NAME:______________________________

STUDENT NUMBER:______________________________

TUTORIAL GROUP: Day of the Week:__________________________Time:_____________

Time: 60 minutes

This is a closed-book exam worth a total of 60 points. Please answer all questions.

**ONLY THE FOLLOWING ITEMS ARE ALLOWED ON YOUR DESK DURING THE EXAM:**

1. **THIS EXAM BOOK –** It contains this cover page, three question pages and a formula sheet (which may be torn off). Make sure you start by putting your **NAME, ID NUMBER,** and **TUTORIAL GROUP** on the front (cover) page of the exam.
   <u>The entire exam book **will be handed in** at the end of the exam and marked</u>.
   a. **No annotated FORMULA SHEET** allowed.
2. **A CALCULATOR**, which can be anything that does NOT connect to wifi and does NOT communicate with other devices. ACCEPTABLE calculators include programmable and graphing calculators, scientific calculators. UNACCEPTABLE calculators include: cell phones, tablets and laptops.
3. **A PEN OR PENCIL.**
4. **YOUR STUDENT ID CARD**, used to check your identity.

If you are missing anything from the above items, raise your hand and ask an exam supervisor to supply what is missing. If you are missing an item that should have been brought by you (e.g., calculator, pen/pencil) there is a limited supply of extras and are on a first come, first served basis.

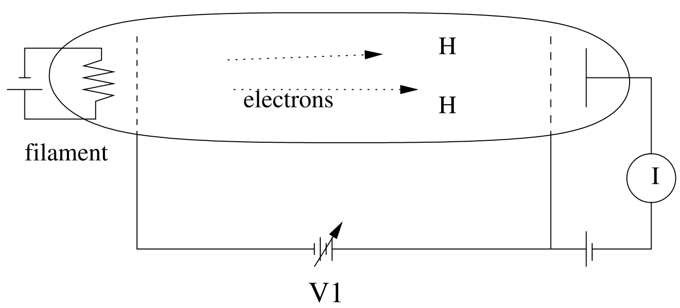
**COMPLETE SOLUTION INCLUDES:**

a) A sketch with a coordinate system, where appropriate. Where a diagram is provided, additional markings can be added to it.
b) Clear and logical workings, including quoting the equations that were used, proper substitutions of numbers (when necessary) with all steps clearly indicated.
c) The final answer with units and two significant figures.

| FOR OFFICE USE ONLY | | | | |
|---|---|---|---|---|
| **QUESTION** | **I** | **II** | **III** | **TOTAL** |
| **MARK** | | | | |
| **MAXIMUM** | **20** | **20** | **20** | **60** |

## Question I

1. In an experiment of the Franck-Hertz type shown below, atomic hydrogens are used instead of mercury atoms. It is found that the current $I$ is sharply reduced when the potential $V_1$ has the values 1.9 V, 3.8 V, and 5.7 V.




3 pts for recognition of 1.9 V as the energy difference

(a) The reduction in the current occurs when electrons collide inelastically with hydrogen atoms and lose some of their energy, which is used to excite hydrogen atoms from the initial state to the excited state. What are the quantum numbers, $n$, of the initial and excited states of the hydrogen atom involved in this inelastic collision? [8 points]

Hydrogen atomic levels have energy given by

3 pts $E_n = \frac{-13.6\text{ eV}}{n^2}$ = $-13.6$ eV, $(n=1)$; $-3.4$ eV $(n=2)$; $-1.5$ eV $(n=3)$

difference 10.2 eV (between $n=1$ and $n=2$); difference 1.9 eV (between $n=2$ and $n=3$).

Since the energy loss happen in multiples of 1.9 eV, the transition involved here is from $n=2$ to $n=3$

1 pt 1 pt.

(b) If you put a radiation detector next to the glass vacuum tube in this experiment, you will be able to observe radiation emitted during this process. What is the wavelength of this radiation (in nm)? [7 points]

When the excited state falls down back to the ground state $(n=3 \rightarrow n=2)$ radiation corresponding to the energy difference will be emitted.

correct calc. 1 pt

formula 2 pts $E = \frac{hc}{\lambda} \rightarrow \lambda = \frac{hc}{E} = \frac{4.14\times10^{-15}\ \cancel{eV}\cdot\cancel{s}\cdot 3\times10^{8}\ m/\cancel{s}}{1.9\ \cancel{eV}} = 6.5\times10^{-7}$ m

use of 1.9 eV (3 pts)

$\therefore$ 650 nm

correct conversion 1 pt.

