MAT292 – Fall 2022

Term Test – November 17, 2022

Time alotted: 110 minutes

Full Name ________________________________

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DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This test contains 9 pages (including this title page). Once the midterm starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages **??**–9 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

	True/False	Short answer	Long answer				
Question	Q1-Q5	Q6	Q7	Q8	Q9	Q10	**Total**
Marks	5	2	8	6	6	6	**33**

SECTION I No justification necessary.

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

1. **(1 mark)** - An n-th order linear ODE must satisfy the principle of superposition.

   Solution: False, if the n-th order linear ODE is not homogenous then it does not satisfy the principle of superposition.

2. **(1 mark)** Given an initial value problem, the local truncation error of improved Euler must necessarily be lower than that of Euler.

   Solution: False. The **bound** on the local truncation error of improved Euler is lower than the **bound** on the local truncation error of Euler's method.

3. **(1 mark)** For an n-th order linear ODE, a set of solutions that are linearly independent necessarily construct a fundamental set.

   Solution: False. The number of elements in the set must be n.

4. **(1 mark)** A fundamental set for $\vec{x}' = A\vec{x}$ necessarily consists of functions of the form $e^{\lambda t}\vec{v}$, where λ is an eigenvalue and $\vec{v}$ is its corresponding eigenvector.

   Solution: False. The fundamental set could also contain linear combinations of those solutions as long as those are linearly independent.

5. **(1 mark)** For the ODE $\vec{x}' = A\vec{x}$, the special fundamental matrix is unique.

   Solution: True, it is equal to e^{At}.

SECTION II Justify all your answers.

6. **(2 marks)** Find a second-order initial value problem whose solution is $y(t) = te^t$. Explain how you found it.

   Solution: Based on the solution, we deduce that the characteristic equation must have two eigenvalues and both have to be equal to one. So our characteristic equation is

   $$(\lambda - 1)^2 = 0.$$

   The second-order ODE is therefore

   $$y'' - 2y + 1 = 0$$

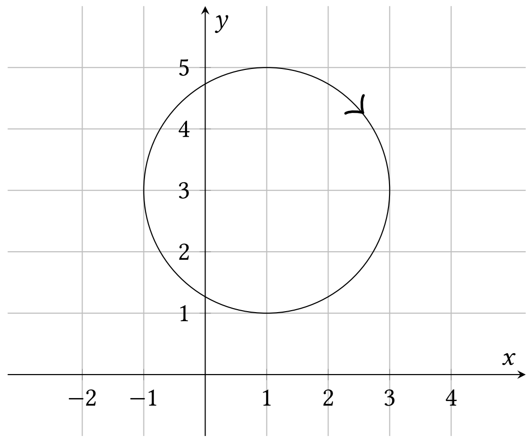
   and its solution is

   $$y(t) = c_1e^t + c_2te^t$$

   We deduce $c_1 = 0$ and $c_2 = 1$.

SECTION III **Justify all your answers.**

7. A physicist measured the trajectory of a particle in a lab. The trajectory and its direction are plotted below.



The physicist is interested in writing an ODE that describes the particle's trajectory. A MAT292 student proposed the following parameterization for the particle's trajectory:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2\cos(t) + 1 \\ 2\sin(-t) + 3 \end{bmatrix}$$

The student also proposed that the particle's trajectory satisfies the following ODE:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \right),$$

where a, b, c, d, e, f, g are some scalars.

(a) **(1 mark)** The student claims $f = 1$ and $g = 3$. Explain why.

Solution: Notice that the ODE also satisfies

$$\begin{bmatrix} (x-f)' \\ (y-g)' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} f \\ g \end{bmatrix} \right),$$

The above is an ODE with a equilibrium point $(1, 3)$. Therefore, $f = 1$ and $g = 3$.

(b) **(1 mark)** The student claims $\sqrt{(x'(t))^2 + (y(t)')^2} = k$ for some fixed constant k. Explain why.

Solution: Deriving the parametric equation we get

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2\sin(t) \\ -2\cos(t) \end{bmatrix}$$

and so $\sqrt{(x'(t))^2 + (y'(t))^2} = 2$.

(c) **(1 mark)** The student claims that the following equation must be satisfied

$$\begin{bmatrix} 0 \\ -k \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Explain why the above equation has to be correct.

Solution: At the point $(3,3)$ the derivative has to be $(0,-k)$. Therefore,

$$\begin{aligned} \begin{bmatrix} 0 \\ -k \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2a \\ 2c \end{bmatrix}. \end{aligned}$$

The equations imply $a = 0$ and $c = -k/2$.

(d) **(1 mark)** The student claims that the following equation must be satisfied

$$\begin{bmatrix} -k \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Explain why the above equation has to be correct.

