AER210 VECTOR CALCULUS and FLUID MECHANICS

Midterm Test # 1

Duration: 1 hour, 50 minutes

20 October 2022

Closed Book, no aid sheets, no calculators

Instructor: Prof. Alis Ekmekci

Family Name: Solutions

Given Name:

Student #:

TA Name/Tutorial #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	18	
2	12	
3	17	
4	10	
5	10	
6	12	
7	13	
8	10	
TOTAL	102	

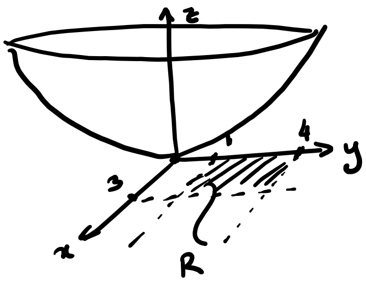
Note the following integrals may be useful:

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \quad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C; \quad \sin^2\theta + \cos^2\theta = 1$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Name: ____________________

1) a) [5 points] Find the volume beneath the surface $z = x^2 + y^2$ and above the rectangular region R in the xy plane, where $0 \le x \le 3$ and $1 \le y \le 4$, forming the appropriate <u>double integral</u> and calculating it.



$$V = \int_{y=1}^{4}\int_{x=0}^{3}(x^2+y^2)\,dx\,dy = \int_{y=1}^{4}\left[\frac{x^3}{3}+y^2x\right]\Bigg|_{x=0}^{x=3} dy$$

$$= \int_{y=1}^{4}(9+3y^2)\,dy = (9y+y^3)\Big|_{y=1}^{y=4}$$

$$= 9.4+4^3-9-1$$

$$= 90$$

b) [5 points] Evaluate the triple integral

$$\iiint_Q f(x,y,z)dV$$

where $f(x,y,z) = 2xy - 3xz^2$ and $Q = \{(x,y,z)|\ 0 \le x \le 2, -1 \le y \le 1, 0 \le z \le 2\}$

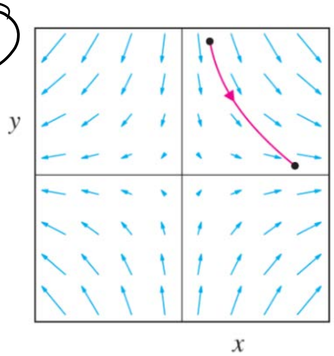
$$\iiint_Q f(x,y,z)\,dV = \int_{z=0}^{2}\int_{y=-1}^{1}\int_{x=0}^{2}(2xy-3xz^2)\,dx\,dy\,dz = \int_{0}^{2}\int_{-1}^{1}\left[x^2y-\frac{3x^2}{2}z^2\right]\Bigg|_{x=0}^{2} dy\,dz$$

$$= \int_{0}^{2}\int_{-1}^{1}\left[4y-6z^2\right]dy\,dz = \int_{0}^{2}(2y^2-6z^2y)\Big|_{y=-1}^{1}dz = \int_{0}^{2}(\cancel{2}-6z^2-\cancel{2}-6z^2)\,dz$$

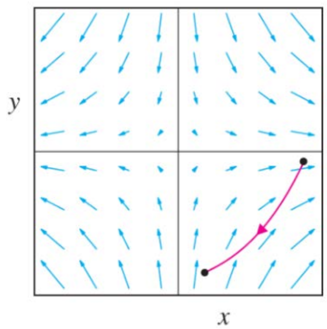
$$= -12\int_{0}^{2}z^2\,dz = -12\cdot\frac{z^3}{3}\Bigg|_{z=0}^{2} = -12\cdot\frac{8}{3} = -32$$

c) [2 points] Below, in each plot, a force vector field and the path of a particle are shown. Based on the visual inspection of these graphs, select the cases where the work done by the force field is *positive*.

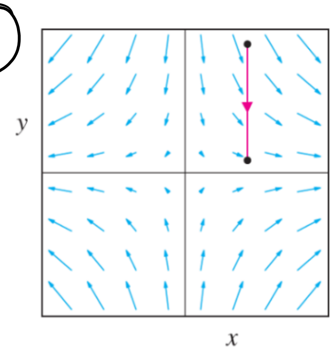
a)



b)



c)



d) [6 points] Circle those that are **true**:

i) $\int_C f(x,y)ds = -\int_{-C} f(x,y)ds$ where C and $-C$ are curves that are reverse of each other.

ii) $\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$ where C and $-C$ are curves that are reverse of each other.

