AER210 VECTOR CALCULUS and FLUID MECHANICS

Quiz 3

Duration: 65 minutes

11 November 2019

Closed Book, no aid sheets

Non-programmable calculators allowed

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Family Name: Alis Ekmekci

Given Name: Solutions

Student #: ____________________

TA Name/Tutorial #: ____________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	12	
2	8	
3	10	
4	10	
5	10	
TOTAL	50	/50

Hints: $E_V = -\dfrac{dP}{d\forall/\forall}$ $\tau = \mu \dfrac{du}{dy}$ $-\nabla p + \rho \vec{g} = \rho \vec{a}$

(Gravitational acceleration: g = 10 m/s^2)

1) (a) Give short explanations to the questions below or calculate the required quantity:

- What is a Newtonian fluid? (**1 mark**)

Fluids for which shear stress varies linearly with the rate of angular deformation ($\frac{d\alpha}{dt}$) (or in other words, the velocity gradient du/dy) are called Newtonian fluids.

- What does no slip condition mean? (**1 mark**)

It means fluid in contact with a solid boundary has the same velocity at the point of contact.

- How does the viscosity of liquids change with temperature? Explain why? (**2 marks**)

Viscosity of liquids decreases with an increase in its temperature. This is because viscosity in liquids is caused by intermolecular forces between molecules, and at higher temperature, molecules possess more energy to overcome intermolecular forces.

- Find the dimensions of dynamic viscosity in MLT system. (**1 mark**)

$$\tau = \mu \frac{du}{dy} \Rightarrow [\mu] = \frac{[\tau]}{[du/dy]} = \frac{[F/A]}{[\frac{du}{dy}]} = \frac{\frac{M}{L^{\cancel{2}}} \cdot \frac{\cancel{L}}{T^{\cancel{2}}}}{\frac{\cancel{L}}{\cancel{T}} \cdot \frac{1}{\cancel{L}}} = \frac{M}{LT}$$

- A pressure rise of 2 × 10^6 Pa is applied to water that initially filled a 2000 cm^3 volume. The bulk modulus of elasticity of water is E_v = 2.2 × 10^9 Pa. Estimate its volume after the pressure rise. (**1 mark**)

$$E_v = -\frac{\Delta P}{\frac{\Delta V}{V}} \Rightarrow \Delta V = -\frac{\Delta P}{E_v} V = -\frac{2\times10^6}{2.2\times10^9} \cdot 2000 = -1.82\ cm^3$$

$$V_{final} = 2000 - 1.82 \cong 1998\ cm^3$$

- What is gage pressure? (**1 mark**)

Pressure with respect to the local atmospheric pressure.

(b) Three spheres of same diameter are submerged in the same body of water. One sphere is steel, one is a spherical balloon filled with water, and one is a spherical balloon filled with air.

- Which sphere has the largest buoyant force? (**1 mark**)

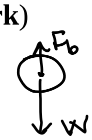
All three spheres have the same buoyant force (because $F_b = \rho_{fluid}\, g \cdot V$ ← volume, and they all have same volume).

- If you move the steel sphere from a depth of 1 m to 10 m, what happens to the magnitude of the buoyant force acting on that sphere? (**1 mark**)

When you move the steel sphere deeper, the buoyant force does not change.

- If all spheres are released from a cage at a depth of 1 m, what happens to the 3 spheres assuming weight of the balloon's rubber is negligible? (**3 mark**)

Steel sphere: $\rho_{sphere} > \rho_{air}$



$F_b < W \Rightarrow$ the steel sphere sinks down.

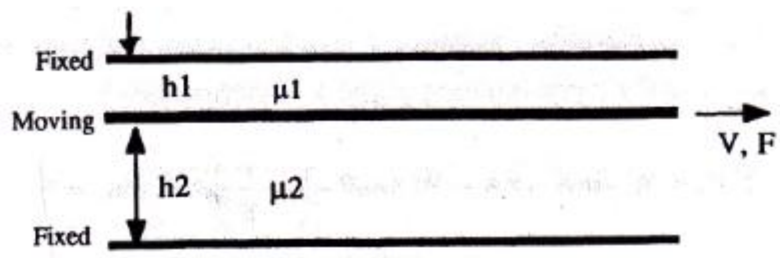
Water-filled sphere: $\rho_{sphere} = \rho_{water}$

$F_b = W \Rightarrow$ the water-filled sphere stays at the same depth.

Air-filled sphere $\rho_{sphere} < \rho_{water}$

$F_b > W \Rightarrow$ air-filled balloon moves upward.

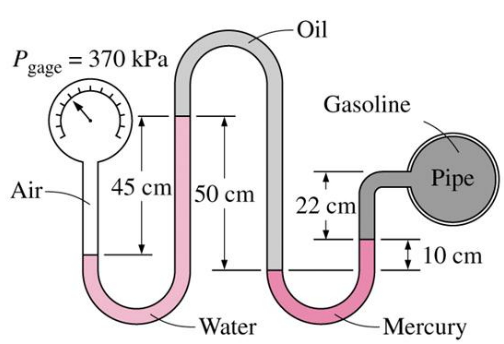
2) a) A thin plate is separated from two fixed plates by two liquids with viscosity μ_1 and μ_2 as shown in the figure below. The plate spacings h_1 and h_2 are unequal, as shown. The contact area is A between the center plate and each fluid. <u>Assuming a linear velocity distribution</u> in each fluid, derive the force F required to pull the plate at a constant velocity V. (**4 marks**)



Assuming a linear velocity distribution on each side of the plate, we obtain:

$$F = \tau_1 A + \tau_2 A = \mu_1 \frac{V}{h_1} + \mu_2 \frac{V}{h_2}$$

b) A gasoline containing pipe is connected to a pressure gage through a double-U manometer, as shown in the figure below. If the reading of the pressure gage is 370 kPa, determine the gage pressure of the gasoline in the pipe. Densities are as follows: $\rho_{water} = 1000\ kg/m^3$, $\rho_{oil} = 790\ kg/m^3$, $\rho_{mercury} = 13600\ kg/m^3$, $\rho_{gasoline} = 700\ kg/m^3$. The gravitational acceleration can be taken as $g = 10\ m/s^2$. (**4 marks**)

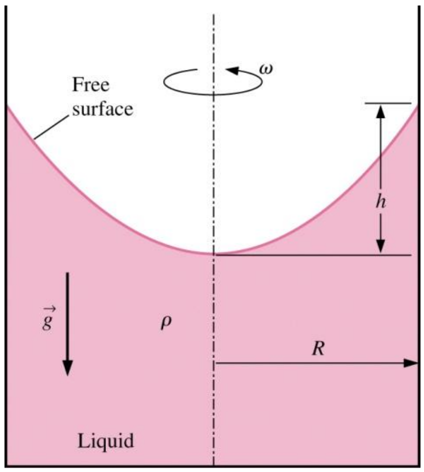


$$P_{gage} - \rho_{water} g (0.45) + \rho_{oil} g (0.50) - \rho_{mercury} g (0.10) - \rho_{gasoline} g (0.22) = P_{gasoline}$$

$$370\,000 - 1000 . 10 (0.45) + 790 . 10 . 0.50 - 13600 . 10 (0.10) - 700 . 10 . 0.22 = P_{gasoline}$$

$$P_{gasoline} = 354,310\ Pa$$

3) Consider a liquid in a cylindrical container in which both the container and the liquid are rotating as a rigid body. The elevation difference h between the center of the liquid surface and the rim of the liquid surface is a function of angular velocity ω, fluid density ρ, gravitational acceleration g, and radius R of the container. Use the method of repeating variables selecting ω, ρ, and R as the repeating variables, and find a dimensionless relationship between the parameters. Show all your work. (**10 marks**)



$h = f_1(\omega, \rho, g, R)$

$$\left.\begin{aligned}[h] &= L \\ [\omega] &= \frac{1}{T} \\ [\rho] &= \frac{M}{L^3} \\ [g] &= \frac{L}{T^2} \\ [R] &= L\end{aligned}\right\}$$

# of variables = 5

# of reference dimensions = 3 (M, L, T)

Buckingham-Pi Thrm: (# of Π terms) = (# of variables) − (# of reference dimensions)

# of Π terms = 5 − 3 = 2 ⇐ 2 Π terms to be determined.

Repeating variables: ω, ρ, R

$\Pi_1 = h\,\omega^a \rho^b R^c$

$M^0L^0T^0 = L \frac{1}{T^a} \frac{M^b}{L^{3b}} L^c$

$= L^{1+c-3b}\, T^{-a}\, M^b$

$\boxed{a=0}$

$\boxed{b=0}$

$1+c-3b=0 \Rightarrow \boxed{c=-1}$

$\boxed{\Pi_1 = \frac{h}{R}}$

$\Pi_2 = g\,\omega^a \rho^b R^c$

$M^0L^0T^0 = \frac{L}{T^2} \frac{1}{T^a} \frac{M^b}{L^{3b}} L^c$

$= L^{1-3b+c}\, T^{-2-a}\, M^b$

$\boxed{b=0}$

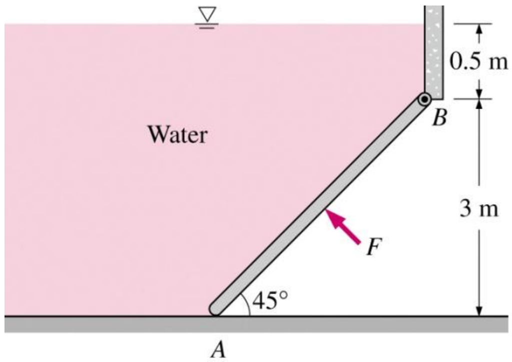
$-2-a=0 \Rightarrow \boxed{a=-2}$

$1-3b+c=0 \Rightarrow \boxed{c=-1}$

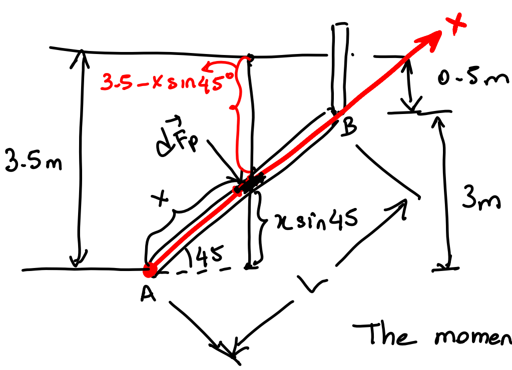
$\boxed{\Pi_2 = \frac{g}{\omega^2 R}}$

$\therefore \Pi_1 = f_2(\Pi_2) \Rightarrow \frac{h}{R} = f_2\left(\frac{g}{\omega^2 R}\right)$

4) The 200-kg-mass, 5-m-wide rectangular gate shown in the figure below is hinged at B and leans against the floor at A making an angle of 45° with the horizontal. The gate is to be opened by applying a normal force at its center. Determine the minimum force F required to open the water gate. Water density is 1000 kg/m^3 and the gravitational acceleration is 10 m/s^2. (**10 marks**)



Method 1: Integration:



$$|d\vec{F}_p| = dF_p = p \cdot dA$$

$dA = w\,dx$

$p = \rho g (3.5 - x\sin 45°)$

$$L = \frac{3}{\sin 45°}$$

The moment about the hinge (B) induced by water pressure is:

$$M_{water} = \iint_{Area} (L-x)\, p \cdot dA$$

$$= \int_{x=0}^{L} (L-x)\, \rho g\, (3.5 - x\sin 45°)\, w\, dx$$

$$= \rho g w \int_0^L \left(\frac{3}{\sin 45°} - x\right)\left(3.5 - x\sin 45°\right) dx$$

$$= \rho g w \int_0^L \left(\frac{(3)(3.5)}{\sin 45°} \underbrace{- 3x - 3.5x}_{-6.5x} + x^2 \sin 45°\right) dx$$

$$= \rho g w \left[\frac{(3)(3.5)}{\sin 45°} x - 6.5\frac{x^2}{2} + \frac{x^3}{3}\sin 45°\right]_{x=0}^{L}$$

$$M_{water} = \rho g w \left[\frac{(3).(3.5)}{\sin 45°} \cdot \frac{3}{\sin 45°} - \frac{6.5}{2} \cdot \frac{3^2}{\sin^2 45°} + \frac{\cancel{3^3}\,3^2}{\cancel{3}} \cdot \frac{\cancel{\sin 45°}}{\cancel{\sin^3 45°}\ \sin^2 45°} \right]$$

$w = 5m$

$g = 10 m/s^2$

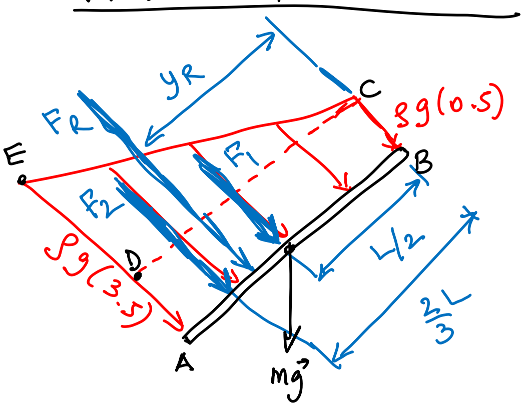
$\rho = 1000 kg/m^3$

$$M_{water} = 1{,}125{,}000 \text{ N.m}$$

$$M_{water} + mg \sin 45 . \frac{L}{2} = F . \frac{L}{2} \Rightarrow F = M_{water} \cdot \frac{2}{L} + mg \sin 45°$$

$$\boxed{F = 531{,}744.3 \text{ N}}$$

Method #2: Pressure Prism



$L = 3/\sin 45°$

$$F_1 = \text{Volume}\Big|_{ABCD} = \rho g (0.5) L w$$

$$F_2 = \text{Volume}\Big|_{DCE} = \rho g \frac{(3.5 - 0.5)}{2} L w = \frac{3}{2} \rho g L w$$

$$F_R = F_1 + F_2 = \frac{1}{2} \rho g L w + \frac{3}{2} \rho g L w =$$

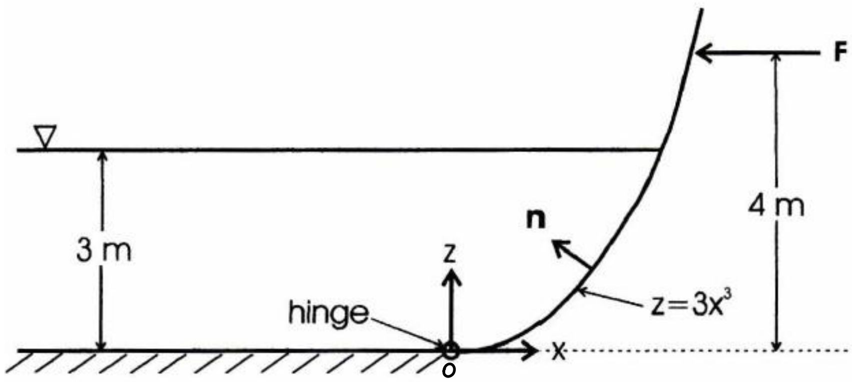
$$\boxed{F_R = 2 \rho g L w}$$

$$F_R \cdot y_R = F_1 \cdot \frac{L}{2} + F_2 \cdot \frac{2L}{3}$$

$$2 \rho g L w \cdot y_R = \frac{1}{2} \rho g L w \cdot \frac{L}{2} + \frac{3}{2} \rho g L w \cdot \frac{2L}{3} \Rightarrow \boxed{y_R = \frac{5}{8} L}$$

$$F_R \cdot y_R + mg \cos 45° \cdot \frac{L}{2} = F \cdot \frac{L}{2} \Rightarrow \boxed{F = 531{,}744\ 3 \text{ kg}}$$

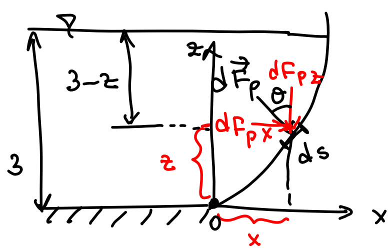
5) A rectangular channel of width 2 m contains water to a depth of 3 m. The gate at the end of the channel is a parabola, $z = 3x^3$, and a horizontal force, $\vec{F}$, is applied 4 m above the hinge to keep the gate closed. Ignoring the mass of the gate, find the magnitude of the force required using the integration method. Density of water is ρ_{water} = 1,000 kg/m^3 and gravitational acceleration is g = 10 m/s^2. (**10 marks**)



Closing moment: $M_{closing} = F \cdot \underbrace{4}_{\text{height}}.$

Opening moment: $M_{opening} = ?$

Method 1:

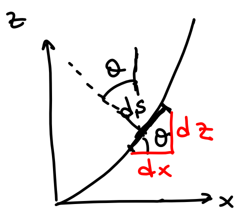


$$dM_o = x\,|dF_{p,z}| + z\,|dF_{p,x}|$$

$$\|d\vec{F}_p\| = dF_p = p\,dA = \underbrace{\rho g (3-z)}_{p}\,\underbrace{w\,ds}_{dA}$$

$$dF_{p,x} = dF_p \sin\theta = \rho g (3-z)\, w \sin\theta\, ds$$

$$dF_{p,z} = dF_p \cos\theta = \rho g (3-z)\, w \cos\theta\, ds$$



$$ds\cos\theta = dx \qquad ds\sin\theta = dz \quad \Rightarrow \quad ds = \frac{dx}{\cos\theta} = \frac{dz}{\sin\theta}$$

$$|dF_{p,x}| = \rho g (3-z)\, w \sin\theta\, ds = \rho g (3-z)\, w\, \cancel{\sin\theta}\, \frac{dz}{\cancel{\sin\theta}} = \rho g (3-z)\, w\, dz$$

$$|dF_{p,z}| = \rho g (3-z)\, w \cos\theta\, d\theta = \rho g (3-z)\, w\, \cancel{\cos\theta}\, \frac{dx}{\cancel{\cos\theta}} = \rho g (3-z)\, w\, dx$$

$$dM_o = x\,|dF_{p,z}| + z\,|dF_{p,x}|$$

$$= x\rho g(3-\underbrace{z}_{z=3x^3})\,w\,dx + z\rho g(3-z)\,w\,dz$$

$$= x\rho g\,(3-3x^3)\,w\,dx + z\rho g\,(3-z)\,w\,dz$$

$$M_{opening} = \int_{x=0}^{x=1} x\rho g\,(3-3x^3)\,w\,dx + \int_{z=0}^{3} z\rho g\,(3-z)\,w\,dz$$

$$= \rho g w\int_{x=0}^{1}(3x-3x^4)\,dx + \rho g w\int_{z=0}^{3}(3z-z^2)\,dz$$

$$= \rho g w\left[\frac{3x^2}{2}-\frac{3x^5}{5}\right]_{x=0}^{1} + \rho g w\left[\frac{3z^2}{2}-\frac{z^3}{3}\right]_{z=0}^{3}$$

$$= \rho g w\left[\frac{3}{2}-\frac{3}{5}\right] + \rho g w\left[\frac{27}{2}-\frac{\overset{9}{\cancel{27}}}{\cancel{3}}\right]$$

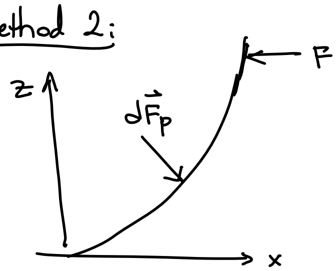
$$= \rho g w\left[15-\frac{3}{5}-9\right] = \frac{27}{5}\rho g w$$

$M_{opening} = M_{closing}$

$$\frac{27}{5}\rho g w = 4F$$

$$\frac{27}{\cancel{5}}\cdot 1000\cdot \overset{5}{\cancel{10}}\cdot \cancel{2} = \overset{2}{\cancel{4}}F \Rightarrow F = 27{,}000\ N$$

<u>Method 2:</u>



$$dM = \vec{r}\times d\vec{F}_p \qquad d\vec{F}_p = -p\,dS\,\vec{n}$$

$$dM_{opening} = dM_y = (\vec{r}\times d\vec{F}_p)\cdot\vec{j}$$

$$= \left[(\vec{r}\times\vec{n})\cdot\vec{j}\right](-p\,dS)$$

Let's parametrize the surface:

$$\left.\begin{array}{l} x = x \\ y = y \\ z = 3x^3 \end{array}\right\} \vec{r}(x,y) = x\vec{i} + y\vec{j} + 3x^3\vec{k} \quad \text{where} \quad \begin{array}{l} 0 \le y \le 2 \text{ (width)} \\ 0 \le x \le 1 \end{array}$$

$$\vec{r}_x = \vec{i} + 9x^2\vec{k}$$

$$\vec{r}_y = \vec{j}$$

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 9x^2 \\ 0 & 1 & 0 \end{vmatrix} = -9x^2\vec{i} + \vec{k}$$

(this $\vec{N}$ is in the $-d\vec{F}_p$ direction. So, we will use this).

$$dM_{opening} = (\vec{r} \times \vec{n}) \cdot \vec{j}\,(-p\,dS)$$

$$= \left[\vec{r} \times \frac{(\vec{r}_x \times \vec{r}_y)}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}}\right] \cdot \vec{j}\,(-p) \underbrace{\cancel{\|\vec{r}_x \times \vec{r}_y\|}\,dx\,dy}_{dS}$$

$$= \left[\vec{r} \times (\vec{r}_x \times \vec{r}_y)\right] \cdot \vec{j}\,(-p\,dx\,dy)$$

$$\vec{r} \times (\vec{r}_x \times \vec{r}_y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & 3x^3 \\ -9x^2 & 0 & 1 \end{vmatrix} = y\vec{i} + (-x - 27x^5)\vec{j} + 9x^2y\,\vec{k}$$

$$dM_{opening} = \underbrace{\left[\vec{r} \times (\vec{r}_x \times \vec{r}_y)\right] \cdot \vec{j}}\,(-p\,dx\,dy)$$

$$= (-x - 27x^5)(-p\,dx\,dy)$$

$$p = \rho g(3 - z) = \rho g(3 - 3x^3)$$

$$z = 3x^3$$

$$dM_{opening} = (x + 27x^5)\,\rho g(3 - 3x^3)\,dx\,dy$$

$$M_{opening} = \iint_S dM_{opening}$$

$$M_{opening} = \int_{x=0}^{1}\int_{y=0}^{2} 3\rho g\left(x - x^4 + 27x^5 - 27x^8\right) dy\, dx$$

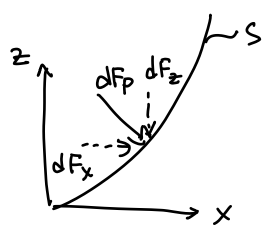
$$= 3\rho g \cdot 2 \cdot \left[\frac{x^2}{2} - \frac{x^5}{5} + \frac{27x^6}{6} - \frac{27x^9}{9}\right]_{x=0}^{1}$$

$$= 6\rho g\Big[\underbrace{\frac{1}{2} - \frac{1}{5} + \frac{\cancel{27}^{\,9}}{\cancel{6}_{\,2}} - \frac{\cancel{27}^{\,3}}{\cancel{9}}}_{\frac{18}{10}}\Big] = 6\rho g \cdot \frac{9}{5}$$

$$M_{opening} = M_{closing}$$

$$6\rho g \cdot \frac{9}{5} = F \cdot 4 \Rightarrow F = 6 \cdot 1000 \cdot 10 \cdot \frac{9}{5} \cdot \frac{1}{4} = 27{,}000\,N$$

Method 3:



Parametric eqn for the surface S is:

$$\vec{r}(x,y) = x\vec{i} + y\vec{j} + 3x^3\vec{k}$$

$$d\vec{F}_p = -p\,\vec{n}\,dS$$

? ?

$$\left.\begin{aligned}\vec{r}_x &= \vec{i} + 9x^2\vec{k}\\ \vec{r}_y &= \vec{j}\end{aligned}\right\} \qquad \vec{r}_x \times \vec{r}_y = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 9x^2\\ 0 & 1 & 0\end{vmatrix} = -9x^2\vec{i} + \vec{k}$$

$$d\vec{F}_p = -p\,\vec{n}\,dS$$

$$= -p\,\overbrace{\frac{(\vec{r}_x \times \vec{r}_y)}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}}}^{\vec{n}}\;\overbrace{\cancel{\|\vec{r}_x \times \vec{r}_y\|}\,dx\,dy}^{dS}$$

$$= -p\,\underbrace{(\vec{r}_x \times \vec{r}_y)}_{-9x^2\vec{i} + \vec{k}}\,dx\,dy$$

$$d\vec{F}_p = -p\left(-9x^2\,\vec{i} + \vec{k}\right)dx\,dy$$

$$p = \rho g(3 - z) = \rho g(3 - 3x^3)$$

$$z = 3x^3$$

$$d\vec{F}_p = \rho g\,(3-3x^3)\left(9x^2\,\vec{i} - \vec{k}\right)dx\,dy$$

$$dF_{px} = 9x^2\rho g\,(3-3x^3)\,dx\,dy$$

$$dF_{pz} = -\rho g\,(3-3x^3)\,dx\,dy$$

$$dM = x\,|dF_z| + z\,|dF_x|$$

$$z = 3x^3$$

$$= x\rho g\,(3-3x^3)\,dx\,dy + z \cdot 9x^2\rho g\,(3-3x^3)\,dx\,dy$$

$$dM = 3\rho g\,(x - x^4)\,dx\,dy + 81\rho g\,(x^5 - x^8)\,dx\,dy$$

$$M_{opening} = \int_{x=0}^{1}\int_{y=0}^{2} 3\rho g\,(x - x^4)\,dy\,dx + 81\rho g\int_{x=0}^{1}\int_{y=0}^{2}(x^5 - x^8)\,dy\,dx$$

$$= 3\rho g \cdot 2 \cdot \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{1} + 81\rho g \cdot 2 \cdot \left[\frac{x^6}{6} - \frac{x^9}{9}\right]_{x=0}^{1}$$

$$= 3\rho g \cdot 2\left(\frac{1}{2} - \frac{1}{5}\right) + \overset{27}{\cancel{81}}\rho g \cdot 2 \cdot \left(\frac{1}{\underset{2}{\cancel{6}}} - \frac{1}{\underset{3}{\cancel{9}}}\right)$$

$$= 6\rho g\left(\frac{1}{2} - \frac{1}{5} + \frac{9}{2} - 3\right)$$

$$M_{opening} = 6\rho g \cdot \left(\frac{9}{5}\right)$$

$$M_{opening} = M_{closing} \Rightarrow 6\rho g \cdot \left(\frac{9}{5}\right) = F \cdot 4 \Rightarrow \quad F = 6 \cdot 1000 \cdot 10 \cdot \frac{9}{5} \cdot \frac{1}{4} = 27{,}000\ \underline{N}$$