University of Toronto
Faculty of Applied Science and Engineering

Final Examination, 12 December 2013

First Year, Program 5

MAT194F Calculus I

Exam Type A

No aids of any kind are permitted.
No calculators of any kind are permitted.

Time allowed: 2 ½ hours.

Each question is worth 10 marks out of a total of 100.

Examiners: P.C. Stangeby and D. Penneys

1. (a) Find the derivative of: $2x$, $2/x^3$, $\sin^{-1}(3\sqrt{x})$, $\ln(3x^{-3})$, $2^{2/\sqrt{x}}$.

   (b) Find the anti-derivative of: $2x$, $3/x^3$, $\sin(3x)$, $x^2e^{x^3}$, 7^x.

2. (a) Provide a $\delta-\varepsilon$ type of proof that $\lim_{x\to-1}(2x+1)=-1$.

   (b) Provide a similarly rigorous proof that $\lim_{x\to\infty}\dfrac{4x-1}{2x+1}=2$.

3. Find the smallest possible area of an isosceles triangle that has a circle of radius r inside it.

4. Let $f(x)=\dfrac{1}{\sqrt[4]{1+x|x|}}$.

   (a) What is the domain of f? Where is f continuous? Where is f differentiable?
   (b) Sketch the curve of $f(x)$, indicating all important features.
   (c) Find the volume of the solid of revolution obtained by rotating the region bounded by $y=f(x)$, $y=0$, $x=-1/\sqrt{2}$ and $x=1$ around the x axis. Simplify your answer as much as possible.

Page 1 of 2 pages. Turn this page over.

5. Calculate the flowing limits:

(a) $\lim_{x\to 0} \cos(x+\sin x)$, (b) $\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3}$, (c) $\lim_{x\to 0} \frac{e^x-1-x}{x^2}$,

(d) $\lim_{x\to -\infty}\left(x^3+\sqrt{x^6+x^3+1}\right)$, (e) $\lim_{x\to\infty}\left(\sqrt[3]{x^3+x^2}-\sqrt[3]{x^3-x^2}\right)$

6. Consider the differential equation $y''+y'+y=xe^{-x}$.
   (a) Find the most general solution y.
   (b) Determine $\lim_{x\to\infty} y$ for any solution y.

7. Compute $\lim_{n\to\infty}\sum_{j=1}^{n}\frac{n}{j^2+n^2}$ any way you can.

8. Let $f(x)=x^3+3x+1$ and $g(x)=\arctan(x)$. How many real roots does $f(g(x))$ have? Justify your answer with a proof.

9. $\mathfrak{R}$ is the region bounded by the 4 curves: $y=3+\frac{1}{\pi}\sin(\pi x)$, $y=4-\frac{1}{\pi}\sin(\pi x)$, $x=-\frac{1}{\pi}\sin(\pi y)$ and $x=1+\frac{1}{\pi}\sin(\pi y)$.
   (a) Sketch $\mathfrak{R}$. You don't need to indicate the important features.
   (b) Calculate the area of $\mathfrak{R}$.
   (c) Calculate the volume obtained by rotating $\mathfrak{R}$ about the line $y=x$.

   Note: if you know the answer to (b), then the answer to (c) can be written down without any need for calculus. If you do that correctly you will get 1 mark. For full marks a correct calculus calculation is required.

10. Let $f(x)=\int_2^x \frac{dt}{\ln t}, x\geq 2$.

   (a) Show that there is a constant b such that $\int_b^{\ln x}\frac{e^t}{t}dt=f(x)$ and find the value of b.

   (b) Let $g(x)=e^4 f(e^{2x-4})-e^2 f(e^{2x-2}), x>3$. Show that $g'(x)=e^{2x}(x^2-3x+2)^{-1}$.

   (c) Express $\int_c^x \frac{e^{2t}}{t-1}dt$ in terms of $f(x)$ where $c=1+\frac{1}{2}\ln 2$.

Page 2 of 2 pages. Turn this page over.