SOLUTIONS

Q1 a)

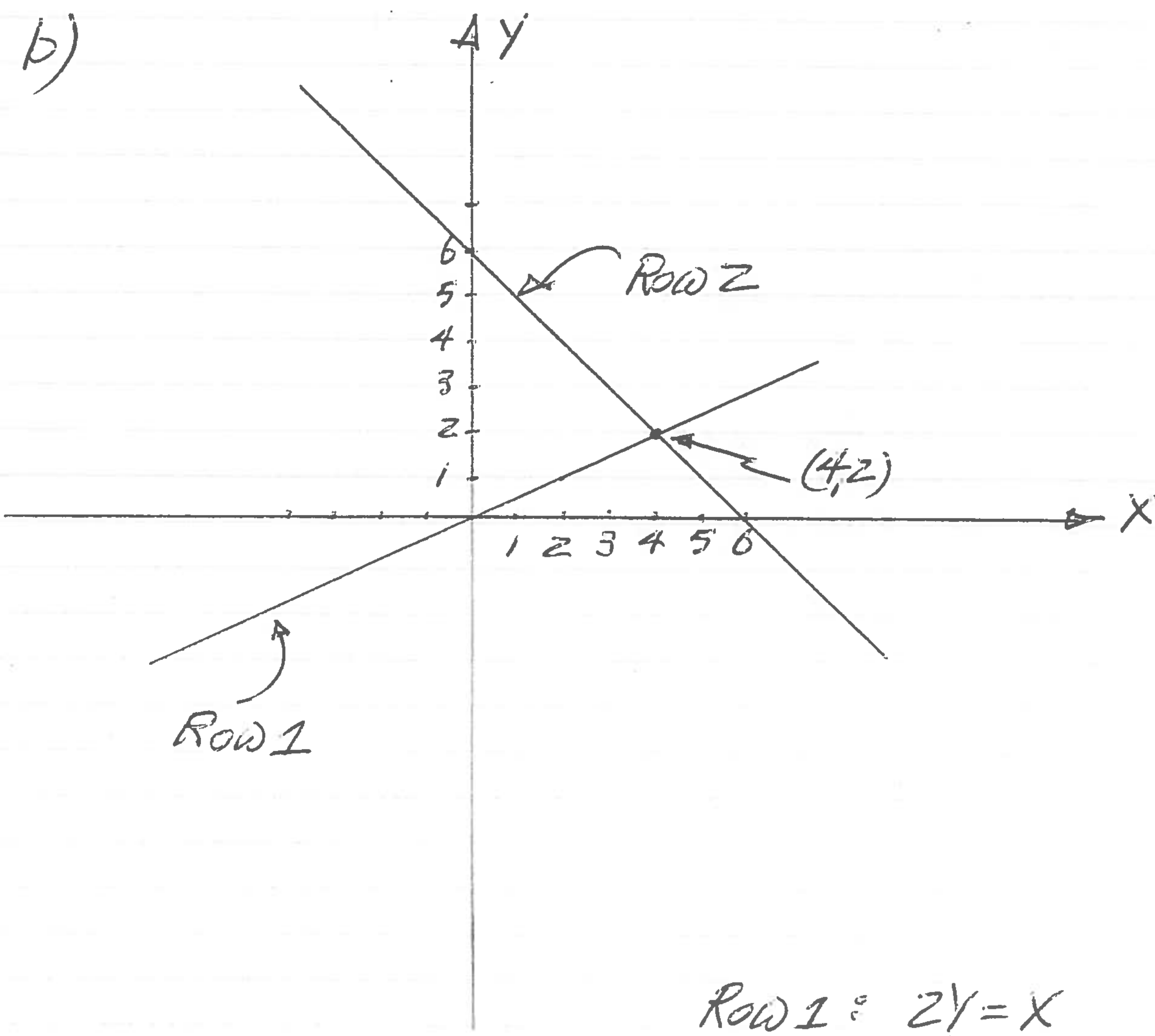
$$x - 2y = 0$$
$$x + y = 6$$

$$x = 2y$$

$$2y + y = 6 \Rightarrow 3y = 6 \Rightarrow y = 2$$

$$\Rightarrow x = 4$$

b)



