PHY294, Winter 2016, Thermal Physics Term Test.

One 8.5 × 11 inches double-sided, hand-written aid sheet allowed. Duration: 75 minutes.

I. *Discovering the "Dulong-Petit law": solids and LARGE molecules*

1. Consider first an n-atomic nonlinear molecule. Find the heat capacity of an ideal gas of such molecules, assuming that all degrees of freedom are thermalized. Then, find the leading term, in the large-n limit ($n \gg 1$), in the expression for the heat capacity of this gas.
2. Next, consider a $3n$-oscillator ($n \gg 1$) Einstein solid in a macrostate with $q \gg 3n$ quanta. Let the frequency of each oscillator be ω. Find the heat capacity of the solid.
3. Compare your findings of the previous two problems. Did you expect the results to be similar or distinct? Explain.

10 points

II. *Spin systems in thermal contact*

Consider two spin systems (the "electronic paramagnet" of class or textbook, consisting of spins that can only take two values, ± 1). Each system is composed of N spins, and the two systems are in thermal contact and isolated from the rest of the universe.

1. Let the total number of up spins $N_{up,total} = N_{up,1} + N_{up,2}$ in the combined system be N ($N_{up,1}$ and $N_{up,2}$ are the number of up spins in the first and second system, respectively). How many *macrostates* of the combined system are there with $N_{up,total} = N$?
2. Find an approximate expression, valid for large-N, for the number of *microstates* of the combined system with $N_{up,total} = N$ (recall the Stirling formula $N! \simeq N^N e^{-N}\sqrt{2\pi N}$).
3. Out of the number of *macrostates* of the combined system you found in 1.), which is the most likely one? What is the multiplicity of the most likely *macrostate*?
4. *Argue* that comparing the multiplicity that you found in 3.) to the total number of *microstates* you found in 2.) gives an idea of the width of the peak of the multiplicity function at the most likely *macrostate*. Make a crude estimate the width of that peak, assuming that the probability distribution has roughly a rectangular shape. Does your result qualitatively agree with what you know so far?

20 points

Turn over, please →

I.

1. $C_V = \frac{f}{2} k N$

$$f = (3 + 3 + (3n - 6) \cdot 2)$$

as $n \to \infty$ ($n \gg \infty$)

$$f \approx 6n\left(1 + \frac{O(1)}{n}\right)$$

a number of order unity

so

$$C_V \approx 3 k N$$

2. in $q \gg 3n$ limit we have equipartition, so

$U \approx (3n) kT$ so $C_V = 3 k n$

(ω is irrelevant here)

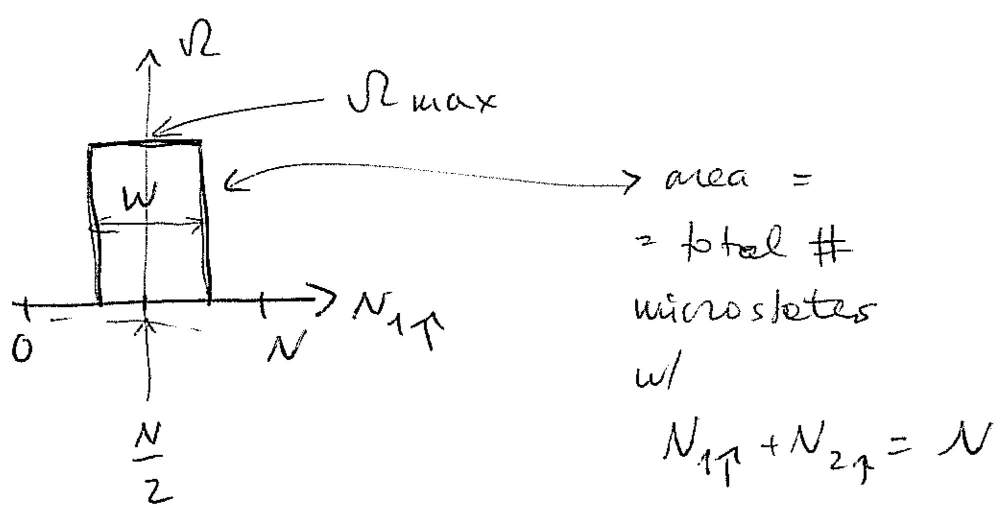
3. <u>a solid (crystal) can be thought as a molecule made out of n atoms</u>; if crystal is at rest transl. & rotations do not contribute –

– so it's natural that C_V for n-atomic molecule & a 3n-oscilator Einstein solid are identical.