Solution: At the point $(1, 1)$ the derivative has to be $(-k, 0)$ for the same scalar k. Therefore,

$$\begin{aligned} \begin{bmatrix} -k \\ 0 \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right) \\ &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -2b \\ -2d \end{bmatrix}. \end{aligned}$$

The equations imply $b = k/2$ and $d = 0$.

(e) **(1 mark)** Use the above items to deduce the ODE.

Solution: Using the previous items, we deduce the ODE is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & k/2 \\ -k/2 & 0 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

(f) **(1 mark)** What should the eigenvalues of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be?

Solution: $\pm ki/2$.

(g) **(1 mark)** Is one able to deduce the particle's velocity based on the physicist's measurements?

Solution: No, because it does not specify velocity.

(h) **(1 mark)** The student proposed that the particle's trajectory satisfies the ODE written above. Is this ODE unique or are there other ODEs that the particle's trajectory satisfies?

Solution: No. The argument to sin and cos can be multiplied by arbitrary constant ω. There are therefore infinite possible parametric equations and infinite possible ODEs that describe them.

8. Let $f(t, y) = e^{-t^2} y \sin(y)$. Consider the ODE:

$$y'(t) = f(t, y)$$

Note: For this question, none of the parts will ask or require you to **explicitly** solve this equation!
Hint for part b): Your bounds should not involve tedious computations. Recall that if $|g(t)| \leq M$ and $|h(t)| \leq N$ then

$$|g(t)h(t)| \leq M \cdot N, \qquad |g(t) + h(t)| \leq M + N$$

(a) **(1 mark)** Does the ODE have any equilibrium solutions? If it does, find an expression for them.

Solution: The equilibrium solutions are $y = k\pi$ where k is an integer. These are precisely the roots of $y\sin(y)$.

(b) **(3 marks)** Let $y_{sol}(t)$ be a solution to the following initial value problem:

$$\begin{cases} y'(t) = f(t, y) \\ y(0) = \frac{1}{2} \end{cases}$$

On the interval $t \in [0, 4]$, upper bound the three quantities:

- $|f(t, y_{sol}(t))|$

  Solution: Observe that for all $t \geq 0$, $e^{-t^2} \leq 1$. Similarly, $|\sin(y)| \leq 1$ for all $y \in \mathbb{R}$. Thus, we know that

  $$|f(t, y_{sol}(t))| \leq |y_{sol}(t)|$$

  So we just need to bound $y_{sol}(t)$. By the initial condition we know that $0 \leq y(0) \leq \pi$. Then since solution curves can not intersect (by existence and uniqueness), we know that $|y_{sol}(t)| \leq \pi$ for all $t \in [0, 4]$. Then

  $$|f(t, y_{sol}(t))| \leq \pi$$

- $|f_y(t, y_{sol}(t))|$

  Solution: Notice that $f_y(t, y) = e^{-t^2}(\sin(y) + y\cos(y))$. By the same logic as above, we can bound this by

  $$|f_y(t, y_{sol}(t))| \leq 1(1 + \pi) = 1 + \pi$$

- $|f_t(t, y_{sol}(t))|$

  Solution: Notice that $f_t(t, y) = -2te^{-t^2} y \sin(y)$. Since we are interested in the interval $[0, 4]$, we know that $t \leq 4$. Using this and what we did above, we get

  $$|f_t(t, y_{sol}(t))| \leq 2 \cdot 4 \cdot 1 \cdot \pi \cdot 1 = 8\pi$$

(c) **(2 marks)** Suppose we found the bounds in part b) to be:

$$|f(t, y_{sol}(t))| \leq M_1, \quad |f_y(t, y_{sol}(t))| \leq M_2, \quad |f_t(t, y_{sol}(t))| \leq M_3$$

Using M_1, M_2, M_3, derive an upper bound on the local truncation error for Euler's method if we use a step size of h for the initial value problem given in part b) on the interval $t \in [0, 4]$.

Solution: We know the local truncation error is bounded by

$$|e_{n+1}| \leq \frac{|y''(\xi)|}{2} h^2$$

So we just need to bound $y''(t)$ on the interval $t \in [0, 4]$. By the Chain Rule and the Law of Total Derivative, we know that $y''(t) = f_t(t, y) + f_y(t, y)f(t, y)$. Using the bounds from the previous part, we can conclude that

$$y''(t) \leq 8\pi + (1 + \pi)\pi = 9\pi + \pi^2$$

for all $t \in [0, 4]$. Thus, the local truncation error is bounded by

$$|e_{n+1}| \leq \frac{9\pi + \pi^2}{2} h^2$$

9. Consider the following initial value problem:

$$\begin{cases} y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y(t) = g(t) \\ y(t_0) = y_0,\ y'(t_0) = y_1,\ y''(t_0) = y_2 \end{cases}$$

