ECE259: Electromagnetism

Term test 1, February 13th, 2024

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SOLUTION

Instructions

- Duration: 1 hour 30 minutes (9:10-10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**
- Please write with a **dark** pen or pencil. This test will be scanned.

Question 1 [20 pt]

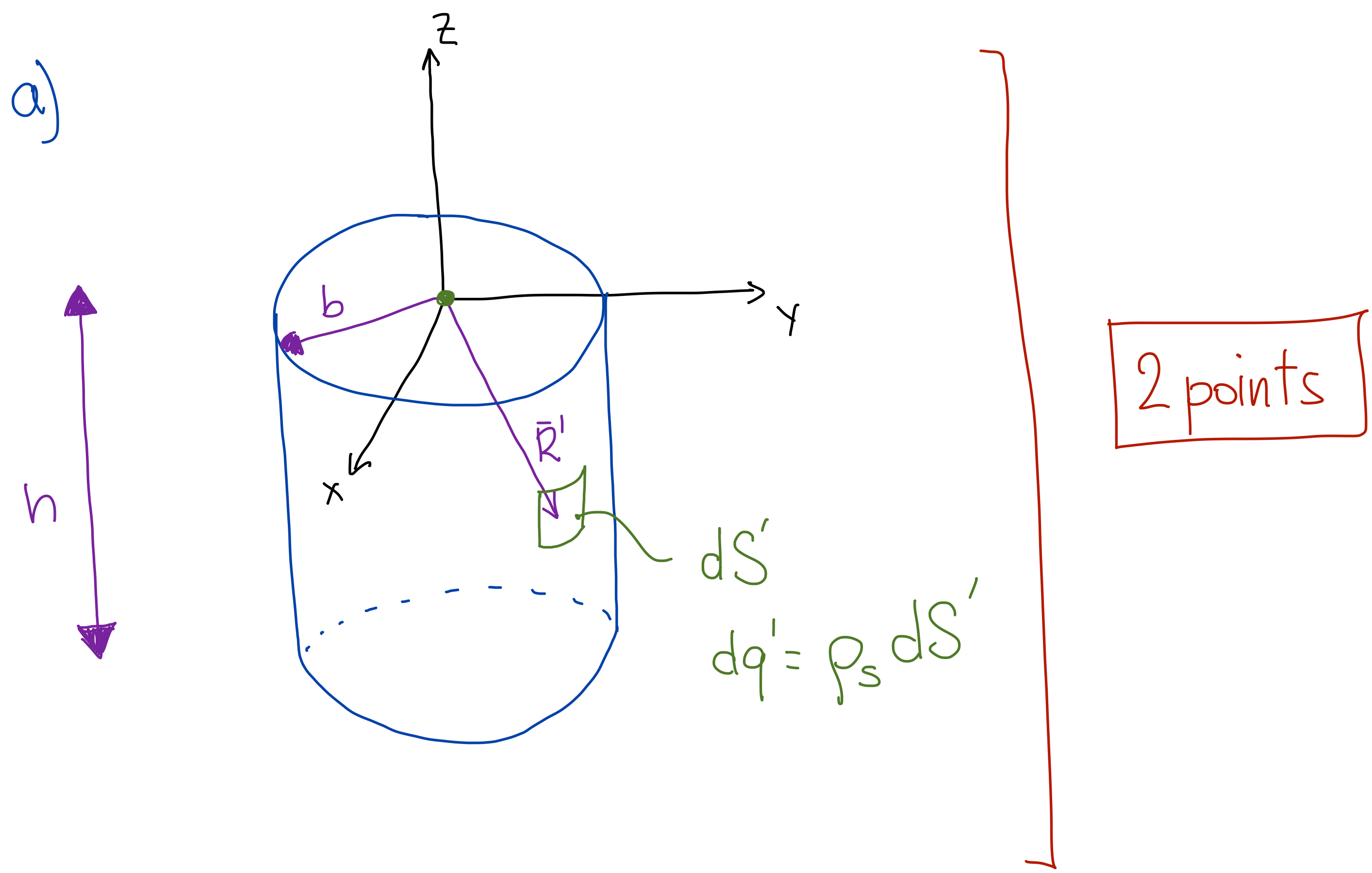
We have a hollow and thin cylindrical shell of height h and radius b, surrounded by free space. The axis of the shell is the z axis. The shell extends from $z = -h$ to $z = 0$. Its surface S is given by

$$S : r = b \quad \varphi \in [0, 2\pi] \quad z \in [-h, 0] \tag{1}$$

The object is charged with uniform surface charge density ρ_S.

a) Draw the object and the coordinate system [2 points];

b) Calculate the electric field $\mathbf{E}$ produced by the charged object at the origin, showing your derivation in detail [18 points].



