UNIVERSITY OF TORONTO
Engineering Science

PHY293, Part A: Waves and Oscillations

Term Test 1, 21 October 2019

Duration: 60 minutes

- *Write your name, student number and tutorial group on top of **all** examination booklets and test pages used.*
- *Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. Not other aid (notes, textbook, dictionary) is allowed. **Communication devices are strictly forbidden. Turn them off and make sure they are in plain sight to the invigilators.***
- *Answer **all** questions. For each question, the mark breakdown for each subsection is listed in square brackets at the beginning of the question.*
- *There are three questions in this mid-term. Partial credit will be given for partially correct answers. So, please show any intermediate calculations that you do and write down, in a clear fashion, any relevant assumptions you are making along the way.*
- *As one question progresses, the difficulty increases. If you are stuck within one of the three questions, try another.*
- *Do not separate the stapled sheets of the question paper. Hand in the question and rough work sheets together with your exam booklet at the end of the test.*
- *This test has 4 pages, and the total number of marks is 100, plus 5 bonus marks.*

Some possibly (but not necessarily!) useful formulas.

If $\ddot{x}+\gamma\dot{x}+\omega_0^2 x=\omega_0^2 A_f\cos(\omega t)$:

	Amplitude	Velocity	Dissipated Power
Peak freq.	$\omega_{max}=\omega_0\sqrt{1-1/(2Q^2)}$	$\omega_{max}=\omega_0$	$\omega_{max}=\omega_0$
Peak value	$A_{max}=\dfrac{QA_f}{\sqrt{1-1/(4Q^2)}}$	$V_{max}=\omega_0 QA_f$	$P_{max}=\dfrac{mA_f^2\omega_0^3 Q}{2}$
Misc.	$A(\omega)=\dfrac{A_f}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}}}$	$V(\omega)=\omega A(\omega)$	$\overline{P}(\omega)=\dfrac{m\gamma V^2(\omega)}{2}$
	$\tan\delta=\dfrac{\omega\gamma}{\omega_0^2-\omega^2}$		$\approx\dfrac{P_{max}}{1+\frac{4(\omega_0-\omega)^2}{\gamma^2}}\ (Q\gg 1)$

$$\mathsf{M}\ddot{\vec{X}}+\mathsf{K}\vec{X}=0;\qquad \det\left(\mathsf{K}-\omega^2\mathsf{M}\right)=0.$$

$$\mathsf{M}^{-1}\mathsf{K}\quad\text{symmetric and}\quad |\vec{Y}_i|=1\quad\Rightarrow\quad \vec{Y}_i\cdot\vec{Y}_j=\delta_{ij}$$

$$\vec{X}(t)=\sum_{n=1}^{N}C_n\vec{Y}_n\cos(\omega_n t+\phi_n),\quad\text{with}\quad C_n\cos\phi_n=\vec{X}_0\cdot\vec{Y}_n\quad\text{and}\quad C_n\sin\phi_n=-\frac{\vec{V}_0\cdot\vec{Y}_n}{\omega_n}.$$

$$\det\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}=a_{11}(a_{22}a_{33}-a_{32}a_{23})-a_{12}(a_{21}a_{33}-a_{31}a_{23})+a_{13}(a_{21}a_{32}-a_{31}a_{22})$$

$$ax^2+bx+c=0\quad\Leftrightarrow\quad x=\frac{1}{2a}\left(-b\pm\sqrt{b^2-4ac}\right)$$

$$\frac{\partial^2 y}{\partial t^2}-v^2\frac{\partial^2 y}{\partial x^2}=0\quad\text{with}\quad v=\sqrt{\frac{T}{\mu}}$$

$$y(x,t)=\sum_{n=1}^{\infty}C_n\cos(\omega_n t+\phi_n)\sin(k_n x)=\sum_{n=1}^{\infty}\left[\alpha_n\cos(\omega_n t)+\beta_n\sin(\omega_n t)\right]\sin(k_n x),$$

$$\text{with}\quad \alpha_n=\frac{2}{L}\int_0^L y(0,x)\sin(k_n x)\mathrm{d}x\quad\text{and}\quad \beta_n=\frac{2}{L\omega_n}\int_0^L \dot{y}(0,x)\sin(k_n x)\mathrm{d}x.$$

$$y(x,t)=A\sin\left(\frac{2\pi}{\lambda}(x-vt)\right)=A\sin\left(k(x-vt)\right)=A\sin\left(kx-\omega t\right)=A\sin\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right].$$

$$\omega=2\pi\nu,\quad \nu=1/T,\quad k=2\pi/\lambda,\quad v=\omega/k=\lambda/T=\lambda\nu.$$

$$\text{Energy Flux}=\frac{1}{2}\mu_i v\omega^2A^2=\frac{1}{2}\sqrt{T\mu_i}\,\omega^2A^2.$$

$$\rho=\frac{A_R}{A_I}=\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}};\qquad \tau=\frac{A_T}{A_I}=\frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}=\rho+1$$

1. [30 marks] The **tension in the A string of a guitar** is adjusted to produce a fundamental frequency of 110 Hz.

