University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test 1
9:10 – 10:55, 17 October 2022
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1. Evaluate the following limits if they exist. Indicate the limit laws used in your solution.

(a) $\lim_{x\to5}\frac{x^2-6x+5}{x-5}$ (b) $\lim_{x\to5}\frac{x^2-5x+6}{x-5}$ (c) $\lim_{h\to0}\frac{(-5+h)^2-25}{h}$

(d) $\lim_{h\to0}\frac{\sqrt{1+h}-\sqrt{1-h}}{h}$ (e) $\lim_{h\to0}\frac{(x+h)^3-x^3}{h}$

a) $\lim_{x\to5}\frac{x^2-6x+5}{x-5} = \lim_{x\to5}\frac{(x-5)(x-1)}{x-5}$ sum, difference & product rules

$= \lim_{x\to5}(x-1)$ cancel common factor

direct substitution

$= 4$

b) $\lim_{x\to5}\frac{x^2-5x+6}{x-5} = \lim_{x\to5}\frac{(x-3)(x-2)}{x-5}$ sum, difference & product rules

quotient rule

$= 6\lim_{x\to5}\frac{1}{x-5}$ direct substitution

infinite limit

$\longrightarrow \infty$

c) $\lim_{h\to0}\frac{(-5+h)^2-25}{h} = \lim_{h\to0}\frac{25-10h+h^2-25}{h}$ sum & difference rules

cancel common factor

$= \lim_{h\to0} -10+h$

direct substitution

$= -10$

d) $\lim_{h\to0}\frac{\sqrt{1+h}-\sqrt{1-h}}{h} = \lim_{h\to0}\frac{(1+h)-(1-h)}{h(\sqrt{1+h}+\sqrt{1-h})}$

cancel common factor

$= \lim_{h\to0}\frac{2h}{h(\sqrt{1+h}+\sqrt{1-h})}$ quotient law

root law

$= \lim_{h\to0}\frac{2}{\sqrt{1+h}+\sqrt{1-h}}$ direct substitution

$= 1$

e) $\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$ sum & product rules

$= \lim_{h \to 0} \frac{(3x^2 + 3xh + h^2)h}{h}$ cancel common factor

$= \lim_{h \to 0} 3x^2 + 3xh + h^2$ power rule

$= 3x^2$ direct substitution

2. Calculate the derivative of the following functions, citing all theorems used:

(a) $f(x) = 2x^3$ (b) $f(x) = 3/x^2$ (c) $f(x) = 2\cos(-3x)$

(d) $f(x) = 3\cos^2(2x^2)$ (e) $f(x) = (1 + x)/(2 - x)^2$

a) $f(x) = 2x^3 \Rightarrow f'(x) = 6x^2$

constant multiplier rule
power rule

b) $f(x) = \frac{3}{x^2} \Rightarrow f'(x) = 3\cdot(-2)x^{-3}$

$= \frac{-6}{x^3}$

constant multiplier
general power rule

c) $f(x) = 2\cos(-3x) \Rightarrow f'(x) = 2.(-\sin(-3x))\cdot(-3)$

$= 6\sin(-3x)$

- chain rule with constant multiplier
- basic trig derivative

d) $f(x) = 3\cos^2(2x^2) \Rightarrow f'(x) = 3\cdot 2\cos(2x^2)(-\sin(2x^2))\cdot 4x$

$= -24x\cos(2x^2)\sin(2x^2)$

- power rule, basic trig derivative, chain rule
- constant multiplier & power rule