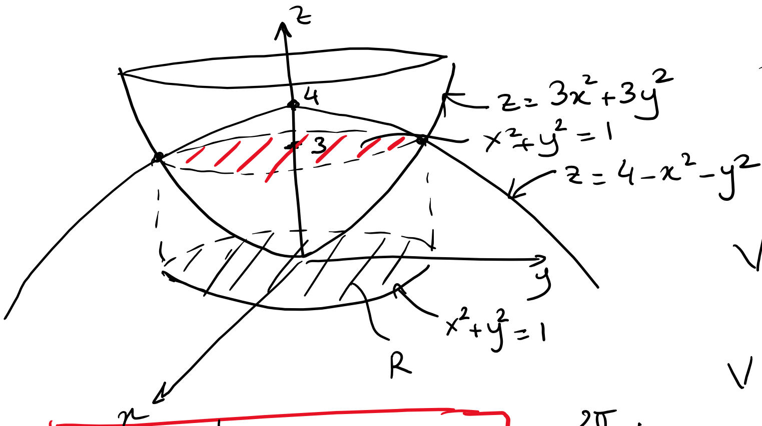
iii) If $\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}$ is the vector equation showing parametrically a smooth surface S, where u and v are parameters from some region D, $\vec{r_u}(u,v)\times\vec{r_v}(u,v)$ shows a vector field tangent to the surface S.

iv) For a definite single integral, $\int_a^b f(x)dx = -\int_b^a f(x)dx$

v) If $\vec{r}(t)$ is the parametric equation of a curve C, then for the curve to be smooth, $\vec{r}'(t)$ should be continuous and $\vec{r}'(t) \neq \vec{0}$ except at the endpoints of C.

vi) In a conservative force vector field, the work done by the force in carrying an object from any point A to a point B is always zero.

2) a) [6 points] Form the double integral in polar coordinates that gives the volume trapped between the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$. Calculate this volume. Provide a sketch of the region.



Intersection:

$$3x^2+3y^2 = 4 - x^2 - y^2$$

$$\Rightarrow \boxed{x^2+y^2=1}$$

$$V = \iint_R \left[(4-x^2-y^2)-(3x^2+3y^2)\right] dx\,dy$$

$x^2+y^2=r$

$$V = \iint_R 4(1-x^2-y^2)\,dx\,dy$$

$$\boxed{V = 4\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1-r^2)\,r\,dr\,d\theta} = 4\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(r-r^3)\,dr\,d\theta = 4\int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1 d\theta = 4\int_0^{2\pi}\frac{1}{4}\,d\theta = 2\pi$$

b) [6 points] Compute the mass m of the helical spring given by the parametric equations $x = cos2t$, $y = sin2t$, $z = t$, where $0 \le t \le \pi$.The length density of this spring (i.e., mass per unit length) is $\rho = z^2$.

$$m = \int_C \rho(x,y,z)\,ds = ?$$

$$\vec{r}(t) = \underbrace{\cos 2t}_{x(t)}\,\vec{i} + \underbrace{\sin 2t}_{y(t)}\,\vec{j} + \underbrace{t}_{z(t)}\,\vec{k}$$

$$\vec{r}\,'(t) = -2\sin 2t\,\vec{i} + 2\cos 2t\,\vec{j} + \vec{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{(-2\sin 2t)^2 + (2\cos 2t)^2 + 1^2}$$

$$= \sqrt{4(\underbrace{\sin^2 2t + \cos^2 2t}_{1}) + 1}$$

$$\boxed{\|\vec{r}\,'(t)\| = \sqrt{5}}$$

$$ds = \|\vec{r}\,'(t)\|\,dt = \sqrt{5}\,dt$$

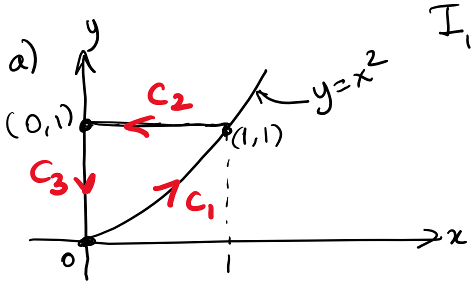
$$m = \int_C \rho(x,y,z)\,ds = \int_{t=0}^{\pi} \rho(x(t),y(t),z(t))\,\|\vec{r}\,'(t)\|\,dt = \int_{t=0}^{t} t^2\sqrt{5}\,dt$$

