CHE 260: THERMODYNAMICS AND HEAT TRANSFER

FINAL FOR HEAT TRANSFER

12th DECEMBER 2014

NAME:

STUDENT ID NUMBER:

Q1	Q2	Q3	Q4	Q5	Q6	Total
15	10	20	15	15	15	90

INSTRUCTIONS

1. This examination is open textbook (the custom textbook for this course) along with one 8.5" x 11" aid sheet (both sides), closed internet, closed all communication devices.
2. All non-communicating calculators are permissible.
3. Please write legibly. If your handwriting is unreadable, your answers will not be assessed.
4. Answers written in pencil will NOT be re-marked. This is University policy.
5. For all problems, you must present the solution process in a step by step fashion for partial marks.
6. **ANSWERS WRITTEN ON THE BLANK, LEFT SIDES OF THE ANSWER BOOK WILL NOT BE ASSESSED. USE THE LEFT SIDES FOR ROUGH WORK ONLY. THIS WILL BE IMPLEMENTED STRICTLY THIS TIME.**

Q.1. [15 points] DEEP-FRIED ICE-CREAM

I recently went to a Thai restaurant, where, on the dessert menu, they had an item called deep-fried ice-cream. It was described as "A combination of hot and cold. The hot crispy crust and creamy filling cold make this dessert a delight!" I ordered it and dissected it, only to discover that deep-fried ice cream is essentially a spherical scoop of ice-cream, coated with a thick layer of batter, and deep fried in hot oil.

I decided to make some deep-fried ice-cream at home for my wife. The initial temperature of the ice-cream was -18ºC. The oil was maintained at a temperature of 185ºC. The ice-cream scoop I took was initially a sphere of 2.0 cm radius, and I coated it with a 0.5 cm thick layer of batter, and immersed it in oil. To ensure that the batter was completely cooked, I allowed the ice-cream to be deep-fried for 20 min.

Use the one-term approximation to solve parts (b), (c) and (d). Comment on the validity of the one-term approximation for these parts. For your calculations, assume the batter to also be at an initial temperature of -18ºC. The convective heat transfer coefficient is 450 W/m^2-K, and the thermal conductivities of the batter and ice-cream are both 1 W/m-K. The density and the specific heat capacity of the batter and ice-cream are 980 kg/m^3 and 4000 J/kg-K respectively.

(a) **[3 points]** What is the rate-controlling step in this heat transfer problem, conduction within the solid, or convection past the solid surface?

Solution:

To determine the rate controlling step, we calculate the Biot number.

$$\mathrm{Bi} = \frac{hR}{k} = \frac{450 \times 2.5 \times 10^{-2}}{1} = 11.25$$

Since Bi>>1, the conductive heat transfer resistance within the fried ice-cream is greater than convective resistance past its surface; conduction is the rate controlling step.

1 point for thinking about the Biot number.

1 point for correctly calculating the Biot number.

1 point for the correct conclusion based on the calculation.

(b) **[4 points]** After my deep fry experiment, did the ice-cream inside melt completely (this would classify as a fried-ice-cream 'fail')? Assume that the ice-cream melts at 0ºC.

Solution:

If the center temperature is above 0ºC after 20 min, the ice-cream has melted.

From table 11-2, for Bi=11.25, λ_1=2.8550 and A_1=1.9315 for a sphere.

1 point for the correct values of lambda1 and A1

$$\theta_0 = \frac{T_0 - T_\infty}{T_i - T_\infty} = A_1 \exp\left(-\lambda_1^2 t^*\right)$$

$$\alpha = \frac{k}{\rho C_p} = \frac{1}{980 \times 4000} = 2.5510 \times 10^{-7} \text{ m}^2\text{/s.}$$

0.5 point for calculating alpha

$$t^* = \frac{t}{R^2 / \alpha} = \frac{20 \times 60}{(0.025)^2 / 2.5510 \times 10^{-7}} = 0.48979 > 0.2$$

0.5 point for calculating t*

One term approximation will work.

0.5 point for commenting on validity of 1 term approximation

$$\theta_0 = 1.9315 \exp\left(-2.8550^2 \frac{20 \times 60}{(0.025)^2 / 2.5510 \times 10^{-7}}\right) = 0.035650$$

$$\frac{T_0 - 185}{-18 - 185} = 0.035650$$

T_0 = 177ºC.

