Solution

University of Toronto; Division of Engineering Science
STA286S: Probability and Statistics Term Test
Monday, February 23, 2015, 9:10-10:50 am

Examiners: B. Donmez and L. Al Labadi

Student Number:______________________________

Family Name:________________________ First Name:________________________

Lecture Section: **(circle one)**

- LEC01 (Prof. Al Labadi)
- LEC02 (Prof. Donmez)

Tutorial Section: **(circle one)**

Tutorial	**Lecture**	**Time**	**Location**	**TA**
TUT01	01	Mon 1-2 pm	BA2159	Hootan Habibkhani
TUT02		Mon 1-2 pm	WB144	Zhenhua Lin
TUT03		Mon 1-2 pm	BA3012	Wayne Giang
TUT04		Wed 1-2 pm	WB144	Victor Veitch
TUT05	02	Mon 2-3 pm	WB144	Hootan Habibkhani
TUT06		Fri 10-11 am	WB144	Wayne Giang
TUT07		Tues 1-2 pm	BA2159	Victor Veitch
TUT08		Tues 1-2 pm	BA2165	Zhenhua Lin

Instructions:

- **Time allowed:** 1 hour and 40 minutes.
- **Aids:** a non-programmable calculator and a one-sided A4 size aid sheet.
- There are six questions. Carefully proportion your time among them. If you do not understand a question, or are having some other difficulty, do not hesitate to ask for clarification.
- There are 9 pages including this page. The last page contains the standard normal table. Please ensure that you are not missing any pages.
- Points for each question are indicated in parenthesis. Total points: 100.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Max	**15**	**15**	**15**	**10**	**15**	**30**	**100**
Score							

GOOD LUCK!

Question 1 (15 pts): Consider the following three events in the experiment of tossing two fair dice.

A= The first die shows an even number [Hint: (2,1), (4,2), etc.]
B = The second die shows an odd number [Hint: (1,1), (2,3), etc.]
C= The sum on the two dice is even [Hint: (1,1), (1,3), etc.]

Show that the three events A, B and C are pairwise independent, but not independent. That is, A and B are independent, A and C are independent B and C are independent, but A, B and C are not independent.

$A = \{ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$B = \{ (1,1), (2,1), (3,1), (4,1), (5,1), (6,1),$
$(1,3), (2,3), (3,3), (4,3), (5,3), (6,3),$
$(1,5), (2,5), (3,5), (4,5), (5,5), (6,5) \}$

$C = \{ (1,1), (1,3), (1,5), (2,2), (2,4), (2,6),$
$(3,1), (3,3), (3,5), (4,2), (4,4), (4,6),$
$(5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \}$

$A \cap B = \{ (2,1), (4,1), (6,1), (2,3), (4,3), (6,3),$
$(2,5), (4,5), (6,5) \}$

$A \cap C = \{ (2,2), (2,4), (2,6), (4,2), (4,4), (4,6),$
$(6,2), (6,4), (6,6) \}$

$B \cap C = \{ (1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3),$
$(5,5) \}$

$A \cap B \cap C = \phi$.

Now,

$$P(A) = P(B) = P(C) = \frac{18}{36} = \frac{1}{2}.$$

$$P(A\cap B) = P(A\cap C) = P(B\cap C) = \frac{9}{36} = \frac{1}{4}$$

$$P(A\cap B\cap C) = 0.$$

$P(A\cap B) = P(A) \times P(B) = \frac{1}{4} \Rightarrow A, B$ are indep.

$P(A\cap C) = P(A) \times P(C) = \frac{1}{4} \Rightarrow A, C$ are indep.

$P(B\cap C) = P(B) \times P(C) = \frac{1}{4} \Rightarrow B, C$ " ".

But

$$P(A\cap B\cap C) = 0 \neq P(A) \times P(B) \times P(C) = \frac{1}{8}$$

$\Rightarrow A, B, C$ are <u>not</u> indep.

Question 2: Suppose that silicon chips are manufactured at two factories, I and II. The number of defects on a chip manufactured at factory I is Poisson distributed with a mean of 0.25 defects/chip while for factory II the number of defects is also Poisson distributed with mean 0.5 defects/chip. Each factory produces an equal number of chips so that the probability that a given chip comes from factory I is 0.5.