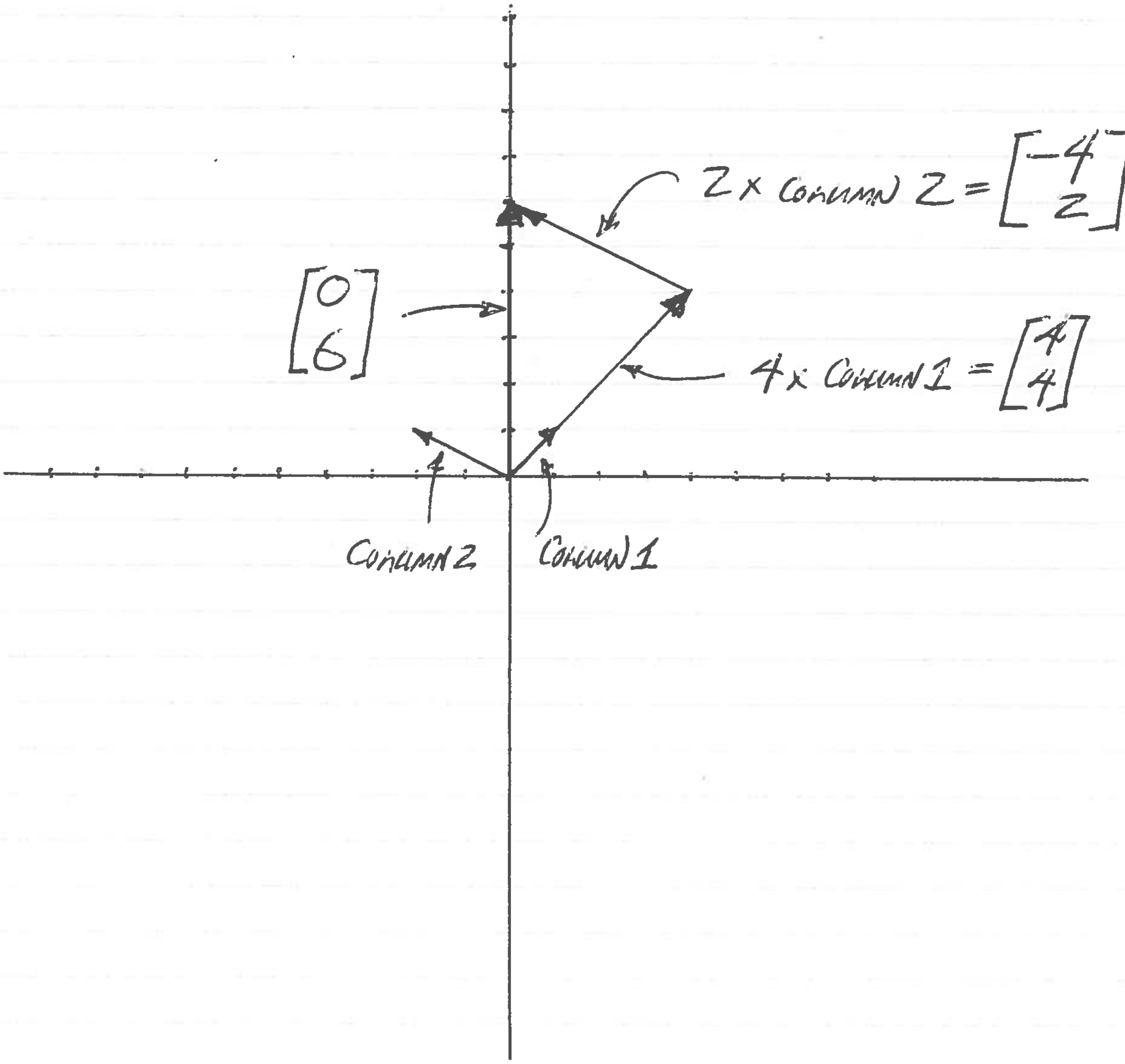
Row 1: $2y = x$
$y = \frac{1}{2}x$

Row 2: $x + y = 6$
$y = -x + 6$

— 2 —

c) $x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}0\\6\end{bmatrix}$

COLUMN 1: $\begin{bmatrix}1\\1\end{bmatrix}$ COLUMN 2: $\begin{bmatrix}-2\\1\end{bmatrix}$



Q2 a) $\vec{u} = \frac{1}{\sqrt{1^2+2^2+2^2}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

b) LET $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$ BE A VECTOR ORTHOGONAL TO $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

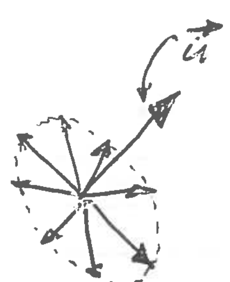
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$$

$$X + 2Y + 2Z = 0$$

LET $Y = 1$ AND $Z = 1 \Rightarrow X = -4$

$$\vec{V} = \frac{1}{\sqrt{(-4)^2+1^2+1^2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{18}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

c) THERE ARE INFINITE POSSIBILITIES FOR $\vec{V}$ BECAUSE THERE ARE INFINITE SOLUTIONS TO $X + 2Y + 2Z = 0$.



ALL UNIT VECTORS ORTHOGONAL TO $\vec{u}$ TRACE OUT A CIRCLE IN R^3 AROUND THE ORIGIN WITH RADIUS $= 1$.

Q3 a) EQUATION OF LINE THAT PASSES THROUGH THE ORIGIN AND $P_1 = (2,2,1)$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = t\vec{d} \qquad t \text{ SCALAR}$$

