2.)

(2.3) Here, following example 2.2, we use equation 2.4.12

$$\Rightarrow \frac{dp}{dz} = -\rho g = -\frac{p}{RT} g_o \quad \text{where} g = g_o \quad \text{at} z = 0$$

Here, we take $T = T_o - \alpha z$,so

$$\frac{dp}{p} = -\frac{g_o dz}{RT_o - \alpha z} = -\frac{g_o}{RT_o}\frac{dz}{(1-\beta z)} \quad ;\beta = \frac{\alpha}{T_o}$$

$$\Rightarrow \ln p = \frac{g_o}{\beta R T_o} \ln(1-\beta z) + C$$

We set $p = P_a$ at $z = 0 \Rightarrow C = P_a$ and

$$\frac{p}{P_a} = (1-\beta z)^{\left\{\frac{g_o}{\beta R T_o}\right\}} = \left(1 - \frac{\alpha z}{T_o}\right)^{\left\{\frac{g_o}{R\alpha}\right\}}$$

If

$$\frac{p}{P_a} = 0.5 \Rightarrow (0.5)^{\frac{R\alpha}{g_o}} = 1 - \frac{\alpha z}{T_o} \Rightarrow z = \frac{T_o}{\alpha}\left\{1 - 0.5^{\frac{R\alpha}{g_o}}\right\}$$

For the stated conditions $\alpha = 1°C/100m = 0.01$

$$\begin{aligned} z &= \frac{298}{0.01}\left\{1 - (0.5)^{\frac{287\times 0.01}{9.804}}\right\} = 29,800\{1 - 0.8164\} \\ &= 5471m \end{aligned}$$

Error in assuming $g = g_o$? At $z = 5471m$

$$\begin{aligned} g &= g_o \frac{Re^2}{(Re+z)^2} = g_o\left\{1 + \frac{z}{Re}\right\}^{-2} \approx g_o\left\{1 - \frac{2z}{Re}\right\} \\ &= 0.9983 g_o \end{aligned}$$

Error from assuming $g = g_o$ is probably less than that associated with temperature decrease assumption.

2.3

(i) With $dp/dz = -\rho g$ and $E_T = \rho\left(\frac{dp}{d\rho}\right)_T$;

$$\frac{dp}{dz} = \frac{dp}{d\rho}\cdot\frac{d\rho}{dz} = \frac{E_T}{\rho}\frac{d\rho}{dz} = -\rho g \Rightarrow -\frac{d\rho}{\rho^2} = \frac{g}{E_T}dz \Rightarrow \frac{1}{\rho} = \frac{gz}{E_T} + C$$

At $z = 0$, $\rho = \rho_0 \Rightarrow \rho^{-1} = \rho_0^{-1} + \frac{gz}{E_T}$ For $z = -H$,

$\rho^{-1} = \rho_0^{-1} - gH/E_T$

(ii) For $E_T = 2.40\times10^9$, $g = 9.81$, $H = 11033$, $\rho_0 = 1025$

$\rho_H = 1074$ kg/m³, a 4.8% increase.

(iii) In $dp/dz = -\rho g$ we use above result to get

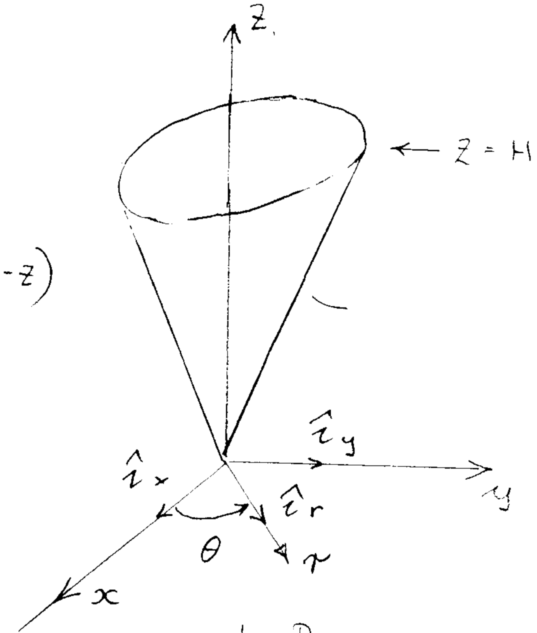
$$\frac{dp}{dz} = -\frac{\rho_0 g}{1+(\rho_0 gz/E_T)} \Rightarrow p_H = p_a - E_T \ln\left[1 - \frac{\rho_0 gH}{E_T}\right]$$

For $H = 11033$ m, $p_H = 113.6$ MPa, which compares with a maximum of about 20 MPa for an industrial gas cylinder.

(iv) If we assume $\rho = \rho_0 \Rightarrow p_H = 1025\times9.81\times11033 = 110.9$ MPa or 2.4% low.

2.7

(i) Cone volume $= \frac{1}{3}\pi R^2 H$
$= \frac{1}{3} H^3 \tan^2\alpha$

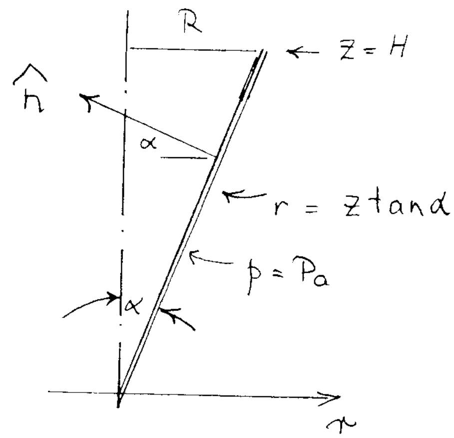


(ii) In the liquid the pressure is $p = P_a + \rho g(H - z)$ so that the resultant pressure force acting on an element δS of the cone's interior surface is

$$\delta \vec{F}_p = -(p - P_0)\hat{n}\,\delta S$$

$$= -\rho g\left(H - \frac{r}{\tan\alpha}\right)\hat{n}\,\delta S$$

since $r = z\tan\alpha$ on cone's surface



(iii) Here, from geometry

$$\hat{n} = -\cos\alpha\,\hat{\imath}_r + \sin\alpha\,\hat{\imath}_z$$

$$= -\cos\alpha\cos\theta\,\hat{\imath}_x - \cos\alpha\sin\theta\,\hat{\imath}_y + \sin\alpha\,\hat{\imath}_z$$

Alternatively, the cone is described by

$$z^2\tan^2\alpha - r^2 = z^2\tan^2\alpha - (x^2 + y^2) = 0,$$

so that $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{-x\hat{\imath}_x - y\hat{\imath}_y + z\tan^2\alpha\,\hat{\imath}_z}{(r^2 + z^2\tan^4\alpha)^{1/2}}$

which, after algebra, reduces to the above result

(iv) To calculate δS, we relate it to the element $r\,\delta r\,\delta\theta$ in the x-y plane: $\delta S = \dfrac{r\,\delta r\,\delta\theta}{|\hat{n}\cdot\hat{\imath}_z|} = \dfrac{r\,\delta r\,\delta\theta}{\sin\alpha}$

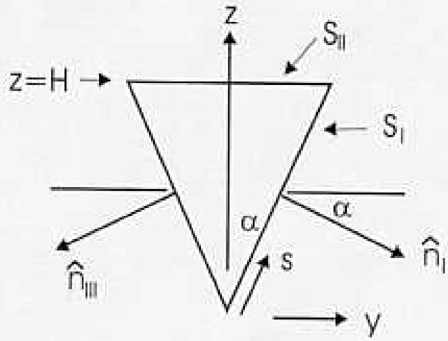
Thus $\delta\vec{F}_p = -\rho g(H - r/\tan\alpha)\{-\cos\alpha\cos\theta\,\hat{\imath}_x - \cos\alpha\sin\theta\,\hat{\imath}_y + \sin\alpha\,\hat{\imath}_z\}\dfrac{r\,\delta r\,\delta\theta}{\sin\alpha}$

(v)
$$\vec{F}_p = -\rho g\left[\int_0^{H\tan\alpha}\left(Hr - \frac{r^2}{\tan\alpha}\right)dr\right]\left[\left\langle -\cot\alpha\int_0^{2\pi}\cos\theta\,d\theta\right\rangle\hat{\imath}_x - \left\langle\cot\alpha\int_0^{2\pi}\sin\theta\,d\theta\right\rangle\hat{\imath}_y + \left\langle\int_0^{2\pi}d\theta\right\rangle\hat{\imath}_z\right]$$

$$= -\left\langle 2\pi\rho g\left[\frac{Hr^2}{2} - \frac{r^3}{3\tan\alpha}\right]_0^{H\tan\alpha}\right\rangle\hat{\imath}_z = -\rho g V\hat{\imath}_z$$

