ESC195 - Final Exam
April 2022

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Closed book, no aid sheets, no calculators
There are 12 questions; each question is worth 10 marks.

1. Evaluate the integrals:

a) $\int \frac{lny}{\sqrt{y}}\, dy$ b) $\int \frac{dt}{t^2\sqrt{t^2-16}}$ c) $\int \tan^5 x\, dx$

2. Find the area that lies inside $r = 3 + 2\cos\theta$ and outside $r = 4$. Provide a sketch of the region.

3. (a) Let $a_n = \max\{\sin 1, \sin 2, \sin 3, \ldots \sin n\}$
Does $\{a_n\}$ converge? What is its limit? Explain.

(b) Find the radius and interval of convergence of the power series: $\displaystyle\sum_{n=2}^{\infty} \frac{5^n}{n} x^n$

4. Proof of the Limit comparison Test. (You may use the basic comparison test in your proof.)

   (a) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ is also convergent.

   (b) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then $\sum a_n$ is also divergent.

5. Assume that f is a non-negative increasing function defined for $x \geq 1$.

(a) Show that $\sum_{k=1}^{n-1} f(k) < \int_1^n f(x)\, dx < \sum_{k=1}^{n} f(k)$

(b) By taking $f(x) = \ln(x)$, obtain the inequality $e\, n^n\, e^{-n} < n! < e\, n^{n+1}\, e^{-n}$ and deduce $\frac{e^{1/n}}{e} < \frac{(n!)^{1/n}}{n} < \frac{e^{1/n}\, n^{1/n}}{e}$.

(c) Determine $\lim_{n \to \infty} \frac{(n!)^{1/n}}{n}$

6. (a) L'Hospital's rule by Taylor Series: Suppose f and g have Taylor series about the point a. If $f(a) = g(a) = 0$ and $g'(a) \neq 0$, evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$ by expanding f and g in their Taylor series. Show that the result is consistent with l'Hospital's rule.

(b) Find all solutions to the equation: $1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots = 0$

7. The motion of a particle is given by $\vec{r}(t) = t\,\hat{\imath} + \dfrac{1}{t}\,\hat{\jmath} + \sqrt{2}\ln(t)\,\hat{k}$, for $t > 0$. Determine the unit tangent vector, the unit normal vector and the tangential and normal components of acceleration of this particle at time $t = 1$. Also find the curvature of its path at $t = 1$.

8. (a) Find the limit if it exists, or show that the limit does not exist: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$

(b) Use the formal definition for the derivative of a multivariable function (the $o(h)$ formulation) to find the gradient of: $f(x,y) = \frac{1}{2}x^2 + 2xy + y^2$. Show that all remainder terms are $o(h)$.

9. Find the absolute maximum and minimum values of $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$ on the set: $D = \{(x, y)|\ 0 \leq x \leq 2, 0 \leq y \leq 3\}$. Provide a sletch of the region, and identify and show the locations of all critical points.

10. Use Lagrange Multipliers to find the extreme values of the function $f(x, y, z) = x^2 + y^2 + 2z$ on the curve of intersection between the plane $x + y + 2z = 2$ and the paraboloid $z = x^2 + y^2$.

11. (a) Use Clairaut's theorem to determine if the vector function: $(e^x \ln z + 2xy)\,\hat{\imath} + (x^2 + z\,\sin y)\,\hat{\jmath} + (\frac{e^x}{z} - \cos y)\,\hat{k}$ is a gradient , $\nabla f(x, y, z)$. If so, find such a function f.

(b) Solve the integral equation: $y(x) = 2 + \int_1^x (y(t))^2\,dt$

12. Use a change of variables to convert the integral: $\displaystyle\int_0^{\pi/4} \sqrt{\tan x}\, dx$ to one amenable to solution by partial fractions, then evaluate the integral.