TUTORIAL 5 — SOLUTIONS

Q1:

A AND B BOTH COMMUTE WITH C

$\therefore$ $AC = CA$
$BC = CB$

TO PROVE AB COMMUTES WITH
WE WANT TO SHOW THAT

$$(AB)C = C(AB)$$

$$(AB)C = A(BC) = A(CB) = (AC)B = (CA)B = C(AB)$$

$\therefore$ AB COMMUTES WITH C.

Q2:

PROVE THAT $P\varphi = \varphi P$ IFF $(P-\varphi)(P+\varphi) = P^2 - \varphi^2$

GIVEN $P\varphi = \varphi P$

THEN $(P-\varphi)(P+\varphi) = P^2 + P\varphi - \varphi P - \varphi^2$

$= P^2 + P\varphi - P\varphi - \varphi^2$

$= P^2 - \varphi^2$

GIVEN $(P-\varphi)(P+\varphi) = P^2 - \varphi^2$

THEN $\cancel{P^2} + P\varphi - \varphi P - \cancel{\varphi^2} = \cancel{P^2} - \cancel{\varphi^2}$

$\therefore$ $P\varphi - \varphi P = 0$

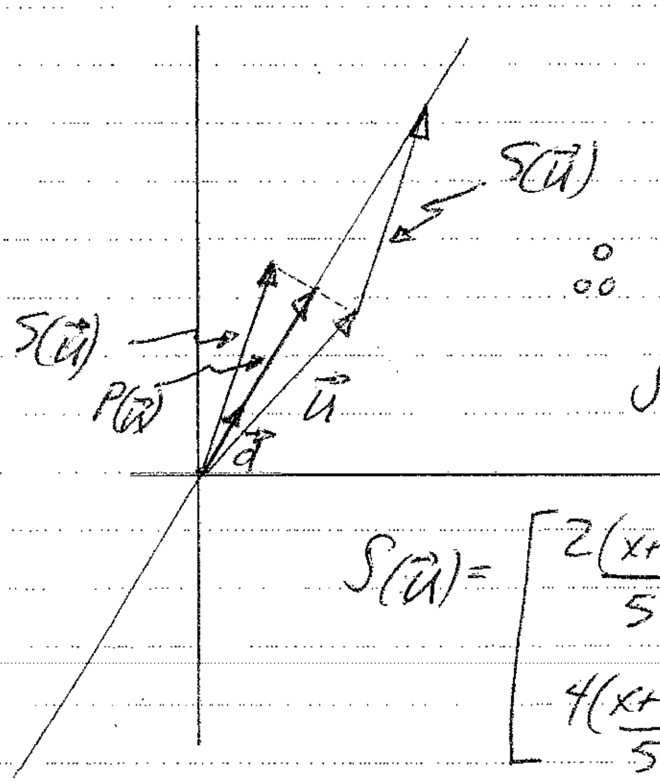
$P\varphi = \varphi P$

Q3:

a) TAKEN FROM LECTURE, THE TRANSFORMATION WHICH ASSIGNS EACH VECTOR $\begin{bmatrix} x \\ y \end{bmatrix}$ TO ITS PROJECTION ON THE LINE WITH DIRECTION VECTOR $\vec{d} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ IS GIVEN BY:

$$P(\vec{u}) = \text{proj}_{\vec{d}}\, \vec{u} = \begin{bmatrix} \left(\frac{x+2y}{5}\right) \\ 2\left(\frac{x+2y}{5}\right) \end{bmatrix} \qquad \text{WHERE } \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$$

b)



$$\therefore\ 2P(\vec{u}) = \vec{u} + S(\vec{u})$$

$$S(\vec{u}) = 2P(\vec{u}) - \vec{u}$$

$$S(\vec{u}) = \begin{bmatrix} 2\left(\frac{x+2y}{5}\right) - x \\ 4\left(\frac{x+2y}{5}\right) - y \end{bmatrix}$$

$$S(\vec{u}) = \begin{bmatrix} \frac{-3x+4y}{5} \\ \frac{4x+3y}{5} \end{bmatrix}$$

