Grading Scheme for final exam of 2013

1. (a) The wave equation is $\partial^2 y/\partial x^2 = (1/c^2)\, \partial^2 y/\partial t^2$ [3 marks]. (Get 1 mark only with one gets the constant term $(1/c^2)$ wrong.)
   (b) Let Y = X(x) T(t). Then $c^2 X''/X = T''/T = -\omega^2$ which is a constant [1 mark].
   Solving, we get $y(x, t) = (A \cos kx + B \sin kx) \cos (wt + \alpha)$ Eq. (1) [1 mark] where $k = \omega/c$.
   The boundary conditions: $y(0, t) = 0$ and $y(L, t) = 0$ implies $A = 0$
   And $\sin kL = 0$. i.e., $kL = n\pi$. i.e., $k_n = n\pi/L$ where n = 1, 2, 3, … [1 mark]
   So, standing wave solutions are
   $y_n(x, t) = B \sin(n\pi x/L) \cos(n\pi ct/L + \alpha)$ Eq. (2) [1 mark]

   (c) $I = \int_0^L \sin(k_n x) \sin(k_m x)\, dx = 0$ if $m \neq n$ [3 marks.]

   (d) Eq. (2) can be re-written as

   $y(x, t) = \sum_n \left[C_n \sin k_n x \cos w_n t + D_n \sin k_n x \sin w_n t \right]$ Eq. (3). [1 marks]

   The initial conditions are $y(x, 0) = s(x)$ Eq. (4)

   and $\partial y/\partial x|_{(x,0)} = v(x) = 0$ Eq. (5) [1 mark]

   Let us use the notation $f(x)\, \Theta\, g(x)$ to denote the dot product between the two functions f(x) and g(x) (i.e., $\int_0^L f(x) g(x)\, dx$).

   Taking dot product of Eq. (4) with $\sin k_m x$ and using the orthogonality condition, we get

   $C_m = \sin k_m x\, \Theta\, s(x) / \sin k_m x\, \Theta\, \sin k_m x$ Eq. (6) [1 mark]

   And

   $D_m = \sin k_m x\, \Theta\, v(x) / \sin k_m x\, \Theta\, \sin k_m x$ Eq. (7)

   i.e., $D_m = 0$ Eq (8). [1 mark]

   Eq. (6) implies

   $C_m = (2/L) \sin k_m x\, \Theta\, s(x)$ Eq. (9). [1 mark]

   So, $C_m = I_1 + I_2$ where

$$I_1 = (2/L) \int_0^{L/3} (3x/L) \sin(m \pi x / L) \quad \text{Eq. (10)}$$

And

$$I_2 = (2/L) \int_{L/3}^{L} \left[(3/2) - (3x/2L) \right] \sin(m \pi x / L) \quad \text{Eq. (11)}$$ [1 mark]

Using integration by parts, we find the final expression for C_m. [5 marks].

[Partial credits for partially correct calculation for the expression C_m.]