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

$$\text{proj}_{\vec{d}}\vec{v} = \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d}$$

$$\vec{v} \cdot \vec{d} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 6+8+4 = 18$$

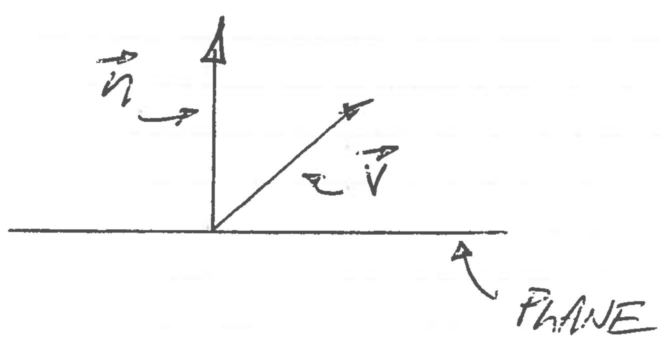
$$\|\vec{d}\| = \sqrt{2^2+2^2+1^2} = 3$$

$$\text{proj}_{\vec{d}}\vec{v} = \frac{18}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$$

6) PLANE THAT PASSES THROUGH THE ORIGIN AND CONTAINS BOTH P_1 AND $P_2 = (1,0,0)$ IS PARALLEL TO BOTH $\vec{OP_1}$ AND $\vec{OP_2}$. THEREFORE A NORMAL TO THIS PLANE CAN BE FOUND BY TAKING THE CROSS PRODUCT OF $\vec{OP_1}$ AND $\vec{OP_2}$.

$$\vec{OP_1} \times \vec{OP_2} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

LET $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$



FIND $\text{proj}_{\vec{n}} \vec{v} = \dfrac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n}$

$$\vec{v} \cdot \vec{n} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 0 + 4 - 8 = -4$$

$$\|\vec{n}\| = \sqrt{0^2 + 1^2 + (-2)^2} = \sqrt{5}$$

$$\text{proj}_{\vec{n}}\vec{v} = -\frac{4}{5}\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