(a) (8 pts) What is the probability that a given chip has 0 defects?

Let X = # of defects on a chip produced in factory I. Then, $X \sim$ Poisson(λ=0.25).

Let Y = # of defects on a chip produced in factory II. Then, $Y \sim$ Poisson(λ=0.5).

A: The event that a given chip has 0 defects
B: The event that a given chip is from factory I, P(B) = 0.5
C: The event that a given chip is from factory II, P(C) = 0.5

$$P(A|B=0) = P(X=0) = \frac{e^{-0.25}0.25^0}{0!} = e^{-0.25} \cong 0.7788$$

$$P(A|C=0) = P(Y=0) = \frac{e^{-0.5}0.5^0}{0!} = e^{-0.5} \cong 0.6065$$

$$P(A) = P(A \cap B) + P(A \cap C) = P(A|B)P(B) + P(A|C)P(C) = 0.7788 \times 0.5 + 0.6065 \times 0.5 = 0.6927$$

(b) (7 pts) Suppose a chip is found to have 0 defects. What is the probability that it was produced by factory II?

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(A|C)P(C)}{0.6927} = \frac{0.6065 \times 0.5}{0.6927} \cong 0.4378$$

Question 3: Let X be a random variable which takes the values 1, 2, 3. If $P(X = 1) = 0.5$ and $E(X) = 1.7$, find

(a) (5 pts) $P(X = 3)$

Let $P(X=3) = K$

x	1	2	3
$f(x)$	0.5	$0.5-K$ 0.3	K 0.2

$$E(X) = 1 \times 0.5 + 2 \times (0.5 - K) + 3K = 1.7$$

$$\Rightarrow 0.5 + 1 - 2K + 3K = 1.7$$

$$K = 0.2$$

(b) (5 pts) $Var(X)$

$$E(X^2) = 1 \times 0.5 + 4 \times 0.3 + 9 \times 0.2$$

$$= 3.5$$

$$Var(X) = EX^2 - (EX)^2 = 3.5 - (1.7)^2 = 0.61$$

(c) (5 pts) $P(X > 2 | X > 1)$

$$P(X > 2 | X > 1) = \frac{P(X > 1, X > 2)}{P(X > 1)}$$

$$= \frac{P(X > 2)}{P(X > 1)} = \frac{P(X = 3)}{P(X = 2) + P(X = 3)}$$

$$= \frac{0.2}{0.5}$$

$$= \frac{2}{5} \; (= 0.4)$$

Question 4: On a given day, a lecture may be cancelled due to bad weather with probability 0.05. Lecture cancellations on different days are independent.

(a) (5 pts) If there are 15 lectures left in the semester, find the probability that at least 2 of them get cancelled.

Let X = no. of canceled classes out of 15 remaining.

$X \sim \text{Binomial}(n=15, p=0.05)$

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - P(X=0) - P(X=1)$$

$$= 1 - \binom{15}{0} 0.05^0 (0.95)^{15} - \binom{15}{1}(0.05)^1 (0.95)^{14}$$

$$= 1 - (0.95)^{15} - 15(0.05)(0.95)^{14}$$

$$= 0.171$$

(b) (5 pts) Compute the probability that the tenth class this semester is the third class that gets cancelled.

Let Y = the no. of classes until three classes are canceled.

$\Rightarrow Y \sim \text{Negative Binomial}(k=3, p=0.05)$

$$\therefore P(Y=3) = \binom{10-1}{3-1} (0.05)^3 (0.95)^7$$

$$= \frac{9!}{2!\,7!} (0.05)^3 (0.95)^7$$

$$= \frac{(9)(8)}{2} (0.05)^3 (0.97)^7 = 0.00314$$

Question 5: The time needed to complete an exam is normally distributed with mean 40 minutes and standard deviation 4 minutes.

(a) (7 pts) What is the proportion of students who complete the exam in less than 44 min?

X : the time needed to complete the exam.

$X \sim N(\mu = 40, \sigma = 4)$

$P(X < 44) = P\left(Z < \frac{44-40}{4}\right) = P(Z < 1) = 0.8413.$

(b) (8 pts) If 50 students are selected at random, what is the probability that at least 10 of them will complete the exam in less than 44 min? [**Hint:** Use an appropriate approximation.]