c) $$P(\vec{u}) = M_P \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+2y}{5} \\ \frac{2(x+2y)}{5} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}}_{M_P} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$S(\vec{u}) = M_S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-3x+4y}{5} \\ \frac{4x+3y}{5} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}}_{M_S} \begin{bmatrix} x \\ y \end{bmatrix}$$

d) $|M_p| = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right)$

$= \frac{4}{25} - \frac{4}{25}$

$= 0$ $\therefore$ NO INVERSE FOR M_p

$|M_s| = \left(\frac{-3}{5}\right)\left(\frac{3}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{4}{5}\right)$

$= \frac{-9}{25} - \frac{16}{25}$

$= -1$ $\therefore$ INVERSE FOR M_s EXISTS

FROM LECTURE NOTES

$N_s M_s = I$

$$N_s = \frac{1}{|M_s|}\begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

Q4:

a) $M_A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$

IF $\vec{u}$ IS ANY VECTOR IN R^2 WHICH LIES ON THE LINE THROUGH THE ORIGIN WITH DIRECTION VECTOR $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

THEN $\vec{u} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ t SCALAR

$$M_A\vec{u} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} t \\ t \end{bmatrix}$$

$$= \begin{bmatrix} t-t \\ 2(t-t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) IF $\vec{u}$ IS ANY VECTOR IN R^2

$$M_A\vec{u} = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x-y \\ 2x-2y \end{bmatrix}$$

$$= (x-y)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

WHICH IS A VECTOR PARALLEL TO $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ FOR ANY X AND Y VALUE.

$\therefore$ $M_A \vec{u}$ LIES ON THE LINE THROUGH THE ORIGIN WITH DIRECTION VECTOR $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q5:

a) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$ $ax + by = \lambda x$ ①
$cx + dy = \lambda y$ ②

① $by = \lambda x - ax$
$y = \frac{(\lambda - a)}{b} x$

① → ② $cx + \frac{d(\lambda - a)}{b} x = \lambda \frac{(\lambda - a)}{b} x$

$\times b$ $cbx + d\lambda x - ad\,x = \lambda^2 x - a\lambda x$

$\div x$ $cb + d\lambda - ad = \lambda^2 - a\lambda$

$\lambda^2 - a\lambda - d\lambda + ad - bc = 0$

$\lambda^2 + (-a-d)\lambda + ad - bc = 0$

COMPARE TO $\lambda^2 + \alpha\lambda + \beta = 0$

$\therefore$ $\alpha = -(a+d)$ $\beta = ad - bc$

b) $M_P = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$ $\alpha = -\left(\frac{1}{5} + \frac{4}{5}\right) = -1$

$\beta = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = 0$

SOLVE $\lambda^2 - \lambda + 0 = 0$

$$\lambda(\lambda - 1) = 0$$

$$\lambda = 0, 1$$

$M_S = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ $\alpha = -\left(\frac{-3}{5} + \frac{3}{5}\right) = 0$

$\beta = \left(\frac{-3}{5}\right)\left(\frac{3}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) = -1$

