UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION: April 20, 2023 6:30-9:00 PM

DURATION: 2½ hours

Second Year - Engineering Science

ECE259H1S - Electromagnetism

First name (please write as legibly as possible within the boxes)

Last name

Student Number

Calculator Type: 2 (All non-programmable calculators permitted)

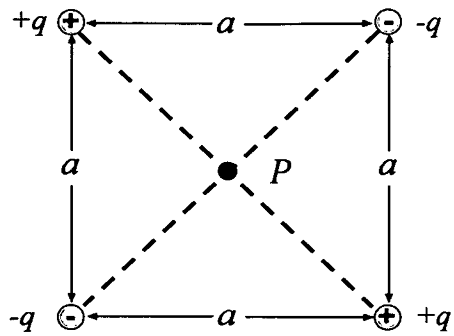
Exam Type: B (Only the aid sheet provided with the exam is allowed)

Examiner: M. Stickel

1. DO NOT disconnect any pages. Leave this booklet stapled.
2. Where appropriate, include your final answers for the questions in the box provided. Make sure to include units if your answer is numeric.
3. An aid sheet is provided for you separately.
4. Ensure you respect all academic integrity policies and guidelines. Your work must be entirely your own.
5. Take a deep breath and relax!

Question	Marks
Q1	10
Q2	10
Q3	20
Q4	20
Q5	20
Q6	20
BONUS	5
Total	**100**

(*4 marks*) 1. (a) Four charges of equal magnitude are organized into a square with a side length a, as shown. The electric scalar potential at the center of the square (point P) is:



(i) $V(P) = \frac{\sqrt{2}q}{4\pi\epsilon_0 a}$

(ii) $V(P) = \frac{-\sqrt{2}q}{4\pi\epsilon_0 a}$

(iii) $V(P) = 0$

(iv) $V(P) = \frac{2\sqrt{2}q}{4\pi\epsilon_0 a}$

Briefly justify your answer.

(6 marks) 1. (b) Below is given an electric flux density vector field and an electric potential scalar field. For the given statements select if they are **True** or **False**. Briefly justify your choice.

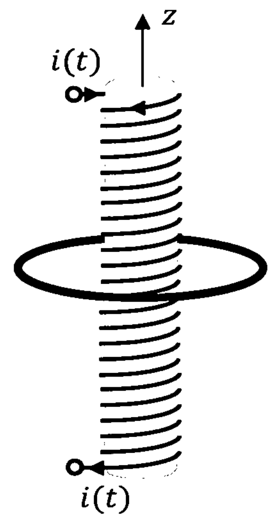
$$\mathbf{D} = \frac{2}{R}\hat{\mathbf{a}}_R - 4\cos\theta\,\hat{\mathbf{a}}_\theta \ [\text{C/m}^2] \qquad V = \frac{6\cos\varphi}{r^2}\ [\text{V}]$$

True or **False** (i) **D** represents an electric field associated with a static charge distribution.

True or **False** (ii) At the point $P(x,y,z) = (0,1,0)$, **D** will only have an $\hat{\mathbf{a}}_y$-component.

True or **False** (iii) V represents an electric scalar potential field within a region with a homogeneous dielectric and in which there is no free volume charge density.

[10 marks] 2. Consider the case in which a metallic loop with resistance R is placed around a very long air-filled solenoid as shown in the figure to the right. The solenoid has an initial current of $i(t=0) = I_0$ (flowing in the $-\hat{\mathbf{a}}_\varphi$ direction). The current in the solenoid then linearly ***decreases*** to half its value between $t = 0$ and $t = t_0$, and then is held constant.



For each of the four statements below state if it is **True** or **False**, and briefly justify your choice.

True or **False** (i) Between $t = 0$ and $t = t_0$, the induced current in the metallic loop is in the $\hat{\mathbf{a}}_\varphi$ direction.

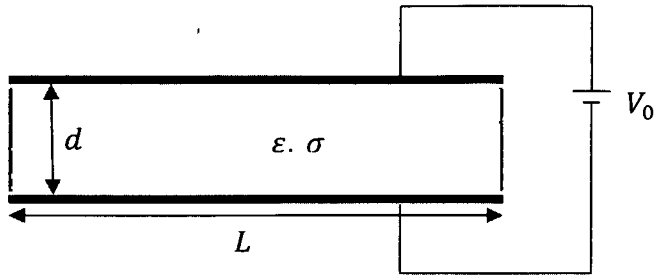
True or **False** (ii) If radius of the metallic loop is doubled, the magnitude of the induced emf in the loop will increase (as compared to the case with the original loop radius).

