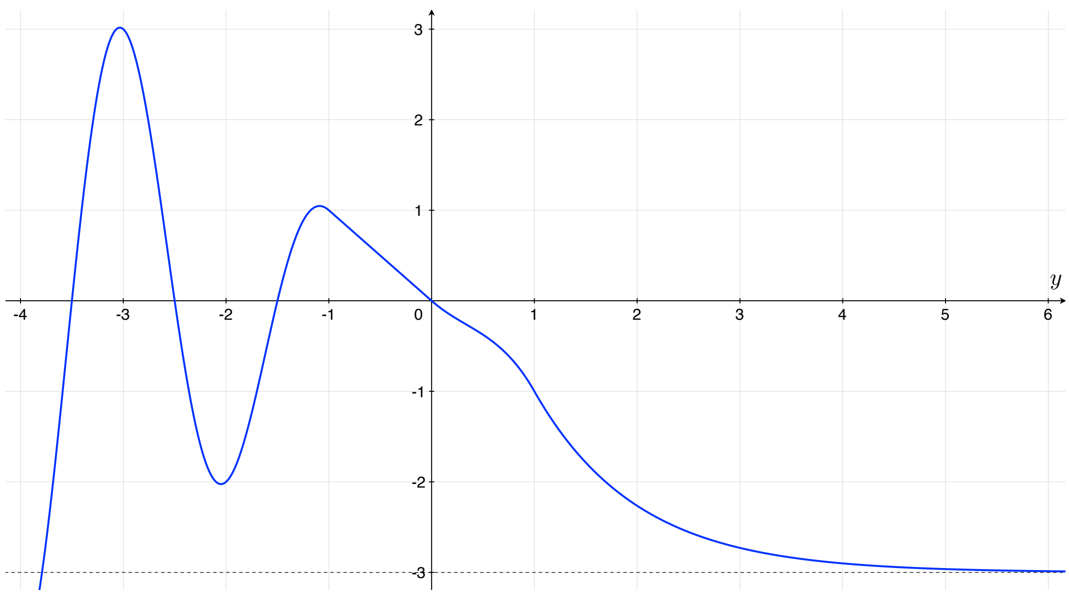
MAT292 - Calculus III - Fall 2013

Solution of Term Test - October 17, 2013

Time allotted: 90 minutes. Aids permitted: None.

PART I (8 marks)

Assume $y(t)$ is a solution of the differential equation $\frac{dy}{dt} = f(y)$ with some initial condition and $f(y)$ defined by the graph below.



1. If $y(0) = 1$, then the solution $y(t)$ as a horizontal asymptote at -3. True $\boxed{\text{False}}$

2. If $y(0) = 1$, then the solution $y(t)$ as a slant asymptote with slope -3. True $\boxed{\text{False}}$

3. If $y(0) = 1$, then $\lim_{t\to\infty} y(t) = 0$. $\boxed{\text{True}}$ False

4. If $y(0) = -3$, then the solution has a maximum at $y = -\frac{5}{2}$. True $\boxed{\text{False}}$

5. If $y(0) = -3$, then the solution has a maximum at $y = -3$. True $\boxed{\text{False}}$

6. The points $-\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}$, and 0 are equilibrium points. $\boxed{\text{True}}$ False