1 point for calculation of T_0 using the 1 term approximation

The ice cream has melted (burned!). This was a fried-ice-cream fail.

0.5 point for conclusion

(c) **[4 points]** What is the maximum duration of deep fry over which at least some fraction of the ice-cream would have been in a solid state?

Solution:

<u>2 points for recognizing that the temperature at the center needs to be taken as zero (melting point) to estimate the maximum duration of deep fry.</u>

$$t^* = \frac{1}{\lambda_1^2}\ln\left[A_1\left(\frac{T_i - T_\infty}{T_0 - T_\infty}\right)\right] = \frac{1}{2.8550^2}\ln\left[1.9315\left(\frac{-18-185}{0-185}\right)\right] = 0.09215$$

Note that this t*<0.2, so the one term approximation will not be accurate, and an error greater than 2% should be expected.

<u>1 point for comment on validity of one term approximation</u>

$$t = t^*\frac{R^2}{\alpha} = 0.09215\frac{(0.025)^2}{2.5510\times10^{-7}} = 225.8\,\text{seconds, or } 3.76\text{ min}$$

<u>1 point for the correct time</u>

(d) **[4 points]** What is the maximum duration of deep fry over which all the ice-cream remains frozen?

Solution:

<u>2 points for realizing that temperature at r=2.0 needs to be set to 0 to find onset of melting</u>

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} = A_1\exp\left(-\lambda_1^2 t^*\right)\frac{\sin\left(\lambda_1 r^*\right)}{\lambda_1 r^*}$$

$$r^* = \frac{2.0}{2.5} = 0.8$$

<u>0.5 point for the calculation of *r**</u>

$$t^* = \frac{1}{\lambda_1^2}\ln\left[\frac{\sin\left(\lambda_1 r^*\right)}{\lambda_1 r^*}A_1\left(\frac{T_i - T_\infty}{T - T_\infty}\right)\right]$$

$$= \frac{1}{2.8550^2}\ln\left[\frac{\sin\left(2.8550\times 0.8\right)}{2.8550\times 0.8}1.9315\left(\frac{-18-185}{0-185}\right)\right] = -4.345 < 0$$

<u>0.5 point for calculation of t*</u>

This is too short a time for the one term approximation to even provide a physically realistic answer.

<u>1 point for comment on one term approximation's validity.</u>

Q.2. [10 points] DRAG ON A PLATE

For flow over a flat plate with an extremely rough surface, convection heat transfer effects are known to be correlated by the expression: $\mathrm{Nu}_x = 0.04\,\mathrm{Re}_x^{0.9}\,\mathrm{Pr}^{1/3}$, where Nu_x is the local Nusselt number at a distance x from the leading edge of the plate, Re_x is a Reynolds number based on the length x, and Pr is the Prandtl number. For flow of a fluid over the plate at 10 m/s, what is the surface shear stress at $x = 1$ m from the plate's leading edge? Take the fluid density and viscosity to be 1000 kg/m^3 and 10^{-3} Pa-s respectively.

Solution:

<u>5 points for realizing that Colburn analogy needs to be used</u>

According to the Colburn analogy,

$$\frac{C_{f,x}}{2} = \frac{\mathrm{Nu}_x}{\mathrm{Re}_x\,\mathrm{Pr}^{1/3}}$$

$$\frac{C_{f,x}}{2} = \frac{0.04\,\mathrm{Re}_x^{0.9}\,\mathrm{Pr}^{1/3}}{\mathrm{Re}_x\,\mathrm{Pr}^{1/3}} = 0.04\,\mathrm{Re}_x^{-0.1}$$

$$C_{f,x} = 0.08\,\mathrm{Re}_x^{-0.1}$$

<u>2 points for simplification to eliminate Pr</u>

$$C_{f,x} = 0.08\left(\frac{Ux}{\nu}\right)^{-0.1} = 0.08\left(\frac{10\times 1}{10^{-3}/1000}\right)^{-0.1} = 0.015962$$

$$\frac{\tau_{w,x}}{\frac{1}{2}\rho U^2} = 0.015962$$

$$\tau_{w,x} = \frac{1}{2}1000\times 10^2\times 0.015962 = 798.10\ \mathrm{N/m^2}.$$

<u>3 points for the calculations</u>

Q. 3. [20 points] HEAT LOSSES FROM A STEAM PIPE

An uninsulated steam pipe is used to transport high temperature steam from one building to another. The pipe is 0.5 m in diameter, has a surface temperature of 150ºC, and is exposed to ambient air at -10ºC. The air moves in cross-flow fashion over the pipe with a velocity of 5 m/s.

