UNIVERSITY OF TORONTO
ENGINEERING SCIENCE

2017 FINAL EXAM

MAT185

Duration - 2.5 hours

No Aids Allowed

Name: ______________________ Student Number: ______________

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page. Write clearly.
- You may ask us questions, but we cannot answer math questions or questions like "have I shown enough work?"
- There will be partial credit awarded for some questions, so show your work.
- If you need extra room to write a solution, use the back of the pages of the exam. Make sure you write "*CONTINUED ON BACK*" so that the grader knows where to look.
- Please try to write neatly and to express your ideas clearly. We cannot give points for solutions that we cannot read or understand.
- You may leave early. Give your completed exam to an invigilator and enjoy your summer.

Grades:

Question 1: __________ (out of 12)

Question 2: __________ (out of 9)

Question 3: __________ (out of 14)

Question 4: __________ (out of 15)

Question 5: __________ (out of 12)

Question 6: __________ (out of 20)

Question 7: __________ (out of 18)

Total: __________ (out of 100)

Question 1

For questions (a-c) below, decide whether the statement is true or false. (2 points each)

(a) Let V be the vector space (over the field $\mathbb{R}$) of continuous functions, with the usual function addition and usual scalar multiplication. The mapping

$$\begin{aligned} T : V &\longrightarrow \mathbb{R}^2 \\ f &\longmapsto (f(0), f(1) + 1) \end{aligned}$$

is a linear transformation.

☐ True ☐ False

(b) If $\mathbf{A}, \mathbf{B} \in {}^n\mathbb{R}^n$ are similar matrices, then they have the same eigenvalues and the same eigenvectors.

☐ True ☐ False

Let $\mathbf{A} \in {}^m\mathbb{R}^n$ be a matrix with $m > n$ and let $\text{rank}(\mathbf{A}) = n$. For questions (c-f) below, decide whether each statement is true (i.e., the statement *must* be true, given the information provided) or false. (2 points each)

(c) $\mathbf{A}^T\mathbf{A}$ is invertible. ☐ True ☐ False

(d) $\dim(\text{null}(\mathbf{A}))$ equals the number of zero rows in the RREF of $\mathbf{A}$. ☐ True ☐ False

(e) The columns of $\mathbf{A}$ are linearly independent. ☐ True ☐ False

(f) $\text{row}(\mathbf{A}) = \mathbb{R}^n$. ☐ True ☐ False

Question 2

Suppose $\mathbf{A}$ is a square matrix with characteristic polynomial $c_{\mathbf{A}}(\lambda) = \lambda^6 + 5\lambda^5 + 6\lambda^4$.

(a) Calculate all the eigenvalues of $\mathbf{A}$ and their algebraic multiplicities. (2 points)

(b) What are the possible values for the size of $\mathbf{A}$? (i.e. how many rows/columns could $\mathbf{A}$ have, given the information provided) (2 points)

(c) What are the possible values for $\text{rank}(\mathbf{A})$? (3 points)

(d) What are the possible values for $\text{tr}(\mathbf{A})$? (2 points)

Question 3

(a) State the definition of a *Hermitian* Matrix. (2 points)

(b) Write one special property of the eigenvalues of a Hermitian matrix, and one special property of the eigenvectors of a Hermitian matrix. (3 points)

(c) State the *subspace test.* (3 points)

(d) State the definition of the *dimension* of a vector space. (2 points)

(e) Suppose $\mathbf{A} \in {}^7\mathbb{R}^3$, and $\mathbf{b} \in {}^7\mathbb{R}$ with the property that $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = 2$. How many solutions does $\mathbf{A}x = \mathbf{b}$ have? (4 points)

Question 4

For this question, recall that the set of *integers* is the set

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\},$$

and the notation ${}^n\mathbb{Z}^n$ denotes the set of n-by-n matrices of integers. Let $\mathbf{A} \in {}^n\mathbb{Z}^n$ be lower triangular, and $\mathbf{B} \in {}^n\mathbb{Z}^n$ be upper triangular. The diagonal entries of $\mathbf{A}$ and $\mathbf{B}$ are all ones. That is,

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ a_{21} & 1 & 0 & & 0 \\ a_{31} & a_{32} & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & 1 & b_{23} & & b_{2n} \\ 0 & 0 & 1 & & b_{3n} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

where $a_{ij}, b_{ij} \in \mathbb{Z}$ for all i, j. Finally, let $\mathbf{C} = \mathbf{AB}$. For this entire problem, you may use the fact (without proving it) that if all entries of a matrix are integers, then the determinant of that matrix is also an integer. Each part of this question has an elegant solution: if your proof involves unpleasant calculations, look for a simpler one!

