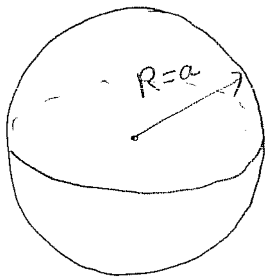
Question 1 [25 pts]

A. A spherical surface of radius R=α supports a non-uniform charge distribution, with surface charge density $\rho_s(\theta,\phi) = \rho_0 \sin(\theta)$, where ρ_0 is a positive constant and $0 \le \theta \le \pi$.

1. Compute the electric potential V(R=0) at the center of the sphere. You may find the following integral useful: $\int_0^{\pi} \sin^2\theta \, d\theta = \frac{\pi}{2}$. **[10 points]**



$$dQ' = \rho_s ds' = \rho_0 \sin\theta' \cdot a^2 \sin\theta' d\theta' d\varphi' \quad [3 \text{ pts}]$$

$$\bar{R} = 0, \; \bar{R}' = a\,\bar{a}_R \Rightarrow |\bar{R} - \bar{R}'| = a \quad [2 \text{ pts}]$$

$$dV = \frac{dQ'}{4\pi\epsilon_0 a} = \frac{\rho_0 a^{\cancel{2}} \sin^2\theta'}{4\pi\epsilon_0 \cancel{a}} d\theta' d\varphi' \quad [3 \text{ pts}]$$

$$\Rightarrow V = \frac{\rho_0 a}{4\pi\epsilon_0} \underbrace{\underbrace{\int_0^{\pi} \sin^2\theta' d\theta'}_{\pi/2} \underbrace{\int_0^{2\pi} d\varphi'}_{2\pi}}_{\pi^2} = \frac{\rho_0 \pi a}{4\epsilon_0} \quad [2 \text{ pts}]$$

$$\left(\text{units: } \frac{C}{\cancel{m^2}} \times \frac{\cancel{m}}{F/\cancel{m}} = V\right) \cdot \checkmark$$

2. Is it possible to use Gauss law to compute the electric field outside the sphere?

[5 points]

No, because $\rho_s = \rho_s(\theta)$

No symmetry in charge distribution $\Rightarrow$

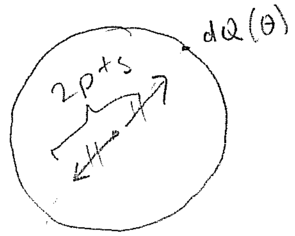
not possible to find a surface with

$\bar{E} \cdot d\bar{s} = |\bar{E}||d\bar{s}|$ or 0 $\Rightarrow$ Gauss Law ~~sta~~ valid but

NOT useful for field calculation.

3. A point charge Q = 1 nC is placed at the center of the sphere. Find the Coulomb force **F** that will be applied to the point charge. No integral calculations are needed.

[5 points]

