EXAM 2013 SOLUTIONS

Q1

a) GIVEN $AX_1 = 0$ AND $AX_2 = B$

THEN $A(X_2 - X_1) = AX_2 - AX_1 = B - 0 = B$

$\therefore$ $X_2 - X_1$ IS ANOTHER SOLUTION TO $AX = B$.
(SO IS $X_2 + X_1$)

b) GIVEN $AX_1 = B$ AND $AX_2 = B$ AND $X_1 \neq X_2$

THEN $A(X_1 - X_2) = AX_1 - AX_2 = B - B = 0$

$\therefore$ $X_1 - X_2$ IS A NON-TRIVIAL SOLUTION TO $AX = 0$.
(SO IS $X_2 - X_1$)

c) SUPPOSE WE USE THE GAUSSIAN ELIMINATION ALGORITHM TO TAKE $[A \mid 0]$ TO ITS REDUCED NORMAL FORM.

LET r BE THE NUMBER OF NONZERO ROWS IN THIS REDUCED NORMAL FORM (OR NUMBER OF LEADING 1's).

THEN $r \leq m < n$

$\therefore$ $n - r > 0$ ($n-r$ IS NUMBER OF FREE VAR.)

$\therefore$ THE SOLUTION HAS AT LEAST ONE FREE VARIABLE AND HENCE THERE ARE INFINITE SOLUTIONS.

Q2

a) ORTHOGONALITY CONDITIONS

$$\begin{bmatrix} \alpha \\ 1 \\ \beta \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad (1)\ 2\alpha + \beta = 0$$

$$\begin{bmatrix} \alpha \\ 1 \\ \beta \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = 0 \qquad (2)\ -\alpha + 1 + 2\beta = 0$$

$$\therefore \quad \beta = -2\alpha \qquad (1)$$

$$-\alpha + 1 + 2(-2\alpha) = 0 \qquad (1) \to (2)$$

$$5\alpha = 1$$

$$\boxed{\begin{aligned} \alpha &= 0.2 \\ \beta &= -0.4 \end{aligned}}$$

Q2

b) SOLVE LEAST SQUARES PROBLEM ASSOCIATED WITH:

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \\ 4 & -1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$X_{LS} = (A^TA)^{-1} A^TB$$

$$A^TA = \begin{bmatrix} 1 & 1 & 4 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 0 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 \\ 0 & 17 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{18} & 0 \\ 0 & \frac{1}{17} \end{bmatrix}$$

$$A^TB = \begin{bmatrix} 1 & 1 & 4 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 \\ 2 \end{bmatrix}$$

Q2 b) CONT'D

$$X_{LS} = \begin{bmatrix} \frac{1}{18} & 0 \\ 0 & \frac{1}{17} \end{bmatrix} \begin{bmatrix} 12 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{12}{18} \\ \frac{2}{17} \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/17 \end{bmatrix}$$

BEST APPROXIMATION IS : $\frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + \frac{2}{17}\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$

THE FACT THAT $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ AND $\begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}$

ARE ORTHOGONAL MAKES FOR AN "EASY TO INVERT" (A^TA) MATRIX.

Q3 a) LOOKING FOR $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ THAT SATISFIES THE FOLLOWING EQUATIONS:

POINT $(2,-9)$ $\quad -9 = a(2^2) + b(2) + c = 4a + 2b + c$

POINT $(-1,0)$ $\quad 0 = a(-1^2) + b(-1) + c = a - b + c$

HOR. TANGENT AT $(2,-9)$ $\quad \frac{dy}{dx} = 0 = 2a(2) + b = 4a + b$

$$\begin{bmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -9 \\ 0 \\ 0 \end{bmatrix}$$

$\Downarrow$ RNF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -5 \end{array}\right]$$

$\therefore \; a = 1, \; b = -4, \; c = -5$

$$y = x^2 - 4x - 5$$

Q3 b) LOOKING FOR $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ THAT SATISFIES

THE FOLLOWING EQUATIONS:

POINT $(-1,0)$ $\quad 0 = a(-1^2) + b(-1) + c = a - b + c$

HOR. TANGENT AT $x=2$ $\quad \frac{dy}{dx} = 0 = 2a(2) + b = 4a + b$

$$\begin{bmatrix} 1 & -1 & 1 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow \text{ RNF}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 0 \\ 0 & 1 & -4/5 & 0 \end{array}\right]$$

LEADING: a, b
FREE: c

$$a = -\tfrac{1}{5}c, \quad b = \tfrac{4}{5}c, \quad c = c$$

$$\checkmark \; y = -\tfrac{1}{5}cx^2 + \tfrac{4}{5}cx + c$$

$$\checkmark \; y = -\tfrac{1}{5}c\,(x^2 - 4x - 5)$$

Q4 a) GIVEN B AND AB ARE INVERTIBLE.