(a) **[10 points]** What is the heat loss per unit length of pipe?

(b) **[10 points]** What is the heat loss per unit length after the pipe is insulated with a rigid urethane foam (k = 0.026 W/mºC) of 10 cm thickness?

Use an appropriate correlation from the textbook to determine the heat transfer coefficient. Get air properties from table A-22.

Solution:

(a) In this problem, we will employ the correlation 12-47 in the textbook for flow across a cylinder.

This requires fluid properties at the film temperature.

$$T_f = \frac{T_s + T_\infty}{2} = \frac{150 - 10}{2} = 70\ ^{\circ}\text{C}.$$

<u>1 point for calculation of film temperature</u>

From table A-22, k = 0.02881 W/m-K, ρ = 1.028 kg/m^3, μ = 2.052 x 10^{-5} kg/m-s, α = 2.780 x 10^{-5} m^2/s at 70ºC.

<u>2 points for getting required properties from table A-22</u>

$$\nu = \frac{\mu}{\rho} = \frac{2.052\times10^{-5}}{1.028} = 1.9961\times10^{-5}\ \text{m}^2\text{/s}.$$

$$\text{Re} = \frac{UD}{\nu} = \frac{5\times0.5}{1.9961\times10^{-5}} = 1.2524\times10^{5}$$

<u>1 point for calculation of Re</u>

$$\text{Pr} = \frac{\nu}{\alpha} = \frac{1.9961\times10^{-5}}{2.780\times10^{-5}} = 0.7180$$

RePr>0.2, so the following correlation can be used.

<u>1 point Verification of condition for correlation e.g. RePr>0.2</u>

$$\mathrm{Nu}_{\mathrm{cyl}} = 0.3 + \frac{0.62\,\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(0.4/\mathrm{Pr}\right)^{2/3}\right]^{1/4}}\left[1+\left(\frac{\mathrm{Re}}{282{,}000}\right)^{5/8}\right]^{4/5}$$

$$\mathrm{Nu}_{\mathrm{cyl}} = 0.3 + \frac{0.62\left(1.2524\times10^{5}\right)^{1/2}\left(0.7180\right)^{1/3}}{\left[1+\left(0.4/0.7180\right)^{2/3}\right]^{1/4}}\left[1+\left(\frac{1.2524\times10^{5}}{282{,}000}\right)^{5/8}\right]^{4/5} = 252.02$$

<u>Calculation of Nu: 2 points</u>

Thus, $h = \frac{k}{D}252.02 = \frac{0.02881}{0.5}\times 252.02 = 14.52\ \mathrm{W/m^2\,{}^{o}C}$

<u>Calculation of h: 1 point</u>

Heat loss per unit length $= h\pi D\left(T_s - T_\infty\right) = 14.52\times\pi\times0.5\times\left(150+10\right) = 3649\ \ \mathrm{W/m}$.

<u>Calculation of heat loss: 2 points</u>

(b)

Diameter of insulated object $D = 0.5 + 2\times10\times10^{-2} = 0.7\ \ \mathrm{m}$.

<u>1 point for calculation of new outer diameter</u>

We need a new estimate of the ‘surface’ temperature for the film temperature and therefore the air properties. To do this, we use the heat transfer coefficient from the previous calculation.

$$\frac{T_s - T_\infty}{\frac{1}{2\pi k_{\mathrm{ins}}}\ln\left(\frac{D_{\mathrm{ins}}}{D_{\mathrm{pipe}}}\right)+\frac{1}{h\pi D_{\mathrm{ins}}}} = \frac{T_{\mathrm{ins}} - T_\infty}{\frac{1}{h\pi D_{\mathrm{ins}}}}$$

$$T_{\mathrm{ins}} = T_\infty + \frac{T_s - T_\infty}{\frac{hD_{\mathrm{ins}}}{2k_{\mathrm{ins}}}\ln\left(\frac{D_{\mathrm{ins}}}{D_{\mathrm{pipe}}}\right)+1} = -7.6\,{}^{o}\mathrm{C}$$

<u>Estimation of surface temperature: 2 points</u>

The film temperature is, therefore, very close to the ambient temperature due to the thick insulation.