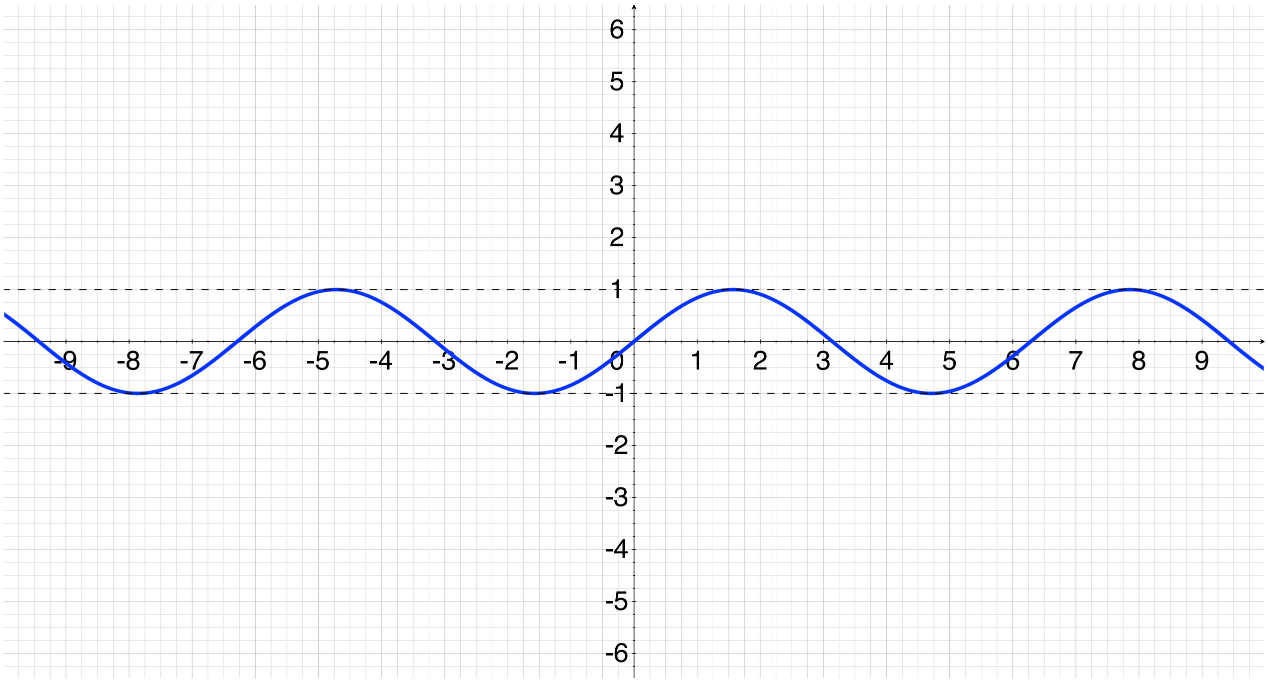
7. The solution $y(t) = -\frac{5}{2}$ is stable. $\boxed{\text{True}}$ False

8. The solution $y(t) = -\frac{3}{2}$ is stable. True $\boxed{\text{False}}$

PART II Answer the following questions. **Justify** your answers.

1. Consider the equation $\frac{dy}{dt} = f(y) = \sin(y)$.

(a) Sketch the graph of $f(y)$ versus y.



(b) Determine the critical (equilibrium) points, and classify each one as asymptotically stable, unstable, or semistable.

Solution. The critical points are the zeros of $f(y)$: $k\pi$ for all integers k.

The critical points $k\pi$ where k is odd are asymptotically stable, since y near y_0, $f(y) > 0$ for $y < y_0$ and $f(y) < 0$ for $y > y_0$.

The critical points $k\pi$ where k is even are asymptotically unstable, since y near y_0, $f(y) < 0$ for $y < y_0$ and $f(y) > 0$ for $y > y_0$.

