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Theory of a Zeeman Ring Laser—Part II: Special Cases

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Abstract—**We apply the Zeeman ring laser theory to the case of four-mode operation in an inhomogeneously broadened medium in the Doppler limit. The coupling parameters between the four laser modes are studied quantitatively. Reductions to the standing wave laser and the scalar ring laser cases are made. The case of operation in a homogeneously broadened medium is also treated.**

I. Introduction

In 1964, Lamb [1] developed a semi-classical theory of the laser using two-level atoms and a classical scalar multimode standing-wave electric field. Experience has shown that the approach he took was well conceived, not only in its immediate predictions, e.g., the Lamb dip and mode locking, but also in the ease with which it can be extended to treat more complicated laser configurations. In particular, a number of authors [2]-[7] have treated scalar ring laser operation with two or more modes and weak or arbitrarily intense electric fields. In addition, various papers [8]-[10] have appeared on Zeeman lasers whose atoms have magnetic sublevels and whose cavities are anisotropic thus, requiring vector fields. Many of these problems are discussed in the textbook by Sargent, Scully, and Lamb [11].

The scalar ring laser and Zeeman laser problems were combined by Hanson and Sargent [12] (referred to as Paper I), the first paper in the present series. That paper gives the theory of a ring laser subject to a uniform axial dc magnetic field in an extension of the two-mirror standing-wave treatment of Sargent, Lamb, and Fork [9]. The active medium consists of moving atoms that have two electronic levels with arbitrary angular momenta. The electric field for the circularly polarized traveling waves is treated classically. Losses due to backscattering and cavity anisotropy are included. In addition, the results of a generalized treatment are given, which includes an arbitrarily oriented magnetic field, a given electric field polarization, varying isotopic abundance, and hyperfine structure. The Lamb self-consistency requirement is used to obtain amplitude and frequency determining equations for multimode operation as functions of laser parameters. Particularly useful in the analysis is the "perturbation tree" and its associated tables. The Zeeman ring laser theory presented is the most general and comprehensive (and, therefore, the most complicated) Lamb-theory extension to date.

In the present paper, we consider the simplest two- and four-mode special cases of the general theory. The discussion is valuable both in illustrating the theory and in providing bases for Zeeman ring laser applications [13] and polarized-field saturation spectroscopy [14]. Specifically, the saturation coefficients (9)-(17) may be used to derive saturation ratios $\theta_{\perp}/\theta_{11}$ in [14]. Section II gives tables for the four-mode coefficients (two counterrunning waves, each of which has two polarizations) for the Doppler broadened medium. We show that unlike the situation in simpler problems, a relative phase angle appears in the polarization of the medium. This presents a complication in the analysis.

A third paper [13] to be published at a later date will deal with the degree to which the four-mode problem can be reduced to four weakly coupled single-mode problems, and applied the results to the DILAG (differential laser gyro). Section III examines the degree of coupling between any two modes. Graphs illustrating the variations of the coupling parameters and intensities as functions of detuning and magnetic field are

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TABLE I
DEFINITION OF MODE SUBSCRIPTS IN (1) IN TERMS OF RUNNING-WAVES DIRECTION AND CLASSICAL FIELD POLARIZATIONS. ONE RING LASER CONSISTS OF MODES 2 AND 3, i.e., CONSISTS OF LEFT CIRCULAR POLARIZATIONS, AND THE OTHER IS MADE UP OF MODES 1 AND 4.

Mode subscript	Direction	Unit vector	Classical polarization
1	A	$\hat{\varepsilon}_-$	Anticlockwise, right circularly polarized
2	A	$\hat{\varepsilon}_+$	Anticlockwise, left circularly polarized
3	C	$\hat{\varepsilon}_-$	Clockwise, left circularly polarized
4	C	$\hat{\varepsilon}_+$	Clockwise, right circularly polarized

presented. Appendix A gives the reductions of the Zeeman ring laser four-mode problem to the two-mode standing-wave theory and the scalar ring laser theory. In Appendix B, we present the four-mode coefficient for a homogeneously broadened medium and we compare the coupling parameters for this case with those of an inhomogeneously broadened medium.

II. BIDIRECTIONAL FOUR-WAVE OPERATION IN THE STRONG DOPPLER LIMIT

In this section we consider two oppositely directed traveling waves, each with two orthogonal circular polarizations. This is basically the configuration for the differential laser gyro (DILAG) [16], [17]. Mathematically, this is a four-"mode" problem. The electric field is then given by

$$\begin{aligned}E(s,t) &= \tfrac{1}{2}\{E_1\hat{e}_- \exp[-i(\nu_1 t+\phi_1)]\\&\quad + E_2\hat{e}_+\exp[-i(\nu_2 t+\phi_2)]\}\exp(ik_A s)\\&\quad + \tfrac{1}{2}\{E_3\hat{e}_-\exp[-i(\nu_3 t+\phi_3)]\\&\quad + E_4\hat{e}_+\exp[-i(\nu_4 t+\phi_4)]\}\exp(-ik_C s) + \text{c.c.} \qquad (1)\end{aligned}$$

which induces the polarization of the medium

$$\begin{aligned}P(s,t) &= \tfrac{1}{2}\{P_1\hat{e}_- \exp[-i(\nu_1 t+\phi_1)]\\&\quad + P_2\hat{e}_+\exp[-i(\nu_2 t+\phi_2)]\}\exp(ik_A s)\\&\quad + \tfrac{1}{2}\{P_3\hat{e}_-\exp[-i(\nu_3 t+\phi_3)]\\&\quad + P_4\hat{e}_+\exp[-i(\nu_4 t+\phi_4)]\}\exp(-ik_C s) + \text{c.c.} \qquad (2)\end{aligned}$$

Here the subscript $C(A)$ denotes clockwise (anticlockwise) traveling waves, and s is the displacement along the axis in the ring cavity. The $\hat{e}_\pm$ (circularly polarized) unit vectors are defined by $\hat{e}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}$. Table I defines this notation in terms of that used in classical optics.

The electric field amplitude and frequency determining equations for the general case, which are (72) and (73) in Paper I, are

$$\dot{E}_n = E_n\left\{\alpha_n - \sum_{m=1}^{4}\theta_{nm}E_m^2 - \text{Im}\,[(\theta_{nijk}+\theta_{nkji}) \cdot \exp(-ip_n d_n\psi)]E_n^{-1}E_iE_jE_k\right\} \qquad (3)$$

and

$$\nu_n + \dot{\phi}_n = \Omega_n + \sigma_n - \sum_{m=1}^{4}\tau_{nm}E_m^2 - \text{Re}\,[(\theta_{nijk}+\theta_{nkji}) \cdot \exp(-ip_n d_n\psi)]E_n^{-1}E_iE_jE_k \qquad (4)$$

where the relative phase angle $\psi = (\nu_1 - \nu_2 - \nu_3 + \nu_4)t + \phi_1 - \phi_2 - \phi_3 + \phi_4$, Ω_n's are the passive cavity frequencies, α_n is the nth node net gain, σ_n is the nth mode frequency pulling coefficient, the Q_n's and the frequency biasing are absorbed into the α_n's and the Ω_n's, θ_{nm}'s are the cross saturation coefficients for $n \neq m$ and the self-saturation coefficient for $n = m$, τ_{nm} are the cross-mode pushing coefficients for $n \neq m$ and the self-mode pushing coefficient for $n = m$, and θ_{nijk}'s are the relative phase angular coefficients which are nonzero only for the combinations of n, i, j, and k listed in Part I, Table IV, and $p_n = +(-)$ for $n = 2, 4\ (1, 3)$ and $d_n = +(-)$ for $n = 1, 2\ (3, 4)$.

