PHY293 term test 2019 Solutions

November 10, 2019

1 Guitar A string

1.1 (a) The harmonics are:

- second harmonic: $\nu_2 = 2\nu_1 = 220$ Hz.
- Third harmonic: $\nu_3 = 3\nu_1 = 330$ Hz.

The wave velocity is $v = \sqrt{T/\mu}$, with T the tension in the string and μ the linear mass. **None of these quantities depend on the frequency.**

1.2 (b) Number of harmonics

$$\frac{12,000 \text{ Hz}}{110 \text{ Hz}} \approx 109.1.$$

Therefore, the guitarist **can hear up to 109 harmonics**, fundamental included.

1.3 (c) Note C

For the note A, $L_1 = 65$ cm and $\nu_1 = v/(2L_1) = 110$ Hz. Let the new length be L', and the new frequency $\nu' = 131$ Hz.

What does not change between the two cases is the wave velocity:

$$v = 2L_1\nu_1 = 2L'\nu' \quad \Rightarrow \quad L' = \frac{L_1\nu_1}{\nu'}.$$

```
In [3]: nu1 = 110   # Hz
        nup = 131   # Hz
        L1 = 65   # cm
        print("L' = {:.1f} cm approximately".format(L1*nu1/nup))
```

```
L' = 54.6 cm approximately
```

2 Torsional harmonic oscillator

2.1 (a) Natural frequency

Dividing the equation of motion by I,

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 = \frac{\tau(t)}{I},$$

with $\gamma = C/I$ and

$$\boxed{\omega_0 = \sqrt{\frac{\kappa}{I}}}.$$

2.2 (b) Light damping range

Damping is critical when $\omega_0^2 = \dfrac{\gamma^2}{4}$, i.e., when

$$\frac{\kappa}{I} = \frac{C_c^2}{4I^2} \Leftrightarrow C_c = 2\sqrt{\kappa I}.$$

Therefore, we have light damping when $\boxed{C < 2\sqrt{\kappa I}}$.

2.3 (c) Pseudo-oscillation frequency

From the lecture notes (formula is to be remembered!),

$$\boxed{\omega_d = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = \sqrt{\frac{\kappa}{I} - \frac{C}{4I^2}}}.$$

Both versions should count as correct if all quantities have been properly defined before.

2.4 (d) Constant torque

At equilibrium, $\ddot{\theta} = \dot{\theta} = 0$. Therefore, $\kappa\theta_{eq} = \tau = FL$

$$\Rightarrow \boxed{\theta_{eq} = \frac{FL}{\kappa}}.$$

2.5 (e) Oscillatory forcing

```
In [1]: # Entering the numbers from the text
        I = 2  # moment of inertia, kg m**2
        kappa = 200  # rotational spring constant, N m rad**-1
        C = 3  # rotational damping, kg m**2 rad**-1
        a = 0.05  # forcing amplitude, rad
```

First, we need to convert the equation of motion into the canonical form

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \omega_0^2 A_f \cos(\omega t).$$

Dividing again by the moment of inertia I, we obtain

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2 = \frac{\kappa a}{I}\cos(\omega t) = \omega_0^2 a\cos(\omega t).$$

We can therefore readily use the formula sheet, with

$$A(\omega) = \frac{a}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}}},$$

and $Q = \omega_0/\gamma = \sqrt{\kappa I}/C$.

Now for the phase:

$$\delta = \arctan\left(\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right).$$

```
In [2]: # Defining this function (or just plugging in the numbers in a calculator)
        from numpy import sqrt, pi, arctan

        omega0 = sqrt(kappa/I)
        gamma = C/I
        Q = omega0/gamma

        def A(omega): return a/sqrt((1-(omega/omega0)**2)**2 + (omega/omega/Q)**2)
        def delta(omega): return arctan(omega*gamma/(omega0**2 - omega**2))

        print("For omega = 3pi rad, A = {:.3f} rad".format(A(3*pi)))
        print("             and delta = {:.3f} rad".format(delta(3*pi)))
```

```
For omega = 3pi rad, A = 0.267 rad
             and delta = 0.902 rad
```

There you go: $\boxed{A(3\pi \text{ rad}) \approx 0.267 \text{ rad}}$ and $\boxed{\delta(3\pi) \approx 0.902 \text{ rad}}$.