$$= \sqrt{5}\,\frac{t^3}{3}\Big|_{t=0}^{\pi} = \frac{\sqrt{5}}{3}\pi^3$$

3) [17 points = (part a: 12 points, part b: 5 points)]

Evaluate the following line integral by two methods**: a) directly and b) using the Green's theorem**. C consists of the arc of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ and the line segments from $(1,1)$ to $(0,1)$ and from $(0,1)$ to $(0,0)$:

a)



$$I_1 = \oint_C x^2y^2dx + xydy = \oint_C \vec{F}(x,y)\cdot d\vec{r} = ?$$

$$\vec{F}(x,y) = x^2y^2\,\vec{i} + xy\,\vec{j}$$

$$C_1:\ y = x^2 \Rightarrow x = t,\ y = t^2 \Rightarrow \boxed{\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} \quad \text{where } 0 \le t \le 1}$$

$$I_1 = \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^1 \left((t^6\,\vec{i} + t^3\,\vec{j})\cdot(\vec{i} + 2t\,\vec{j})\right)dt$$

$$= \int_0^1 (t^6 + 2t^4)\,dt = \frac{t^7}{7} + \frac{2t^5}{5}\Big|_{t=0}^{1} = \frac{1}{7} + \frac{2}{5} = \frac{19}{35}$$

$$C_2:\ \left.\begin{array}{l}\vec{r}_1 = (1,1)\\ \vec{r}_2 = (0,1)\end{array}\right\}\ \vec{r} = \vec{r}_1 + t(\vec{r}_2 - \vec{r}_1) \Rightarrow \vec{r} = (1,1) + t((0,1) - (1,1))$$

$$\vec{r} = (1,1) + t(-1,0)$$

$$\vec{r}(t) = (1-t,\ 1)$$

$$\boxed{C_2:\ \vec{r}(t) = (1-t)\,\vec{i} + \vec{j} \quad \text{where } 0 \le t \le 1}$$

$$I_2 = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_0^1 \left[(1-t)^2\,\vec{i} + (1-t)\,\vec{j}\right]\cdot(-\vec{i})\,dt$$

$$= \int_0^1 -(1-t)^2\,dt = \int_0^1 (-1 + 2t - t^2)\,dt = \left(-t + t^2 - \frac{t^3}{3}\right)\Big|_{t=0}^{1} = \cancel{-1} + \cancel{1} - \frac{1}{3} = -\frac{1}{3}$$

$$C_3:\ \vec{r} = (0,1) + t((0,0) - (0,1)) = (0,1) + t(0,-1) = (0,1-t)$$

$$\boxed{C_3:\ \vec{r}(t) = (1-t)\,\vec{j} \quad \text{where } 0 \le t \le 1}$$

EXTRA PAGE

$$I_3 = \int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = 0$$

$$\vec{F}(\vec{r}(t)) = 0\hat{i} + 0\hat{j} = \vec{0}$$

$$I = I_1 + I_2 + I_3 = \frac{19}{35} - \frac{1}{3} + 0 = \frac{22}{105}$$

b) Green's theorem:

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right) dA$$

$$\oint_C \underbrace{x^2y^2}_{P(x,y)}\,dx + \underbrace{xy}_{Q(x,y)}\,dy = \iint_R \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2y^2)\right] dA = \iint_R (y - 2x^2y)\,dA$$

$$= \boxed{\int_{x=0}^{1}\int_{y=x^2}^{1} (y - 2x^2y)\,dy\,dx}$$

$$= \int_0^1 \left(\frac{y^2}{2} - x^2y^2\right)\Bigg|_{y=x^2}^{y=1} dx = \int_0^1 \left(\frac{1}{2} - x^2 - \frac{x^4}{2} + x^6\right) dx$$