In the present paper, we simplify the complicated amplitude and frequency determining (3) and (4) by making the following assumptions. We take the strong Doppler limit, i.e., the Doppler width, $Ku \gg$ decay rates, and detuning where K is the laser field wavenumber and u is the average atomic velocity. The $T_{\tau w}$ integrals, which contribute to the nonlinear coefficients, in this limit are given in Part I, Table IV. In addition, we assume that the decay rates $\gamma_{a'b'} = \gamma_{a''b'} = \gamma_{a''b''} = \gamma_{a'b''} = \gamma_{ab'}$, $\gamma_{a'a''} = \gamma_a$, $\gamma_{b''b'} = \gamma_b$, and $|N_{\tau w}| = 1/L\int_0^L ds\,(\lambda_a/\gamma_a - \lambda_b/\gamma_b) \equiv \bar{N}$ where the excitation rates for the different "a" sublevels are equal and those for all the "b" sublevels are also equal (see Part I, Fig. 2 for level structure). Written explicitly, the T integrals are

$$\begin{aligned}\text{Type 1:}\quad T_{21} + T_{31} &= i\sqrt{\pi}\,\bar{N}/2\,D(\omega_{a'b'}/2 - \omega_{a''b''}/2\\&\quad -\nu_\mu/2 + \nu_\rho - \nu_\sigma/2)\,[D_a(\omega_{a'a''} + \nu_\rho - \nu_\sigma)\\&\quad + D_b(\omega_{b''b'} + \nu_\rho - \nu_\mu)] \qquad (5)\end{aligned}$$

$$\begin{aligned}\text{Type 2:}\quad T_{41} &= i\sqrt{\pi}\,\bar{N}\,D(\omega_{a'b'}/2 + \omega_{a''b'}/2 - \nu_\mu\\&\quad + \nu_\rho/2 - \nu_\sigma/2)D_b(\omega_{b''b'} + \nu_p - \nu_\mu) \qquad (6)\end{aligned}$$

$$\begin{aligned}\text{Type 3:}\quad T_{11} &= i\sqrt{\pi}\,\bar{N}\,D(\omega_{a'b'}/2 + \omega_{a'b''}/2 - \nu_\mu/2\\&\quad + \nu_\rho/2 - \nu_\sigma)D_a(\omega_{a'a''} + \nu_\rho - \nu_\sigma) \qquad (7)\end{aligned}$$

where

$$D(\Delta\omega) = (\gamma_{ab} - i\Delta\omega)^{-1}$$
$$D_\alpha(\Delta\omega) = (\gamma_\alpha + i\Delta\omega)^{-1}$$
$$\alpha = a, b$$

and all other T's are zero. Furthermore, we assume (for the moment) that there is no backscattering or $x - y\ Q$ and phase anisotropy for the different traveling waves. This leads to a diagonal $\overleftrightarrow{G}$ matrix [12] where the $\text{Re}(g_{nn}) = 1/Q_n$ and $\text{Im}(g_{nn})$ accounts for the frequency biasing introduced into the cavity.

The conservation of angular momentum ($a' = b' \pm 1$), wave vector ($k_n = k_\mu - k_\rho + k_\sigma$), and energy (rotation wave approximation in the interaction potential) reduces the nonlinear contribution to 36 terms. These are summarized in Tables II and III where we show the magnetic sublevel transitions involved for each term, the type of integral encountered, and the coefficient in the electric field amplitude and fre-

TABLE II

SUMMARY OF THE $\sigma_{n\mu\rho\sigma}$ CALCULATION WHICH YIELDS THE COEFFICIENTS GIVEN IN (9)–(17). THE DIRECTION AND CHANGE IN MAGNETIC QUANTUM NUMBER ARE INDICATED BY n_n AND p_n. THE MAGNETIC SUBLEVEL TRANSITIONS ARE GIVEN BY THE DASHES IN COLUMN EIGHT. A LEFT (RIGHT) SLANTING DASH, (\) (/), INDICATES A CHANGE IN THE MAGNETIC QUANTUM NUMBER OF −1 (+0). THE NINTH COLUMN INDICATES THE θ_{nm} TO WHICH THE PARTICULAR $\sigma_{n\mu\rho\sigma}$ BELONGS.

$\vec{n}\vec{\mu}\vec{\rho}\vec{\sigma}$	$d_{\vec{\mu}}$	$d_{\vec{\rho}}$	$d_{\vec{\sigma}}$	Integral Type	$d_{\vec{n}}$	$p_{\vec{n}}$	Transitions	$\theta_{\vec{n}\vec{m}}$
1111	+	+	+	1	+	-	\\\	11
2222	+	+	+	1	+	+	///	22
3333	-	-	-	1	-	-	\\\	33
4444	-	-	-	1	-	+	///	44
1122	+	+	+	1	+	-	//\	12
1221	+	+	+	1	+	-	\//	12
2211	+	+	+	1	+	+	/\\	21
2112	+	+	+	1	+	+	\\/	21
3344	-	-	-	1	-	-	//\	34
3443	-	-	-	1	-	-	\//	34
4433	-	-	-	1	-	+	/\\	43
4334	-	-	-	1	-	+	\\/	43
1133	+	-	-	3	+	-	\\\	13
1331	-	-	+	2	+	-	\\\	13
3311	-	+	+	3	-	-	\\\	31
3113	+	+	-	2	-	-	\\\	31
2244	+	-	-	3	+	+	///	24
2442	-	-	+	2	+	+	///	24
4422	-	+	+	3	-	+	///	42
4224	+	+	-	2	-	+	///	42
2233	+	-	-	3	+	+	/\\	23
2332	-	-	+	2	+	+	\\/	23
3322	-	+	+	3	-	-	//\	32
3223	+	+	-	2	-	-	\//	32
1144	+	-	-	3	-	-	//\	14
1441	-	-	+	2	-	-	\//	14
4411	-	+	+	3	+	+	/\\	41
4114	-	-	+	2	+	+	\\/	41

TABLE III

SUMMARY OF THE COEFFICIENTS OF THE TERMS CONTAINING PHASE ANGLES

$\vec{n}\vec{\mu}\vec{\rho}\vec{\sigma}$	$d_{\vec{\mu}}$	$d_{\vec{\rho}}$	$d_{\vec{\sigma}}$	Integral Type	$d_{\vec{n}}$	$p_{\vec{n}}$	Transitions	Phase Angle
1243	+	-	-	3	+	-	\//	$+\psi$
1342	-	-	+	2	+	-	//\	
2134	+	-	-	3	+	+	\\/	$-\psi$
2431	-	-	+	2	+	+	/\\	
3421	-	+	+	3	-	-	\//	$-\psi$
3124	+	+	-	2	-	-	//\	
4312	-	+	+	3	-	+	\\/	$+\psi$
4213	+	+	-	2	-	+	/\\	

quency determining equations [see (3) and (4)] to which the term contributes.