Write down intermediate formula and numbers, e.g., ω^2/ω_0^2, Q, etc., for partial marks.

```
In [15]: print(A(3*pi)/pi)   # out of curiosity
         print(delta(3*pi)/pi)   # out of curiosity
```

```
0.08509031932519269
0.2871016343379431
```

```
In [17]: print("For omega = omega0, A = {:.3f} rad".format(A(omega0)))
         print("            and delta = {:.3f} rad".format(delta(omega0)))
```

```
For omega = omega0, A = 0.333 rad
            and delta = 1.571 rad
```

```
/Users/NicoG/anaconda/lib/python3.5/site-packages/ipykernel/__main__.py:9: RuntimeWarning: divi
```

```
In [18]: print(A(omega0)/pi)   # out of curiosity
         print(delta(omega0)/pi)   # out of curiosity
```

```
0.10610329539459691
0.5
```

```
/Users/NicoG/anaconda/lib/python3.5/site-packages/ipykernel/__main__.py:9: RuntimeWarning: divi
```

At the natural frequency, $\boxed{A(\omega_0) \approx 0.267 \text{ rad}}$ and $\boxed{\delta(\omega_0) \approx 1.57 \text{ rad} \approx \pi/2}$ *(only need to get one result correct for full marks).*

3 Mass(es) and a spring

3.1 (a) New oscillation frequency

The new oscillation frequency now accounts for twice the mass:

$$\omega_a = \sqrt{k/(2m)} = \omega_b/\sqrt{2}.$$

Either expression works.

3.2 (b) ϕ, v_b and v_a

We have $x_b = A\cos(\omega_b t + \phi)$ and therefore $v_b(t) = -\omega_b A \sin(\omega_b t + \phi)$.

At $t = 0$, $x_b = A/2 = A\cos\phi$. Therefore, $\cos\phi = 1/2$, meaning that $\phi = \pm\pi/3$. But we also know that $v_b(t=0) > 0$, along with $A > 0$ and $\omega_b > 0$. It implies that $\sin\phi < 0$. Therefore, $\boxed{\phi = -\pi/3}$.

In this case, $\boxed{v_b(t=0) = -\omega_b A \sin\phi = +\frac{\sqrt{3}}{2}\omega_b A}$.

Finally, $\boxed{v_a(t=0) = \frac{1}{2}v_b(t=0) = \frac{\sqrt{3}}{4}\omega_b A}$.

3.3 (c) Position after

$$x_a(t) = B\cos(\omega_a t) + C\sin(\omega_a t) \quad \text{and} \quad v_a(t) = -\omega_a B\sin(\omega_a t) + \omega_a C\cos(\omega_a t).$$

At $t = 0$, $x_a = A/2 = B$ and $v_a = \sqrt{3}\omega_b A/4 = \omega_a C = \frac{1}{\sqrt{2}}\omega_b C$.

Therefore, $B = A/2$ and $C = \frac{\sqrt{6}}{4}A$, and we have

$$\boxed{x_a(t) = \frac{A}{2}\left(\cos(\omega_a t) + \sqrt{\frac{3}{2}}\sin(\omega_a t)\right)}.$$

3.4 (d) Amplitude of the new motion

Too long, turned into a bonus question.

First derivation: We need to know when $\dot{x}_a = v_a = 0$ and compute x_a at this instant.

$$v_a = \frac{A}{2}\left(-\omega_a \sin(\omega_a t) + \sqrt{\frac{3}{2}}\omega_a \cos(\omega_a t)\right).$$

The velocity above is zero when

$$\tan(\omega_a t_M) = \sqrt{3/2}.$$

Remembering the derivation for the phase of the forced oscillator,

$$\cos(\omega_a t_M) = \frac{\sqrt{2}}{\sqrt{\sqrt{3}^2 + \sqrt{2}^2}} = \sqrt{\frac{2}{5}} \quad \text{and} \quad \sin(\omega_a t_M) = \sqrt{\frac{3}{5}}.$$

Therefore,

$$A_a = |x_a(t_M)| = \frac{A}{2}\left(\sqrt{\frac{2}{5}} + \sqrt{\frac{3}{2}\cdot\frac{3}{5}}\right) = \frac{A}{2}\frac{\sqrt{4}+\sqrt{9}}{\sqrt{10}} = \frac{A}{2}\frac{5}{\sqrt{10}} = \frac{A}{2}\sqrt{\frac{5}{2}},$$

or, more simply,

$$\boxed{A_a = \sqrt{\frac{5}{8}}A} \approx 0.79A.$$

Alternative derivation: The expression for $x_a(t)$ could be written in the form $x_a(t) = A_a\cos(\omega_a t + \phi_a)$. To turn one into the other, we just have to write the sum of cosines and sines into a sum of products of cosines and sines, i.e.,

$$\cos(\omega_a t) + \frac{\sqrt{6}}{2}\sin(\omega_a t) = D\left(\cos\theta\cos(\omega_a t) + \sin\theta\sin(\omega_a t)\right).$$

The trick is that we don't need to determine θ: we just need $\cos^2\theta + \sin^2\theta = 1$ for it to work. So, what we really need to determine is

$$D = \sqrt{1^2 + \left(\sqrt{6}/2\right)^2} = \sqrt{10/4} = \sqrt{5/2},$$

and the new amplitude is $\boxed{A_a = \sqrt{\frac{5}{8}}A} \approx 0.79A$.

```
In [32]: sqrt(5/8)
```

```
Out[32]: 0.7905694150420949
```