UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Final Exam

MAT185H1S — Linear Algebra

Examiners: S Uppal & G M T D'Eleuterio

26 April 2019

Student Name:

Last Name First Names

Student No: e-Address:

Signature:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution. The total number of marks available is **100**.
3. Write solutions *only* in the boxed space provided for each question. *Do not* write solutions on the reverse side of pages. These will *not* be scanned and therefore will *not* be marked.
4. Three blank pages are provided at the end for rough work. Work on these back pages will *not* be marked unless clearly indicated; in such cases, clearly indicate on the question page(s) that the solution(s) is continued on a back page(s).
5. *Do not* write over the QR code on the top right-hand corner of each page.
6. *No* aid is permitted.
7. The duration of this exam is 2 hours and 30 minutes.
8. There are 16 pages and 6 questions in this test paper.

A Note on Notation:

1. ${}^m\mathbb{R}^n = M_{m\times n}(\mathbb{R})$, the former notation is used in the Notes and the latter in Nicholson.

A. Definitions and Statements

Fill in the blanks.

1(a). The *span* of a set of vectors $\{v_1, v_2 \cdots v_n\}$ is defined as

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1(b). A *subspace* of a vector space is defined as

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/3

1(c). State the *dimension formula*, *i.e.*, the *rank-nullity theorem.*

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/3

1(d). The *eigenspace* of a matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

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/3

1(e). State the *Diagonalization Test* (*i.e.*, the necessary and sufficient conditions for the diagonalizability of a matrix).

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/3

B. Possible or Impossible

For each of the following, give an example if possible or explain why it is impossible.

2(a). A subspace $\mathcal{S}$ of $\mathbb{R}$ such that $\mathcal{S} \neq \{\mathbf{0}\}$ and $\mathcal{S} \neq \mathbb{R}$.

/3

2(b). Two square matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\operatorname{rank} \mathbf{A} = \operatorname{rank} \mathbf{B}$ but $\operatorname{rank} \mathbf{A}^2 \neq \operatorname{rank} \mathbf{B}^2$.

/3

2(c). A 3×3 matrix whose image space and null space are both two-dimensional.

/3

2(d). A matrix $\mathbf{A} \neq \mathbf{O}$ such that $\operatorname{adj} \mathbf{A} = \mathbf{O}$.

/3

2(e). A matrix that is neither diagonalizable nor invertible.

/3

C. Proving Ground

In each of the following questions, two statements are given. Determine the relation between the two and indicate your answer in the box provided. There are four options:

If there is no relation	...indicate by...	"✗"
If the left statement implies the right	...indicate by...	"$\Rightarrow$"
If the left statement is implied by the right	...indicate by ...	"$\Leftarrow$"
If the left statement implies and is implied by the right	...indicate by...	"$\Leftrightarrow$"

The value of each question is 3 marks. For 3 marks the complete answer is required while a partially correct answer will earn 2 marks.

3(a). Let $v \in \mathcal{V}$, a vector space.

$\{v\}$ is linearly dependent $\square$ $v = 0$

3(b). Let $x, y, z \in \mathcal{V}$, a vector space.

$\{x, y, z\}$ is linearly independent $\square$ each of $\{x, y\}, \{y, z\}, \{z, x\}$ is linearly independent

3(c). Let $\mathbf{A} \in {}^n\mathbb{R}^n$.

$\mathbf{A} = -\mathbf{A}^T$ $\square$ $\det \mathbf{A} = 0$

3(d). Let $\mathbf{A} \in {}^3\mathbb{R}^3$.

$\mathbf{A}$ has eigenvalues $1, 2, 3$ $\square$ $\mathbf{A}^2$ has eigenvalues $1, 4, 9$

3(e). Let $\mathbf{A} \in {}^n\mathbb{R}^n$.

$\mathbf{A}^2 = \mathbf{1}$ $\square$ $\mathbf{A}$ is diagonalizable and has only ± 1 as eigenvalues

D. Just the Answers

You have been provided space for rough work but you will be graded on just the answers in the boxes.

4. Consider the following system of first-order ordinary differential equations:

$$\begin{aligned} \dot{x}_1 &= \quad + x_2 + x_3 \\ \dot{x}_2 &= x_1 \quad\;\; + x_3 \\ \dot{x}_3 &= x_1 + x_2 \end{aligned}$$

4(a). What are the eigenvalues of the system?

/3

4(b): What are the eigenvectors of the system?

/6

4(c). Solve for the response of the system given $x_1(0) = 0, x_2(0) = 1, x_3(0) = 2$.

/6

E. Problems

5. Let $\{\mathbf{x}, \mathbf{y}\} \in {}^2\mathbb{R}$ be linearly independent and let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be invertible. Show that $\{a\mathbf{x} + c\mathbf{y}, b\mathbf{x} + d\mathbf{y}\}$ is a basis for ${}^2\mathbb{R}$.

...cont'd

5. . . . cont'd

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6. Matrices $\mathbf{A}, \mathbf{B} \in {}^n\mathbb{R}^n$ are said to be *simultaneously diagonalizable* if there exists an invertible matrix $\mathbf{S} \in {}^n\mathbb{R}^n$ such that both $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ are diagonal.

6(a). If $\mathbf{A}, \mathbf{B}$ are simultaneously diagonalizable, explain why there exists a basis for ${}^n\mathbb{R}$ in which each vector is an eigenvector of both $\mathbf{A}$ and $\mathbf{B}$. (Two or three sentences should suffice.)

/5

6(b). Show that if $\mathbf{A}, \mathbf{B}$ are simultaneously diagonalizable then $\mathbf{AB} = \mathbf{BA}$.

...cont'd

6. . . . cont'd

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