Decomposição de grafos livres de tri-
gulo e 5-regular em caminhos de tamanho 5 ↳ todo vértice tem grau 5

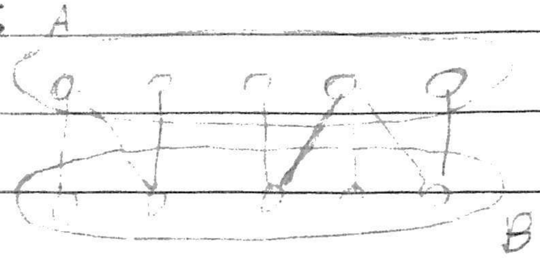
Teorema 1.1 (Lovász). Every n-vertex graph can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles

P_K (resp. C_K) a path (resp. cycle) of length K, that is, with K edges.

P_K-decomposition. partition the edges of a graph into paths of length K. Dividing the set of edges of the graph into disjoint subsets, where each subset forms a path of length K

spiral

A bipartite graph is a type of graph where the set of vertices can be divided into two distinct and disjoint sets, such that no two vertices within the same set are adjacent. All edges in the graph connect vertices from one set to vertices in the other set



Mixed-graph: algumas arestas podem ter orientação:

uma tupla $\bar{G} = (V, E, A)$

$V(\bar{G}), E(\bar{G}), A(\bar{G})$: conjunto de vertices arestas e arcos, respectivamente

$\hat{A}(\bar{G})$: o conjunto de arestas obtido ao remover a direção dos arcos em $A(\bar{G})$

Reler último parágrafo das definições

G the underlying graph of $\bar{G}$, i.e., G is the graph s.t. $V(G) = V(\bar{G})$ and $E(G) = E(\bar{G}) \cup \hat{A}(\bar{G})$

Canonical $\{P_5, T_5\}$-decomposition

2-factor: spanning subgraph where each vertex has a degree of exactly 2

P_K any path of length K
T_K: the graph (trail) that is obtained by a path $v_0 v_1 \dots v_{K-1}$ by the addition of the edge $v_{K-1} v_1$ ▶ recorrendo a tri-
lha euleriana geramos uma orientação D*
In an eulerian orientation mixed graph a copy $v_0 v_1 \dots v_5$ of P_5 (resp. a copy $v_0 v_1 \dots v_4 v_1$ of T_5) is canonical if it's directed edges are precisely $v_1 v_0$ and $v_4 v_5$ (resp. $v_1 v_0$ and $v_2 v_1$)

*de G (um grafo conexo com grau par tem uma trilha euleriana)

Theorem 3.1 (Petersen) Every $2K$-regular graph contains a 2-factor

Theorem 3.2 (Kotzig). Every 3-regular graph containing a perfect matching admits a P_3-decomposition

spiral

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$\hat{A}(\bar{G})$ set of edges obtained by removing the orientation of the directed edges in $\vec{A}(\bar{G})$

Complete graph: todo mundo conectado por uma aresta

3.1 Couples in a $\{P_5, T_5\}$-decomposition

We say that a copy $\bar{B}$ of T_5 is well-oriented if we can label the vertices of $\bar{B}$ such that $\bar{B} = b_0 b_1 b_2 b_3 b_4 b_1$ and either $A(\bar{B}) = \{b_2 b_1\}$ or $A(\bar{B}) = \{b_2 b_1, b_1 b_0\}$. In this case, the vertex b_4 is called the connection-vertex of $\bar{B}$, and is denoted by $cv(\bar{B})$