SOLVE $\lambda^2 - 0\lambda - 1 = 0$

$$\lambda^2 - 1 = 0$$

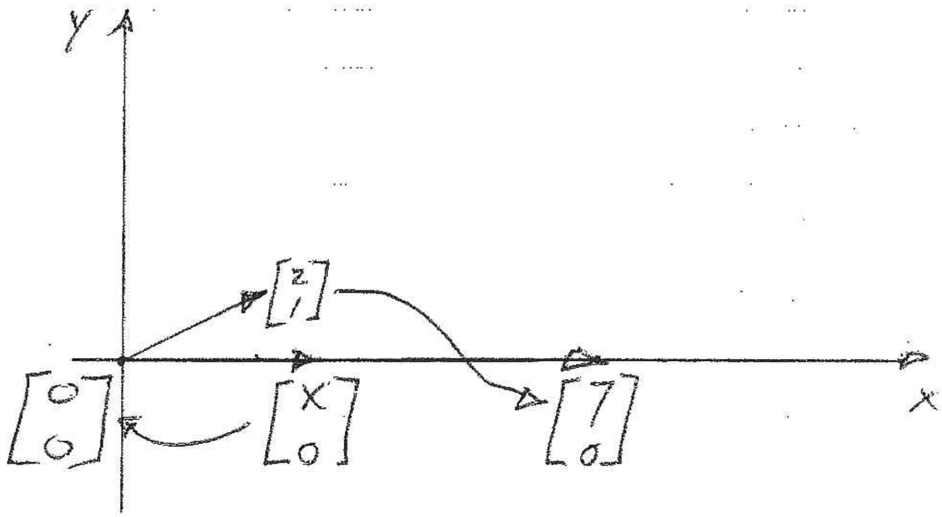
$$\lambda = +1, -1$$

Q6:

EIGENVALUES AND EIGENVECTORS OF MATRIX M SATISFY:

$$M\begin{bmatrix}x\\y\end{bmatrix} = \lambda\begin{bmatrix}x\\y\end{bmatrix} \qquad \lambda \text{ SCALAR}$$

a) $$\begin{bmatrix}0 & 7\\0 & 0\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}7\\0\end{bmatrix}$$

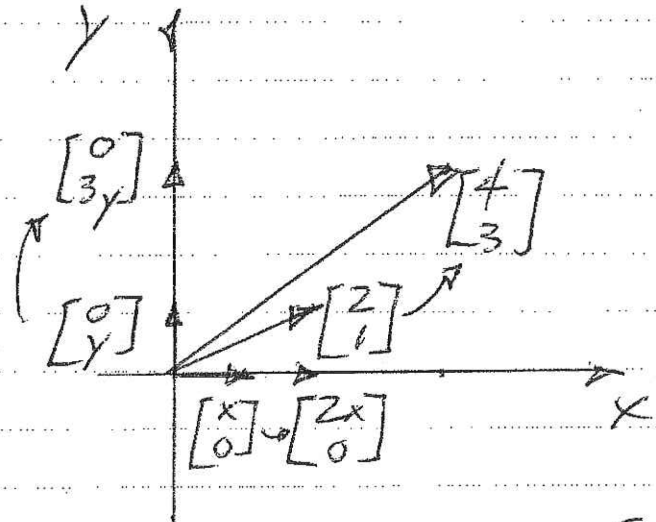


WHAT HAPPENS WHEN $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\0\end{bmatrix}$?

$$\begin{bmatrix}0 & 7\\0 & 0\end{bmatrix}\begin{bmatrix}x\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}x\\0\end{bmatrix}$$

EIGENVALUE IS ZERO AND CORRESPONDING EIGENVECTORS ARE ALL VECTORS PARALLEL TO THE X-AXIS.

b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



WHAT HAPPENS WHEN $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \end{bmatrix} = 2\begin{bmatrix} x \\ 0 \end{bmatrix}$$

WHAT HAPPENS WHEN $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$?

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3y \end{bmatrix} = 3\begin{bmatrix} 0 \\ y \end{bmatrix}$$

ONE EIGENVALUE IS 2 AND CORRESPONDING EIGENVECTORS ARE ALL VECTORS PARALLEL TO THE X-AXIS.

ONE EIGENVALUE IS 3 AND CORRESPONDING EIGENVECTORS ARE ALL VECTORS PARALLEL TO THE Y-AXIS.

Q7:

LET $N = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

SOLVE $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 5 & 5 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$5a + b = 1 \qquad \therefore \quad a = 0$$
$$5a = 0 \qquad b = 1$$
$$a + c = 0 \qquad c = 0$$

$$5d + e = 0 \qquad \therefore \quad d = 1/5$$
$$5d = 1 \qquad e = -1$$
$$d + f = 0 \qquad f = -1/5$$

$$5g + h = 0 \qquad \therefore \quad g = 0$$
$$5g = 0 \qquad h = 0$$
$$g + i = 1 \qquad i = 1$$

$\therefore$

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 1/5 & -1 & -1/5 \\ 0 & 0 & 1 \end{bmatrix}$$