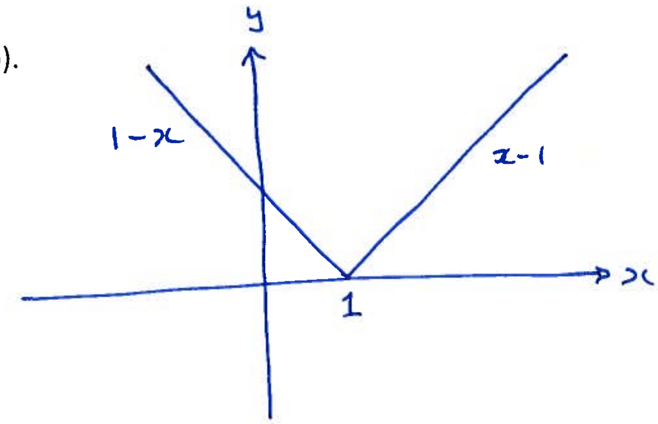
e) $f(x) = \frac{1+x}{(2-x)^2} \Rightarrow f'(x) = \frac{(2-x)^2\cdot 1 - (1+x).2(2-x)(-1)}{(2-x)^4}$

$= \frac{2-x+2(1+x)}{(2-x)^3} = \frac{4+x}{(2-x)^3}$

- quotient rule, power rule, chain rule

3. For $f(x) = |1-x|$
   a) Sketch $f(x)$.
   b) What is the value of $\lim_{x\to 1} f(x)$?
   c) Use a $\varepsilon - \delta$ type of proof to justify your answer to b).

a) $f(x) = |1-x| = \begin{cases} 1-x, & x<1 \\ x-1, & x \geq 1 \end{cases}$



b) $\lim_{x\to 1^-} |1-x| = \lim_{x\to 1^-} (1-x) = 0$

$\lim_{x\to 1^+} |1-x| = \lim_{x\to 1^+} (x-1) = 0$

$\therefore \lim_{x\to 1} |1-x| = 0$

c) $\lim_{x\to 1^-} |1-x|$: $x<1$ $\therefore |1-x| = 1-x$

given $\epsilon > 0$, find $\delta > 0$ st $|f(x) - 0| < \epsilon$ for $0 < 1-x < \delta$

$\Rightarrow | |1-x| - 0 | = 1-x < \epsilon$ $\Rightarrow$ choose $\delta = \epsilon$

$\Rightarrow$ given $1-x < \delta$ $\therefore |f(x)-0| = |1-x| = 1-x < \delta = \epsilon$

$\therefore$ by the definition of a one-sided limit:

$\lim_{x\to 1^-} |1-x| = 0$

$\lim_{x\to 1^+} |1-x|$: $x>1$ $\therefore |1-x| = x-1$

given $\epsilon > 0$, find $\delta > 0$ s.t. $|f(x) - 0| < \epsilon$ for $0 < x-1 < \delta$

$\Rightarrow | |1-x| - 0 | = x-1 < \epsilon$ $\Rightarrow$ choose $\delta = \epsilon$

$\Rightarrow$ given $x-1 < \delta$ $\therefore |f(x)-0| = |1-x| = x-1 < \delta = \epsilon$

$\therefore$ by the definition of a one-sided limit.

$\lim_{x\to 1^+} |1-x| = 0$

4. Solve:

a) $|x-3| < |x+5|$

b) $\frac{x-1}{x+1} > 1$

a) $|x-3| < |x+5|$

$\Rightarrow x > 3 \Rightarrow x-3 < x+5 \Rightarrow -3 < 5$ always Ok

$\Rightarrow -5 < x < 3 \Rightarrow 3-x < x+5 \Rightarrow -2x < 2 \Rightarrow x > -1$

