University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus I
Final Exam
December 2022

No calculators or aids

There are 12 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1) a) Suppose that $\lim_{t\to\infty} f(t) = \infty$. Can $f(t)$ ever decrease? If so, sketch a graph of an example. If not, why not?

b) Suppose $\frac{4t-3}{t} \le f(t) \le \frac{4t^5+7}{t^5-3}$ when $t > 92$. Determine $\lim_{t\to\infty} f(t)$.

c) Design a rational function for which $\lim_{t\to 4} f(t)$ is a $\frac{0}{0}$ indeterminate form, but the actual value of the limit is $\sqrt{2}$

d) Sketch graphs of function f and g $(\neq 0)$ so that $\lim_{t\to 0} f(t)g(t)$ exists, but $\lim_{t\to 0} f(t)$ does not.

e) Suppose $|f(x) - 7| < 10^{-3}$ whenever $0 < |x - 5| < 0.0001$. What do we know about $\lim_{x\to 5} f(x)$?

2) Find the limits (Do not use l'Hospital's rule):

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + 2x - 3}$

(b) $\lim_{x \to 1^+} \frac{x^2 - 2}{x^2 + 2x - 3}$

(c) $\lim_{x \to 2^+} \frac{2 - x}{|2 - x|}$

(d) $\lim_{x \to 0} \sin(x - 1 + \cos x)$

(e) $\lim_{x \to 0} \frac{2 - \sqrt{4 - x^2}}{x}$

3) Calculate $\frac{dy}{dx}$ for:

(a) $y = 3x^2$ (b) $y = 3/x^2$ (c) $y = \frac{3+x^2}{2-x}$

(d) $y = sin^2(x^3)$ (e) $x^2y^2 + xcosy = 2.$

4) Provide a rigorous proof (i.e. a $\delta - \varepsilon$ proof) that $\lim_{x \to 3} \left(x^2 - x - 6\right) = 0.$

5) Find the dimensions of the rectangle of largest area that has its base on the *y*-axis and its other two vertices on the right side of the *y*-axis and lying on the parabola $x = 9 - y^2$.

6) Sketch the graph of the function $f(x) = 1 + \frac{1}{x} + \frac{1}{x^2}$ noting any maximum and minimum points, points of inflection, vertical tangents, and asymptotes, as well as intervals of increase and decrease, convexity and concavity.

7) Let $\Re$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 3x - 2x^2$. Provide a sketch of the region, and calculate the following quantities:

(i) The area of $\Re$

(ii) The volume obtained by rotating $\Re$ about the *x*-axis; use the washer method.

(iii) The volume obtained by rotating $\Re$ about the *y*-axis; use the shell method.

8) a) Find the particular solution to the differential equation: $x^2y' = y - xy,\ \ y(-1) = -1$

b) Evaluate the integrating factor, and use it to solve the differential equation: $xy' + y = \sqrt{x}$

9) Use the method of undetermined coefficients to find the general solution of the differential equation:

$$y'' + 6y' + 9y = 16\, e^{-x} \cos 2x$$

10) A clock's minute hand has length 4 and its hour hand has length 3. What is the distance between the tips of the two hands at the moment when the distance is increasing most rapidly?

11) a) Given a differentiable function $f(x)$ which has a bounded derivative $|f'(x)| \leq k$, show that:

$$\left| \int_0^1 f(x)dx - \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \cdot \frac{1}{n} \right| \leq \frac{k}{2n}$$

b) Given a function $f(x)$ with a continuous 1st derivative, and $f(0) = 0$. If $|f'(x)| \leq |f(x)|$ for all x, show that f must be a constant.

12) let $f(x)$ be an increasing real-valued function defined for $x \geq 0$, for which $f(0) = 0$. Let its inverse be $f^{-1}(x)$. Given a and b in the domains of f and f^{-1}, respectively, show that:

$$\int_0^a f(x)dx + \int_0^b f^{-1}(x)dx \geq ab$$