A tedious calculation of the atomic polarization to third order gives the following expressions for the coefficients appearing on the right-hand side (RHS) of (3) and (4). They are, for the linear net gain and mode pulling coefficients,

$$\alpha_n = F_1 \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} \cdot |\wp_{a'b'}|^2 Z_i [\gamma_{ab} + i(\omega_{a'b'} - \nu_n)] - \nu/(2Q_n) \tag{8}$$

and

$$\sigma_n = F_1 \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} \cdot |\wp_{a'b'}|^2 Z_r [\gamma_{ab} + i(\omega_{a'b'} - \nu_n)] \tag{9}$$

where $a'b'$ denotes the laser upper and lower levels, $F_1 = \nu \overline{N}(\hbar K u \epsilon_0)^{-1}$, $\nu = \sum_{i=1}^{4} \nu_i/4$, Z_r and Z_i are real and imaginary parts of the plasma dispersion function [1], [15], and so far the results are also valid for the non-Doppler limit case. $\wp_{a'b'}$ is the dipole matrix element between levels a' and b'. For brevity, the third-order coefficients are given only for the strong Doppler limit case. These are the self-saturation coefficients

$$\theta_{nn} = 4F_3 \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\wp_{a'b'}|^4 \tag{10}$$

$$\tau_{nn} = 0 \tag{11}$$

and the cross-saturation coefficients which can be divided to three groups. There are coefficients arising from coupling between unidirectional modes ($nm = 12, 21, 34, 43$),

$$\begin{aligned} \theta_{nm} = {} & 2F_3/\overline{\gamma} \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\wp_{a'b'}|^2 |\wp_{a'-2p_n,b'}|^2 \gamma_a L(\delta_a) \\ & * \{1 + \gamma_b/\gamma_a [1 - 2\delta_a^2/(\gamma_a \gamma_{ab})] L_a(2\delta_a)\} \\ & + \text{same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longrightarrow \delta_b \\ & \text{and } \wp_{a'-2p_n,b'} \longrightarrow \wp_{a',b'+2p_n} \end{aligned} \tag{12}$$

and

$$\begin{aligned} \tau_{nm} = {} & 2F_3/\overline{\gamma} \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\wp_{a'b'}|^2 |\wp_{a'-2p_n,b'}|^2 \\ & \cdot \gamma_a L(\delta_a)(p_n \delta_a/\gamma_{ab}) * [1 + \gamma_b/\gamma_a(1 + 2\gamma_{ab}/\gamma_a) \\ & \cdot L_a(2\delta_a)] + \text{same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longrightarrow \delta_b \\ & \text{and } \wp_{a'-2p_n,b'} \longrightarrow \wp_{a',b'+2p_n} \end{aligned} \tag{13}$$

where

$$\begin{aligned} F_3 &= \tfrac{1}{4} \pi^{1/2} (\hbar^2 \gamma_a \gamma_b \gamma_{ab})^{-1} \overline{\gamma} F_1 \\ \overline{\gamma} &= (\gamma_a + \gamma_b)/2 \\ L(\Delta\omega) &= \gamma_{ab}^2 [\gamma_{ab}^2 + \Delta\omega^2]^{-1} \\ L_\alpha(\Delta\omega) &= \gamma_\alpha^2 [\gamma_\alpha^2 + \Delta\omega^2]^{-1} \\ \delta_\alpha &= \mu_B H g_\alpha/\hbar + (\nu_- - \nu_+)/2. \end{aligned}$$

+, − refers to the modes which cause the transitions $a' = b' + 1$, $a' = b' - 1$ (e.g., ν_1 is ν_- while ν_4 is ν_+), μ_B is the Bohr magneton, H is the external axial magnetic field, and g_α is the Lande g factor for level α. The coupling between bidirectional modes of the same polarization contributes to the coefficients ($nm = 14, 41, 23, 32$)

$$\begin{aligned} \theta_{nm} = {} & 2F_3/\overline{\gamma} \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\wp_{a'b'}|^2 |\wp_{a'-2p_n,b'}|^2 \\ & \cdot \gamma_a L(\omega_{a'b'} - p_n \delta_a - \nu_n) \\ & + \text{same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longrightarrow \delta_b \\ & \text{and } \wp_{a'-2p_n,b'} \longrightarrow \wp_{a',b'+2p_n} \end{aligned} \tag{14}$$

and

$$\tau_{nm} = 2F_3/\bar{\gamma} \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a'-2p_n,b'}|^2$$
$$\cdot \gamma_a(\omega_{a'b'} - p_n\delta_a - \nu_n)/\gamma_{ab} \, L(\omega_{a'b'} - p_n\delta_a - \nu_n)$$
$$+ \text{ same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longrightarrow \delta_b$$
$$\text{and } \mathcal{P}_{a'-2p_n,b'} \longrightarrow \mathcal{P}_{a'b'+2p_n}. \tag{15}$$

Next, we have the coefficients due to the coupling between bidirectional oppositely polarized modes ($nm = 13, 31, 24, 42$):

$$\theta_{nm} = 4F_3 \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\mathcal{P}_{a'b'}|^4 L[\omega_{a'b'} - (\nu_n + \nu_m)/2] \tag{16}$$

and

$$\tau_{nm} = 4F_3 \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\mathcal{P}_{a'b'}|^4$$
$$\cdot [\omega_{a'b'} - (\nu_n + \nu_m)/2]/\gamma_{ab} \, L[\omega_{a'b'} - (\nu_n + \nu_m)/2]. \tag{17}$$

Finally, for the combinations of n, i, j, k given in Table III,

$$\theta_{nijk} + \theta_{nkji} = i2F_3/\bar{\gamma} \sum_{a'} \sum_{b'} \delta_{a',b'+p_n} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a'-2p_n,b'}|^2$$
$$\cdot \gamma_a\gamma_b\gamma_{ab} \, D[\omega_{a'b'} - (\nu_i - \nu_j)/2 - \nu_k]$$
$$\cdot D_a(2\delta_a p_n) + \text{ same with } \gamma_a \longleftrightarrow \gamma_b,$$
$$\delta_a \longrightarrow \delta_b \text{ and } \mathcal{P}_{a'-2p_n b'} \longrightarrow \mathcal{P}_{a',b'+2p_n}. \tag{18}$$

The frequency determining equation (4) may be recast into an equation of motion for ψ.