$$= \left(\frac{1}{2}x - \frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{7}\right)\Bigg|_{x=0}^{1} = \frac{1}{2} - \frac{1}{3} - \frac{1}{10} + \frac{1}{7}$$

$$= \frac{22}{105}$$

Name: ____________________

4) [10 points] Prove that the vector field $\vec{F}(x,y) = xy^2\vec{i} + x^2y\vec{j}$ is a conservative vector field. Then, finding a potential function $f(x,y)$ of the vector field, evaluate $\int_C \vec{F}(x,y)\cdot d\vec{r}$ over the curve C given by the parametric equation $\vec{r}(t) = t\sin(\pi t)\vec{i} + \cos(\pi t^2)\vec{j}$ for $0 \le t \le 1$.

$$\vec{F}(x,y) = \underbrace{xy^2}_{P(x,y)}\vec{i} + \underbrace{x^2y}_{Q(x,y)}\vec{j}$$

$$P(x,y) = xy^2 \rightarrow \frac{\partial P(x,y)}{\partial y} = \frac{\partial}{\partial y}(xy^2) = 2xy$$

$$Q(x,y) = x^2 \rightarrow \frac{\partial}{\partial x}Q(x,y) = \frac{\partial}{\partial x}(x^2y) = 2xy$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F} \text{ is conservative}$$

$$\vec{F} = \nabla f \Rightarrow P(x,y)\vec{i} + Q(x,y)\vec{j} = \frac{\partial f(x,y)}{\partial x}\vec{i} + \frac{\partial f(x,y)}{\partial y}\vec{j}$$

$$P(x,y) = \frac{\partial f(x,y)}{\partial x} \quad \& \quad Q(x,y) = \frac{\partial f(x,y)}{\partial y}$$

$$xy^2 = \frac{\partial}{\partial x}f(x,y) \Rightarrow f(x,y) = \int xy^2\,dx = f(x,y) = \frac{x^2y^2}{2} + g(y)$$

$$xy^2 = \frac{\partial}{\partial y}(f(x,y)) \Rightarrow x^2y = \frac{\partial}{\partial y}\left(\frac{x^2y^2}{2} + g(y)\right)$$

$$x^2y = x^2y + g'(y) \Rightarrow g'(y) = 0$$

$$\boxed{f(x,y) = \frac{x^2y^2}{2}}$$

$$\vec{r}(t=0) = 0\vec{i} + \cos 0\vec{j} = \vec{j} \Rightarrow (0,1)$$

$$\vec{r}(t=1) = \sin\pi\,\vec{i} + \cos\pi\,\vec{j} = -\vec{j} \rightarrow (0,-1)$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f\cdot d\vec{r} = f(\vec{r}(t=1)) - f(\vec{r}(t=0)) = f(0,-1) - f(0,1) = 0$$

5) Suppose that the surface S is the portion of the cylindrical surface $x^2 + y^2 = 1$ with $x \geq 0$ and z between $z = 1$ and $z = 2$.

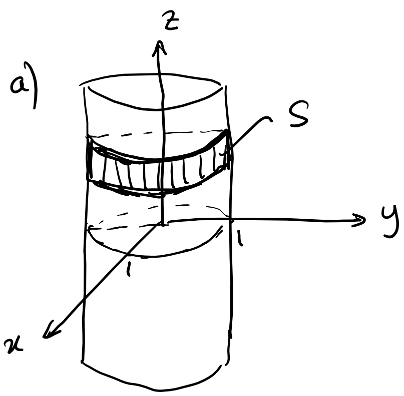
a) [1 points] Sketch the surface S

b) [3 points] Parametrize the surface

c)[6 points] Evaluate the following surface integral by converting it to a double integral in terms of its parameters.

$$\iint_S z\,dS$$

a)



b) Parametric form of S:

$$\left.\begin{array}{l} x = \cos\theta \\ y = \sin\theta \\ z = z \end{array}\right\}$$

Parameters: θ & z

$-\pi/2 \leq \theta \leq \pi/2$

$1 \leq z \leq 2$

In vector form:

$\vec{r}(\theta, z) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + z\,\vec{k}$

$-\pi/2 \leq \theta \leq \pi/2$, $1 \leq z \leq 2$

c) $\vec{r}_\theta(\theta, z) = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$

