ESC194 - Midterm Test #1

October 15, 2020

9:10 - 11:50 am, EST

The following materials are considered to be acceptable aids during the writing of this test:

- The Stewart textbook and the student solution manuals
- The Stangeby/Barbeau ESC194 Supplement
- Any course notes or problem solutions prepared by the student
- Any handouts or other materials posed on the ESC194 course website

All questions are worth 10 marks

J W Davis

Solutions

1. Find $f'(x)$ for $f(x) =$

a) $x^2 + 3x + 2$ b) $x^4 + \sin x$ c) $x^4 \sin x$ d) $\frac{1}{x+1}$, $x \neq -1$ e) $\frac{2 - \sin x}{2 - \cos x}$

a) $f(x) = x^2 + 3x + 2 \Rightarrow f'(x) = 2x + 3$

b) $f(x) = x^4 + \sin x \Rightarrow f'(x) = 4x^3 + \cos x$

c) $f(x) = x^4 \sin x \Rightarrow f'(x) = 4x^3 \sin x + x^4 \cos x$

d) $f(x) = (x+1)^{-1} \Rightarrow f'(x) = (-1)(x+1)^{-2} = \frac{-1}{(x+1)^2}$ $x \neq -1$

e) $f(x) = \frac{2 - \sin x}{2 - \cos x} \Rightarrow f'(x) = \frac{(-\cos x)(2 - \cos x) - (2 - \sin x)(\sin x)}{(2 - \cos x)^2}$

$= \frac{\cos^2 x - 2\cos x + \sin^2 x - 2\sin x}{(2 - \cos x)^2}$

$= \frac{1 - 2(\sin x + \cos x)}{(2 - \cos x)^2}$

2. Evaluate the following limits if they exist. Indicate the limit laws used in your solution.

a) $\lim_{x \to 2} (3x^3 - x^2 - 4)$ b) $\lim_{x \to 0} \frac{(x-1)^2 - 1}{x}$ c) $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 4x + 3}$

d) $\lim_{x \to -1} \frac{x^2 - x - 2}{|x+1|}$ e) $\lim_{t \to 0} \left[\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right]$

a) $\lim_{x \to 2} 3x^3 - x^2 - 4 = \lim_{x \to 2} 3x^3 - \lim_{x \to 2} x^2 - \lim_{x \to 2} 4$ sum/difference rule

$= 3\left(\lim_{x \to 2} x\right)^3 - \left(\lim_{x \to 2} x\right)^2 - 4$ product, power rules, limit of a constant

$= 3 \cdot 2^3 - 2^2 - 4 = 16$ limit of x rule

b) $\lim_{x \to 0} \frac{(x-1)^2 - 1}{x} = \lim_{x \to 0} \frac{x^2 - 2x + 1 - 1}{x} = \lim_{x \to 0} \frac{x(x-2)}{x}$

$= \lim_{x \to 0} (x - 2)$ $x \neq 0$

$= \lim_{x \to 0} x - \lim_{x \to 0} 2$ difference rule, limit of a constant

$= 0 - 2 = -2$ limit of x rule

c) $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{(x-3)(x+2)}{(x-3)(x-1)}$

$= \lim_{x \to 3} \frac{x+2}{x-1}$ $x \neq 3$

$= \frac{\lim_{x \to 3} x + \lim_{x \to 3} 2}{\lim_{x \to 3} x - \lim_{x \to 3} 1}$ quotient rule, sum/difference rule

$= \frac{3+2}{3-1} = \frac{5}{2}$ limit of a constant rule, limit of x rule

2d) $\lim_{x\to -1} \frac{x^2-x-2}{|x+1|}$:

$$\lim_{x\to -1^-} \frac{x^2-x-2}{|x+1|} = \lim_{x\to -1^-} \frac{(x+1)(x-2)}{-(x+1)} = \lim_{x\to -1^-} -(x-2) = 3$$

$$\lim_{x\to -1^+} \frac{x^2-x-2}{|x+1|} = \lim_{x\to -1^+} \frac{(x+1)(x-2)}{+(x+1)} = \lim_{x\to -1^+} (x-2) = -3$$