2. continued

True or **False** (iii) If the solenoid is filled with a magnetic core with $\mu_r = 500$, the current induced in the closed loop will be increased (as compared to the case with an air-filled core.)

True or **False** (iv) If the metallic loop is open circuited there will be no induced emf.

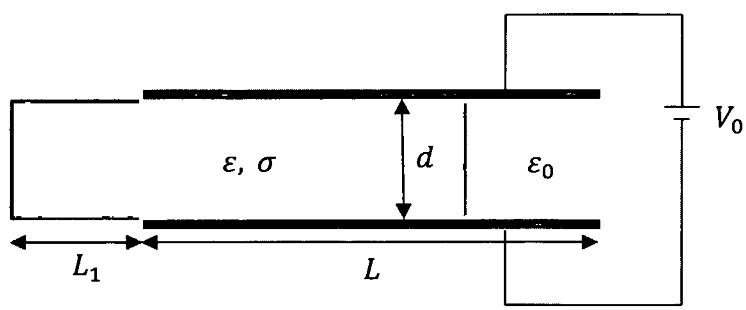
[20 marks] 3. Consider two parallel perfectly conducting plates with width $W = 12$ cm (i.e., into the page) and length $L = 12$ cm, as shown. The plates are connected to a voltage source $V_0 = 6$ V and entirely filled with dielectric material with conductivity $\sigma = 4 \times 10^{-13}$ S/m and permittivity $\varepsilon = 5\varepsilon_0$. The distance between the plates is $d = 3$ mm. You can ignore the effects of fringing fields for this entire question. Recall, $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.



(*4 marks*) (a) Find the resistance between the plates. Briefly explain why the answer you get makes sense.

$R =$ ________________

(*8 marks*) (b) Now consider the case where the dielectric material is pulled out of the capacitor by a distance $L_1 = 5$ cm, while the battery remains attached to the capacitor. How much work is required to move this dielectric? What is the source of this work, meaning does this come from an external agent, or from the stored energy in the original capacitor? The dielectric still fills the entire width (W) of the capacitor (i.e., into the page), and the piece of dielectric has a total length of L.



3. (b) continued

Work Required = ____________________

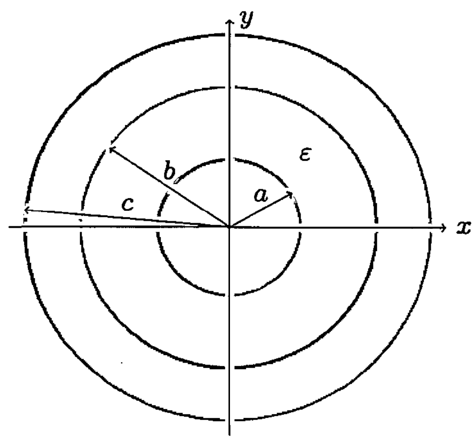
(*8 marks*) 3. (c) Determine the amount the dielectric piece should be pulled out (i.e., the length L_1) such that the energy dissipated in the resistance of the dielectric in one minute is equivalent to energy stored in the capacitor.

3. (c) continued

$L_1 =$ ____________________

[20 marks] 4. Consider an infinitely long system of coaxial cylinders as shown in the figure below. This system consists of an inner cylinder with radius a and a constant volume charge density ρ_{v0}, and an outer cylinder with inner radius b, outer radius c, and a constant volume charge density $-\rho_{v0}$. A perfect dielectric occupies the region $r \in [a, b]$ and has permittivity $\varepsilon = \varepsilon_r \varepsilon_0$. Elsewhere, permittivity is ε_0. There is no metal in the system at all.

(10 marks) (a) Determine the electric field intensity, **E**, everywhere.



4. (a) continued

(*5 marks*) 4. (b) What is the electric scalar potential difference between $r = a$ and $r = c$? Meaning what is $\Delta V = V_c - V_a$? You can use the values of $\rho_{v0} = 4\ \mu\text{C/m}^3$, $a = 1$ cm, $b = 2$ cm, and $c = 3$ cm. Briefly explain why your answer makes sense.