(c) Your measured spectral line width of the radiation from part (b) is 0.11 eV. Estimate the lifetime of the excited state of the hydrogen atom in seconds. [5 points]

Using the time-energy uncertainty principle

$\Delta t \cdot \Delta E \geq \frac{\hbar}{2}$ $\qquad \Delta t \sim \frac{6.59\times10^{-16}\ \cancel{eV}\cdot s}{2\cdot 0.11\ \cancel{eV}} = 3\times10^{-15}$ s

3 pt 2 pt.

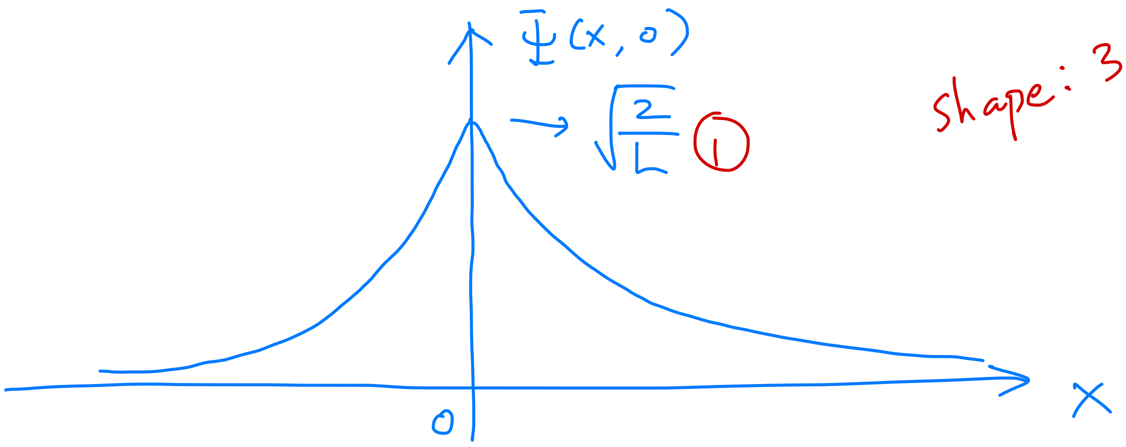
Different numerical answer due to factor of 2 or $2\pi$ are acceptable. (full mark)

**Question II**

1. Consider a free electron with the initial wave function

$$\Psi(x,0) = \frac{1}{\sqrt{L}} e^{-|x|/L}$$

(a) Plot $\Psi(x,0)$ as a function of $x$. Clearly label your graph and indicate x- and/or y-intercepts, if any. [7 points]




shape: 3 pts

label 2pts,

(b) If 1000 electrons have this wave function, how many are expected to be found in the region $x \geq L$? (Note that the wavefunction is already normalized.) [8 points]

Use the probability distribution $\boxed{P(x) = |\psi(x,0)|^2}$ 1

$$P(x \geq L) = \boxed{\int_L^\infty |\psi(x,0)|^2 dx} = \frac{1}{L}\int_L^\infty e^{-2x/L} dx$$ — 2 correct integral

Correct integrand and limits 3

$$= \frac{1}{\cancel{L}}\left[-\frac{\cancel{L}}{2} e^{-2x/L}\right]_L^\infty = \left[0 + \frac{1}{2}e^{-2}\right] = 0.0677$$ 1 numeric

$\therefore 10^3 \cdot P(x \geq L) \cong$ 68 particles 1

**Question II continued on the next page.**

(c) Assume $L = 1.0$ nm and estimate the minimum velocity of an electron described by this wave function in m/s. [5 points]

Let's estimate that the uncertainty in the particle position is L, then the zero-point momentum can be found using the Heisenberg uncertainty principle

1 1

$$mv = p \sim \Delta p \sim \frac{\hbar}{2\Delta x} = \frac{\hbar}{2L} \leftarrow 1$$

$$\therefore \quad v = \frac{\hbar}{2mL} = \frac{1.055 \times 10^{-34} (J \cdot s)}{2 \cdot 9.1 \times 10^{-31} (kg) \cdot 10^{-9} (m)} = 5.8 \times 10^{4} \text{ m/s}$$

1

numeric 1

Any value between $10^4$ to $10^6$ will be accepted.

recognition of zero point momentum

recognition of Heisenberg uncertainty principle.

## Question III

1. A particle of mass $m$, which moves freely inside an infinite potential well of length $a$, has the following initial wave function at $t = 0$:

$$\Psi(x, 0) = A\left[2\sin\left(\frac{\pi x}{a}\right) + \sin\left(\frac{3\pi x}{a}\right)\right],$$

where $A$ is a real constant.

