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UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Term Test

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This is a closed book test. No calculators are permitted.

All six questions are of equal value.

This test contains 20 pages including the cover page 1, this information page 2, and page 20 that is for rough work only. The test is printed two-sided.

Do <u>not</u> tear any pages from this test.

Present complete solutions in the space provided.

Given information:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

Q1: Consider the scalar equation of a plane given by,

$$x + 2y + 3z = 0$$

4 i) What vector is produced by projecting $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on this plane?

Solution:

This vector $\vec{v}$ is not parallel to the plane because it is not orthogonal to the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Therefore, we will begin by projecting the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$,

$$proj_{\vec{n}}\vec{v} = \frac{6}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{3}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Since $\vec{v}$, the $proj_{\vec{n}}\vec{v}$ and the projection of $\vec{v}$ on the plane form a right-angle triangle, we can solve for the vector produced by projecting $\vec{v}$ on the plane as,

$$\vec{v} - proj_{\vec{n}}\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{7}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$$

3 ii) What vector is produced by projecting $\vec{v} = \begin{bmatrix} -1.5 \\ -3 \\ -4.5 \end{bmatrix}$ on this plane?

Solution:

This vector $\vec{v}$ is parallel to the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and therefore it is orthogonal to the plane.

Hence, the vector produced by projecting $\vec{v} = \begin{bmatrix} -1.5 \\ -3 \\ -4.5 \end{bmatrix}$ on this plane is the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

3 iii) What vector is produced by projecting $\vec{v} = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$ on this plane?

Solution:

This vector $\vec{v}$ is parallel to the plane. Therefore, the vector produced by projecting $\vec{v} = \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$ on this plane is the vector $\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$.

Q2:

2 i) What is the length of the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ in R^4?

Solution:

$||\vec{v}|| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$

2 ii) Find a vector $\vec{u}$ of length 4 that is parallel to $\vec{v}$ but in the opposite direction.

Solution:

A unit vector that is parallel to $\vec{v}$ but is in the opposite direction is given by

$\frac{-1}{||\vec{v}||}\vec{v} = \frac{-1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, a vector $\vec{u}$ of length 4 that is parallel to $\vec{v}$ but is in the opposite direction is given by $4\left(\frac{-1}{2}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \\ -2 \end{bmatrix}$.

2 iii) Using dot product, find the angle θ between $\vec{u}$ and $\vec{v}$.

Solution:

$$cos\,\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-8}{4\sqrt{4}} = -1$$

$\therefore \theta = \pi$

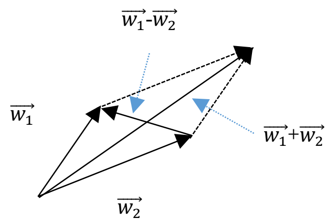
4 iv) Consider two nonzero, nonparallel vectors $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$ in R^4. Show that the sum of the squared diagonal lengths of the parallelogram formed by $\overrightarrow{w_1}$ and $\overrightarrow{w_2}$,

$$\|\overrightarrow{w_1} + \overrightarrow{w_2}\|^2 + \|\overrightarrow{w_1} - \overrightarrow{w_2}\|^2$$

add up to the sum of the four squared side lengths of the parallelogram,

$$2\|\overrightarrow{w_1}\|^2 + 2\|\overrightarrow{w_2}\|^2$$

Solution:



(Diagram not required)

$$\begin{aligned}\|\overrightarrow{w_1} + \overrightarrow{w_2}\|^2 + \|\overrightarrow{w_1} - \overrightarrow{w_2}\|^2 &= (\overrightarrow{w_1} + \overrightarrow{w_2}) \cdot (\overrightarrow{w_1} + \overrightarrow{w_2}) + (\overrightarrow{w_1} - \overrightarrow{w_2}) \cdot (\overrightarrow{w_1} - \overrightarrow{w_2}) \\ &= \|\overrightarrow{w_1}\|^2 + 2\overrightarrow{w_1} \cdot \overrightarrow{w_2} + \|\overrightarrow{w_2}\|^2 + \|\overrightarrow{w_1}\|^2 - 2\overrightarrow{w_1} \cdot \overrightarrow{w_2} + \|\overrightarrow{w_2}\|^2 \\ &= 2\|\overrightarrow{w_1}\|^2 + 2\|\overrightarrow{w_2}\|^2\end{aligned}$$

Q3:

2 i) Which value(s) for $a_{3,3}$ would leave matrix A with just two independent columns?

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & a_{3,3} \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 \\ 9 \\ a_{3,3} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$x_1 = 2$

$3x_1 + x_2 = 9 \, , \therefore \, x_2 = 9 - 3(2) = 3$

$5x_1 = a_{3,3} \, , \therefore \, a_{3,3} = 5(2) = 10$

Therefore, we can express column 3 as a linear combination of columns 1 and 2 when $a_{3,3} = 10$,

$$\begin{bmatrix} 2 \\ 9 \\ 10 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

leaving matrix A with just two independent columns.

