The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 2 - Thursday March 16, 2017

Instructor: Prof. Piero Triverio

Last name: SOLUTION

First name:

Student number:

Tutorial section number:

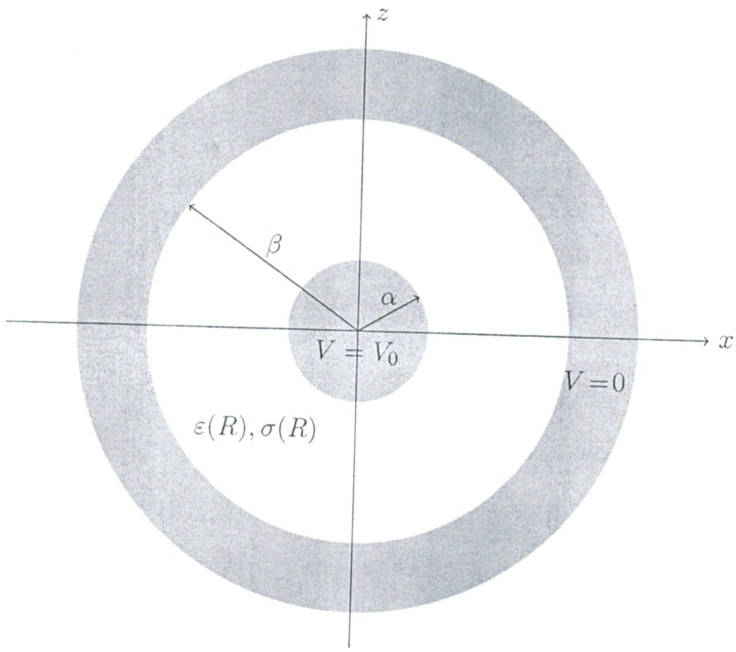
Section	Day	Time	Room	TA name
TUT0101	Wednesday	13:00-14:00	WB 144	Ayman
TUT0102	Wednesday	13:00-14:00	BA 2159	Antoine
TUT0103	Wednesday	13:00-14:00	BA 3008	Utkarsh
TUT0104	Wednesday	13:00-14:00	BA 3012	Neeraj
TUT0105	Monday	15:00-16:00	BA 2159	Ayman
TUT0106	Monday	15:00-16:00	BA 3008	Antoine
TUT0107	Monday	15:00-16:00	BA 3012	Utkarsh
TUT0108	Monday	15:00-16:00	BA 3116	Neeraj

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil are typically *not* eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

Marks:	Q1: /26	Q2: /20	Q3: /14	TOTAL: /60

Question 1



Consider the **spherical** capacitor shown in the figure above. The capacitor consists of two perfect conductors of radii α and β separated by an imperfect dielectric with absolute permittivity

$$\varepsilon(R) = \varepsilon_0 \frac{\beta}{R},$$

and conductivity

$$\sigma(R) = \sigma_0 \frac{\beta}{R}.$$

The inner conductor is held at potential $V = V_0$. The outer conductor is taken as reference for potentials.

1. Use Poisson's equation to find the potential $V(R)$ for $R \in [\alpha, \beta]$. You can assume $\rho_v = 0$ for $\alpha < R < \beta$ [8 points].

Poisson's equation $\nabla \cdot (\varepsilon \nabla V) = 0$

$$\cancel{\frac{1}{R^2}} \frac{\partial}{\partial R}\left[R^{\cancel{2}} \cancel{\varepsilon_0} \frac{\cancel{\beta}}{\cancel{R}} \frac{\partial V}{\partial R}\right] = 0$$

$$\frac{\partial}{\partial R}\left[R \frac{\partial V}{\partial R}\right] = 0$$

[3pt]

$$R\frac{\partial V}{\partial R} = C_1 \quad ; \quad \frac{\partial V}{\partial R} = \frac{C_1}{R}$$

$$V(R) = C_1 \ln R + C_2$$

Boundary conditions:

$$\begin{cases} V(\alpha) = V_0 \\ V(\beta) = 0 \end{cases} \qquad \begin{cases} C_1 \ln \alpha + C_2 = V_0 \\ C_1 \ln \beta + C_2 = 0 \end{cases}$$

[2pt]

$$C_1 \ln\frac{\alpha}{\beta} = V_0 \quad ; \quad C_1 = \frac{V_0}{\ln \alpha/\beta}$$

$$C_2 = -\frac{V_0}{\ln \alpha/\beta} \ln \beta$$

$$V(R) = \frac{V_0}{\ln \alpha/\beta} \ln R - \frac{V_0}{\ln \alpha/\beta} \ln \beta = \frac{V_0}{\ln \frac{\alpha}{\beta}} \ln R/\beta =$$

$$= V_0 \frac{\ln \beta/R}{\ln \beta/\alpha}$$

[3pt]