1) Coordinate system: cylindrical [1pt]

$\varphi' \in [0, 2\pi]$ $z' \in [-h, 0]$ $r' = b$

2) $dq' = \rho_s dS' = \rho_s b\, d\varphi' dz'$ [1pt]

3) $\bar{R} = 0$ [2pt]

$\bar{R}' = b\bar{a}_r' + z'\bar{a}_z$ [2pt]

$\bar{R} - \bar{R}' = -b\bar{a}_r' - z'\bar{a}_z = -b\underbrace{\cos\varphi'}\bar{a}_x - b\underbrace{\sin\varphi'}\bar{a}_y - z'\bar{a}_z$ [2pt]

$|\bar{R} - \bar{R}'| = \sqrt{b^2 + (z')^2}$ [2pt]

will integrate to zero

since $\int_{\varphi'=0}^{2\pi} \cos\varphi' d\varphi' = 0$

4) $$d\bar{E} = -\frac{\rho_s b\, d\varphi' dz'}{4\pi\varepsilon_0} \cdot \frac{b\cos\varphi'\,\bar{a}_x + b\sin\varphi'\,\bar{a}_y + z'\,\bar{a}_z}{\left[(z')^2 + b^2\right]^{3/2}}$$

5) Integration

$$\bar{E} = \underbrace{\int_{\varphi'=0}^{2\pi}\int_{z'=-h}^{0}}_{[1\,pt]} \overbrace{\left(-\frac{\rho_s b\, d\varphi' dz'}{4\pi\varepsilon_0}\right) \cdot \frac{b\cos\varphi'\,\bar{a}_x + b\sin\varphi'\,\bar{a}_y + z'\,\bar{a}_z}{\left[(z')^2 + b^2\right]^{3/2}}}^{[1pt]} =$$

Integration Limits

integrate to zero since

$$\int_{\varphi'=0}^{2\pi} \cos\varphi'\, d\varphi' = 0$$

and same for $\sin\varphi'$

$$= -\frac{\rho_s b}{4\pi\varepsilon_0} \int_{\varphi'=0}^{2\pi} \int_{z'=-h}^{0} \frac{z' \, \bar{a}_z}{\left[(z')^2 + b^2\right]^{3/2}} d\varphi' dz' =$$

$$= -\frac{\rho_s b}{4\pi\varepsilon_0} \, 2\pi \, \bar{a}_z \, \frac{1}{2} \int_{z'=-h}^{0} \frac{2z'}{\left[(z')^2 + b^2\right]^{3/2}} dz' =$$

$$\int \frac{1}{x^{3/2}} dx = \int x^{-3/2} dx = \frac{x^{-1/2}}{\left(-\frac{1}{2}\right)} = -\frac{2}{\sqrt{x}}$$

$$= -\frac{\rho_s b}{4\varepsilon_0} \bar{a}_z \left[-\frac{2}{\sqrt{(z')^2 + b^2}} \right]_{z'=-h}^{0} =$$

$$= -\frac{\rho_s b}{4\varepsilon_0} \bar{a}_z \left[-\frac{2}{b} + \frac{2}{\sqrt{b^2 + h^2}} \right] =$$

$$= \frac{\rho_s b}{2\varepsilon_0} \left[\frac{-1}{\sqrt{b^2+h^2}} + \frac{1}{b} \right] \bar{a}_z =$$

[4pt]

$$= \boxed{\frac{\rho_s}{2\varepsilon_0} \left(1 - \frac{b}{\sqrt{b^2+h^2}} \right) \bar{a}_z}$$

Final result
If correct: 2pt

if wrong: 0pt or 1pt
(irrespective of carryover)

