Last name: First name: ID number:

ECE 286
Midterm exam

April 5, 2022
6:30 – 7:30 pm

Circle your lecture section:

LEC0101 (Tuesday 11-12)

LEC0102 (Monday 12-1)

Guidelines:

- Write your answer in the space provided for each question. Show your work and use the back side of sheets as needed.
- The exam is Type D. You may use a non-programmable calculator and a one-sided, 8.5 by 11 handwritten note sheet (prepared by you).
- Only exams written in pen will be considered for regrades.
- **You will find tables for the normal and t distributions in the back of the exam. You may remove them.**

Problem	Score
1	/10
2	/5
3	/5
Total	/20

1. The number of buses that arrive each hour is described by a Poisson process with parameter $\lambda = 2$. Answer parts (a)-(d) below. Justify your answers.

   (a) (2 points) How many buses arrive on average between 8am and 10am? *Solution: The mean of a Poisson RV is λ. Therefore, we have $2\lambda = 4$.*

(b) (2 points) What is the variance of the number of buses that arrive between 8am and 10am? *Solution: There are two ways to think about this. The variance of a Poisson RV is* λ. *If we say* $\lambda = rt$, *then* $r = 2$ *buses per hour. So for two hours, we have* $2 \cdot 2 = 4$. *Alternatively, we can think of the number of buses in two consecutive hours as the sum of two Poisson RVs, in which case the variances add: 2+2=4.*

(c) (2 points) Three buses came in the last ten minutes. What is the expected time until the next bus arrives? *Solution: If buses arrive according to a Poisson process with* $\lambda = rt$*, then the time between arrivals is described by the exponential distribution with* $\beta = 1/r$*. Due to the memoryless property of the exponential distribution, the time until the next arrival does not depend on the past. Therefore, the expected time until the next arrival is the mean of the exponential distribution,* $\beta = 1/r = 1/2$ *hours.*

(d) Each day for 30 days you go to another bus stop. On average over the 30 days you waited 10 minutes. Answer parts (i)-(ii) below.

i. (2 points) Estimate 95% confidence intervals for the true mean wait time. *Solution: Because* $n = 30$, *we can use the Central Limit Theorem instead of the* t *distribution. Because we know wait time is modeled by the exponential distribution, we can approximate the variance as* $\sigma^2 = 10^2$. *Define the statistic*

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\bar{X} - \mu}{10/\sqrt{30}}.$$

Because we want to use CLT based confidence intervals with $\alpha = 0.05$, *we have* $z_{\alpha/2} = z_{0.025} = 1.96$. *Therefore,*

$$0.95 = P(-1.96 \le Z \le 1.96).$$

Using realized values

$$z = \frac{\bar{x} - \mu}{10/\sqrt{30}} = \frac{10 - \mu}{10/\sqrt{30}},$$

we obtain the confidence interval $[8.92, 11.07]$ *minutes.*

ii. (2 points) Estimate a 95% prediction intervals for how long you will wait for the next bus you take. If the lower limit is negative, discuss how you might modify your answer to make it more useful. *Solution: We can again use the Central Limit Theorem with $\sigma^2 = 10^2$. Define the statistic*

$$Z = \frac{\bar{X} - \mu}{\sigma\sqrt{1 + 1/n}} = \frac{\bar{X} - \mu}{10\sqrt{1 + 1/30}}.$$

This has a standard normal distribution. Because we want to use CLT based confidence intervals with $\alpha = 0.05$, we have $z_{\alpha/2} = z_{0.025} = 1.96$. Therefore,

$$0.95 = P(-1.96 \leq Z \leq 1.96).$$

Using realized values

$$z = \frac{\bar{x} - \mu}{10\sqrt{1 + 1/30}} = \frac{10 - \mu}{10\sqrt{1 + 1/30}},$$

we obtain the confidence interval $[-9.92, 29.92]$ minutes. One can clearly not wait -9.92 minutes for a bus. We could instead truncate the distribution at zero, and recompute the associated probabilities.

2. A factory produces a component, A, that should weigh 30 pounds. To estimate the true weight, they fabricate and weigh component A 100 times. From this experiment, they conclude that the 95% confidence interval for component A's mean weight is $[29.8, 30.1]$. Answer parts (a)-(b) below. Justify your answers.

   (a) (2 points) Compute the standard deviation of component A's weight. *Solution: Because it's a 95%, confidence interval, we have* $\alpha = 0.05$ *and* $z_{\alpha/2} = 1.96$*. The center of the interval is 29.95 and* $n = 100$*. Therefore,* $0.15 = 1.96\sigma/\sqrt{100}$*. From this we have* $\sigma = 10 \cdot 0.15/1.96 = 0.77$*.*

(b) Component B's weight is also uncertain. Because it is more expensive to produce, it is only weighed five times. The sample mean is $\bar{x} = 7$ pounds, and the sample variance is $s^2 = 1$ pound. Answer parts (i)-(ii) below.

i. (1 point) If you wanted to compute confidence intervals, what further assumption would you need to use the t distribution? *Solution: We must assume that the component's weight is normally distributed.*

ii. (2 points) Compute 95% confidence intervals using the t distribution. *Solution: Since $n = 5$, we use the t distribution with $v = 4$. For $\alpha = 0.05$, we have $t_{\alpha/2} = t_{0.025} = 2.776$. We therefore have confidence intervals*

$$\left[\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}\right] = \left[7 - 2.776\frac{1}{\sqrt{5}}, 7 + 2.776\frac{1}{\sqrt{5}}\right] = [5.76, 8.24].$$

3. X is a random variable with uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases} .$$

Suppose we observe realizations $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = 4$. We want to estimate a and b from these observations. Answer parts (a)-(b) below. Justify your answers.

(a) (2 points) Write the likelihood function for a and b. *Solution: The likelihood function is*

$$L(x_1, ..., x_4; a, b) = \begin{cases} \frac{1}{(b-a)^4} & \textit{if } a \le x_i \le b, \ i = 1, ..., 4 \\ 0 & \textit{otherwise} \end{cases} .$$

(b) (3 points) Find the maximum likelihood estimates of a and b. *Solution: The likelihood function is maximized when* $b - a$ *is minimized and* $a \leq x_i \leq b,\ i = 1, ..., 4$. *Therefore, the maximum likelihood estimate is* $a = 1$ *and* $b = 4$.