University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – April, 2019

FIRST YEAR – ENGINEERING SCIENCE

MAT195S CALCULUS II

Examiners: F. Al Faisal and J. W. Davis

First name (please write as legibly as possible within the boxes)

Last name

Student number

Instructions:
(1) Closed book examination; no calculators, no aids are permitted
(2) Answer as many questions as you can. Parts of questions may be answered.
(3) Do not separate or remove any pages from this exam booklet.

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Question	Marks	Earned	Question	Marks	Earned
1	9		7	10	
2	9		8	10	
3	9		9	12	
4	11		10	11	
5	12		11	10	
6	10		12	8	
			Total	121	

1) Evaluate the integrals:

a) $\int \frac{\sqrt{x^2-9}}{x^3}dx$

b) $\int \sqrt{1-e^x}\,dx$

(9 marks)

2) a) Find the length of the curve: $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \le t \le 1$

(4 marks)

b) If two parallel planes intersect a sphere, calculate the surface area of the part of the sphere that lies between the two planes. Hence, deduce that the area depends only on the radius of the sphere and the distance between the two planes, but not on where the planes intersect the sphere.

(5 marks)

3) Determine whether the following integrals converge or diverge. If they converge, find their value.

a) $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}\,(x+1)}$ b) $\displaystyle\int_1^{e} \frac{dx}{x(\ln x)^2}$ c) $\displaystyle\int_2^{\infty} \frac{x}{\sqrt{x^4-1}}\,dx$

(9 marks)

4) Sketch the region indicated, and find an integral representing the area of the region. Do not evaluate the integrals.

a) The region enclosed by one loop of the curve $r = \sin 3\theta$.

b) The region that lies inside both $r = \sin 2\theta$ and $r = \cos 2\theta$.

c) The region that lies inside the circle $r = 2\sin\theta$ but outside $r = \frac{3}{2} - \sin\theta$.

(11 marks)

5) Determine whether the following series converge or diverge:

a) $\sum_{n=1}^{\infty} \frac{n^3}{(\ln 2)^n}$ b) $\sum_{n=1}^{\infty} \left(1+\frac{2}{n}\right)^n$ c) $\sum_{k=2}^{\infty} \frac{1}{k(1+\ln k)}$ d) $\sum_{n=1}^{\infty} \frac{n^{\left(n+\frac{1}{n}\right)}}{\left(n+\frac{1}{n}\right)^n}$

(12 marks)

6) a) Suppose that the power series $\sum c_n (x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$ exists, then it is equal to the radius of convergence of the power series.

(5 marks)

b) Determine the radius and interval of convergence for the series: $\sum_{k=2}^{\infty}\frac{(x+2)^k}{2^k \ln k}$

(5 marks)

7) If the function $f(x) = (1 - 2\mu x + x^2)^{-1/2}$ is expanded in a Maclaurin series in x:

$$\frac{1}{\sqrt{1 - 2\mu x + x^2}} = \sum_{n=0}^{\infty} P_n(\mu) x^n$$

The coefficients $P_n(\mu)$ are called the *Legendre polynomials.*

a) Find $P_0(\mu)$, $P_1(\mu)$, $P_2(\mu)$ and $P_3(\mu)$. (The coefficients may be found in several ways, including by direct differentiation, and by expanding f as a binomial series.)

b) Show that in the cases where $\mu = \pm 1$, the coefficients can be evaluated using the geometric series.

(10 marks)

8) On our first midterm test, we defined the gamma function $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt$, for $z > 0$.

a) Let $I(s) = \int\limits_0^{\infty} \frac{x^s}{e^x - 1}dx$, $s > 0$. Show that $I(s) = \Gamma(s+1)\cdot\sum\limits_{n=1}^{\infty}\frac{1}{n^{s+1}}$

b) Find $\int\limits_0^{\infty} \frac{x}{e^x - 1}dx$ by evaluating $I(1)$. Express your answer in the form π^2/k.

Hint 1: On our first midterm test we saw that $\Gamma(4) = 3!$. One can similarly show in general that $\Gamma(n+1) = n!$ for any positive integer n. You may use this result without proof.
Hint 2: A result from the second midterm test may be helpful in part b).

(10 marks)

9) The motion of a particle is given by: $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + \ln(1+t)\,\hat{k}$, for $t > 0$. Determine the tangential and normal components of acceleration of this particle, and the curvature of its path at time $t = 1$.

Hint: There are several ways of solving this problem without finding the derivative of the unit tangent vector.

(12 marks)

10) a) Find the unit direction vector(s) $\hat{u}$ for which the directional derivative for $f(x,y) = ye^{-xy}$ at the point $(0,2)$ has value 1. That is, find $\hat{u}$ so that $D_{\hat{u}}f(0,2) = 1$.

(6 marks)

b) Are there any points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 1$? If so, find them.

(5 marks)

11) Find the maximum value of $f(x,y) = xy - x^3y^2$ on the square $0 \le x \le 1, \quad 0 \le y \le 1$. Provide a sketch of the region, and identify and show the locations of all critical points.

(10 marks)

12) Use Lagrange multipliers to find the extreme values of the function $f(x,y,z) = xz + xy$ subject to the two constraints: $x^2 + z^2 = 2$ and $xy = 1$.

(8 marks)