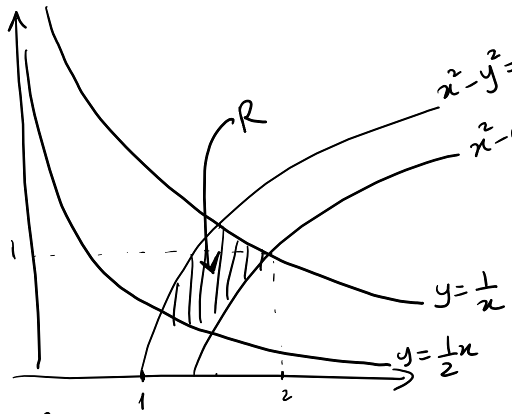
$\vec{r}_z(\theta, z) = \vec{k}$

$$\vec{r}_\theta(\theta, z) \times \vec{r}_z(\theta, z) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

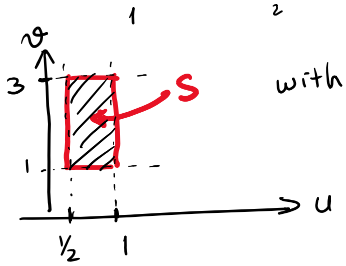
$$\|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\iint_S z\,dS = \int_{\theta=-\pi/2}^{\pi/2} \int_{z=1}^{2} z\,\underbrace{\|\vec{r}_\theta \times \vec{r}_z\|}_{1}\,dz\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{z^2}{2}\Big|_{z=1}^{2} d\theta = \frac{3}{2}\,\theta\Big|_{\theta=-\pi/2}^{\pi/2} = \frac{3}{2}\pi$$

6) [12 points] Let R be the plane region with density $\rho(x,y) = x^2y^2$ [kg/m^2] that is bounded by the hyperbolas: $xy = ½$, $xy = 1$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 3$. Use an appropriate coordinate transformation to find the polar moment of inertia $I_o = \iint_R (x^2 + y^2)\rho dxdy$ of this region. Sketch the region in both the xy plane and the plane of new variables.



$$\left.\begin{aligned} xy &= \tfrac{1}{2} \\ xy &= 1 \\ x^2-y^2 &= 1 \\ x^2-y^2 &= 3 \end{aligned}\right\} \begin{aligned} u &= xy \\ v &= x^2-y^2 \end{aligned} \implies \begin{aligned} u &= \tfrac{1}{2} \\ u &= 1 \\ v &= 1 \\ v &= 3 \end{aligned}$$



with $u = xy$
$v = x^2 - y^2$

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \det\begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix} = -2(x^2+y^2) \Rightarrow \boxed{\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{2(x^2+y^2)}}$$

$$I_o = \iint_R (x^2+y^2)\,\rho(x,y)\,dx\,dy = \iint_S (x^2+y^2)\cdot x^2y^2 \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_S \cancel{(x^2+y^2)}\cdot x^2y^2 \cdot \frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv = \int_{v=1}^{3}\int_{u=\frac{1}{2}}^{1} \frac{u^2}{2}\,du\,dv = \int_{v=1}^{3} \left.\frac{u^3}{6}\right|_{u=\frac{1}{2}}^{u=1} dv$$

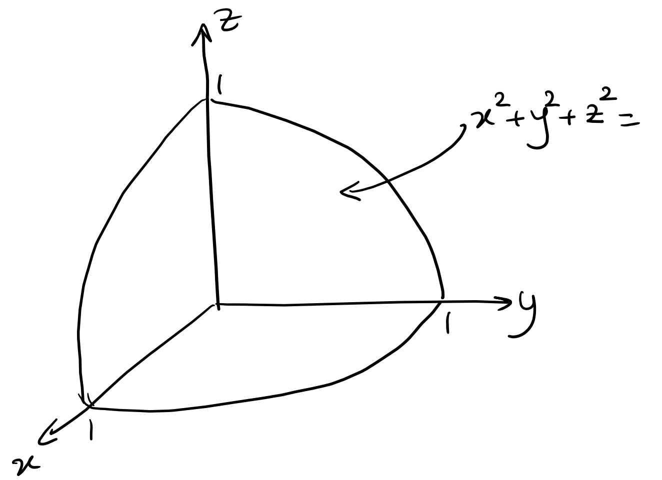
$$= \int_{v=1}^{3} \frac{1}{6}\left(1-\frac{1}{8}\right) = \frac{7}{48}\,v\Big|_{v=1}^{v=3} = \frac{7}{48}(3-1) = \frac{7}{24}$$

7) a) [8 points] Sketch the solid whose volume is given by the following triple integral, and then re-write the iterated integral in the integration order of first y, then z and then x.