II.

1. Combined spin sytem.

   1. Macrostates $(N_{1\uparrow}, N_{\uparrow,2})$ with $N_{1\uparrow} + N_{\uparrow,2} = N$ are $(0, N)$, $(1, N-1)$, ..., $(N, 0)$, so in total there are $N+1$ such states.
   2. Consider the combined system as one system with $2N$ spins. Then the multiplicity for a state with $N_{\text{tot}\uparrow} = N_{1\uparrow} + N_{\uparrow,2} = N$ is $\Omega_{\text{tot}} = \binom{2N}{N} = (2N)!/(N!)^2 \approx 2^{2N}/\sqrt{\pi N}$, where we have used the Stirling formula given in the problem.
   3. The most likely macrostate with $N_{1\uparrow} + N_{\uparrow,2} = N$ is $(N/2, N/2)$. Its microscopic multiplicity is $\Omega_{\text{max}} = \binom{N}{N/2}\binom{N}{N/2} \approx 2^{2N+1}/(\pi N)$, where the Stirling approxiation has been employed again.
   4. Each macrostate in 1. comes with its own microscopic multiplicity. Plotting the microscopic multiplicity against the macrostates (specified by say $N_{1\uparrow}$ $(= 0, .., N)$) gives a peaked distribution that is centered about the most likely macrostate (determined in 2.) and takes the value 1 at the edges (for the states $N_{1,\uparrow} = 0$ and N). The total number of microstates is Ω_{tot} while the peak height is given by Ω_{max}. Approximating the distribution by a rectangle, one can estimate the with w via $\Omega_{\text{tot}} = \Omega_{\text{max}} \cdot w$. Thus $w = \left(2^{2N}/\sqrt{\pi N}\right)/\left(2^{2N+1}/(\pi N)\right) = \sqrt{\pi N}/2$. Comparing w with the whole width of the distribution $\sim N$ gives $w/N \sim 1/\sqrt{N}$, which implies that the peak drastically sharpens up in the thermodynamic limit $(N \to \infty)$.



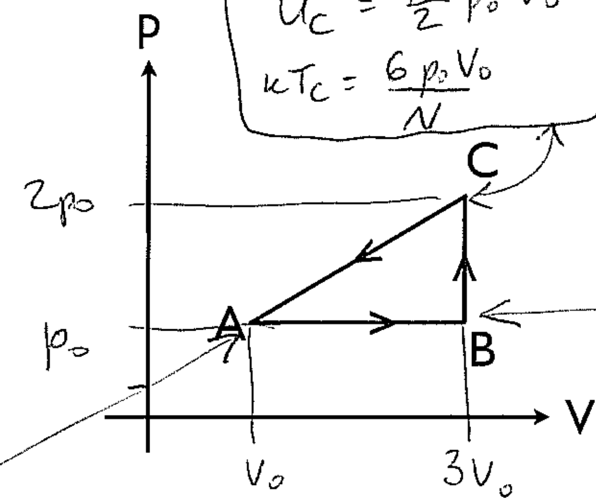
as N goes up: $w/N \sim \frac{1}{\sqrt{N}}$, peak sharper.

III. *A cyclic process*

An monatomic ideal gas made of N atoms is made to undergo the cyclic process shown on the figure below, in p-V coordinates. The points A, B, C have (p, V) coordinates as follows: $A = (p_0, V_0)$, $B = (p_0, 3V_0)$ and $C = (2p_0, 3V_0)$. For each of the steps, AB, BC, CA, determine the work done on the gas, the change of the energy of the gas, the heat added to the gas. Then find these quantities for the entire process. What does this cyclic process accomplish?