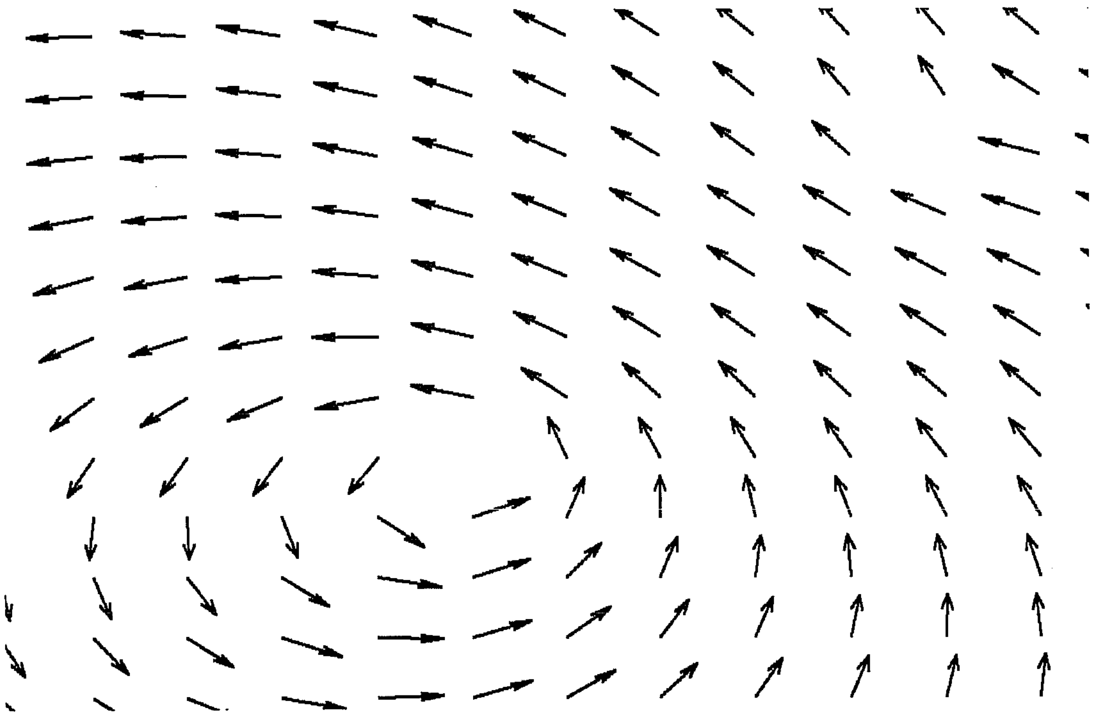


$$\underbrace{dQ(\pi-\theta) = dQ(\theta)}_{3pts}$$

Forces cancel in pairs as shown, due to the $\sin^2\theta$ dependence of

$$dQ = \rho_s \cdot ds = \rho_0 \sin\theta \, a^2 \sin\theta \, d\theta \, d\varphi$$

B. Consider the vector field **F** shown below. Can **F** be the electric field generated by an electrostatic charge distribution? Explain. **[5 points]**



No. $\oint \bar{F} \cdot d\bar{l} \neq 0$, clearly $\bar{F}$ has circulation.

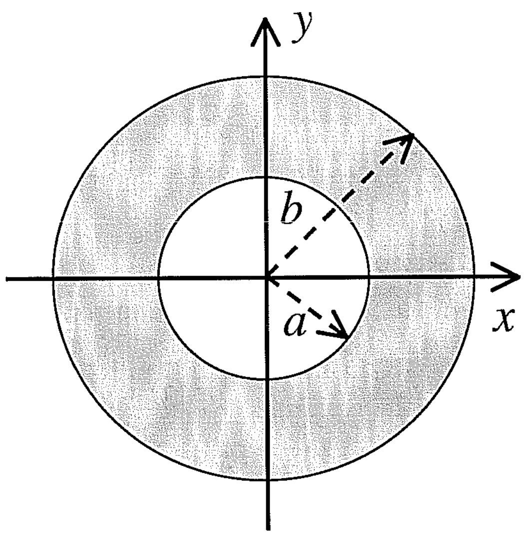
--- End of Question 1

Question 2 [25 pts]

The figure below shows the cross section of a hollow cylinder. The cylinder is infinitely long along the z-direction and is uniformly charged. The volume charge density is:

$$\rho_\upsilon(\mathrm{r}) = \begin{cases} \rho_0, & a < \mathrm{r} < \mathrm{b} \\ 0, & \text{elsewhere} \end{cases}$$

where ρ_0 is a positive constant (in C/m^3).



1. Find the electric field in any position in space (r>0). **[15 pts]**

$\bar{E} = E_r(r)\,\bar{a}_r$ [2pts]

due to cylindrical symmetry

Gauss Law applicable on a cylinder [2pts]

$r<a$: $Q_{enclosed} = 0 \Rightarrow E_r = 0$ [1pt]

$a<r<b$: $Q_{enclosed} = \underbrace{\rho_0 \pi (r^2-a^2)\cdot L}_{[2pts]} \Rightarrow \underbrace{E_r \cdot 2\pi r \cdot \cancel{L} = \frac{\rho_0 \pi (r^2-a^2)\cdot \cancel{L}}{\epsilon_0}}_{[2 pts]}$

$$E_r = \frac{\rho_0 (r^2-a^2)}{\underbrace{2\epsilon_0 r}_{1\,pt}}$$

$\underline{r>b}$: $Q_{enclosed} = \rho_o \pi (b^2 - a^2) \cdot L$ [2pts]

$$E_r \cdot 2\cancel{\pi} r \cancel{L} = \frac{\rho_o \cancel{\pi} (b^2 - a^2) \cdot \cancel{L}}{\epsilon_0}$$

$$E_r = \frac{\rho_o (b^2 - a^2)}{2 \epsilon_0 r}$$ [1pt]

2. Does the electric potential increase/decrease or stay the same as a function of distance r, in the regions r<a, a< r<b, r>b? Explain. **[6 pts]**

$r<a$: V stays constant because $\bar{E}=0$ as no electric field exists (no charges) [2pts]