We will employ film properties at the film temperature (-10-7.6)/2 = -8.8ºC.

New film temperature: 1 point

At this temperature, from table A-22, k = 0.02297 W/m-K, ν = 1.262 x 10^{-5} m^2/s , α = 1.711 x 10^{-5} m^2/s.

2 points for getting required properties from table A-22

$$\mathrm{Re} = \frac{UD}{\nu} = \frac{5\times 0.7}{1.262\times 10^{-5}} = 2.7734\times 10^{5},$$

$$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{1.9961\times 10^{-5}}{2.780\times 10^{-5}} = 0.7376$$

1 point for Re

RePr>0.2, so the following correlation can be used.

$$\mathrm{Nu}_{\mathrm{cyl}} = 0.3 + \frac{0.62\,\mathrm{Re}^{1/2}\,\mathrm{Pr}^{1/3}}{\left[1+\left(0.4/\mathrm{Pr}\right)^{2/3}\right]^{1/4}}\left[1+\left(\frac{\mathrm{Re}}{282,000}\right)^{5/8}\right]^{4/5} = 450.60$$

Thus, $h = \frac{k}{D}450.60 = \frac{0.02297}{0.7}\times 450.60 = 14.79\ \mathrm{W/m^2\,{}^{o}C}$

Calculation of Nu: 1 point

Calculation of h: 1 point

Heat loss per unit length $= \dfrac{T_s - T_\infty}{\dfrac{1}{2\pi k_{\mathrm{ins}}}\ln\left(\dfrac{D_{\mathrm{ins}}}{D_{\mathrm{pipe}}}\right) + \dfrac{1}{h\pi D_{\mathrm{ins}}}} = 76.54\ \ \mathrm{W/m}.$

Calculation of heat loss: 1 point

Q.4. [15 points] THERMOCOUPLE JUNCTION

A thermocouple junction, which may be approximated as a sphere, is to be used for a temperature measurement in a fluid stream. The convection coefficient between the junction surface and the gas is h = 500 W/m^2°C. The junction thermophysical properties are k = 10 W/m°C, C_p = 400 J/kg°C, ρ = 8500 kg/m^3.

(a) **[8 points]** Determine the junction diameter required for the thermocouple to have a time constant of 0.5 seconds in response to temperature changes. [If F is a variable of interest that decays exponentially with time t as $F = F_0 \exp\left(-t/\tau\right)$, then τ is called its time constant.] **Hint**: What Bi regime would you want a 'fast' thermocouple to work in?

Solution:

Assuming the lumped system analysis to be valid, we have

$$\frac{T_m - T_\infty}{T_i - T_\infty} = \left(-\frac{hA}{\rho C_p V} t\right)$$

Realizing that the lumped system analysis needs to be used for a fast thermocouple: 3 points

The time constant of the lumped system is, therefore,

$$\tau = \frac{\rho C_p V}{hA} = \frac{\rho C_p D}{6h}$$

Time constant of lumped system: 2 points

The desired time constant is 0.5 s. The diameter that will result in this time constant is

$$D = \frac{6h\tau}{\rho C_p} = \frac{6 \times 500 \times 0.5}{8500 \times 400} = 4.412 \times 10^{-4} \text{ m} = 441.2 \text{ microns.}$$

Calculation of diameter: 1 point

It needs to be ensured that we are in the low Biot number regime.

$$\text{Bi} = \frac{hD}{2k} = \frac{500 \times 4.412 \times 10^{-4}}{2 \times 10} = 0.01 < 0.1.$$

The lumped system analysis is, therefore, valid.

Verifying that lumped system analysis is valid

(b) **[7 points]** The junction, initially at 25ºC, is suddenly placed in a fluid at 150ºC. How long does it take for the temperature to reach 149ºC?