□

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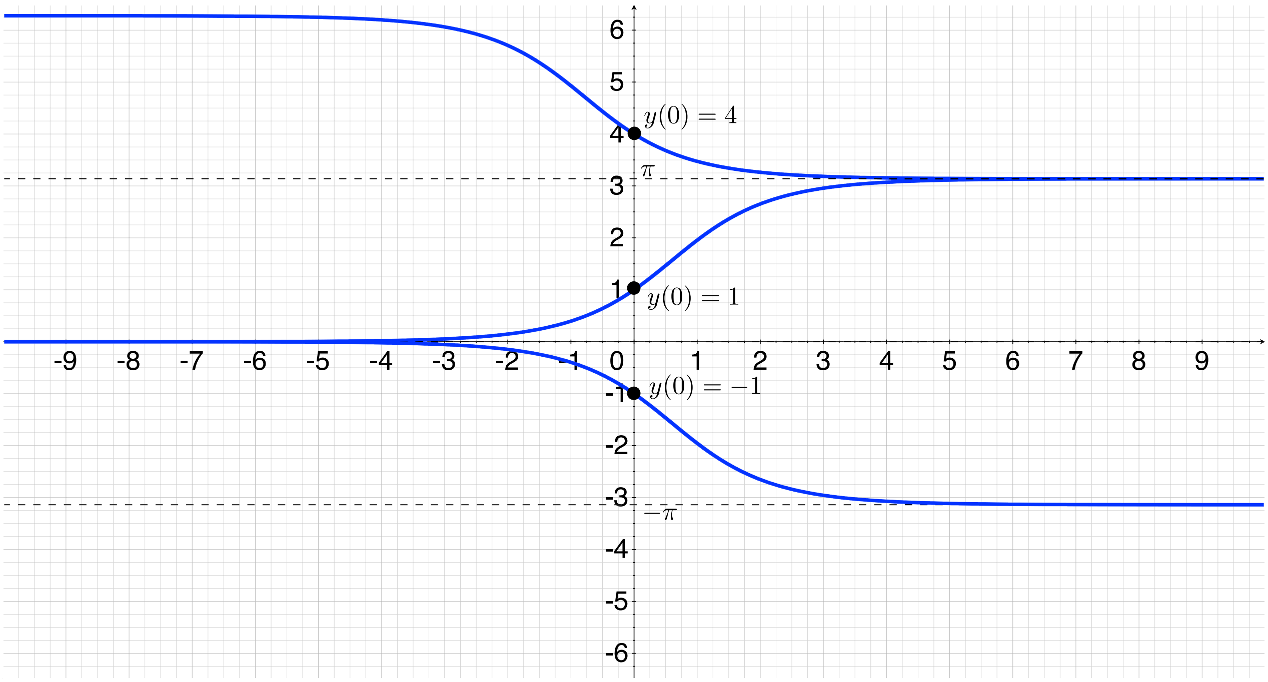
(c) On the same set of axes, sketch the graphs of the solutions to $\frac{dy}{dt} = f(y)$ with initial conditions:

$$y(0) = 1 \quad , \quad y(0) = 4 \quad , \text{ and } \quad y(0) = -1.$$

Note: Be sure to clearly label which initial condition each curve corresponds to.



(d) Is there a solution $y = \phi(t)$ to $\frac{dy}{dt} = \sin(y)$ such that $\phi(0) < -1$ and $\phi(1) > 2$? Justify your answer.

Solution. Solutions of autonomous differential equations cannot cross critical points. So, there cannot be such solution since there is a critical point $y = 0$ between $\phi(0)$ and $\phi(1)$.

□

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2. Solve the initial value problem,

$$\begin{cases} x^2y\dfrac{dy}{dx} = -2xy - y^2\dfrac{dy}{dx} \\ y(0) = 1 \end{cases}$$

Solution. Write the equation as $2xy + (x^2y + y^2)\,\frac{dy}{dx} = 0$
Then, $M = 2xy$, $N = x^2y + y^2$
$M_y = 2x$, $N_x = 2xy$. $M_y \neq N_x$ so the equation is not exact.
$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N) \implies 2x\mu + \mu_y 2xy = 2xy\mu + \mu_x N$
Assume $\mu_x = 0$. Then

$$\begin{aligned} \mu_y &= \frac{2xy - 2x}{2xy}\mu \\ \mu_y &= \left(1 - \frac{1}{y}\right)\mu \\ \ln(\mu) &= (y - \ln(y)) \\ \mu &= \frac{e^y}{y} \end{aligned}$$

Multiplying the equation by μ yields

$$2xe^y + \left(x^2e^y + ye^y\right)\frac{dy}{dx} = 0$$

an exact equation.
$\Psi_x = 2xe^y \implies \Psi = x^2e^y + g(y)$
$x^2e^y + ye^y = \Phi_y = x^2e^y + g'(y) \implies g'(y) = ye^y \implies g(y) = (y-1)e^y$
$\implies \Psi = x^2e^y + (y-1)e^y$
So, $e^y(x^2 + y^2 - 1) = c$ solves the equation.
$y(0) = 1 \implies c = 0$ so the solution to the IVP is $e^y(x^2 + y^2 - 1) = 0$.
Dividing by e^y yields $x^2 + y^2 = 1$.