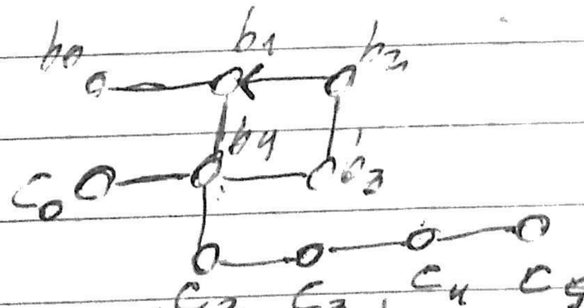
Let $\bar{P} = v_0 v_1 v_2 v_3 v_4 v_5$ be a copy of P_5 in a mixed graph $\bar{G}$. We say that $\bar{P}$ is a roofed path in $\bar{G}$ if $v_4 v_1$ is an edge of $E(\bar{G}) \cup \hat{A}(\bar{G})$. Furthermore, we say that $v_4 v_1$ is the roof of $\bar{P}$

Def.: Let $(\bar{B}, \bar{C})$ be a pair of elements of $\mathcal{D}$, where $\bar{B}$ is a well-oriented copy of T_5, say $b_0 b_1 b_2 b_3 b_4 b_1$, and $\bar{C}$ is a copy of P_5, say $c_0 c_1 c_2 c_3 c_4 c_5$. We say that $(\bar{B}, \bar{C})$ is a TP-couple of $\mathcal{D}$ if $cv(\bar{B}) \in \{c_1, c_2, c_3, c_4\}$. If $cv(\bar{B}) \in \{c_1, c_4\}$, then we say that $(\bar{B}, \bar{C})$ is a TP-couple in position 1, and if $cv(\bar{B}) \in \{c_2, c_3\}$, then we say that $(\bar{B}, \bar{C})$ is a TP-couple in position 2

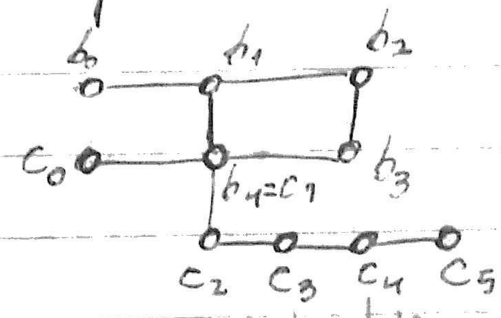
Def. 3.4 A $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of a mixed graph G is called complete if the following three conditions hold. (definition of completeness)

(i) Every copy of T_5 in $\mathcal{D}$ is well oriented
(ii) Every directed edge of $\bar{G}$ is internal to $\mathcal{D}$
(iii) Every TP-couple of $\mathcal{D}$ is solvable

Def. Let $\bar{G}$ be a mixed 5-regular graph and $\mathcal{D}$ a $\{P_5, T_5\}$-decomposition of $\bar{G}$. Let $\bar{B} = b_0 b_1 b_2 b_3 b_4 b_1$ and $\bar{C} = c_0 c_1 c_2 c_3 c_4 c_5$ be elements of $\mathcal{D}$ s.t. $cv(\bar{B}) = b_4 = c_1$. We say that the TP-couple $(\bar{B}, \bar{C})$ is solvable if $\bar{B}^+ = b_0 b_1 b_2 b_3 b_4 c_0$ and $\bar{C}^+ = b_1 c_1 c_2 c_3 c_4 c_5$ are edge-disjoint paths (of length five) in $\bar{G}$. (Note that $\bar{B}^{+}$ is a roofed path in $\bar{G}$ and $(\bar{B}, \bar{C})$ is a TP-couple in position 1.)



well-oriented



solvable
$b_4 b_1$ roofed path

Lemma 3.5 Let G be a triangle-free 5-regular graph containing a perfect matching. If G is the underlying

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graph of a mixed graph $\bar{G}$ such that $A(\bar{G})$ induces a 2-factor with an Eulerian orientation $\mathcal{E}$, and $\bar{G}$ admits an $\mathcal{E}$-canonical $\{P_5, T_5\}$-decomposition $\mathcal{D}_{\mathcal{E}}$, then $\mathcal{D}_{\mathcal{E}}$ is complete

Corollary 3.6 Let G be a triangle-free 5-regular graph containing a perfect matching. Then G is the underlying graph of a mixed graph $\bar{G}$ s.t. $A(\bar{G})$ induces a 2-factor with an Eulerian orientation, and $\bar{G}$ admits a complete $\{P_5, T_5\}$-decomposition.

