The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term Test 1 - Monday February 3, 2020
Instructors: Profs. Micah Stickel and Piero Triverio

Last name: Solutions / Marking

First name:

Student number:

Tutorial section number:

Section	**Day**	**Time**	**Room**
TUT0101	Monday	14:00-15:00	BA1230
TUT0102	Monday	14:00-15:00	BA2175
TUT0103	Monday	14:00-15:00	BA2135
TUT0104	Monday	14:00-15:00	BA2159
TUT0105	Wednesday	13:00-14:00	BA2165
TUT0106	Wednesday	13:00-14:00	BA2195
TUT0107	Wednesday	13:00-14:00	BA1230
TUT0108	Wednesday	13:00-14:00	BAB024

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

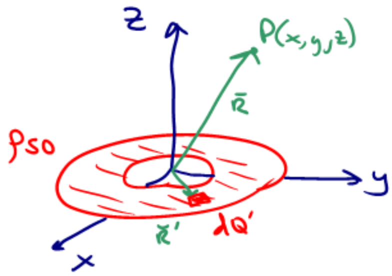
Marks:

Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1

A uniform surface charge density, ρ_{S0} lies in the xy-plane, centered about the origin, and exists between $r = a$ and $r = b$ ($b > a$).

(a) Draw a picture of this situation. Clearly indicate on the figure and state separately the expressions for dQ', **R**, and **R**$'$. [4 points]



$$dQ' = \rho_{s0}\, ds' = \rho_{s0}\, r'\, d\phi'\, dr'$$

$$\bar{R} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z = r\hat{a}_r + z\hat{a}_z = R\hat{a}_R$$

↳ for part (b) below, $\bar{R} = z\hat{a}_z$

$$\bar{R}' = r'\hat{a}_r$$

Diagram [1]
↳ correct geometry
↳ correct orientation
dQ' [1]
$\bar{R}$ [1]
↳ General is acceptable
$\bar{R}'$ [1]

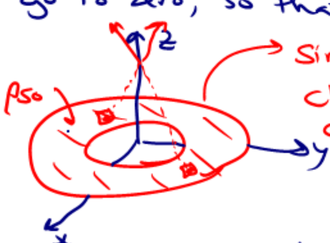
(b) For this charge distribution, determine the expression for the electric field intensity at any point on the positive z-axis. [10 points]

General guidelines: − 0.5 for simple mathematical or copy errors
↳ Don't carry through mistakes

$$\bar{E} = \iint_S d\bar{E}' = \iint_S \frac{dQ'(\bar{R}-\bar{R}')}{4\pi\varepsilon_0|\bar{R}-\bar{R}'|^3} = \int_a^b\int_0^{2\pi} \frac{\rho_{s0}\, r'\, d\phi'\, dr'}{4\pi\varepsilon_0[(r')^2+z^2]^{3/2}}(-r'\hat{a}_r + z\hat{a}_z)$$

Correct integral [3]
↳ Limits
↳ $|\bar{R}-\bar{R}'|$
↳ $\bar{R}-\bar{R}'$

From symmetry we can argue that the $\hat{a}_r$ component of $\bar{E}$ will go to zero, so that $\bar{E} = E_z\hat{a}_z$:



Since ρ_s is uniform, each small piece of charge has a partner charge which will cancel the $\hat{a}_r$ - component & add to the $\hat{a}_z$ - component.

$\hat{a}_r \to 0$ [3]
↳ Either through clearly stated symmetry argument or through math.

$$\therefore \bar{E} = \frac{\rho_{s0}\, z\hat{a}_z}{4\pi\varepsilon_0}\int_a^b\int_0^{2\pi}\frac{r'\, d\phi'\, dr'}{[(r')^2+z^2]^{3/2}} = \frac{\cancel{2\pi}\rho_{s0}\, z\hat{a}_z}{\cancel{4\pi}\varepsilon_0\ 2}\int_a^b\frac{r'\, dr'}{[(r')^2+z^2]^{3/2}}$$

ϕ, r integration [4]
↳ [2] each integral

Let $u = (r')^2 + z^2 \to du = 2r'\, dr$

$$\bar{E} = \frac{\rho_{s0}\, z\hat{a}_z}{2\varepsilon_0}\int_{a^2+z^2}^{b^2+z^2}\frac{1}{2}u^{-3/2}\, du = \frac{\rho_{s0}\, z\hat{a}_z}{2\varepsilon_0}\left[\left(\frac{1}{2}\right)(-2)u^{-\frac{1}{2}}\Big|_{a^2+z^2}^{b^2+z^2}\right]$$

$$\therefore \bar{E} = \frac{\rho_{s0} z}{2\varepsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]\hat{a}_z$$