$\square$

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3. An old TV (in the shape of a cube of side r) with mass m is thrown into the ocean. Let $v(t)$ be its vertical velocity (positive upwards). Consider the following three forces acting on the object while it is completely under water:

- Gravity: proportional to the mass (with constant g)
- Resistance: proportional (with constant γ) to the square of the instantaneous velocity;
- Buoyancy: given by Archimedes' Principle (proportional to the volume of the object with constant A).

(a) Determine a differential equation for the velocity of the TV.

Solution. Using Newton's Laws of Motion: $F = m \cdot a$, so

$$m\,v' = -mg - \text{sign}(v)\gamma\, v^2 + A\,r^3.$$

This means that the DE is

$$m\,v' = -mg - \gamma\, v^2 + A\,r^3, \tag{U}$$

when $v > 0$ – when the TV is surfacing, and

$$m\,v' = -mg + \gamma\, v^2 + A\,r^3, \tag{D}$$

when $v < 0$ – when the TV is sinking.

Note. We considered as correct solutions either differential equations (U) or (D), as long as on the following parts, everything was done consistently.

□

(b) Assume that $r = \frac{1}{2}m$ (27" TV), with $m = 30kg$, a drag coefficient $\gamma = 1$, and $A = 301 \cdot 2^3$, and approximate the gravitational constant $g \approx 10$. If the initial velocity is $-\frac{1}{2}$, solve the initial value problem.

Solution. The initial-value problem we want to solve is

$$30\,v' = 1 - \text{sign}(v)v^2$$
$$v(0) = -\frac{1}{2}$$

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Initially, the TV is sinking ($v < 0$), so we first solve the differential equation:

$$\frac{v'}{1+v^2} = \frac{1}{30}.$$

We get

$$\begin{aligned}\arctan v &= \frac{t}{30} + C \\ v &= \tan\left(\frac{t}{30} + C\right)\end{aligned}$$

We now have to find C using the initial condition: $v(0) = -\frac{1}{2}$:

$$\begin{aligned}-\frac{1}{2} &= v(0) = \tan(C) \\ C &= -\arctan\frac{1}{2}\end{aligned}$$

The solution is

$$v(t) = \tan\left(\frac{t}{30} - \arctan\frac{1}{2}\right)$$

We can then find the time T when $v = 0$:

$$v(T) = 0 \qquad \Leftrightarrow \qquad T = 30\arctan\frac{1}{2}.$$

At that point, the TV "wants" to start surfacing, so the DE is satisfies is

$$\begin{aligned}30v' &= 1 - v^2 \\ v(T) &= 0\end{aligned}$$

We now solve the DE:

$$\begin{aligned}\frac{v'}{1-v^2} &= \frac{1}{30} \\ \frac{v'}{(1-v)(1+v)} &= \frac{1}{30}\end{aligned}$$

We can write

$$\frac{1}{(1-v)(1+v)} = \frac{1}{2}\frac{1}{1-v} + \frac{1}{2}\frac{1}{1+v},$$

so we get

$$\begin{aligned}\frac{v'}{1-v} + \frac{v'}{1+v} &= \frac{1}{15} \\ -\ln|1-v| + \ln|1+v| &= \frac{t}{15} + C \\ \ln\left|\frac{1+v}{1-v}\right| &= \frac{t}{15} + C \\ \frac{1+v}{1-v} &= Ke^{\frac{t}{15}} \qquad (\star) \\ v &= \frac{K - e^{-\frac{t}{15}}}{K + e^{-\frac{t}{15}}}\end{aligned}$$

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We now have to find K using the initial condition: $v(T) = 0$. We use equation $(\star)$ to simplify finding K:

$$\begin{aligned}\frac{1+v(T)}{1-v(T)} &= Ke^{\frac{T}{15}}\\ 1 &= Ke^{\frac{T}{15}}\\ K &= e^{-\frac{T}{15}}\end{aligned}$$