Question 2 [20 pt]

A cylinder is infinitely long and has radius α. The cylinder is centered around the z axis, and is surrounded by vacuum. Charge is present throughout the volume of the cylinder, with density

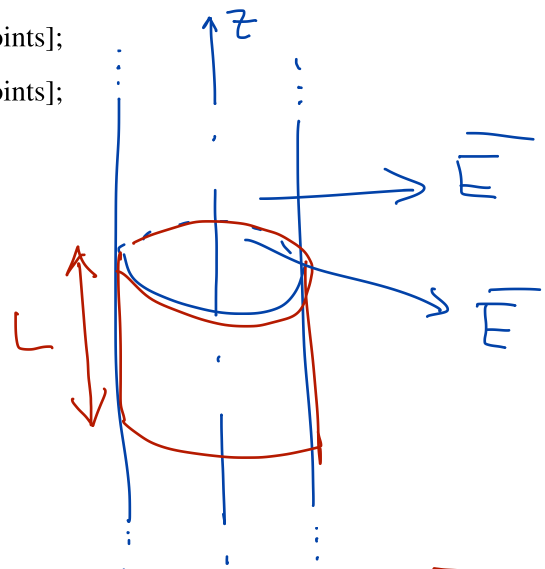
$$\rho_v(r) = \rho_0 \frac{r^2}{\alpha^2} \quad \text{for } r \leq \alpha\,.$$

a) Calculate the charge per unit length Q' present in the cylinder [4 points]

b) Calculate the electric field $\mathbf{E}$ outside the cylinder ($r > \alpha$) [6 points];

c) Calculate the electric field $\mathbf{E}$ inside the cylinder ($r \leq \alpha$) [10 points];

Please justify all your answers.



a) Change in section of length L

$$Q^{*} = \int_{z=0}^{L}\int_{r=0}^{\alpha}\int_{\varphi=0}^{2\pi} \rho_0 \frac{r^2}{\alpha^2}\, r\,dr\,d\varphi\,dz = \frac{L\rho_0}{\alpha^2}\,2\pi \int_0^{\alpha} r^3 dr = \frac{L\rho_0 \cancel{2}\pi}{\alpha^2}\,\frac{\alpha^4}{\cancel{4}\,2} =$$

$$= \frac{L\rho_0 \pi \alpha^2}{2}$$

[4pt]

Charge per unit length $Q' = \frac{Q}{L}$

$$\boxed{Q' = \frac{Q}{L} = \frac{\rho_0 \pi \alpha^2}{2}}$$

b) Due to symetry, we expect field to be in the form $\bar{E} = E(r)\,\bar{a}_r$ [1pt]

Gaussian surface: cylinder, height $= L$, radius $r > a$ [2pt]

Gauss' Law

$$\int \bar{E} \cdot \bar{dS} = \frac{Q' \cdot L}{\varepsilon_0}$$

$$2\pi r L \cdot E(r) + 0 + 0 = \frac{\rho_0 \pi a^2}{\varepsilon_0 2} L$$

No flux through top & bottom faces since $\bar{E} \perp \bar{dS}$ [1pt]

$$2\pi r \cancel{L} E_r(r) = \frac{\rho_0 \pi a^2}{\varepsilon_0 2} \cancel{L}$$

$$E_r(r) = \frac{\rho_0 \cancel{\pi} a^2}{4 \cancel{\pi} r \varepsilon_0}$$

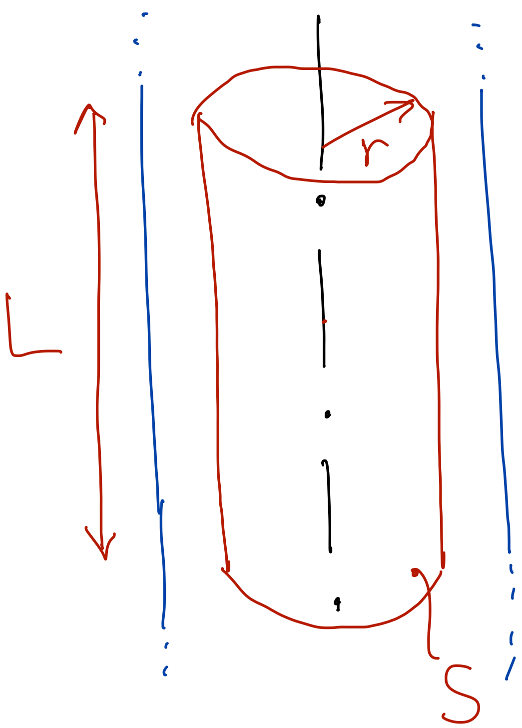
$$\boxed{\bar{E} = \frac{\rho_0 a^2}{4 r \varepsilon_0} \bar{a}_r}$$ [2pt]

