| Family Name, Given Name (Please print) | Student Number | Tutorial Leader's Name |
|---|---|---|

# PHY293 – Oscillations – Midterm Test (Solutions)

Friday, October 7, 2010

Duration - 50 minutes

**PLEASE read carefully the following instructions.**

**Aids allowed:** A non-programmable calculator without text storage.

Before starting, please **print** your name, tutorial group, and student number **at the top of this page and at the top of the answer sheet**.

There are three questions on this midterm test. Each question is worth one-third of the total grade.

Partial credit will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion**, any relevant assumptions you are making along the way.

POSSIBLY USEFUL EQUATIONS:

| | Amplitude | Velocity | Power |
|---|---|---|---|
| Peak Frequency | $\omega = \omega' = \omega_0\sqrt{1 - 1/(2Q^2)}$ | $\omega = \omega_0$ | $\omega = \omega_0$ |
| Peak Value | $a_m = \frac{a_0 Q}{\sqrt{1-\frac{1}{4Q^2}}}$ | $v_m = a_0\omega_0 Q$ | $P_m = \frac{1}{2} m a_0^2 \omega_0^3 Q$ |
| General | $a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}}$ | $v(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}}$ | $< P(\omega) >= P_{\mathrm{m}} \frac{\gamma^2}{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}$ |
| | $\tan\delta = \frac{\omega\gamma}{(\omega_0^2-\omega^2)}$ | | $< P >= P_{\mathrm{m}} \frac{\gamma^2/4}{(\omega_0-\omega)^2+\gamma^2/4} \quad Q \gg 1$ |

**Do no separate the two stapled sheets of the question paper. Hand in the question sheets with your exam booklet at the end of the test.**

Good luck!

1. Explain succinctly (ie. in three sentences or less) the meaning *and* significance of each of the following, in the context of harmonic oscillations we've discussed in this class. Your answer should make clear not only what the term, or concept, *is*, but also put it in the context of this course and make it clear why it is *important*.

The whole question was worth [10] points. [2.5] points for each of the definitions/descriptions requested. These solutions include comments on common mistakes or shortcoming with solutions seen on last year's real midterm test.

   (a) Velocity resonance;

   In a forced, damped simple harmonic oscillator the peak velocity occurs at $\omega = \omega_0$. At this frequency the driver is able to impart its force at just the right time to push the oscillator faster and faster – until, at equilibrium, the mass is moving at its maximum velocity – the peak of the velocity resonance. Many people copied the $v_m$ formula from the front page which didn't get them much more credit unless they went on to explain that $Q$ amplifies the driver velocity $a_0\omega$ or some other additional statement. An application of the velocity resonance occurs in an LRC circuit where the current (the electrical analog of mechanical velocity) is maximised at resonance. Tuning and LRC circuit to resonance produced the maximum current that can be used to deliver, say, peak brightness in a light bulb, in a non-linear fashion. Many of you made the connection between velocity resonance and power resonance, which was reasonable, but really belonged in the answer to part d). Thus this tended not to get you much more credit as it was bordering on 'write anything down that might be related to resonance and hope for the best'.

   (b) Forced oscillations in the stiffness regime;

   This regime occurs at very low driving frequencies ($\omega \ll \omega_0$) where the amplitude of the forced oscillator matches that of the driving force. Many just said "low frequency" but for many objects the natural frequency is 1-10Hz, which is already pretty low and this phenomenon is only prevalent for 'much lower' frequencies than that – typically 1/10th or less of $\omega_0$. So I was really looking for you to set the scale with $\omega_0$ (which by the way is proportional to $\sqrt{k}$...) The oscillator follows the phase of the driver exactly in this limit, that is it moves in perfect synchronisation. We call this the stiffness dominated regime because at these low frequencies the spring does not deform – acting more as a bar between the driver and the mass. The amplitude of the mass approaches $a_0$ and the relative phase between them goes to 0. We say the spring is too "stiff" to respond at in this regime so the mass just tracks the force. This limit is one often sought by when designing mechanical structures for, if it is feasible/cost-effective to keep well below resonance one can predict reliably what a structure will do. Several people tried to tell me that the frequency of the mass followed $omega$ (that of the driver) but this is **always** the case for the steady-state solution (the mass takes up the oscillation frequency of the driver) so this wasn't much help in defining the stiffness regime specifically. No credit was given for just explaining what "forced oscillations" were in this question. Sorry.

