University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – December, 2018

SECOND YEAR – ENGINEERING SCIENCE
Program 5
AER210F VECTOR CALCULUS and FLUID MECHANICS
Examiners: A. Ekmekci and J. W. Davis

Instructions:
(1) Closed book examination; except for a non-programmable calculator, no aids are permitted.
(2) Write your name and student number in the space provided below.
(3) Answer as many questions as you can. Parts of questions may be answered.
(4) Questions are NOT assigned equal marks.
(5) Use the overleaf side of pages for additional or preliminary work.
(6) Do not separate or remove any pages from this exam booklet.
(7) You may use $g = 10\ \text{m/s}^2$, $\rho_{water} = 1000\ \text{kg/m}^3$ where appropriate.

Family Name:______________________________

Given Name:______________________________

Student Number:______________________________

FOR MARKER USE ONLY		
Question	Mark	Earned
1	12	
2	12	
3	10	
4	8	
5	8	
6	10	
7	10	
8	13	
9	10	
10	9	
11	10	
12	7	
13	10	
TOTAL	130	

The following integrals and formulae may be useful:

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \quad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C;$$

$$\int cos^4\theta d\theta = \frac{1}{4}cos^3\theta sin\theta + \frac{3}{4}\int cos^2\theta d\theta + C$$

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \quad \oiint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\vec{F}\, dV; \quad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

$$\frac{T_o}{T} = 1 + \frac{(\gamma-1)}{2}M^2; \quad \frac{P_o}{P} = \left[1 + \frac{(\gamma-1)}{2}M^2\right]^{\gamma/(\gamma-1)}; \quad \frac{\rho_o}{\rho} = \left[1 + \frac{(\gamma-1)}{2}M^2\right]^{1/(\gamma-1)}; \quad p = \rho RT; \quad c = \sqrt{\gamma RT}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z}; \quad -\nabla p + \rho\vec{g} = \rho\vec{a}; \quad \frac{p}{\rho} + \frac{V^2}{2} + gz = constant; \quad \nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{\partial}{\partial t}\iiint_{\forall} \rho d\forall + \oiint \rho\vec{V}\cdot\overrightarrow{dA} = 0; \quad \sum \vec{F}_{CV} = \frac{d}{dt}\iiint_{CV} \vec{V}(\rho d\forall) + \oiint_{CS} \vec{V}(\rho\vec{V}\cdot d\vec{A}); \quad \tau = \mu\frac{du}{dy}$$

1) a) [4 marks] Evaluate the integral $\int_0^1 \int_{\sqrt{x}}^1 \sin\left(\frac{y^3+1}{2}\right)\, dy\, dx$ by reversing the order of integration. Provide a sketch of the region.

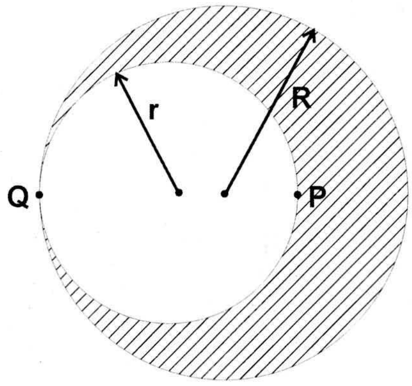
b) [5 marks] Find the volume of the solid bounded by the paraboloid of revolution $x^2 + y^2 = az$, the xy-plane and the cylinder $x^2 + y^2 = 2ax$. Provide a sketch of the volume.
Note: Integrals on the cover page may be useful.

c) [3 marks] Evaluate the line integral $\int_C xe^{yz}\, ds$, where C is the line segment from (0, 0, 0) to (1, 2, 3).

2) [12 marks] Verify Green's theorem for the integral $\oint_C (2xy - x^2)dx + (x + y^2)dy$, where C is the closed curve forming the boundary of the region bounded by $y = x^2$ and $y^2 = x$.

3) [10 marks] A disk of radius r is removed from a larger disk of radius R to form an earring (see figure). Assume the earring is a thin plate of uniform density.

a) Find the centre of mass of the earring in terms of r and R. Hint: set the origin at the point Q.
b) Show that the ratio R/r such that the centre of mass lies at point P (on the edge of the inner disk) is the golden mean: $(1+\sqrt{5})/2$.