c) We still expect due to symmetry

$$\bar{E} = E(r)\,\bar{a}_r$$

[1pt]

Gaussian surface: cylindrical surface with radius $r<\alpha$ [2pt]



Charge Q inside S

$$Q = \int_{z=0}^{L}\int_{r'=0}^{r}\int_{\varphi=0}^{2\pi} \rho_0 \frac{(r')^2}{\alpha^2}\, r'dr'd\varphi\, dz = \frac{L\rho_0}{\alpha^2}\,2\pi \int_0^r (r')^3 dr'$$

[4pt]

$$= \frac{L\rho_0}{\alpha^2} 2\pi \frac{r^4}{4} = \frac{\pi L \rho_0 r^4}{2\alpha^2}$$

Gauss' law

$$\int \bar{E} \cdot \overline{dS} = Q/\varepsilon_0$$

No flux top/bottom: 1pt

$$\cancel{2\pi} r \cancel{L} E_r(r) + 0 + 0 = \frac{\cancel{L} \rho_0 \cancel{2\pi} r^4}{\alpha^2 \cancel{4}_2}$$

Calculations [2pt]

$$E_r(r) = \frac{\rho_0 r^4}{2\alpha^2 2r\varepsilon_0} = \frac{\rho_0 r^3}{4\alpha^2 \varepsilon_0}$$

$$\boxed{\bar{E} = \frac{\rho_0}{4\alpha^2 \varepsilon_0} r^3 \bar{a}_r}$$

Question 3.1 [5 points]

We have an electrostatic system. Initially, all potentials are defined with reference to infinity. What happens when the point $(x = 1, y = 1, z = 1)$ is adopted as a new reference point for potentials?

1. Both potential V and electric field $\mathbf{E}$ remain the same;
2. Electric field $\mathbf{E}$ changes, potential V remains the same;
3. Potential V changes, electric field $\mathbf{E}$ remains the same; [2pt] right answer
4. Both potential V and electric field $\mathbf{E}$ change.

Justify your answer.

$V(x,y,z)$

Reference point 1

$V(\infty)=0$

Reference point 2

$V(1,1,1)$

$$\underbrace{V(x,y,z)-V(\infty)}_{\text{potential with respect to } \infty} = \underbrace{V(x,y,z)-V(1,1,1)}_{\text{potential with respect to } (1,1,1)} + \underbrace{V(1,1,1)-V(\infty)}_{\text{constant}}$$

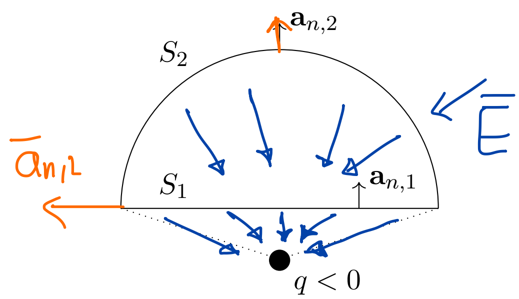
Potential changes

Justification of why V changes [2pt]

$\bar{E} = -\nabla V$ does not since gradient of constant is $\emptyset$

Justification $\bar{E}$ same [1pt]

Question 3.2 [5 points]



Consider the closed surface shown in the figure above, which consists of:

- the circular base S_1, with normal $\mathbf{a}_{n,1}$;
- the half spherical surface S_2, with normal $\mathbf{a}_{n,2}$.