LET $C = AB$

$\therefore A = CB^{-1}$

SINCE A IS THE PRODUCT OF INVERTIBLE MATRICES, THEN A IS INVERTIBLE AND ITS INVERSE IS GIVEN BY:

$$A^{-1} = (CB^{-1})^{-1} = BC^{-1}$$

(NOTE: QUESTION ONLY ASKED TO SHOW A IS INVERTIBLE)

b)

$$(A^T B^T)^{-1} = (B^T)^{-1}(A^T)^{-1}$$

$$= (B^{-1})^T (A^{-1})^T$$

$$= (A^{-1}B^{-1})^T$$

$$\therefore (A^T B^T)^{-1} = (A^{-1}B^{-1})^T$$

IS A VALID EQUATION.

Q4 c) IF $A^4 = I$

THEN $(A^3)A = A(A^3) = I$

$\therefore A^{-1} = A^3$

d) $A + B = AB$

$A^{-1}(A+B) = A^{-1}(AB)$

$A^{-1}A + A^{-1}B = (A^{-1}A)B$

$I + A^{-1}B = B$

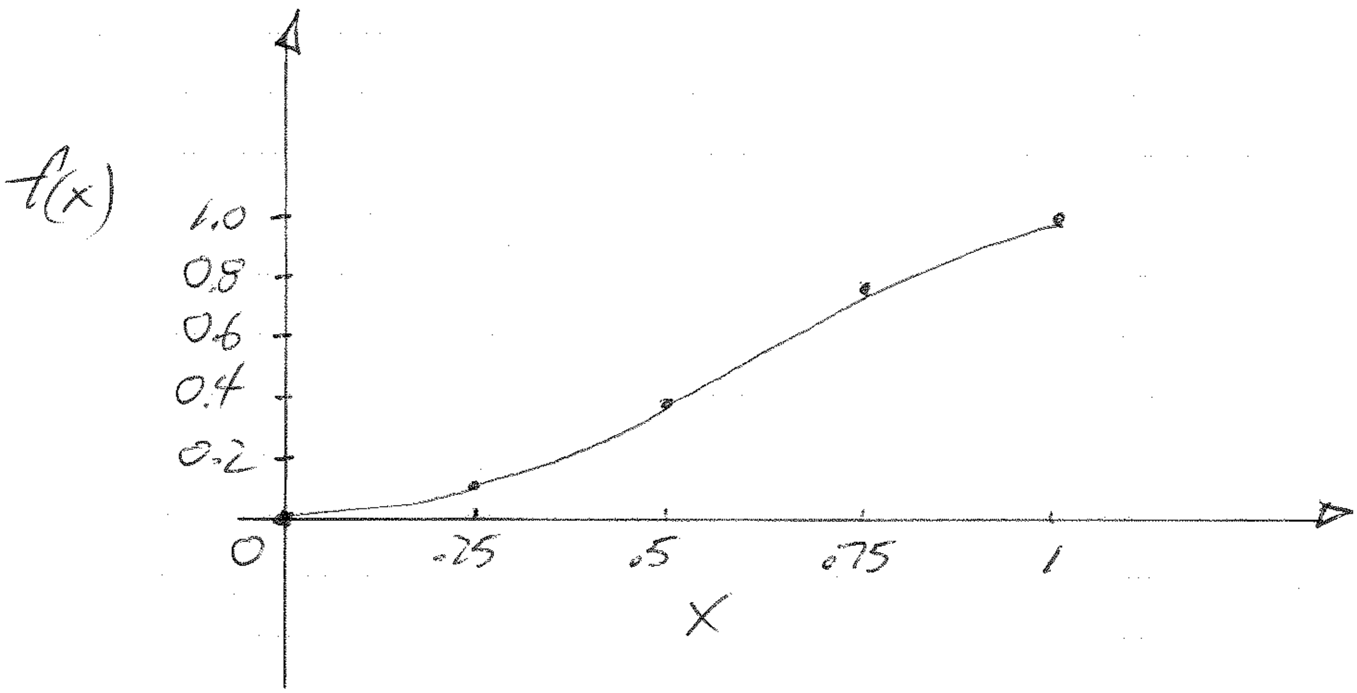
$(I + A^{-1}B)B^{-1} = BB^{-1}$

$B^{-1} + A^{-1}(BB^{-1}) = I$

$\therefore B^{-1} + A^{-1} = I$

Q5 a) $f(x) = \sin\left(\frac{\pi x^2}{2}\right)$

x	0	0.25	0.5	0.75	1
$f(x)$	0	.098	.383	.773	1



(NOTE: SKETCH MAY MAKE USE OF OTHER POINTS)

Q5 b) $T_4 = \sum_{i=1}^{4} \frac{f(x_{i-1}) + f(x_i)}{2} \Delta x$

$\Delta x = \frac{1-0}{4} = 0.25$

$T_4 = \frac{0.25}{2}\left(f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right)$

$= \frac{0.25}{2}\left(0 + 2(.098) + 2(.383) + 2(.773) + 1\right)$

$= 0.439$

c) START BY DETERMINING $K \geq |f''(x)|$

$f'(x) = \pi x \cos\left(\frac{\pi x^2}{2}\right)$

$f''(x) = \pi \cos\left(\frac{\pi x^2}{2}\right) - \pi^2 x^2 \sin\left(\frac{\pi x^2}{2}\right)$

BASED ON THE SKETCH OF $f(x)$ IN PART a) OR BY LOOKING AT $f''(x)$, IT WOULD APPEAR THAT THE MAXIMUM VALUE OF $|f''(x)|$ IS AT $x=0$ OR $x=1$.