(c) Use the results of part (b) above to show that the electric field intensity for an infinitely large plate charged with a uniform charge density ρ_{S0} is given by $\mathbf{E}(0,0,z) = \frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_z$ for $z > 0$. Make sure to clearly describe your process and reasoning. [2 points]

From: $\bar{E} = \frac{\rho_{s0} z}{2\varepsilon_0}\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]\hat{a}_z$

Let $a \to 0$ and $b \to \infty$. In this case $\frac{1}{\sqrt{a^2+z^2}} \to \frac{1}{z}$ for $z>0$

and $\frac{1}{\sqrt{b^2+z^2}} \to 0$

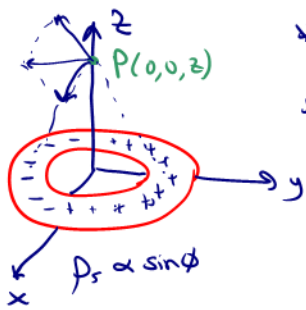
Right limits [1]
$a \to 0$
$b \to \infty$

$\therefore \bar{E}(0,0,z) = \frac{\rho_{s0} \cancel{z}}{2\varepsilon_0}\left(\frac{1}{\cancel{z}}\right)\hat{a}_z$

$= \frac{\rho_{s0}\,\hat{a}_z}{2\varepsilon_0}$

Right conclusions [1]

(d) For the charge distribution geometry described above, identify a non-uniform ρ_S that would result in the electric field intensity along the positive z-axis to only have a y-component. Briefly justify your answer. [4 points]



* We want $\bar{E}(0,0,z) = E_y\hat{a}_y$ for $z>0$.

* One way to consider this is to look back at the original integral, but replace ρ_{s0} with the unknown ρ_s:

$$\bar{E} = \int_a^b\int_0^{2\pi} \frac{\rho_s r' d\phi' dr'}{4\pi\varepsilon_0\left[(r')^2+z^2\right]^{3/2}}\left(-r'\hat{a}_r + z\hat{a}_z\right)$$

In this case $\hat{a}_r = \cos\phi'\hat{a}_x + \sin\phi'\hat{a}_y$

Justification [2]

If we let $\rho_s \propto \sin\phi \to$ then the integrand includes

$$-(r')^2\sin\phi'\cos\phi'\hat{a}_x - (r')^2\sin^2\phi'\hat{a}_y + z\sin\phi'\hat{a}_z$$

In this case the $\hat{a}_x$ & $\hat{a}_z$ components will integrate to zero as we integrate ϕ' from 0 to 2π, but $\hat{a}_y$ will be non-zero.

$\therefore \rho_s = \underline{\underline{\rho_{s0}\sin\phi}}$ will result in $\bar{E}(0,0,z) = E_y\hat{a}_y$

Correct conclusion [2]

$\Rightarrow$ Note this can also be argued by considering the vectoral superposition of the fields for a $\rho_s = \rho_{s0}\sin\phi$ distribution.

Question 2

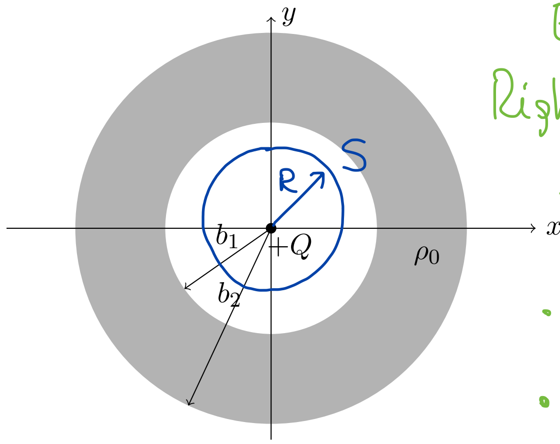
Marking scheme

$\bar{E} = E(R)\bar{a}_R$ [3 pt]

Right Gaussian surface [2 pt]

For each region:

- Correct Q_{enc} [2 pt]
- Correct flux calculation [1 pt]
- Correct $\bar{E}$ calculation [2 pt]



Consider the system made by

- a point charge $+Q$ located at the origin;
- a uniform charge distribution in the **spherical** shell $b_1 < R < b_2$, where volume charge density is equal to ρ_0.

A cross section of the system is shown in the figure above.

i) Use Gauss' law to find the electric field **E** in the region $R < b_1$ [[redacted]].