2.8

Take $p = p_0 - \rho g z$, where $p_0 = p(z=0)$



A) on S_I, $\hat{n} = \cos\alpha\,\hat{i}_y - \sin\alpha\,\hat{i}_z$ $\quad \delta\vec{F}_P = -(p_0 - \rho g z)\cos\alpha\,\hat{i}_y - \sin\alpha\,\hat{i}_z L\delta s$
With $z = s\cos\alpha$

$$\begin{aligned}\vec{F}_P|_I &= \{L\int_0^{H/\cos\alpha}(-p_0\cos\alpha + \rho g s\cos^2\alpha)ds\}\hat{i}_y + \{L\int_0^{H/\cos\alpha}(p_0\sin\alpha - \rho g s\cos\alpha\sin\alpha)ds\}\hat{i} \\ &= \{-p_0LH + \rho g\frac{H^2L}{2}\}\hat{i}_y + \{p_0LH\tan\alpha - \rho g\frac{LH^2\tan\alpha}{2}\}\hat{i}_z\end{aligned}$$

B) on S_{III}, $\hat{n} = -\cos\alpha\,\hat{i}_y - \sin\alpha\,\hat{i}_z \Rightarrow$

$$\vec{F}_P|_{III} = \{p_0LH - \rho g\frac{H^2L}{2}\}\hat{i}_y + \{p_0LH\tan\alpha - \rho g\frac{LH^2}{2}\tan\alpha\}\hat{i}_z$$
$$\Rightarrow \vec{F}_P|_I + \vec{F}_P|_{III} = \{2p_0LH\tan\alpha - \rho g LH^2\tan\alpha\}\hat{i}_z$$

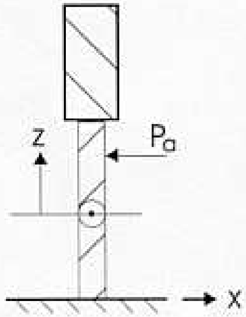
C) On S_{II}, $\hat{n} = +\hat{i}_z$, $\quad p = p_0 - \rho g H$

$$\begin{aligned}\Rightarrow \vec{F}_P|_{II} &= \{-(p_0 - \rho g H)(2LH\tan\alpha)\}\hat{i}_z \\ &= \{-2p_0LH\tan\alpha + 2\rho g LH^2\tan\alpha\}\hat{i}_z\end{aligned}$$

D) $\vec{F}_P = \rho g LH^2\tan\alpha\,\hat{i}_z = \rho g V\hat{i}_z$

2.10

(2.2) Choose $z = 0$ to correspond with gate hinge.



Then $p = Pa + \rho g(H - D - Z)$. On a strip of width δz, the resultant pressure force is $\delta \vec{F}_P = (p - Pa)W\delta z\hat{i}_x$. With counterclockwise moments positive, this has a moment about the hinge δM_H given by

$$\delta M_H = -(p - Pa)Wz\delta z = -\rho gW(H - D - z)z\delta z$$

Thus

$$M_H = -\rho gW\int_{-D}^{+D}\{(H-D)z - z^2\}dz = -\rho gW\left\{(H-D)\frac{z^2}{2} - \frac{z^3}{3}\right\}_{-D}^{+D} = +\frac{2}{3}\rho gWD^3$$

Note: Since p is linearly dependent on z, we would not expect H to be a factor in the moment, although it is of course a factor in the hinge force

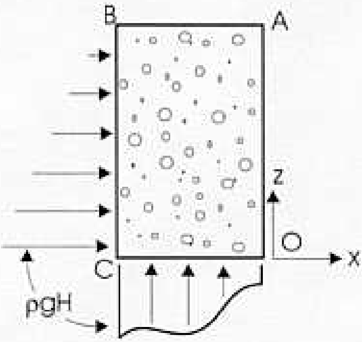
$$|\vec{F}_P| = F_H = \rho gW\int_{-D}^{+D}\{(H-D) - Z\}dz = \rho gW = 2\rho gW(H-D)D$$

The average pressure is $p_{av} = Pa + \rho g(H - D)$. The area is $2WD$; hence the above result.

2.12 (2.12) 1. Pressure Distributions Along Side and Bottom

(a) along CB $p = \rho_w g(H - z)$

(b) along CO assume $p = p_b(x)$ $0 \leq x \leq -B$ such that $p(-B) = \rho_w g H$



2. Moment of Pressure Force on CB: (unit depth)(clockwise $+ve$)

$$\delta M_{CB} = p\delta z \cdot z = \rho_w g(H - z)z\delta z$$

$$M_{CB} = \rho_w g \int_0^H (H - z)z dz = \rho_w g \left\{ \frac{Hz^2}{2} - \frac{z^3}{3} \right\} \Big|_0^H = \rho_w g \frac{H^3}{6}$$

3. Moment of Pressure Force on CO:

$$M_{OC} = \int_0^{-B} p_b(x) x dx: \quad \text{Let } \xi = -Bx \quad \text{and put } p_b = \rho_w g H f(\xi)$$

$$\text{where we expect } f(1) = 1 \longrightarrow M_{OC} = \rho_w g H B^2 \int_0^1 f(\xi)\xi d\xi$$

Put $M_{OC} = \rho_w g H B^2 \mathcal{K}$

$$\boxed{\mathcal{K} = \int_0^1 f(\xi) d\xi}$$

4. Examples If $f(\xi) = 1$ as asked in the problem, $\mathcal{K} = \int_0^1 \xi d\xi = \frac{1}{2}$ If $f(\xi) = \xi$, corresponding to linear fall-off in pressure as suggested by the theory of flow through porous media, then $\mathcal{K} = \frac{1}{3}$.

5. Moment Balance: $M_{BC} + M_{OC} - \rho_C g B H \frac{B}{2} = 0$

$$\rho_w g \frac{H^3}{6} + \rho_w g H B^2 \mathcal{K} - \rho_c g \frac{B^2}{2} = 0$$

6. Solution: Solve for B, with $\sigma_c = \frac{\rho_c}{\rho_w}$, the relative density,

$$\frac{H^2}{6} + B^2 \mathcal{K} - \frac{\sigma_c B^2}{2} = 0$$

$$\text{or} \quad \boxed{\frac{B}{H} = \frac{1}{\sqrt{3\ sigma_c - 6\mathcal{K}}}}$$

$$\text{For conservative design case} \quad \mathcal{K} = \frac{1}{3} \quad \boxed{\frac{B}{H} = \{3\,(\sigma_c - 1)\}^{-1/2}}$$

$$\text{For "porous media" case} \quad \mathcal{K} = \frac{1}{3} \quad \boxed{\frac{B}{H} = \left\{3\left(\sigma_c - \frac{2}{3}\right)\right\}^{-1/2}}$$

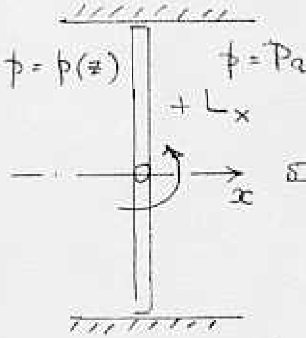
NOTE For concrete $\sigma_c \approx 2.37$

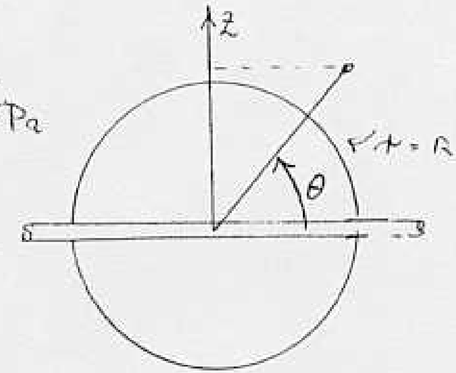
2.14

(i) On valve disc,

$p = \rho g(H - z) + P_a$

regardless of shape of reservoir.





(ii) On an element of disc

$\delta \vec{F}_p = (p(z) - P_a)\, r\, \delta r\, \delta\theta$, and $\delta L_x = -(p(z) - P_a)\, z\, r\, \delta r\, \delta\theta$

where the minus sign accounts for the fact that, for $z > 0$ contributions to L_x in the counterclockwise sense are $-$ve.