$U_C = \frac{18}{2} p_0 V_0$

$kT_C = \frac{6 p_0 V_0}{N}$



$U_B = \frac{9}{2} p_0 V_0$

$kT_B = \frac{3 p_0 V_0}{N}$

$kT_A = \frac{p_0 V_0}{N}$

$U_A = \frac{3}{2} p_0 V_0$

15 points

Total number of points: $10 + 20 + 15 = 45$.

$\underline{AB}$: W_{AB} = work done by gas = $2p_0V_0$

$U_B - U_A = Q_{AB} - W_{AB}$

$(\frac{9}{2} - \frac{3}{2})p_0V_0 = 3p_0V_0 = Q_{AB} - 2p_0V_0 \Rightarrow Q_{AB} = \cancel{6p_0V_0}$ $= 5p_0V_0$

$W_{AB} = 2p_0V_0$ ← work by gas

$Q_{AB} = 5p_0V_0$ ← heat absorbed by gas

$U_B - U_A = \frac{6}{2}p_0V_0$ ← change of energy

III . A

<u>BC</u> :

$$W_{BC} = 0 \quad \text{since} \quad V = \text{const}$$

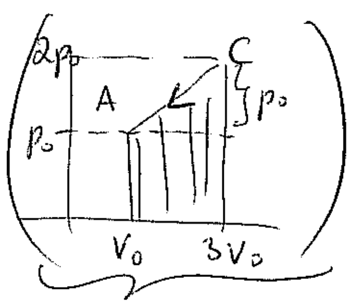
$$U_C - U_B = Q_{BC}$$

$$\left(\frac{18}{2} - \frac{9}{2}\right) p_0 V_0 = Q_{BC}$$

$$\begin{cases} Q_{BC} = \frac{9}{2} p_0 V_0 \rightarrow \text{heat absorbed } \underline{\text{by}} \text{ gas} \\ W_{BC} = 0 \rightarrow \text{work done } \underline{\text{by}} \text{ gas} \\ U_C - U_B = \frac{9}{2} p_0 V_0 \rightarrow \text{change of energy} \end{cases}$$

<u>CA</u> :

$$W_{CA} = - \underbrace{\left(\quad \right)}_{\text{area of shaded part}} = -3 p_0 V_0$$



work done by gas is < 0

$$\left.\begin{matrix} \frac{2 p_0 V_0}{2} \\ 2 p_0 V_0 \end{matrix}\right\} 3 p_0 V_0$$

$$U_A - U_C = -W_{CA} + Q_{CA}$$

$$\left(\frac{3}{2} - \frac{18}{2}\right) p_0 V_0 = 3 p_0 V_0 + Q_{CA} \Rightarrow Q_{CA} = -\left(\frac{15}{2} + 3\right) p_0 V_0 = -\frac{21}{2} p_0 V_0$$

so

$$\begin{cases} W_{CA} = -3\,p_0 V_0 & \leftarrow \text{gas does negative work (work done } \underline{\text{on}} \text{ gas)} \\ Q_{CA} = -\frac{21}{2}\,p_0 V_0 & \leftarrow \text{gas absorbs negative heat (i.e. gives away heat)} \\ U_A - U_C = -\frac{15}{2}\,p_0 V_0 & \leftarrow \text{energy decreases} \end{cases}$$

<u>to sum up:</u>

$A \to B \to C$ cycle:

work <u>by</u> gas: $W_{AB} + W_{BC} + W_{CA}$

$$= [\,2 + 0 - 3\,]\,p_0 V_0$$

$$= -p_0 V_0$$

; over one cycle work done <u>on</u> gas > 0 (work <u>by</u> gas < 0)

heat absorbed <u>by</u> gas:

$$Q_{AB} + Q_{BC} + Q_{CA}$$

$$= \left(5 + \frac{9}{2} - \frac{21}{2}\right) p_0 V_0$$

$$= \frac{10+9-21}{2}\, p_0 V_0 = -p_0 V_0$$

negative, gives away heat

over cycle gas gives away heat $p_0 V_0$

III C

So over the entire cycle

work equal to $p_0 V_0$ is done on the gas

&

heat equal to $p_0 V_0$ is released by the gas — its energy remains same,

of course, since this is a cycle.