UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test I

MAT185H1S — Linear Algebra

Examiners: S Uppal & G M T D'Eleuterio

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Last Name First Names

Student No:

e-Address:

Signature:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution. The total number of marks available is **50**.
3. Write solutions *only* in the boxed space provided for each question. *Do not* write solutions on the reverse side of pages. These will *not* be scanned and therefore will *not* be marked.
4. Two blank pages are provided at the end for rough work. Work on these back pages will *not* be marked unless clearly indicated; in such cases, clearly indicate on the question page(s) that the solution(s) is continued on a back page(s).
5. *Do not* write over the QR code on the top right-hand corner of each page.
6. *No* aid is permitted.
7. The duration of this test is 90 minutes.
8. There are **??** pages and 5 questions in this test paper.

A Note on Notation:

1. ${}^m\mathbb{R}^n = M_{m\times n}(\mathbb{R})$, the former notation is used in the Notes and the latter in Nicholson.

A. Definitions and Statements

Fill in the blanks.

1(a). The *span* of a set of vectors is defined as

The span of $\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\} \subset \mathcal{V}$ is

$$\operatorname{span}\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\} = \left\{ \boldsymbol{v} \;\middle|\; \boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i,\ \lambda_i \in \Gamma \right\}$$

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1(b). The *dimension* of a vector space is defined as

The dimension of a vector space is the number of vectors in any of its bases.

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1(c). A *subspace* of vector space is defined as

A subset of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if it is a vector space itself over the same field and under the same vector addition and scalar multiplication as for $\mathcal{V}$.

[The subspace test is not the definition of a subspace.]

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1(d). The *row space* of a matrix $\mathbf{A}$ is defined as

The row space of $\mathbf{A}$ is

$$\operatorname{row} \mathbf{A} = \operatorname{span}\{\mathbf{r}_1 \cdots \mathbf{r}_m\}$$

where $\mathbf{r}_1 \cdots \mathbf{r}_m$ are the rows of $\mathbf{A}$.

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1(e). State the *dimension formula* also known as the *rank-nullity theorem*.

For any $\mathbf{A} \in {}^m\mathbb{R}^n$,

$$\dim \operatorname{null} \mathbf{A} = n - \operatorname{rank} \mathbf{A}$$

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B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. A correct answer earns 2 marks but 1 mark will be deducted for an incorrect answer; the minimum total mark for this section is 0.*

2(a). If $\boldsymbol{x}_1, \boldsymbol{x}_2$ are vectors in a vector space $\mathcal{V}$, then $\text{span}\{\boldsymbol{x}_1\} \cup \text{span}\{\boldsymbol{x}_2\} = \text{span}\{\boldsymbol{x}_1 + \boldsymbol{x}_2\}$. $\boxed{F}$

2(b). If $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ are linearly independent vectors in a vector space $\mathcal{V}$, then $\text{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\} \cap \text{span}\{\boldsymbol{x}_3, \boldsymbol{x}_4\} = \{\boldsymbol{0}\}$. $\boxed{T}$

2(c). If B_1 is a basis for a vector space $\mathcal{U}_1$ and B_2 is a basis for a vector space $\mathcal{U}_2$ then $B_1 \cup B_2$ is a basis for $\mathcal{U}_1 \cup \mathcal{U}_2$. $\boxed{F}$

2(d). The dimension of the vector space of all 2×2 (real) magic squares is 1. (Recall that a magic square is an $n \times n$ matrix in which all rows, all columns and both diagonals sum to the same number.) $\boxed{T}$

2(e). Given an $m \times n$ matrix $\mathbf{A}$ and $m \times m$ matrix $\mathbf{U}$, if $\text{rank}\,\mathbf{UA} = \text{rank}\,\mathbf{A}$ then $\mathbf{U}$ is invertible. $\boxed{F}$

C. Problems

3. Let V be the set of all 2×2 matrices with real entries of the form,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

Define vector addition and scalar multiplication as follows:

$$\textit{Vector addition: } \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \text{ (ordinary matrix multiplication)}$$

$$\textit{Scalar multiplication: } c\mathbf{A} = \begin{bmatrix} 1 & 0 \\ ca & 1 \end{bmatrix} \text{ for all } c \in \mathbb{R}$$

You are granted that (*i*) V is closed under vector addition, (*ii*) vector addition is associative, (*iii*) V is closed under scalar multiplication, (*iv*) scalar multiplication is associative, (*v*) $(c+d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ and (*vi*) $1\mathbf{A} = \mathbf{A}$.

Does V over $\mathbb{R}$ with vector addition and scalar multiplication defined above constitute a vector space? Explain.