2. Calculate the resistance R between the conductors [8 points].

$$\bar{E} = -\nabla V = -\frac{\partial}{\partial R}\left[V_0 \frac{\ln \beta/R}{\ln \beta/\alpha} \right] \bar{a}_R =$$

$$= -\frac{V_0}{\ln \beta/\alpha} \frac{\partial}{\partial R}\left[\ln \beta - \ln R \right] \bar{a}_R = + \frac{V_0}{\ln \beta/\alpha} \frac{1}{R} \bar{a}_R$$

[2pt]

$$\bar{J} = \sigma \bar{E} = \frac{V_0 \sigma_0 \beta}{\ln \beta/\alpha} \frac{1}{R^2} \bar{a}_R$$

[2pt]

$$I = \int_S \bar{J} \cdot d\bar{S} = J \cdot 4\pi R^2 = \frac{V_0 \sigma_0 \beta}{\ln \beta/\alpha} 4\pi$$

[2pt]

since $\bar{J} \parallel d\bar{S}$

$$R = \frac{V_0}{I} = \frac{\ln \beta/\alpha}{4\pi \sigma_0 \beta}$$

[2pt]

3. Calculate the capacitance C between the conductors [6 points].

Need charge Q on inner conductor

From boundary condition on PECs:

[2pt] $\begin{cases} D_n = \rho_s \\ \rho_s = D_n(R=\alpha) = \varepsilon(\alpha)\, E_n(\alpha) = \varepsilon_0 \dfrac{\beta}{\alpha} \dfrac{V_0}{\ln \beta/\alpha} \dfrac{1}{\alpha} = \dfrac{\varepsilon_0}{\alpha^2} \dfrac{\beta V_0}{\ln \beta/\alpha} \end{cases}$

[2pt] $Q = \rho_s \cdot 4\pi\alpha^2 = \dfrac{\varepsilon_0 \beta V_0}{\ln \beta/\alpha} 4\pi$

[2pt] $C = Q/V_0 = \dfrac{\varepsilon_0 \beta 4\pi}{\ln \beta/\alpha}$

4. The dielectric between the conductors has a dielectric strength of $E_{br} = 20\,\mathrm{kV/mm}$. Find the maximum voltage $V_{0,max}$ that the capacitor can sustain without suffering any damage. Express the result in kV using $\alpha = 1\,\mathrm{mm}$ and $\beta = 3\,\mathrm{mm}$ [4 points].

$\left.\begin{array}{l} |E| < E_{br} \text{ at any point} \\ E \text{ is maximum for } R=\alpha \end{array}\right\}$ [2pt]

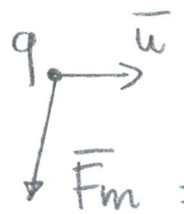
$\left.\begin{array}{l} E(\alpha) = \dfrac{V_0}{\ln \beta/\alpha} \dfrac{1}{\alpha} < E_{br} \\ V_0 < E_{br}\, \alpha \cdot \ln \beta/\alpha = 20 \cdot 1 \cdot \ln 3/1 = 21.97\,\mathrm{kV} \end{array}\right\}$ [2pt]

Question 2.1

A point charge q is travelling with velocity $\mathbf{u}$ when it enters a region where a magnetic flux density $\mathbf{B}$ is present. The magnetic flux density can change:

1. neither the magnitude nor the direction of $\mathbf{u}$;
2. only the magnitude of $\mathbf{u}$, but not the direction of $\mathbf{u}$;
3. only the direction of $\mathbf{u}$, but not the magnitude of $\mathbf{u}$; [Right answ: 2pt]
4. both the magnitude and the direction of $\mathbf{u}$.

Briefly justify your answer [4 points].



$$\bar{F}_m = q\,\bar{u} \times \bar{B}$$

$\bar{F}_m$ always $\perp$ to $\bar{u}$ by definition of cross product

$\Rightarrow$ can steer the charge's trajectory

$\Rightarrow$ can't accelerate/decelerate the charge

Expl: 2pt

Question 2.2

We have a parallel plate capacitor connected to a voltage source. In the initial state ~~,~~ the dielectric between the plates is air, and the energy stored in the capacitor is W_e. The voltage source is disconnected, and then the capacitor is filled with a dielectric with relative permittivity $\varepsilon_r = 3$. What is the energy W_e' stored in the capacitor in its final state?