(a) Prove that $\det(\mathbf{C}) = 1$. (3 points)

Continued on next page

Question 4 (continued)

(b) Let $\mathbf{b} \in {}^n\mathbb{Z}$. Show that the solution to $\mathbf{C}x = \mathbf{b}$ also has integer entries. (6 points)

(c) Show that $\mathbf{C}$ is invertible, and that $\mathbf{C}^{-1}$ has integer entries. (6 points)

Question 5

Let $\mathbf{A} = [a_{ij}] \in {}^n\mathbb{C}^n$ be a complex-valued matrix.

(a) Consider an eigenvector $\mathbf{x} = (x_1, \ldots, x_n)^T \in {}^n\mathbb{C}$ of $\mathbf{A}$ associated with the eigenvalue λ. Let m be the index of the largest modulus in $\mathbf{x}$, that is,

$$|x_j| \leq |x_m| \quad \text{for any } j = 1, \ldots, n.$$

Write the m^{th} component of the eigenproblem $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. (2 points)

(b) Prove that

$$|\lambda - a_{mm}| \leq \sum_{\substack{j=1 \\ j \neq m}}^{n} |a_{mj}|$$

Hint: You may use the *triangle inequality*, which states that for any complex numbers $z_1, \ldots, z_\ell$,

$$|z_1| + |z_2| + \cdots + |z_\ell| \geq |z_1 + z_2 + \cdots + z_\ell|.$$

(5 points)

Continued on next page

Question 5 (continued)

(c) Deduce that any eigenvalue of $\mathbf{A}$ is located in one of the closed discs of the complex plane centered at a_{mm} and having the radius $R_m = \sum_{j \neq m}^{n} |a_{mj}|$. (5 points)

Question 6

A matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ is called *idempotent* if $\mathbf{A}^2 = \mathbf{A}$.

(a) Prove that the only invertible idempotent matrix is the identity matrix. (3 points)

(b) Show that if λ is an eigenvalue of an idempotent matrix, then $\lambda = 0$ or $\lambda = 1$. (4 points)

Continued on next page

Question 6, (continued)

(c) Show that for any idempotent matrix $\mathbf{A} \in {}^n\mathbb{R}^n$, every nonzero column of $\mathbf{A}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 1. (4 points)

(d) Show that for any idempotent matrix $\mathbf{A}$, $E_{\lambda=1} = \text{col}(\mathbf{A})$. (4 points)

Continued on next page

Question 6, (continued)

(e) Show that every idempotent matrix is diagonalizable. (Hint: calculate the dimension of its eigenspaces). (5 points)

Question 7

Dante and Beatrice measure their feelings for each other with numbers: zero represents a neutral feeling, positive numbers represent affection, and negative numbers represent disaffection. Let $x_1(t)$ denote Dante's feelings for Beatrice at time t, and $x_2(t)$ denote Beatrice's feelings for Dante at time t. Dante grows more attracted to people who love him, while Beatrice grows more attracted to people who hate her. Specifically, $\dot{x}_1 = 4x_2$, and $\dot{x}_2 = -1x_1$. At $t = 0$, $x_1(0) = x_2(0) = 10$.

(a) Let $\mathbf{x}(t) = (x_1(t)\ x_2(t))^T$. Write an equation of the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ that describes how Dante and Beatrice's affection changes with time.(2 points)

(b) Find the eigenvalues of $\mathbf{A}$. (4 points)

Continued on next page

(c) For each eigenvalue you found in part (b), find a basis for the corresponding eigenspace. (4 points)

(d) Write down the *general form* of a solution to the equation you found in part (a). You should not yet use the initial conditions to solve for your arbitrary constants in your solution. (4 points)

Continued on next page

Question 7, continued

(e) Write down the *specific* solution to the equation you found in part (a), by using the initial conditions $x_1(0) = x_2(0) = 10$ to solve for the constants in your answer to part (d). (3 points)

(f) Predict how Dante and Beatrice will feel for each other at $t = 5$. You do not need to simplify your answer; any correct expression is worth full marks. (1 point)

You're done with Math 185! To protect the exam, please restrict your fallen tears of joy to this box. $\longrightarrow$ $\boxed{}$