To prove that V is a vector space, we must show that all the axioms of a vector space are satisfied. We are given that (i)-(vi) hold. What remains therefore are Axioms AIII, AIV and MIII(b).

AIII [Zero]. The zero element is

$$\mathbf{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

because

$$\mathbf{A} + \mathbf{O} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} = \mathbf{A}$$

for all $\mathbf{A}$.

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3. *. . . cont'd*

AIV [Negative]. *The negative of any* **A** *is*

$$-\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix}$$

as

$$\mathbf{A} + (-\mathbf{A}) = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{O}$$

MIII(b) [Distribution].

$$\begin{aligned} c(\mathbf{A} + \mathbf{B}) &= c\left(\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\right) \\ &= c\begin{bmatrix} 1 & 0 \\ a+b & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ c(a+b) & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ ca+cb & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ ca & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ cb & 1 \end{bmatrix} \\ &= \left(c\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}\right)\left(c\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\right) \\ &= c\mathbf{A} + c\mathbf{B} \end{aligned}$$

which proves distribution over vector addition.

Hence V is a vector space.

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4. Consider $p_1, p_2, p_3 \in \mathbb{P}_2$ such that

$$\begin{array}{lll} p_1(0) = 1 & p_2(0) = 0 & p_3(0) = 0 \\ p_1(1) = 0 & p_2(1) = 1 & p_3(1) = 0 \\ p_1(2) = 0 & p_2(2) = 0 & p_3(2) = 1 \end{array}$$

(a) Show that $B = \{p_1, p_2, p_3\}$ is linearly independent.

(b) Is B a basis for $\mathbb{P}_2$? Explain.

(a) Consider

$$\lambda_1 p_1(x) + \lambda_2 p_2(x) + \lambda_3 p_3(x) = 0$$

Now at $x = 0$, $p_1(0) = 1, p_2(0) = p_3(0) = 0$ so

$$\lambda_1 = 0$$

At $x = 1$, we likewise find $\lambda_2 = 0$ and, at $x = 2$, we have $\lambda_3 = 0$. Hence, $B = \{p_1, p_2, p_3\}$ is linearly independent.

(b) As $B \subset \mathbb{P}_2$, containing three vectors, is linearly independent and $\dim \mathbb{P}_2 = 3$, B must be a basis by Chapter 6, Theorem VIII, Part 1. [It's not necessary to quote chapter and verse by number.]

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5. A vector $\mathbf{x} \in \mathbb{R}^n$ is *symmetric* if $x_k = x_{n-k+1}, k = 1 \cdots n$; it is *antisymmetric* (or *skew-symmetric*) if $x_k = -x_{n-k+1}, k = 1 \cdots n$. That is, $\mathbf{x}$ is symmetric or antisymmetric, respectively, if

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_2 & x_1 \end{bmatrix}$$

or

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & -x_2 & -x_1 \end{bmatrix}$$

Now let

$$\mathcal{U} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is symmetric}\}$$
$$\mathcal{W} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is antisymmetric}\}$$

(a) Both $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathbb{R}^n$. Choose one and show that it is indeed a subspace.

(b) Show that $\dim \mathbb{R}^n = \dim \mathcal{U} + \dim \mathcal{W}$.

(a) Let's show that $\mathcal{U}$ is a subspace of $\mathbb{R}^n$. We use the Subspace Test:

SI. The zero $\mathbf{0}$, where all $x_i = 0$, of $\mathbb{R}^n$ is clearly symmetric and hence in $\mathcal{U}$.

SII. Let $\mathbf{x}, \mathbf{y}$ be any two vectors in $\mathcal{U}$ and consider $\mathbf{z} = \mathbf{x} + \mathbf{y}$. Then

$$z_k = x_k + y_k = x_{n-k+1} + y_{n-k+1} = z_{n-k+1}$$

which proves that $\mathbf{z}$ is symmetric. So $\mathbf{x} + \mathbf{y} \in \mathcal{U}$ and closure under vector addition is satisfied.

SIII. Let $\mathbf{x}$ be any vector in $\mathcal{U}$ and consider $\mathbf{z} = \lambda\mathbf{x}$. Then

$$z_k = \lambda x_k = \lambda x_{n-k+1} = z_{n-k+1}$$

which proves that $\lambda\mathbf{x}$ is symmetric. Accordingly, $\lambda\mathbf{x} \in \mathcal{U}$ and closure under scalar multiplication is satisfied.

Thus $\mathcal{U}$ is a subspace of $\mathbb{R}^n$. In a similar fashion, $\mathcal{W}$ can also be shown to be a subspace of $\mathbb{R}^n$.