   (a) [10] What are the frequencies of the second and third harmonics? Does the wave velocity change in going to these harmonics?

   (b) [10] After many years of listening to loud music, the hearing range of the guitarist extends to 12 kHz. What is the total number of harmonics of the A string the guitarist can hear?

   (c) [10] If the guitar A string is 65 cm long, how far from the end of the string should the guitarist place their finger to play the note C (131 Hz)?

2. [40 marks] A **torsional harmonic oscillator** (cf. figure 1a) is described by the following equation of motion:

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + C\frac{\mathrm{d}\theta}{\mathrm{d}t} + \kappa\theta = \tau(t), \tag{1}$$

   where θ is the angle of twist from its equilibrium position in radians, I is the moment of inertia, C is the rotational damping, κ is the torsion spring constant and τ is the drive torque.

   (a) [5] What is its natural angular frequency ω_0?

   (b) [5] Which range of values of C corresponds to the light damping case?

   (c) [5] For the unforced light damping case, what is the angular frequency of oscillations?

   (d) [5] Suppose a force F, constant in time, is applied at a distance L from the axis so that the torque is $\tau = FL$. When the oscillatory motion of the transient dies out, what is the resulting angle of twist, θ_{eq}?

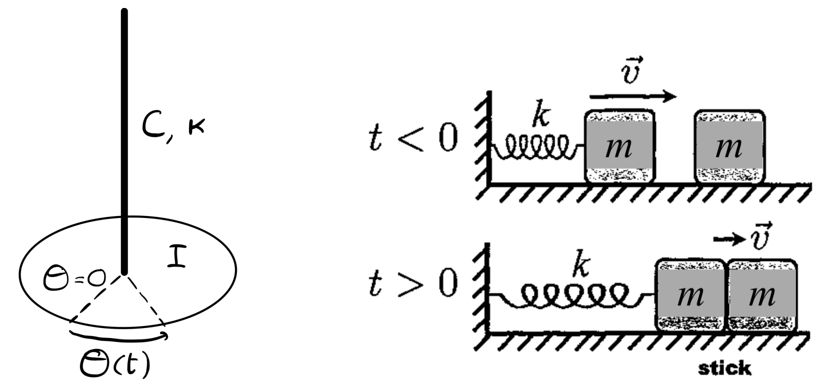
   (e) [20] Suppose that $I = 2.0\ \mathrm{kg\,m^2}$, $\kappa = 200\ \mathrm{N\,m\,rad^{-1}}$, and $C = 3.0\ \mathrm{kg\,m^2\,s^{-1}\,rad^{-1}}$. Suppose also that a driving force, $F(t)$, with an angular frequency, ω, is applied at a distance L from the axis, thus resulting in a torque,

$$\tau(t) = F(t)L = \kappa a \cos(\omega t), \tag{2}$$

   where $a = 0.050$ rad and $\omega = 3\pi\ \mathrm{rad\,s^{-1}}$. Determine the amplitude and relative phase of the steady state (i.e., post-transient) oscillation of the torsional harmonic oscillator. If the driving angular frequency, ω, is changed to the natural angular frequency ω_0, what will be the amplitude of oscillation?

   Hint: clearly indicate intermediate reasoning/results/numbers for partial marks.

3. [30 + 5 marks] A **block of mass *m* oscillates on a spring** with spring constant k. The surface has no friction. The oscillating motion initially has amplitude A. At $t = 0$, when the mass is at position $+A/2$ and moving to the right (positive velocity) as shown in the upper part of figure 1b, it collides with a second block of equal mass



(a) Torsional harmonic oscillator (Q2)

(b) Collision of two masses (Q3)

Figure 1: Sketches for questions 2 and 3.

m. The blocks stick together to form a new block of mass $2m$ and continue to the right as shown in the lower part of figure 1b. We will denote quantities before the collision with a subscript "1" and quantities after with subscript "2".

We assume that momentum is conserved in the collision, i.e., $mv_1 = 2mv_2$ and we have $v_2 = v_1/2$ the instant after the collision.

(a) [5] Let $\omega_1 = \sqrt{k/m}$ be the oscillation frequency before the collision. What is the new oscillation frequency ω_2 after the collision?

(b) [13] Suppose the motion before the collision is given by $x_1(t) = A\cos(\omega_1 t + \phi)$. Find ϕ, and then an expression for $v_1(t=0)$, followed by $v_2(t=0)$ the instant after the collision. Except for ϕ, express all results in terms of ω_1 and A.

Hint: $\cos(\pi/3) = 1/2$ *and* $\sin(\pi/3) = \sqrt{3}/2$.

(c) [12] Find an expression for $x_2(t)$, the subsequent motion of the joined masses after the collision.

Hint: Since the collision is at $t = 0$, velocity v_2 and the collision position $x = +A/2$ constitute initial conditions on the subsequent motion. You may want to use the general form $x(t) = B\cos(\omega t) + C\sin(\omega t)$, where B and C are constants.

(d) [*5 bonus marks*] What is the new amplitude of the subsequent undamped motion, expressed in terms of A?

Hint: Remember that the new amplitude is just the maximum of $x_2(t)$.

THIS IS THE END OF THE TEST.