Relembrando: decomposição de um grafo G ~~(é uma coleção de arestas disjuntas que formam subgrafos $H_1, H_2, \ldots, H_n$ onde cada aresta pertence a exatamente)~~ é um conjunto disjunto de subgrafos onde a união desses subgrafos formam o grafo original, G.

4. Disentanglement of couples

Def.: Given a $\{P_5, T_5\}$-decomposition

$\mathcal{D}$ of a mixed 5-regular graph $\bar{G}$, we denote by $\tau(\mathcal{D})$ the number of copies of T_5 in $\mathcal{D}$

Def. Let $k \geq 3$ and let $\bar{B}_1, \ldots, \bar{B}_k$ be copies of T_5 in $\mathcal{D}$. We say that $\bar{B}_1 \ldots \bar{B}_k$ is a sequence of couples of $\mathcal{D}$ if $(\bar{B}_i, \bar{B}_{i+1})$ is a TT-couple for $1 \leq i \leq k-1$. If $(\bar{B}_k, \bar{B}_1)$ is also a TT-couple, then we say that $\bar{B}_1 \ldots \bar{B}_k$ is a cycle of couples of $\mathcal{D}$. Furthermore, if such a sequence (resp. cycle) is composed only by TT-couples in position i, then we say that it is in position i, for $i = 1, 2$. A sequence (resp. cycle) of couples is called mixed if it contains couples in positions 1 and 2

4.1 TP-couples in position 1

Lemma 4.1 Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\bar{G}$. Let $(\bar{B}, \bar{C})$ be a TP-couple of $\mathcal{D}$ in position 1. If $\bar{C}$ is a roofed path in $\hat{G}$ s.t. its roof is not in $\bar{B}$, then $(\bar{B}, \bar{C})$ is solvable

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4.2 TT-couples in position 1

Lemma 4.2 Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, \vec{P}_5\}$-decomposition of $\bar{G}$. If $\bar{B}_1 \dots \bar{B}_k$ is a cycle of TT-couples of $\mathcal{D}$ in position 1, then $\bigcup_{i=1}^{k} \bar{B}_i$ admits a P_5-decomposition $\mathcal{P}$ such that all directed edges of $\bigcup_{i=1}^{k} B_i$ are internal to $\mathcal{P}$

4.3. TT-couples in position 2

Lemma 4.3 Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. Then, for every vertex v, there is exactly one element $B \in \mathcal{D}$ s.t. $d_B(v)$ is odd

Lemma 4.4. Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $(\bar{B}, \bar{C})$ is a TT-couple of $\mathcal{D}$ in position 2, where $\bar{B} = b_0 b_1 b_2 b_3 b_4 b_1$ and $\bar{C} = c_0 c_1 c_2 c_3 c_4 c_1$

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then $\bar{B}' = b_0 b_1 b_2 b_3 b_4 c_1$ and $C' = c_0 c_1 c_4 c_3 c_2 b_1$ are roofed paths in $\bar{B} \cup \bar{C}$. Furthermore, all directed edges of B' and C' are internal.

Lemma 4.5 Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a (~~couple~~) complete $\{P_5, T_0\}$-decomposition of $\bar{G}$. Let $\bar{B}_1 \bar{B}_2 \bar{B}_3$ be a sequence of TT-couples of $\mathcal{D}$ in position 2. Then, in each of the following cases, $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ admits a P_5-decomposition $\mathcal{P}$ s.t. all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}$

(a) when the pending vertex of $\bar{B}_1$ is the connection-vertex of $\bar{B}_2$;