$$\lim_{x\to -1^-} f \neq \lim_{x\to -1^+} f \quad \therefore \lim_{x\to -1} \frac{x^2-x-2}{|x+1|} \text{ DNE}$$

e) $$\lim_{t\to 0} \left[\frac{1}{t\sqrt{1+t}} - \frac{1}{t}\right] = \lim_{t\to 0} \left[\frac{1-\sqrt{1+t}}{t\sqrt{1+t}}\right] \times \frac{1+\sqrt{1+t}}{1+\sqrt{1+t}}$$

$$= \lim_{t\to 0} \frac{1-(1+t)}{t\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$= \lim_{t\to 0} \frac{-1}{\sqrt{1+t}\,(1+\sqrt{1+t})}$$

$$= -\frac{1}{2}$$

3. a) Let $f(x) = \dfrac{x^2 - 7x + 12}{x - a}$

i) For what values of a, if any, does $\lim_{x \to a^+} f(x)$ equal a finite number?

ii) For what values of a, if any, does $\lim_{x \to a^+} f(x) = \infty$?

iii) For what values of a, if any, does $\lim_{x \to a^+} f(x) = -\infty$?

b) Prove using $\delta - M$ arguments: $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = \infty$

a) $f(x) = \dfrac{x^2 - 7x + 12}{x - a} = \dfrac{(x-3)(x-4)}{x-a}$

i) the limit will exist for $a = 3$ or $a = 4$

ii) for $+^{ve}$ numerator: $x < 3$ and $x > 4$

iii) for $-^{ve}$ numerator: $3 < x < 4$

b) Prove $\lim_{x \to -3} \dfrac{1}{(x+3)^4} = \infty$

i) Find $\delta > 0$ st. for $0 < |x+3| < \delta$ $\quad \dfrac{1}{(x+3)^4} > M$

$\dfrac{1}{(x+3)^4} > M \Rightarrow (x+3)^4 < \dfrac{1}{M} \Rightarrow |x+3| < \left(\dfrac{1}{M}\right)^{1/4}$ choose δ

2) Proof: given $M > 0$, let $\delta = M^{-1/4}$

if $0 < |x+3| < \delta$ then $|x+3| < M^{-1/4} \Rightarrow (x+3)^4 < \dfrac{1}{M}$

or $\dfrac{1}{(x+3)^4} > M$

$\therefore$ By the definition of an infinite limit:

$$\lim_{x \to -3} \frac{1}{(x+3)^4} = \infty$$

4. Prove that if there is a number B such that $\left|\frac{f(x)}{x}\right| \leq B$ for all $x \neq 0$, then $\lim_{x\to 0} f(x) = 0$

Consider $x > 0$ $\therefore$ $|f(x)| \leq Bx$ or $-Bx \leq f(x) \leq Bx$

since $\lim_{x\to 0^+} -Bx = \lim_{x\to 0^+} Bx = 0$

$\Rightarrow \lim_{x\to 0^+} f(x) = 0$ by pinching th'm

Consider $x < 0$ $\therefore$ $|f(x)| \geq Bx$ or $-Bx \geq f(x) \geq Bx$

since $\lim_{x\to 0^-} -Bx = \lim_{x\to 0^-} Bx = 0$

$\Rightarrow \lim_{x\to 0^-} f(x) - 0$ by pinching th'm

$\lim_{x\to 0^-} f(x) = 0 = \lim_{x\to 0^+} f(x)$ $\therefore$ $\lim_{x\to 0} f(x) = 0$

5. a) Verify that the function $f(x) = 3x^2 - 2x + 2$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find all numbers c that satisfy the conclusions of the Mean Valus Theorem.

b) Let $f(x) = \dfrac{1}{(x-1)^2}$. Show that there is no value of c in $(0, 3)$ such that $f(3) - f(0) = f'(c)(3 - 0)$. Why does this not contradict the Mean Valus Theorem?

