University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus I
Midterm Test 1
9:10 – 10:55, 19 October 2023
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1. Evaluate the following limits if they exist. Indicate the limit laws used in your solution.

(a) $\lim\limits_{x\to 2}\frac{x^2-4}{x^2+2x-3}$ (b) $\lim\limits_{x\to 1^+}\frac{x^2-2}{x^2+2x-3}$ (c) $\lim\limits_{x\to 2^+}\frac{2-x}{|2-x|}$

(d) $\lim\limits_{x\to 0}\sin(x-1+\cos x)$ (e) $\lim\limits_{x\to 0}\frac{2-\sqrt{4-x^2}}{x}$

a) $\lim\limits_{x\to 2}\frac{x^2-4}{x^2+2x-3} = \frac{2^2-4}{2^2+4-3}$

$= \frac{0}{5} = 0$

sum, difference & quotient rules
direct substitution

b) $\lim\limits_{x\to 1^+}\frac{x^2-2}{x^2+2x-3} = \lim\limits_{x\to 1^+}\frac{x^2-2}{(x+3)(x-1)}$

$= \frac{-1}{4}\lim\limits_{x\to 1^+}\frac{1}{x-1}$

$= -\infty$

sum, difference, product & quotient rules
direct substitution
infinite limit

c) $\lim\limits_{x\to 2^+}\frac{2-x}{|2-x|} = \lim\limits_{x\to 2^+}\frac{2-x}{-(2-x)}$

$= \lim\limits_{x\to 2^+} -1$

$= -1$

cancel common factor

d) $\lim\limits_{x\to 0}\sin(x-1+\cos x) = \sin(0-1+\cos(0))$

$= \sin(0-1+1)$

$= \sin(0)$

$= 0$

composite function rule
direct substitution

e) $\lim_{x\to 0} \frac{2-\sqrt{4-x^2}}{x} = \lim_{x\to 0} \frac{4-(4-x^2)}{x(2+\sqrt{4-x^2})}$

$= \lim_{x\to 0} \frac{x^2}{x(2+\sqrt{4-x^2})}$ cancel common factor

$= \lim_{x\to 0} \frac{x}{2+\sqrt{4-x^2}}$ quotient law

root law

direct substitution

$= \frac{0}{4} = 0$

2. Calculate the derivative of the following functions, citing all theorems used:

(a) $f(x) = 3x^2$ (b) $f(x) = 2/x^3$ (c) $f(x) = \frac{3+x^2}{2-x}$

(d) $f(x) = \sin^2(x^3)$ (e) $f(x) = \tan(\sqrt{x})$

a) $f(x) = 3x^2 \Rightarrow f'(x) = 6x$

constant multiplier rule
power rule

b) $f(x) = 2/x^3 \Rightarrow f'(x) = 2\cdot(-3)x^{-4}$

$= -6/x^4$

constant multiplier
general power rule

c) $f(x) = \dfrac{3+x^2}{2-x} \Rightarrow f'(x) = \dfrac{(2-x)(2x) - (3+x^2)(-1)}{(2-x)^2}$

$= -\dfrac{x^2+4x+3}{(2-x)^2}$

quotient rule
power rule

d) $f(x) = \sin^2(x^3) \Rightarrow f'(x) = 2\sin(x^3)\cdot\cos(x^3)\cdot(3x^2)$

$= 6x^2\sin(x^3)\cos(x^3)$

power rule, basic trig derivative, chain rule
constant multiplier & power rule

e) $f(x) = \tan\sqrt{x} \Rightarrow f'(x) = \sec^2\sqrt{x}\cdot\frac{1}{2}x^{-1/2}$

$= \dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$

basic trig derivative, general power rule

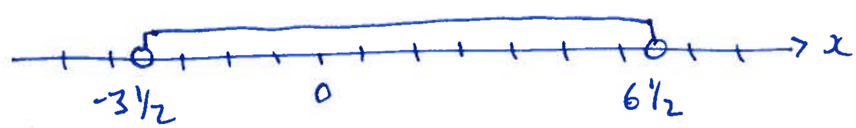
3. Plot the range of x specified by the inequalities:

a) $|3-2x| < 10$ b) $\frac{3x+7}{2x^2+6x-8} > 0$

a) boundary values: $|3-2x| = 10 \Rightarrow$ $3-2x = 10 \Rightarrow x = -\frac{7}{2}$

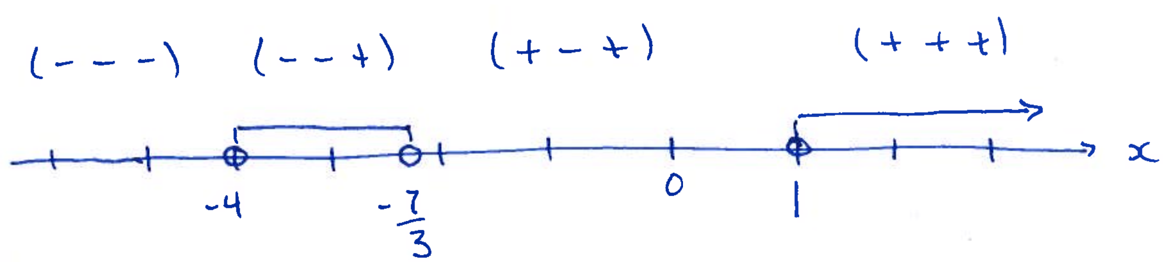
$3-2x = -10 \Rightarrow x = \frac{13}{2}$

$$\left.\begin{array}{lll} x < -\frac{7}{2} & \Rightarrow & |3-2x| > 10 \\ -\frac{7}{2} < x < \frac{13}{2} & \Rightarrow & |3-2x| < 10 \\ x > \frac{13}{2} & \Rightarrow & |3-2x| > 10 \end{array}\right\} \quad x \in \left(-3\tfrac{1}{2}, 6\tfrac{1}{2}\right)$$



b) $\frac{3x+7}{2x^2+6x-8} = \frac{3x+7}{2(x-1)(x+4)} > 0 \iff (3x+7)(x-1)(x+4) > 0$

zeros: $-4, -\frac{7}{3}, 1$



4) Use a $\varepsilon - \delta$ type of proof to prove the limits:

a) $\lim\limits_{x\to 1} 2x+3 = 5$ b) $\lim\limits_{x\to -1} \frac{(x+1)^2}{x^3} = 0$

a) prove $\lim\limits_{x\to 1} 2x+3 = 5$

1) Find $\delta > 0$ st. for $0 < |x-1| < \delta$ $|(2x+3)-5| < \epsilon$

$\Rightarrow |(2x+3)-5| = 2|x-1| < \epsilon \Rightarrow |x-1| < \frac{\epsilon}{2}$ $\therefore$ choose $\delta = \frac{\epsilon}{2}$

2) Proof: given $\epsilon > 0$, let $\delta = \frac{\epsilon}{2}$

thus $|2x+3-5| = 2|x-1| < 2\delta = \epsilon$ for $0 < |x-1| < \delta = \frac{\epsilon}{2}$

$\therefore$ by the definition of a limit $\lim\limits_{x\to 1} 2x+3 = 5$

b) prove $\lim\limits_{x\to -1} \frac{(x+1)^2}{x^3} = 0$

1) Find $\delta > 0$ st. for $|x+1| < \delta$, $\left|\frac{(x+1)^2}{x^3} - 0\right| < \epsilon$

now, take $\delta < \frac{1}{2}$ $\therefore$ $|x+1| < \delta \Rightarrow -\frac{1}{2} < x+1 < \frac{1}{2} \Rightarrow -\frac{3}{2} < x < -\frac{1}{2}$

$\Rightarrow$ for $-\frac{3}{2} < x < -\frac{1}{2}$, $\frac{8}{27} < \left|\frac{1}{x^3}\right| < 8$

$\therefore \left|\frac{(x+1)^2}{x^3}\right| < 8|x+1|^2 \Rightarrow |x+1|^2 < \frac{\epsilon}{8} \Rightarrow |x+1| < \sqrt{\frac{\epsilon}{8}}$

$\therefore$ choose $\delta = \min\left\{\frac{1}{2}, \sqrt{\frac{\epsilon}{8}}\right\}$

2) Proof: given $\epsilon > 0$, let $\delta = \min\left\{\frac{1}{2}, \sqrt{\frac{\epsilon}{8}}\right\}$

thus $\left|\frac{(x+1)^2}{x^3} - 0\right| = \left|\frac{(x+1)^2}{x^3}\right| < 8|x+1|^2 < 8\,\frac{\epsilon}{8} = \epsilon$

$\therefore \left|\frac{(x+1)^2}{x^3} - 0\right| < \epsilon$ whenever $0 < |x+1| < \delta = \min\left\{\frac{1}{2}, \sqrt{\frac{\epsilon}{8}}\right\}$

$\therefore$ by the definition of a limit, $\lim\limits_{x\to -1} \frac{(x+1)^2}{x^3} = 0$

5. a) Can the graph of a polynomial have vertical or horizontal asymptotes? Explain.

b) Sketch the graph of a function that satisfies all of the given conditions:

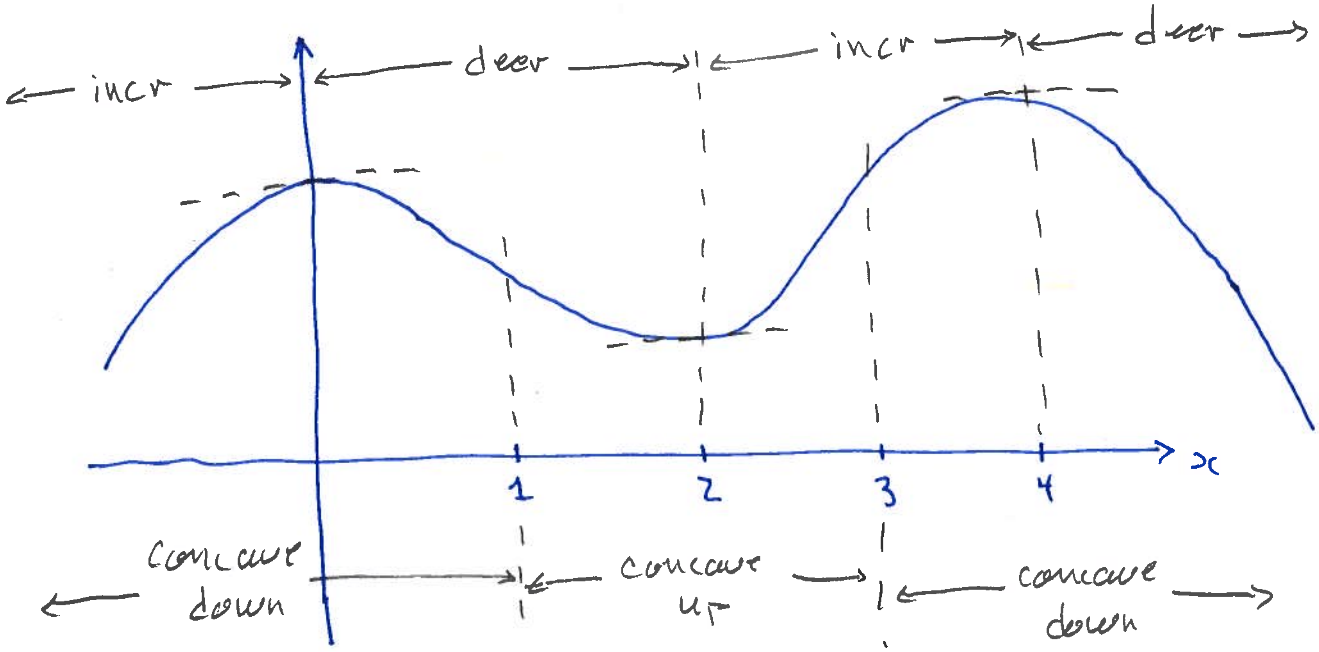
$$f'(0) = f'(2) = f'(4) = 0$$
$$f'(x) > 0 \text{ if } x < 0 \text{ or } 2 < x < 4$$
$$f'(x) < 0 \text{ if } 0 < x < 2 \text{ or } x > 4$$
$$f''(x) > 0 \text{ if } 1 < x < 3$$
$$f''(x) < 0 \text{ if } x < 1 \text{ or } x > 3$$

c) Suppose a continuous function f is concave up on $(-\infty, 0)$ and $(0, \infty)$. Assume f has a local maximum at $x = 0$. What, if anything, do you know about $f'(0)$? Explain with an illustration.

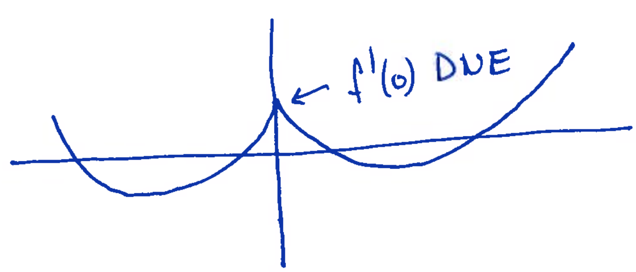
a) No: - The domain of any polynomial is $(-\infty, \infty)$; there are no vertical asymptotes.

- also, $\lim_{x \to \pm\infty} p(x) = \pm\infty$; $\therefore$ no horizontal asymptotes.

b)



c)



6. A spherical balloon is inflated at a rate of 10 cm^3/min. At what rate is the diameter of the balloon increasing when the balloon has a diameter of 5 cm?

$$V = \frac{4}{3}\pi r^3 = \frac{\pi}{6} D^3$$

$$\frac{dV}{dt} = \frac{\pi}{6} \cdot 3D^2 \cdot \frac{dD}{dt} = \frac{\pi D^2}{2} \frac{dD}{dt}$$

$\frac{dV}{dt} = 10 \text{ cm}^3/\text{min}$ when $D = 5$ cm

$$\Rightarrow 10 = \frac{\pi (5)^2}{2} \cdot \frac{dD}{dt} \Rightarrow \frac{dD}{dt} = \frac{4}{5\pi} \text{ cm/min}$$

7. Use implicit differentiation to find dy/dx for:

a) $x^4 + y^4 = 2$

b) $x^2y^2 + x\cos y = 2$

a) $4x^3 + 4y^3 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x^3}{4y^3} = -\frac{x^3}{y^3}$

b) $2xy^2 + 2x^2y\frac{dy}{dx} + \cos y - x\sin y \cdot \frac{dy}{dx} = 0$

$(2x^2y - x\sin y)\frac{dy}{dx} = -2xy^2 - \cos y$

$\Rightarrow \frac{dy}{dx} = \frac{2xy^2 + \cos y}{x\sin y - 2x^2y}$

8. For the function: $f(x) = \sqrt{1-x} + \sqrt{1+x}$

i) Determine the domain of f, the x and y intercepts, and identify any symmetry.
ii) Find the intervals in which f increases or decreases.
iii) Find the extreme values.
iv) Determine the concavity of the graph.
v) Sketch the graph specifying the points of inflection, asymptotes and vertical tangents, if any.

i) domain: $x \le 1$ & $x \ge -1 \Rightarrow x \in [-1,1]$

intercepts: $f(0) = 2$; $f(x) \neq 0$

symmetry: $f(-x) = f(x)$ $\therefore$ symmetric about y-axis

ii) $f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1) + \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}} - \frac{1}{2\sqrt{1-x}} = \frac{\sqrt{1-x} - \sqrt{1+x}}{2\sqrt{1-x^2}}$

$\Rightarrow f'(x) > 0 \Rightarrow \sqrt{1-x} > \sqrt{1+x} \Rightarrow x \in [-1,0 \Rightarrow$ incr

$f'(x) < 0 \Rightarrow \sqrt{1-x} < \sqrt{1+x} \Rightarrow x \in (0,1] \Rightarrow$ decr

iii) $f'(x) = 0 \Rightarrow \sqrt{1-x} = \sqrt{1+x} \Rightarrow 1-x = 1+x \Rightarrow x = 0$

$f(0) = 2$ is a local and absolute maximum

end points: $f(\pm 1) = \sqrt{2}$ = local and absolute minima

iv) $f''(x) = \frac{1}{4}(1-x)^{-3/2}(-1) - \frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4(1-x)^{3/2}} - \frac{1}{4(1+x)^{3/2}} < 0$

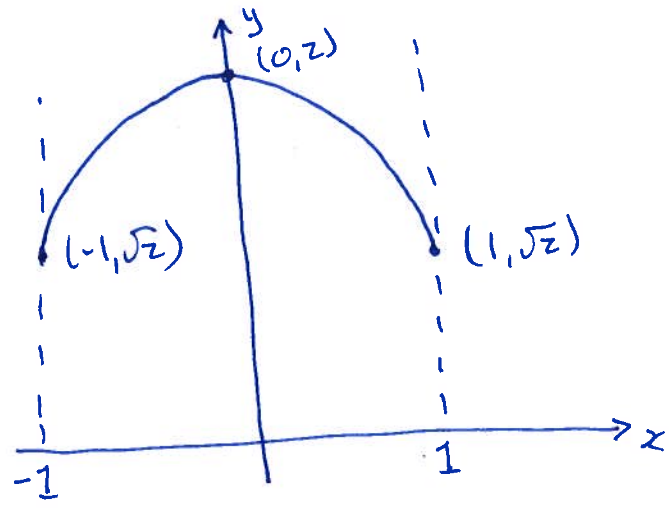
$\therefore$ the graph is concave down $x \in [-1,1]$

v) $f'(-1) \to +\infty$

$f'(1) \to -\infty$

$\therefore$ vertical tangents

no asymptotes or points of inflection



9. Let $p(x)$ be a real polynomial for which $p(a) = b$ and $p(b) = a$. Prove that there exists a point u, with $a < u < b$ and a polynomial $q(x)$ for which:

$$p'(x) + 1 = (x - u)\, q(x)$$

Hint: Use the Mean Value Theorem.

MVT : $p'(c) = \dfrac{p(b) - p(a)}{b - a} = \dfrac{a - b}{b - a} = -1$

$\therefore p'(c) + 1 = 0$ for some pt. c, $a < c < b$

Now $p(x)$ is a real polynomial

$\therefore p'(x)$ is a real polynomial

$\therefore p'(x) + 1$ is a real polynomial $\rightarrow$ call $Q(x)$

Now $Q(c) = 0$ $\therefore x = c$ is a root of $Q(x)$

$\therefore x - c$ is a factor of $Q(x)$

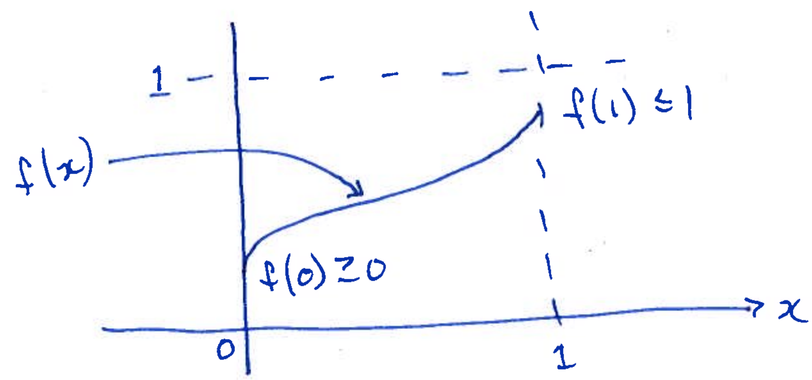
$\therefore Q(x) = (x - c) \cdot q(x)$

where $q(x)$ is a polynomial

$\therefore p'(x) + 1 = (x - u) \cdot q(x)$ with $a < u < b$

and setting $u = c$

10. Show that if $f(x)$ is a continuous function on $[0, 1]$ and $0 \le f(x) \le 1$ for each x, then for some number $c \in [0,1]$, $f(c) = c^2$.



Case ①: If $f(0) = 0 \Rightarrow f(0) = 0^2 = 0 \quad \therefore c = 0$ works.

Case ②: If $f(1) = 1 \Rightarrow f(1) = 1^2 = 1 \quad \therefore c = 1$ works.

Case ③: $f(0) > 0$ and $f(1) < 1$

$\Rightarrow$ let $g(x) \equiv f(x) - x^2$

$\therefore g(0) = f(0) - 0^2 > 0$

$g(1) = f(1) - 1^2 < 0$

$\therefore$ by the Intermediate Value Theorem, there is some number $c \in (0,1)$ such that $g(c) = 0$

$\therefore f(c) - c^2 = 0$ or $f(c) = c^2$