# Final Exam solutions 2012

**Problem 1** (**4 marks**)
Write an equation that represents the Newton's law of universal gravitation in vector notation. Describe all quantities you are using in this equation.

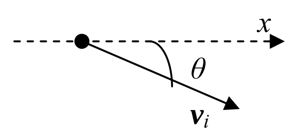
**Solution**

$$\vec{\mathbf{F}}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}_{12}$$

where $\mathbf{F}_{12}$ is force exerted by the point mass 1 on the point mass 2 via gravitational field;
$G$ is the universal gravitational constant;
$m_1$ and $m_2$ are the masses of interacting point masses;
$r$ is the distance between the two point masses; and
$\mathbf{r}_{12}$ is the position vector of the point mass 2 relative to the position of the point mass 1.

**Problem 2** (**5 marks**)
A ball is tossed from an upper-story window of a building. The ball is given an initial velocity of 8.00 m/s at an angle of 20.0° below the horizontal. It strikes the ground 3.00 s later.
(a) How far horizontally from the base of the building does the ball strike the ground?
(b) Find the height from which the ball was thrown.
Use unit-vector notation $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ to obtain expressions for
(c) the ball's position vector $\boldsymbol{r}$ as a function of time;
(d) the ball's velocity vector $\boldsymbol{v}$ as a function of time; and
(e) the ball's acceleration vector $\boldsymbol{a}$ as a function of time.

**Solution**




(a) $x_f = v_{xi}\, t = v_i \cos\theta\, t = (8.00\cdot \cos 20.0^{\circ} \cdot 3.00)$ m = 22.6 m

(b) Taking $y$ positive downwards,

$$y_f = v_{yi}t + \frac{1}{2}gt^2 = v_i \sin\theta\ t + \frac{1}{2}gt^2 = (8.00\cdot 0.342\cdot 3.00 + 0.5\cdot 9.80\cdot 9.00)\text{ m} = 52.3$$

m

or taking $y$ positive upwards, $y_f$ = - 52.3 m

Answer: The height is 52.3 m.

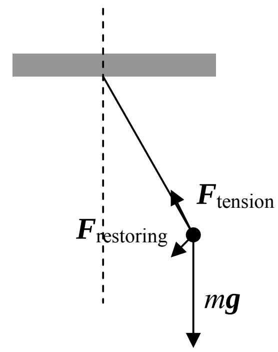
(c) Taking $y$ positive downwards, $\boldsymbol{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = [7.52t\,\hat{\mathbf{i}} + (2.74t + 4.90t^2)\,\hat{\mathbf{j}}\,]$ m
or taking $y$ positive upwards, $\boldsymbol{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = [7.52t\,\hat{\mathbf{i}} + (-\,2.74t - 4.90t^2)\,\hat{\mathbf{j}}\,]$ m

(d) $\boldsymbol{v} = (\mathrm{d}x/\mathrm{d}t)\,\hat{\mathbf{i}} + (\mathrm{d}y/\mathrm{d}t)\,\hat{\mathbf{j}} = [7.52\,\hat{\mathbf{i}} + (2.74 + 4.90\,t)\,\hat{\mathbf{j}}\,]$ m/s
or $= [7.52\,\hat{\mathbf{i}} + (-2.74 - 4.90\,t)\,\hat{\mathbf{j}}\,]$ m/s

(e) $\boldsymbol{a} = 9.80\,\hat{\mathbf{j}}$ m/s$^2$
or = $-\,9.80\,\hat{\mathbf{j}}$ m/s$^2$

**Problem 3** **(7 marks)**

A simple pendulum has a mass of 0.250 kg and a length of 1.00 m. It is displaced from the vertical through an angle of 5.00° and then released. Find
(a) the period of oscillation,
(b) the maximum speed,
(c) the maximum angular acceleration, and
(d) the maximum restoring force. Show the free-body diagram for the pendulum not at the position of equilibrium.




**Solution**

(a) $\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.80}{1.00}}\frac{rad}{s} = 3.13s^{-1}$ $T = \frac{2\pi}{\omega} = 2.01$ s

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Possible correct solution 1:

(b) $v_{max} = A\omega = L\,\theta_{max}\,\omega = [1.00\,(5.00^o/180^o)\,\pi\,3.13]$ m/s = 0.273 m/s

(c) Maximum linear aceleration $a_{\max} = A\omega^2 = (0.273 \cdot 3.13)$ m/s$^2$ = 0.855 m/s$^2$

→ maximum angular acceleration $\alpha_{\max} = a_{\max} / L = 0.855$ rad/s$^2$
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Possible correct solution 2
(b) Law of conservation of energy gives

$$\frac{mv_{\max}^2}{2} = mgh_{\max} = mgL\left(1-\cos 5.00^\circ\right) = m9.80\cdot 3.8053\cdot 10^{-3} \Rightarrow$$

$$v_{\max} = \sqrt{2\cdot 9.80\cdot 3.8053\cdot 10^{-3}}\,m/s =$$
$$= 0.273 \text{ m/s}$$