a) $\left.\begin{array}{l} f(x) = 3x^2 - 2x + 2 \\ f'(x) = 6x - 2 \end{array}\right\}$ f is continuous on $[0,2]$ and differentiable on $(0,2)$ since polynomials are continuous and differentiable for $x \in \mathbb{R}$

$$\Rightarrow f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{10 - 2}{2} = 4$$

$$\Rightarrow 6c - 2 = 4 \Rightarrow c = 1 \in (0, 2)$$

b) $f(x) = (x-1)^{-2}$ $\qquad f'(x) = -2(x-1)^{-3} = \dfrac{-2}{(x-1)^3}$

find c st.: $f'(c) = \dfrac{-2}{(c-1)^3} = \dfrac{f(3) - f(0)}{3 - 0} = \dfrac{1/4 - 1}{3} = -\dfrac{1}{4}$

$$\Rightarrow \frac{-2}{(c-1)^3} = -\frac{1}{4} \Rightarrow 8 = (c-1)^3 \Rightarrow c - 1 = 2$$

$$\therefore c = 3$$

but $c \notin (0, 3)$

This does not contradict the MVT since f is not continuous at $x = 1$.

6. Let $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ and $g(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) Show that f and g are both continuous at 0.

b) Show that f is not differentiable at 0.

c) Show that g is differentiable at 0 and give $g'(0)$.

a) Since $\left| \sin \frac{1}{x} \right| \leq 1$

$\Rightarrow \quad -x \leq x \sin \frac{1}{x} \leq x \quad$ and $\quad -x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$

Thus: $\lim\limits_{x \to 0} f(x) = 0 = f(0)$ and $\lim\limits_{x \to 0} g(x) = 0 = g(0)$

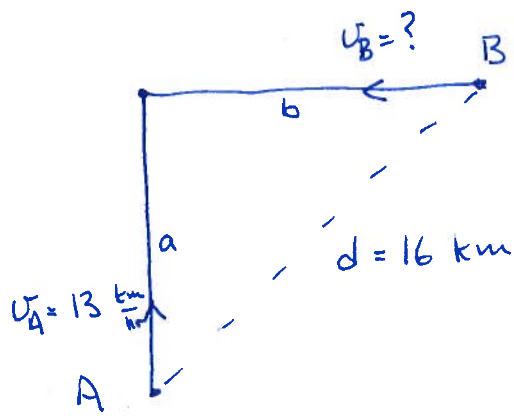
by pinching theorem.

b) $\lim\limits_{h \to 0} \dfrac{h \sin \frac{1}{h} - 0}{h} = \lim\limits_{h \to 0} \sin \frac{1}{h}$ DNE

c) $\lim\limits_{h \to 0} \dfrac{h^2 \sin \frac{1}{h} - 0}{h} = \lim\limits_{h \to 0} h \sin \frac{1}{h} = 0$

$\therefore$ $g(x)$ is differentiable at $x = 0$ and $g'(0) = 0$

7. Two boats are racing with constant speed towards a finish marker; boat A sailing from the south at 13 km/hr and boat B approaching from the east. When both boats are the same distance from the marker, the boats are 16 km apart and the distance between them is decreasing at a rate of 17 km/hr. Which boat will win the race?



$$a^2 + b^2 = 2b^2 = 16^2$$
$$\Rightarrow b = 8\sqrt{2}$$

$$a = 8\sqrt{2} - v_A t = 8\sqrt{2} - 13t \qquad a' = -13$$
$$b = 8\sqrt{2} - v_B t \qquad b' = -v_B$$

$$d^2 = a^2 + b^2 \Rightarrow 2d \cdot d' = 2aa' + 2bb'$$
$$\therefore 2 \cdot 16 \cdot (-17) = 2 \cdot 8\sqrt{2}(-13) + 2 \cdot 8\sqrt{2}(-v_B)$$
$$\Rightarrow 34 = 13\sqrt{2} + \sqrt{2}\, v_B$$
$$\therefore v_B = \frac{34 - 13\sqrt{2}}{\sqrt{2}} = 11.04 \text{ km/hr}$$

$\therefore$ Boat A will win the race

8. Prove using $\epsilon - \delta$ arguments: $\lim_{x \to 1} \frac{\sqrt{|x-1|}}{(2+x)^2} = 0$

Given $\epsilon > 0$, find a $\delta > 0$ s.t. for $0 < |x-1| < \delta$, $\left| \frac{\sqrt{|x-1|}}{(x+2)^2} - 0 \right| < \epsilon$

