AER210 VECTOR CALCULUS and FLUID MECHANICS

Quiz 1

Duration: 75 minutes

7 October 2019

Closed Book, no aid sheets

Non-programmable calculators allowed

Instructor: Prof. Alis Ekmekci

Family Name: Alis Ekmekci

Given Name: Solutions

Student #: ________

TA Name/Tutorial #: ________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	12	
2	9	
3	10	
4	10	
5	10	
6	9	
TOTAL	60	/55

 Name: ____________________

1) a) (3 marks) Given R = {(x, y)| 0 ≤ x ≤ 2 and 1 ≤ y ≤ 4}, evaluate:

$$\iint_R (6x^2 + 4xy^3)dA$$

Method 1

$$= \int_1^4 \int_0^2 (6x^2 + 4xy^3)\,dx\,dy$$

$$= \int_1^4 \left[6\frac{x^3}{3} + 4\frac{x^2 y^3}{2} \right]_{x=0}^{x=2} dy$$

$$= \int_1^4 \left(2x^3 + 2x^2y^3\right)\Big|_{x=0}^{x=2} dy$$

$$= \int_1^4 (16 + 8y^3)\,dy$$

$$= \left. 16y + \frac{8y^4}{4} \right]_{y=1}^{y=4}$$

$$= \left[16\cdot(4) + 2\cdot(4)^4\right] - \left[16\cdot(1) + 2\cdot(1)^4\right]$$

$$= 558$$

Method 2

$$= \int_0^2 \int_1^4 (6x^2 + 4xy^3)\,dy\,dx$$

$$= \int_0^2 \left[6x^2y + 4\frac{xy^4}{4} \right]_{y=1}^{y=4} dx$$

$$= \int_0^2 \left(6x^2y + xy^4\right)\Big|_{y=1}^{y=4} dx$$

$$= \int_0^2 \left[6\cdot x^2(4) + 4^4x\right] - \left[6x^2(1) + x(1)\right] dx$$

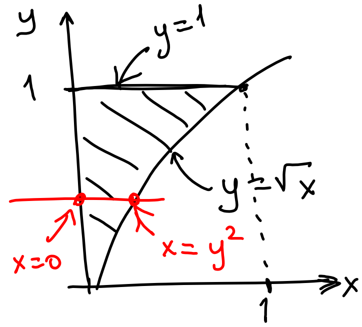
$$= \int_0^2 (24x^2 + 256x - 6x^2 - x)\,dx$$

$$= \int_0^2 (18x^2 + 255x)\,dx$$

$$= \left. \frac{\cancel{18}^{\,6} x^3}{\cancel{3}_{\,1}} + \frac{255x^2}{2} \right]_{x=0}^{x=2} = 558$$

(b) (5 marks) Evaluate the following integral by reversing the order of integration. Provide a sketch of the region, over which integration is performed.

$$\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\,dy\,dx$$



$$\int_{y=0}^{y=1} \int_{x=0}^{x=y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^1 \left. \sqrt{y^3+1}\,x \right]_{x=0}^{x=y^2} dy$$

$$= \frac{1}{3}\int_0^1 \underbrace{\sqrt{y^3+1}}_{\sqrt{u}}\ \underbrace{3y^2\,dy}_{du} = \frac{1}{3}\cdot\frac{2}{3}\left(y^3+1\right)^{3/2}\Big]_0^1 = \boxed{\frac{2}{9}\left(2^{3/2} - 1\right)}$$

$$y^3 + 1 = u \Rightarrow du = 3y^2\,dy$$

c) [4 marks] Use Leibnitz's rule to find f ' (x) for:

$$f(x) = \int_{\sin x}^{e^x} xy\,dy$$

$$f'(x) = \int_{\sin x}^{e^x} \underbrace{\frac{d}{dx}(xy)}_{y}\,dy + e^x\, x\, e^x - x \sin x \cos x$$