(iii) But $z = r\sin\theta$, so $L = -\int_0^{2\pi}\int_0^{R} \rho g(H - r\sin\theta)\, r^2 \sin\theta\, dr\, d\theta$

$$= -\rho g H \int_0^{2\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta + \rho g \int_0^{2\pi}\int_0^{R} r^3 \sin^2\theta\, dr\, d\theta$$

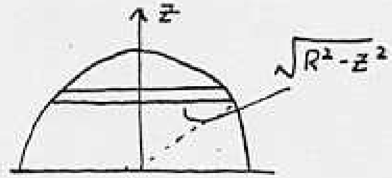
$$= -\rho g \frac{HR^3}{3}\int_0^{2\pi} \sin\theta\, d\theta + \rho g \frac{R^4}{4}\int_0^{2\pi} \frac{1}{2}\left[1 - \cos 2\theta\right] d\theta = \frac{\pi \rho g R^4}{4}$$

(iv) Mounting the shaft vertically gives zero L_x owing to symmetry of pressure distribution about shaft axis

2.17

a) $\nabla p = \rho \vec{g} \Rightarrow \frac{\partial p}{\partial z} = -\rho g, \; \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$

$\Rightarrow \; p = P_a + \rho g\,(H - z)$



(b) Two methods of integration

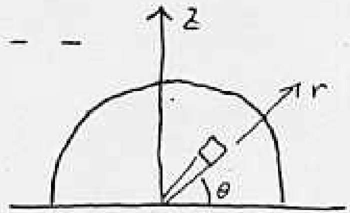
I: Strips of width $2\sqrt{R^2 - z^2}$ at constant z

$\delta F_p = 2(p - P_a)\sqrt{R^2 - z^2}\,\delta z \Rightarrow \delta M_{BB} = -2\rho g\,(H - z)\sqrt{R^2 - z^2}\; z\,dz$

$\Rightarrow |M_{BB}| = 2\rho g \left\{ H\int_0^R (R^2 - z^2)^{1/2} z\,dz - \int_0^R (R^2 - z^2)^{1/2} z^2\,dz \right\}$

$= 2\rho g \left\{ H\left[-\frac{1}{3}(R^2 - z^2)^{3/2}\right]_0^R - \frac{\pi R^4}{16} \right\} = \rho g R^3 \left[\frac{2}{3}H - \frac{\pi R}{8}\right] \Rightarrow$

$\Rightarrow \; F = |M_{BB}|/R = \underline{\rho g R^2 \left[\frac{2}{3}H - \pi R/8\right]}$



II : Polar co-ordinates

$\delta F_p = (p - P_a)\, r\,\delta r\,\delta\theta \, , \; z = r\sin\theta$

$\Rightarrow \; \delta M_{BB} = -\rho g\,(H - r\sin\theta)\cdot r\sin\theta \cdot r\,\delta r\,\delta\theta$

$\Rightarrow |M_{BB}| = \rho g H \int_0^\pi \int_0^R r^2 \sin\theta\,d\theta - \rho g \int_0^\pi \int_0^R r^3 \sin^2\theta\,d\theta$

$= \rho g H \frac{R^3}{3} \int_0^\pi \sin\theta\,d\theta - \rho g \frac{R^4}{4} \int_0^\pi \sin^2\theta\,d\theta$

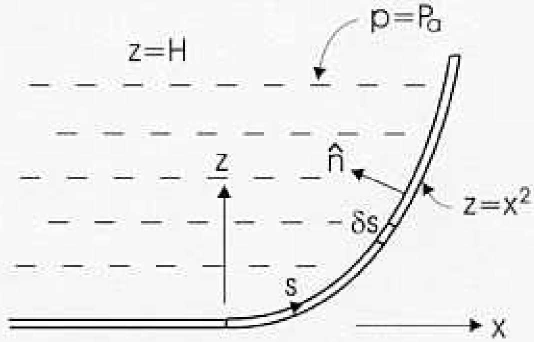
$= \rho g \frac{HR^3}{3}\left[-\cos\theta\right]_0^\pi - \frac{\pi \rho g}{8} R^4 = \rho g R^3 \left[\frac{2}{3}H - \frac{\pi R}{8}\right]$

2.19

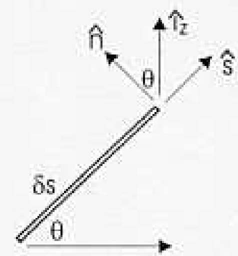
(2.4) 1. Start with

$$\vec{F}_p = -\int_S p\hat{n}\,dS$$

Here we take δS to be a rectangular strip of width δs and length W. [here "s" is the distance along the wall]; $\delta S = W\delta s$

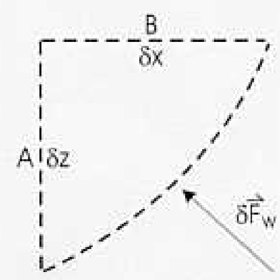


2. For the wall, $\hat{n}$ points to the interior of the liquid and is $\hat{n} = -\sin\theta\,\hat{\imath}_x + \cos\theta\,\hat{\imath}_y$



3. Then $\hat{n}\delta S = -W\sin\theta\delta s\,\hat{\imath}_x + W\cos\theta\delta s\,\hat{\imath}_y$
   But $\sin\theta = \frac{\delta x}{\delta s}$, $\cos\theta = \frac{\delta x}{\delta s}$ so
   $-p\hat{n}\delta S = pW\,\{\delta z\,\hat{\imath}_x - \delta x\,\hat{\imath}_z\}$

4. This result can be obtained by a physical argument which is basically the same as that used to obtain Pascal's principle [$\mathcal{T}$1.1]. Consider the forces acting on the element of fluid adjacent to the element δs of the wall and depicted in the diagram.



If p_A and p_B are the average pressures on the vertical and horizontal sides, and if $\delta\vec{F}_w$ is the force exerted on the fluid by the wall, equilibrium requires

$$p_A W\delta z\,\hat{\imath}_x - p_B W\delta x\,\hat{\imath}_z + \rho g W\frac{\delta x\delta z}{2}\,\hat{\imath}_z = \vec{0}$$

or, in the limit as $\delta s \to 0$,

$$\delta \vec{F}_w = pW\delta x\,\hat{\imath}_z - pW\delta z\,\hat{\imath}_x$$

and the force exerted by the fluid on the wall is equal and opposite to $\delta \vec{F}_w$, as obtained above.

5. The pressure is obtained by noting that $\nabla p = \rho \vec{g}$ reduces here to

$$\frac{\partial p}{\partial z} = -\rho g \quad \Rightarrow \quad p = P_A = \rho g(H - z)$$

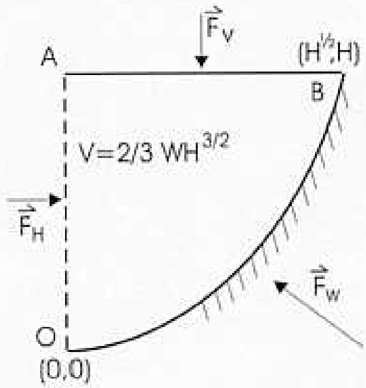
6. We thus obtain

$$\begin{aligned}\frac{1}{W}\vec{F}_p &= \left\{\int_0^H [P_a + \rho g(H - Z)]\,dz\right\}\hat{\imath}_x - \left\{\int_0^{sqrtH} [P_a + \rho g(H - z)]\,dx\right\}\hat{\imath}_z \\ &= \left\{P_a H + \rho g\frac{H^2}{2}\right\}\hat{\imath}_x - \left\{P_a\sqrt{H} + \rho g\int_0^{sqrtH}(H - x^2)\,dx\right\}\hat{\imath}_z\end{aligned}$$

Note that $\quad W\int_0^{\sqrt{H}}(H - z)\,dx = W\int_0^{\sqrt{H}}(H - x^2)\,dx = \frac{2}{3}WH\sqrt{H}$

is the volume V of the liquid above the curved surface.

7. To interpret this, note that we can consider the forces acting on the finite volume of liquid above the curved surface.

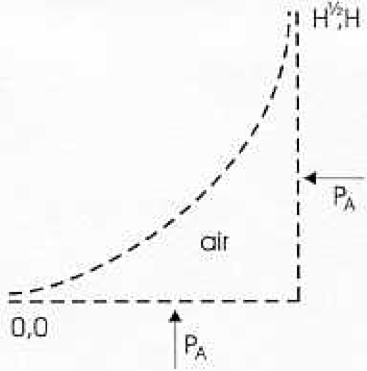


Thus for this body of matter.

$$\vec{F}_H + \vec{F}_V + \vec{F}_W = \rho g V\,\hat{\imath}_z = \vec{0}$$

Where we have $\vec{F}_V = -P_a\sqrt{H}W\,\hat{\imath}_z$ and $\vec{F}_H = p_{av}HW\,\hat{\imath}_x$
where p_{av} is the average pressure on OA so that $p_{av} = P_a + \rho g\frac{H}{2}$
Hence $\vec{F}_H = \left\{P_a HW + \rho g H^2\frac{W}{2}\right\}\hat{\imath}_x$
from which we obtain $\vec{F}_p = -\vec{F}_W = \vec{F}_H + \vec{F}_V - \rho g V\,\hat{\imath}_z$
which is the same as obtained by direct integration.

8. The net force on the wall can be obtained as follows:



Consider the forces on a volume of air outside the wall and directly below it, and repeat the argument immediately above. However, this time, since $\frac{\rho_{water}}{\rho_{air}}$ is about 830, we take $p = P_A$ everywhere it follows that the above argument gives the atmospheric force $\vec{F}_A$ on the underside of the wall is

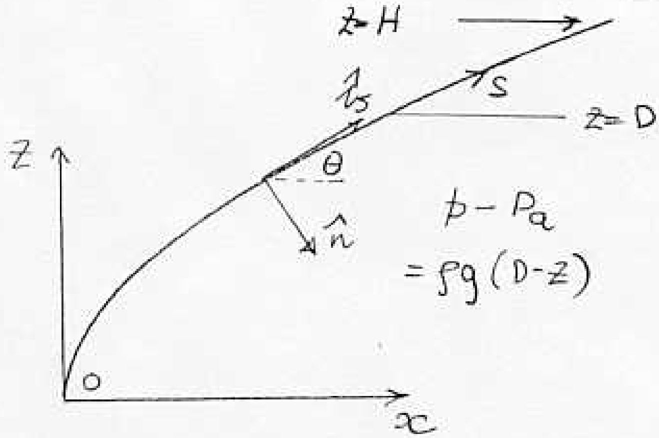
$$\vec{F}_A = P_A \left\{ -W H\, \hat{\imath}_x + W\sqrt{H}\, \hat{\imath}_z \right\}$$

This gives the force on the wall due to the liquid as

$$\vec{F}_P - \vec{F}_A = \rho g W \left\{ \frac{H^2}{2}\, \hat{\imath}_x - \frac{2}{3} H^{\frac{3}{2}}\, \hat{\imath}_z \right\}$$

which is that obtained by assuming P_A to be zero.

2.20



1) Relevant unit normal:

For forces acting on the wall by the water is oriented as shown; with

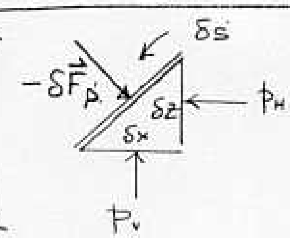
$$\hat{\imath}_s = \cos\theta\,\hat{\imath}_x + \sin\theta\,\hat{\imath}_z$$
$$= dx/ds\,\hat{\imath}_x + dz/ds\,\hat{\imath}_z$$

$$\hat{n} = \cos\left(\theta - \frac{\pi}{2}\right)\hat{\imath}_x + \sin\left(\theta - \pi/2\right)\hat{\imath}_z = \frac{dz}{ds}\hat{\imath}_x - \frac{dx}{ds}\hat{\imath}_z$$

2) Force on element $\delta S = W\delta s \Rightarrow \delta\vec{F}_p = -(p - P_a)\hat{n}\,\delta S$

$$= -(p - P_a)\left\{\frac{dz}{ds}\hat{\imath}_x - \frac{dx}{ds}\hat{\imath}_z\right\}W\delta s = \rho g W(D - z)\left\{-\delta z\,\hat{\imath}_x + \delta x\,\hat{\imath}_z\right\}$$

Note: This can be arrived at directly by considering the forces acting the small triangular prism depicted to the right. Since according to Theorem 1.1, the pressure forces must balance directly, regardless of the mass in the volume, we have, with $\{-\delta\vec{F}_p\}$ being the force exerted by the wall



$$-\delta\vec{F}_p + p_v W\delta x\,\hat{\imath}_z - p_H W\delta z\,\hat{\imath}_x = \vec{0}. \text{ If } p_v = p_H \Rightarrow \delta\vec{F}_p = Wp\{-\delta z\,\hat{\imath}_x + \delta x\,\hat{\imath}_z\}$$

3) Moments about O: From the geometry $\delta L = x\,\delta F_{pz} - z\,\delta F_{px}$ if positive δF_{px} and δF_{pz} act to the right and upwards respectively

$$\Rightarrow \delta L = \rho g W\left\{(D - z)x\,\delta x + (D - z)z\,\delta z\right\} = \rho g W\left\{(D - z)\cdot Kz^3\cdot 3Kz^2\delta z + (D - z)z\delta z\right\}$$

where we have used $x = Kz^3$ on the wall

$$\Rightarrow L = \rho g W\left\{\int_0^D \left\{3K^2\left[Dz^5 - z^6\right] + (Dz - z^2)\right\}dz = \rho g W\left\{\frac{K^2 D^7}{14} + \frac{D^3}{6}\right\}\right.$$

4) Applied Force: F_A: Since this horizontal, it has moment arm H, and

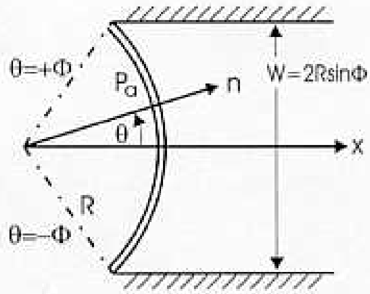
$$F_A = \frac{\rho g W}{H}\left\{\frac{K^2 D^7}{14} + \frac{D^3}{6}\right\}$$

Final Note: In calculating moments, we can use $\delta\vec{L} = \vec{r} \times \delta\vec{F}$

$$\Rightarrow \delta\vec{L} = \begin{bmatrix} \hat{\imath}_x & \hat{\imath}_y & \hat{\imath}_z \\ x & 0 & z \\ \delta F_x & 0 & \delta F_z \end{bmatrix} = (-x\delta F_z + z\delta F_x)\hat{\imath}_y,$$ which is the right sign since ↻ about O is in the −ve y direction

2.21

(2.14) [I changed notation, x points towards the barrier.]

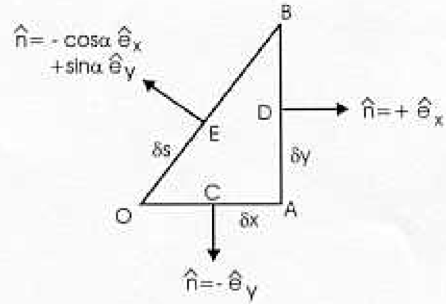


On barrier $\delta\vec{F}_P = -(p - Pa)\hat{n}\delta S$
Here
(i) $p = Pa + \rho g(H - z)$
(ii) $\hat{n} = \cos\theta\hat{i}_x + \sin\theta\hat{i}_y$
(iii) $\delta S = R\delta\theta\delta z$

$$\begin{aligned}
\Rightarrow \vec{F}_P &= -\int_{-\Phi}^{+\Phi}\int_0^H \rho g(H-z)\{\cos\theta\hat{i}_x + \sin\theta\hat{i}_y\}Rdzd\theta \\
&= -\{\rho g R\int_0^H (H-z)dz\int_{-\Phi}^{+\Phi}\cos\theta d\theta\}\hat{i}_x - \{\rho g R\int_0^H (H-z)ds\int_{-\Phi}^{+\Phi}\sin\theta d\theta\}\hat{i}_y \\
&= -\{\rho g R\frac{H^2}{2}\Big[\sin\theta\Big]_{-\Phi}^{+\Phi}\}\hat{i}_x + -\{\rho g R\frac{H^2}{2}\Big[\cos\theta\Big]_{-\Phi}^{+\Phi}\}\hat{i}_y \\
&= -\rho g R H^2\sin\Phi\hat{i}_x \\
\Rightarrow \vec{F}_P &= -\rho g\frac{H^2}{2}W\hat{i}_x \qquad \Big\{\text{NB } \rho g\frac{H}{2} \text{ is mean } (p - Pa)\text{, HW is projected area}\Big\}
\end{aligned}$$

122

(2.5) I use $p = p(x, y)$ [not as in question].



If volume is small enough

$$p(x, y) = p_0 + \left.\frac{\partial p}{\partial x}\right|_0 \delta x + \left.\frac{\partial p}{\partial y}\right|_0 \delta y + \cdots$$

To calculate pressure forces, apply hydrostatic axiom to three sides, using the average pressure on the three faces: (i.e. at midpoints C, D, and E)

$$\begin{aligned}
v\delta\vec{F}_P &= \delta\vec{F}_{P\ OA} + \delta\vec{F}_{P\ AB} + \delta\vec{F}_{P\ BO} = -p_C\hat{n}\delta S\Big|_{OA} - p_D\hat{n}\delta S\Big|_{AB} - p_E\hat{n}\delta S\Big|_{BO} \\
&= -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)(-\hat{e}_y)\delta x\delta y \\
&\quad -\left(p_0 + \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)(+\hat{e}_x)\delta y\delta z \\
&\quad -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2} + \frac{\partial p}{\partial y}\frac{\delta y}{2}\right)(-\cos\alpha\,\hat{e}_x + \sin\alpha\,\hat{e}_y)\delta y\delta z
\end{aligned}$$

Adding together $\hat{e}_x$ and $\hat{e}_y$ components gives

$$\begin{aligned}
\delta\hat{F}_P &= \Bigg\{-\left(p_0\delta y\delta z + \frac{\partial p}{\partial x}\delta x\delta y\delta z + \frac{\partial p}{\partial y}\frac{\partial y^2\partial z}{2}\right) + \Bigg(p_0\cos\alpha\,\delta s\delta z + \frac{\partial p}{\partial x}\cos\alpha\,\frac{\delta x\delta s\delta z}{2} \\
&\quad + \frac{\partial p}{\partial y}\cos\alpha\frac{\delta y\delta s\delta z}{2}\Bigg)\Bigg\}\hat{e}_x \\
&\quad \Bigg\{+\left(p_0\delta x\delta z + \frac{\partial p}{\partial x}\frac{\delta x^2\delta y}{2}\right) - \Bigg(p_0\sin\alpha\,\delta s\delta z + \frac{\partial p}{\partial x}\sin\alpha\,\frac{\delta x\delta s\delta z}{2} \\
&\quad + \frac{\partial p}{\partial y}\sin\alpha\frac{\delta y\delta s\delta z}{2}\Bigg)\Bigg\}\hat{e}_y
\end{aligned}$$

But $\delta x = \delta\sin\alpha$, $\delta y = \delta s\cos\alpha$, so that

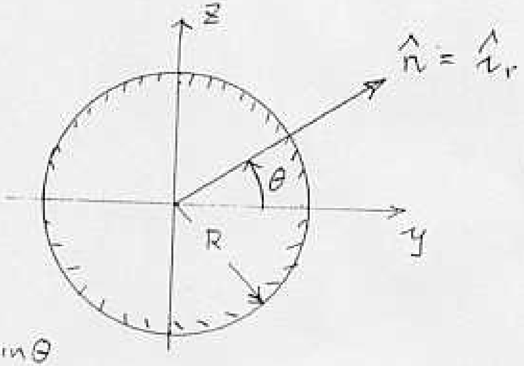
$$\begin{aligned}
\delta\vec{F}_P &= -\left.\frac{\partial p}{\partial x}\right|_0\frac{\delta x\delta y\delta z}{2}\hat{e}_x - \left.\frac{\partial p}{\partial y}\right|_0\frac{\delta x\delta y\delta z}{2}\hat{e}_y \\
&= -\nabla p\delta V
\end{aligned}$$

2.23

(i) On cylinder surface

$$\hat{n} = \hat{i}_r = \cos\theta\,\hat{i}_y + \sin\theta\,\hat{i}_z$$

$$\Rightarrow \delta\vec{F}_p = -p\left\{\cos\theta\,\hat{i}_y + \sin\theta\,\hat{i}_z\right\} RL\,\delta\theta$$



(ii) But here $p = p_0 + ay + bz$ and on surface, $p = p_0 + aR\cos\theta + bR\sin\theta$

(iii) $$\vec{F}_p = -\left[RL\int_{\theta=0}^{2\pi}\left\{p_0\cos\theta + aR\cos^2\theta + bR\cos\theta\sin\theta\right\}d\theta\right]\hat{i}_y$$

$$-\left[RL\int_{\theta=0}^{2\pi}\left\{p_0\sin\theta + aR\cos\theta\sin\theta + bR\sin^2\theta\right\}d\theta\right]\hat{i}_z$$

$$= -R^2L\left[\left\{a\int_0^{2\pi}\cos^2\theta\,d\theta\right\}\hat{i}_y + \left\{b\int_0^{2\pi}\sin^2\theta\,d\theta\right\}\hat{i}_z\right]$$

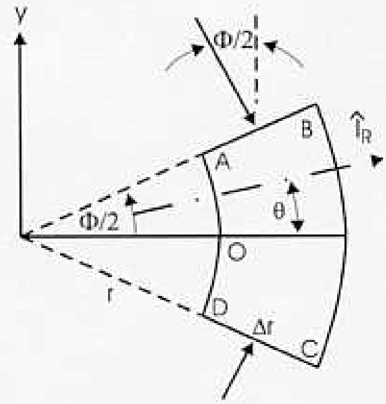
But $\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\sin^2\theta\,d\theta = \pi \Rightarrow \vec{F}_p = -\pi R^2 L\left\{a\hat{i}_y + b\hat{i}_z\right\}$

(iv) Finally, $\nabla p = a\hat{i}_x + b\hat{i}_y$ here and $V_{cyl} = \pi R^2 L \Rightarrow$

$$\vec{F}_p = -\nabla p\, V_{cyl}$$

2.24

(2.13) I have simplified this problem to illustrate the point I want to make by assuming that $p = p(r)$ alone and have re-oriented the volume so that it has an included angle Φ about the x-axis (and unit depth)



(1) With $\hat{i}_r = \cos\theta\hat{i}_x + \sin\theta\hat{i}_y$ calculate the resultant force on AD $\vec{F}_P \equiv \vec{F}_{P1}$

$$\begin{aligned}
\delta\vec{F}_P &= +p(r)r\delta\theta\{\cos\theta\hat{i}_x + \sin\theta\hat{i}_y\} \\
\Rightarrow \vec{F}_{P1} &= p(r)r\int_{-\frac{\Phi}{2}}^{+\frac{\Phi}{2}}(\cos\theta\hat{i}_x + \sin\theta\hat{i}_y) \\
&= p(r)r\Big[\sin\theta\hat{i}_x - \cos\theta\hat{i}_y\Big]_{-\frac{\Phi}{2}}^{+\frac{\Phi}{2}} \\
\vec{F}_{P1} &= 2p(r)r\sin\frac{\Phi}{2}\hat{i}_x
\end{aligned}$$

2) By same arguement, on BC, the force $\vec{F}_{P2}$ is

$$\vec{F}_{P2} = -2p(r+\Delta r)(r+\Delta r)\sin\frac{\Phi}{2}\hat{i}_x$$

3) Thus, as $\Delta r \to 0$, we put $p(r+\Delta r) \to p_0 + \frac{dp}{dr}|_0\Delta r$,
and as $\Phi \to 0$ we put $\sin\frac{\Phi}{2} \to \frac{\Phi}{2}$ and

$$\begin{aligned}
\vec{F}_{P1} + \vec{F}_{P2} &\to 2\Big\{p_0 r - (p_0 + \frac{dp}{dr}\Big|_0\Delta r)(r+\Delta r)\Big\}\frac{\Phi}{2}\hat{i}_x \\
&\to \Big\{p_0 r - p_0 r - \frac{dp}{dr}\Big|_0 r\Delta r - p_0\Delta r - \frac{dp}{dr}\Big|_0\Delta r^2\Big\}\Phi\hat{i}_x \\
&\to -\Big(\frac{dp}{dr}\Big|_0 r + p_0\Big)\Delta r\Phi\hat{i}_x
\end{aligned}$$

4) But the pressure on the sides AB and CD have components in the $\hat{i}_x$ direction; with the force on AB being $\vec{F}_{P3}$, as $\Delta r \to 0$, taking the average pressure on AB as $(p_0 + \frac{dp}{dr}|_0\frac{\Delta r}{2})$

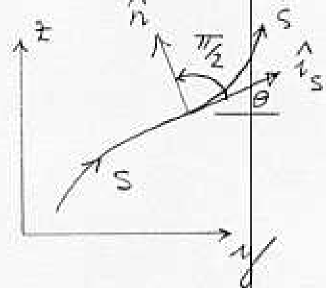
$$\vec{F}_{P3} \to (p_0 + \frac{dp}{dr}\Big|_0\frac{\Delta r}{2})\Delta r\{\sin\frac{\Phi}{2}\hat{i}_x - \cos\frac{\Phi}{2}\hat{i}_y\}$$

As $\Delta r \to 0$ and $\Phi \to 0$ thus has component $\vec{F}_{P3}\cdot\hat{i}_x \to p_0\Delta r\frac{\Phi}{2}$.

5) Adding all contributions we have v

$$\vec{F}_P\cdot\hat{i}_x \to (\frac{dp}{dr}\Big|_0 r + p_0)\Delta r\Phi + 2p_0\Delta r\frac{\Phi}{2} = -\frac{dp}{dr}\Big|_0 r\Delta r\Phi = -\frac{dp}{dr}\Big|_0\Delta V$$

2.25



(i) $\hat{i}_s = \cos\theta\,\hat{i}_y + \sin\theta\,\hat{i}_z$

$\Rightarrow \hat{n} = \cos\left(\theta+\frac{\pi}{2}\right)\hat{i}_y + \sin\left(\theta+\frac{\pi}{2}\right)\hat{i}_z$

$= -\sin\theta\,\hat{i}_y + \cos\theta\,\hat{i}_z$

But $\cos\theta = \frac{dy_w}{ds}$ $\sin\theta = \frac{dz_w}{ds}$

$\Rightarrow \hat{n} = -\frac{dz_w}{ds}\hat{i}_y + \frac{dy_w}{ds}\hat{i}_s$

(ii) In liquid $p = P_a + \rho g(H-z) \Rightarrow$ on z_s

$\delta\vec{F}_p = -(p-P_a)\hat{n}W\delta s = -\rho g(H-z_w)\left\{-\frac{dz_w}{ds}\hat{i}_y + \frac{dy_w}{ds}\hat{i}_z\right\}W\delta s$

With $0 \leq s \leq s_L$

$\vec{F}_p = \left\{\rho g W\int_0^{s_L}(H-z_w)\frac{dz_w}{ds}ds\right\}\hat{i}_y - \left\{\rho g W\int_0^{s_L}(H-z_w)\frac{dy_w}{ds}\right\}\hat{i}_z$

(iii) By removing the parametrized forms these can be written as

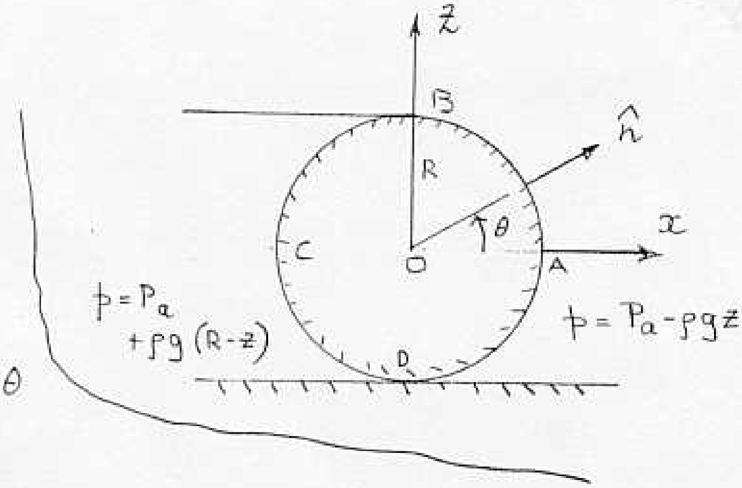
$\vec{F}_p = \left\{\rho g W\int_0^H(H-z)dz\right\}\hat{i}_y - \left\{\rho g W\int_0^L(H-z_w(y))dy\right\}\hat{i}_z$

If V = volume of ABCD, $V = \int_0^L(H-z_w)dy$, so

$\vec{F}_p = \{\rho g W H^2/2\}\,\hat{i}_y - \{\rho g W V\}\,\hat{i}_z$

The first term is precisely the force acting on the line AD as if it were the wall and the second term

2.26



With $\hat{n} = \cos\theta\, \hat{i}_x + \sin\theta\, \hat{i}_z$ on cylinder surface

$$\delta \vec{F}_p = -p\hat{n}\,\delta S$$
$$= -p\{\cos\theta\,\hat{i}_x + \sin\theta\,\hat{i}_z\} RB\,\delta\theta$$

$$\Rightarrow F_{pz} = -RB\int_{\theta=0}^{2\pi} p\sin\theta\, d\theta$$

Here on cyl'r:
(1) $p = P_a, \quad 0 \leq \theta \leq \pi/2$
(2) $p = P_a + \rho g(R - z) = P_a + \rho g R\{1 - \sin\theta\}, \quad \pi/2 \leq \theta \leq 3\pi/2$
(3) $p = P_a - \rho g z = P_a - \rho g R\sin\theta, \quad 3\pi/2 \leq \theta \leq 2\pi$

We write $F_{pz} = -RB\int_{\theta=0}^{2\pi}(P_a + \Delta p)\sin\theta\, d\theta$

$$= -RBP_a\int_{\theta=0}^{2\pi}\sin\theta\, d\theta - RB\int_{\theta=0}^{2\pi}\Delta p\sin\theta\, d\theta = -RB\int_{\theta=0}^{2\pi}\Delta p\sin\theta\, d\theta$$

$$\Rightarrow F_{pz} = -\rho g R^2 B\left\{\int_{\theta=\pi/2}^{3\pi/2}(\sin\theta - \sin^2\theta)\, d\theta - \int_{\theta=3\pi/2}^{2\pi}\sin^2\theta\, d\theta\right\}$$

$$= \rho g R^2 B\left\{\int_{\theta=\pi/2}^{2\pi}\sin^2\theta\, d\theta - \int_{\theta=\pi/2}^{3\pi/2}\sin\theta\, d\theta\right\}$$

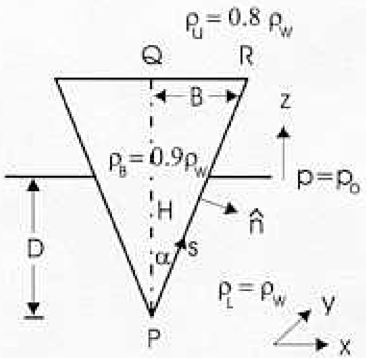
$$= \rho g R^2 B\left\{\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_{\pi/2}^{2\pi} + \left[\cos\theta\right]_{\pi/2}^{3\pi/2}\right\} = \frac{3\pi}{4}\rho g R^2 B$$

But $V_{cyl} = \pi R^2 B \Rightarrow F_{pz} = \frac{3}{4}\rho g V = W$

This result can be obtained by use of Stevin's principle of solidification and Archimedes' principle. Separating the body into the 1/4 cyl'r OAD and the 1/2 cyl'r BCD, one can use these principles to determine the forces on the two parts as if they were separated, and then add the results, obtaining the above formula ...

2.2a (2.10)

(a) Use symmetry to simplify: resultant pressure forces on PR and QR balance the weight in PQR, and use unit length in y-direction



(b) With $z=0$ at in interface;

$$\begin{aligned} p &= p_0 - \rho_U g z, \qquad z \geq 0 \\ p &= p_0 - \rho_L g z, \qquad z \geq 0 \end{aligned}$$

(c) On face PR, $\hat{n} = \cos\alpha\,\hat{i}_x - \sin\alpha\,\hat{i}_z$

$$\begin{aligned} \delta\vec{F}_P &= -p\hat{n}\delta S = -p\{\cos\alpha\,\hat{i}_x - \sin\alpha\,\hat{i}_z\}\delta S \\ \delta F_{pz} &= \delta\vec{F}_P\cdot\hat{i}_z = p\sin\alpha\,\delta S = p\sin\alpha\,\frac{\delta z}{\cos\alpha} = p\tan\alpha\,\delta z \end{aligned}$$

With $\tan\alpha = B/H$,

$$\begin{aligned} F_{pz} &= \frac{B}{H}\Big\{\int_{-D}^{0}(p_0-\rho_L g z)dz + \int_0^{H-D}(p_0 - \rho_U g z)dz\Big\} \\ &= \frac{B}{H}\Big\{p_0 H + \rho_L g\frac{D^2}{2} - \rho_U g\frac{(H-D)^2}{2}\Big\} \qquad \text{on PQ} \end{aligned}$$

(d) On face QR, $p = p_0 - \rho_U g(H-D) \Rightarrow F_{pz} = -p_0 B + \rho_U B g(H-D)$ on QR

(e) With the weight of the body in PQR being $\rho_B g\frac{BH}{2}$ we have

$$\rho_B g\frac{BH}{2} = \frac{B}{H}\Big\{p_0 H + \rho_L g\frac{D^2}{2} - \rho_U g\frac{(H-D)^2}{2}\Big\} - p_0 B + \rho_U B g(H-D)$$

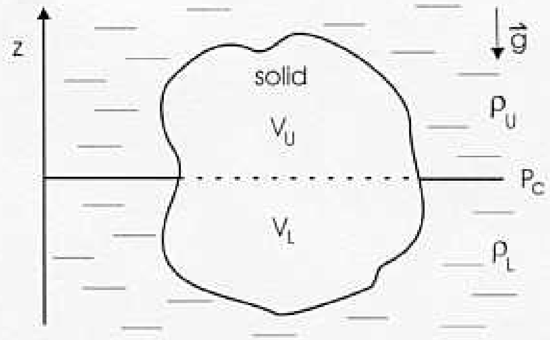
(f) Solve:

$$0.9\frac{H}{2} = \frac{D^2}{2H} - 0.8\frac{(H-D)^2}{2H} + 0.8(H-D)$$

$$\Rightarrow \frac{D}{H} = \frac{1}{\sqrt{2}}$$

2.30

(2.16) First derive the equivalent of Archimedes' principle for a solid floating at the interface, which we set at $z = 0$. We do this by noting that, according to $\mathcal{P}$II the force on the solid is the same as that on a lump of liquid having the same shape and floating in the same position, in particular for a liquid lump consisting of the upper liquid in that part of V above $z = 0$, V_U, and the lower liquid in $V_L = V - V_U$.

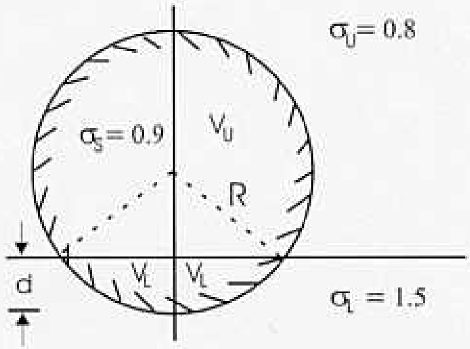


Hence
$$\vec{F}_p = -\int_S p\hat{n}dS = -\int_V \nabla \; dV = -\int_{V_U} dV - \int_{V_L} \nabla p dV$$

But $\nabla p = -\rho_U g$ in V_U and $\nabla p = -\rho_L g = V_L$ so

$$\vec{F}_p = (\rho_U g V_U + \rho_L g V_L)\,\hat{e}_z = \boxed{g\left\{\rho_U V_U + \rho_L (V - V_U)\right\}\hat{e}_z = \vec{F}_p}$$

Now consider the cylinder.



With $\rho = \sigma\rho_w$, where ρ_w is the density of water, the equilibrium condition is,

$$\sigma_u \rho_w g V_U + \sigma_L \rho_w g V_L - \sigma_S \rho_w g(V_U + V_L) = 0 \text{or} 0.8V_U + 1.5V_L - 0.9(V_U + V_L) = 0$$

This can be solved for $\frac{V_L}{V} = \eta \Rightarrow 0.8(1-\eta) + 1.5\eta = 0.9$ or $\boxed{\eta = \frac{1}{7}}$ Thus, regardless of its shape, an object having $\sigma_S = 0.9$ must float at the interface satisfying this requirement. For cylinder, we cannot obtain an explicit solution:

$$\frac{V_L}{V} = \frac{R^2\left\{\theta - \sin\theta\cos\theta\right\}}{\pi R^2} \Rightarrow \boxed{\frac{\theta - \sin\theta\cos\theta}{\pi} = \frac{1}{7}}$$

so we must solve numerically. A bisection technique gives $\theta = 0.9285$, and $\frac{d}{R} = 1 - \cos\theta = 0.401$

2.32

(2.8) 1. *Equation of Motion*

Given $\delta\vec{f}_p + \delta\vec{f}_b = \delta m\vec{a}$; for a particle in arbitrarily shaped δV; with $\delta m = \rho\delta V$ and by Theorem 3.1 and definition of a body force:

$$-\nabla p\delta V + \rho\vec{g}\delta V = \rho\delta V\vec{a} \quad \Rightarrow \quad \frac{-\nabla p}{\rho} + \vec{g} = \vec{a}$$

$$\text{or} \qquad \nabla p = \rho(\vec{g} - \vec{a})$$

2. *Implementation* Here $\vec{g} = -g\hat{\imath}_z$, $\vec{a} = a_x\hat{\imath}_z - a_z\hat{\imath}_z$, so

$$\frac{\partial p}{\partial x}\hat{\imath}_x + \frac{\partial p}{\partial y}\hat{\imath}_y + \frac{\partial p}{\partial z}\hat{\imath}_z = \rho\left(-a_x\hat{\imath}_x - a_y\hat{\imath}_y - a_z\hat{\imath}_z\right)$$

$$\begin{array}{rcl} \frac{\partial p}{\partial x} = & -\rho a_x & \rightarrow p = -\rho a_x x + f(y,z) \\ \frac{\partial p}{\partial y} = & 0 & \rightarrow p = f(x,z) \\ \frac{\partial p}{\partial z} = & -\rho(a_z+g) & \rightarrow p = -\rho(a_z+g)z + f(x,y) \end{array}$$

$$\Rightarrow p = -\rho\left\{a_x x + (a_z+g)\,z\right\} + c$$

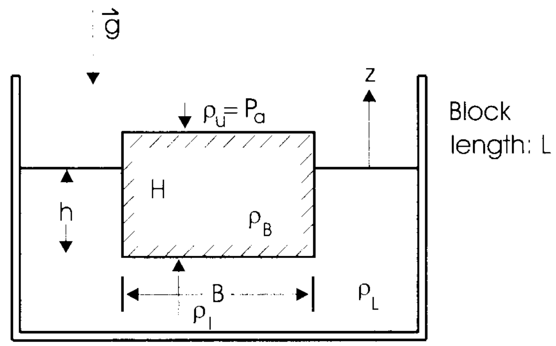
3. *Fixing constant*: $p = p_a$ at $x = z = L \Rightarrow$

$$p_a = -\rho\left(a_x + a_z + g\right)L + C \qquad \Rightarrow \qquad C + p_a + \rho\left(a_x + a_z + g\right)L$$

$$\boxed{p = p_a + \rho\left(a_x\left[L - x\right] + \left(a_z + g\right)\left[L - z\right]\right)}$$

4.35

(2.15) Consider the case when accelerating upwards since the non-accelerating case can be calculated by setting acceleration $a = 0$.



Equation of motion for block in $\hat{e}_x$ direction:

$$p_e BL - P_A BL - \rho_B BLHg = \rho_B BLHa$$
$$p_e - Pa = \rho_B(a+g)H$$

Equation of motion for a fluid particle of arbitrary shape δV

$$\delta\vec{F}_p + \delta\vec{F}_b = \delta m_p \vec{a}$$
$$-\nabla p\delta V + \rho_L \delta V \vec{g} = \rho_L \delta V \vec{a}$$
$$-\frac{\nabla p}{\rho_L} + \vec{g} = \vec{a}$$

or, here $$-\frac{\nabla p}{\rho_L} + g\hat{e}_z = a\hat{e}_z \Rightarrow \frac{\partial p}{\partial z} = -\rho_L(a+g)$$

so $$p = Pa - \rho_L(a+g)z \qquad \text{for } z < 0$$

Combine these two, $p = p_e$ at $z = -h$, so

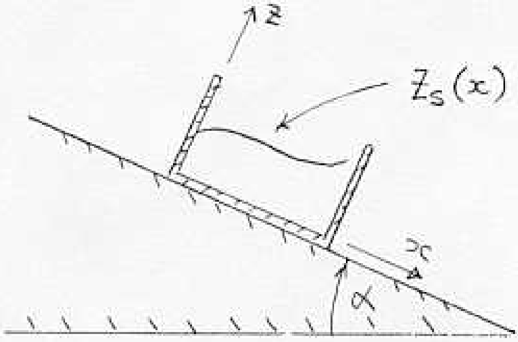
$$p_e - Pa = \rho(a+g)h$$
$$\rho_L(a+g)h = \rho_B(a+g)H \quad \rightarrow \quad \frac{h}{H} = \frac{\rho_B}{\rho_L}$$

for all accelerations. Thus h is independent of a.