$$\dot{\psi} = a + b \sin\psi + c \cos\psi \tag{19}$$

where

$$a = \sum_{n=1}^{4} \left[-p_n d_n \left(\Omega_n + \sigma_n - \sum_{m=1}^{4} \tau_{nm} E_m^2 \right) \right] \tag{20}$$

$$b = \operatorname{Im} \{E_2E_3 [(\theta_{1243} + \theta_{1342})E_4/E_1 + (\theta_{4213} + \theta_{4312})E_1/E_4]$$
$$+ E_1E_4 [(\theta_{2134} + \theta_{2431})E_3/E_2 + (\theta_{3124} + \theta_{3421})E_2/E_3]\} \tag{21}$$

$$c = \operatorname{Re}\{-E_2E_3 [(\theta_{1243} + \theta_{1342})E_4/E_1$$
$$+ (\theta_{4213} + \theta_{4312})E_1/E_4] + E_1E_4[(\theta_{2134} + \theta_{2431})E_3/E_2$$
$$+ (\theta_{3124} + \theta_{3421})E_2/E_3]\}. \tag{22}$$

The time derivative of ψ vanishes for two special cases: 1) when we have standing waves in the cavity, i.e., $\nu_1 = \nu_3$ and $\nu_2 = \nu_4$; and 2) when we have linearly polarized running waves in the cavity, i.e., $\nu_1 = \nu_2$ and $\nu_3 = \nu_4$. For these two cases, the terms containing the exponential of ψ in (3) and (4) may not be assumed to be rapidly varying in time and, therefore, cannot be discarded. In particular, the reduction to the cases of standing wave operation in a two-mirror cavity and scalar ring laser operation (Appendix A) will not be possible without the phase angle terms.

In order to neglect the phase angle terms in (3) and (4), the frequency differences of all four modes must be significant. This is the case for the DILAG [13], [16], [17]. In this decoupled approximation, (3) becomes

$$\dot{I}_n = 2I_n \left(\alpha_n - \sum_{n=1}^{4} \theta_{nm} I_m \right) \tag{23}$$

where the dimensionless intensity of the nth mode is

$$I_n = (\mathcal{P}E_n/\hbar)^2/(2\gamma_a\gamma_b)$$

and

$$\mathcal{P} = \langle n_a J_a \| er \| n_b J_b \rangle.$$

A method for solving (23) is given in Sargent *et al.* [11] and in O'Bryan and Sargent [6]. Before proceeding further with obtaining the solutions (23), we will examine more quantitatively the coupling between the different laser modes.

III. Two-Mode Coupling Parameter

In this section we examine the two-mode coupling parameters appearing in the intensity equations (23). Graphs of coupling parameters and intensities versus detuning and magnetic field strength are presented.

At steady state, the coupling between modes is characterized by the set of parameters [11]

$$C_{nm} = \theta_{nm}\theta_{mn}/(\theta_{nn}\theta_{mm}). \tag{24}$$

The summation over b' in (10)-(16) can easily be performed. Assuming that the gyromagnetic ratios for levels a and b are equal, the resulting expressions for the θ's are products of Lorentzians with multiplicative factors M_1, M_2, M_3, or M_4 (defined below). These sums can be readily evaluated using (27) Part I. Thus,

$$M_1 = \sum_{a'} \sum_{b'} \delta_{a',b' \pm 1} |\mathcal{P}_{a'b'}|^4$$
$$= \mathcal{P}^4/60 \begin{cases} J(J+1)(2J+1)(2J^2+2J+1) & \Delta J = 0 \\ (J+1)(2J+1)(2J+3)(6J^2+12J+5) & \Delta J = \pm 1 \end{cases} \tag{25}$$

$$M_2 = 2 \sum_{a'} \sum_{b'} \delta_{a',b' \pm 1} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a',b' \pm 2}|^2$$
$$= \mathcal{P}^4/60 \begin{cases} J(J+1)(2J+1)(2J-1)(2J+3) & \Delta J = 0 \\ (J+1)(J+2)(2J+1)(2J+3)(2J+5) & \Delta J = \pm 1 \end{cases} \tag{26}$$

$$M_3 = 2 \sum_{a'} \sum_{b'} \delta_{a',b' \pm 1} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a' \mp 2,b'}|^2$$
$$= \mathcal{P}^4/60[J(J+1)(2J+1)(2J-1)(2J+3)] \qquad \Delta J = 0, \pm 1 \tag{27}$$

and

$$M_4 = \sum_{a'} \sum_{b'} \delta_{a',b' \pm 1} |\mathcal{P}_{a'b'}|^2 \, [|\mathcal{P}_{a' \mp 2,b'}|^2 + |\mathcal{P}_{a',b' \pm 2}|^2]$$

$$= \wp^4/60 \begin{cases} J(J+1)(2J+1)(2J-1)(2J+3) & \Delta J = 0 \\ (J+1)(2J+1)(2J+3)(2J^2+4J+5) & \Delta J = \pm 1 \end{cases} \quad (28)$$

where $J = J_a$.

Using (12)-(18) and (25)-(28) in (24), one has for the coupling between modes of different polarizations and the same direction

$$C_{12} = C_{21} = C_{34} = C_{43}$$
$$= \frac{1}{(2\bar{\gamma}M_1)^2} \left\{ L(\delta_a) \left[\gamma_a + \gamma_b \left(1 - \frac{2\delta_a^2}{\gamma_a \gamma_{ab}} \right) L_a(2\delta_a) \right] \frac{M_3}{2} + L(\delta_b) \left[\gamma_b + \gamma_a \left(1 - \frac{2\delta_b^2}{\gamma_b \gamma_{ab}} \right) L_b(2\delta_b) \right] \frac{M_2}{2} \right\}. \quad (29a)$$

The maximum value for these coupling parameters occurs when δ_a and δ_b are zero. Therefore, there is a magnetic field strength where these coupling parameters will have the maximum values

$$C_{12} = C_{21} = C_{34} = C_{43}$$
$$= \begin{cases} \left[\dfrac{(2J-1)(2J+3)}{(2J^2+2J+1)} \right]^2 & \Delta J = 0 \\ \left[\dfrac{2J^2+4J+5}{6J^2+12J+5} \right]^2 & \Delta J = \pm 1 \end{cases} \quad (29b)$$

Similarly, one has for the coupling parameters between modes of different polarizations and different directions

$$C_{\substack{13\\24}} = C_{\substack{31\\42}} = L^2 \left[\omega_0 \mp \mu_B H g/\hbar - (\nu_{\substack{1\\2}} + \nu_{\substack{3\\4}})/2 \right] \quad (30)$$

and between modes of the same polarization and different directions

$$C_{\substack{23\\14}} = C_{\substack{32\\41}} = [(\gamma_a M'' + \gamma_b M')/\bar{\gamma}]^2 \, L^2 \left[\omega_0 - (\nu_{\substack{2\\1}} + \nu_{\substack{3\\4}})/2 \right] \quad (31)$$

where

$$M' = \frac{1}{4} \frac{M_2}{M_1} = \begin{cases} \dfrac{(2J-1)(2J+3)}{4(2J^2+2J+1)} & \Delta J = 0 \\ \dfrac{(J+2)(2J+5)}{4(6J^2+12J+5)} & \Delta J = \pm 1 \end{cases} \quad (32)$$

$$M'' = \frac{1}{4} \frac{M_3}{M_1} = \begin{cases} \dfrac{(2J-1)(2J+3)}{4(2J^2+2J+1)} & \Delta J = 0 \\ \dfrac{J(2J-1)}{4(6J^2+12J+5)} & \Delta J = \pm 1. \end{cases} \quad (33)$$