(a) Find $A$ so that $\Psi(x, 0)$ is normalized. [8 points]

Recognizing $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$, → 2pts the wave fn. can be written as

$$\Psi(x,0) = A\sqrt{\frac{a}{2}}\left(2\psi_1(x) + \psi_3(x)\right) \rightarrow \text{2pts.}$$

$$1 = \int |\Psi(x,0)|^2 dx = A^2 \cdot \frac{a}{2} \int \left(4\psi_1(x) + 4\psi_1\psi_3 + \psi_3^2\right) dx \quad 1$$

$$= A^2 \cdot \frac{a}{2} \cdot (4 + 0 + 1) = A^2 \cdot \frac{5a}{2} \rightarrow \boxed{A = \sqrt{\frac{2}{5a}}} \quad 1$$

recognition (using) orthonormal properties of $\psi_n$'s 2

See next page for alternate sol.

Question III continued on the next page.

QIII (a) alternate solution

You can normalize this wave fn. the hard way.

$$1 = \int |\Psi(x,0)|^2 dx = A^2 \int_0^a \left[ 4\sin^2\left(\frac{\pi x}{a}\right) + 4\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right) + \sin^2\left(\frac{3\pi x}{a}\right) \right] dx$$ (1)

$$4\int_0^a \sin^2\left(\frac{\pi x}{a}\right) dx = 4 \cdot \int_0^a \frac{1-\cos\left(\frac{2\pi x}{a}\right)}{2} dx$$

$$= 4 \cdot \left[ \frac{x}{2} - \frac{a}{2\pi} \cdot \frac{1}{2} \sin\left(\frac{2\pi x}{a}\right) \right]_0^a$$

$$= 4 \cdot \frac{a}{2}$$ (2)

$$\int_0^a \sin^2\left(\frac{3\pi x}{a}\right) dx = \int_0^a \frac{1-\cos\left(\frac{6\pi x}{a}\right)}{2} dx$$

$$= \left[ \frac{x}{2} - \frac{a}{6\pi} \cdot \frac{1}{2} \sin\left(\frac{6\pi x}{a}\right) \right]_0^a$$

$$= \frac{a}{2}$$ (2)

$$4\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right) dx = 4\int_0^a \frac{\cos\left(\frac{2\pi x}{a}\right) - \cos\left(\frac{4\pi x}{a}\right)}{2} dx$$

$$= 2\left[ \frac{a}{2\pi}\sin\left(\frac{2\pi x}{a}\right) - \frac{a}{4\pi}\sin\left(\frac{4\pi x}{a}\right) \right]_0^a$$

$$= 0$$ (2)

$$\therefore 1 = A^2 \left(5 \cdot \frac{a}{2}\right) \longrightarrow A = \sqrt{\frac{2}{5a}}$$ (1)

(b) What is the expectation value of the particle's energy in eV? Assume that the particle is an electron and the length of the well is $a = 1.0$ nm. [7 points]

Note that the wave function has the form

$$\Psi(x,0) = \frac{2}{\sqrt{5}}\psi_1 + \frac{1}{\sqrt{5}}\psi_3 \quad - 2$$

Then the expectation value can be calculated by

$$1 \quad \langle E \rangle = \sum |c_n|^2 E_n = \frac{4}{5}E_1 + \frac{1}{5}E_3 = \frac{4E_1 + E_3}{5} \quad 1$$

where $E_1 = \frac{\hbar^2\pi^2}{2ma^2} = \frac{(6.626\times10^{-34})^2}{8 \cdot 9.1\times10^{-31}\cdot(10^{-9})^2}\frac{1}{1.6\times10^{-19}} = 0.377\,\text{eV}$ 1

$E_3 = 9 \cdot E_1 = 3.4\,\text{eV}$ 1

$\therefore \langle E \rangle = 0.8 \cdot 0.377 + 0.2 \cdot 3.4 = 0.98\,\text{eV}$ 1

(c) What is the probability of finding that the particle's energy is 1.51 eV? [5 points]

Note that $E_2 = 4 \cdot E_1 = 1.51\,\text{eV}$. — 1

The probability of finding the energy $E_2$ is just given by $|c_2|^2$. But the wave function is a linear combination of only $n=1$ & $n=3$ states. $|c_2|^2 = 0$. 2

Therefore probability of finding 1.51eV is zero.