2 ii) Consider matrix A given below,

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \end{bmatrix}$$

Factor matrix A into CR.

Solution:

$C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}$ because columns 3 and 4 are just linear combinations of the first two columns.

$$R = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

2 iii) Give a word description of the column space of matrix A in part (ii).

Solution:

The column space of matrix A is all linear combinations of the two column vectors in matrix C. This represents a plane in R^3 through the origin.

2 iv) Give a vector equation that represents the column space in part (iii).

Solution:

A vector equation of this column space in part (iii) is given by,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad c, d \text{ are scalars}$$

2 v) Give a scalar equation that represents the column space in part (iii).

Solution:

A normal vector to this plane may be found by taking the cross product,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

A scalar equation of this column space in part (iii) is given by,

$$0x - 2y + 2z = 0$$

Q4: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Show that the vector $AB\vec{x}$ is in the column space of matrix A.

Solution:

Start by calculating the vector $AB\vec{x} = (AB)\vec{x}$,

2.5 $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$

2.5 Therefore, $(AB)\vec{x} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \end{bmatrix} = (x_1 + x_2) \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

The column space of matrix A is defined as all linear combinations of the column vectors of matrix A,

2.5 $y_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

One such linear combination is when $y_1 = y_2 = y$. In this case,

2.5 $y_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = y \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} y + 2y \\ 3y + 4y \end{bmatrix} = \begin{bmatrix} 3y \\ 7y \end{bmatrix} = y \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Therefore, the vector $AB\vec{x}$ is in the column space of matrix A.

Q5:

5 i) Consider the system of linear equations,

$$\begin{aligned} 2x_1 - 3x_2 &= 3 \\ 4x_1 - 5x_2 + x_3 &= 7 \\ 2x_1 - x_2 - 3x_3 &= 10 \end{aligned}$$

After placing this system in the form $A\vec{x} = \vec{b}$, use elimination on the corresponding augmented matrix to bring it to its upper triangular form $U\vec{x} = \vec{c}$.

Solution:

$$\left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 10 \end{array}\right] \xrightarrow[R2-2R1]{} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -3 & 10 \end{array}\right] \xrightarrow[R3-R1]{} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 7 \end{array}\right]$$

$$\xrightarrow[R3-2R2]{} \left[\begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 5 \end{array}\right]$$

Therefore, we have arrived at the desired form $U\vec{x} = \vec{c}$, where

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$$

2 ii) Using back substitution, solve for x_1, x_2, x_3 in part (i).

Solution:

By back substitution,

$-5x_3 = 5 \quad \rightarrow \quad x_3 = -1$

$x_2 + x_3 = 1 \quad \rightarrow \quad x_2 = 1 - (-1) = 2$

$2x_1 - 3x_2 = 3 \quad \rightarrow \quad x_1 = \frac{1}{2}\big(3 + 3(2)\big) = 4.5$

2 iii) What is the row picture of the solution in part (ii)? What is the column picture of the solution in part (ii)?

Solution:

Row picture: 3 planes in R^3 that intersect at a single point $(4.5, 2, -1)$

Column picture: one linear combination of the column vectors of matrix A that equals $\vec{b}$,

$$(4.5)\begin{bmatrix}2\\4\\2\end{bmatrix}+(2)\begin{bmatrix}-3\\-5\\-1\end{bmatrix}+(-1)\begin{bmatrix}0\\1\\-3\end{bmatrix}=\begin{bmatrix}3\\7\\10\end{bmatrix}$$

1 iv) Consider a 5x5 matrix A (different from matrix A in part (i)) where the last corner entry, $a_{5,5}$, is equal to 11. After performing elimination on this matrix, the last pivot of the matrix U, $u_{5,5}$, turns out to be equal to 4. What different entry for $a_{5,5}$ would have resulted in $u_{5,5}$ being equal to 0? Explain your answer.

Solution:

To produce the last pivot in row 5, rows above row 5 are added or subtracted from row 5 during elimination. Therefore, if $a_{5,5} = 11$ resulted in $u_{5,5} = 4$, then $a_{5,5} = 7$ would have resulted in $u_{5,5} = 0$.

Q6:

3 i) Suppose you have a 3x3 elimination matrix E that replaces row 2 with row 2 minus row 1. Determine matrix E.

Solution:

Starting with the identity matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Take row 2 and subtract row 1 to produce E,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3 ii) Suppose you have a 3x3 permutation matrix P that exchanges rows 2 and 3. Determine matrix P.

Solution:

Starting with the identity matrix,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exchange rows 2 and 3 to produce P,

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

4 iii) Determine a single 3x3 matrix that first replaces row 2 with row 2 minus row 1 and then exchanges rows 2 and 3.

Solution:

A matrix that first takes row 2 and subtracts row 1 and then exchanges rows 2 and 3 may be found by finding the product PE,

$$PE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

This page is for rough work only.