ESC103F Engineering Mathematics and Computation: Tutorial #1

Question 1: Let $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ be the vectors in standard position of two points P_1 and P_2.

a) Using vector addition, derive an expression for the vector $\overrightarrow{P_1P_2}$ in terms of $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. (Hint: draw a diagram that locates the origin, P_1 and P_2.)

b) If the point M is $1/3^{rd}$ of the way from P_1 to P_2, derive a general expression for the vector $\overrightarrow{OM}$ in terms of $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$. (Hint: draw a diagram that locates the origin, P_1, P_2 and M.) From this general expression, determine M if P_1=(1,2,3) and P_2=(4,5,6).

Solution:

a) $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$
$\therefore \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$

b) $\overrightarrow{OP_1} + \overrightarrow{P_1M} = \overrightarrow{OM}$

$$\overrightarrow{OP_1} + \tfrac{1}{3}\overrightarrow{P_1P_2} = \overrightarrow{OM}$$

$$\overrightarrow{OP_1} + \frac{1}{3}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) = \overrightarrow{OM}$$

$$\therefore \overrightarrow{OM} = \frac{2}{3}\overrightarrow{OP_1} + \frac{1}{3}\overrightarrow{OP_2}$$

$$\overrightarrow{OM} = \frac{2}{3}\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\5\\6\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

$$\therefore M = (2,3,4)$$

Question 2: Given points P(2,-1,4), Q(3,-1,2), A(0,2,1) and B(1,3,0), determine if $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ are parallel.

Solution:

$$\overrightarrow{PQ} = \begin{bmatrix}3-2\\-1-(-1)\\2-4\end{bmatrix} = \begin{bmatrix}1\\0\\-2\end{bmatrix}, \overrightarrow{AB} = \begin{bmatrix}1-0\\3-2\\0-1\end{bmatrix} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Since $\overrightarrow{AB}$ cannot be expressed as a scalar multiple of $\overrightarrow{PQ}$, then the two vectors are <u>not</u> parallel.

Question 3: Let the points A, B, C, and D in the plane form a quadrilateral ABCD (a four-sided figure). Let E, F, G, and H be the midpoints of each side of the quadrilateral. Using a vector method approach, prove that the quadrilateral EFGH is a parallelogram. (Hint: draw a diagram showing all eight points.)

Solution:

Assuming E is chosen to be the midpoint between A and B, and F is chosen to be the midpoint between B and C:

$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BF} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}$$

$\therefore$ $\overrightarrow{EF}$ and $\overrightarrow{AC}$ are parallel.

Assuming G is chosen to be the midpoint between C and D, and H is chosen to be the midpoint between D and A, we can show in a similar way that $\overrightarrow{HG}$ is parallel to $\overrightarrow{AC}$.
$\therefore$ $\overrightarrow{EF}$ is parallel to $\overrightarrow{HG}$.

Similarly, we can show that $\overrightarrow{EH}$ is parallel to $\overrightarrow{FG}$.

$\therefore$ EFGH is a parallelogram.

Question 4: Points A(-3, 2), B(1, -2) and C(7, 1) are given.

Find the coordinates of point D so that ABCD forms a parallelogram. (Hint: plot the 3 points A, B, and C and show the approximate location of point D to form the parallelogram.)

Solution:

We want to find point D such that $\overrightarrow{BC} = \overrightarrow{AD}$ and $\overrightarrow{BA} = \overrightarrow{CD}$.

With these two equalities in place, the following equality must hold:

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} + \overrightarrow{BA} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

Now we are able to determine $\overrightarrow{OD}$:

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$\therefore$ point D is located at (3,5).

Question 5: The linear combination of two vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ define a plane that goes through the origin. Sketch these two vectors in the xyz space.

i) Consider linear combinations $c\vec{v} + d\vec{w}$. Write an expression for a single vector in terms of *c* and *d* that defines the plane.

ii) Using your result from part (i), find a vector that is **not** in the plane.

Solution:

i) $c\vec{v} + d\vec{w} = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix}$

ii) For all vectors in this plane, the y-component is always equal to the sum of the x- and z-components.

$\therefore \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, for example, is not in the plane because $2 \neq 1 + 3$.

Question 6: Find two different linear combinations of the three vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ that produce $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? If you take **any** three vectors $\vec{u}, \vec{v}$ and $\vec{w}$, will there always be two different linear combinations that produce $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Solution:

$c \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 7 \end{bmatrix} + e \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Writing out the two equivalent scalar equations:

$c + 2d + e = 0$

$3c + 7d + 5e = 1$

From the first equation:

$\therefore c = -2d - e$

Substituting into the second equation:

$3(-2d - e) + 7d + 5e = 1$

$\therefore d = 1 - 2e$

If we choose $e = 1$, then $d = -1$ and $c = 1$.

If we choose $e = -1$, then $d = 3$ and $c = -5$.

This would not be true for just **any** three vectors $\vec{u}, \vec{v}$ and $\vec{w}$. For example, if the three vectors are parallel to each other but not parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then no combinations will produce $\vec{b}$.