1. $W_e' = W_e$;
2. $W_e' = 3W_e$;
3. $W_e' = 9W_e$;
4. $W_e' = \frac{1}{3}W_e$; [2pt]
5. $W_e' = \frac{1}{9}W_e$;

Briefly justify your answer [4 points].

$$W_e = \frac{1}{2} C V^2 = \frac{1}{2}\frac{Q^2}{C}$$

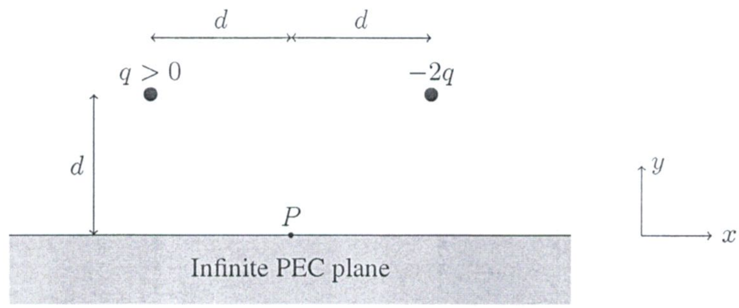
Q remains constant since source disconnected

C increases $C' = 3C$

$$W_e' = \frac{1}{2}\frac{Q^2}{3C} = \frac{1}{3} W_e$$

[2pt]

Question 2.3

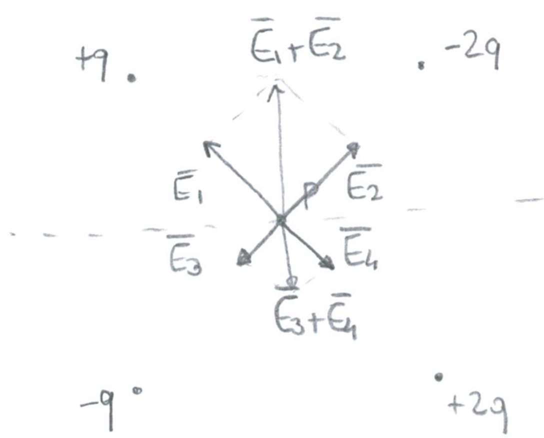


A positive point charge $q > 0$ and a negative point charge $-2q$ are placed above an infinitely wide PEC plane (PEC: perfect electric conductor). Point P is on the boundary of the PEC plane, as shown in the figure. The electric field $\mathbf{E}$ at P is in the direction of

1. $+\mathbf{a}_x$;
2. $-\mathbf{a}_x$;
3. $+\mathbf{a}_y$; [2pt]
4. $-\mathbf{a}_y$;
5. $2\mathbf{a}_x - \mathbf{a}_y$;
6. $-2\mathbf{a}_x + \mathbf{a}_y$.

Briefly justify your answer [4 points].

Image theory

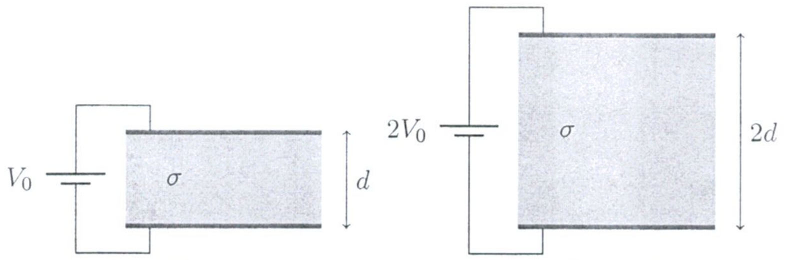


$\bar{E}_i$: field produced by each charge

The total field is in the $+\bar{a}_y$ direction

[2pt]

Question 2.4



System A   System B

The figure shows two systems. System A (left panel) consists of two parallel conducting plates separated by a material with conductivity σ. The plates are held at voltage V_0. System B (right panel) is like system A, but the following parameters are doubled: distance between the plates and voltage applied.

What is the correct relation between the total power dissipated in system A and in system B?

1. $P_A = P_B$;
2. $P_A = 2P_B$;
3. $P_A = \frac{1}{2}P_B$; [2pt]
4. $P_A = 4P_B$;
5. $P_A = \frac{1}{4}P_B$;

Briefly justify your answer [4 points].

The resistance of system B is twice the resistance of A

$$R_B = 2R_A$$

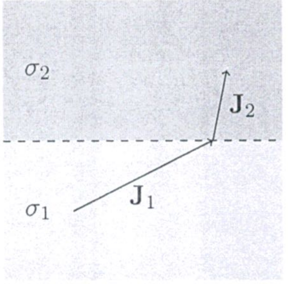
$$P_A = \frac{V_0^2}{R_A}$$

$$P_B = \frac{(2V_0)^2}{2R_A} = 2\frac{V_0^2}{R_A} = 2P_A$$

[2pt]

$$P_A = \frac{1}{2}P_B$$

Question 2.5



The figure shows the interface between two conductive materials. Which material is a better conductor?

