University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – April, 2018

FIRST YEAR – ENGINEERING SCIENCE

MAT195S CALCULUS II

Examiners: F. Al Faisal and J. W. Davis

First name (please write as legibly as possible within the boxes)

Last name

Student number

Instructions:
(1) Closed book examination; no calculators, no aids are permitted
(2) Answer as many questions as you can. Parts of questions may be answered.
(3) Do not separate or remove any pages from this exam booklet.

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Question	Marks	Earned	Question	Marks	Earned
1	13		7	10	
2	8		8	12	
3	9		9	8	
4	10		10	10	
5	10		11	12	
6	8		12	10	
			Total	120	

1) Evaluate the integrals:

a) $\int \frac{dx}{\sqrt{2x - x^2}}$

b) $\int \tan^5 x \, dx$

c) $\int \frac{4x}{x^3 + x^2 + x + 1} dx$

(13 marks)

2) Find the area of region outside $r = \cos(2\theta)$ and inside $r = 1/2$. Provide a sketch of the region.

(8 marks)

3) Sketch the parametric curve: $x = t^3 - 3t, \quad y = t^2$

(9 marks)

4) a) Determine whether the sequence converges or diverges. If it converges, find its limit.

i) $a_n = n^2 e^{-n}$ ii) $a_n = \ln(n+1) - \ln(n)$

(5 marks)

b) Determine the radius and interval of convergence for the series: $\sum_{k=1}^{\infty} \frac{(5x-4)^k}{k^3}$

(5 marks)

5) Prove the Alternating Series Test for series convergence:

Let $\{a_n\}$ be a sequence of positive numbers. If $a_{k+1} < a_k$, and $a_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges. (A diagram is most helpful in formulating this proof.)

(10 marks)

6) Let $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Show that $\sum_{n=2}^{\infty} \frac{1}{n^2\, a_n\, a_{n-1}}$ converges, and find its sum.

Hint: $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$

(8 marks)

7) A bee and two trains: One day a bee was frightened by a train (train A) travelling at 20 km/hr eastward along a straight stretch of track. The bee flies away from the train, going east at 30 km/hr. At that exact moment, a westbound train (train B) is exactly 10 km away, travelling toward train A at 20 km/hr. The bee flies east until it meets train B, at which point it turns around and flies westward (again at 30 km/hr) back towards train A. Each time the bee meets a train, it turns around and flies the other way. Eventually, the trains meet, and the bee falls down dead from exhaustion. How far has the bee flown? The simple answer is found by finding how long it takes for the trains to meet, and noting that the bee is always flying at 30 km/hr. (There are no marks for this solution.) The harder way is to derive and sum the infinite series formed by the distance travelled by the bee on each of the eastbound and westbound legs of its journey. Find this series and its sum.

(10 marks)

8) Find the unit tangent vector, the principal normal vector and an equation in x, y, z for the osculating plane at the point $t = 1$ on the curve: $\bar{r}(t) = t\,\hat{i} + 2t\,\hat{j} + t^2\,\hat{k}$

(12 marks)

9) Suppose that $z = f(x,y)$ has continuous second order partial derivatives. Suppose also that $x = s^2 - t^2$ and $y = 2st$. Show that $\frac{d^2z}{ds^2} + \frac{d^2z}{dt^2} = g(s,t)\left(\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2}\right)$ for some function $g(s,t)$ – and determine this function explicitly.

(8 marks)

10) Let $f(x.y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & for \quad (x,y) \neq (0,0) \\ 0 & for \quad (x,y) = (0,0) \end{cases}$

a) Find $f_x(a,b)$ for all (a,b) in $\mathbb{R}^2$.
b) Show that f_x is continuous at $(0,0)$.
c) Without doing any extra work, find $f_y(a,b)$ and explain why Find f_y is continuous at $(0,0)$.
d) Hence, or otherwise, find the directional derivative $D_u f(0,0)$ where $\hat{u}$ is an arbitrary unit vector.

(10 marks)

11) Find the absolute max/min values of $f(x,y) = y^2 - 2y - 3x^2y$ on the closed and bounded set enclosed by the curve $x = \sqrt{y}$ and the lines $x = 1$ and $y = 0$. Provide a sketch of the region, and indicate the locations of the maximum and minimum.

(12 marks)

12) The plane $x+2y+3z=30$ intersects the paraboloid $z=(x^2+y^2)/3$ in an ellipse. Use Lagrange multipliers to find the points on the ellipse that are furthest away from and closest to the origin.

(10 marks)