(c) $I\,\alpha_{max} = \tau_{max}$

$\alpha_{max} = \tau_{max} / I = mgL\sin\theta_{max} / (mL^2) = g\sin\theta_{\max} / L = 0.854$ rad/s$^2$

$a_{\max} = L\alpha_{\max} = 0.854$ m/s$^2$
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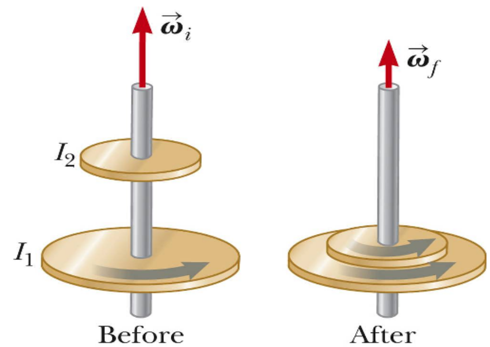
(d) Possible correct answer 1: $F_{\max} = ma_{\max} = 0.250 \cdot a_{max}$ N = 0.214 N

Possible correct answer 2: $F_{\max} = mg\sin\theta_{max} = 0.250 \cdot 9.80 \cdot 0.087155743$ N = 0.214 N

**Problem 4 (8 marks)**

A disk with moment of inertia $I_1$ rotates about a vertical frictionless axle. At an instant $t_1$ when the angular speed of the rotating disk is $\omega_i$, a second disk having moment of inertia $I_2$ and initially not rotating, drops onto the first disk. The two disks eventually reach the same angular speed $\omega_f$ at an instant $t_2$.




(a) Find $\omega_f$ in terms of given quantities.

(b) What work is produced by kinetic friction on the first disk during the time interval $t_2 - t_1$? Is it positive or negative?

(c) What work is produced by kinetic friction on the second disk during the time interval $t_2 - t_1$? Is it positive or negative?

(d) What is the total work produced by kinetic friction in this collision? Is the total work positive, negative or zero? Justify your answer

**Solution**

(a) From conservation of angular momentum for a system of two cylinders:

$$I_1\omega_i = (I_1 + I_2)\omega_f \quad \Rightarrow \quad \omega_f = \omega_i \frac{I_1}{I_1 + I_2} < \omega_i \quad (4.1)$$

(b) $W_1 = \Delta K_1 = \dfrac{I_1\omega_f^2}{2} - \dfrac{I_1\omega_i^2}{2} < 0$

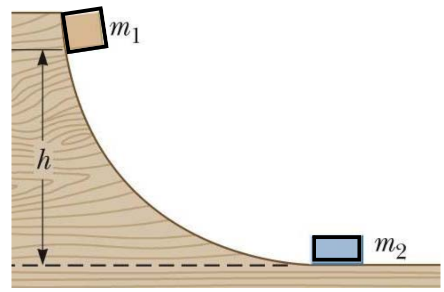
(c) $W_2 = \Delta K_2 = K_{2f} = \dfrac{I_2\omega_f^2}{2} > 0$

(d) $W = W_1 + W_2 = \dfrac{1}{2}\left[(I_1 + I_2)\omega_f^2 - I_1\omega_i^2\right] =$ (after substituting solution (4.1) $-\dfrac{I_1 I_2 \omega_i^2}{I_1 + I_2} < 0$

or because friction is a non-conservative force, the total kinetic energy must decrease, thus making total work negative.

**Problem 5** **(9 marks)**

Two blocks are free to slide along the frictionless wooden track as shown in the figure to the right. The block of mass $m_1$ = 5.00 kg is released from the height $h$ =5.00 m above the flat part of the track. Protruding from its front end is the north pole of a strong magnet, which repels the north pole of an identical magnet embedded in the back end of the block of mass $m_2$ = 10.0 kg, initially at rest. The two blocks never touch. The size of the blocks can be neglected.




(a) Calculate the maximum height to which $m_1$ rises after the elastic collision.
(b) Find the speed of the second block right after the collision
(c) Is the total mechanical energy conserved in this collision?

**Solution**

The magnets work to prevent contact and loss of mechanical energy due to deformation of blocks in contact. Besides, the answer to the question does not depend on whether the block 2 starts moving to the right before the block 1 changes the direction of its velocity. We can consider the problem is about a perfectly elastic collision and forget about the magnets.

(a) To calculate the maximum height, one must first find the velocities the two blocks acquire after the collision. This can be found by applying laws of conservation of energy and linear momentum for perfectly elastic collision.

