ESC195 - Midterm Test #1
February 9, 2023
9:10 - 10:50 am
Instructor: J. W. Davis

Closed book, no aid sheets, no calculators
There are 7 questions worth 10 marks.
Plus a bonus question worth 5 marks.

1. Use l'Hospital's rule to evaluate the following limits:

a) $\lim_{x\to e} \dfrac{\ln x - 1}{x - e}$ b) $\lim_{x\to 0^+} (\sin x)\sqrt{\dfrac{1-x}{x}}$ c) $\lim_{x\to\infty} (\tanh x)^x$

a) $\lim_{x\to e} \dfrac{\ln x - 1}{x - e} \overset{*}{=} \lim_{x\to e} \dfrac{\frac{1}{x}}{1} = \dfrac{1}{e}$

0/0 type

b) $\lim_{x\to 0^+} (\sin x)\sqrt{\dfrac{1-x}{x}} = \lim_{x\to 0^+} \sqrt{1-x}\cdot\dfrac{\sin x}{\sqrt{x}} = \lim_{x\to 0^+}\sqrt{1-x}\cdot\lim_{x\to 0^+}\dfrac{\sin x}{\sqrt{x}}$

$0\cdot\infty$ 0/0

$= 1\cdot\lim_{x\to 0^+}\dfrac{\sin x}{\sqrt{x}} \overset{*}{=} \lim_{x\to 0^+}\dfrac{\cos x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to 0^+} 2\sqrt{x}\cos x = 0$

c) consider $\lim_{x\to\infty} \ln(\tanh x)^x = \lim_{x\to\infty} x\ln(\tanh x) = \lim_{x\to\infty}\dfrac{\ln(\tanh x)}{\frac{1}{x}} \to \dfrac{0}{0}$

$\overset{*}{=} \lim_{x\to\infty}\dfrac{\frac{(\operatorname{sech} x)^2}{\tanh x}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\dfrac{\frac{1}{\cosh x\sinh x}}{\frac{-1}{x^2}} = \lim_{x\to\infty}\dfrac{-x^2}{\cosh x\sinh x} \to \dfrac{\infty}{\infty}$

$\overset{*}{=} \lim_{x\to\infty}\dfrac{-2x}{\cosh^2 x+\sinh^2 x} \overset{*}{=} \lim_{x\to\infty}\dfrac{-2}{4\cosh x\sinh x} = 0$

$\therefore \lim_{x\to\infty}(\tanh x)^x = e^0 = 1$

(alternate) consider $\ln\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^x = x\ln\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right) = \dfrac{\ln\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)}{\frac{1}{x}} \to \dfrac{0}{0}$

$\lim_{x\to\infty}\dfrac{\ln\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)}{1/x} \overset{*}{=} \lim_{x\to\infty}\dfrac{\frac{e^x+e^{-x}}{e^x-e^{-x}} - \frac{e^x-e^{-x}}{e^x+e^{-x}}}{-1/x^2}$

$= \lim_{x\to\infty}\dfrac{\frac{e^{2x}+2+e^{-2x}-(e^{2x}-2+e^{-2x})}{e^{2x}-e^{-2x}}}{-1/x^2} = \lim_{x\to\infty}\dfrac{\frac{4}{e^{2x}-e^{-2x}}}{-1/x^2} = \lim_{x\to\infty}\dfrac{-4x^2}{e^{2x}-e^{-2x}} \to \dfrac{\infty}{\infty}$

$\overset{*}{=} \lim_{x\to\infty}\dfrac{-8x}{2e^{2x}+2e^{-2x}} \overset{*}{=} \lim_{x\to\infty}\dfrac{-8}{4e^{2x}-4e^{-2x}} = 0$

$\therefore \lim_{x\to\infty}\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)^x = \lim_{x\to\infty}(\tanh x)^x = e^0 = 1$