Solution:

Using the lumped system analysis,

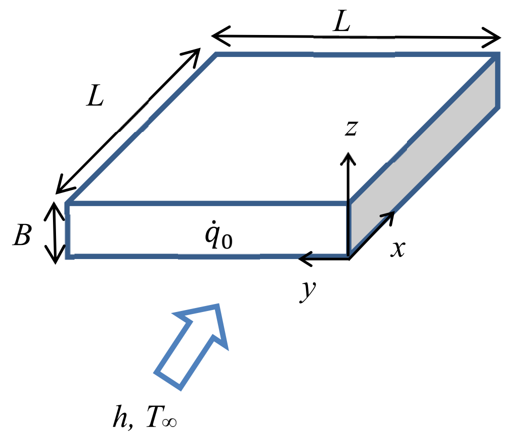
$$\frac{T_m - T_\infty}{T_i - T_\infty} = \left(-\frac{6h}{\rho C_p D} t \right).$$

$$t = \frac{\rho C_p D}{6h} \ln\left(\frac{T_i - T_\infty}{T_m - T_\infty} \right) = \frac{8500 \times 400 \times 441.2 \times 10^{-6}}{6 \times 500} \ln\left(\frac{25 - 150}{149 - 150} \right) = 2.41 \text{ s.}$$

It takes about 2.4 seconds.

<u>Calculation: 7 points</u>

Q.5 [15 points] TEMPERATURE DISTRIBUTION IN A SQUARE-SHAPED SOLID



Consider a square-shaped solid (thermal conductivity k) of side L in the x and y directions, and thickness B in the z direction. A constant source of heat $\dot{q}_0$ (W/m^3) is present everywhere within the solid. The solid is placed in an ambient fluid at a temperature T_∞. The heat transfer coefficient corresponding to convective heat transfer past the surface of the plate is h. Answer the following questions:

(a) **[4 points]** Write down the governing equation for the temperature distribution under steady state conditions for a constant thermal conductivity. Specify the boundary conditions of each face. Note that the temperature will be, in general, a function of x, y and z co-ordinates.

Solution:

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)=-\dot{q}_0$$

<u>1 point for governing equation</u>

$$k\left.\frac{dT}{dx}\right|_{x=0}=h\left(\left.T\right|_{x=0}-T_\infty\right),\qquad -k\left.\frac{dT}{dx}\right|_{x=L}=h\left(\left.T\right|_{x=L}-T_\infty\right),$$

$$k\left.\frac{dT}{dx}\right|_{y=0}=h\left(\left.T\right|_{y=0}-T_\infty\right),\qquad -k\left.\frac{dT}{dx}\right|_{y=L}=h\left(\left.T\right|_{y=L}-T_\infty\right),$$

$$k\left.\frac{dT}{dx}\right|_{z=0}=h\left(\left.T\right|_{z=0}-T_\infty\right),\qquad -k\left.\frac{dT}{dx}\right|_{z=B}=h\left(\left.T\right|_{z=B}-T_\infty\right),$$

<u>3 points for boundary conditions</u>

(b) **[5 points]** Render the governing equations and the boundary conditions dimensionless. The spatial scales are given; keep the scale for the temperature undetermined as ΔT_c . Identify the spatial scales and the scale for temperature. There should be three dimensionless parameters appearing in the dimensionless equations: the Biot number $\mathrm{Bi} = \dfrac{hB}{k}$, the geometric aspect ratio, $\varepsilon = \dfrac{B}{L}$, and another parameter that involves the unknown temperature scale. Write the equations in terms of these parameters.

Solution:

Define $\theta = \dfrac{T - T_\infty}{\Delta T_c}$, $x^* = \dfrac{x}{L}$, $y^* = \dfrac{y}{L}$, $z^* = \dfrac{z}{B}$

$$k\left(\frac{\Delta T_c}{L^2}\frac{\partial^2\theta}{\partial x^{*2}} + \frac{\Delta T_c}{L^2}\frac{\partial^2\theta}{\partial y^{*2}} + \frac{\Delta T_c}{B^2}\frac{\partial^2\theta}{\partial z^{*2}}\right) = -\dot{q}_0$$

$$k\frac{\Delta T_c}{B^2}\left(\frac{B^2}{L^2}\frac{\partial^2\theta}{\partial x^{*2}} + \frac{B^2}{L^2}\frac{\partial^2\theta}{\partial y^{*2}} + \frac{\partial^2\theta}{\partial z^{*2}}\right) = -\dot{q}_0$$

$$\frac{B^2}{L^2}\frac{\partial^2\theta}{\partial x^{*2}} + \frac{B^2}{L^2}\frac{\partial^2\theta}{\partial y^{*2}} + \frac{\partial^2\theta}{\partial z^{*2}} = -\frac{\dot{q}_0 B^2}{k\Delta T_c}$$