$Y \sim \text{Binomial}(n = 50, p = 0.8413)$, Y = no. of students who complete the exam in less than 44 min.

$P(Y \geq 10) = 1 - P(Y \leq 9)$

$\approx 1 - P(Y \leq 9.5)$ Using the normal approximation to the binomial

$= 1 - P\left(Z \leq \frac{9.5 - 42.065}{2.584}\right)$

$\mu = np = 42.065$

$\sigma = \sqrt{np(1-p)} = 2.584$

$= 1 - P(Z \leq -12.6025)$

$= 1 - \text{zero}$

$= \underline{\underline{1}}$

Question 6: Suppose X and Y are continuous random variables with joint probability density function (pdf)

$$f(x,y) = \begin{cases} k(6-x-y), & 0 \le x \le 2 \text{ and } 2 \le y \le 4 \\ 0, & \text{elsewhere.} \end{cases}$$

In addition: $E(X) = \frac{5}{6}$, $E(Y) = \frac{17}{6}$, $E(X^2) = 1$, $E(Y^2) = \frac{25}{3}$ and $E(XY) = \frac{7}{3}$.

(a) (5 pts) Find k.

$$\int_2^4 \int_0^2 k(6-x-y)\,dx\,dy = k\int_2^4 \left[6x - \frac{x^2}{2} - xy\right]_0^2 dy = 1$$

$$\Rightarrow k\int_2^4 (12-2-2y)\,dy = 1$$

$$\Rightarrow k\int_2^4 (10-2y)\,dy = 1 \Rightarrow k\left(10y - y^2\right)\Big|_2^4 = 1$$

$$\Rightarrow k(24-16) = 1$$

$$\Rightarrow 8k = 1 \Rightarrow \boxed{k = \frac{1}{8}}$$

(b) (8 pts) Find the marginal pdf of X and Y.

- $g(x) = \int_2^4 \frac{1}{8}(6-x-y)\,dy = \frac{1}{8}\left(6y - xy - \frac{y^2}{2}\right)\Big|_2^4 = \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right]$

$$= \frac{6-2x}{8}, \quad 0 \le x \le 2$$

- $h(y) = \int_0^2 \frac{1}{8}(6-x-y)\,dx = \frac{1}{8}\left[12-2-2y\right] = \frac{10-2y}{8}, \quad 2 \le y \le 4.$

(c) (4 pts) Are X and Y independent? **Mathematically** prove or disprove.

Since

$f(x,y) \neq g(x)h(y) \Rightarrow$ X and Y are not indep.

(d) (7 pts) Find $Var(3X + 2Y + 2)$. Hint: use the additional information given in the question.

$$Var(3X+2y+2) = Var(3X+2Y) = 9\,Var(X) + 4\,Var(Y) + 12\,cov(X,Y)$$

- $Var(X) = EX^2 - (EX)^2 = 1 - \frac{25}{36} = \frac{11}{36}$
- $Var(Y) = EY^2 - (EY)^2 = \frac{25}{3} - \left(\frac{17}{6}\right)^2 = \frac{11}{36}$
- $COV(X,Y) = EXY - EXEY = \frac{7}{3} - \left(\frac{5}{6}\right)\left(\frac{17}{6}\right) = \frac{-1}{36}$

$$\therefore\ Var(3X+2Y+2) = 9\left(\frac{11}{36}\right) + 4\left(\frac{11}{36}\right) + 12\left(\frac{-1}{36}\right)$$

$$= \frac{131}{36}.$$

(e) (6 pts) Find $Corr(-3X + 2, 2Y - 1)$. Hint: use the additional information given in the question.

$$Corr(X,Y) = \frac{COV(X,Y)}{\sigma_X\,\sigma_Y} = \frac{\frac{-1}{36}}{\sqrt{\frac{11}{36}}\cdot\sqrt{\frac{11}{36}}} = \frac{\frac{-1}{36}}{11/36} = \frac{-1}{11}.$$

Now,

$$Corr(-3X+2, 2Y-1) = \frac{1}{11}.$$

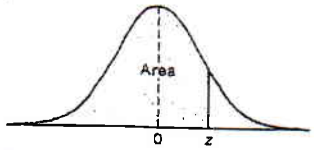

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

END!