4) [8 marks] Liquid lithium, with density 512 kg/m^3 flows with velocity $\vec{v} = y\hat{\imath} + x\hat{\jmath} + z\hat{k}$. Find the mass flux (kg/s) of the flow upward through the paraboloid $z = 8 - (x^2 + y^2)/2, x^2 + y^2 \leq 24$.

5) [8 marks] For a smooth surface S, and a fixed vector $\vec{A}$, prove that:

$$2\int_S \vec{A}\cdot\hat{n}dS = \int_C \vec{A}\times\vec{r}\cdot d\vec{r}$$

Where C is the boundary curve of the region S, and where $\vec{r}(x,y,z) = (x,y,z)$.

6) [10 marks] Verify the divergence theorem for the vector field $\vec{F} = xy\hat{\imath} + yz\hat{\jmath} + xz\hat{k}$, where V is the unit cube $0 \leq x \leq 1, 0 \leq y \leq 1,\ 0 \leq z \leq 1$.

7) [**10 marks**] Circle the true statements in the following:

- The speed of sound in a truly incompressible fluid is:

  a) zero **b)** infinitely large **c)** cannot be defined

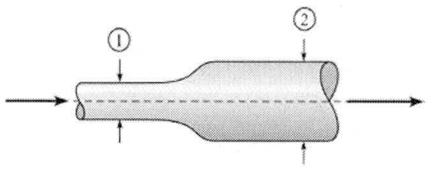
- A gas enters a diverging duct. If the flow speed decreases at the end of the duct, circle the possible situations:

  a) The gas at the entrance of the duct is supersonic.
  b) The gas at the entrance of the duct is subsonic.
  c) The gas at the entrance of the duct is sonic.

  Hint: $\frac{dA}{A} = -\frac{dV}{V}(1 - M^2)$, where M shows the Mach number, V shows the flow speed and A shows the area.

- Air is flowing from a ventilation duct (cross section 1) as shown in the figure, and is expanding to be released into a room at cross-section 2. The area at cross section 2, A_2, is 3 times A_1. Assume that the density of air remains constant. The relation between the volume flow rates $\dot{V}_1$ and $\dot{V}_2$ is:

  a) $\dot{V}_2 = (1/3)\dot{V}_1$
  b) $\dot{V}_2 = \dot{V}_1$
  c) $\dot{V}_2 = 3\dot{V}_1$
  d) $\dot{V}_2 = 9\dot{V}_1$

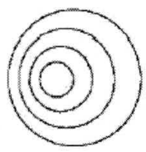


- The law of conservation of mass for a system requires that the mass of the system

  a) remains constant
  b) is zero
  c) changes at a time rate equal to the net outflow rate of mass from the system
  d) changes as the system moves to flow regions of varying density

- If a pebble thrown into water creates the following wake pattern, which one is true:

  a) the water flow is subcritical
  b) the water flow is supercritical
  d) the water flow is critical
  c) the water is still



- The speed of sound is

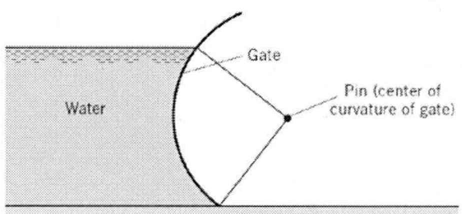
  a) higher in the winter than in the summer
  b) lower in the winter than in the summer
  c) the same in the winter and summer

- Which one is the different term between the Euler equations and the Navier-Stokes equations?

  a) Net pressure force per unit volume term
  b) Net body force per unit volume term
  c) Net viscous force per unit volume term
  d) Acceleration terms

- Water is held back by the radial gate shown below. The resultant of the pressure forces acting on the gate:

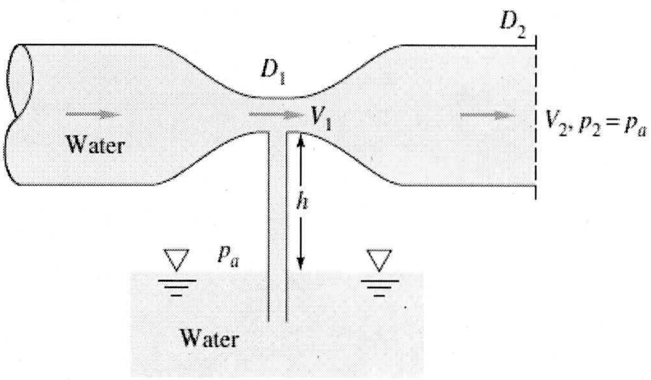
  a) passes above the pin
  b) passes through the pin
  c) passes below the pin



- Stream function is defined in a way that it automatically satisfies

  a) energy equation **b)** momentum equation **c)** continuity equation **d)** Bernoulli

8) a) [**7 marks**] In a pipe flow, a necked-down (throat) section in a pipe flow develops a low throat pressure which can draw fluid upward from a reservoir that contains still water, as shown in the figure below. The free surface of the water in the reservoir is exposed to the atmospheric pressure. Also, the pressure at cross-section 2 is atmospheric pressure. Assuming no losses and one-dimensional flow in the venturi, derive an expression in terms of h, D_1 and D_2 for the velocity V_1 which is just sufficient to bring reservoir fluid into the throat.

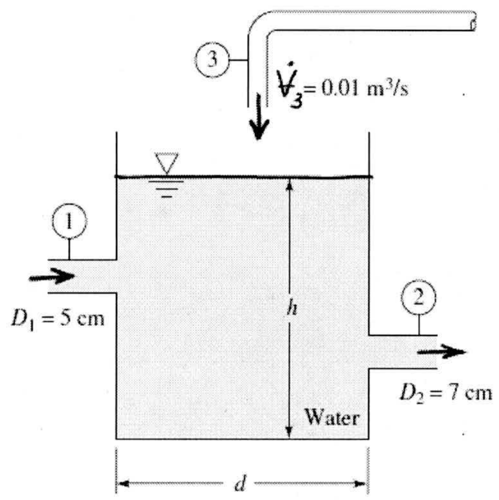


8) b) [**6 marks**] An airflow at $M = 0.85$ passes through a pipe with a cross-sectional area of 60 cm^2. The stagnation pressure is 360 kPa and the stagnation temperature is 10°C. Calculate the mass flow rate through the conduit. For air, the specific heat ratio is $\gamma = 1.4$, the gas constant $R = 287$ J/kg-K. *Note:* You may find some useful formulae in the front page. The temperature conversion from Celsius to Kelvin can be found by [K] = [°C] + 273.

9) [**10 marks**] The general integral form of the continuity equation, applicable to a control volume is given as:

$$\frac{\partial}{\partial t}\iiint_{\forall} \rho d\forall + \oiint \rho\vec{V}\cdot\vec{dA} = 0$$

where the symbol $\forall$ shows the volume of the control volume and $\vec{V}$ is velocity vector.
The cylindrical tank with a diameter d contains water as shown in the figure below. The tank is being filled through sections 1 and 3, while being emptied through section 2. Assume that all sections (section 1, 2 and 3) have circular cross-sections.



a) [**6 marks**] Derive an analytic expression for the water-level change *dh/dt* in terms of volume flow rates ($\dot{\forall}_1$, $\dot{\forall}_2$, $\dot{\forall}_3$) and the tank diameter d.

<u>Hint:</u> First, select an appropriate control volume, and then apply the integral form of the continuity equation to this control volume.

b) [**4 marks**] If the water level h is constant, the velocity at section 1 is $V_1 = 3$ m/s and the volume flow rate at section 3 is $\dot{\forall}_3 = 0.01\ m^3/s$, determine the exit velocity V_2. Note that the diameters of the sections 1 and 2 are $D_1 = 5$ cm and $D_2 = 7$ cm, respectively.