<u>Scaling the governing equation: 2 points</u>

The boundary conditions become

$$\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=0} = \left(\frac{hL}{k}\right)\theta\big|_{x^*=0}, \qquad -\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=1} = \left(\frac{hL}{k}\right)\theta\big|_{x^*=1},$$

$$\left.\frac{\partial\theta}{\partial y^*}\right|_{y^*=0} = \left(\frac{hL}{k}\right)\theta\big|_{y^*=0}, \qquad -\left.\frac{\partial\theta}{\partial y^*}\right|_{y^*=1} = \left(\frac{hL}{k}\right)\theta\big|_{y^*=1},$$

$$\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=0} = \left(\frac{hB}{k}\right)\theta\big|_{z^*=0}, \qquad -\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=1} = \left(\frac{hB}{k}\right)\theta\big|_{z^*=1}.$$

<u>Scaling the boundary conditions: 2 points</u>

Rewriting the above equations in terms of the dimensionless parameters that have been specified,

$$\varepsilon^2 \frac{\partial^2\theta}{\partial x^{*2}} + \varepsilon^2 \frac{\partial^2\theta}{\partial y^{*2}} + \frac{\partial^2\theta}{\partial z^{*2}} = -\left(\frac{\dot{q}_0 B^2}{k\Delta T_c}\right)$$

The boundary conditions become

$$\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=0} = \frac{\mathrm{Bi}}{\varepsilon}\theta\Big|_{x^*=0}, \qquad -\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=1} = \frac{\mathrm{Bi}}{\varepsilon}\theta\Big|_{x^*=1},$$

$$\left.\frac{\partial\theta}{\partial y^*}\right|_{y^*=0} = \frac{\mathrm{Bi}}{\varepsilon}\theta\Big|_{y^*=0}, \qquad -\left.\frac{\partial\theta}{\partial y^*}\right|_{y^*=1} = \frac{\mathrm{Bi}}{\varepsilon}\theta\Big|_{y^*=1},$$

$$\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=0} = \mathrm{Bi}\,\theta\Big|_{z^*=0}, \qquad -\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=1} = \mathrm{Bi}\,\theta\Big|_{z^*=1}.$$

<u>Rewriting in terms of dimensionless parameters: 1 point</u>

(c) **[2 points]** When the geometric aspect ratio $\varepsilon = \frac{B}{L}$ is much less than 1, i.e. when the solid is basically a plate, the governing equation simplifies to just two terms. Write this equation down. Hence identify the temperature scale.

Solution:

For ε << 1, $\varepsilon^2 \frac{\partial^2\theta}{\partial x^{*2}} + \varepsilon^2 \frac{\partial^2\theta}{\partial y^{*2}} + \frac{\partial^2\theta}{\partial z^{*2}} = -\left(\frac{\dot{q}_0 B^2}{k\Delta T_c}\right)$ simplifies to

$$\frac{\partial^2\theta}{\partial z^{*2}} = -1$$

<u>1 point for governing equation</u>

The temperature scale is $\Delta T_c = \frac{\dot{q}_0 B^2}{k}$

<u>1 point for temperature scale</u>

(d) **[4 points]** Integrate the equation from (c), apply suitable boundary conditions, and get the temperature distribution. In which regions of the plate is this solution likely to fail?

Solution:

$$\frac{\partial^2\theta}{\partial z^{*2}} = -1$$

Integrating this once,

$$\frac{\partial\theta}{\partial z^*} = -z^* + c_1$$

Integrating this once more,

$$\theta = -\frac{z^{*2}}{2} + c_1 z^* + c_2$$

<u>Integration 1.5 points</u>

The two boundary conditions to be used are the ones in z*.

$$\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=0} = \mathrm{Bi}\,\theta\big|_{z^*=0}, \qquad -\left.\frac{\partial\theta}{\partial z^*}\right|_{z^*=1} = \mathrm{Bi}\,\theta\big|_{z^*=1}.$$

The first b.c. gives

$$-0 + c_1 = \mathrm{Bi}\left(-0 + 0 + c_2\right)$$

$$c_1 = \mathrm{Bi}\ c_2$$

The second b.c. gives

$$1 - c_1 = \mathrm{Bi}\left(-\frac{1}{2} + c_1 + c_2\right)$$

$$1 - \mathrm{Bi}\ c_2 = \mathrm{Bi}\left(-\frac{1}{2} + \mathrm{Bi}\ c_2 + c_2\right)$$

$$c_2 = \frac{\left(1 + \frac{\mathrm{Bi}}{2}\right)}{2\mathrm{Bi} + \mathrm{Bi}^2} = \frac{1}{2\mathrm{Bi}} \quad \Rightarrow \quad c_1 = \frac{1}{2}.$$

<u>Determination of constants: 2 points</u>

The temperature distribution is

$$\theta = -\frac{z^{*2}}{2} + \frac{z^*}{2} + \frac{1}{2\mathrm{Bi}} = \frac{1}{2\mathrm{Bi}} + \frac{1}{2} z^*\left(1 - z^*\right)$$

This expression is likely to fail near the edges of the slab.

