ESC103F Engineering Mathematics and Computation: Tutorial #3

Question 1: Consider the plane that intersects the coordinate axes at $x = a$, $y = b$ and $z = c$, where a, b, c are nonzero scalars. Using the vector cross product, derive and show that the scalar equation of this plane is given by:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Solution:

We are given 3 points that lie in the plane, $P_1(a,0,0)$, $P_2(0,b,0)$ and $P_3(0,0,c)$.

This allows us to define two vectors that are parallel to the plane:

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -a \\ b \\ 0 \end{bmatrix}, \overrightarrow{P_1P_3} = \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix}$$

Taking the cross product of these two vectors will produce a normal to the plane:

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{bmatrix} bc \\ ac \\ ab \end{bmatrix}$$

This allows us to write the scalar equation of this plane:

$$bcx + acy + abz + d = 0$$

We can now substitute any known point in the plane to solve for d:

$$bc(a) + ac(0) + ab(0) + d = 0$$
$$\therefore d = -abc$$
$$\therefore bcx + acy + abz = abc$$

Given that a, b, c are nonzero scalars:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Question 2: Find the vector equation for the line through (-2,5,0) that is parallel to the planes $2x + y - 4z = 0$ and $-x + 2y + 3z + 1 = 0$.

Solution:

Any line that is parallel to these two planes has a direction vector that is orthogonal to normal vectors associated with these planes.

$$\overrightarrow{n_1} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \overrightarrow{n_2} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Use cross product to find a vector that is orthogonal to these normal vectors:

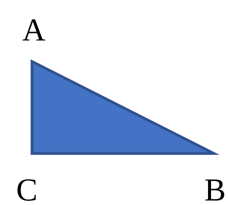
$$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix}$$

Therefore, a direction vector for the line is $\begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix}$. Given a point on the line is (-2,5,0), then a vector equation of this line is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 11 \\ -2 \\ 5 \end{bmatrix} \; where\ t\ is\ a\ scalar$$

Question 3: Let A, B and C(2,-1,1) be the vertices of a triangle where $\overrightarrow{AB}$ is parallel to $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\overrightarrow{AC}$ is parallel to $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, and angle $C = 90^0$. Find the vector equation of the line through B and C.

Solution:



Sketch a triangle ABC with a right angle at C:

We are given the following information:

$$\overrightarrow{AB} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \overrightarrow{AC} = s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \; where\ t\ and\ s\ are\ scalars$$

By vector addition:

$$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$$

$$\therefore \overrightarrow{CB} = \overrightarrow{AB} - \overrightarrow{AC} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - s \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} t - 2s \\ -t \\ t + s \end{bmatrix}$$

We also know that $\overrightarrow{CB}$ is orthogonal to $\overrightarrow{AC}$.

$$\begin{bmatrix} t-2s \\ -t \\ t+s \end{bmatrix} \cdot \begin{bmatrix} 2s \\ 0 \\ -s \end{bmatrix} = (t-2s)(2s) + (t+s)(-s) = 0$$

$$\therefore t = 5s$$

$$\therefore \overrightarrow{CB} = \begin{bmatrix} 3s \\ -5s \\ 6s \end{bmatrix} = s \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$$

This gives us a direction vector for the line through B and C. With C(2,-1,1) being on the line, we can write the following vector equation of the line through B and C:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix} \textit{ where } s \textit{ is a scalar}$$

Question 4: Find all unit vectors parallel to the *yz*-plane that are orthogonal to the vector $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

Solution:

A vector parallel to the yz-plane is orthogonal to a normal of this plane, and a normal to the yz-plane is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Now we need to find a vector orthogonal to both $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, and we will use the cross product of these two vectors to find such a vector:

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

Two unit vectors can be defined from $\vec{v}$:

$\frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$ and $\frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

Question 5: The volume of a tetrahedron is given by:

1/3(area of base)(height)

Use this result to prove that the volume of a tetrahedron with sides defined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by $\frac{1}{6}|\ \vec{a} \cdot (\vec{b} \times \vec{c})|$, taking the base to be defined by vectors $\vec{b}$ and $\vec{c}$. (Note: $\vec{a} \cdot (\vec{b} \times \vec{c})$ is referred to as the scalar triple product.)

Solution:

Area of the tetrahedron base is equal to the area of the triangle defined by $\vec{b}$ and $\vec{c}$. This is equal to ½ of the area of the parallelogram defined by $\vec{b}$ and $\vec{c}$, that is equal to $\frac{1}{2}\left\|\vec{b} \times \vec{c}\right\|$.

The height of the tetrahedron can be obtained by projecting $\vec{a}$ onto a vector that is orthogonal to the base, e.g. $\vec{b} \times \vec{c}$:

$$\left\|proj_{\vec{b}\times\vec{c}}\vec{a}\right\| = \left\|\frac{\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)}{\left\|\vec{b} \times \vec{c}\right\|^2}(\vec{b} \times \vec{c})\right\| = \frac{\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|}{\left\|\vec{b} \times \vec{c}\right\|}$$

Therefore, volume of the tetrahedron is given by:

$$\frac{1}{3}\left(\frac{1}{2}\left\|\vec{b} \times \vec{c}\right\|\right)\left(\frac{\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|}{\left\|\vec{b} \times \vec{c}\right\|}\right) = \frac{1}{6}\left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$$

Question 6: Find the distance between the point P(-3,1,3) and the plane $5x + z = 3y - 4$.

