MAT292 – Fall 2021
Quiz – October 4, 2021

Time alotted: 50 minutes

Full Name ______________________

Student Number ______________________

Email ______________________ @mail.utoronto.ca

Signature ______________________

DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This quiz contains 7 pages (including this title page). Once the quiz starts, make sure you have all of them.

In the first section, only answers and sometimes brief justifications are required.

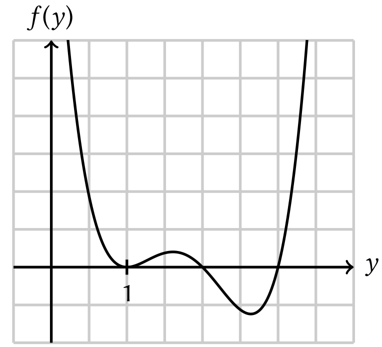
In the second section, justify your answers fully.

You can use pages 6–7 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

	Short answer				Long answer		
Question	Q1	Q2	Q3	Q4	Q6	Q7	**Total**
Marks	3	1	2	4	12	12	**34**

GOOD LUCK! YOU GOT THIS!

SECTION I Provide the final answer. Justify briefly only when asked.



1. **(3 marks)** Find and classify the equilibrium points of the differential equation $\frac{dy}{dt} = f(y) = (y-1)^2(y-2)(y-3)$ whose phase plot is given on the right. You do **NOT** need to justify your answers.

   Solution: The equilibrium points are $y = 1$, $y = 2$, and $y = 3$.
   Either use the phase plot to analyze (e.g. with arrows), or use formulas:

   $$f'(y) = 2(y-1)(y-2)(y-3) + (y-1)^2(y-3) + (y-1)^2(y-2).$$

   Since $f'(2) = -1 < 0$, $y = 2$ is an asymptotically stable equilibrium. Since $f'(3) = 4 > 0$, $f = 3$ is an asymptotically unstable equilibrium. Since $f'(1) = 0$, we look at the sign of f. Notice that $f(t) > 0$ for t slightly above and slightly below 1. Therefore f is a semi-stable equilibrium.

2. **(1 mark)** Consider again the ODE $\frac{dy}{dt} = f(y) = (y-1)^2(y-2)(y-3)$ from the previous question. If $y(t)$ is a solution and $y(0.5) = 2.8$, write the limit in the box on the right. You do **NOT** need to justify.

   $\lim_{t\to\infty} y(t) = \boxed{2}$

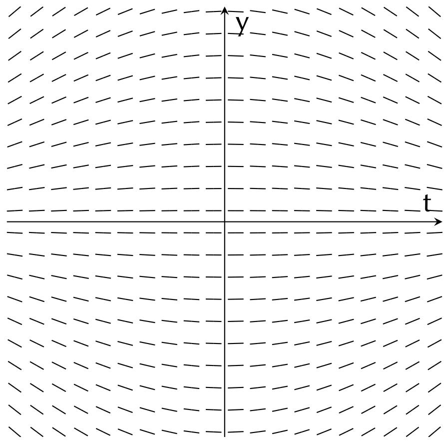
   Solution: Since at the initial value $y' = f(2.8) < 0$, the solution is decreasing. y' stays negative and the solution continues to decrease, approaching the stable equilibrium at $y = 2$.

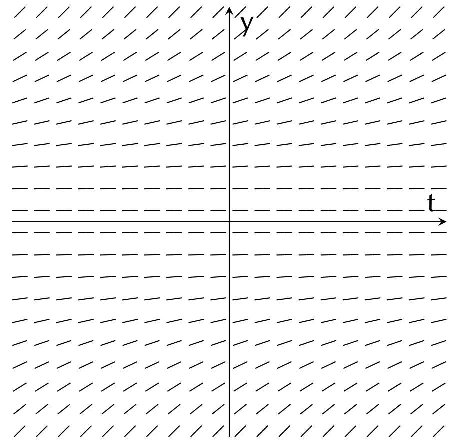
3. **(2 marks)** Assume $\phi(t)$ and $\psi(t)$ both solve the same first-order non-homogeneous linear ODE and that $a \in \mathbb{R}$. Then $\phi(t) + a\psi(t)$ is also a solution to this ODE. This statement is... ○ TRUE ○ FALSE

   Choose true or false above, then justify:

   Solution: Consider, for example, the non-homogeneous linear ODE $y' + ty = 1$. If ϕ and ψ both solve the ode, we have $\phi' + t\phi = 1$ and $\psi' + t\psi = 1$ and therefore $(\phi + a\psi)' + t(\phi + a\psi) = \phi' + t\phi + a(\psi' + t\psi) = 1 + a$. This means that if $a \neq 0$, the function$y = \phi + a\psi$ does not fulfil the ODE (since we don't get 1 on the right hand side). Therefore the statement is false.