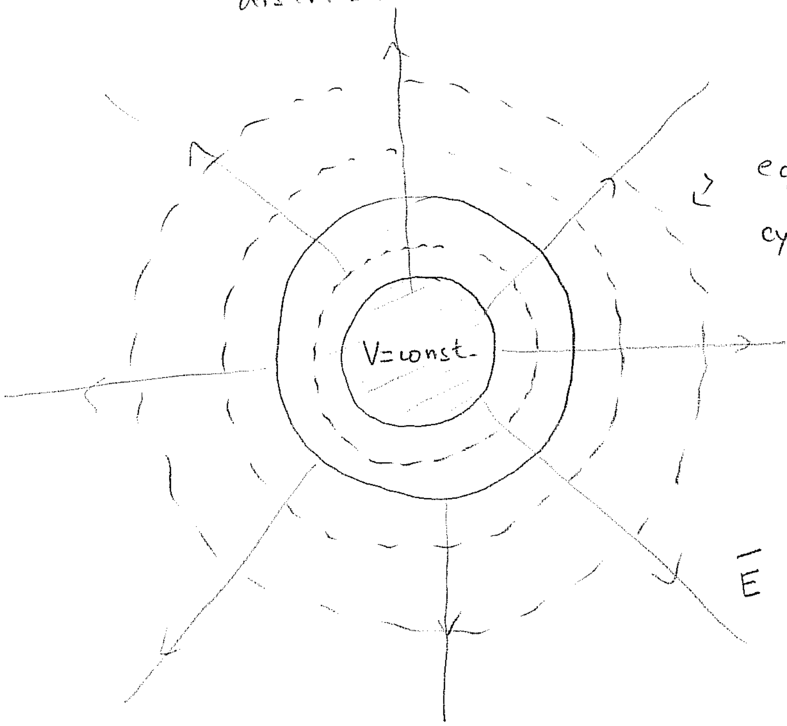
$a<r<b$: electric field lines in the $\bar{a}_r$ direction due to symmetry and $\rho_o > 0$ $\Rightarrow$ potential <u>decreases</u> as r increases because $\bar{E}$ points in the direction of decreasing potential. [2pts]

$r>b$: Same as $a<r<b$ [2pts]

3. Draw the equipotential surfaces for all three sub-regions and propose an appropriate reference point with V=0. Can the reference point be at $r \to \infty$?

[4 pts]

Reference cannot be at ∞, as this charge distribution is infinite [1 pt].



equipotentials are cylinders around z-axis (parallel to the boundaries of ρ)

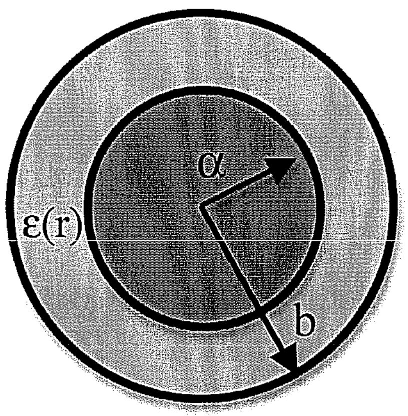
r=a can be the reference point/surface ("ground").

--- End of Question 2

Question 3 [25 pts]

A. Consider a coaxial (cylindrical) capacitor formed by two perfectly conducting cylinders of radii α and b, respectively. The area $\alpha \leq r \leq b$ between the cylinders is filled by a dielectric with dielectric permittivity $\varepsilon(r) = \frac{\varepsilon_0 r}{\alpha}$.

A voltage source keeps the potential difference between the conductors at V(r=a)-V(r=b)=V_0. The cross-section of the capacitor is shown below:



1. Show that the electric flux density between the conductors is: $\bar{D} = \frac{\varepsilon_0 V_0 b}{r(b-a)} \bar{\alpha}_r$.

   Discuss both the magnitude and the direction of this vector. **[10 pts]**

$\oint_S \bar{D} \cdot d\bar{s} = Q_{\text{enclosed, FREE}}$ for S = cylinder at a < r < b

$\boxed{\text{From cylindrical symmetry } (\epsilon = \epsilon(r)) \Rightarrow \bar{D} = D_r(r)\, \bar{a}_r}$ [2pts]

$\Rightarrow \int D_r \cdot \bar{a}_r \cdot \bar{a}_r \, ds = \rho_s \cdot 2\pi a L \Rightarrow$ [2pts]

$D_r \cdot 2\pi r L = \rho_s 2\pi a L \Rightarrow D_r = \rho_s a / r$ [1pt]

with ρ_s an assumed charge distribution on r = a.