Comment on failure of expression: 0.5

Q. 6. [15 points] THERMAL AND MOMENTUM BOUNDARY LAYERS FOR FLOW PAST A WEDGE

Consider a fluid at a temperature T_∞ that impinges against a stationary wedge of angle θ, as shown in the figure below. The wedge is maintained at a constant temperature $T_S > T_\infty$. As the fluid flows past the wedge, momentum and thermal boundary layers of thicknesses δ and δ_t respectively, are developed near the surface of the wedge. These thicknesses are functions of the position *x* measured along the surface of the wedge.

Above the wedge, outside the momentum boundary layer, the velocity, u_e, is not constant as in the flat plate problem, but a function of the position *x*:

$$u_e = c\,x^m, \quad \text{where } c \text{ is a constant, and } m = \theta / (2\pi - \theta).$$

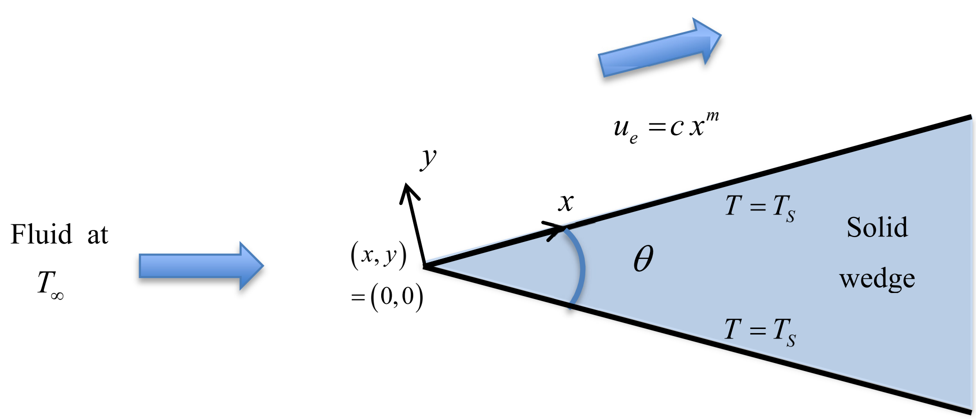
If the fluid properties are such that the thermal boundary layer is much thicker than the momentum boundary layer, determine, up to an undetermined prefactor,

(a) **[5 points]** the thickness of the momentum boundary layer, $\delta(x)$,

(b) **[5 points]** the thickness of the thermal boundary layer, $\delta_t(x)$, and

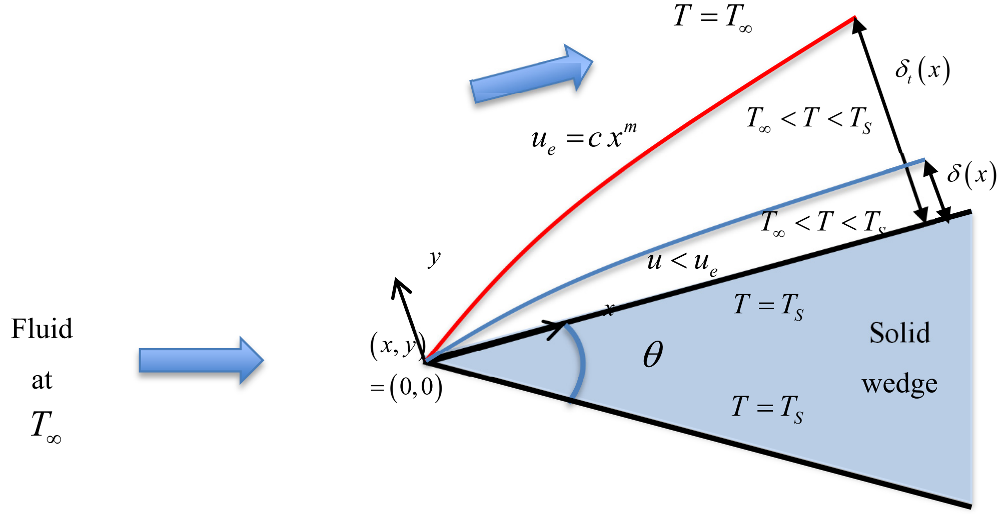
(c) **[3 points]** the Nusselt number for heat transfer, $Nu_x = hx / k_f$,