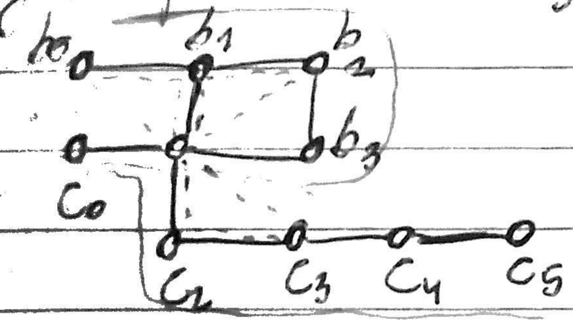
(b) when the pending vertex of $\bar{B}_3$ is the connection-vertex of $\bar{B}_1$;

(c) when the pending vertex of $\bar{B}_2$ is not the connection-vertex of $\bar{B}_3$, and the pending vertex of $\bar{B}_3$ is not the connection-vertex of $\bar{B}_1$

 def. K-regular is a graph where

every vertex has degree equal to k

Lemma 4.6 Let $\bar{G}$ be a triangle-free 5-regular mixed graph and let $\mathcal{D}$ be a complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. If $\bar{B}_1 \dots \bar{B}_k$ is a cycle of couples of $\mathcal{D}$ in position 2, then $\bigcup_{i=1}^{k} \bar{B}_i$ admits a P_5-decomposition $\mathcal{P}$ s.t. all directed edges of $\bar{B}_1 \cup \bar{B}_2 \cup \bar{B}_3$ are internal to $\mathcal{P}$



drawing of lemma 4.1 containing its triangles that causes a contradiction

I. Main result

Def

Given a mixed graph $\bar{G}$ and a complete $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of $\bar{G}$, we say that a copy of T_5 in $\mathcal{D}$ is an initial element of $\mathcal{D}$ if it is not the base of any couple in $\mathcal{D}$. If $\mathcal{D}$ has the least number of copies of T_5 among all complete $\{P_5, T_5\}$-decompositions of $\bar{G}$, then $\mathcal{D}$ is called a minimal complete $\{P_5, T_5\}$-decomposition of $\bar{G}$. Furthermore, if there is at least one copy of T_5 in $\mathcal{D}$, that is, $\tau(\mathcal{D}) \neq 0$, then we say that $\mathcal{D}$ is nontrivial

Lemma 5.1 Let G be a triangle-free 5-regular graph. Suppose that G is the underlying graph of a mixed graph $\bar{G}$ s.t $A(\bar{G})$ induces a 2-factor with an Eulerian orientation. If there is a nontrivial minimal complete $\{P_5, T_5\}$-decomposition $\mathcal{D}$ of $\bar{G}$, then the following properties hold

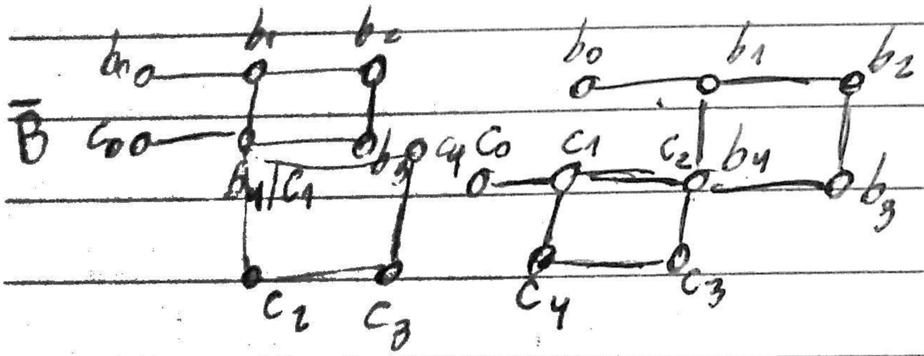
(a) Every copy of T_5 in $\mathcal{D}$ is the top of exactly one couple of $\mathcal{D}$

(b) $\mathcal{D}$ contains no initial element

(c) Every copy of T_5 in $\mathcal{D}$ is the base of exactly one couple of $\mathcal{D}$. Furthermore, every copy of P_5 in $\mathcal{D}$ is not the base of any couple of $\mathcal{D}$

definition of top and base

Let $\bar{B}$ be a well-oriented copy of T_5, we say that, if $(\bar{B}, \bar{x})$ is a couple then $\bar{B}$ and $\bar{x}$ are respectively the top and the base of $(\bar{B}, \bar{x})$