The charge distribution has spherical symmetry

$\Rightarrow \quad \bar{E} = E(R)\,\bar{a}_R$

Gaussian surface: spherical surface of radius R

Gauss' law: $$\int_S \bar{E} \cdot d\bar{S} = \frac{Q_{enc}}{\varepsilon_0}$$

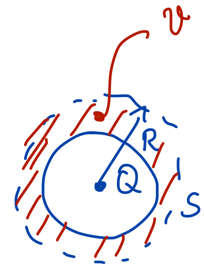
For $R < b_1 \Rightarrow Q_{enc} = Q$

$$\int_S E(R)\,\underbrace{\bar{a}_R \cdot \bar{a}_R}_{1}\, dS_R = \frac{Q}{\varepsilon_0}$$

$$E(R)\,4\pi R^2 = \frac{Q}{\varepsilon_0} \;; \quad E(R) = \frac{Q}{4\pi\varepsilon_0 R^2}$$

$$\boxed{\bar{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\,\bar{a}_R}$$

ii) Use Gauss' law to find the electric field $\mathbf{E}$ in the region $b_1 < R < b_2$ [redacted].



If $R \in [b_1, b_2]$

$$Q_{enc} = Q + \int_v \rho_v \, dv = Q + \rho_0 \frac{4}{3}\pi (R^3 - b_1^3)$$

In Gauss law, the left hand side remains the same

$$E(R)\, 4\pi R^2 = \frac{Q + \rho_0 \frac{4}{3}\pi (R^3 - b_1^3)}{\varepsilon_0}$$

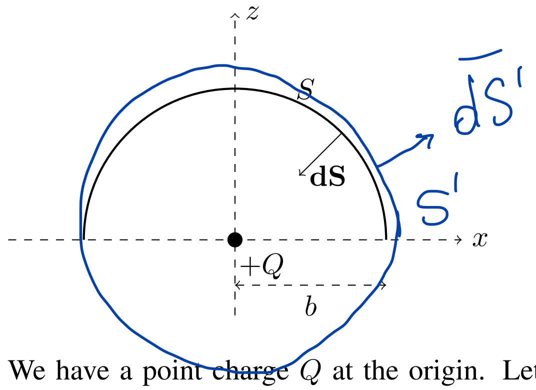
$$\boxed{\bar{E} = \frac{Q + \rho_0 \frac{4}{3}\pi (R^3 - b_1^3)}{4\pi \varepsilon_0 R^2}\, \bar{a}_R}$$

iii) Use Gauss' law to find the electric field $\mathbf{E}$ in the region $R > b_2$ [redacted].

For $R > b_2$, $Q_{enc} = Q + \rho_0 \frac{4}{3}\pi (b_2^3 - b_1^3)$

$$\boxed{\bar{E} = \frac{Q + \rho_0 \frac{4}{3}\pi (b_2^3 - b_1^3)}{4\pi \varepsilon_0 R^2}\, \bar{a}_R}$$

Question 3.1



We have a point charge Q at the origin. Let S be the hemispherical surface of radius b shown in the figure ($S : R = b, \varphi \in [0, 2\pi], \theta \in [0, \pi/2]$), with normal oriented inwards. The flux

$$\int_S \mathbf{E} \cdot \mathbf{dS}$$

is equal to:

(a) $-\frac{Q}{2\varepsilon_0}$

(b) $-\frac{Q}{\varepsilon_0}$

(c) 0

(d) $+Q/\varepsilon_0$

(e) $+Q/2\varepsilon_0$

[1pt] for right answer (a)

Briefly justify your answer. [5 points]

[4pt] for justification

We use Gauss law. We need a closed surface $S' \rightarrow$ spherical surface of radius b

$$\Phi' = \underbrace{\int_{S'} \overline{E} \cdot \overline{dS'}}_{\text{outward flux}} = \frac{Q}{\varepsilon_0}$$

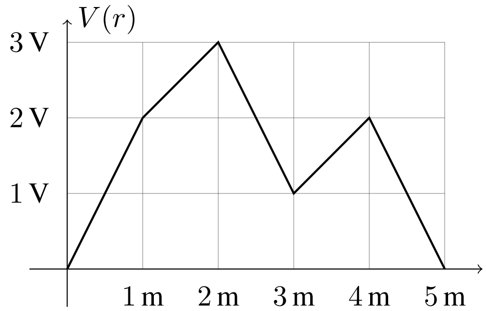
$$\int_S \overline{E} \cdot \overline{dS} = -\frac{\Phi'}{2} = -\frac{Q}{2\varepsilon_0}$$

because it is inward flux

because S is a half sphere

Question 3.2

An electrostatic potential $V(r)$ is present in the xy plane and is function of r only. The following graph depicts