2

## Modern Physics Formulae

**Useful constants:**
$h = 6.626 \times 10^{-34}$ Js $= 4.14 \times 10^{-15}$eV $\cdot$ s (Planck constant)
$c = 3 \times 10^8$ m/s (speed of light)
$e = 1.602 \times 10^{-19}$ C (electron charge)
$m_e = 9.1 \times 10^{-31}$ kg $= 0.511$ MeV/$c^2$ (electron mass)
$m_p = 1.67 \times 10^{-27}$ kg $= 936.7$ MeV/$c^2$ (proton mass) $m_p/m_e = 1836$
$k = 8.99 \times 10^9$ Nm$^2$/C$^2$ (Coulomb constant)
$k_B = 1.38 \times 10^{-23}$ J/K $= 8.6 \times 10^{-5}$ eV/K (Boltzmann constant)
$N_A = 6.02 \times 10^{23}$ (Avogadro number)
$R_H = 1.096776 \times 10^7$ m$^{-1}$ (Rydberg constant)

**Quantum Mechanics:**

$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1}$ $\quad E_{ph.el.} = hf - \phi$ $\quad E = hf = \hbar\omega$

$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$ (Compton) $\quad \lambda_C = \frac{h}{mc} = 2.4263 \times 10^{-12}$m $\quad n\lambda = 2d\sin\theta$ (Bragg's law)

$\frac{1}{\lambda} = R_H\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$ $\quad 1$ Ry. $= -13.6$ eV $\quad F_{cent} = \frac{mv^2}{r}$

$r_n = \frac{(n\hbar)^2}{kme^2}$ $\quad r_1 = 0.053$nm (Bohr radius) $\quad E_n = -\frac{m(kZ^2e^2)^2}{2\hbar^2}\frac{1}{n^2}$

$\lambda = \frac{h}{p}$ $\quad \Delta x\Delta p \geq \frac{\hbar}{2}$ $\quad \Delta E\Delta t \geq \frac{\hbar}{2}$

$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$ $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x)$ $\quad E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$

$\langle x\rangle = \sum xP(x)$ (discrete) $\quad \langle x\rangle = \int xP(x)dx$ (continuous) $\quad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar}$ $\quad c_n = \int \psi_n^*(x)\Psi(x,0)dx$ $\quad \langle E\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$

$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int \phi(k)e^{ikx}dk$ $\quad \phi(k) = \frac{1}{\sqrt{2\pi}}\int \Psi(x,0)e^{-ikx}dx$

$T = \frac{16E(U_0 - E)}{U_0^2}e^{-2\alpha a}$ $(E \ll U_0,\ \ \alpha^2 = 2m(U_0 - E)/\hbar^2)$

$T = \exp\left[-\frac{4\sqrt{2m}}{3\hbar}\frac{W^{3/2}}{e\varepsilon}dx\right]$ (Field Emission) $\quad \ln\tau_\alpha \propto 1/\sqrt{E}$

**Special relativity:**

$\Delta t' = \gamma\Delta t_0,$ $\quad l' = l_0/\gamma,$ $\quad \gamma = (1 - v^2/c^2)^{-1/2}$ $\quad \beta = v/c$

$$\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') & u_x' &= \frac{u_x - v}{1 - u_xv/c^2} & u_x &= \frac{u_x' + v}{1 + u_x'v/c^2} \\ y' &= y & y &= y' & u_y' &= \frac{u_y}{\gamma(1 - u_xv/c^2)} & u_y &= \frac{u_y'}{\gamma(1 + u_x'v/c^2)} \\ z' &= z & z &= z' & u_z' &= \frac{u_z}{\gamma(1 - u_xv/c^2)} & u_z &= \frac{u_z'}{\gamma(1 + u_x'v/c^2)} \\ t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2) \end{aligned}$$

$f_{obs} = f_{sce}\sqrt{\frac{1 \mp \beta}{1 \pm \beta}}$ (upper sign moving away) $\quad E^2 = (pc)^2 + (mc^2)^2$

$\vec{p} = \gamma_p m\vec{u},$ $\quad E = \gamma_p mc^2,$ $\quad K = (\gamma_p - 1)mc^2,$ $\quad \gamma_p = (1 - u^2/c^2)^{-1/2}$

**Math formulae:**

$c = a + ib$ $\quad c^* = a - ib$ $\quad |c|^2 = c^*c = a^2 + b^2$ $\quad e^{i\theta} = \cos\theta + i\sin\theta$

$\sin x = \frac{e^{+ix} - e^{-ix}}{2i}$ $\quad \cos x = \frac{e^{+ix} + e^{-ix}}{2}$ $\quad \int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{\alpha}$ $\quad \int_{-\infty}^{\infty} x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$

$\sigma_y^2 = \langle y^2\rangle - \langle y\rangle^2$ $\quad$ For $x \ll 1$, $e^x \approx 1 + x$, $\ln(1 + x) \approx x$, $\sin x \approx x$, $\cos x \approx 1 - x^2/2$.