Solution:

Start by finding a point in the plane:

$P_1(0,0,-4)$

Now let’s find the vector $\overrightarrow{P_1P}$:

$$\overrightarrow{P_1P} = \begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix}$$

Now let’s project $\overrightarrow{P_1P}$ onto a normal to the plane $\vec{n} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$, and the length of this vector will give us the distance from P to the plane:

$$\|proj_{\vec{n}}\overrightarrow{P_1P}\| = \frac{|\overrightarrow{P_1P} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{11}{\sqrt{35}}$$

Question 7: Show that the planes $3x - y + 6z = 7$ and $-6x + 2y - 12z = 1$ are parallel, and find the distance between the planes.

Solution:

The planes are parallel because normal vectors to these two planes are parallel:

$$(-2)\begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix}$$

Now let us find a vector $\vec{v}$ defined by connecting two points, one on each plane:

$3x - y + 6z = 7$ contains the point (0,0,7/6)

$-6x + 2y - 12z = 1$ contains the point (0,0,-1/12)

$$\vec{v} = \begin{bmatrix} 0 - 0 \\ 0 - 0 \\ \frac{7}{6} - (\frac{-1}{12}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5/4 \end{bmatrix}$$

Now let's project $\vec{v}$ onto a normal to the two planes $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$, and the length of this vector will give us the distance between the two planes:

$$\|proj_{\vec{n}}\vec{v}\| = \frac{|\vec{v} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{15}{2\sqrt{46}}$$

Question 8: Find the vector equation for the line in R^3 that contains the point P(-1,6,0) and is orthogonal to the plane $4x - z = 5$.

Solution:

A line that is orthogonal to this plane will have a direction vector parallel to a normal to the plane:

$$\vec{d} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$$

Therefore, a vector equation for this line that contains P(-1,6,0) is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \text{ where } t \text{ is a scalar}$$

Question 9: Find the scalar equation for the plane that is represented by the vector equation:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

Solution:

To find a normal to the plane, we can take the cross product of two vectors parallel to the plane:

$$\vec{n} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

The scalar equation for the plane is:

$$3x + 6y + 2z + d = 0$$

Substituting into this equation one point in the plane allows us to solve for d:

$$3(-1) + 6(5) + 2(6) + d = 0$$

$$d = -39$$

Therefore, the scalar equation for the plane is given by:

$$3x + 6y + 2z = 39$$

Question 10: The equation $ax + by = 0$ represents a line through the origin in R^2 if a and b are not both zero. What does this equation represent in R^3 if you think of it as $ax + by + 0z = 0$? Explain.

Solution:

This equation in R^3 represents a plane that passes through the line $ax + by = 0$ in the xy-plane with a normal vector $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ having a zero z-component.

Question 11: Consider the following planes in R^3:

$1x + 0y + 0z = 2$
$0x + 1y + 0z = 3$
$0x + 0y + 1z = 4$

i) Construct a mental image of the row picture.
ii) Express the 3 equations as a single vector equation.
iii) Construct a mental image of the column picture.
iv) What does the solution to these 3 equations represent in R^3?

Solution:

i) The row picture consists of 3 planes not through the origin with different normal vectors ($\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$) and we are looking for all points that are common to these 3 planes.

ii) $x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}$

iii) The column picture consists of linear combinations of 3 vectors and we are looking for all combinations that add up to $\begin{bmatrix}2\\3\\4\end{bmatrix}$.

iv) The solution can be easily seen by examining the column picture and recognizing that there is only one linear combination that adds up to $\begin{bmatrix}2\\3\\4\end{bmatrix}$, namely, $x = 2$, $y = 3$, $z = 4$.

This solution represents the point (2,3,4) in R^3. In terms of the row picture, this is the point where the 3 planes intersect.

Question 12: Consider the following planes in R^3:

$x + y + z = 2$
$x + 2y + z = 3$
$2x + 3y + 2z = 5$

i) Express these 3 equations as a single vector equation.
ii) Find two linear combinations of the column vectors on the left-hand side that equal the right-hand side vector.

iii) Show that it is only possible to find linear combinations of the column vectors on the left-hand side when the right-hand side vector is equal to $\begin{bmatrix} 4 \\ 6 \\ c \end{bmatrix}$ and c is equal to what value?

iv) Express these 3 equations in the form $A\vec{v} = \vec{b}$.

Solution:

i) $$x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

ii) By inspection, two combinations that satisfy the above equation are:

$x = 1\,, y = 1\,, z = 0$ and $x = 0\,, y = 1\,, z = 1$

iii) Consider solutions to the following vector equation:

$$x\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ c \end{bmatrix}$$

Equation 2 minus Equation 1 gives:

$y = 2$

$\therefore x + z = 2$

Substituting into Equation 3 allows us to solve for c:

$2(x + z) + 3y = 4 + 6 = 10 = c$

iv) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$