Regular Path decompositions of odd regular graphs

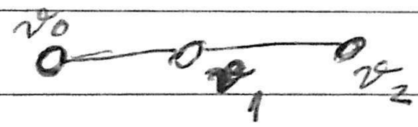
2. Terminology

def. of a trail

A trail of an undirected graph G is a walk $W = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ of G with no repeated edges.

$$e_j = v_{j-1} v_j \quad \text{for } 1 \leq j \leq k$$

The trail is closed if $v_0 = v_k$



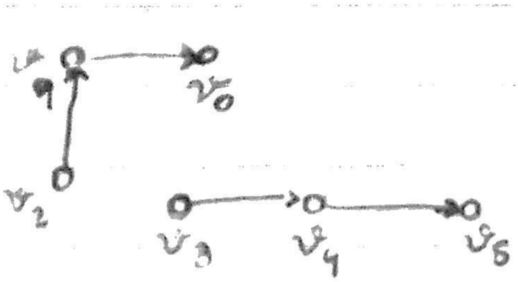
An eulerian orientation of a graph G is an orientation of its edges s.t. each vertex has an equal number of incoming and outgoing edges. A graph G admits an eulerian orientation iff all its vertices are of even degree

An r-factor of a graph G is an r-regular spanning subgraph of G.
In this article, when we subtract a 1-factor M from a d-regular graph G, it is understood to be an edge subtraction so that the result $G-M$ is $(d-1)$-regular.

3. Trail and path decompositions

def: A k-trail decomposition (k-TD) of a graph G is an edge-partition of G into trails of length k. In the particular case where all the trails of the decomposition are paths, we call it a k-path decomposition (k-PD) of G.

Given a k-trail or k-path decomposition T of a graph G with k odd, we define an edge-coloring $c: E(G) \to \{0, .., (k-1)/2\}$ in the following way: given a trail $t = v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$ in T, let $c(e_i) = \left| \frac{(k+1)}{2} - i \right|$. The edges of color i will be called the edges of type i induced by the k-trail decomposition



edges e_1, e_2, e_4, e_5

On the article in question we study the existence of a d-TD or a d-PD T when G is an odd d-regular graph.

Proposition 1 let G be an odd d-regular graph with a d-TD T. Then any bridge of G must be of type 0, no vertex is incident with more than $\frac{d-1}{2}$ bridges and for any maximal 2-edge-connected subgraph H of G, the order of H is not less than the number of bridges linking H to G-H

Def. a 1-factor M in an odd d-regular graph G is extendable to a d-trail decomposition (respectively to a d-path decomposition, or simply extendable) if there exists a d-trail decomposition (respectively a d-path decomposition) of G for which the edges of type 0 are the edges of M.

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Proposition 2 Let G be an odd d-regular graph with a 1-factor M. Then G admits a d-TD T s.t. the edges of M are the edges of type 0

Theorem 1 In an odd d-regular graph of girth $\geq d$, all 1-factors are extendable

def. Let G be a graph (directed or undirected (?)) the girth of G is the length of its shortest cycle

def. a d-trail decomposition of an odd d-regular graph G, is said to be rainbow-making if the edges of type i induce a 2-factor for each $i \neq 0$

Theorem 2 a 5-trail decomposition of a 5-regular graph G is rainbow-making iff the edges of type 0 form a 1-factor of G. Furthermore, the orientation of each 2-factor induced by a rainbow-making 5-TD is eulerian

4 Graphs with non-extendable 1-factors

half-trail.

Def. Given u a vertex of G, e the edge of the 1-factor covering u, and t the trail of the trail decomposition whose central edge is e, we call half-trail of origin u the part of $t-e$ containing the outgoing edge of type 1 oriented away from u.