   (c) A high quality (Q) resonator;

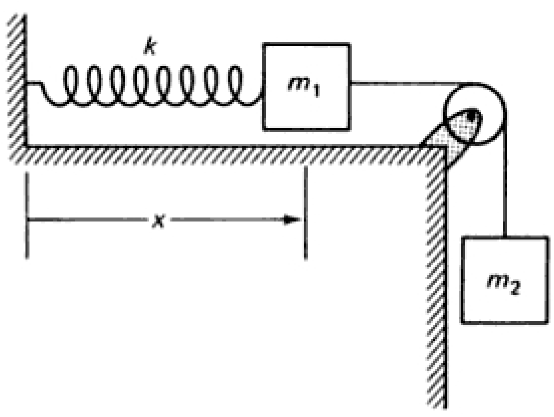
   The $Q$ of an oscillator is a measure of how many cycles an un-driven oscillator will go through before the damping reduces its energy by a factor $e^{-1}$, so a high quality oscillator goes through many more oscillations than a low quality one. The $Q$ of an oscillator is given by $\omega_0/\gamma$, so an oscillator will go through many more oscillations if the damping coefficient is small compared to the natural frequency of a system. In driven oscillator systems with large $Q$ an amplification of the motion/velocity/power of the driver occurs near $\omega = \omega_0$ such that the oscillating object has a larger distance amplitude/velocity/power than the driver by a factor of $Q$. A real-world application of this phenomenon includes the tuning of an electronic amplifier, which is just a LRC circuit that can have a $Q$ of 1000 or more and thus responds with **much** larger amplitude only at the resonant frequency (ie. the frequency of a signal, like a radio station, we are trying to tune into). There were lots of other examples of high $Q$ (the pendulum in a grandfather clock that swings for a long time with very little input of energy etc.) as a result you tended to get at most 1.5/2.5 without a mention of at least one of them.

(d) Power absorption in a forced, damped oscillator.

Power can be most efficiently absorbed from the driver in a damped oscillating circuit if the driver forces the oscillations is 90-degrees out of phase with the oscillations the circuit is making. This is because $P \sim \vec{F} \cdot \vec{v}$ and the velocity vector of the oscillator is exactly 90-degrees out of phase with the position vector ($\dot{x} \sim \sin(\omega t)$ if $x \sim \cos(\omega t)$). It also peaks at resonance as many of you pointed out. For optimum power transfer through an oscillating system – eg. a hydro-electric turbine converting gravitational potential energy into electric potential energy (ie. voltage) – care must be taken to design the system such that the phase relationship between the driver and the oscillator is kept as near 90-degrees as practical. In practice this means tuning the damping, relative to the fixed natural frequencies of the system to keep $\delta = \tan^{-1} \frac{\omega\gamma}{\omega_0^2 - \omega^2}$ near 90-degrees.

Very little (no?) credit was given here (or in any other part) for statements like "as can be seen from the formula for $P(\omega)$ on the cover of the exam". If you were going to copy a formula from the front you'd better actually put it in your answer **and** add some explanation of what at least one of the factors actually means and how it relates to the concept you are trying to explain in order for me to give you credit for it in your answer. Also little credit was given for saying "Bridge" or "LRC circuit" makes concept 'xxx' important. We were looking for an explanation of why 'xxx' would manifest itself in a bridge, a circuit, etc.

2. Consider a mass $m_1$ attached to a spring (of un-stretched length $d$) and pulled by constant force $F_2$, $F_2 = m_2 g$ as shown in the following figure




(a) Suppose that the system is in equilibrium when $x = L$. Is $L > d$ or is $L < d$?