1) find δ : $\left| \frac{\sqrt{|x-1|}}{(x+2)^2} - 0 \right| = \frac{\sqrt{|x-1|}}{(x+2)^2}$

let $\delta < 1$ $\therefore$ $\left.\begin{array}{l} |x-1| < 1 \\ -1 < x-1 < 1 \\ 0 < x < 2 \end{array}\right\}$ $\therefore$ $2+x > 2$, $\frac{1}{2+x} < \frac{1}{2}$, $\frac{1}{(2+x)^2} < \frac{1}{4}$

$\therefore \frac{\sqrt{|x-1|}}{(2+x)^2} < \frac{1}{4}\sqrt{|x-1|} \overset{\text{want}}{<} \epsilon$

$\sqrt{|x-1|} < 4\epsilon$

$|x-1| < (4\epsilon)^2$ $\underset{=}{\text{choose}}$ δ

$\Rightarrow \delta = \min\{1, (4\epsilon)^2\}$

2) proof : given $\delta = \min\{1, (4\epsilon)^2\}$

$\therefore |x-1| < \delta \Rightarrow \sqrt{|x-1|} < \sqrt{\delta} \Rightarrow \frac{1}{4}\sqrt{|x-1|} < \frac{1}{4}\sqrt{\delta}$

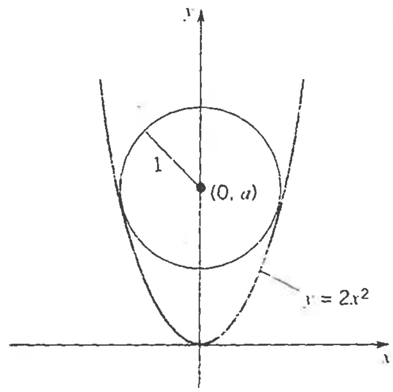
for $\delta < 1$ $\frac{1}{(2+x)^2} < \frac{1}{4}$ an $\frac{\sqrt{|x-1|}}{(2+x)^2} < \frac{1}{4}\sqrt{\delta}$

$\therefore \left| \frac{\sqrt{|1-x|}}{(2+x)^2} - 0 \right| < \frac{1}{4}\sqrt{\delta} \leq \frac{1}{4}\sqrt{(4\epsilon)^2} = \epsilon$

$\therefore$ By the definition of a limit $\lim_{x \to 1} \frac{\sqrt{|x-1|}}{(2+x)^2} = 0$

9. a) Find all values for a, b and c such that the curves $y_1 = ax^3 + bx + c$ and $y_2 = cx^2 + bx$ have a common tangent-line.

b) A circle with radius 1 with centre on the y-axis is inscribed in the parabola $y = 2x^2$ as shown in the figure. Find the points of intersection.



a) $\left.\begin{aligned} y_1 &= ax^3 + bx + c \\ y_2 &= cx^2 + bx \end{aligned}\right\}$ To have a common tangent line there must be a common slope at a point of intersection.

$\left.\begin{aligned} y_1' &= 3ax^2 + b \\ y_2' &= 2cx + b \end{aligned}\right\}$ $y_1' = y_2' \Rightarrow 3ax^2 + b = 2cx + b$ $\Rightarrow x = 0$

$3ax = 2c$ $\Rightarrow a = c = 0$

or $x = \frac{2}{3}\frac{c}{a}$

case ① $x = 0$: $y_1(0) = c$, $y_2(0) = 0$ $\therefore$ no intersection except when $c = 0$

case ② $a = c = 0$ $\Rightarrow y_1(x) = bx = y_2(x)$ $\therefore$ common tangents for all x