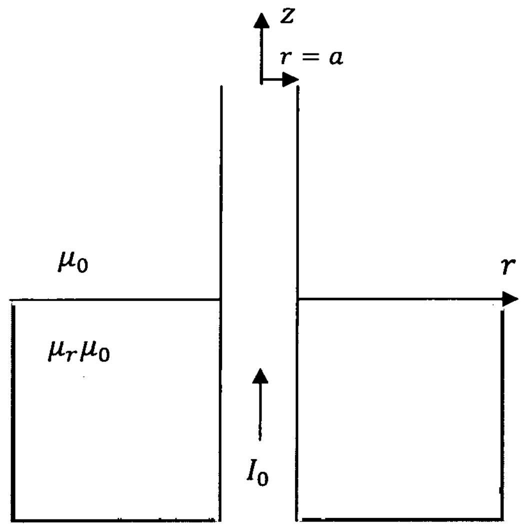
$\Delta V = V_c - V_a =$ ________________

(*5 marks*) 4. (c) What are the bound surface current densities, if they exist, at $r = a$ and $r = b$?

$\rho_{sb}(r = a) =$ ________________

$\rho_{sb}(r = b) =$ ________________

[20 marks] 5. An infinitely long solid straight wire with outer radius $r = a$ carries a total current of I_0 in the $+z$-direction. The current is non-uniformly distributed in the wire according to a volume current density given by $\mathbf{J} = A_0 r^2 \hat{\mathbf{a}}_z$. The bottom half of the wire is buried in a homogeneous magnetic material with a magnetic permeability of μ_r. You can assume there is no free current density in the xy-plane.



(3 marks) (a) Determine the expression for the constant A_0 in terms of I_0, a, and π. State the units for A_0.

$A_0 =$ ____________________

(7 marks) (b) Use Ampère's law to determine the magnetic field intensity, $\mathbf{H}$, within and outside of the wire in the air region.

5. (b) continued

$\mathbf{H}_{wire}$ = ______________________

$\mathbf{H}_{air}$ = ______________________

(*2 marks*) (c) Find the expressions for the magnetic flux density in the air and magnetic material regions, i.e., Air: $z > 0, r > a$ and Magnetic Material: $z < 0, r > a$.

$\mathbf{B}_{wire}$ = ______________________

$\mathbf{B}_{air}$ = ______________________

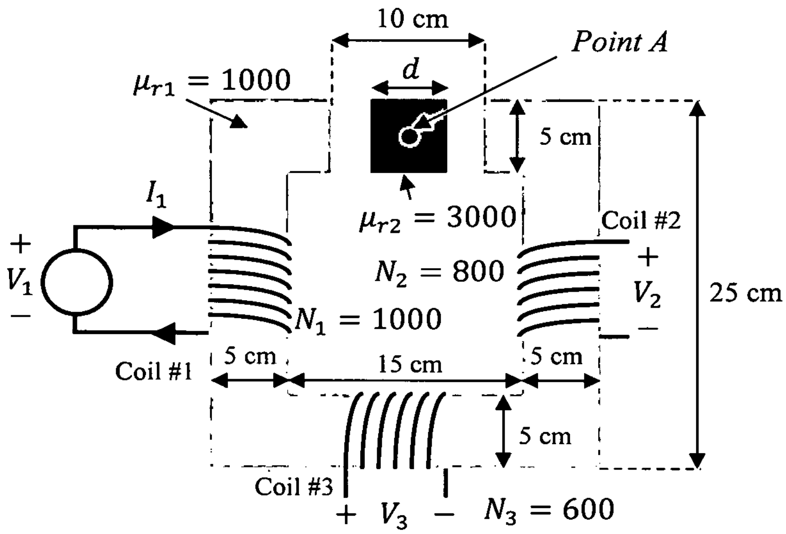
(*4 marks*) 5. (d) Find the expression for the bound surface current that may exist on the surface $z = 0$. If it doesn't exist, briefly explain why.

$\mathbf{J}_{ms} =$ ____________________

(*4 marks*) 5. (e) Find the total magnetic energy stored within the region described by $r \in [a, 2a]$, $z \in [-1,1]$, $\varphi \in [0,2\pi)$.

$W_m =$ ____________________

[20 marks] 6. The magnetic circuit illustrated below has a 10 cm wide air gap cut into the main magnetic core and a second magnetic material of width d ($d \leq 10$ cm) partially fills this air gap. All sections of the core within this circuit have a square cross-sectional area given by $S = 5$ cm x 5 cm. For the purposes of this problem you can ignore the effects of fringing fields. Recall, the permeability of free space is given as $\mu_0 = 4\pi \times 10^{-7}$ H/m.