2. Evaluate the integrals:

a) $\int x\tan^2 x\,dx$ b) $\int \frac{dx}{x^4\sqrt{9x^2-1}}$ c) $\int \frac{3x+27}{(x-1)(x^2+9)}\,dx$

a) $\int x\tan^2 x\,dx$

let $u = x$ $\quad dv = \tan^2 x\,dx$

$du = dx$ $\quad = (\sec^2 x - 1)\,dx$

$v = \tan x - x$

$= x(\tan x - x) - \int (\tan x - x)\,dx$

$= x\tan x - x^2 + \frac{x^2}{2} - \ln|\sec x| + C = x\tan x - \frac{x^2}{2} - \ln|\sec x| + C$

b) $\int \frac{dx}{x^4\sqrt{9x^2-1}}$

let $3x = \sec\theta$

$3dx = \sec\theta\tan\theta\,d\theta$

$\sqrt{9x^2-1} = \tan\theta$

$\sin\theta = \frac{\sqrt{9x^2-1}}{3x}$

$= \int \frac{\frac{1}{3}\sec\theta\tan\theta\,d\theta}{\frac{1}{3^4}\sec^4\theta\tan\theta} = 3^3\int\frac{d\theta}{\sec^3\theta} = 3^3\int\cos^3\theta\,d\theta$

$= 3^3\int(1-\sin^2\theta)\cos\theta\,d\theta = 27\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + C$

$= 9\,\frac{\sqrt{9x^2-1}}{x} - \frac{(9x^2-1)^{3/2}}{3x^3} + C$

c) $\frac{3x+27}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$ $\Rightarrow 3x+27 = Ax^2+9A+Bx^2-Bx+Cx-C$

$x^2:\ A+B=0$

$x:\ -B+C=3$

$1:\ 9A-C=27$

$A=3$

$B=-3$

$C=0$

$\therefore \int\frac{3x+27}{(x-1)(x^2+9)}\,dx = \int\frac{3}{x-1}\,dx - \int\frac{3x}{x^2+9}\,dx$

$= 3\ln|x-1| - \frac{3}{2}\ln(x^2+9) + C$

3. Determine whether the integral is convergent or divergent. Evaluate the integrals that are convergent.

a) $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$ b) $\displaystyle\int_1^\infty \frac{3x^2+1}{x^3+x}\,dx$ c) $\displaystyle\int_{-\infty}^\infty \frac{x^3}{1+x^8}\,dx$

a) $\displaystyle\int_0^8 x^{-1/3}\,dx = \lim_{a\to 0} \int_a^8 x^{-1/3}\,dx = \lim_{a\to 0}\left[\frac{3}{2}x^{2/3}\right]_a^8$

$\displaystyle = \lim_{a\to 0} \frac{3}{2}\left(4 - a^{2/3}\right) = 6$

b) $\displaystyle\int_1^\infty \frac{3x^2+1}{x^3+x}\,dx = \lim_{b\to\infty}\int_1^b \frac{3x^2+1}{x^3+x}\,dx = \lim_{b\to\infty}\left[\ln(x^3+x)\right]_1^b$

$\displaystyle = \lim_{b\to\infty}\left(\ln(b^3+b) - \ln 2\right) \longrightarrow \infty$ diverges

c) $\dfrac{x^3}{1+x^8}$ is an odd function $\therefore \displaystyle\int_{-b}^0 \frac{x^3}{1+x^8}\,dx = -\int_0^b \frac{x^3}{1+x^8}\,dx$

i) For $-1 \le x \le 1$, $\dfrac{x^3}{1+x^8}$ takes on finite values $\therefore \displaystyle\int_0^1 \frac{x^3}{1+x^8}\,dx$ exists

ii) For $x > 1$, $\dfrac{x^3}{1+x^8} < \dfrac{1}{x^5} \Rightarrow \displaystyle\int_1^\infty \frac{dx}{x^5}$ converges $\left(\frac{1}{x^p},\ p>1\right)$

$\therefore \displaystyle\int_1^\infty \frac{x^3}{1+x^8}\,dx$ converges

iii) given $\displaystyle\int_0^\infty \frac{x^3}{1+x^8}\,dx$ converges and $\displaystyle\int_{-\infty}^0 \frac{x^3}{1+x^8}\,dx = -\int_0^\infty \frac{x^3}{1+x^8}\,dx$

we conclude $\displaystyle\int_{-\infty}^\infty \frac{x^3}{1+x^8}\,dx = 0$

4. (a) Determine the length of the curve: $x(t) = \cos t + t\sin t, \quad t \in [0, \pi]$
$y(t) = \sin t - t\cos t$

$$s = \int \sqrt{(x')^2 + (y')^2}\, dt$$

$$x' = -\sin t + \sin t + t\cos t = t\cos t$$

$$y' = \cos t - \cos t + t\sin t = t\sin t$$

$$\therefore s = \int_0^{\pi} \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \int_0^{\pi} t\, dt = \left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2}$$