PROJECTION OF $\vec{v}$ ON THE PLANE IS GIVEN BY:

$$\vec{v} - \text{proj}_{\vec{n}}\vec{v}$$

$$= \begin{bmatrix}3\\4\\4\end{bmatrix} - \left(-\frac{4}{5}\right)\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}3\\4\\4\end{bmatrix} + \begin{bmatrix}0\\4/5\\-8/5\end{bmatrix}$$

$$= \begin{bmatrix}3\\24/5\\12/5\end{bmatrix}$$

c) $$\begin{bmatrix}3\\4\\4\end{bmatrix} = \begin{bmatrix}0\\-4/5\\8/5\end{bmatrix} + \begin{bmatrix}3\\24/5\\12/5\end{bmatrix}$$

Q4 a) $A\vec{\omega} = \lambda\vec{\omega}$

$$A(A\vec{\omega}) = A(\lambda\vec{\omega})$$

$$(AA)\vec{\omega} = \lambda(A\vec{\omega})$$

$$A^2\vec{\omega} = \lambda(\lambda\vec{\omega})$$

$$A^2\vec{\omega} = \lambda^2\vec{\omega}$$

$\therefore$ $EIG(A^2) = (EIG\,A)^2$

b) $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ $\quad A\vec{\omega} - \lambda_1\vec{\omega} = \vec{0}$

$$(A - \lambda_1 I)\vec{\omega} = \vec{0}$$

$$A - \lambda_1 I = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{bmatrix} = \begin{bmatrix} 1-\lambda_1 & 0 \\ 1 & 2-\lambda_1 \end{bmatrix}$$

$$\det(A - \lambda_1 I) = (1-\lambda_1)(2-\lambda_1) - (1)(0)$$

$$= (1-\lambda_1)(2-\lambda_1)$$

$$\det(A - \lambda_1 I) = 0$$

$\therefore$ $\lambda_1 = 1, 2$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$$

$$A^2\vec{\omega} - \lambda_2\vec{\omega} = \vec{0}$$

$$(A^2 - \lambda_2 I)\vec{\omega} = \vec{0}$$

$$A^2 - \lambda_2 I = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1-\lambda_2 & 0 \\ 3 & 4-\lambda_2 \end{bmatrix}$$

$$\det(A^2 - \lambda_2 I) = (1-\lambda_2)(4-\lambda_2) - (3)(0)$$

$$= (1-\lambda_2)(4-\lambda_2)$$

$$\det(A^2 - \lambda_2 I) = 0$$

$$\therefore \lambda_2 = 1, 4$$

$$\lambda_2 = \lambda_1^2 \quad (1 = 1^2 \text{ AND } 4 = 2^2)$$

Q5 a)

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{= P} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

b)

$$P^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

c) IF THE TRANSFORMATION IS APPLIED FOUR TIMES, THE VECTOR $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$ WILL RETURN TO $\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

$\therefore\ P^4 = P^3 P = I$

$\therefore\ n = 3$

C) CONT'D

$$P^3 = P^2 P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

TO SHOW THAT P^3 IS THE INVERSE OF P, ONE MUST SHOW THAT $P^3 P = I$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Q6 a) $\left[\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 1 & -1 & 1 & 4 \end{array}\right]$

$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & -2 & -2 & -2 \end{array}\right]$ $R2-R1$

$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 0 & 1 & 1 & 1 \end{array}\right]$ $R2 * \frac{1}{-2}$

$\to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 1 & 1 \end{array}\right]$ $R1-R2$

RNF

b) $x + 2z = 5$ $\Rightarrow$ $x = 5 - 2z$
$y + z = 1$ $\quad$ $y = 1 - z$
$z = z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

c) LINE IN R^3 (NOT THROUGH THE ORIGIN)

d) $y = 2 = 1 - z \Rightarrow z = -1$

$\therefore$ $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ -1 \end{bmatrix}$$