in terms of the flow conditions and the fluid properties: density ρ, specific heat capacity = C_p, viscosity = μ, and thermal conductivity k_f. Derive the answers for a general *m*, but write the solutions specifically for the case of the flow past a flat plate $[m = 0 \ (\theta = 0)]$ and the flow against a perpendicular flat plate $[m = 1 \ (\theta = \pi)]$. Assume that the flow is laminar and that the fluid properties are constant. Sketch the momentum and thermal boundary layers **[2 points].**



(Note that while the boundary layers have been shown only on the upper side of the wedge, identical boundary layers will exist on the bottom surface).

Solution



<u>Sketch of boundary layers.: 2 points</u>

Momentum boundary layer thickness

The momentum boundary layer is a result of diffusion of viscous momentum normal to the surface of the wedge, and the convection of momentum in the flow direction.

Let us consider the thickness of the momentum boundary layer at a distance x from the origin to be δ. If ν is the momentum diffusivity, then the characteristic time for momentum to diffuse over a length δ is,

$$t_c \sim \frac{\delta^2}{\nu}$$

<u>1 point for time scale</u>

Since the fluid moves with a characteristic velocity $u_e = cx^m$ in the boundary layer, the distance travelled in a time t_c is

$$u_e t_c \sim cx^m \frac{\delta^2}{\nu},$$

and this must scale as the distance x in the flow direction corresponding to boundary layer thickness δ.

$$x \sim cx^m \frac{\delta^2}{\nu}$$

<u>1 point for equating convection in flow direction with diffusion in y direction</u>

The above equation can be rearranged to provide a measure of the thickness of the momentum boundary layer as a function of the co-ordinate x along the wedge.

$$\delta \sim \sqrt{\frac{\nu x^{1-m}}{c}}$$

<u>2 points for rearranging and getting boundary layer thickness</u>

When m=0 (flat plate),

$$\delta \sim \sqrt{\frac{\nu x}{c}}$$

and when m=1 (wall perpendicular to the flow),

$$\delta \sim \sqrt{\frac{\nu}{c}}$$

Note that in the case of the wall perpendicular to the flow, the momentum boundary layer thickness does not change with the axial position; it is a constant.

<u>Solutions for m=0 and m=1</u>

Thermal boundary layer thickness

$$t_c \sim \frac{\delta^2}{\alpha}$$

Since the thermal boundary layer is much thicker than the momentum boundary layer, the fluid moves with a characteristic velocity $u_e = cx^m$ in the thermal boundary layer, the distance travelled in a time t_c is

$$u_e t_c \sim cx^m \frac{\delta^2}{\alpha},$$

and this must scale as the distance x in the flow direction corresponding to boundary layer thickness δ_t.

$$x \sim cx^m \frac{\delta_t^2}{\alpha}$$

The above equation can be rearranged to provide a measure of the thickness of the thermal boundary layer as a function of the co-ordinate x along the wedge.

$$\delta_t \sim \sqrt{\frac{\alpha x^{1-m}}{c}}$$

When m=0 (flat plate),

$$\delta_t \sim \sqrt{\frac{\alpha x}{c}}$$

and when m=1 (wall perpendicular to the flow),

$$\delta_t \sim \sqrt{\frac{\alpha}{c}}.$$

Similar grading scheme as part (a)

(c) Nusselt number:

$h \sim \dfrac{k}{\delta_t}$ and $Nu_x \sim \dfrac{x}{\delta_t} \sim \dfrac{x}{\sqrt{\dfrac{\alpha x^{1-m}}{c}}} \sim \sqrt{\dfrac{cx^{m+1}}{\alpha}}$

Correct formula for Nu: 1 point
Calculation of Nu: 1 point

For m=0, $Nu_x \sim \sqrt{\dfrac{cx}{\alpha}}$

For m=1, $Nu_x \sim x\sqrt{\dfrac{c}{\alpha}}$

Solutions for m=0 and m=1