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} dzdydx$$

b) [5 points] Calculate the volume. Note that while calculating if the change of variables makes the evaluation of this integration easy, you can do so.

a)



$$V = \int_{x=0}^{1} \int_{z=0}^{\sqrt{1-x^2}} \int_{y=0}^{\sqrt{1-x^2-z^2}} dydzdx$$

b)

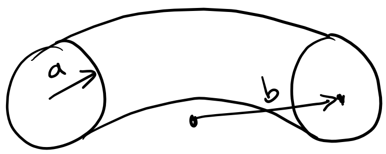
$$V = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\rho^3}{3} \sin\phi \Big|_{\rho=0}^{\rho=1} d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{1}{3} \sin\phi \, d\theta d\phi = \int_{\phi=0}^{\pi/2} \frac{1}{3} \sin\phi \, \theta \Big|_{\theta=0}^{\pi/2} d\phi = \frac{1}{3} \cdot \frac{\pi}{2} \int_{\phi=0}^{\pi/2} \sin\phi \, d\phi$$

$$= \frac{\pi}{6} (-\cos\phi) \Big|_{\phi=0}^{\pi/2} = \frac{\pi}{6} \cdot \Big(\underbrace{-\cos\frac{\pi}{2}}_{0} + \underbrace{\cos 0}_{1} \Big) = \frac{\pi}{6} \ //$$

Name: ____________________

8) [10 points] Find the surface area of the torus, which is given by the following parametric equations: $x = (b + a\cos\phi)\cos\theta$, $y = (b + a\cos\phi)\sin\theta$, $z = a\sin\phi$, where a and b are constants and the parameters vary as follows: $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$.



$$\left.\begin{aligned} x &= (b+a\cos\phi)\cos\theta \\ y &= (b+a\cos\phi)\sin\theta \\ z &= a\sin\phi \end{aligned}\right\} \quad \begin{aligned} &0 \le \theta \le 2\pi \\ &0 \le \phi \le 2\pi \end{aligned}$$

$$x_\theta = -(b+a\cos\phi)\sin\theta \qquad x_\phi = -a\sin\phi\cos\theta$$

$$y_\theta = (b+a\cos\phi)\cos\theta \qquad y_\phi = -a\sin\phi\sin\theta$$

$$z_\theta = 0 \qquad z_\phi = a\cos\phi$$

$$\vec{N} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(b+a\cos\phi)\sin\theta & (b+a\cos\phi)\cos\theta & 0 \\ -a\sin\phi\cos\theta & -a\sin\phi\sin\theta & a\cos\phi \end{bmatrix}$$

$$= \vec{i}\left[a(b+a\cos\phi)\cos\phi\cos\theta\right] + \vec{j}\left[a(b+a\cos\phi)\cos\phi\sin\theta\right]$$

$$+\vec{k}\left[\underbrace{a(b+a\cos\phi)\sin\phi\sin^2\theta + a(b+a\cos\phi)\sin\phi\cos^2\theta}_{a(b+a\cos\phi)\sin\phi}\right]$$

$$\|\vec{N}\| = a(b+a\cos\phi)\sqrt{\cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi} = a(b+a\cos\phi)$$

$$S = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} \|\vec{N}\|\, d\theta\, d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi} a(b+a\cos\phi)\, d\theta\, d\phi = \int_{\phi=0}^{2\pi} a(b+a\cos\phi)\,\theta\Big|_0^{2\pi} d\phi$$

$$= \int_{\phi=0}^{2\pi} a(b+a\cos\phi)\cdot 2\pi\, d\phi = 2\pi\left[ab\phi + a^2\sin\phi\right]_{\phi=0}^{2\pi} = 2\pi ab\cdot 2\pi + 2\pi a^2(\underbrace{\sin 2\pi}_{0} - \underbrace{\sin 0}_{0})$$

$$= 2\pi a \cdot 2\pi b$$