SOLUTIONS

Q1

a)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad x_1 - 2x_3 - 3x_4 = 0$$
$$x_2 - 2x_3 - x_4 = 0$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATRIX A

b) $B = \begin{bmatrix} A & A \end{bmatrix} \quad (4 \times 8)$

APPLYING GAUSSIAN ELIMINATION TO B YIELDS

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

x_1, x_2, x_3, x_4 LEADING

x_5, x_6, x_7, x_8 FREE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_8 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Q2 a) $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 40 \end{bmatrix}$$

NO SOLUTION

i) USE NORMAL EQUATIONS

$\vec{x}_{LS} = (A^TA)^{-1}A^T\vec{b}$

$A^TA = 10$

$A^T\vec{b} = 40$

$\vec{x}_{LS} = \frac{1}{10}(40) = 4$

$\therefore\ y = 4$

ii) $y = 20$ BECAUSE ANY OTHER VALUE OF C WILL PRODUCE $|e_i|$ VALUES GREATER THAN 20, WHEREAS WITH C=20, ALL $|e_i|$ VALUES ARE EQUAL TO 20.

iii) $\left.\begin{aligned} |e_1| + \cdots + |e_9| &= 9C \\ |e_{10}| &= 40 - C \end{aligned}\right\}$ $\therefore\ |e_1| + \cdots + |e_{10}| = 8C + 40$

$y = 0$ BECAUSE $8C + 40$ INCREASES AS C INCREASES.

Q2 b) $A\vec{x} = \vec{b}$ $\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

USE NORMAL EQUATIONS

$\vec{x}_{LS} = (A^TA)^{-1} A^T\vec{b}$

$A^TA = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$

$(A^TA)^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$A^T\vec{b} = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$

$\vec{x}_{LS} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 4/6 \\ 4/2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 2 \end{bmatrix}$

LINEAR COMBINATION CLOSEST TO $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ IS

$\frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

Q3 a)

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

USING G.E. $\rightarrow$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[I \mid A^{-1} \right]$$

b)

$$\vec{x} = A^{-1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\2\\1\end{bmatrix}$$

c)

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} 1-\lambda & -1 & 1 & -1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)^4$$

$$= 0$$

EIGENVALUES $\lambda = 1,1,1,1$

FOR EIGENVECTORS, NEED TO SOLVE

$$(A-\lambda I)\vec{\omega} = \vec{0}$$

USING $\lambda = 1$.

$$\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{G.E.}]{\text{USING}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3, x_4 LEADING

x_1 FREE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

EIGENVECTOR ASSOCIATED WITH $\lambda = 1$

Q4 a)

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} \qquad L = \begin{bmatrix} \cdot & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} R1 * \tfrac{1}{a} \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix} R2 - a*R1 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix} R3 - a*R1 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & \cdot & 0 & 0 \\ a & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} R4 - a*R1 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & \cdot & 0 & 0 \\ a & \cdot & \cdot & 0 \\ a & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix} R2 * \tfrac{1}{b-a} \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & \cdot & \cdot & 0 \\ a & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix} R3 - (b-a)R2 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & \cdot & 0 \\ a & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix} R4-(b-a)R2 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & \cdot & 0 \\ a & b-a & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & c-b & d-b \end{bmatrix} R3 * \frac{1}{c-b} \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & c-b & 0 \\ a & b-a & \cdot & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & d-c \end{bmatrix} R4-(c-b)R3 \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & c-b & 0 \\ a & b-a & c-b & \cdot \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R4 * \frac{1}{d-c} \qquad L = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & c-b & 0 \\ a & b-a & c-b & d-c \end{bmatrix}$$

U $\qquad\qquad\qquad\qquad$ L

$$LU = \begin{bmatrix} a & 0 & 0 & 0 \\ a & b-a & 0 & 0 \\ a & b-a & c-b & 0 \\ a & b-a & c-b & d-c \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = A \quad \checkmark$$

b) $a \neq 0$

$b-a \neq 0 \Rightarrow a \neq b$

$c-b \neq 0 \Rightarrow b \neq c$

$d-c \neq 0 \Rightarrow c \neq d$

Q5 a) i) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$

FULL RANK AND SQUARE. $\therefore$ A IS INVERTIBLE.
$\therefore$ ALL $\vec{b}$ ARE IN THE COLUMN SPACE OF A.

ii) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

NOT FULL RANK. $\therefore$ 0 OR ∞ SOLUTIONS.

$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 2 & 4 & 8 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 4 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_2 \end{array}\right]$

$\therefore$ IF $b_3 - 2b_2 = 0$, THEN $\vec{b}$ IS IN THE COLUMN SPACE OF A.

IF $b_3 - 2b_2 \neq 0$, THEN THERE ARE NO $\vec{b}$ IN THE COLUMN SPACE OF A.

b) $\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 4 & b_2 \\ 2 & 5 & b_3 \\ 3 & 9 & b_4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 - 2b_3 + 4b_1 \\ 0 & 1 & b_3 - 2b_1 \\ 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 3b_1 - 3b_3 + b_4 \end{array}\right]$

FOR THE SYSTEM TO BE SOLVABLE

$$b_2 - 2b_1 = 0$$
$$3b_1 - 3b_3 + b_4 = 0$$

UNDER THESE CONDITIONS

$$x_1 = b_1 - 2b_3 + 4b_1 = 5b_1 - 2b_3$$
$$x_2 = b_3 - 2b_1$$

Q6 a) $y'(t) = -2t^3 + 12t^2 - 20t + 8.5 = f(t, y)$

$\Delta t = 0.5 \quad ; \quad y(t=0) = 1$

$y_{0+\frac{1}{2}} = y_0 + \left(\frac{0.5}{2}\right) f(0, y_0)$

$= 1 + (.25)(8.5) = 3.125$

$y'_{0+\frac{1}{2}} = f\left(\frac{0.5}{2}, 3.125\right) = 4.21875$

$y_1 = y_0 + (0.5)(4.21875) = 3.109375$

EXACT VALUE:

$y(t=0.5) = 3.21875$

b) $y_{1,n+1} = y_{1,n} + \Delta t\,(-0.5\,y_{1,n})$

$y_{2,n+1} = y_{2,n} + \Delta t\,(4 - 0.3\,y_{2,n} - 0.1\,y_{1,n})$