Since $m_2 > 0$ the spring will be stretched beyond its equilibrium length and we'll have $L > d$.[1]

(b) If $L$ and $d$ are known, what is the spring constant $k$?

By making a simple free-body diagram for $m_1$ we'll have a spring force pulling to the left: $F_{\text{spring}} = k(x - d)$ (the force vanishes at $x = d$ and will pull to the left for $x > d$) while we'll have a force of $F_{\text{pulley}} = m_2 g$ pulling over the pulley to the right. At equilibrium $x = L$ these will balance to give: $k(L - d) = m_2 g$. Re-arranging them we get: $k = \frac{m_2 g}{(L-d)}$. Of course at equilibrium we have $L > d$ so this will never give an un-defined (or negative) spring constant. [3]

(c) If the system is at rest in the position $x = L$ and the mass $m_2$ is suddenly removed (for example, but cutting the string that connects $m_1$ and $m_2$), the what is the period and amplitude of the oscillations that $m_1$ will start to execute?

The equation of motion will be $m_1\ddot{x} + \frac{m_2 g}{(L-d)}[x-d] = 0$. This leads to the solution of the form $[x-d] = A\cos(\omega_0 t)$ [1] where $\omega_0 = \sqrt{\frac{m_2 g}{m_1(L-d)}}$. The period is $T = 2\pi/\omega_0 = 2\pi\sqrt{\frac{m_1(L-d)}{(m_2 g}}$[3]. And the amplitude must be $L-d$ at $t = 0$, but since the cosine will be be 1 at $t = 0$ then we have $A = L-d$ and $m_1$ will oscillate back-and-forth with an amplitude of $(L-d)$ [2] for all time. We actually changed the grading scheme for this problem as a result of looking at how many/most of your did the problem. While the equation of motion wasn't asked for in the problem we needed to see some explanation of how you got the period and amplitude. Just saying "I know the answers are 'X' got you the 3+2 points, but still lost you the first of the points here.

3. A block of mass $m$ is connected to a spring, the other end of which is fixed. The block is immersed in viscous damping medium. The following observations have been made of the system:

   1. If the block is pushed horizontally with a force $mg$, the spring length is reduced by $h$;
   2. The viscous resistive force is equal to $mg$ if the block moves with a certain speed: $u$.

(a) For this complete system (spring and mass in damping medium), in the absence of any driving force, write down the differential equation governing horizontal oscillations of the mass in terms of $m$, $g$, $h$ and $u$;

We can deduce $kh = mg$ from the first observation and $bu = mg$ from the second. Our usual definition of $\gamma$ is $b/m$ so the second observation gives $\gamma = g/u$. This allows us to write down the un-driven oscillator the equation of motion:

$$\ddot{x} + g/u\dot{x} + g/hx = 0[2]$$

Check the units. Note that many of you asked if the mass was suspended vertically from the spring because of the way the force was specified. This doesn't make any difference for the equation of motion – well you might have ended up with an $x_0$ offset – for the un-driven oscillator. I assume the mass is lying horizontally on an otherwise frictionless surface (only damping coming from the viscous medium specified). This makes part e) much easier to do.

Answer the following for the case $u = 3\sqrt{gh}$:

In this case $\gamma = g/u = 1/3\sqrt{g/h}$ and $\omega_0 = \sqrt{g/h}$. We'll need those below.

(b) What is the angular frequency of the damped oscillations?

We know $\omega^2 = \omega_0^2 - \gamma^2/4$ for these under-damped oscillations (only kind of oscillations that can happen!). So we get $\omega^2 = g/h - (1/36)g/h = (35/36)g/h$ or $\omega = 1/6\sqrt{35g/h}$ ... a bit weird. [1]

(c) After what time, in multiples of $\sqrt{h/g}$, is the energy of the oscillator reduced by $1/e$?