(b) Find the area of the surface of revolution generated by revolving the curve

$$2x = y\sqrt{y^2-1} + \ln|y - \sqrt{y^2-1}|, \quad y \in [2, 5]$$

about the x-axis.

$$A = \int 2\pi y \sqrt{1 + (x')^2}\, dy$$

$$2x' = \frac{1}{2}\frac{y \cdot 2y}{\sqrt{y^2-1}} + \sqrt{y^2-1} + \frac{1}{y - \sqrt{y^2-1}}\left(1 - \frac{1}{2}\frac{2y}{\sqrt{y^2-1}}\right)$$

$$= \frac{y^2+y^2-1}{\sqrt{y^2-1}} + \frac{\sqrt{y^2-1} - y}{(y-\sqrt{y^2-1})(\sqrt{y^2-1})} = \frac{2y^2-1}{\sqrt{y^2-1}} - \frac{1}{\sqrt{y^2-1}}$$

$$= \frac{2(y^2-1)}{\sqrt{y^2-1}} = 2\sqrt{y^2-1} \implies x' = \sqrt{y^2-1}$$

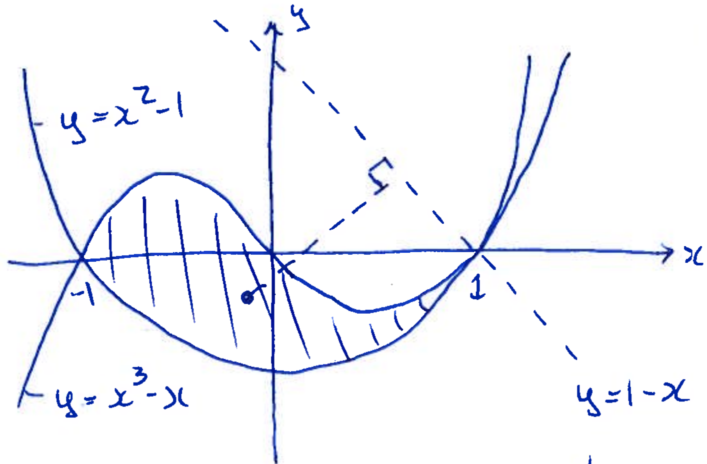
$$\therefore A = \int_2^5 2\pi y\sqrt{1+(y^2-1)}\, dy = \int_2^5 2\pi y^2\, dy = 2\pi\left[\frac{y^3}{3}\right]_2^5$$

$$= 2\pi\left(\frac{125-8}{3}\right) = 78\pi$$

5. (a) Find the centroid of the region bounded by the curves: $y = x^3 - x$, $y = x^2 - 1$
Provide a sketch of the region indicating the location of the centroid.

(b) Find the volume obtained by rotating this region about the line: $y = 1 - x$

a)



Intersection: $x^3 - x = x(x^2-1) = x^2 - 1 \Rightarrow x = \pm 1$

$A = \int_{-1}^{1} [(x^2-x)-(x^2-1)]\,dx = \int_{-1}^{1} (1-x^2)\,dx$

(odd terms go to zero by symmetry)

$= \left[x - \frac{x^3}{3}\right]_{-1}^{1} = 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}$

$\bar{x}A = \int_{-1}^{1} x(x^3 - x - x^2 + 1)\,dx = \int_{-1}^{1}(x^4 - x^2 - x^3 + x)\,dx = \int_{-1}^{1}(x^4 - x^2)\,dx$

$= \left[\frac{x^5}{5} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{5} - \frac{1}{3} + \frac{1}{5} - \frac{1}{3} = \frac{-4}{15} \Rightarrow \bar{x} = \frac{3}{4} \cdot \frac{-4}{15} = -\frac{1}{5}$

$\bar{y}A = \int_{-1}^{1} \frac{1}{2}\left[(x^3-x)^2 - (x^2-1)^2\right]dx = \frac{1}{2}\int_{-1}^{1}(x^6 - 2x^4 + x^2 - x^4 + 2x^2 - 1)\,dx$

$= \frac{1}{2}\int_{-1}^{1}(x^6 - 3x^4 + 3x^2 - 1)\,dx = \frac{1}{2}\left[\frac{x^7}{7} - \frac{3x^5}{5} + \frac{3x^3}{3} - x\right]_{-1}^{1}$

$= \frac{1}{7} - \frac{3}{5} + 1 - 1 = \frac{-16}{35} \quad \therefore \bar{y} = \frac{3}{4} \cdot \frac{-16}{35} = \frac{-12}{35}$

b) slope of line $\perp$ to $y = 1-x$ $= +1$; point on line $= \left(-\frac{1}{5}, \frac{-12}{35}\right)$