$\Rightarrow x < -5 \Rightarrow 3-x < -x-5 \Rightarrow 3 < -5$ never Ok

$\Rightarrow$ test $x = 3 \Rightarrow 0 < 8$ Ok

$\therefore \; x \in (-1, \infty)$

b) $\frac{x-1}{x+1} > 1 \Rightarrow \frac{x-1}{x+1} - 1 > 0$

$\Rightarrow \frac{x-1 \;\; -x-1}{x+1} > 0$

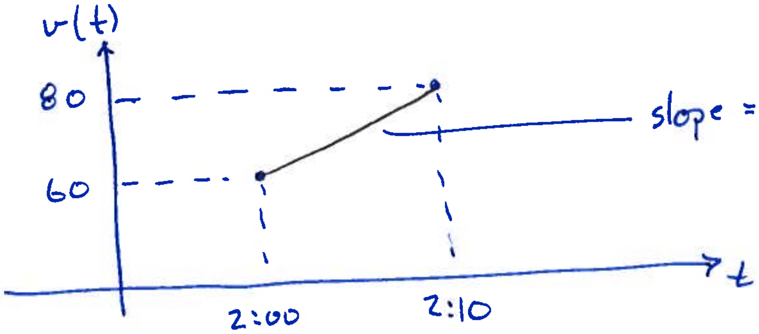
$\Rightarrow \frac{-2}{x+1} > 0 \Rightarrow$ true for $x+1 < 0$

or $x < -1$

5. a) At 2:00 pm a car's speedometer reads 60 km/hr. At 2:10 pm it reads 80 km/hr. Show that at some time between 2:00 pm and 2:10 pm the acceleration is exactly 120 km/hr^2.

b) Suppose f is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

a)



slope $= \dfrac{80-60 \text{ km/hr}}{1/6 \text{ hr}} = 120 \text{ km/hr}^2$

- The question describes a physical system, thus we assume that $v(t)$ is continuous and differentiable.
- $\therefore$ by MVT there is a time $t_0 \in (2{:}00, 2{:}10)$ where $v'(t) =$ average slope $= 120 \text{ km/hr}^2$.
- Given $v'(t) =$ acceleration, we have $a(t_0) = 120 \text{ km/hr}^2$

b) The function f satisfies the requirements of the MVT:

$\therefore$ there is a number c in $(-b, b)$ where:

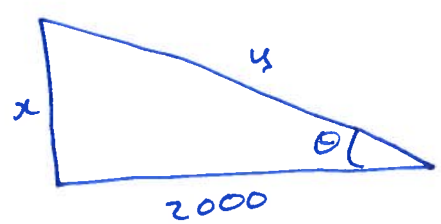
$$f'(c) = \frac{f(b) - f(-b)}{b-(-b)}$$

Given f is an odd function : $f(-b) = -f(b)$

$$\therefore f'(c) = \frac{f(b)}{b}$$

6. A television camera is positioned 2000 m from the base of a rocket launch pad. A rocket rises vertically, and its speed is 200 m/s at the point when it has risen 1000 m from the ground.
   a) How fast is the distance from the camera to the rocket changing at that moment?
   b) It the camera is always directed at the rocket, how fast is the camera's angle of elevation changing at that same moment?

a)



Given $\frac{dx}{dt} = 200$ m/s, find $\frac{dy}{dt}$

$$y = \sqrt{x^2 + 2000^2}$$

$$y^2 = x^2 + 2000^2 \Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{2x}{2y}\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt}(x=1000) = \frac{1000}{\sqrt{1000^2 + 2000^2}} \cdot 200 = \frac{200}{\sqrt{5}} \text{ m/s}$$

b) $$\tan\theta = \frac{x}{2000} \Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{2000}\frac{dx}{dt}$$

$$\Rightarrow \frac{d\theta}{dt}(x=1000) = \frac{200}{2000} \cdot \cos^2\theta = \frac{1}{10}\left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{50} \text{ rad/s}$$