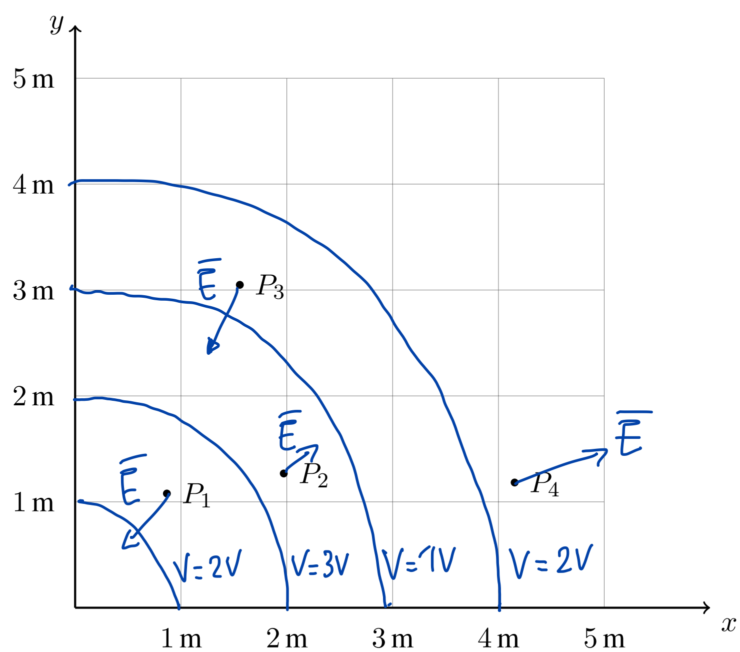


$V(r)$ as a function of r.

Using the axis at the bottom of this page:

- sketch the equipotential lines that pass through the points $(1\,\mathrm{m}, 0)$, $(2\,\mathrm{m}, 0)$, $(3\,\mathrm{m}, 0)$, $(4\,\mathrm{m}, 0)$;
- Indicate with an arrow the direction of the electric field $\mathbf{E}$ at points P_1, P_2, P_3 and P_4.

Briefly justify your answer. [5 points]



[1.5pt] for equipot lines

[1.5pt] for correct $\bar{E}$ direction

V is constant on the surfaces r = const

$\Rightarrow$ equipotential surfaces are cylindrical surfaces in the xy plane they are circles [1pt]

$\bar{E}$ is $\perp$ to equipotential lines [0.5pt]

$\bar{E}$ points in the direction of decreasing V [0.5pt]

Question 3.3

A very large plate has a constant charge density of $\rho_S = \rho_{SA}$ and is located at $z = -a$, where $a > 0$. A sphere of radius $R = b = \frac{a}{2}$ is centered about the origin and is uniformly charged with a charge density of $\rho_S = \rho_{SB}$. Which of the statements below represents the total electric field intensity at $P(x, y, z) = (0, a, 0)$. Note: You may assume the electric field from the very large plate is the same as it would be for an infinitely-large plate.

(a) $\mathbf{E}_{total} = \left(\frac{\rho_{SA}}{2\varepsilon_0} + \frac{\rho_{SB}}{2\varepsilon_0}\right)\mathbf{a}_z$

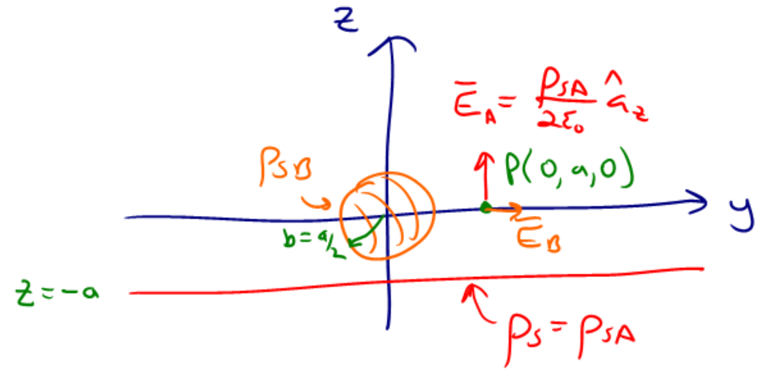
(b) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z$

(c) 0

(d) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z + \frac{\rho_{SB}}{4\varepsilon_0}\mathbf{a}_y$

(e) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z + \frac{\rho_{SB}}{2b\varepsilon_0}\mathbf{a}_R$

(f) None of the above.