(*14 marks*) (a) If $d = 6$ cm and $I_1 = 4$ A, determine the values needed to complete the table below. Make sure to include a clearly labeled equivalent magnetic circuit and show all your work.

Quantity	**Result With Units**
The magnetic flux flowing through Coil #2, Φ_2	
The magnitude of the magnetic flux density at *Point A*, B_A	
The magnitude of the magnetic flux intensity at *Point A*, H_A	
The self-inductance of Coil #1, L_{11}	
The mutual-inductance between Coil #1 and Coil #2, L_{12}	
The mutual-inductance between Coil #1 and Coil #3, L_{13}	
The total magnetic energy stored in the system, W_m	

6. (a) continued

(*6 marks*) 6. (b) For the magnetic circuit above, if $I_1 = 3\cos(200t)$ A, find the value of d (0 cm $\leq d \leq$ 10 cm) required to produce a peak value of the induced emf for Coil #2 of 20.16 V.

$d =$ ________________

(5 marks) ***Bonus Question***

The electric field intensity in a spherical capacitor is given as $\mathbf{E} = \frac{E_0 \cos(\omega t)}{R^2} \hat{\mathbf{a}}_R$. The capacitor is filled with a homogenous and lossy dielectric with a conductivity of σ and relative permittivity of ε_r. Show that the ratio of the magnitudes of the conduction current to the displacement current is given as: $\frac{|J|}{|J_d|} = \frac{\sigma}{\omega \varepsilon_r \varepsilon_0}$.

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

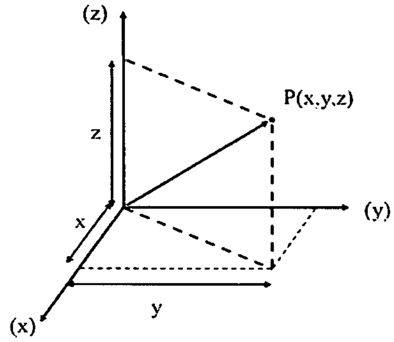
Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = rdrd\phi dz$

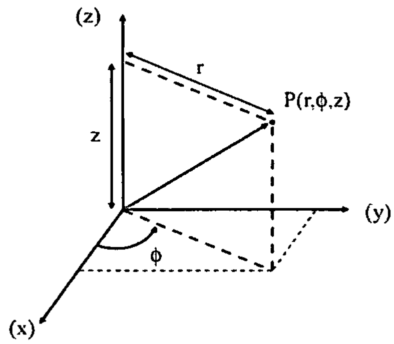
1.3 Spherical coordinates

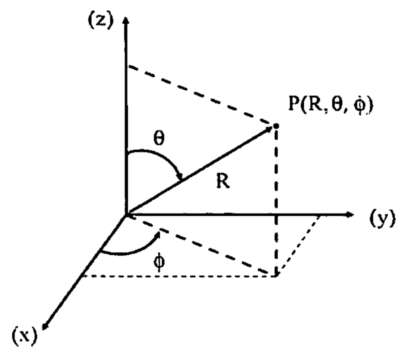
Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ







3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	-$A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

5. Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z$$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z$$

$$= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta$$

$$+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi$$

6. Electromagnetic formulas

Table 1 Electrostatics

$$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3}dQ'$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{D} = \rho_v$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

$$\mathbf{E} = -\nabla V \qquad V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{|\mathbf{R} - \mathbf{R}'|}$$

$$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

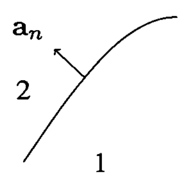
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E} \qquad \mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$$

$$\rho_{p,v} = -\nabla \cdot \mathbf{P}$$

$$\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

$$E_{1,t} = E_{2,t}$$



$$Q = CV \qquad W_e = \frac{1}{2}QV$$

$$W_e = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D} \cdot \mathbf{E} dv$$

$$\nabla \cdot (\varepsilon \nabla V) = -\rho_v \qquad \nabla \cdot (\varepsilon \nabla V) = 0$$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu I$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{d\mathbf{l}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot d\mathbf{l} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S}_2$
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$	$V_{emf} = \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot d\mathbf{S}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$