The solution for $t \geq T$ is

$$v(t) = \frac{e^{-\frac{T}{15}} - e^{-\frac{t}{15}}}{e^{-\frac{T}{15}} + e^{-\frac{t}{15}}}.$$

The solution is

$$v(t) = \begin{cases} \tan\left(\frac{t}{30} - \arctan\frac{1}{2}\right) & \text{if } 0 \leq t \leq T,\\ \dfrac{e^{-\frac{T}{15}} - e^{-\frac{t}{15}}}{e^{-\frac{T}{15}} + e^{-\frac{t}{15}}} & \text{if } t \geq T.\end{cases}$$

□

(c) Does the TV go up or down underwater? What is the terminal velocity of the TV?

Solution. The initial velocity is negative, so the TV is sinking at the beginning, but at time $T = 30\arctan\frac{1}{2}$ it starts surfacing and

$$\lim_{t\to\infty} v(t) = \frac{e^{-\frac{T}{15}}}{e^{-\frac{T}{15}}} = 1,$$

so the terminal velocity for the TV (underwater) is 1m/s upwards. □

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(d) Assume that $r = \frac{1}{2}m$ (27" TV), with $m = 30kg$, a drag coefficient $\gamma = 1$, and $A = 300 \cdot 2^3$, and approximate the gravitational constant $g \approx 10$. If the initial velocity is -1, solve the initial value problem.

Solution. The initial-value problem we want to solve is

$$\begin{aligned} 30\, v' &= -\text{sign}(v) v^2 \\ v(0) &= -\frac{1}{2} \end{aligned}$$

Initially, the TV is sinking ($v < 0$), so we first solve the differential equation:

$$\frac{v'}{v^2} = \frac{1}{30}.$$

We first solve the differential equation:

$$\begin{aligned} v^{-2}\, v' &= \frac{1}{30} \\ -v^{-1} &= \frac{t}{30} + C \\ v &= -\frac{1}{\frac{t}{30} + C} \\ v &= -\frac{30}{t + K} \end{aligned}$$

We now have to find K using the initial condition: $v(0) = -1$:

$$\begin{aligned} -1 = v(0) &= -\frac{30}{K} \\ K &= 30 \end{aligned}$$

The solution is

$$v(t) = -\frac{30}{t + 30}$$

In this case, the speed v is never 0, so this is the final solution. □

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(e) Does the TV go up or down underwater? What is the terminal velocity of the TV?

Solution. The initial velocity is negative, so the TV is sinking at the beginning, but

$$\lim_{t\to\infty} v(t) = 0,$$

so the TV keeps sinking (even though it slows down).

The limit is 0, so the terminal velocity for the TV (underwater) is 0m/s. □

(f) Assume that the initial position of the TV is 0 (the surface of the ocean). Will the TV hit the bottom of the ocean?

Solution. Since the speed will slow down, we cannot tell without finding the position $y(t)$ of the TV, which solves the differential equation

$$y' = v = -\frac{30}{t+30}.$$

This implies that

$$y = -30\ln(t+30) + C.$$

Using the initial condition $y(0) = 0$, we obtain $C = 30\ln 30$, so

$$y = -30\ln(t+30) + 30\ln 30 = 30\ln\frac{30}{t+30}.$$

This means that as time passes, the depth of the TV converges to

$$\lim_{t\to\infty} y(t) = (\ln 0^+) = -\infty.$$

So independently of the depth of the ocean, the TV will hit the bottom. □

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4. Consider the initial value problem,

$$\begin{cases} y' = 2ty \\ y(0) = 1 \end{cases}$$

(a) Calculate the integral form of the initial value problem.

(**Hint.** Pay attention to the initial condition)

Solution.

$$\begin{aligned} \int_0^t y'(s)ds &= \int_0^t 2sy(s)ds \\ y(t) - y(0) &= \int_0^t 2sy(s)ds \\ y(t) &= \int_0^t 2sy(s)ds + 1 \end{aligned}$$

□

(b) Let $\phi_0(t) = 1$. Compute the next **four** Picard approximations for the solutions of the initial value problem.

