Problems for Chapter 5: Compressible flows

1. For compressible one-dimensional, inviscid, adiabatic, steady channel flow of a perfect gas with constant specific heat ratio γ

(a) show that, in the presence of terrestrial gravity, the energy equation simplifies to:

$$1 + \frac{\gamma - 1}{2} M^2 = \left(1 - \frac{\gamma - 1}{\gamma} \frac{gz}{RT_0}\right) \frac{T_0}{T} \ .$$

(b) For air flowing from a reservoir at $T = 300\ K$ through a 10 meter long nozzle which has its axis vertically positioned, find the temperature at the nozzle exit if $M = 1$ at the exit.

(c) Repeat part (b) assume that the influence of gravity is negligible, i.e., $g = 0$. Compare the results with those associated with (b).

2. The Bernoulli equation for steady, inviscid, compressible, adiabatic flow along a streamline for which gravity is neglected takes the form of:

$$e + \frac{P}{\rho} + \frac{V^2}{2} = const.$$

Consider an ideal gas undergoing an isentroic process. Show that the relationship for area change in a channel after a section of minimum area (a throat), A^*, where $M = 1$ is given by:

$$\frac{A}{A^*} = \frac{1}{M} \frac{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}}{\left(1 + \frac{\gamma - 1}{2}\right)^{\frac{1}{\gamma - 1}}}$$

Problem: For compressible one0dimensional frictionless adiabatic steady channel flow of a perfect gas with p=ρRT and constant specific heat ratio γ show that, in the presence of terrestrial gravity, the energy equation takes the form

$$1+\frac{\gamma-1}{2}M^2=\left[1-\frac{\gamma-1}{\gamma}\frac{gz}{RT_o}\right]\frac{T_o}{T}$$

For air flowing from a reservoir at T_o = 300 °K through a 10 meter long nozzle which has its axis vertical, find the error in T at the nozzle exit caused by omitting gravity if the flow Mach number M = 1 at the exit.

Energy eqn: $\overbrace{e+\frac{p}{\rho}}^{h}+\frac{V^2}{2}+gz=\text{const}$

→ From thermodynamics, remember: $\frac{p}{\rho}=RT$, $e=C_vT$ and $C_v=\frac{R}{\gamma-1}$

$$e+\frac{p}{\rho}=h \Rightarrow C_vT+RT=h \Rightarrow \frac{RT}{\gamma-1}+RT=h$$

$C_v = \frac{R}{\gamma-1}$

$$\frac{RT(\cancel{1}+\gamma-\cancel{1})}{\gamma-1}=h$$

$$\boxed{\frac{\gamma RT}{\gamma-1}=h}$$

→ Assume that $V=0$ (adiabatic stagnation of the flow) occurs at $z=0$. (This assumption is nothing more than selecting the reference frame such that $z=0$ corresponds to the point of stagnation)

At the stagnation: $h_o+\cancel{\frac{V^2}{2}}+g\cancel{z}=\text{const}$

$\qquad\qquad\qquad\qquad\qquad 0 \qquad 0$

Then, if we write the compressible Bernoulli between a point and the stagnation point in the flow:

$$h+\frac{V^2}{2}+gz=h_o \Rightarrow \frac{\gamma RT}{\gamma-1}+\frac{V^2}{2}+gz=\frac{\gamma RT_o}{\gamma-1}$$

Multiply both sides of the last eqn with $\frac{\gamma-1}{\gamma RT}$

$$1 + \frac{\gamma-1}{2}\frac{V^2}{\underbrace{\gamma RT}_{c^2}} + \frac{(\gamma-1)}{\gamma}\frac{gz}{RT} = \frac{T_0}{T}$$

$$V^2/c^2 = M^2$$

$$1 + \frac{\gamma-1}{2}M^2 + \frac{(\gamma-1)}{\gamma}\frac{gz}{RT} = \frac{T_0}{T}$$

$$\boxed{1 + \frac{\gamma-1}{2}M^2 = \frac{T_0}{T}\left\{1 - \frac{(\gamma-1)}{\gamma}\frac{gz}{RT_0}\right\}}$$

(b) For flow at $M=1$, with $g=0$, $\gamma = 1.4$

Using $\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$,

we obtain:

$$\frac{T_0}{T} = 1 + \frac{1.4-1}{2}1^2 = 1.2$$

For $T_0 = 300K$, $T = \frac{300}{1.2} = 250K$

(c) The error in omitting gravity for $M=1$

$$T = \frac{T_0}{1 + \frac{\gamma-1}{2}M^2}\left\{1 - \frac{(\gamma-1)}{\gamma}\frac{gz}{RT_0}\right\} = \frac{300}{1.2}\left\{1 - \frac{0.4}{1.4}\times\frac{9.81\times 10}{287\times 300}\right\}$$

$$T = \frac{300}{1.2}\left\{1 - 0.000325\right\} K$$

Thus T is reduced by 0.0325 percent!

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<u>Note:</u> Remember that when deriving $T_o = T + \frac{V^2}{2C_p}$ and from which deriving $\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2} M^2$ in the class (see class notes) we assummed potential energy to be negligible compared to enthalpy and kinetic energy in the compressible Bernoulli eqn. That is, we assumed $h + \frac{V^2}{2} + \cancel{gz} = \text{const} \Rightarrow h + \frac{V^2}{2} = \text{const}$ and when $V=0 \Rightarrow h + \frac{V^2}{2} = h_o$. From which, we obtaind. $T_o = T + V^2/2C_p$ and $\frac{T_o}{T} = 1 + \frac{\gamma-1}{2} M^2$. This problem shows the error made by neglecting the gravitational effects (in other words potential energy) to be very small!

② Conservation of mass

$$\rho A V = \rho^* A^* V^*$$

$V^* = c \rightarrow$

$$\frac{A}{A^*} = \frac{\rho^*}{\rho} \cdot \frac{V^*}{V} = \frac{c}{V} \cdot \frac{\rho^*}{\rho} = \frac{1}{M} \cdot \frac{\rho^*}{\rho}$$

(I) $\frac{\rho_o}{\rho} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}}$ $\leftarrow$ from isentropic flow relations

(II) $\frac{\rho_o}{\rho^*} = \left[1 + \frac{\gamma - 1}{2}\right]^{\frac{1}{\gamma - 1}}$ $\leftarrow$ at the throat $M = 1$

Divide (I) and (II) $\rightarrow$

$$\frac{\rho^*}{\rho} = \frac{\left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}}}{\left[1 + \frac{\gamma - 1}{2}\right]^{\frac{1}{\gamma - 1}}} \rightarrow \frac{A}{A^*} = \frac{1}{M} \cdot \frac{\left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{1}{\gamma - 1}}}{\left[1 + \frac{\gamma - 1}{2}\right]^{\frac{1}{\gamma - 1}}}$$