$f''(x=0) = \pi$

$f''(x=1) = -\pi^2$

Q5

c) CONT'D

$\therefore$ CHOOSE $K = \pi^2$

$$\therefore |E| \leq \frac{\pi^2 (1-0)^3}{12 \cdot 4^2}$$

$$= \frac{\pi^2}{12 \cdot 16}$$

$$= 0.0514$$

d) WITH DIFFERENT n, THE BOUND ON $|E|$ WILL CHANGE BY:

$$\frac{4^2}{n^2}$$

FIND n SUCH THAT

$$\frac{4^2}{n^2} \leq 0.1 \quad \text{(90\% REDUCTION)}$$

FOR EXAMPLE, $n = 12 \Rightarrow \frac{4^2}{12^2} = 0.11$

$n = 13 \Rightarrow \frac{4^2}{13^2} = .095$

$\therefore$ n NEEDS TO BE INCREASED FROM 4 TO 13.

Q6 a) $$\frac{d^2T}{dx^2} = -h(x)$$

USING FINITE DIFFERENCES AND $\Delta x = 2.5$:

$$\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{(\Delta x)^2} = -h(x)$$

$$T(x+\Delta x) - 2T(x) + T(x-\Delta x) = -h(x)(\Delta x)^2$$

WRITING THIS EQUATION AT THE 3 INTERIOR POINTS:

$x = 2.5$
$$T(5) - 2T(2.5) + T(0) = -h(x)(2.5)^2$$

$x = 5$
$$T(7.5) - 2T(5) + T(2.5) = -h(x)(2.5)^2$$

$x = 7.5$
$$T(10) - 2T(7.5) + T(5) = -h(x)(2.5)^2$$

BOUNDARY CONDITIONS:

$$T(0) = 40$$
$$T(10) = 200$$

GIVEN $h(x) = 25$

Q6

a) CONT'D

$$T(5) - 2T(2.5) + 40 = -156.25$$

$$T(7.5) - 2T(5) + T(2.5) = -156.25$$

$$200 - 2T(7.5) + T(5) = -156.25$$

OR

$$T(5) - 2T(2.5) = -196.25$$

$$T(7.5) - 2T(5) + T(2.5) = -156.25$$

$$-2T(7.5) + T(5) = -356.25$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} T(2.5) \\ T(5) \\ T(7.5) \end{bmatrix} = \begin{bmatrix} -196.5 \\ -156.25 \\ -356.25 \end{bmatrix}$$

$$A \qquad X \quad = \quad B$$

Q6 b) NEED TO FIND A^{-1}

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow \text{ G.E.}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/4 & -1/2 & -1/4 \\ 0 & 1 & 0 & -1/2 & -1 & -1/2 \\ 0 & 0 & 1 & -1/4 & -1/2 & -3/4 \end{array}\right]$$

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

$$X = A^{-1}B$$

$$= \begin{bmatrix} -3/4 & -1/2 & -1/4 \\ -1/2 & -1 & -1/2 \\ -1/4 & -1/2 & -3/4 \end{bmatrix} \begin{bmatrix} -196.25 \\ -156.25 \\ -356.25 \end{bmatrix}$$

b) CONT'D

$$X = \begin{bmatrix} 314.375 \\ 432.500 \\ 394.375 \end{bmatrix}$$

x	0	.25	.5	.75	1
$T(x)$	40	314.375	432.500	394.375	200

c) $h(x)$ IS NO LONGER CONSTANT BUT CHANGES WITH X.

$$h(x=.25) = .12(.25)^3 - 2.4(.25)^2 + 12(.25)$$

$$= 16.875$$

$$h(x=.5) = 15.000$$

$$h(x=.75) = 5.625$$

Q6 c) CONT'D

$x = 2.5$

$$T(5) - 2T(2.5) + \cancel{T(0)}^{40} = (-16.875)(2.5)^2$$

$x = 5$

$$T(7.5) - 2T(5) + T(2.5) = (-15)(2.5)^2$$

$x = 7.5$

$$\cancel{T(10)}^{200} - 2T(7.5) + T(5) = (-5.625)(2.5)^2$$

OR

$$T(5) - 2T(2.5) = -145.46875$$

$$T(7.5) - 2T(5) + T(2.5) = -93.75$$

$$-2T(7.5) + T(5) = -235.15625$$

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} T(2.5) \\ T(5) \\ T(7.5) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -145.46875 \\ -93.75 \\ -235.15625 \end{bmatrix}}_{B}$$

$$X = A^{-1}B$$

→ ONLY B HAS CHANGED

→ A (AND A^{-1}) HAVE REMAINED UNCHANGED.