We know that $E(t) = e^{-\gamma t}E_0$ so the energy will fall to $1/e$ of its original value when $\gamma t = 1$ or $t = 1/\gamma$. This gives $t = 3\sqrt{h/g}$. So the multiple we were seeking was 3 [2].

(d) What is the $Q$ of this oscillator?

$Q = \omega_0/\gamma$ so in this case it is $\sqrt{g/h}/(\sqrt{g/h}/3) = 3$ [2].

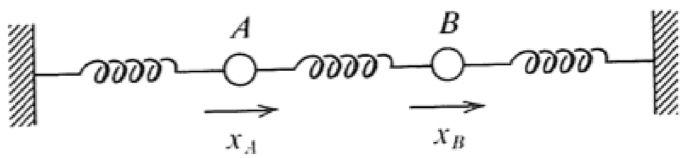
(e) If the oscillator is driven with a force $mg\cos(\omega t)$, where $\omega = \sqrt{2g/h}$, what is the steady-state amplitude of the resulting oscillations of the mass?

From the formula sheet we know:

$$a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

There are many ways to work out $a_0$ but the simplest is to note that the driving force is $mg$ and this was just the force it took to displace the mass a distance $h$ (observation 1). Finding this was worth [1] point. So $a_0 = h$. Plugging $\omega = \sqrt{2g/h}$ into the equation for $a(\omega)$ I get: $a(\omega = \sqrt{2g/h}) = h\sqrt{9/11} \approx 0.90h$. Getting the right formula for this was worth [1] point and getting the right answer another [1] point.

4. Two equal masses, $m$, are on a frictionless surface, held between rigid supports by three identical, massless springs, with spring constant $k$, as shown in the figure. The displacements from equilibrium, along the line the springs are described by the coordinates $x_A$ and $x_B$, as shown. If either of the masses is clamped, the period for one complete oscillation of the other mass (the one that is left free) is 3s (you can use $m = 1$kg if you feel this will help).




(a) If both masses are free, what are the periods of the two normal modes of the system?

- Find the full spring and mass matrices from free-body diagrams for each mass separately to get:

$$m\ddot{x_A} = -kx_A + k(x_B - x_A)$$
$$m\ddot{x_B} = -k(x_B - x_A) - kx_B$$

- Which we can re-arrange on the LHS to:

$$m\ddot{x_A} + 2kx_A - kx_B = 0$$
$$m\ddot{x_B} + 2kx_B - kx_A = 0$$

- We can re-write these equations of motion in matrix form [2] as:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \vec{x} = 0$$

- Find the natural frequencies by assuming harmonic solutions $\ddot{\vec{x}} = -\omega^2\vec{x}$ and finding determinant

$$\det\left| -\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \right| = 0$$
$$\det\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} = 0$$
$$(2k - m\omega^2)(2k - m\omega^2) - (-k)^2 = 0$$
$$m^2\omega^4 - 6km\omega^2 + 3k^2 = 0$$

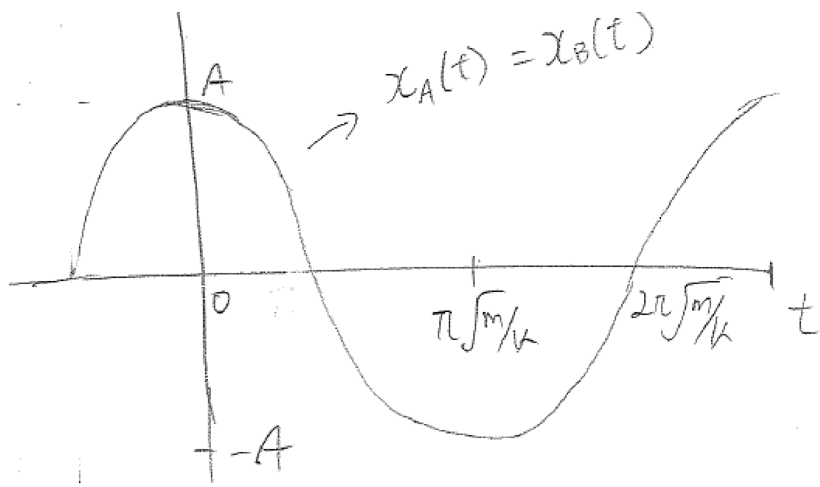
- Once again we have a quadratic equations for $\omega^2$ which has solutions

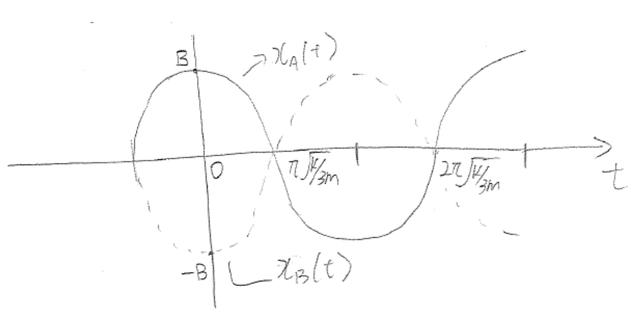
$$\omega^2 = \frac{1}{2m^2}\left[-(-4mk) \pm \sqrt{(-4mk)^2 - 4(m^2)(3k^2)}\right]$$
$$= k/m(2 \pm 1)$$

- This gives two roots: $\omega_1 = \sqrt{k/m}\ \mathrm{s}^{-1}$ and $\omega_2 = \sqrt{3k/m}\ \mathrm{s}^{-1}$
- Can convert this to the periods by taking $T_i = 2\pi/\omega_i$ giving $T_1 = 2\pi\sqrt{\frac{m}{k}}$ and $T_2 = 2\pi\sqrt{\frac{m}{3k}}$ [2] (note: you'd lose points for not including the units or only giving the frequencies, since the question asked for the period)

(b) Sketch graphs of $x_A$ and $x_B$ versus $t$ in each mode.

- The easiest way to sketch these is to determine the normal modes
- For $\omega_1$ the eigenvetor is $\xi_1 = (1, 1)$ [1]
- This means that the two masses track each other or $x_a = x_b$
- For $\omega_2$ the eigenvector is $\xi_2 = (1, -1)$ [1]
- This means the two masses oscillate in opposition to each other or $x_a = -x_b$
- Of course this happens at a frequency that is $\sqrt{3}$ faster than for the first mode [0.5] .
- Putting this together, with some attempt made to show the faster oscillation frequency in the second mode looks like:




(c) At $t = 0$ mass $A$ is held at its normal resting position and mass $B$ is pulled aside a distance of 5 cm. The masses are released from rest at this instant:

- Write an equation for the subsequent positions of each mass as a function of time
  - At $t = 0$ mass $A$ is at rest position while mass $B$ is 5 cm away
  - This gives an initial position vector of $\vec{x} = (0, 5)$ cm (would also have accepted (0,-5)cm and the slightly numerical result that would have followed (see below))

- The masses are both at rest at $t = 0$ so $\dot{\vec{x}} = 0$
- The general solution is:

$$\vec{x} = C(1,1)\cos(\omega_1 t + \delta_1) + D(1,-1)\cos(\omega_2 + \delta_2)$$

- Differentiating we get:

$$\dot{\vec{x}} = -\omega_1 C(1,1)\sin(\omega_1 t + \delta_1) - \omega_2 D(1,-1)\sin(\omega_2 + \delta_2)$$

- But with $\dot{\vec{x}} = 0$ the only possible solution is $\delta_1 = \delta_2 = 0$
- From the former equation (the general solution for $\vec{x}$) we find:

$$C(1,1) + D(1-1) = (0,5) \text{ cm}$$

- From the first row of this equation we conclude C=-D
- From the second row of this equation we find C=2.5 cm
- So the full solution is:

$$\vec{x} = 2.5 \text{ cm}(1,1)\cos(\omega_1 t) - 2.5 \text{ cm}(1,-1)\cos(\omega_2)[2.5]$$

- Of course you can (and should, though you won't get extra marsk for it) readily check that this gives the specified positions at $t = 0$

* What length of time (in s) characterises the periodic transfer of motion from $B$ to $A$ and back again? After one complete cycle, is the situation at $t = 0$ exactly re-produced?
  - In fact the system never returns to exactly the same configuration because we can never have a time when both $\omega_1 = 2n\pi$ *and* $\omega_2 = 2n'\pi$ are equal since $\omega_2/\omega_1 = \sqrt{3}$ is not an integer [1] .
  - However we can still use the trigonometric trick we saw in class for the first coupled oscillator solution (see notes from September 26) to convert :

$$x_{A,B} = 2.5\text{cm}[\cos(\omega_1 t) \mp \cos(\omega_2 t)]$$

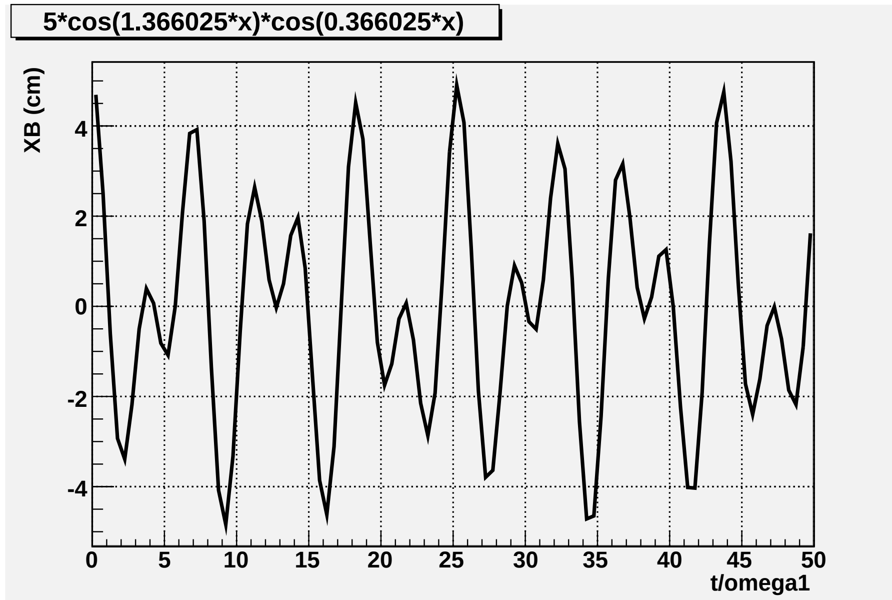
  - to

$$x_B = 2.5 \text{ cm}[\cos(\omega_1 t) + \cos(\omega_2 t)]$$
$$x_B = 5 \text{ cm}[\cos(\frac{\omega_1 + \omega_2}{2}t)\cos(\frac{\omega_1 - \omega_2}{2}t)]$$

  - Plugging in the numbers we get:

$$x_B = 5 \text{ cm}[\cos(1.36\omega_1 t)\cos(0.36\omega_1 t)]$$

    which never returns to exactly 5 cm but comes *close* after $\tau = 2\pi/(0.36\omega_1) \approx 17/\omega_1$,
    * This is when the second cosine returns the to $+1$ for the first time.
    * The other problem with this example is that $\omega_1$ and $\omega_2$ are not all that close together,
    * So they don't produce the classic *beats* pattern we saw in class on September 26.
  - Instead the motion of mass $B$ described by the equations above looks like:




- There is a peak, near 5 cm, at $t = 17/\omega_1$ but there are many others that are nearly as large, some of them quite near by...
- This wasn't the best choice of constants to have used for a 'simple' or specific answer to part e).
- A question like part d) of this one won't find it's way onto a test or final exam (this was a problem set problem from last year), but if it did the first answer – that it never gets exactly back to the starting point – would be more than good enough.