Briefly justify your answer. [5 points]

From Q1: $\bar{E}_A = \frac{\rho_{SA}}{2\varepsilon_0}\hat{a}_z$

From Gauss's Law: $\varepsilon_0 \oint\!\!\oint_S \bar{E}_B \cdot \overline{dS} = Q_{enc}$

$\varepsilon_0 E_B (\cancel{4\pi} R^2) = \rho_{SB}(\cancel{4\pi} b^2)$ for $R > b$

$\therefore \bar{E}_B = \frac{\rho_{SB} b^2}{\varepsilon_0 R^2}\hat{a}_R = \frac{\rho_{SB} a^2}{4\varepsilon_0 R^2}\hat{a}_R$

At $P(0,a,0) \rightarrow \bar{E}_B = \frac{\rho_{SB}\cancel{a^2}}{4\varepsilon_0 \cancel{a^2}}\hat{a}_y = \frac{\rho_{SB}}{4\varepsilon_0}\hat{a}_y$

$\therefore \bar{E}_{TOTAL} = \bar{E}_A + \bar{E}_B = \underline{\underline{\frac{\rho_{SB}}{4\varepsilon_0}\hat{a}_y + \frac{\rho_{SA}}{2\varepsilon_0}\hat{a}_z}}$

Justification [4]
↳ 1 - correct orientation
1 - correct $\bar{E}_A$
1 - correct $\bar{E}_B$
1 - proper vector addition

Correct conclusion [1]

Question 3.4

Consider the three situations shown below, and in particular the total force on charge Q_0. Match one of the eight options given to each of these situations. Note, $b > 0$ and F_x and F_y can take any positive value and don't represent a fixed value for all eight options. As well, you can use any of these options more than once if needed. Justify your answer by drawing the individual and total force vectors on each of the diagrams. [5 points]

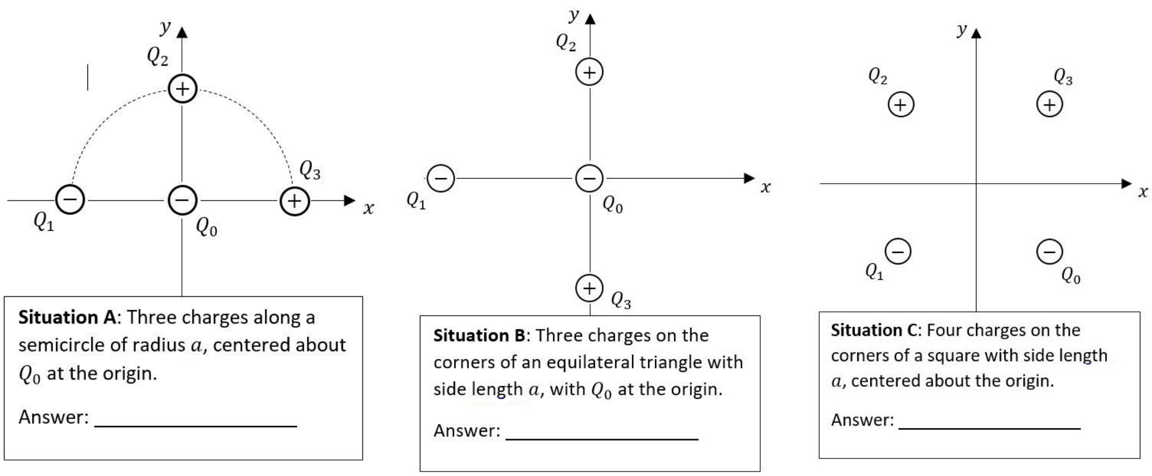


Figure 1

(a) $\mathbf{F}_{total} = F_x\mathbf{a}_x$

(b) $\mathbf{F}_{total} = F_y\mathbf{a}_y$

(c) $\mathbf{F}_{total} = F_x\mathbf{a}_x + F_y\mathbf{a}_y$

(d) $\mathbf{F}_{total} = F_x\mathbf{a}_x + bF_y\mathbf{a}_y$

(e) $\mathbf{F}_{total} = bF_x\mathbf{a}_x + F_y\mathbf{a}_y$

(f) $\mathbf{F}_{total} = \mathbf{0}$

(g) $\mathbf{F}_{total} = -F_y\mathbf{a}_y$

(h) $\mathbf{F}_{total} = -F_x\mathbf{a}_x$

THIS QUESTION WAS NOT PROPERLY STATED

-> All students to receive 5 points