there is unique $\vec{x} \in \mathbb{R}^m$ s.t. $T(\vec{x}) = \vec{b}$.

<u>Thm</u>. Any lin. trans. is a matrix trans.

<u>Ex</u>. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ T lin. trans.

$\vec{x} \in \mathbb{R}^2$. $\quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \vec{e_1} + x_2 \vec{e_2}$

$$T(\vec{x}) \overset{\text{linearity}}{=\!=} x_1 T(\vec{e_1}) + x_2 T(\vec{e_2})$$

$$= A\vec{x}. \quad A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}$$

Pf: $T : \mathbb{R}^n \to \mathbb{R}^m$. lin. trans.

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \vec{e_n} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\vec{x} = x_1 \vec{e_1} + \cdots + x_n \vec{e_n}$$

$$T(\vec{x}) = x_1 T(\vec{e_1}) + \cdots + x_n T(\vec{e_n})$$

$$= A\vec{x}.$$

$$A = \begin{bmatrix} T(\vec{e_1}) & T(\vec{e_2}) & \cdots & T(\vec{e_n}) \end{bmatrix}$$

$\uparrow$

standard matrix of T.

$\square$

Fact: $T: \mathbb{R}^n \to \mathbb{R}^m$ is lin. trans.

$$\Rightarrow \quad T(\underset{\mathbb{R}^n}{\underset{\cup}{\vec{0}}}) = \underset{\mathbb{R}^m}{\underset{\cup}{\vec{0}}}$$

for any $\vec{u} \in \mathbb{R}^n$.

$$T(0 \cdot \vec{u}) = T(\underbrace{\vec{0}}_{\mathbb{R}^n})$$

$$\|$$

$$\underbrace{0}_{\text{scalar}} \cdot T(\vec{u}) = \underbrace{\vec{0}}_{\mathbb{R}^m}$$

Ex. translation.

$T: \mathbb{R}^n \to \mathbb{R}^2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + c_1 \\ x_2 + c_2 \end{bmatrix}$$

is NOT a lin. trans.

$T(\vec{0}) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ shifted away from $\vec{0}$