1. Material 1; [2pt]
2. Material 2;
3. The two materials have the same conductivity;
4. More information is needed to answer this question.

Briefly justify your answer [4 points].

Boundary conditions for $\bar{J}$

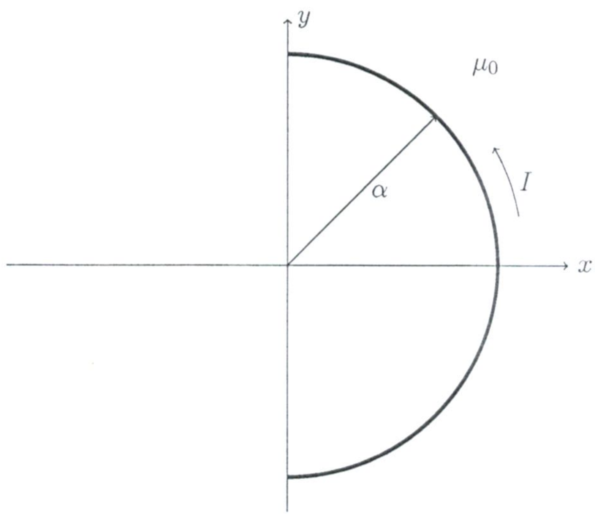
$$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$$

$$\frac{J_{2,t}}{J_{1,t}} = \frac{\sigma_2}{\sigma_1}$$ since $J_{2,t} < J_{1,t}$

$\Rightarrow \sigma_2 < \sigma_1$

[2pt]

Question 3



The semi-circular contour shown in the figure above is part of a closed circuit where a DC current I flows. The contour has radius α and is in the xy plane.

1. Find the magnetic flux density B produced by the current in the semi-circular contour at a point $z = h > 0$ on the positive z axis [14 points].

Biot-Savart law

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int \frac{\bar{dl}' \times (\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|^3}$$

Cylindrical coordinates [1 pt]

contour : $r' = \alpha \quad \varphi' \in [-\frac{\pi}{2}, \frac{\pi}{2}] \quad z' = 0$ [1 pt]

$\bar{dl}' = \alpha \, d\varphi' \, \bar{a}_{\varphi}'$ [1 pt]

$\bar{R} = h \, \bar{a}_z$ [1 pt]

$\bar{R}' = \alpha\,\bar{a}_{r'}$ [1pt]

$\bar{R} - \bar{R}' = h\,\bar{a}_z - \alpha\,\bar{a}_{r'}$ [1pt]

$|\bar{R} - \bar{R}'| = \sqrt{h^2 + \alpha^2}$ [1pt]

$$\left.\begin{aligned} d\bar{l}' \times (\bar{R} - \bar{R}') &= \alpha\, d\varphi'\, \bar{a}_{\varphi'} \times \left[h\,\bar{a}_z - \alpha\,\bar{a}_{r'} \right] = \\ &= \alpha h\, d\varphi'\, \bar{a}_{r'} + \alpha^2\, d\varphi'\, \bar{a}_z = \end{aligned}\right\} \text{[2pt]}$$

↳ function of φ'

$$= \alpha h\, d\varphi' \cos\varphi'\, \bar{a}_x + \underbrace{\alpha h\, d\varphi' \sin\varphi'\, \bar{a}_y}_{\substack{\text{will integrate} \\ \text{to zero since} \\ \int_{\varphi'=-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\varphi'\, d\varphi' = 0}} + \alpha^2 d\varphi'\, \bar{a}_z$$

[2pt]

$$\bar{B} = \frac{\mu_0 I}{4\pi} \int_{\varphi'=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha h \cos\varphi' d\varphi' \bar{a}_x + \alpha^2 d\varphi' \bar{a}_z}{\left(\sqrt{\alpha^2+h^2}\right)^3} =$$

$$= \frac{\mu_0 I}{\cancel{4}_2 \pi} \frac{\alpha h}{\left(\sqrt{\alpha^2+h^2}\right)^3} \cdot \cancel{2}^{1}\, \bar{a}_x + \frac{\mu_0 I}{4\cancel{\pi}} \frac{\alpha^2 \bar{a}_z}{\left(\sqrt{\alpha^2+h^2}\right)^3} \cancel{\pi} =$$

$$= \frac{\mu_0 I \alpha}{2\left[\alpha^2+h^2\right]^{3/2}} \cdot \left[\frac{h}{\pi} \bar{a}_x + \frac{\alpha}{2} \bar{a}_z \right]$$

[3pt]