These results agree with those of Fradkin and Khayutin [18]. Equation (30) indicates that the coupling parameters for modes of different polarizations and opposite directions are independent of J values and are always neutral ($C = 1$) or weak ($C < 1$). Fig. 1 is a plot of (28), and it shows that for modes of different polarizations and the same direction, the coupling is always weak for $\Delta J = \pm 1$. However, the $\Delta J = 0$ transitions

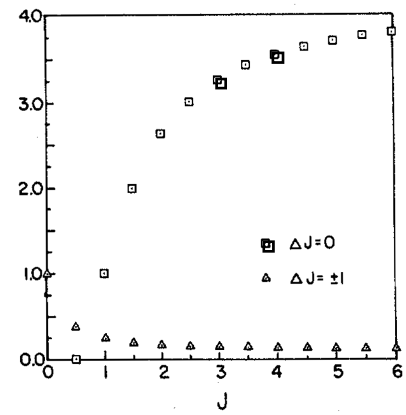

Fig. 1. C_{max} versus J: the maximum value of coupling parameter for waves with different polarizations traveling in the same direction (1-2 and 3-4) is plotted as a function of upper level angular momentum J. $\gamma_a = 18$ MHz and $\gamma_b = 40$ MHz.

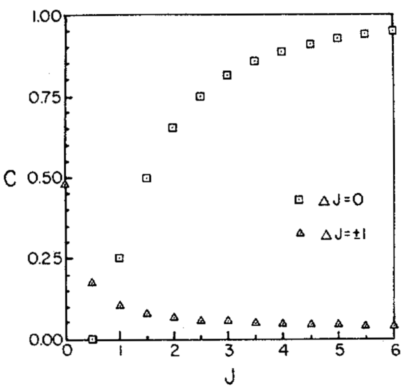

Fig. 2. C_{max} versus J: the maximum value of coupling parameter for waves of same polarization and opposite directions (1-3, 2-4) is plotted as a function of upper level angular momentum J. $\gamma_a = 18$ MHz and $\gamma_b = 40$ MHz.

lead to strong coupling when $J > 1$. Equation (30), for no detuning, is plotted in Fig. 2. We see that the coupling is always weak for modes of the same polarization and opposite directions.

The magnetic field dependence of the coupling parameters is shown in Figs. 3-5 for the transitions ($J_a \to J_b$), $1 \to 0$, $1 \to 2$, and $2 \to 2$. One can see that the only strongly coupled case occurs between unidirectional waves for the $2 \to 2$ transition. The coupling between bidirectional waves of different polarizations (1-3) has the same form for different J transitions and has a maximum value of 1. Coupling between bidirectional waves of the same polarization (1-4) is always weak and is independent of the magnetic field.

Fig. 6(a)-(c) are plots of intensities versus detuning for the $J_a = 1$ to $J_b = 0$ transition. As expected, a dip appears only for the bidirectional cases. Since bidirectional waves of different polarizations (1-3) interact with the same magnetic sublevels, the dip for this case is deeper than that from bidirectional waves of the same polarizations (1-4).

Finally, as shown in Appendix A, the standing wave Zeeman laser is also covered by our theory. Fig. 7 shows the intensities of the two Zeeman laser modes versus detuning. Superimposed on this plot are the mode intensities of a unidirectional ring laser.

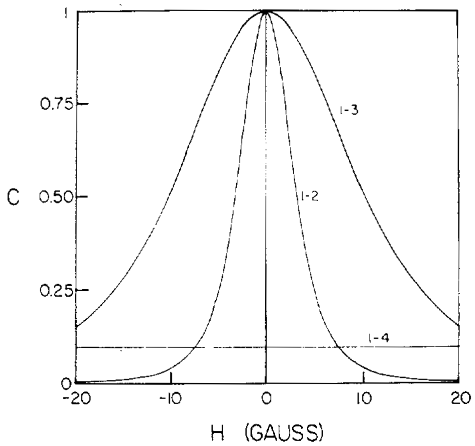

Fig. 3. C versus axial magnetic field under central tuning for $J_a = 1$, $J_b = 0$, and mode combinations 1-2, 1-3, and 1-4. The other laser parameters are $\gamma_a = 18$ MHz, $\gamma_b = 40$ MHz, $\gamma = 29$ MHz, $Ku = 1010$ MHz, $g_a = g_b = 1.295$, and $N/N_{\text{threshold}} = 1.20$.

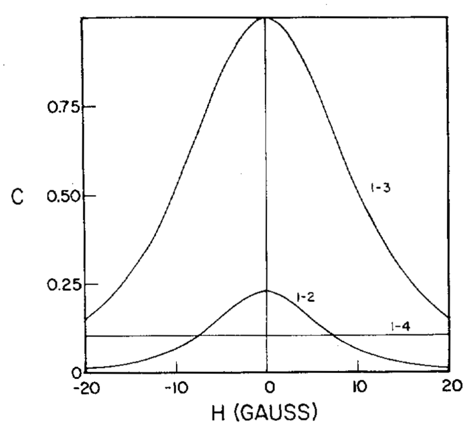

Fig. 4. C versus axial magnetic field under central tuning for $J_a = 1$, $J_b = 2$, and mode combinations 1-2, 1-3, 1-4. The other laser parameters are the same as those of Fig. 3.

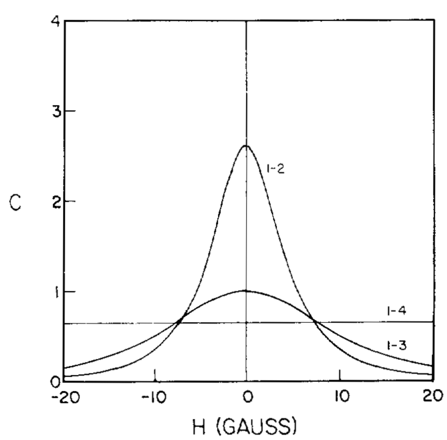

Fig. 5. C versus axial magnetic field under central tuning for $J_a = 2$, $J_b = 2$, and mode combinations 1-2, 1-3, 1-4. The other laser parameters are the same as those of Fig. 3.