4. **(4 marks)** Below are two direction fields of two ODEs. For each plot and each row, make a choice. You do **NOT** need to justify. “Can’t tell” means that by visual inspection it is impossible to tell.





The 1st-order ODE generating this direction field is...			The 1st-order ODE generating this direction field is...		
○ autonomous	○ non-autonomous	○ can’t tell	○ autonomous	○ non-autonomous	○ can’t tell
○ linear	○ non-linear	○ can’t tell	○ linear	○ non-linear	○ can’t tell

Solution: Plot on the left: The ODE is not autonomous, since the slope y' depends on t and not just on y. We can’t tell if the ODE is linear. The increase in slope *might* be linear, but it’s impossible to tell by inspection. A justification/example was not necessary, but to see why “can’t tell” is the right answer, compare the direction fields of $f(t, y) = -t \sin y$ and $f(t, y) = -ty$ in the range $-1 \le t \le 1$, $-1 \le y \le 1$. Their direction fields look virtually the same in the given range, but one of these ODEs is non-linear and the other one is linear.

Plot on the right: The ODE is autonomous, since the slope y' only depends on the value of y, not on the value of t. If it were also linear, note that every autonomous first order linear ODE is of the form $ay' + by = c$. This would require that the slope is monotonic in y (either slopes have to always increase as y increases, or they have to always decrease). This is not the case in the picture (since the slopes first decrease and then increase again), so the ODE is non-linear.

SECTION II **Justify** all your answers.

5. For this question, consider the ODE $(t-2)y' + g(y) = 1$.
Assume that $g(y)$ has a continuous derivative for all values of y.

(a) **(3 marks) For this part only**, you can assume that $g(y) = 2y$. Find the solution such that $y(0) = 1$.

Solution: The ODE $(t-2)y' + 2y = 1$ is linear. After computing an integrating factor (or by "staring and then guessing a factor"), we can use product rule

$$(t-2)^2y' + 2(t-2)y = t-2 \Longrightarrow [(t-2)^2y]' = t-2 \Longrightarrow (t-2)^2y = \frac{1}{2}t^2 - 2t + C \Longrightarrow y = \frac{1}{(t-2)^2}\left[\frac{1}{2}t^2 - 2t + C\right]$$

For the initial value, we get $1 = y(0) = \frac{1}{4}C$, so $C = 4$.

$$\boxed{\text{Solution to the IVP: } y(t) = \frac{1}{(t-2)^2}\left[\frac{1}{2}t^2 - 2t + 4\right]}$$

For parts b/c/d, you do NOT know $g(y)$. You only know that it has a continuous derivative for all y.

True means that you can guarantee it is true. **False** means that you can guarantee it is false. **Undecidable** means that, given the information available to you, it is impossible to tell if the statement is true or false.

The correct choice alone is not worth points. All points are given for the explanation.

(b) **(4 marks)** "Given an initial value $y(0) = c$, we can find a unique solution to the IVP."

Solution: To answer this, rearrange the ODE to standard form

$$y' = f(t,y) = \frac{1-g(y)}{(t-2)} \Longrightarrow f_y(t,y) = \frac{-g'(y)}{(t-2)}$$

Around the point $(t,y) = (0,c)$, we have

- $f(t,y)$ is continuous since $g(y)$ is differentiable (and therefore continuous) and since we are not close to the pole $t = 2$.
- $f_y(t,y)$ is continuous since $g'(y)$ is continuous by assumption and since we are not close to the pole $t = 2$.

Therefore, by the general E-U-Theorem, the solution to the IVP exists and is unique. The statement is true.

(c) **(3 marks)** "Given an initial value $y(0) = c$, we can find $y(1)$."

Solution: Since we do not know $g(y)$ the ODE could be nonlinear. We could only use the general E-U-Theorem. Even though the conditions are fulfilled for the initial time and value (see part (a)), the general E-U-Theorem only tells us that for *some* range $-h < t < h$, the unique solution does exist. We don't know if the solution exists until $t = 1$. It doesn't matter that "the ODE is continuous until $t = 1$". We would need to know $g(y)$ and c to decide this question. The statement is therefore undecidable.