Lemma 1. Let A be an edge-cut of G induced by a partition $V(G) = V_1 \cup V_2$. For $i = 1, \dots, (d-1)/2$, let a_i (respectively b_i) be the number of edges of type i in A oriented by T from V_1 to V_2 (respectively from V_2 to V_1). Then we have

$$\sum_{i=1}^{(d-1)/2} (d-2i)(b_i - a_i) = 0$$

spiral

Theorem 3. Let τ be a d-TD of an odd d-regular graph G s.t. the edges of type 0 induce a 1-factor of G. Let A be an edge-cut of G induced by a partition $V(G) = V_1 \cup V_2$. For $i = 1, \dots, (d-1)/2$, let a_i (respectively b_i) be the number of edges of type i in A oriented by τ from V_1 to V_2 (respectively from V_2 to V_1). Then we have

$$\sum_{i=1}^{(d-1)/2} a_i = \sum_{i=1}^{(d-1)/2} b_i$$

$$\sum_{i=1}^{(d-1)/2} i a_i = \sum_{i=1}^{(d-1)/2} i b_i$$

In particular if A contains exactly 2 non-zero edges, they are of the same type with opposite orientations

Corollary 1. Let M be a 1-factor of a simple odd d-regular graph G and suppose that M is extendable to a d-path decomposition P of G. If H is a subgraph of G isomorphic to K_d s.t. $|E(H) \cap E(M)| = 1$, and A is the edge-cut induced by the partition $V(G) = V(H) \cup (V(G) \setminus V(H))$, then the two edges of A of non-zero type are of type 1

Corollary 2 Let M be a 1-factor of a simple 5-regular graph G and suppose that M is extendable to a 5-path decomposition $\mathcal{T}$ of G. If H is an induced subgraph of G isomorphic to $K_{4,4}$ s.t. $|E(H) \cap E(M)| = 1$, and A is the edge-cut induced by the partition $V(G) = V(H) \cup (V(G) \setminus V(H))$, then the two edges of A of non-zero type are of type 1

Theorem 4 A cubic graph admits a 3-TD iff it admits a 1-factor

Proposition 3: A necessary condition for a 5-regular graph to have a 5-TD is to have a 2-factor

* necessary condition for a 5-TD when $d = 5$

spiral

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Sufficient conditions for a 1-factor of a 5-regular graph to be extendible to a 5-PD

Def: In a simple 5-regular graph containing a 1-factor M, an M-triangle (respectively an M-square) is a cycle of length 3 (respectively 4) containing exactly one edge of M. M-triangles and M-squares will be referred to as M-cycles.

Def: Let G be a simple 5-regular graph with a 1-factor M. A pair of M-cycles C_1 and C_2 is said to be a bad pair if they have at least one edge in common and the graph whose set is $(E(C_1) \cup E(C_2)) \setminus M$ contains a vertex of degree 3

Def: Let G be a simple 5-regular graph with a 1-factor M and an eulerian orientation of $G-M$. An M-triangle $uevе_1we_2u$ with $e \in M$ is balanced if e_1 and e_2 have opposite orientati-

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ons with respect to the end-vertices of e (one is incoming, the other is outgoing). An M-square $u e v e_1 w e_2 x e_3 u$ with $e \in M$ is balanced if e_1 and e_3 have opposite orientations with respect to the end vertices of e. The eulerian orientation is M-balanced if every M-triangle and every M-square is balanced

Theorem 5. Given an eulerian orientation $\mathcal{O}$ of an even r-regular graph G, there exists a decomposition of G into $r/2$ 2-factors s.t. $\mathcal{O}$ induces eulerian orientations in each 2-factor

Theorem 6. If a 5-regular simple graph G has a 1-factor M inducing no bad pair of M-cycles then M is extendable to a 5-PD

Theorem 7. Let G be a 5-regular simple graph with a 1-factor M. Consider the graph H whose vertex set is the set of

 M-cycles of G belonging to some

bad pair and with an edge between two M-cycles whenever they form a bad pair. If H is a star then M is extendable to a 5-PD