$\Rightarrow E_r(r) = \frac{D_r(r)}{\epsilon_0 r / a} = \frac{\rho_s a^2}{\epsilon_0 r^2}$ [1 pt]

$$\int_{r=a}^{b} \bar{E} \cdot d\bar{l} = V_0 \Rightarrow \int_a^b \frac{\rho_s a^2}{\epsilon_0 r^2} dr = V_0 \qquad [2 \text{ pts}]$$

$$\Rightarrow \frac{\rho_s a^2}{\epsilon_0} \left(\frac{1}{a} - \frac{1}{b} \right) = V_0$$

$$\Rightarrow \rho_s = \frac{\epsilon_0 V_0}{a^{\cancel{2}} \frac{b-a}{\cancel{a} b}} = \frac{\epsilon_0 V_0 \cdot b}{a (b-a)} \qquad [1 \text{ pt}]$$

$$\Rightarrow D_r = \frac{\frac{\epsilon_0 V_0 b}{\cancel{a} (b-a)} \cdot \cancel{a}}{r} = \frac{\epsilon_0 V_0 b}{r (b-a)} \qquad [1 \text{ pt}]$$

2. Find the capacitance and the stored energy per unit length (i.e. for a length L=1m) of this capacitor. You can use that $\overline{D}(\alpha < r < b) = \frac{\varepsilon_0 V_0 b}{r(b-a)} \overline{\alpha}_r$.

[5 pts]

$$\rho_s(r=a) = \bar{a}_r \cdot \bar{D}(r=a) = \frac{\epsilon_0 V_0 \cdot b}{a \cdot (b-a)} \quad [2 pts]$$

$$\Rightarrow Q = \int \rho_s ds = \frac{\epsilon_0 V_0 b}{a \cdot (b-a)} \cdot 2\pi a L = \frac{2\pi \epsilon_0 V_0 b}{b-a} L$$

$$\Rightarrow Q/V_0 = C = \frac{2\pi \epsilon_0 b}{b-a} L \Rightarrow$$

$$C' = \frac{C}{L} = \frac{2\pi \epsilon_0 b}{b-a} \quad [2 pts]$$

$$W_e = \frac{1}{2} C V_0^2 = \frac{1}{2} \left(\frac{2\pi \epsilon_0 b}{b-a} \cdot L \right) \cdot V_0^2$$

$$\Rightarrow \frac{W_e}{L} = \frac{1}{2} \frac{2\pi \epsilon_0 b}{b-a} V_0^2 \quad [1 pt].$$

B. A parallel-plate capacitor filled with air is charged until the electric field inside is 1000 V/m; then it is disconnected from the battery. When a dielectric is inserted between the plates, the electric field is reduced to 200 V/m.

1. Does the electric energy stored in the capacitor increase, decrease or stay the same when the dielectric is inserted? If it increases/decreases, by what factor?

[3 pts]

$W_e = \frac{Q^2}{2C}$ and $C = \epsilon A/h$

Since the source is disconnected, charge is conserved, so Q remains the same, while C increases by a factor of 5 (same as field decreases) $\Rightarrow$

$\boxed{W_e \text{ reduces by } 5}$

(dissipated to polarize the dielectric!)

2. Does the potential difference between the plates increase, decrease or stay the same when the dielectric is inserted? If it increases/decreases, by what factor?

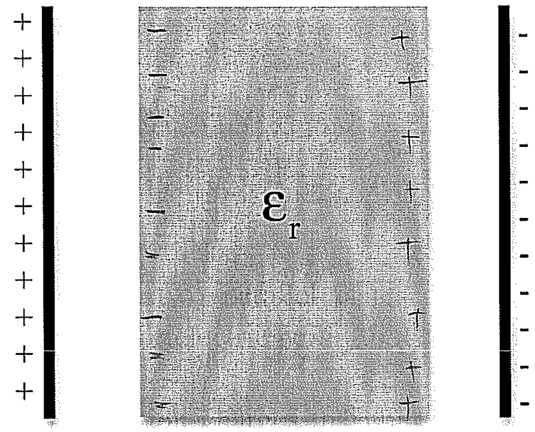
[2 pts]

$V = \int_+^- \bar{E} \cdot d\bar{l} \Rightarrow$ decreases by 5.

C. The gap between the two parallel plates of a capacitor is partially filled with a dielectric with relative dielectric permittivity $\varepsilon_r > 1$.

1. Draw the polarization charges that may exist in the dielectric and explain. You can disregard edge effects. **[3 pts]**

1 pt   1 pt.



No volume - pol charges as $\nabla \cdot \bar{P} = 0$ [1 pt]

2. With reference to the previous figure, is the electric field in the capacitor stronger inside or outside the dielectric? How about the electric flux density? Again, you can disregard edge effects. **[2 pts]**

$\bar{D}$ $\perp$ to boundaries $\Rightarrow$ stays constant

everywhere $\Rightarrow$ $\bar{E} = \bar{D}/\epsilon$ is stronger OUTSIDE

the dielectric.

--- End of Question 3