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5. . . . cont'd

(b) To solve this part, we can establish a basis for $\mathcal{U}$ and $\mathcal{W}$. There are two cases we would need to consider, n even and n odd. Let's take n even first: Consider the vectors

$$\mathbf{e}_i = [0 \;\; \cdots \;\; 0 \;\; \underset{\substack{\uparrow \\ i\text{th entry}}}{1} \;\; \cdots \;\; \underset{\substack{\uparrow \\ (n-i+1)\text{th entry}}}{1} \;\; \cdots \;\; 0 \cdots \;\; 0]$$

for $i = 1 \cdots \frac{1}{2}n$. These are linearly independent and span $\mathcal{U}$; they accordingly form a basis for $\mathcal{U}$. The vectors

$$\mathbf{f}_i = [0 \;\; \cdots \;\; 0 \;\; \underset{\substack{\uparrow \\ i\text{th entry}}}{1} \;\; \cdots \;\; \underset{\substack{\uparrow \\ (n-i+1)\text{th entry}}}{-1} \;\; \cdots \;\; 0 \cdots \;\; 0]$$

for $i = 1 \cdots \frac{1}{2}n$, are linearly independent and span $\mathcal{W}$; they form a basis for $\mathcal{W}$. Thus

$$\dim \mathcal{U} + \dim \mathcal{W} = \frac{n}{2} + \frac{n}{2} = n = \dim \mathbb{R}^n$$

Now n odd: For the basis of $\mathcal{U}$, we need to add, as a basis vector,

$$\mathbf{e}_{(n+1)/2} = [0 \;\; \cdots \;\; 0 \;\; \cdots \;\; \underset{\substack{\uparrow \\ (n+1)/2\text{th entry}}}{1} \;\; \cdots \;\; \cdots \;\; 0 \cdots \;\; 0]$$

which has an nonzero entry only in the middle entry. Therefore the numbers of vectors in the basis becomes $(n+1)/2$. For $\mathcal{W}$, the structure of the basis vectors can stay the same but as n is now odd, the total number of vectors in the basis is $(n-1)/2$. (Note that the middle entry will always have to be 0 in the vectors of $\mathcal{W}$.) So

$$\dim \mathcal{U} + \dim \mathcal{W} = \frac{n+1}{2} + \frac{n-1}{2} = n = \dim \mathbb{R}^n$$

The result therefore holds in general.

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5. *...cont'd*

Another approach is to note that any vector in $\mathbb{R}^n$ *can be written as the sum of a vector in* $\mathcal{U}$ *and a vector in* $\mathcal{W}$*: Let* $\bar{\mathbf{x}}$ *be* $\mathbf{x}$ *written from back to front. If* $\mathbf{x}$ *is symmetric,* $\bar{\mathbf{x}} = \mathbf{x}$ *and, if it is antisymmetric,* $\bar{\mathbf{x}} = -\mathbf{x}$*. Now consider any* $\mathbf{x} \in \mathbb{R}^n$*. This may be written as*

$$\mathbf{x} = \frac{1}{2}(\mathbf{x} + \bar{\mathbf{x}}) + \frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})$$

Clearly $\frac{1}{2}(\mathbf{x}+\bar{\mathbf{x}}) \in \mathcal{U}$ *and* $\frac{1}{2}(\mathbf{x}-\bar{\mathbf{x}}) \in \mathcal{W}$*. So we may say that* $\mathbb{R}^n = \mathcal{U}+\mathcal{W}$*.*

Moreover, $\mathcal{U} \cap \mathcal{W} = \{\mathbf{0}\}$*. The latter claim is supported by the fact that for any vector in both subspaces,* $x_k = x_{n-k+1}$ *and* $x_k = -x_{n-k+1}$*; that is,* $x_{n-k+1} = -x_{n-k+1}$*, which implies that* $x_k = 0$*.*

A basis $B_{\mathcal{U}}$ *for* $\mathcal{U}$ *and a basis* $B_{\mathcal{W}}$ *for* $\mathcal{W}$ *must together then be linearly independent and together they must span* $\mathbb{R}^n$*; that is, the union of a basis for* $\mathcal{U}$ *and a basis for* $\mathcal{W}$ *must form a basis for* $\mathbb{R}^n$*. Therefore, the number of vectors in* $B_{\mathcal{U}}$ *plus the number of vectors in* $B_{\mathcal{W}}$ *must be* n *and we can conclude*

$$\dim \mathbb{R}^n = \dim \mathcal{U} + \dim \mathcal{W}$$

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