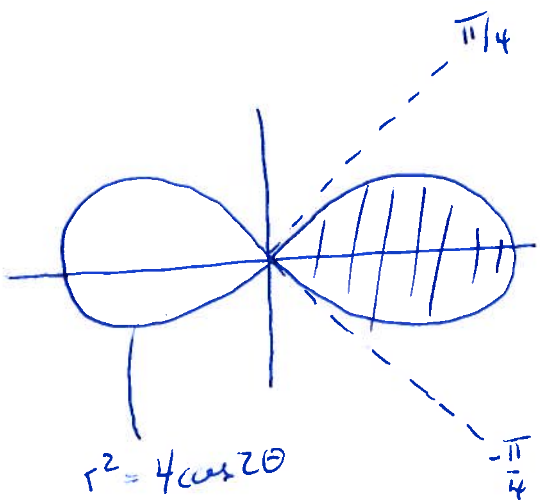
$\therefore y = mx + b \Rightarrow \frac{-12}{35} = 1 \cdot \left(-\frac{1}{5}\right) + b \Rightarrow b = \frac{-5}{35} = -\frac{1}{7} \Rightarrow y = x - \frac{1}{7}$

Intersection: $x - \frac{1}{7} = 1 - x \Rightarrow 2x = \frac{8}{7} \Rightarrow x = \frac{4}{7} \therefore y = \frac{3}{7}$

Radius of rotation: $R = \sqrt{\left(\frac{4}{7}+\frac{1}{5}\right)^2 + \left(\frac{3}{7}+\frac{12}{35}\right)^2} = \sqrt{\left(\frac{27}{35}\right)^2 + \left(\frac{27}{35}\right)^2} = \sqrt{2}\,\frac{27}{35}$

Pappus: $V = 2\pi R \cdot A = 2\pi \cdot \sqrt{2}\,\frac{27}{35} \cdot \frac{4}{3} = \frac{18\sqrt{2}\pi}{35}$

6. (a) Find the area of the region enlcolsed by one loop of the curve: $r^2 = 4\cos 2\theta$
Identify the region on a sketch.



$$A = \int \frac{1}{2} r^2 = \int_{-\frac{\pi}{4}}^{\pi/4} 2\cos 2\theta$$

$$= 2 \cdot \frac{1}{2} \left[\sin\theta\right]_{-\frac{\pi}{4}}^{\pi/4} = 2$$

(b) Find the length of the polar curve: $r = e^{\theta/2} \quad 0 \leq \theta \leq \frac{\pi}{2}$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\pi/2} \sqrt{e^\theta + \frac{e^\theta}{4}}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{5}}{2} e^{\theta/2}\, d\theta$$

$$= \frac{\sqrt{5}}{2}\left[2e^{\theta/2}\right]_0^{\pi/2} = \sqrt{5}\left(e^{\pi/4} - 1\right)$$

7. Sketch a graph of the parametric curve: $x = \dfrac{3t}{1+t^3}$, $y = \dfrac{3t^2}{1+t^3}$

(a) Show that the curve is symmetric about the line $y = x$ by showing that if (a, b) lies on the curve, then (b, a) also lies on the curve. Indicate the points where the curve and the line intersect.

(b) Find all vertical and horizontal tangents. You may make use of symmetry in finding such points.

(c) Show that the line $y = -x - 1$ is a slant asymptote.

a) symmetry about $y=x$: $\dfrac{3t_1}{1+t_1^3} = a$ & $\dfrac{3t_1^2}{1+t_1^3} = b \Rightarrow t_1 \cdot a = b$

let $t_2 = \frac{1}{t_1}$ $\therefore$

$$\left.\begin{aligned} x &= \frac{3t_2}{1+t_2^3} = \frac{3\frac{1}{t_1}}{1+1/t_1^3} = \frac{3t_1^2}{t_1^3+1} = b \\ y &= \frac{3t_2^2}{1+t_2^3} = \frac{3\,{}^1/t_1^2}{1+{}^1/t_1^3} = \frac{3t_1}{t_1^3+1} = a \end{aligned}\right\} \begin{aligned} &(x(t_1), y(t_1)) = (a, b) \\ &\Rightarrow (x(\tfrac{1}{t_1}), y(\tfrac{1}{t_1})) = (b, a) \end{aligned}$$