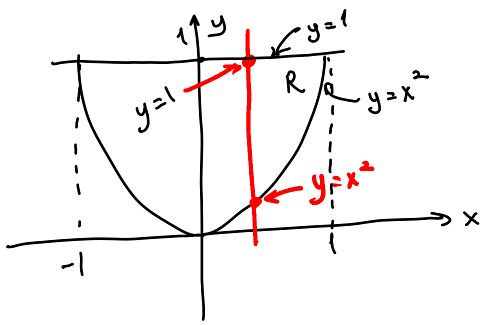
$$= \left.\frac{y^2}{2}\right]_{y=\sin x}^{y=e^x} + x e^{2x} - x \sin x \cos x$$

$$f'(x) = \frac{1}{2}\left(e^{2x} - \sin^2 x\right) + x e^{2x} - x \sin x \cos x$$

2) A two-dimensional plate has the shape of the region bounded by the graphs of $y = x^2$ and $y = 1$. If its mass density function is given by $\rho(x, y) = 1 + 2y + 6x^2$, which has the dimension of mass per unit area.

a) [3 marks] find the total mass of the plate,

b) [6 marks] find the coordinates of the center of mass of the plate,



a) $m = \iint_R \rho(x,y)\, dA =$

$$m = \int_{x=-1}^{x=1} \int_{y=x^2}^{y=1} (1+2y+6x^2)\, dy\, dx$$

$$m = \int_{x=-1}^{1} \Big[y + y^2 + 6x^2 y\Big]_{y=x^2}^{y=1} dx$$

$$= \int_{-1}^{1} \Big[(1+1+6x^2) - (x^2 + x^4 + 6x^4)\Big]\, dx$$

$$= \int_{-1}^{1} (2 + 5x^2 - 7x^4)\, dx$$

$$= 2x + 5\frac{x^3}{3} - \frac{7x^5}{5}\Big]_{x=-1}^{1}$$

$$= \left[2 + \frac{5}{3} - \frac{7}{5}\right] - \left[-2 - \frac{5}{3} + \frac{7}{5}\right]$$

$$= 2\left[\underset{(15)}{2} + \underset{(5)}{\frac{5}{3}} - \underset{(3)}{\frac{7}{5}}\right] = 2\left[\frac{30+25-21}{15}\right] = 2 \cdot \frac{34}{15} = \frac{68}{15}$$

$$\boxed{m = \frac{68}{15}}$$

(b) $m\bar{x} = M_y = \iint_R x\,\rho(x,y)\, dA \Rightarrow \bar{x} = \frac{1}{m} \iint_R x\,\rho(x,y)\, dA = \frac{1}{m} \int_{x=-1}^{1} \int_{y=x^2}^{1} x(1+2y+6x^2)\, dy\, dx$

$$\bar{x} = \frac{1}{m} \int_{-1}^{1} \int_{x^2}^{1} (x + 2xy + 6x^3)\, dy\, dx = \frac{1}{m} \int_{-1}^{1} \Big[xy + xy^2 + 6x^3 y\Big]_{y=x^2}^{y=1} dx$$

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$$= \frac{1}{m}\int_{-1}^{1}\left(\left[x + x + 6x^3\right] - \left[x^3 + x^5 + 6x^5\right]\right)dx = \frac{1}{m}\int_{-1}^{1}\left(2x + 5x^3 - 7x^5\right)dx$$

$$= \frac{1}{m}\left[x^2 + \frac{5x^4}{4} - \frac{7x^6}{6}\right]_{x=-1}^{1} = \frac{1}{m}\left[\left(1 + \frac{5}{4} - \frac{7}{6}\right) - \left(1 + \frac{5}{4} - \frac{7}{6}\right)\right] = 0$$

$\boxed{\bar{x} = 0}$ $\Rightarrow$ This should not surprise you since both the region and the density function are symmetric with respect to y axis (Notice that $\rho(-x, y) = \rho(x, y)$).

$$m\bar{y} = M_x = \iint_R y\,\rho(x,y)\,dA \Rightarrow \bar{y} = \frac{1}{m}\iint_R y\,\rho(x,y)\,dA$$

$$\bar{y} = \frac{1}{m}\int_{x=-1}^{1}\int_{y=x^2}^{1} y\left(1 + 2y + 6x^2\right)dy\,dx = \frac{1}{m}\int_{-1}^{1}\int_{x^2}^{1}\left(y + 2y^2 + 6x^2y\right)dy\,dx$$

$$= \frac{1}{m}\int_{-1}^{1}\left[\frac{y^2}{2} + \frac{2y^3}{3} + 3x^2y^2\right]_{y=x^2}^{y=1} dx$$

