PHY293 Modern Physics –
Quiz # 4 Solutions

Revised: 7 December 2021

Version A We started with the time-dependent Schrödinger equation in one dimension

$$i\hbar\frac{d\psi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}, \tag{1}$$

where $\hbar = 6.582 \times 10^{-16}$ eV s and m is the mass of the particle.

(a) Using the separation of variables technique, derive a differential equation for the space-dependent part of the wave function $\psi(x,t)$. What is the nature of the resulting differential equation?

(b) We have an infinite well of width $\pm a = \pm 0.2$ nm that contains a proton with mass $m_p = 0.938$ GeV/c^2. The time and space dependence of the two lowest energy states are

$$\begin{aligned} \psi_1(x) &= \exp(-iE_1 t/\hbar)\frac{1}{\sqrt{a}}\cos(k_1 x) && (2) \\ \psi_2(x) &= \exp(-iE_2 t/\hbar)\frac{1}{\sqrt{a}}\sin(k_2 x), && (3) \end{aligned}$$

where

$$E_n = \frac{\hbar^2\pi^2 n^2}{8ma^2}. \tag{4}$$

Use the boundary conditions to determine k_1 and k_2. Calculate the energies of the two states.

(c) Write an expression for the probability of finding the particle at a position x and time t in the ground state? Simplify it as much as possible.

(d) Sketch the spatial dependence of the probability of the proton in its lowest-energy state.

Solution:

(a) Separation of variables starts by assuming that $\psi(x,t) = X(x)T(t)$. Substituting that into Schrödinger's equation, we find that doing the differentiation and dividing by ψ gives us

$$\begin{aligned} \left(i\hbar X\frac{dT}{dt}\right)\frac{1}{XT} &= \left(-\frac{\hbar^2}{2m}T\frac{d^2X}{dx^2}\right)\frac{1}{XT} && (5) \\ \Rightarrow i\hbar\frac{1}{T}\frac{dT}{dt} &= -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}. && (6) \end{aligned}$$

Since this has to true for all x and t, and the LHS only depends on t and the RHS only depends on x, it means that they both must be equal to a constant, which we denote E. This gives us an equation for X:

$$EX = -\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}. \tag{7}$$

This equation has solutions that for $E > 0$ are linear combinations of $\sin(kx)$ and $\cos(kx)$, where $k = \sqrt{2mE}/\hbar$.

(b) The boundary conditions require that $\psi(ka) = \psi(-ka) = 0$. For $\psi_1(x)$, it requires that $k_1 a = n\pi$, for $n = 1, 3, 5, \ldots$, so that $k_1 = \pi/a$ if it is to be the ground state. Similarily, for $\psi_2(x)$, the boundary conditions are satisfied if $k_2 a = n\pi$, for $n = 2, 4, 5, \ldots$, so that $k_2 = 2\pi/a$ for it to be the lowest energy state above the ground state. The energies of the two states are

$$E_1 = \frac{\hbar^2\pi^2}{8ma^2} = \frac{(6.582 \times 10^{-16})^2\pi^2(3 \times 10^8)^2}{8(9.38 \times 10^8)(2 \times 10^{-10})^2} = 0t.00128 \text{ eV} \quad (8)$$

$$E_2 = 4E_1 = 0.00512 \text{ eV}, \quad (9)$$

where the powers of c are needed to convert from mass units to energy.

(c) The probability of finding the particle in the ground state is given by

$$p_1(x,t) = |\psi_1(x,t)|^2 \quad (10)$$

$$= |\exp(-iE_1 t/\hbar)\frac{1}{\sqrt{a}}\cos(k_1 x)|^2 \quad (11)$$

$$= \frac{\cos^2(k_1 x)}{a}. \quad (12)$$

Version B

1. A proton with mass $m_n = 0.9383$ GeV/c^2 is in a 1-dimensional potential well of size $d = 5 \times 10^{-15}$ m, or the typical size of an atomic nucleus, with a potential barrier $V_0 = 1$ GeV. You can use $\hbar = 6.582 \times 10^{-16}$ eV s.

   (a) The energy of the n^{th} quantum state is $(\hbar^2\pi^2 n^2)/(8ma^2) \times F$, where $F = 0.7$ is the ratio of the solution to the ground state energy for a finite well and that of an infinite well. What is the energy of the ground state?

   (b) How does this energy compare with the potential well the proton is in?

   (c) Is there a sizeable probability (say $> 50\%$) that we would find the proton in its ground state outside the well?

   (d) Calculate the momentum of the proton in the ground state assuming that the ground state energy is the proton's kinetic energy and the proton is non-relativistic.

   (e) What is the de Broglie wavelength of the proton? How does it compare with the size of the potential well?

Solution:

(a) The energy of the ground state is when $n = 1$,

$$E_1 = (\hbar^2\pi^2)/(8ma^2) \times F = \frac{(6.582 \times 10^{-16})^2\pi^2(3 \times 10^8)^2}{8(9.383 \times 10^8)(2.5 \times 10^{-15})^2} \times (0.7) = 5.74 \text{ MeV}. \quad (13)$$

(b) This energy is about 0.5% of the height of the potential well.

(c) We would expect the proton to be almost always inside the well given this large energy gap.

(d) The momentum of the proton in the ground state would be given by

$$p_1 = \sqrt{2m_p E_1} = \sqrt{2(9.383 \times 10^8)(5.74 \times 10^6)} = 104 \text{ MeV/c}. \quad (14)$$

(e) The de Broglie wavelength is

$$\lambda_p = \frac{2\pi\hbar}{p_1} = \frac{2\pi(6.582 \times 10^{-16})(3 \times 10^8)}{1.04 \times 10^8} = 1.19 \times 10^{-14} \text{ m}. \tag{15}$$

This is about twice the size of the potential well, but of the same order of magnitude.