Appendix A
Reduction to Standing-Wave (Two-Mirror) and Scalar Ring Lasers

A test for our theory will be to use it to describe a two-mirror laser (circular polarizations, standing waves) and a scalar ring laser (linear polarizations, traveling waves) and see if the predictions are in agreement with those of earlier works

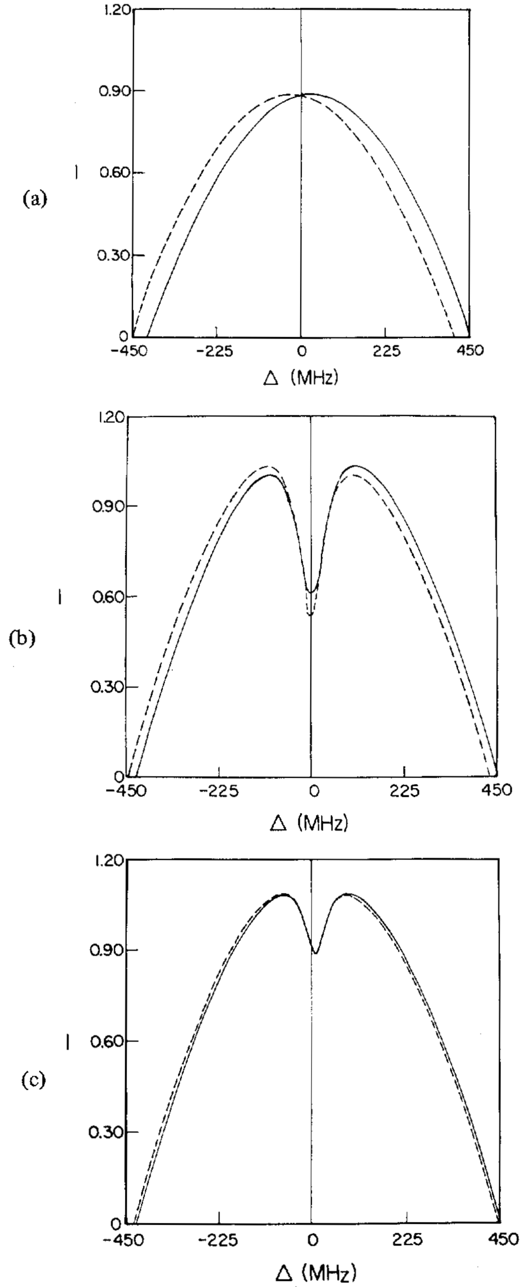

Fig. 6. Intensity versus detuning for $J_a = 1$, $J_b = 0$, and mode combinations (a) 1-2, (b) 1-3, and (c) 1-4. Other laser parameters are the same as those of Fig. 3.

[6], [8]. The amplitude- and frequency-determining equations for the standing-wave two-mirror laser may be obtained if we let

$$k_r = k_l = k \tag{A1}$$

$$\nu_2 = \nu_4 = \nu_+; \qquad \nu_1 = \nu_3 = \nu_- \tag{A2}$$

$$E_2 = E_4 = E_+/2; \quad E_1 = E_3 = E_-/2 \tag{A3}$$

$$\phi_1 = \phi_- + \pi/2, \phi_3 = \phi_- - \pi/2; \quad \phi_2 = \phi_+ + \pi/2,$$
$$\phi_4 = \phi_+ - \pi/2 \tag{A4}$$

where +(−) indicates right (left) circularly polarized light. The electric field given in (1) then becomes

$$E(s,t) = \tfrac{1}{2}\{E_+\hat{e}_+ \exp[-i(\nu_+ t + \phi_+)] + E_-\hat{e}_- \exp[-i(\nu_- t + \phi_-)]\}\sin(ks) + \text{c.c.} \tag{A5}$$

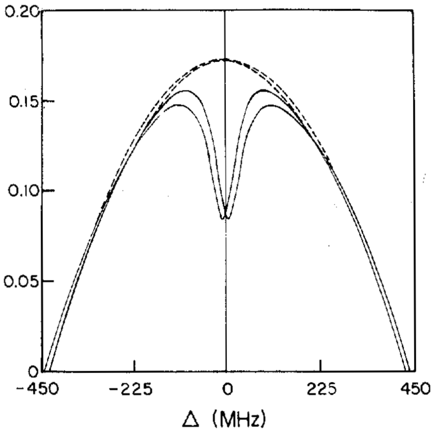

Fig. 7. Intensity versus detuning for the two standing-wave modes of a Zeeman laser (solid curve) and the two unidirectional modes of a ring laser. Other parameters are the same as those of Fig. 3.

and (3) and (4) reduce to

$$\dot{E}_\pm = E_\pm(\alpha_\pm - \beta_\pm E_\pm^2 - \theta_{\pm\mp} E_\mp^2) \tag{A6}$$

and

$$\nu_\pm + \dot{\phi}_\pm = \Omega_\pm + \sigma_\pm - \rho_\pm E_\pm^2 - \tau_{\pm\mp} E_\mp^2 \tag{A7}$$

where

$$\underset{2}{\Omega_1} = \underset{4}{\Omega_3} = \underset{+}{\Omega_-}, \underset{2}{Q_1} = \underset{4}{Q_3} = \underset{+}{Q_-},$$

$$\chi_\pm = \underset{1}{\alpha_2} = \underset{3}{a_4} = F_1 \sum_{a'} \sum_{b'} \delta_{a',b'\pm 1} |\mathcal{P}_{a'b'}|^2$$
$$\cdot Z_i[\gamma_{ab} + i(\omega_{a'b'} - \nu_\pm)] - \tfrac{1}{2}\nu/Q_\pm \tag{A8}$$

$$\sigma_\pm = \underset{1}{\sigma_2} = \underset{3}{\sigma_4} = F_1 \sum_{a'} \sum_{b'} \delta_{a',b'\pm 1} |\mathcal{P}_{a'b'}|^2$$
$$\cdot Z_r[\gamma_{ab} + i(\omega_{a'b'} - \nu_\pm)] \tag{A9}$$

$$\beta_\pm = \tfrac{1}{4}(\underset{11}{\theta_{22}} + \underset{13}{\theta_{24}}) = \tfrac{1}{4}(\underset{33}{\theta_{44}} + \underset{31}{\theta_{42}})$$
$$= F_3 \sum_{a'} \sum_{b'} \delta_{a',b'\pm 1} |\mathcal{P}_{a'b'}|^4 [1 + L(\omega_{a'b'} - \nu_\pm)] \tag{A10}$$

$$\theta_{\pm\mp} = \tfrac{1}{4}[\underset{12}{\theta_{21}} + \underset{14}{\theta_{23}} + \mathrm{Im}(\underset{1243}{\theta_{2134}} + \underset{1342}{\theta_{2431}})]$$
$$= \tfrac{1}{4}[\underset{34}{\theta_{41}} + \underset{32}{\theta_{43}} + \mathrm{Im}(\underset{3124}{\theta_{4213}} + \underset{3421}{\theta_{4312}})]$$
$$= F_3/(2\overline{\gamma}) \sum_{a'} \sum_{b'} \delta_{a',b'\pm 1} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a'\mp 2,b'}|^2$$
$$\cdot \{\gamma_a[L(\delta_a) + L(\omega_{a'b'} \mp \delta_a - \nu_\pm)]$$
$$+ \gamma_b L_a(2\delta_a)[L(\delta_a)(1 - 2\delta_a^2/(\gamma_a\gamma_{ab}))$$
$$+ L(\omega_{a'b'} - \nu_\pm)(1 \mp 2\delta_a(\omega_{a'b'} - \nu_\pm)/(\gamma_a\gamma_{ab}))]\}$$
$$+ \text{the same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longleftrightarrow \delta_b,$$
$$\text{and } \mathcal{P}_{a'\mp 2,b'} \longrightarrow \mathcal{P}_{a'b'\pm 2} \tag{A11}$$