A negative point charge q is located below the surface, as shown. Let Φ_1 and Φ_2 be the flux of the electric field through the surfaces S_1 and S_2, respectively

$$\Phi_1 = \int_{S_1} \mathbf{E} \cdot \mathbf{a}_{n,1} dS_1 \qquad \Phi_2 = \int_{S_2} \mathbf{E} \cdot \mathbf{a}_{n,2} dS_2$$

Which statement is correct? Justify your answer.

1. Φ_1 and Φ_2 have different sign;
2. Φ_1 and Φ_2 are both positive, and $|\Phi_1| = |\Phi_2|$;
3. Φ_1 and Φ_2 are both positive, and $|\Phi_1| > |\Phi_2|$;
4. Φ_1 and Φ_2 are both positive, and $|\Phi_1| < |\Phi_2|$;
5. Φ_1 and Φ_2 are both negative, and $|\Phi_1| = |\Phi_2|$; Right answer: [2pt]
6. Φ_1 and Φ_2 are both negative, and $|\Phi_1| > |\Phi_2|$;
7. Φ_1 and Φ_2 are both negative, and $|\Phi_1| < |\Phi_2|$.

Outward flux through whole surface: $\Phi = \int_{S_2} \bar{E} \cdot \bar{a}_{n,2} dS_2 - \int_{S_1} \bar{E} \cdot \bar{a}_{n,1} dS_1 =$ Justif. $\Phi_1 = \Phi_2$ [2pt]

since $\bar{a}_{n,1}$ inwards

$= \Phi_2 - \Phi_1$

From Gauss' law

$\Phi_2 - \Phi_1 = 0$ since no charge inside

So $\Phi_2 = \Phi_1$

$\Phi_1 < 0$ since $\bar{E} \cdot \bar{a}_{n,1} < 0$ on any point on S_1

Justification sign [1pt]

Question 3.3 [5 points]

a) Explain what an electric dipole is;

b) Drawn an electric dipole, and sketch the electric field **E** distribution that it produces.

a) An electric dipole is made by:

[1pt]

- two equal and opposite changes [1pt]
- at a distance [1pt]

b)

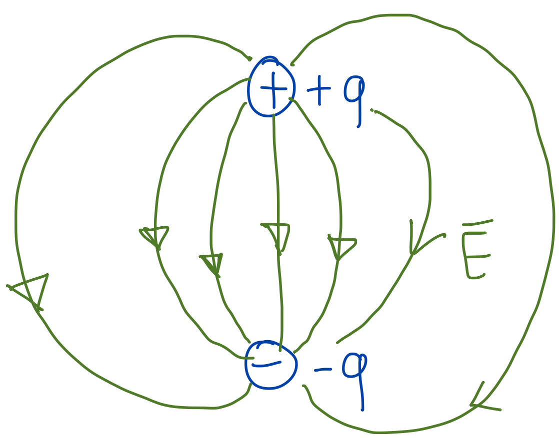


Diagram [2pt]

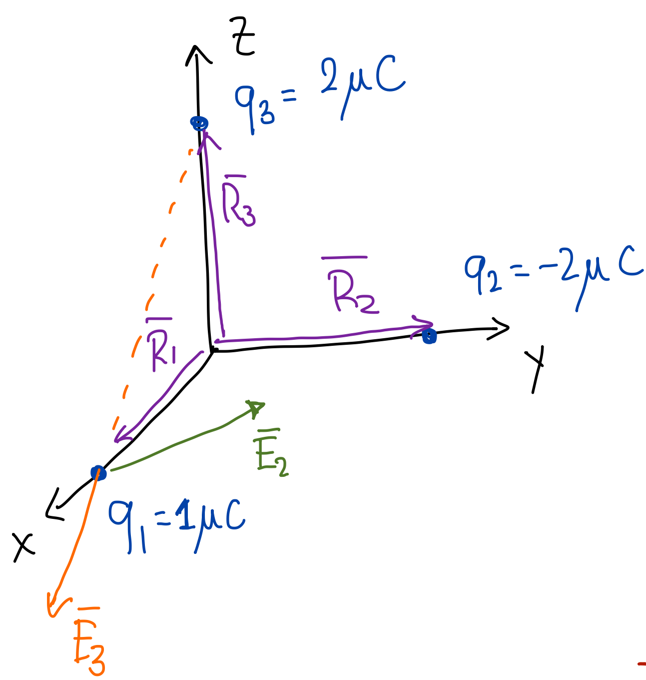
Question 3.4 [5 points]

We have three point charges situated in vacuum:

- charge $q_1 = 1\,\mu$C in position (1 m, 0, 0);
- charge $q_2 = -2\,\mu$C in position (0, 1 m, 0);
- charge $q_3 = +2\,\mu$C in position (0, 0, 1 m);

All charge positions are expressed in Cartesian coordinates. Remember that $\varepsilon_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$.

