UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

Second Year — Engineering Science

STA286H1 S — Probability and Statistics

Term Test

Monday, February 13, 2017

50 Minutes

POSSIBLE SOLUTIONS

1. **(30 marks total)** A mining company operates mines at three locations in British Columbia. Each mine uses a fleet of haul trucks. If a haul truck engine fails it is immediately replaced with a spare engine from the storage facility at that location, if one is available. If a spare engine is not available, the mine has to order one from the supplier.

Assume that all engine failures are independent of each other.

The following table contains information about the three locations:

Location	Number of engines in use	For each engine, chance of that engine failing in the next year	Number of spare engines in storage
A	10	0.10	3
B	8	0.08	2
C	7	0.13	1

a) **(12 marks)** An engine fails within one year. What is the probability that it was being operated at Location A?

$$\frac{0.1 \cdot \frac{10}{25}}{0.1 \cdot \frac{10}{25} + 0.08 \cdot \frac{8}{25} + 0.13 \cdot \frac{7}{25}} = 0.392$$

b) **(8 marks)** What is the probability that Location C will run out of spare engines in the next year and be forced to order more from the supplier?

Denote by X the number of failures at location C in the next year. We need $P(X \geq 2)$. Only way to calculate is to use $1-P(X < 2) = 1-P(X = 0)-P(X = 1) = 1-0.87^7-7(0.87)^6 0.13 = 0.228$.

c) **(10 marks)** For an engine at Location A the distribution of the random failure time T (measured in years) can be modelled by a probability density function of the following form:

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t} & : \quad t > 0 \\ 0 & : \quad t \leq 0 \end{cases}$$

Determine the correct value of λ, using the fact from the previous page that the probability of a particular engine at Location A failing within one year is 0.1.

$$0.1 = \int_0^1 \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda}$$

so $\lambda = -\log 0.9 = 0.1054$.

2. **(22 marks total)** Suppose X and Y are continuous random variables with joint probability density function (p.d.f.) given by:

$$f(x, y) = \begin{cases} \frac{3}{5}(x^2 + xy) & \text{for } 0 < x < 1 \text{ and } 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

a) **(7 marks)** Calculate the density function for the conditional distribution of Y given $X = \frac{1}{2}$.

$$f(x) = \int_0^2 \frac{3}{5}(x^2 + xy)\, dy = \frac{6}{5}(x^2 + x)$$

$$f(y|x = 1/2) = \frac{\frac{3}{5}(1/4 + y/2)}{\frac{6}{5}(1/4 + 1/2)} = \frac{2}{3}(1/4 + y/2)$$

b) **(5 marks)** Determine if X and Y are independent.

X and Y cannot be independent, since the joint density cannot be factored into a function of x and a function of y.

Alternatively, the conditional for Y given $X = x$ changes with the value of x.

c) **(10 marks)** Calculate $E(X - Y)$.

$$\int_0^1 \int_0^2 (x - y)\, \frac{3}{5}\left(x^2 + xy\right)\, dy\, dx = -\frac{1}{2}$$

3. **(8 marks total)** Prove that A and B are independent if and only if $P(A|B) = P(A|B^c)$ (as long as $0 < P(B) < 1$ so that the conditional probabilities exist).

If $A \perp B$ then $P(A|B) = P(A)$ and $P(A|B^c) = P(A)$, so $P(A|B) = P(A|B^c)$.

If $P(A|B) = P(A|B^c)$, then $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = P(A|B)P(B) + P(A|B)P(B^c) = P(A|B)(P(B) + P(B^c)) = P(A|B)$, so that $A \perp B$. Other (likely messier) proofs are possible.