10) [**9 marks**] A thin layer of particles rests on the bottom of a horizontal tube as shown in the figure below. When an incompressible fluid flows through the tube, it is observed that at some critical velocity, the particles will rise and be transported along the tube. The critical velocity V_c is known to depend on the pipe diameter D, particle diameter d, the fluid density ρ, the fluid viscosity μ, the density of the particles ρ_P, and the gravitational acceleration g. Thus,

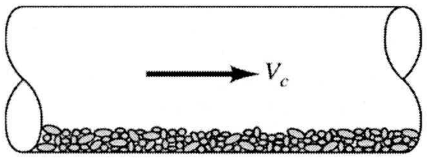
$$V_c = \text{function}\ (D, d, \rho, \mu, \rho_P, g)$$

a) [**5 marks**] By dimensional analysis, **choosing the repeating variables as ρ, D and μ**, determine the dimensionless (π) groups for this problem, and re-write the original dimensional relationship in dimensionless terms.

b) [**2 marks**] A 1/2 scale model is to be used to determine this functional relationship experimentally. If fluid densities and the gravitational acceleration in the small-scale model test and the full-scale prototype test are identical, what should be the viscosity ratio of the model fluid and the prototype fluid (that is, $\mu_{model}/\mu_{prototype}$) in order to insure similarity?

c) [**2 marks**] Assuming all similarity requirements are satisfied between the model and the prototype tests, what is the ratio of the critical velocity of the model and prototype experiments (that is, $(V_c)_{model}/(V_c)_{prototype}$)?

Hint: The dimension of viscosity can be deduced from the following relationship: $\tau = \mu \frac{du}{dy}$.

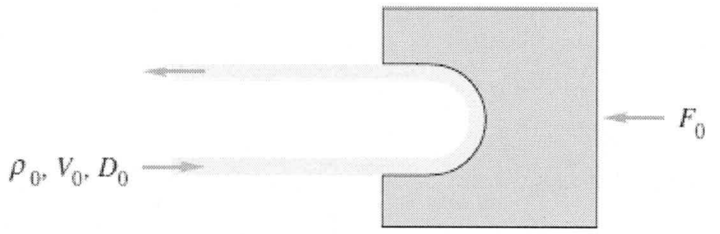


11) [**10 marks**] Consider the two-dimensional velocity field given by

$$\vec{V} = x\vec{\imath} - y\vec{\jmath}$$

a) [1 mark] Is the flow steady or unsteady?
b) [2 marks] Determine the acceleration field.
c) [2 marks] Determine the vorticity of the flow field. Is this a rotational or irrotational flow?
d) [2 marks] Is this an incompressible flow or compressible flow? Provide the proof of your answer to this question.
e) [1 marks] If there are stagnation points in the flow, find those points.
f) [1 marks] If the stream function ψ is defined as u(x,y) = $\partial\psi/\partial y$ and v(x,y) = $-\partial\psi/\partial x$, find the stream function.
g) [1 marks] Sketch the streamline that passes from point (1,1), showing its direction with an arrow.

12) [**8 mark**] As shown in the figure below, the stationary vane turns the water jet completely around. If the maximum possible support force is F_o, find an expression for the maximum jet velocity V_o in terms of F_o, ρ_o, and D_o. Neglect gravity and friction. The water jet has a circular cross-section.



13) [**10 mark**] A pump injects water jet at velocity U_1 through a small pipe, which has a diameter d. The cross-section that this water jet enters the large pipe is called section 1. This jet entrains the water flowing at velocity U_2 in the annular region around the small pipe (let's call this annular region as section 2). After a highly-viscous mixing region, the two flows become fully mixed L distance downstream of the inlet at section 3, where U_3 is approximately constant. At the inlet (at sections 1 and 2), all conditions (pressure, water density, velocity) are known. Assume that $p_1 \approx p_2$ at the inlet, the flow is steady, and the frictional and gravitational effects are negligible. If the diameter ratio of the small and large pipes is d/D $= \beta$,

a) derive U_3 as a function of U_1, U_2 and β.

b) derive p_3 at section 3 in terms of ρ, p_1, β, U_1, U_2 and U_3.