Note: $\cos\theta = \frac{2000}{\sqrt{1000^2 + 2000^2}} = \frac{2}{\sqrt{5}}$

7. For some parameters a and b, define:

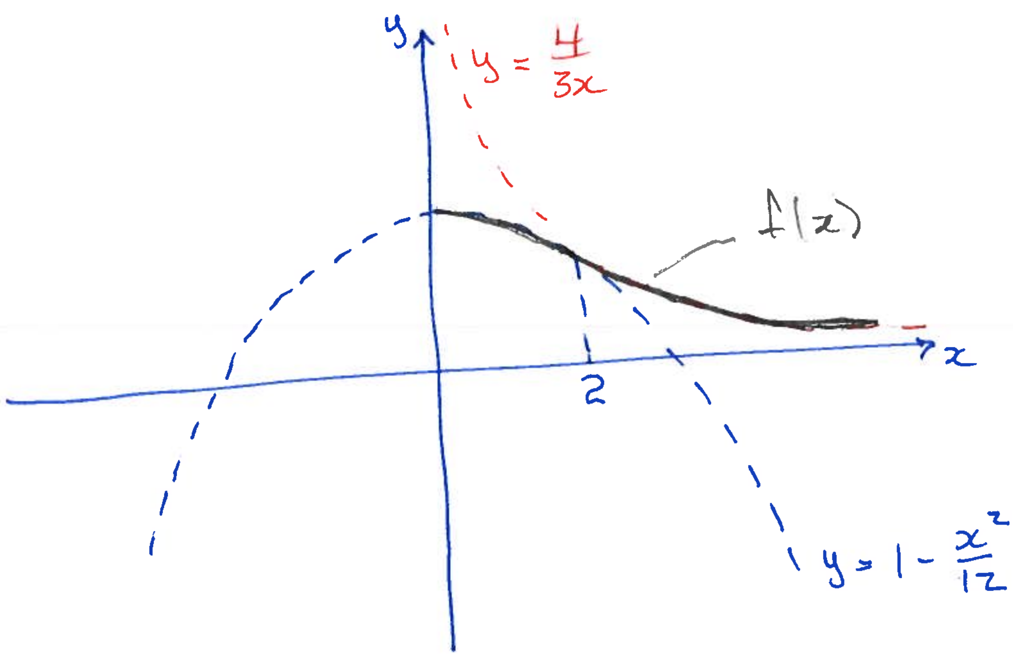
$$f(x) = \begin{cases} 1 - ax^2 & (0 \le x \le 2) \\ b/x & (2 < x) \end{cases}$$

How should a and b be chosen so that $f(x)$ is continuous and differentiable at $x = 2$? Sketch the graph of $y = f(x)$ for these values of the parameters.

continuous: $1 - a \cdot 2^2 = b/2 \Rightarrow b = 2 - 8a$

differenticble: $-2ax = -b/x^2 \Rightarrow -4a = -b/4 \Rightarrow b = 16a$

$16a = 2 - 8a \Rightarrow a = \frac{1}{12} \therefore b = \frac{16}{12} = \frac{4}{3}$



8. Use the definition of the derivative to prove the quotient rule: (Do not use the Reciprocal Function Derivative Theorem.)

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g\prime(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{h \cdot g(x) \cdot g(x+h)}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h\, g(x)\, g(x+h)}$$

$$= \lim_{h\to 0} \left[\frac{g(x)}{g(x)g(x+h)} \cdot \frac{f(x+h)-f(x)}{h} - \frac{f(x)}{g(x)g(x+h)} \cdot \frac{g(x+h)-g(x)}{h}\right]$$

$$= \left[\lim_{h\to 0} \frac{1}{g(x)g(x+h)}\right]\left[g(x) \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - f(x) \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right]$$

$$= \frac{1}{[g(x)]^2}\left(g(x)f'(x) - f(x)g'(x)\right)$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

9. Let $f(x) = x^p(1-x)^q$, where $p \geq 2$ and $q \geq 2$ are integers.

a) Show that the critical numbers of f are: $x = 0,\ p/(p+q)$ and 1.

b) Show that if p is even, then f has a local minimum at 0.

c) Show that if q is even, then f has a local minimum at 1.

d) Use the second derivative test to show that f has a local maximum at $p/(p+q)$ for all p and q.

a) $f'(x) = p x^{p-1}(1-x)^q - x^p \cdot q(1-x)^{q-1}$

$f'(x) = 0 \Rightarrow p x^{p-1}(1-x)^q = x^p q (1-x)^{q-1}$ $\Rightarrow$ sol'ns $x=0$ or $x=1$

$x \neq 0, 1 \Rightarrow p(1-x) = q \cdot x \Rightarrow x = \frac{p}{p+q}$

b) consider z: $0 < z < \frac{p}{p+q}$

$\Rightarrow f(0) = 0$; $f(-\frac{1}{2}) = (-\frac{1}{2})^p(\frac{3}{2})^q > 0$ p even