2.36

(i) With the co-ordinate system attached to the box as shown

$\vec{g} = -g\cos\alpha\,\hat{i}_z + g\sin\alpha\,\hat{i}_x$



(ii) Solving for the box-liquid combination as if it were a single body, if $F_N \hat{i}_z$ is the reaction force that the plane exerts on the box, then the retarding force acting up the plane due to friction is $-k_f F_N \hat{i}_x$. Thus we have

$$\left.\begin{aligned} \hat{i}_x &: \; Mg\sin\alpha - k_f F_s = M a_{box} \\ \hat{i}_z &: \; -Mg\cos\alpha + F_N = 0 \end{aligned}\right\} \quad a_{box} = g(\sin\alpha - k_f\cos\alpha)$$

(iii) We assume that all the particles accelerate at a_{box}, so

$$-\frac{\nabla p}{\rho} + \vec{g} = \vec{a} \Rightarrow -\frac{1}{\rho}\left\{\frac{\partial p}{\partial x}\hat{i}_x + \frac{\partial p}{\partial y}\hat{i}_y + \frac{\partial p}{\partial z}\hat{i}_z\right\} + g(\sin\alpha\,\hat{i}_x - \cos\alpha\,\hat{i}_z)$$
$$= g(\sin\alpha - k_f\cos\alpha)\hat{i}_x$$

$$\Rightarrow \quad \hat{i}_x : -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\alpha = g\sin\alpha - g k_f\cos\alpha \Rightarrow p = \rho g k_f \cos\alpha\, x + f_1(y,z)$$

$$\hat{i}_y : \partial p/\partial y = 0 \Rightarrow p = f_2(x,z) \text{ only}$$

$$\hat{i}_z : -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\cos\alpha = 0 \Rightarrow p = -\rho g\cos\alpha\, z + f_3(x,y)$$

$$\Rightarrow \quad p = \rho g\cos\alpha\{k_f x - z\} + C$$

(iv) $p = P_a$ at $z = Z_s(x) \Rightarrow P_a = \rho g\cos\alpha\{k_f x - Z_s(x)\} + C$

$$\Rightarrow \quad Z_s(x) = k_f x + D, \quad D = (P_a - C)/(\rho g\cos\alpha)$$

To find D, assuming no spillage

$$B\int_0^L Z_s(x)\,dx = \frac{BLH}{2} \Rightarrow D = \frac{H - k_f L}{2}$$

$$\Rightarrow \boxed{Z_s = k_f x + \frac{H - k_f L}{2}}$$

2.37 (2.23)

$$\delta \mathbf{F}_p + \delta \mathbf{F}_g = \delta m_p \mathbf{a}$$
$$-\nabla p \delta \mathcal{V} + \rho \mathbf{g} \delta \mathcal{V} = \rho \mathbf{a} \delta \mathcal{V}$$

The acceleration due to the rotating table is $\mathbf{a} = -r\Omega^2 \hat{\mathbf{n}}_r$. Owing to cylindrical symmetry, $\dfrac{\partial f}{\partial \theta} = 0$,

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}\hat{\mathbf{n}}_r - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{\mathbf{n}}_z - g\hat{\mathbf{n}}_z = -r\Omega^2 \hat{\mathbf{n}}_r.$$

Integrate to find the pressure distribution:

$$\left.\begin{array}{lll} \dfrac{\partial p}{\partial r} = \rho r \Omega^2 & \Rightarrow & p(r,z) = \rho \dfrac{r^2}{2}\Omega^2 + f(z) \\ \dfrac{\partial p}{\partial z} = -\rho g & \Rightarrow & p(r,z) = -\rho g z + f(r) \end{array}\right\} \quad p(r,z) = \rho \frac{r^2}{2}\Omega^2 - \rho g z + C.$$

The pressure is atmospheric at the surface in the column. Since the diameter of the tube is small, we can ignore the shape of the surface.

$$p(0, z = H + D) = p_a = -\rho g(H + D) + C \quad \Rightarrow \quad C = p_a + \rho g(H + D).$$

Therefore, the pressure distribution is given by

$$p(r,z) = p_a + \rho\left[g\big(H + D - z\big) + \frac{r^2\Omega^2}{2}\right].$$

The pressure at the points defined above are:

$$A : p(R, 0) = p_a + \rho\left[g\big(H + D\big) + \frac{R^2\Omega^2}{2}\right],$$
$$B : p(R, H) = p_a + \rho\left[gD + \frac{R^2\Omega^2}{2}\right],$$
$$C : p(0, H) = p_a + \rho g D.$$

2.38

Regardless of the shape of the vessel, the pressure distribution in such a rotating system is given by

$$p = \rho\left[\frac{\omega^2 r^2}{2} - gz\right] + C$$

In the leg at radius 2R, $p = P_a$ at $r = 2R$, $z = h_2$

$$\Rightarrow P_a = \rho\left[2\omega^2 R^2 - g h_2\right] + C$$

Similarly, in the leg at radius R, $p = P_a$ at $r = R$, $z = h_1$

$$\Rightarrow P_a = \rho\left[\frac{\omega^2 R^2}{2} - g h_1\right] + C$$

Thus

$$\rho\left[2\omega^2 R^2 - g h_2\right] + C = \rho\left[\frac{\omega^2}{2} R^2 - g h_1\right] + C$$

$$\Rightarrow \boxed{h_2 - h_1 = \frac{3}{2}\frac{R^2\omega^2}{g}}$$

But, assuming the vertical legs of the tube to have uniform bore of the same diameter, since the volume of the liquid in the horizontal portion and in the elbows does not change as a result of the motion, if no spillage occurs

$$\boxed{h_1 + h_2 = 2H}$$

Solving these two equations gives

$$h_2 = H + \frac{3}{4}\frac{R^2\omega^2}{g} \ ; \ h_1 = H - \frac{3}{4}\frac{R^2\omega^2}{g}$$

2.39 (2.11) 1. <u>Equation of Motion: Both Fluids</u>
From T@A for arbitrary δV

$$\begin{aligned} \delta\vec{F}_p + \delta\vec{F}_b &= \delta m\vec{a} \\ -\nabla p\delta V + \rho\delta V\vec{g} &= \rho\delta V\vec{a} \end{aligned}$$

$$\boxed{-\frac{\nabla p}{\rho} + \vec{g} = \vec{a}}$$

Here because of expected axial symmetry $\nabla p = \vec{\partial p \partial r}\hat{e}_r + \frac{\partial p}{\partial z}\hat{e}_z$

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}\hat{e}_r - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{e}_z - g\hat{e}_z = -r\Omega^2\hat{e}_r$$

$$\boxed{\frac{\partial p}{\partial r} = \rho r\Omega^2 \quad , \quad \frac{\partial p}{\partial z} = -\rho g}$$

2. <u>Pressure Field</u>
$p = \frac{\rho r^2\Omega^2}{2} + f_1(z)$, $p = -\rho g z + f_2(r)$, $f_1(z) = f_2(r) = C$

$$p = \rho r\Omega^2 - \rho g z + C$$

Hence in upper liquid $p_u = \rho_u\Omega_u^2 r^2 - \rho_u g z + C_u$
lower liquid $p_l = \rho_l\Omega_l^2 r^2 - \rho_l g z + C_l$

3. <u>Equation of Interface</u> At $z = Z_s(r)$, $p_u = p_l$

$$\rho_u\Omega_u^2 r^2 - \rho_u g Z_s + C_u = \rho_l\Omega_l^2 r^2 - \rho_l g Z_s + C_l$$

$$Z_s(r) = \frac{1}{2}\left(\frac{\rho_l\Omega_l^2 - \rho_u\Omega_u^2}{(\rho_l - \rho_u)}\right) r^2 + \frac{C_l - C_u}{(\rho_l - \rho_u)\, g}$$

or $Z_s(r) = \Gamma^2 + C_1$ where $\Gamma = \left\{\frac{\rho_l\Omega_l^2 - \rho_u\Omega_u^2}{2(\rho_l - \rho_u)g}\right\}$

4. <u>Constant of Integration</u>
Volume of lower fluid remains fixed.

$$\int_0^R 2\pi\left[\Gamma r^2 + C_1\right] r\, dr = \pi R^2 H \quad \rightarrow \quad \frac{\Gamma R^4}{2} + C_1 R^2 = R^2 H$$

$$\boxed{Z_s(r) = \frac{\rho_l\Omega_l^2 - \rho_u\Omega_u^2}{g(\rho_l - \rho_u)}\left\{\frac{r^2}{2} - \frac{R^2}{4}\right\} + H}$$