$$\rho_\pm = \tfrac{1}{4}\underset{13}{\tau_{24}} = \tfrac{1}{4}\underset{31}{\tau_{42}} = F_3 \sum_{a'} \sum_{b'} \delta_{a'b'\pm 1} |\mathcal{P}_{a'b'}|^4$$
$$\cdot (\omega_{a'b'} - \nu_\pm)/\gamma_{ab} L(\omega_{a'b'} - \nu_\pm) \tag{A12}$$

$$\tau_{\pm\mp} = \tfrac{1}{4}[\underset{12}{\tau_{21}} + \underset{14}{\tau_{23}} + \mathrm{Re}(\underset{1243}{\theta_{2134}} + \underset{1342}{\theta_{2431}})]$$
$$= \tfrac{1}{4}[\underset{32}{\tau_{41}} + \underset{34}{\tau_{43}} + \mathrm{Re}(\underset{3124}{\theta_{4213}} + \underset{3421}{\theta_{4312}})]$$
$$= F_3/(2\overline{\gamma}) \sum_{a'} \sum_{b'} \delta_{a'b'\pm 1} |\mathcal{P}_{a'b'}|^2 |\mathcal{P}_{a'\mp 2,b'}|^2$$
$$\cdot \{\gamma_a[\pm\delta_a/\gamma_{ab} L(\delta_a) + (\omega_{a'b'} \mp \delta_a - \nu_\pm)/\gamma_{ab}$$
$$\cdot L(\omega_{a'b'} \mp \delta_a - \nu_\pm)]$$
$$+ \gamma_b L_a(2\delta_a)\{\pm\delta_a/\gamma_{ab}(1 + 2\gamma_{ab}/\gamma_a)L(\delta_a)$$
$$+ L(\omega_{a'b'} - \nu_\pm)[\pm 2\delta_a/\gamma_a + (\omega_{a'b'} - \nu_\pm)/\gamma_{ab}]\}$$
$$+ \text{the same with } \gamma_a \longleftrightarrow \gamma_b, \delta_a \longrightarrow \delta_b,$$
$$\text{and } \mathcal{P}_{a'\mp 2b'} \longrightarrow \mathcal{P}_{a'b'\pm 2}. \tag{A13}$$

The variables are defined in Section II. Since the relative phase angle vanishes, terms containing $\exp(\pm i\psi)$ in (3) and (4) contribute to the saturation coefficients. If these terms had been dropped, the nonlinear coefficients would have been incorrect. The above results agree with [9].

The equations for the scalar ring laser (two waves traveling in opposite directions with the same linear polarization) may also be obtained from Section II if we allow

$$\nu_1 = \nu_2 = \nu_a; \qquad \nu_3 = \nu_4 = \nu_c \tag{A14}$$

$$E_1 = E_2 = E_a/\sqrt{2}; \quad E_3 = E_4 = E_c/\sqrt{2} \tag{A15}$$

$$\phi_1 = \phi_2 = \phi_a; \qquad \phi_3 = \phi_4 = \phi_c \tag{A16}$$

where $a(c)$ denotes anticlockwise (clockwise) traveling wave. The electric field is then polarized along $\hat{\imath}$ and is of the form

$$E(s,t) = E_a/2 \exp[-i(\nu_a t + \phi_a - k_a S)]$$
$$+ E_c/2 \exp[-i(\nu_c t + \phi_c + k_c S)] + \text{c.c.} \tag{A17}$$

The frequency and amplitude determining equations become

$$\underset{c}{\dot{E}_a} = \underset{c}{E_a}(\underset{c}{\alpha_a} - \underset{c}{\beta_a}\underset{c}{E_a^2} - \underset{ca}{\theta_{ac}}\underset{a}{E_c^2}) \tag{A18}$$

$$\underset{c}{\nu_a} + \underset{c}{\dot{\phi}_a} = \underset{c}{\Omega_a} + \underset{c}{\sigma_a} - \underset{ca}{\tau_{ac}}\underset{a}{E_c^2} \tag{A19}$$

where $\underset{3}{\Omega_1} = \underset{4}{\Omega_2} = \underset{c}{\nu_a}$, $\underset{3}{Q_1} = \underset{4}{Q_2} = \underset{c}{Q_a}$, and in the absence of x-y Q and phase anisotropy, we require the magnetic field strength to be zero.

The coefficients in (A18) and (A19) are given by

$$\underset{c}{\alpha_a} = \underset{3}{\alpha_1} = \underset{4}{\alpha_2} = F_1 M_0 Z_i[\gamma_{ab} + i(\omega_0 - \underset{c}{\nu_a})] - \nu/(2\underset{c}{Q_a}) \tag{A20}$$

$$\underset{c}{\sigma_a} = \underset{3}{\sigma_1} = \underset{4}{\sigma_2} = F_1 M_0 Z_R[\gamma_{ab} + i(\omega_0 - \underset{c}{\nu_a})] \tag{A21}$$

$$\underset{c}{\beta_a} = \tfrac{1}{2}(\underset{33}{\theta_{11}} + \underset{34}{\theta_{12}}) = \tfrac{1}{2}(\underset{44}{\theta_{22}} + \underset{43}{\theta_{21}}) = 2F_3(M_1 + M_4) \tag{A22}$$

$$\underset{ca}{\theta_{ac}} = \tfrac{1}{2}[\underset{31}{\theta_{13}} + \underset{32}{\theta_{14}} + \mathrm{Im}(\underset{3124}{\theta_{1342}} + \underset{3421}{\theta_{1243}})]$$
$$= \tfrac{1}{2}[\underset{41}{\theta_{23}} + \underset{42}{\theta_{24}} + \mathrm{Im}(\underset{4213}{\theta_{2134}} + \underset{4312}{\theta_{2431}})]$$
$$= 2F_3\left\{M_1 L\left(\omega_0 - \frac{\nu_a + \nu_c}{2}\right) + M_4 L(\omega_{ab} - \underset{c}{\nu_a})\right\} \tag{A23}$$

$$\underset{ca}{\tau_{ac}} = \tfrac{1}{2}[\underset{31}{\tau_{13}} + \underset{32}{\tau_{14}} + \mathrm{Re}(\underset{3124}{\theta_{1342}} + \underset{3421}{\theta_{1243}})]$$

$$= \tfrac{1}{2}\left[\underset{41}{\tau_{23}} + \underset{42}{\tau_{24}} + \mathrm{Re}\left(\underset{4213}{\theta_{2134}} + \underset{4312}{\theta_{2431}}\right)\right]$$

$$= 2F_3\left[M_1 L\left(\omega_{ab} - \frac{\nu_a+\nu_c}{2}\right)\left(\omega_0 - \frac{\nu_a+\nu_c}{2}\right)\Big/\gamma_{ab} + M_4 \underset{c}{L}(\omega_0 - \nu_a)(\omega_0 - \nu_a)/\gamma_{ab}\right] \quad \text{(A24)}$$

where ω_0 is the energy spacing at zero magnetic field,

$$M_0 = \sum_{a'}\sum_{b'} \delta_{a',b'\pm 1}\,|\wp_{a'b'}|^2 = \frac{\wp^2}{6}\begin{cases} J(J+1)(2J+1) & \Delta J = 0 \\ (J_a+1)(2J_a+1)(2J_a+3) & \Delta J = \pm 1\end{cases}$$

and M_1, M_4 are defined in Section III.