(a) **(2 marks)** Express the initial value problem as a first order linear system.
Solution: Define the variables $x_1(t) = y(t)$, $x_2(t) = y'(t)$ and $x_3(t) = y''(t)$. Then

$$\begin{aligned} x_3'(t) &= -a(t)x_3(t) - b(t)x_2(t) - c(t)x_1(t) + g(t) & x_3(t_0) &= y_2 \\ x_2'(t) &= x_3(t) & x_2(t_0) &= y_1 \\ x_1'(t) &= x_2(t) & x_1(t_0) &= y_0 \end{aligned}$$

We can rewrite this as

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c(t) & -b(t) & -a(t) \end{pmatrix} \vec{x}(t) + \begin{pmatrix} 0 \\ 0 \\ g(t) \end{pmatrix}$$

where $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$

(b) **(1 mark)** Provide sufficient conditions that will guarantee us existence and uniqueness on an interval $I = (\alpha, \beta)$ where $t_0 \in I$
Solution: Using the existence and uniqueness theorems for first order linear systems, we can conclude that we need

- $a(t), b(t), c(t)$ are continuous in I
- $g(t)$ is continuous in I

(c) Consider the ODE:

$$y'''(t) + ay''(t) + by'(t) + cy(t) = 0$$

where a, b, c are constants. Suppose the corresponding matrix that you found in part (a) has a zero eigenvalue with eigenvector $\vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

- **(1 mark)** Under these assumptions, show that $\vec{v}_1$ is a solution to the system found in part (a).
Solution: Let A denote the corresponding matrix for the system. We have that

$$\frac{d}{dt}\vec{v}_1 = \vec{0}$$

and $A\vec{v}_1 = 0\vec{v}_1 = \vec{0}$. Thus, $\frac{d}{dt}\vec{v}_1 = A\vec{v}_1$ and $\vec{v}_1$ is a solution to the system.
- **(2 marks)** Determine as much as you can about the entries of $\vec{v}_1$ denoted by v_1, v_2, v_3.
Solution: We know that a solution to the system would be

$$\begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

But if $y(t) = v_1$ is constant, that implies $y'(t) = y''(t) = 0$. Thus, we can immediately conclude that $v_2 = v_3 = 0$. That means that an eigenvector with eigenvalue 0 **has** to be the of the form

$$\begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix},\ v_1 \in \mathbb{R}$$

10. Consider the second order ODE:

$$y''(t) - 2y'(t) + y(t) = e^{2t} + t^2$$

(a) **(1 marks)** Find a general solution to the homogeneous equation.
Solution: The characteristic polynomial of the homogeneous equation is:

$$(\lambda - 1)^2$$

so we know the system has a repeated eigenvalue and that the general solution to the homogeneous equation is:

$$y_h(t) = c_1 e^t + c_2 t e^t$$

(b) **(1 marks)** Suppose $y_1(t)$ is a solution to the nonhomogeneous ODE:

$$y''(t) + by'(t) + cy(t) = g_1(t)$$

and $y_2(t)$ is a solution to the nonhomogeneous ODE:

$$y''(t) + by'(t) + cy(t) = g_2(t)$$

Write down an ODE that $y_1(t) + y_2(t)$ will solve. Remember that you need to justify your answers.
Solution: $y_1(t) + y_2(t)$ will solve the following ODE:

$$y''(t) + by'(t) + cy(t) = g_1(t) + g_2(t)$$

(c) **(3 marks)** Find a particular solution to the non-homogeneous equation.
Solution: We can either do variation of parameters or use part b) and use method of undetermined coefficients. Let's first find a particular solution to

$$y''(t) - 2y'(t) + y(t) = e^{2t}$$

If we use the guess that $y_p^1(t) = Ae^{2t}$, we can plug this into the above equation to get that

$$Ae^{2t} = e^{2t} \Rightarrow A = 1$$

Now we need to find a particular solution to the equation

$$y''(t) - 2y'(t) + y(t) = t^2$$

We try a solution of the form

$$y_p^2(t) = At^2 + Bt + C$$

If we plug this into the equation we get that:

$$2A - 4At - 2B + At^2 + Bt + C = t^2$$

Solving the coefficients one at a time we get that $A = 1, B = 4, C = 6$. Thus we can conclude that the particular solution is:

$$y_p(t) = e^{2t} + t^2 + 4t + 6$$

(d) **(1 marks)** Form a general solution to the non-homogeneous equation.
Solution: From part a) and part c) we can conclude that the general solution to the non-homogeneous equation is:

$$y(t) = c_1 e^t + c_2 t e^t + e^{2t} + t^2 + 4t + 6$$