ECE259H1: Electromagnetism

Vector Calculus Quiz – Tuesday January 24, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- Your answers for this quiz should be entered on the answer sheet provided at the end. ***DO NOT detach any pages from this quiz***.
- You also need to provide a brief justification for ***each*** of your answers.
- Any non-programmable calculator can be used, and the Vector Calculus Aid Sheet has been provided. No other aids are allowed.
- You can keep the Vector Calculus Aid Sheet provided to you.
- This test has 10 questions, each of which is worth 3 marks each. One mark is for the correct choice, two marks are for the quality of your brief justification.
- The final page is left blank to provide you with space for rough work. It will NOT be marked unless you specifically direct the marker to that page.

Question #1 (3 marks = 1 mark for correct multiple-choice answer + 2 marks for quality of justification)

Where surfaces $y = 2$ and $r = 1$ intersect is

(a) an infinite plane

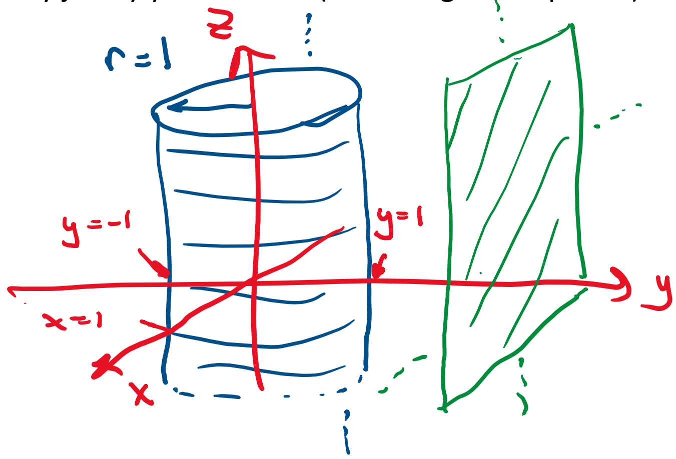
(b) a line

(c) a cylinder

(d) a circle

(e) these surfaces never intersect

Briefly justify your answer (a drawing is acceptable).



These surfaces never intersect.

Question #2 (3 marks)

What is the result of $\mathbf{R} - \mathbf{R}'$ if $\mathbf{R}$ represents the position vector for point $P(R, \theta, \varphi) = (2, 90°, 60°)$ and $\mathbf{R}'$ represents the position vector for the point $P'(R, \theta, \varphi) = (2, 60°, 90°)$?

(a) $\mathbf{R} - \mathbf{R}' = \hat{\mathbf{a}}_x - \hat{\mathbf{a}}_z$

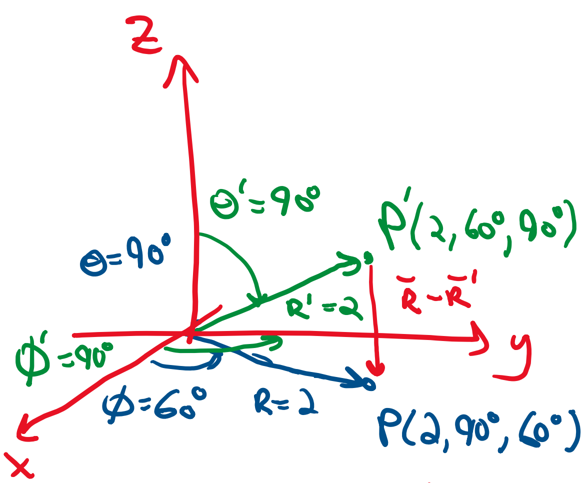
(b) $\mathbf{R} - \mathbf{R}' = \frac{1}{2}\hat{\mathbf{a}}_z$

(c) $\mathbf{R} - \mathbf{R}' = 2\hat{\mathbf{a}}_x + \sqrt{3}\hat{\mathbf{a}}_z$

(d) $\mathbf{R} - \mathbf{R}' = \hat{\mathbf{a}}_x - \sqrt{3}\hat{\mathbf{a}}_z$

(e) $\mathbf{R} - \mathbf{R}' = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}}_x - \hat{\mathbf{a}}_z)$

Briefly justify your answer.



$\hat{a}_z$ component is zero

$$\bar{R} = 2\hat{a}_R$$
$$= 2\sin\theta\cos\phi\,\hat{a}_x + 2\sin\theta\sin\phi\,\hat{a}_y$$
$$= 2\cos(60°)\,\hat{a}_x + 2\sin(60°)\,\hat{a}_y$$
$$= \hat{a}_x + \sqrt{3}\,\hat{a}_y$$

$$\bar{R}' = 2\hat{a}_R$$
$$= 2\sin\theta'\cancel{\cos\phi'}^{\,0}\,\hat{a}_x + 2\sin\theta'\sin\phi'\,\hat{a}_y + 2\cos\theta'\,\hat{a}_z$$
$$= 2\left(\tfrac{\sqrt{3}}{2}\right)(1)\,\hat{a}_y + 2\left(\tfrac{1}{2}\right)\hat{a}_z = \sqrt{3}\,\hat{a}_y + \hat{a}_z$$

$$\therefore \bar{R} - \bar{R}' = \underline{\underline{\hat{a}_x - \hat{a}_z}}$$

Question #3 (3 marks)

What is the magnitude of the vector field $\mathbf{E} = R\hat{\mathbf{a}}_R + R\cos\theta\,\hat{\mathbf{a}}_\varphi$ at the point (x, y, z) = (−5, 3, 4)?

(a) 13 (c) $\sqrt{66}$ (e) -16

(b) $5\sqrt{43}$ (d) 4π

Briefly justify your answer.

$$|E| = \sqrt{E_R^2 + E_\phi^2}$$

$$E_R = R = \sqrt{x^2+y^2+z^2} = \sqrt{25+9+16} = \sqrt{50}$$

$$E_\varphi = R\cos\theta = R\left(\frac{z}{R}\right) = z = 4$$

$$|E| = \sqrt{50+4^2} = \sqrt{66}$$

Question #4 (3 marks)

What is the *x*-component of the vector field $\mathbf{E} = R\hat{\mathbf{a}}_R + R\cos\theta\,\hat{\mathbf{a}}_\varphi$ at the point $P(x, y, z)$ = (−5, 3, 4)? Note: The options provided are expressed up to two decimal places.

(a) $E_x = -3.12$ (c) $E_x = -7.06$ (e) $E_x = -5$

(b) $E_x = 3.12$ (d) $E_x = 7.06$

Briefly justify your answer.

$$E_x = \bar{E}\cdot\hat{a}_x = R\,\underbrace{\hat{a}_R\cdot\hat{a}_x}_{\sin\theta\cos\phi} + R\cos\theta\,\underbrace{\hat{a}_\phi\cdot\hat{a}_x}_{-\sin\phi}$$

From above $R = \sqrt{50}$, $R\cos\theta = 4$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{R} \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}} \quad -\sin\phi = \frac{-y}{\sqrt{x^2+y^2}}$$

$$E_x = (\cancel{\sqrt{50}})\left(\frac{\cancel{\sqrt{x^2+y^2}}}{\cancel{\sqrt{50}}}\right)\left(\frac{x}{\cancel{\sqrt{x^2+y^2}}}\right) + (4)\left(\frac{-y}{\sqrt{x^2+y^2}}\right)$$

$$= -5 - 4\left(\frac{3}{\sqrt{34}}\right) = -5 - \frac{12}{\sqrt{34}} \simeq \underline{\underline{-7.06}}$$

Question #5 (3 marks)

Given that $\mathbf{A} = 2\alpha\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z$ and $\mathbf{B} = \hat{\mathbf{a}}_x + \alpha^2\hat{\mathbf{a}}_y + \hat{\mathbf{a}}_z$. If $\mathbf{A}$ and $\mathbf{B}$ are normal to each other then α is:

(a) $\alpha = \frac{1}{4}$

(b) $\alpha = 0$

(c) $\alpha = 2$

(d) $\alpha = -\frac{1}{2}$

(e) $\alpha = -1$

Briefly justify your answer.

$$\bar{A} \cdot \bar{B} = 2\alpha + \alpha^2 + 1 = 0$$

$$\alpha = \frac{-2 \pm \sqrt{4-4}}{2} = \underline{\underline{-1}}$$

Question #6 (3 marks)

Given that $F = \frac{2}{r}$ what is the value of $\iiint_V F\, dv$, where the volume V is defined to be the region between two concentric cylinders, described by $1 \leq r \leq 2, -4 \leq z \leq 5$, and $0 \leq \varphi \leq 2\pi$?

(a) 20π

(b) 36π

(c) 18π

(d) 60π

(e) Zero

Briefly justify your answer.

$$\int_1^2 \int_0^{2\pi} \int_{-4}^{5} \left(\frac{2}{\cancel{r}}\right) \cancel{r}\, dr\, d\phi\, dz = 2(4)(2\pi)(1)$$

$$= \underline{\underline{36\pi}}$$

Question #7 (3 marks)

The unit vector $\hat{\mathbf{a}}_R$ points in the same direction as $-2\hat{\mathbf{a}}_x + 2\hat{\mathbf{a}}_y$ if

(a) $\varphi = 0, \cancel{\theta = \frac{\pi}{4}}$

(b) $\varphi = 0, \cancel{\theta = 0}$

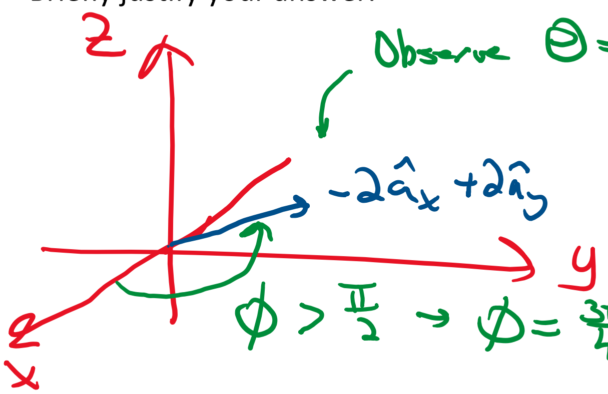
(c) $\cancel{\varphi = \frac{\pi}{4}}, \theta = \frac{\pi}{2}$

(d) $\varphi = \frac{3\pi}{4}, \theta = \frac{\pi}{2}$

(e) $\varphi = \frac{\pi}{2}, \cancel{\theta = 0}$

Briefly justify your answer.

Observe $\theta = \frac{\pi}{2}$



$\phi > \frac{\pi}{2} \rightarrow \phi = \frac{3\pi}{4}$

Question #8 (3 marks)

The surface of a half sphere S is made by the union of $S_1: R = 1, \varphi \in [0, 2\pi], \theta \in \left[0, \frac{\pi}{2}\right]$, $S_2: R \in [0,1], \varphi \in [0, 2\pi], \theta = \frac{\pi}{2}$. The outward flux $\iint_S \varphi \hat{\mathbf{a}}_\theta \cdot \mathrm{d}\mathbf{s}$ is equal to:

(a) $+\pi^2$

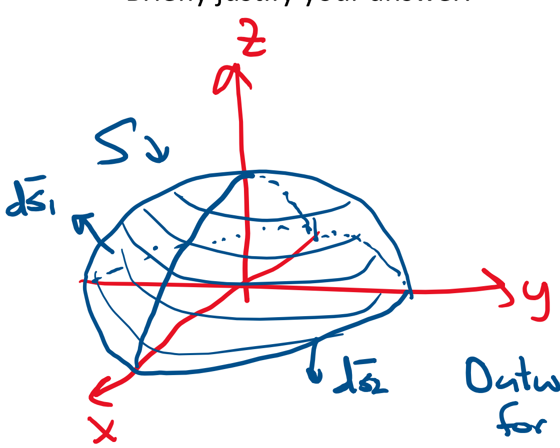
(b) $+\frac{\pi}{2}$

(c) $-\pi^2$

(d) $-\frac{\pi}{2}$

(e) 0

Briefly justify your answer.



$$\iint_S \phi \hat{a}_\theta \cdot ds = \iint_{S_1} \phi \hat{a}_\theta \cdot d\bar{s}_1 + \iint_{S_2} \phi \hat{a}_\theta \cdot d\bar{s}_2$$

$$d\bar{s}_1 = R^2 \sin\theta \, d\phi \, d\theta \, \hat{a}_R$$

$$d\bar{s}_2 = R(1) \, d\phi \, dR \, \hat{a}_\theta$$

$\sin\theta = 1$ for S_2

Outward Flux for S_1 $= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \phi \hat{a}_\theta \cdot R^2 \sin\theta \, d\phi \, d\theta \, \hat{a}_R = 0$

Outward Flux for S_2 $= \int_0^1 \int_0^{2\pi} \phi \hat{a}_\theta \cdot R \, d\phi \, dR \, \hat{a}_\theta = \left(\frac{\phi^2}{2}\Big|_0^{2\pi}\right)\left(\frac{R^2}{2}\Big|_0^1\right)$

$= \left(\frac{4\pi^2}{2}\right)\left(\frac{1}{2}\right) = \underline{\underline{\pi^2}}$

Question #9 (3 marks)

If a vector field is given by $\mathbf{E} = \frac{388}{|\mathbf{R}-\mathbf{R}'|^3}(\mathbf{R} - \mathbf{R}')$, where $\mathbf{R} - \mathbf{R}' = 3y\hat{\mathbf{a}}_x - (x^2 - 4z^2)\hat{\mathbf{a}}_y + xy\hat{\mathbf{a}}_z$. What is the vector field $\mathbf{E}$ at $P(x, y, z) = (3, 2, -1)$?

(a) $\mathbf{E} = \frac{1}{\sqrt{97}}\left[3y\hat{\mathbf{a}}_x - (x^2 + z^2)\hat{\mathbf{a}}_y + xyz\hat{\mathbf{a}}_z\right]$

(b) $\mathbf{E} = \frac{4}{\sqrt{97}}\left[6\hat{\mathbf{a}}_x - 5\hat{\mathbf{a}}_y + 6\hat{\mathbf{a}}_z\right]$

(c) $\mathbf{E} = \frac{4}{\sqrt{97}}\left[-24\hat{\mathbf{a}}_x - 4\hat{\mathbf{a}}_y - 32\hat{\mathbf{a}}_z\right]$

(d) $\mathbf{E} = -24\hat{\mathbf{a}}_x - 32\hat{\mathbf{a}}_z$

(e) None of the above

Briefly justify your answer.

For P $\rightarrow \bar{R} - \bar{R}' = 6\hat{a}_x - (9-4)\hat{a}_y + 6\hat{a}_z$

$= 6\hat{a}_x - 5\hat{a}_y + 6\hat{a}_z$

$\therefore \bar{E} = \frac{388}{(36+25+36)^{3/2}}(6\hat{a}_x - 5\hat{a}_y + 6\hat{a}_z)$

$= \frac{388}{(\sqrt{97})^3}(6\hat{a}_x - 5\hat{a}_y + 6\hat{a}_z) = \frac{4}{\sqrt{97}}(6\hat{a}_x - 5\hat{a}_y + 6\hat{a}_z)$

Question #10 (3 marks)

Use the appropriate expression for the differential surface area ds to determine the area of the surface given by $1 \le r \le 3, \frac{\pi}{4} \le \varphi \le \pi, z = 0$.

(a) $\frac{25\pi}{2}$

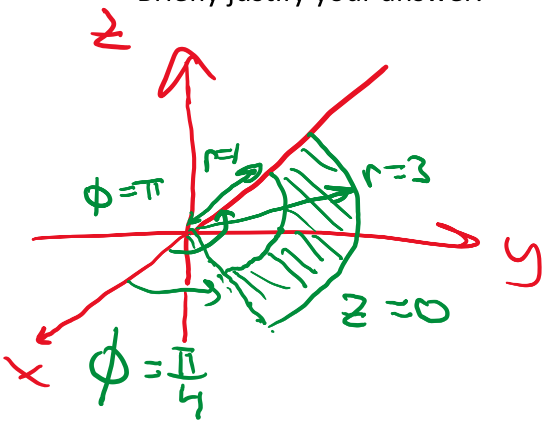
(b) 2π

(c) $\frac{3\pi}{2}$

(d) 3π

(e) Zero

Briefly justify your answer.



Surface area $= \iint ds \leftarrow ds = r\,d\phi\,dr$

$= \int_1^3 \int_{\pi/4}^{\pi} r\,d\phi\,dr = \left(\frac{r^2}{2}\Big|_1^3\right)\left(\phi\Big|_{\pi/4}^{\pi}\right)$

$= \left(\frac{9}{2} - \frac{1}{2}\right)\left(\pi - \frac{\pi}{4}\right) = (4)\left(\frac{3\pi}{4}\right) = \underline{\underline{3\pi}}$