(d) **(2 marks)** "Given an initial value $y(2) = c$, we can find a solution to the IVP."

Solution: Around the initial point $(t,y) = (2,c)$, the functions $f(t,y)$ and $f_y(t,y)$ are not continuous. That means the conditions of the general E-U-Theorem are not fulfilled. This does NOT mean, however, that the solution doesn't exist. The theorem just doesn't help us and dividing by $(t-2)$ was a bad idea in this case. It all depends on what exactly $g(y)$ and c are. The statement is therefore undecidable.

6. There is about to be a zombie outbreak in a remote village in Northern Ontario.
There are 1000 individuals in the village. Every individual is either a zombie or a human.
The number of zombies in the village is denoted by $Z(t)$, where t is measured in days.

(a) (4 marks) At $t = 0$, half the town are zombies. The number of zombies follows logistic growth with rate $r = 1.5$. But there is a second effect, in addition to the effects of the logistic growth: The zombie spell is starting to wear off. One zombie is turning back to human on the first day, two zombies are turning back on the second day, three zombies are turning back on the third day and so on.

Model this scenario as an IVP. Explain the details. You do NOT need to solve the IVP.

Solution: The logistic growth effect can be modeled based on an equation from class and is $1.5Z(t)\left(1 - \frac{Z(t)}{1000}\right)$ since the upper bound is the 1000 individuals (there can't be more zombies than individuals).

The wearing-off effect is time dependent. The simplest function that mirrors the effect is $-t$. Combining the two, and the initial value, we get

$$\text{IVP: } Z'(t) = 1.5Z(t)\left(1 - \frac{Z(t)}{1000}\right) - t \qquad Z(0) = 500$$

We will now consider a different model. **Ignore** anything from part (a) for the rest of the question.

(b) (4 marks) At $t = 0$, there is only one human left in town. As it turns out, she is a magic healer. Her name is Zara and she casts healing spells, turning zombies back into humans. The healing rate is equal to the share of humans (for example if half of all individuals are humans, Zara heals half of the zombies each day; if a quarter of all individuals are humans, Zara heals a quarter of the zombies each day; and so on).

Model this scenario as an IVP. Explain the details. You do NOT need to solve the IVP.

Solution: The healing rate is described by "humans over total population", which is $\frac{1000-Z(t)}{1000}$. We need to multiply the rate by the current number of zombies. And we need to put a minus sign in front since healing *decreases* the number of zombies.

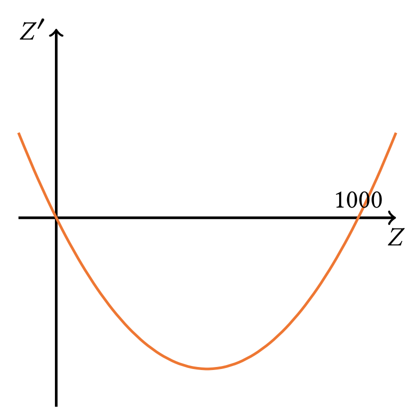
$$\text{IVP: } Z'(t) = -\frac{1000 - Z(t)}{1000} Z(t) \qquad Z(0) = 999$$

(c) (4 marks) Draw a phase plot for the model in part **(b)**. Make sure to label the axes. Using the plot (and without solving the ODE!), answer this question: In the first days, is the number of zombies healed each day increasing or decreasing?

Solution: The plot is a parabola opening upwards with roots (equilibria) $Z = 0$ and $Z = 1000$.

In the beginning, $Z(0) = 999$. We can see that Z is decreasing since Z' is negative in the phase plot for $Z = 999$ (it's below the axis). The number of zombies decreases and we are therefore moving to the left in the phase plot. But, as can be seen in the phase plot, as we are moving to the left and the number of zombies decreases, the magnitude of Z' is increasing. So the number of zombies healed each day increases.

Not asked for the solution: This speeding up happens until the population is 50/50 zombies and humans. At that point, the healing process is slowing down again.



This page is for rough work

Nothing on this page will be marked unless you indicate on the page of a question that your work continues here.

DO NOT TEAR OFF ANY PAGES

This page is for rough work

Nothing on this page will be marked unless you indicate on the page of a question that your work continues here.

DO NOT TEAR OFF ANY PAGES