Calculate the total force $\mathbf{F}$ acting on q_1 (both magnitude and direction).



$$\left.\begin{array}{l} \left.\begin{array}{l} \bar{R}_1 = \bar{a}_x \\ \bar{R}_2 = \bar{a}_y \\ \bar{R}_3 = \bar{a}_z \end{array}\right\} \text{all in m} \end{array}\right] \text{[1pt]}$$

Field at q_1 due to q_2

$$\bar{E}_2 = \frac{q_2}{4\pi\varepsilon_0}\frac{\bar{R}_1 - \bar{R}_2}{|\bar{R}_1 - \bar{R}_2|^3} = \frac{-2\cdot 10^{-6}}{4\pi\varepsilon_0}\frac{\bar{a}_x - \bar{a}_y}{(2)^{3/2}} \ \frac{V}{m}$$

[1pt]

Field at q_1 due to q_3

$$\bar{E}_3 = \frac{q_3}{4\pi\varepsilon_0}\frac{\bar{R}_1 - \bar{R}_3}{|\bar{R}_1 - \bar{R}_3|^3} = \frac{2\cdot 10^{-6}}{4\pi\varepsilon_0}\frac{\bar{a}_x - \bar{a}_z}{(2)^{3/2}} \ \frac{V}{m}$$

[1pt]

Total field at q_1

$$\bar{E} = \bar{E}_2 + \bar{E}_3 = \frac{2\cdot 10^{-6}}{4\pi\varepsilon_0 2\sqrt{2}}\left(\cancel{\bar{a}_x} - \bar{a}_z - \cancel{\bar{a}_x} + \bar{a}_y\right) =$$

$$= \frac{+2\cdot 10^{-6}}{8\sqrt{2}\pi\varepsilon_0}\left(\bar{a}_y - \bar{a}_z\right)$$

Total $\bar{E}$ [1pt]

Force on q_1

$$\bar{F} = q_1\bar{E} = 6.355\cdot 10^{-3}\left(\bar{a}_y - \bar{a}_z\right) = \boxed{6.355\left(\bar{a}_y - \bar{a}_z\right)\ mN}$$

$\bar{F}$ [1pt]

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi drdz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = rdrd\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Differential operators

4.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial R}\vec{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

4.2 Divergence

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

4.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

4.4 Curl

$$\begin{aligned}
\nabla\times\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\vec{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\vec{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\vec{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\vec{a}_\phi
\end{aligned}$$

5. Electromagnetic formulas

Table 1 Electrostatics

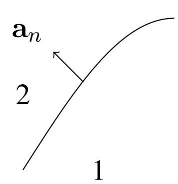
$\mathbf{F} = \frac{1}{4\pi\varepsilon} \frac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_v \frac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3} dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$	$\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon} \int_v \frac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}$	
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$	$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$
$\rho_{p,v} = -\nabla \cdot \mathbf{P}$	
$\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$	
$\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$	
$E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2} QV$
$W_e = \frac{1}{2} \int_v \rho_v V dv = \frac{1}{2} \int_{\mathbb{R}^3} \mathbf{D} \cdot \mathbf{E} dv$	
$\nabla \cdot (\varepsilon \nabla V) = -\rho_v$	$\nabla \cdot (\varepsilon \nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$	
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{B} = \nabla \times \mathbf{A}$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$	$\nabla \cdot \mathbf{A} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\mathbf{A} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$	$\mathbf{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}}{\lvert\mathbf{R} - \mathbf{R}'\rvert} dv'$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$	
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$	
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$	
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$	
$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$	
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$	
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$	

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$