As in the case of a two-mirror cavity, the relative phase angle is zero and, therefore, the terms containing it in (3) and (4) contribute to the nonlinear coefficients. The coupling between the two traveling waves is characterized by the coupling parameter

$$C = \theta_{ac}\theta_{ca}/(\beta_c\beta_a) \simeq L^2\left(\omega_{ab} - \frac{\nu_a+\nu_c}{2}\right) \quad \text{(A26)}$$

which agrees with (28). These results are in agreement with those of [6].

Appendix B
Bidirectional Four-Wave Operation in a Homogeneously Broadened Medium

When the spread in atomic velocities u is much less than the spontaneous lifetimes, we have a homogeneously broadened medium. In this case, the distribution function for the atomic velocities becomes $\delta(v)$ and the linear net gain and mode pulling coefficients are

$$\alpha_n = \nu\bar{N}/(\epsilon_0\hbar\gamma_{ab})\sum_{a'b'}|\wp_{a'b'}|^2\delta_{a',b'+p_n}L(\omega_{a'b'} - \nu_n) \quad \text{(B1)}$$

and

$$\sigma_n = \nu\bar{N}/(\epsilon_0\hbar\gamma_{ab})\sum_{a'b'}|\wp_{a'b'}|^2\delta_{a',b'+p_n}L(\omega_{a'b'} - \nu_n)\cdot(\omega_{a'b'} - \nu_n)/\gamma_{ab}. \quad \text{(B2)}$$

The nonlinear coefficients are

$$\theta_{nn} = 2\nu\bar{N}\bar{\gamma}/(\epsilon_0\hbar^3\gamma_{ab}^2\gamma_a\gamma_b)\sum_{a'b'}|\wp_{a'b'}|^4\cdot L^2(\omega_{a'b'} - \nu_n)\,\delta_{a',b'+p_n} \quad \text{(B3)}$$

$$\theta_{nm} = \nu\bar{N}/(2\epsilon_0\hbar^3\gamma_{ab}^2\gamma_a)\sum_{a'b'}|\wp_{a'b'}|^2\cdot|\wp_{a'-2p_n,b'}|^2\,\delta_{a',b'+p_n}L(\omega_{a'b'} - \nu_n)\cdot\left\{L_a(2\delta_a)[L(\omega_{a'b'} - \nu_n) + L(\omega_{a'-2p_n,b'} - \nu_m)] + 2\frac{\gamma_a}{\gamma_b}L(\omega_{a'-2p_n,b'} - \nu_m)\right\}$$

$$+ \text{the same with } \gamma_a \longleftrightarrow \gamma_b,\ \delta_a \longrightarrow \delta_b, \text{ and } \wp_{a'-2p_n,b'} \longrightarrow \wp_{a',b'+2p_n}. \quad \text{(B4)}$$

$$\tau_{nn} = 2\nu\bar{N}\bar{\gamma}/(\epsilon_0\hbar^3\gamma_{ab}^2\gamma_a\gamma_b)\sum_{a'b'}|\wp_{a'b'}|^4\cdot L^2(\omega_{a'b'} - \upsilon_n)\,\delta_{a',b'+p_n} \quad \text{(B5)}$$

$$\tau_{nm} = \nu\bar{N}/(\epsilon_0\hbar^3\gamma_{ab}^2\gamma_a)\sum_{a'b'}|\wp_{a'b'}|^2\,|\wp_{a'-2,b'}|^2\cdot\delta_{a'b'+p_n}L(\omega_{a'b'} - \nu_n)\cdot\{\gamma_a(\omega_{a'b'} - \nu_n)/(\gamma_b\gamma_{ab})L(\Delta\omega_2) - L_a(2\delta_a)[L(\omega_{a'b'} - \nu_n)(\delta_a/\gamma_a + (\omega_{a'b'} - \nu_n)/\gamma_{ab}) + L(\omega_{a'-2p_n,b'} - \nu_m)(\delta_a/\gamma_a + \delta_a/\gamma_{ab})]\} \quad \text{(B6)}$$

where $n, m = 1, 2, 3, 4$; $n \neq m$, and all the other variables are defined in Section II.

Next we look at central tuning to obtain the maximum values for the coupling parameters. For all two-mode combinations, the maximum coupling parameter is given by

$$C_{nm} \equiv \frac{\theta_{nm}\theta_{mn}}{\theta_{nn}\theta_{mm}}\begin{cases}\left[\dfrac{(2J-1)(2J+3)}{(2J^2+2J+1)}\right]^2 & \text{for } \Delta J = 0 \\[2ex] \left[\dfrac{(2J^2+4J+5)}{(6J^2+12J+5)}\right]^2 & \text{for } \Delta J = \pm 1\end{cases} \quad \text{(B7)}$$

where $J = J_a$.

Thus, the bidirectional coupling parameters for the homogeneously and inhomogeneously broadened cases are equal. For the homogeneously broadened medium, all the coupling parameters are also equal, and this is so because all modes are competing for the same atoms.

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Notes and Lines

Correction to "Diagnostics for the Laser Fusion Program—Plasma Physics on the Scale of Microns and Picoseconds"

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In the above paper[1], the ion charge Z was omitted from the equation in Table I for the electron-ion thermal velocity. The correct form, with Te $\gg$ Ti (as is often the case in the plasma corona), is[2]

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[1] D. T. Attwood, *IEEE J. Quantum Electron.*, vol. QE-14, pp. 909-923, Dec. 1978.
[2] P. C. Clemmow and J. P. Dougherty, *Electro-Dynamics of Particles and Plasmas.* Reading, MA: Addison-Wesley, 1969, sect. 5.5.3 for the case $N_i = N_e/Z$.

$$V_{\mathrm{th},\,e-i} \simeq 0.37\left(\frac{\kappa T}{M}\cdot\frac{Z}{10}\right)^{1/2}\ \mu\mathrm{m/ps}.$$

The numerical form of the equation is normalized to charge state $Z = 10$, as this is a common ionization level in our experiments with glass microballoons. Note that this correction to the electron-ion thermal velocity modifies the expected scale length at the end of p. 912 of the above paper[1]. However, for a pulse length of 30 ps, as used in the interferometric work described, and a plasma expansion velocity of approximately 0.4 μm/ps, one again obtains an expected scale length of about 10 μm, in the absence of strong radiation pressure or thermal gradient effects.

As a further note, the caption to Table I should be corrected to say that temperature (κT) in units of keV, and that the ion charge state Z is relative to 10, as shown explicitly wherever it appears.