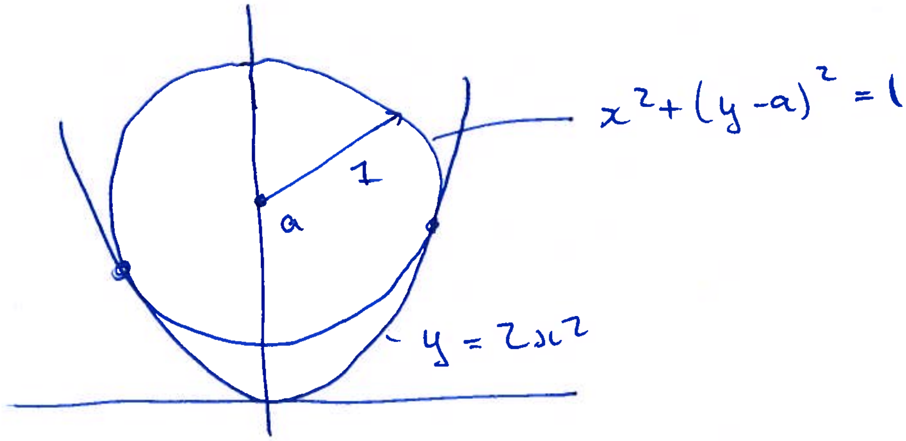
case ③ $x = \frac{2}{3}\frac{c}{a}$ $\Rightarrow a\left(\frac{2}{3}\frac{c}{a}\right)^3 + b\left(\frac{2}{3}\frac{c}{a}\right) + c = c\left(\frac{2}{3}\frac{c}{a}\right)^2 + b\left(\frac{2}{3}\frac{c}{a}\right)$

$\frac{8}{27}\frac{c^3}{a^2} + c = \frac{4}{9}\frac{c^3}{a^2} \Rightarrow \frac{4}{27}\frac{c^3}{a^2} = c \Rightarrow \frac{c}{a} = \pm\frac{3\sqrt{3}}{2}$ $(c \neq 0)$

Answers: 1) $c = 0$; b, a any value

2) $c = \pm\frac{3\sqrt{3}}{2}a$; b, a any value

9b)



At the points of intersection, the two curves will have a common tangent.

circle : $2x+2(y-a)y'=0 \Rightarrow y'=\frac{-x}{y-a}$

parabola: $y'=4x$

$\Rightarrow 4x=\frac{-x}{y-a} \Rightarrow y-a=\frac{1}{4}$ $\quad x \neq 0$

$\therefore y = a-\frac{1}{4}$

($x=0$ would have a common tangent but it is not what we are looking for)

$\Rightarrow x^2+\left(-\frac{1}{4}\right)^2=1 \Rightarrow x=\pm\sqrt{1-\frac{1}{16}}=\pm\sqrt{\frac{15}{16}}$

$\Rightarrow y=2x^2=2\cdot\frac{15}{16}=\frac{15}{8}$

$\therefore$ pts of intersection $\left(\pm\frac{\sqrt{15}}{4}, \frac{15}{8}\right)$

(In this case, $a=\frac{13}{8}$)

10. Calculate enough derivatives of the function $f(x) = \sqrt{1-3x}$ to enable you to guess the general formula for $f^{(n)}(x)$ (i.e., the n^{th} derivative of $f(x)$). Then verify your guess using mathematical induction.

$f(x) = (1-3x)^{1/2}$

$f'(x) = \frac{1}{2}(1-3x)^{-1/2}(-3)$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(1-3x)^{-3/2}(-3)^2$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(1-3x)^{-5/2}(-3)^3$

$f^{iv}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(1-3x)^{-7/2}(-3)^4$

$\Rightarrow f^{(n)}(x) = -\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)}{2^n}\, 3^n (1-3x)^{-\frac{2n-1}{2}}$

for $n \geq 2$

Proof by induction:

1) As shown, formula holds for $n = 2, 3, 4$

2) Assume it holds for $n = k$, show it works for $n = k+1$

$f^{(k)}(x) = -\frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-3)}{2^n}\, 3^n (1-3x)^{-\frac{2k-1}{2}}$

$\left(f^{(k)}(x)\right)' = -\frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-3)}{2^k}\, 3^k \left(-\frac{2k-1}{2}\right)(1-3x)^{-\frac{2k-1}{2}-1}(-3)$

$= -\frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-3)(2k-1)}{2^{k+1}}\, 3^{k+1} (1-3x)^{-\frac{2k+2-1}{2}}$

$= -\frac{1\cdot 3\cdot 5\cdot 7\cdots(2(k+1)-3)}{2^{k+1}}\, 3^{k+1} (1-3x)^{-\frac{2(k+1)-1}{2}} = f^{(k+1)}(x)$