$f(z) = (z)^p(1-z)^q > 0$ all p, q

$\therefore f(0) = 0$ is a local minimum

c) consider w: $\frac{p}{p+q} < w < 1$

$\Rightarrow f(1) = 0$; $f(w) = (w)^p(1-w)^q > 0$ all p, q

$f(\frac{3}{2}) = (\frac{3}{2})^p(-\frac{1}{2})^q > 0$ q even

$\therefore f(1) = 0$ is a local minimum

d) $f''(x) = p(p-1)x^{p-2}(1-x)^q - pq x^{p-1}(1-x)^{q-1} - pq x^{p-1}(1-x)^{q-1} + x^p q(q-1)(1-x)^{q-2}$

$= p(p-1)x^{p-2}(1-x)^q - 2pq x^{p-1}(1-x)^{q-1} + x^p q(q-1)(1-x)^{q-2}$

$$f''\left(\frac{p}{p+q}\right) = p(p-1)\left(\frac{p}{p+q}\right)^{p-2}\left(\frac{q}{p+q}\right)^q - 2pq\left(\frac{p}{p+q}\right)^{p-1}\left(\frac{q}{p+q}\right)^{q-1} + q(q-1)\left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^{q-2}$$

$$= \left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q\left[p(p-1)\left(\frac{p+q}{p}\right)^2 - 2pq\left(\frac{p+q}{p}\right)\left(\frac{p+q}{q}\right) + q(q-1)\left(\frac{p+q}{q}\right)^2\right]$$

$$= \underbrace{(p+q)^2\left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q}_{>0}\ \underbrace{\left[\frac{p-1}{p} - 2 + \frac{q-1}{q}\right]}_{=\left(1 - \frac{1}{p} - 2 + 1 - \frac{1}{q}\right) = -\left(\frac{1}{p} + \frac{1}{q}\right) < 0}$$

$\therefore f''\left(\frac{p}{p+q}\right) < 0$ all p, q $\therefore f\left(\frac{p}{p+q}\right) = \left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q$ is a local maximum

10. Use implicit differentiation to find points (if they exist) on the curve: $x(1-y^2)+y^3=0$ where the tangent line is horizontal or vertical.

$$x(1-y^2)+y^3=0 \Rightarrow x(-2y\cdot y')+(1-y^2)+3y^2\cdot y'=0$$

$$y'(-2xy+3y^2)=y^2-1 \Rightarrow y'=\frac{y^2-1}{3y^2-2xy}$$

$y'=0 \Rightarrow y^2-1=0$ or $y=\pm 1$

$y=1 \Rightarrow x\cdot 0+1=0 \Rightarrow 1=0$
$y=-1 \Rightarrow x\cdot 0-1=0 \Rightarrow -1=0$ $\Big\}$ points not on curve!

$\therefore$ no points with horizontal tangents

Vertical tangents $\Rightarrow x'=\frac{dx}{dy}=0$

$$\Rightarrow x'=[y']^{-1}=\frac{3y^2-2xy}{y^2-1}$$

or $x'(1-y^2)+x(-2y)+3y^2=0 \quad \therefore x'=\frac{3y^2-2xy}{y^2-1}$

$x'=0 \Rightarrow 2xy=3y^2 \Rightarrow y=0$ or $y=\frac{2}{3}x$

$y=0 \Rightarrow x=0$

$y=\frac{2}{3}x \Rightarrow x\left(1-\frac{4}{9}x^2\right)+\frac{8}{27}x^3=0$

$1-\frac{12}{27}x^2+\frac{8}{27}x^2=0$

$x^2=\frac{27}{4} \Rightarrow x=\pm\frac{3\sqrt{3}}{2}$

$\therefore$ vertical tangents at:

$(0,0), \left(\frac{3\sqrt{3}}{2}, \sqrt{3}\right), \left(\frac{-3\sqrt{3}}{3}, -\sqrt{3}\right)$

For reference, the graph looks roughly like this: