ECE259H1: Electromagnetism

Homework Review Quiz – Friday February 3, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The quiz is worth 20 marks and has two questions. Question 1 is worth 5 marks, and Question 2 is work 15 marks.
- Show all of your work.
- The final page has some reference material that you might find helpful.
- Take a deep breath and relax 😌.

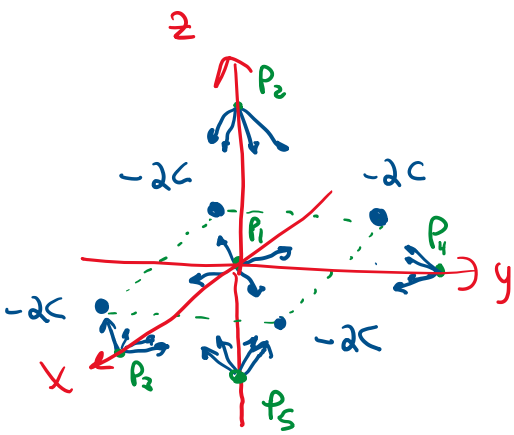
Question #1 (5 marks)

Four equally charged point charges, $Q = -2$ C, sit at the corners of a square that is centered about the z-axis, lies in the xy-plane and has a side length 1 m. The corners of the square are located at points: $(x, y, z) = (0.5, 0.5, 0)$ m, $(-0.5, 0.5, 0)$ m, $(-0.5, -0.5, 0)$ m, and $(0.5, -0.5, 0)$ m.

Consider the question below. Select the correct response and briefly justify your selection. No mathematics are required or needed for this justification, instead a clearly drawn figure with a brief description is sufficient.

At what point(s) will the electric force caused by these four point charges on a point charge $Q_B = +1$ C have only a non-zero $+z$-directed component? You may select more than one answer.

(a) At the origin, $(x, y, z) = (0,0,0)$
(b) At every point along the z-axis
(c) On the z-axis, for positive values of z only (i.e., $z > 0$)
(d) On the z-axis, for negative values of z only (i.e., $z < 0$)
(e) Anywhere in the xy-plane (except the four corners of the square)
(f) Everywhere
(g) Nowhere



For $\bar{F}_E = Q_B \bar{E}_{TOT} = (+1C)\bar{E}_B = +F_z \hat{a}_z$, $\bar{E}_{TOT}$ must be +ve z-directed.
Various example points are shown for E_{TOT}. This only occurs on the negative z-axis. See point P_5.

Question #2 (15 marks)

A partial circle is charged with a uniform linear charge distribution given by $\rho_l = -4$ [C/m]. The wire lies in the xy-plane ($z = 0$), is centered about the z-axis, has a radius of $r = 2$ m and extends from $0 \leq \varphi \leq \frac{5\pi}{4}$. You are to find the electric field intensity at an arbitrary point on the z-axis.

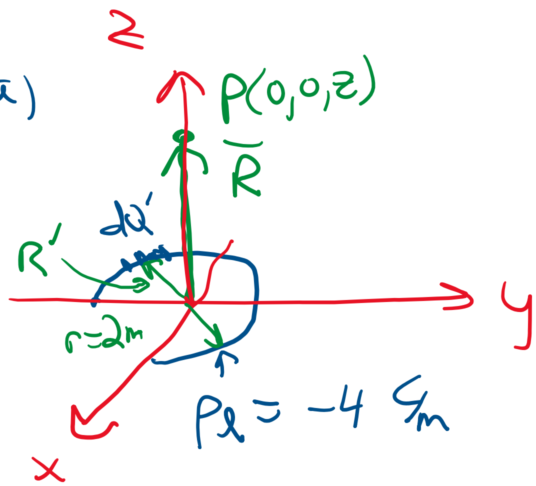
(a) Draw a picture of this situation. Include the position vectors $\mathbf{R}$ and $\mathbf{R}'$, in your figure. [4 marks]

(b) Find the expression for the electric field intensity at an arbitrary point on the z-axis, by:

i. Stating the expressions for dQ', $\mathbf{R}$, and $\mathbf{R}'$. [4 marks]

ii. Integrating over this charge distribution to find $\mathbf{E}(0,0,z)$ [7 marks]

(a)



(b) (i) $dQ' = \rho_\ell \, d\ell' = (-4)(2)\,d\phi' = -8\,d\phi'$

$\bar{R} = z\,\hat{a}_z$

$\bar{R}' = 2\,\hat{a}_r = 2\cos\phi'\,\hat{a}_x + 2\sin\phi'\,\hat{a}_y$

(ii) $$d\bar{E} = \frac{dQ'}{4\pi\varepsilon_0 |\bar{R}-\bar{R}'|^3}(\bar{R}-\bar{R}')$$

$$= \frac{-8\,d\phi'}{4\pi\varepsilon_0 \left[2^2+z^2\right]^{3/2}}\left(-2\cos\phi'\,\hat{a}_x - 2\sin\phi'\,\hat{a}_y + z\,\hat{a}_z\right)$$

Question #2 (continued)

$$\bar{E} = \int d\bar{E} = \int_0^{\frac{5\pi}{4}} \left[\frac{-8\,d\phi'}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \left(-2\cos\phi'\,\hat{a}_x - 2\sin\phi'\,\hat{a}_y + z\,\hat{a}_z\right) \right]$$

$$= \frac{16\,\hat{a}_x}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \int_0^{\frac{5\pi}{4}} \cos\phi'\,d\phi' + \frac{16\,\hat{a}_y}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \int_0^{\frac{5\pi}{4}} \sin\phi'\,d\phi'$$

$$- \frac{8z\,\hat{a}_z}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \int_0^{\frac{5\pi}{4}} d\phi'$$

$$= \frac{16}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \left[\hat{a}_x \left(\sin\phi' \Big|_0^{\frac{5\pi}{4}} \right) + \hat{a}_y \left(-\cos\phi' \Big|_0^{\frac{5\pi}{4}} \right) \right]$$

$$- \frac{8z\,\hat{a}_z}{4\pi\varepsilon_0 (4+z^2)^{3/2}} \left(\frac{5\pi}{4} \right)$$

$$= \frac{4}{\pi\varepsilon_0 (4+z^2)^{3/2}} \left[-\frac{1}{\sqrt{2}}\,\hat{a}_x + \left(1 + \frac{1}{\sqrt{2}}\right)\hat{a}_y \right] - \frac{10}{4\pi\varepsilon_0 (4+z^2)^{3/2}}\,\hat{a}_z$$

This page is left intentionally blank. It can be used for rough work, and will not be marked unless you specifically direct the marker to this page.

Reference Formulae

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

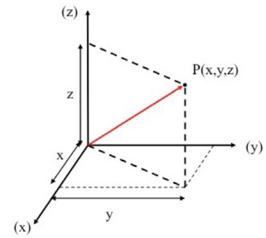
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$

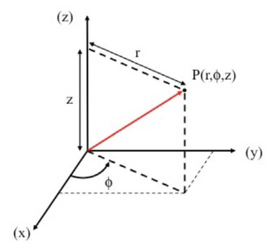
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

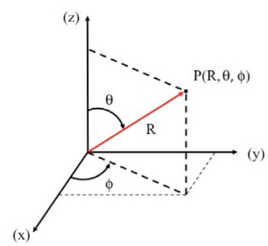
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ







3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	-$A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