$y = x \Rightarrow 3t = 3t^2 \Rightarrow t = 0$ or $t = 1 \Rightarrow (0,0)$ & $(\frac{3}{2}, \frac{3}{2})$

b) HA : $\dfrac{dy}{dt} = 0 \Rightarrow y' = \dfrac{6t}{1+t^3} - \dfrac{3t^2(3t^2)}{(1+t^3)^2} = \dfrac{6t+6t^4-9t^4}{(1+t^3)^2} = \dfrac{3t(2-t^3)}{(1+t^3)^2}$

$y' = 0 \Rightarrow t = 0$ & $t = 2^{1/3} \Rightarrow (0,0)$ & $(2^{1/3}, 2^{2/3})$

From symmetry we have VA at $(0,0)$ & $(2^{2/3}, 2^{1/3})$

c) $t \to -1^+ : x \to -\infty,\ y \to +\infty$

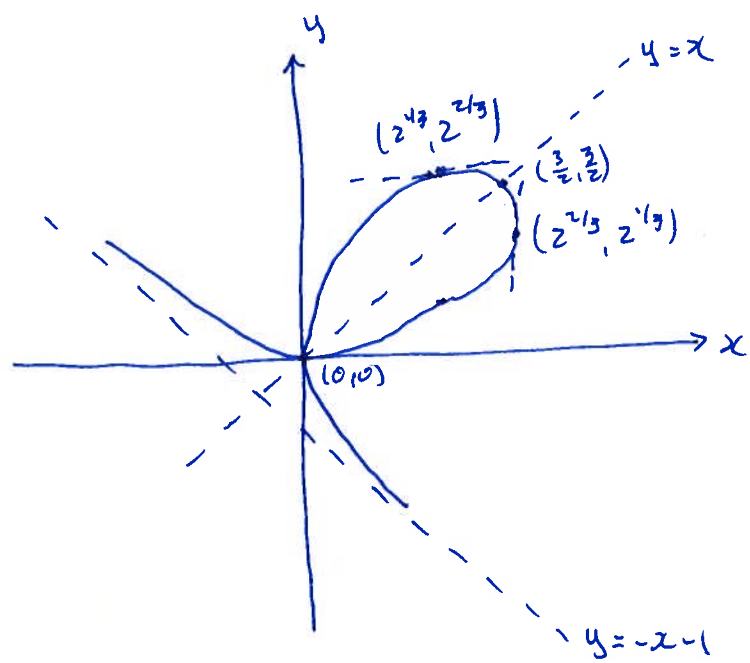
$t \to -1^- : x \to \infty,\ y \to -\infty$

$\therefore$ look for asymptotic behaviour as $t \to -1$

$y - (-x-1) = y + x + 1 = \dfrac{3t + 3t^2 + 1 + t^3}{1+t^3}$

$= \dfrac{(1+t)^3}{1+t^3} = \dfrac{(1+t)^2}{t^2-t+1} \longrightarrow 0$ as $t \to -1$

$\therefore y = -x-1$ is a slant asymptote



8. Bonus Question

Evaluate the improper integral $\displaystyle\int_0^{\pi/2} \ln(\sin x)\, dx$ or show that it doesn't exist.

Hint: you may find one of the double-angle formulas useful.

$$I = \int_0^{\pi/2} \ln(\sin x)\, dx = \int_0^{\pi/2} \ln\left(2 \sin\tfrac{x}{2} \cos\tfrac{x}{2}\right) dx = \frac{\pi}{2}\ln 2 + \int_0^{\pi/2} \ln\left(\sin\tfrac{x}{2}\cos\tfrac{x}{2}\right) dx$$

let $u = \frac{x}{2}$ $\quad du = \frac{dx}{2}$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin u \cos u)\, du = \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin x)\, dx + 2\int_0^{\pi/4} \ln(\cos x)\, dx$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin x)\, dx + 2\int_0^{\pi/4} \ln\left(\sin\left(\tfrac{\pi}{2} - u\right)\right) du$$

let $y = \frac{\pi}{2} - u$ $\quad dy = -du$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin x)\, dx + 2\int_{\pi/2}^{\pi/4} -\ln(\sin y)\, dy$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/4} \ln(\sin x)\, dx + 2\int_{\pi/4}^{\pi/2} \ln(\sin y)\, dy$$

$$= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln(\sin x)\, dx = \frac{\pi}{2}\ln 2 + 2I$$

$$\Longrightarrow I = -\frac{\pi}{2}\ln 2$$