Solution.

$$\begin{aligned} \phi_1(t) &= \int_0^t 2s\phi_0(s)\,ds + 1 = \int_0^t 2s\,ds + 1 = t^2 + 1 \\ \phi_2(t) &= \int_0^t 2s\phi_1(s)\,ds + 1 = \int_0^t 2s(s^2+1)\,ds + 1 = \int_0^t 2s^3 + 2s\,ds + 1 \\ &= \frac{1}{2}t^4 + t^2 + 1 \\ \phi_3(t) &= \int_0^t 2s\phi_2(s)\,ds + 1 = \int_0^t 2s\left(\frac{1}{2}s^4 + s^2 + 1\right)\,ds + 1 \\ &= \int_0^t s^5 + 2s^3 + 2s\,ds + 1 = \frac{1}{6}t^6 + \frac{1}{2}t^4 + t^2 + 1 \\ \phi_4(t) &= \int_0^t 2s\phi_3(s)\,ds + 1 = \int_0^t 2s\left(\frac{1}{6}s^6 + \frac{1}{2}s^4 + s^2 + 1\right)\,ds + 1 \\ &= \int_0^t \left(\frac{1}{3}s^7 + s^5 + 2s^3 + 2s\right)\,ds + 1 = \frac{1}{24}t^8 + \frac{1}{6}t^6 + \frac{1}{2}t^4 + t^2 + 1 \end{aligned}$$

□

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(c) What do the Picard's approximations converge to? Do not justify.
(**Hint.** Write $x = t^2$)

Solution.

$$\phi_n(t) = \sum_{k=0}^{n} \frac{t^{2k}}{k!}$$

Write $x = t^2$, then

$$\phi_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

So

$$\lim_{n\to\infty} \phi_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x = e^{t^2}$$

□

(d) Using the step size $h = \frac{1}{2}$ and Euler's Method, approximate the solution at $t = \frac{3}{2}$.

Solution.

$$\begin{aligned}
y_{n+1} &= y_n + 2ht_ny_n \\
y(0) &\approx y_0 = 1 \\
y\left(\frac{1}{2}\right) &\approx y_1 = 1 \\
y\left(1\right) &\approx y_2 = 1 + 2h^2 = \frac{3}{2} \\
y\left(\frac{3}{2}\right) &\approx y_3 = \frac{3}{2} + 2(2h)\frac{3}{2}h = \frac{3}{2} + 6h^2 = 3
\end{aligned}$$

□

(e) If we need to obtain an error 20 times smaller, which step size h should we choose?

Solution. Since Euler's Method is first order, to obtain an error 20 times smaller, we must divide the step size h by 20. In the case $h = \frac{1}{2}$ we must choose $h = \frac{1}{40}$.

□

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5. (a) State the Existence and Uniqueness Theorem for initial value problems of the form:

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

Solution. Let $p(t)$ and $g(t)$ be continuous in an interval (a, b) containing t_0. Then, for any $y_0 \in \mathbb{R}$, there is a unique solution $y = \phi(t)$ of the initial value problem above for any $t \in (a, b)$.

□

(b) On what interval is the following initial value problem guaranteed to have a unique solution. Justify your answer.

$$\begin{cases} ty' + y = \frac{t}{(t-2)(t+3)} \\ y(4) = 2 \end{cases}$$

Solution. First we need to normalize the differential equation:

$$y' + \frac{1}{t}y = \frac{1}{(t-2)(t+3)}.$$

The functions

$$p(t) = \frac{1}{t} \qquad \text{and} \qquad g(t) = \frac{1}{(t-2)(t+3)}$$

are continuous for $t \in I = \mathbb{R} - \{-3, 0, 2\}$, so by the Theorem stated on (a), a solution exists and is unique for all $t \in (2, \infty)$, the largest interval in I containing $t_0 = 4$.

□

The end.