$$= \frac{1}{m}\int_{-1}^{1}\left[\left(\underset{(3)}{\frac{1}{2}} + \underset{(2)}{\frac{2}{3}} + 3x^2\right) - \left(\frac{x^4}{2} + \frac{2x^6}{3} + 3x^6\right)\right]dx$$

$$= \frac{1}{m}\int_{-1}^{1}\left(\frac{7}{6} + 3x^2 - \frac{x^4}{2} - \frac{11}{3}x^6\right)dx = \frac{1}{m}\left[\frac{7}{6}x + x^3 - \frac{x^5}{10} - \frac{11x^7}{21}\right]_{x=-1}^{1}$$

$$= \frac{15}{68}\left[\left(\frac{7}{6} + 1 - \frac{1}{10} - \frac{11}{21}\right) - \left(-\frac{7}{6} - 1 + \frac{1}{10} + \frac{11}{21}\right)\right]$$

$$= \frac{15}{68}\left(\cancel{2}\cdot\frac{7}{\cancel{6}_3} + 2 - \frac{\cancel{2}}{\cancel{10}_5} - \frac{22}{21}\right) = \frac{15}{68}\left(\underset{35}{\frac{7}{3}} + \underset{105}{2} - \underset{21}{\frac{1}{5}} - \underset{5}{\frac{22}{21}}\right) = \frac{15}{68}\left(\frac{245}{105} + \frac{210}{105} - \frac{21}{105} - \frac{110}{105}\right)$$

$$= \frac{\cancel{15}^{\,\cancel{3}\,1}}{\cancel{68}_{\cancel{34}\,17}}\left(\frac{\cancel{324}^{\,\cancel{162}\,\cancel{81}\,27}}{\cancel{105}_{\cancel{21}\,7}}\right) = \frac{81}{119}$$

$\boxed{\bar{y} = \frac{81}{119}}$

$\therefore \boxed{(\bar{x}, \bar{y}) = \left(0, \frac{81}{119}\right)}$

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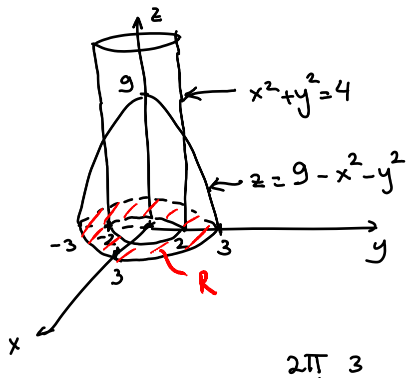
3) [10 marks] Forming the proper <u>double integral in polar coordinates</u>, find the volume that is inside the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 4$ and above the xy-plane.



$$V = \iint_R (9 - x^2 - y^2)\,dA = \int_{\theta=0}^{2\pi}\int_{r=2}^{3} (9 - r^2)\, r\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=2}^{3} (9r - r^3)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Bigg]_{r=2}^{r=3} d\theta = \int_0^{2\pi} \left[\left(\frac{9}{2}\cdot 9 - \frac{81}{4}\right) - \left(\frac{9\cdot 4}{2} - \frac{16}{4}\right)\right] d\theta$$

(2)

$$= \left(\frac{162 - 81}{4} - \frac{72}{4} + \frac{16}{4}\right)\theta\Big|_0^{2\pi}$$

$$= \frac{25}{4}\cdot 2\pi$$

$$= \frac{25\pi}{2}$$

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4) [10 marks] Re-write the following integral as an equivalent iterated integral in the integration order: first x, then y and then z:

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\,dzdydx$$

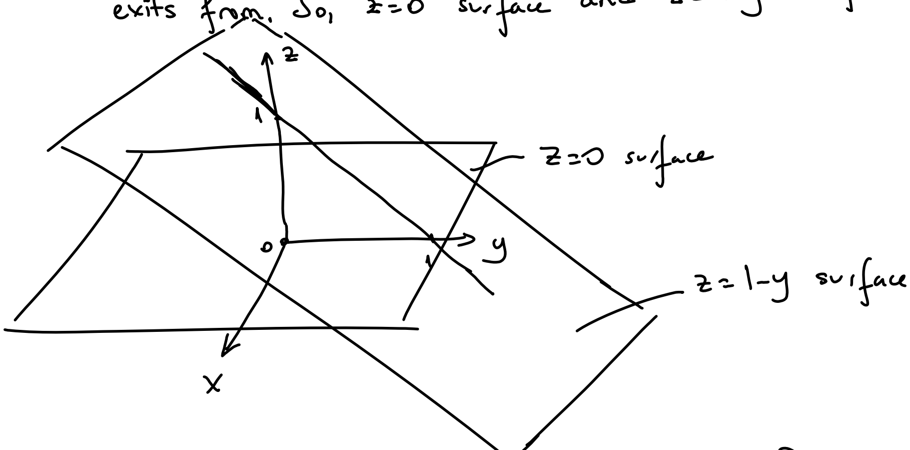
$Q = \{(x,y,z) \mid 0 \le x \le 1 \ , \ \sqrt{x} \le y \le 1 \ , \ 0 \le z \le 1-y\}$

First and foremost, we need to visualize this region:

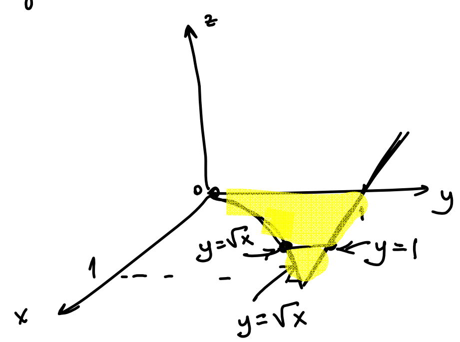
We can write the given triple integral, in the general form:

$$\iint_R \int_{z=g_1(x,y)}^{z=g_2(x,y)} f(x,y,z)\,dzdydx.$$

So, the upper and lower limits of z are functions of x and y, therefore these limits are three dimensional surfaces where a line piercing the volume in the z direction enters and exits from. So, $z=0$ surface and $z=1-y$ surface can be sketched as follows.



The base of the 3-dimensional region R is in the second and third performed integration directions (so, in the xy-plane). There in, $\sqrt{x} \le y \le 1$ and $0 \le x \le 1$. Let's sketch it.



The second integral of the triple integral (which is the first integral to be performed in the double integral over R region (base region)) is in the y direction. So, let's put a line parallel to y direction.

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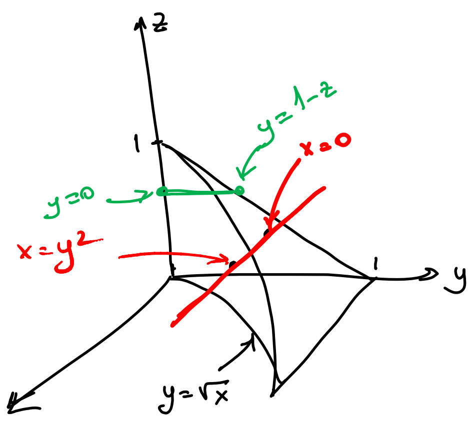
Then, the y line segment varies from $y=\sqrt{x}$ in the lower limit to $y=1$ in the upper limit. Note that x values should always be positive!

x direction limits are given from 0 to 1. So, the line segment only covers the highlighted region above, which is the R region.

Imagining the projections of the base region onto the $z=0$ and $z=1-y$ planes, we can draw the entire 3-dimensional region as:



So, if I write $dx\,dy\,dz$ order, then first we will be piercing the volume in the x direction. So, drawing a line parallel to x, we can determine its limits as

$$0 \le x \le y^2$$

For $dy\,dz$ part: that means in the double integral $\iint_R f\,dy\,dz$, first y will be varying while keeping z constant. So, let's draw a line parallel to y axis in the y-z plane (shown in green colour above). Then, the limits are:

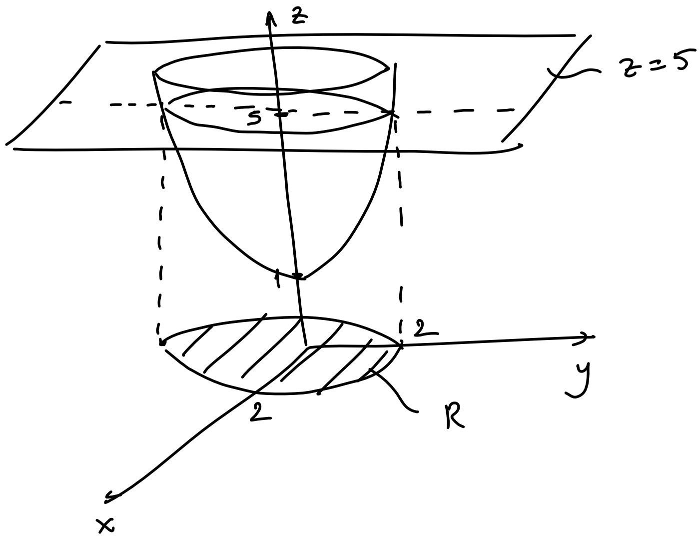
$$0 \le y \le 1-z$$

The y-direction line segment should travel from $z=0$ to $z=1$ to cover the region in y-z plane. So, we have:

$$0 \le z \le 1$$

Then the integral can be written as: $\displaystyle\int_0^1 \int_0^{1-z} \int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$

5) (10 marks) Find the surface area of the portion of the paraboloid $z = 1 + x^2 + y^2$ that lies below the plane $z = 5$.



$$f(x,y) = 1 + x^2 + y^2$$
$$f_x(x,y) = 2x$$
$$f_y(x,y) = 2y$$

@ $z=5$

$5 = 1 + x^2 + y^2$

$x^2 + y^2 = 4$

R region is a circle with radius 2

$$S = \iint_R \sqrt{(f_x(x,y))^2 + (f_y(x,y))^2 + 1}\, dA$$

$$= \iint_R \sqrt{4(x^2+y^2)+1}\, dA$$

Since the R region is circular and the integrand contains the term x^2+y^2, it would be best to use polar coordinates to solve this integral.

$$S = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \frac{1}{8} \underbrace{\sqrt{4r^2+1}}_{\sqrt{u}}\, \underbrace{8r\,dr}_{du}\, d\theta = \frac{1}{8}\int_{\theta=0}^{2\pi} \left[\frac{2(4r^2+1)^{3/2}}{3}\right]_{r=0}^{2} d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$$S = \frac{1}{12}\int_{\theta=0}^{2\pi} (4r^2+1)^{3/2}\Big|_{r=0}^{2} d\theta = \frac{1}{12}\int_{\theta=0}^{2\pi} \left[(4.4+1)^{3/2} - (0+1)^{3/2}\right] d\theta$$

$$= \frac{1}{12}(17^{3/2}-1)\,\theta\Big|_{\theta=0}^{2\pi} = \frac{1}{12}(17^{3/2}-1)\cdot 2\pi = \boxed{\frac{\pi}{6}(17^{3/2}-1) \approx 36.18}$$

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6) (9 marks) Solve the integral equation: $f(x) = 7 - 2x + \int_1^x (-3)e^{3(x-t)} f(t)dt$

$$f(x) = 7 - 2x + \int_1^x (-3)e^{3(x-t)} f(t)\, dt$$

$$f'(x) = -2 + \int_1^x \frac{d}{dx}\left((-3)e^{3(x-t)} f(t)\right) dt - 3e^{3(x-x)} f(x) \cdot \cancel{\frac{dx}{dx}}^{\,1}$$

$$f'(x) = -2 - 3\underbrace{\int_1^x 3e^{3(x-t)} f(t)\, dt}_{7-2x-f(x)} - 3e^{0} f(x)$$

1

$7 - 2x - f(x)$ ← based on the definition of $f(x)$ given in the question.

$$f'(x) = -2 - 3(7 - 2x - f(x)) - 3f(x)$$

$$= -2 - 21 + 6x + 3\cancel{f(x)} - 3\cancel{f(x)}$$

$$f'(x) = 6x - 23$$

$\Downarrow$ Integrating this

$$f(x) = \int f'(x)\,dx = \int (6x - 23)\,dx = 3x^2 - 23x + C$$

$$f(1) = 7 - 2(1) = 5 \Rightarrow \underbrace{f(1)}_{} = 3\cdot(1)^2 - 23\cdot(1) + C$$

$$5 = 3 - 23 + C$$

$$\boxed{C = 25}$$

$$\boxed{f(x) = 3x^2 - 23x + 25}$$