If $v_{1i}$ is the speed of the block 1 just before the collision, then $\frac{m_1 v_{1i}^2}{2} = m_1 g h_i$

$v_{1i} = \sqrt{2gh_i} = \sqrt{2 \cdot 9.80 \cdot 5.00}$ m/s = 9.90 m/s

If $v_{1f}$ is the horizontal component of velocity of the block 1 just after the collision, and $v_{2f}$ is the horizontal component of velocity of the block 2 after the collision, then the laws of conservation give:

$$m_1 v_{1i} + 0 = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{m_1 v_{1i}^2}{2} = \frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2}$$

Two equations with two unknowns have one solution: $v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$ = - 3.30 m/s

which means that the block 1 changes the direction of velocity after the collision,

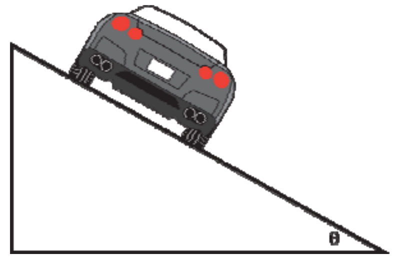
and $v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$ =6.60 m/s Everywhere, the positive horizontal direction is taken to the right.

At the highest point (after the collision) $m_1 g h_f = \frac{m_1 v_{1f}^2}{2} \Rightarrow h_f = \frac{v_{1f}^2}{2g} = 0.556$ m

(c) Yes! The collision is elastic, and two magnets did not permit the blocks come in touch, so that the collision is perfectly elastic! In perfectly elastic collision, the total mechanical energy of the system is conserved.

**Problem 6 (13 marks)**

Due to a smart engineer's design, the car rounds a banked curve as in the figure to the right. The radius of curvature of the road measured from the center of mass of the car is $R$, the banking angle is $\theta$ , and the coefficient of static friction is $\mu_s$.

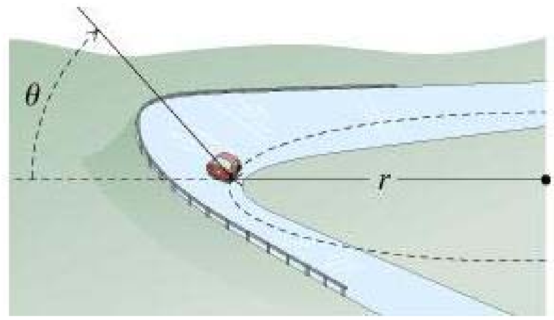



(a) Show the free-body diagram for the moving car on the banked roadway.

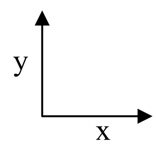
(b) Determine the range of speeds the car can have without slipping up or down the road

**Solution**

The drawing of a car on a banked curve is shown to the right. If the track were not banked, the condition of successful turning on a road with radius of curvature $R$ is given in Lecture 15, slide #4, as $v_{\max\,flat} = \sqrt{\mu_s gR}$ .



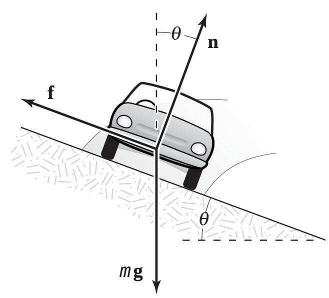

For the car which speed exceeds the $v_{max\,flat}$, the static friction won't be able to keep the car on the circular track. The banked curve gives an opportunity to move faster than with $v_{max\,flat}$ on the surface with same static friction. Therefore, all auto race tracks are banked curves.




If the car is about to slip *down* the incline, the force of static friction $f$ is directed up the incline.

$\Sigma F_y = n\cos\theta + f\sin\theta - mg = 0$, where $f = \mu_s n$

$$n = \frac{mg}{\cos\theta(1+\mu_s\tan\theta)} \quad \Rightarrow \quad f = \frac{\mu_s mg}{\cos\theta(1+\mu_s\tan\theta)}$$



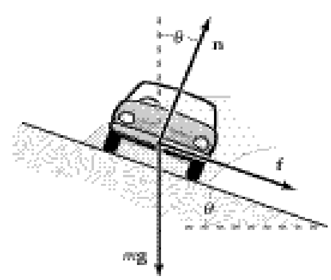

Then, $\Sigma F_x = n\sin\theta - f\cos\theta = m\dfrac{v_{\min}^2}{R}$ , and

$$v_{\min} = \sqrt{\frac{Rg(\tan\theta - \mu_s)}{1+\mu_s\tan\theta}}$$

When the car is about to slip *up* the incline, $f$ is directed down the incline. Then,

$\Sigma F_y = n\cos\theta - f\sin\theta - mg = 0$ with $f = \mu_s n$ yields

$$n = \frac{mg}{\cos\theta(1-\mu_s\tan\theta)} \quad \Rightarrow \quad f = \frac{\mu_s mg}{\cos\theta(1-\mu_s\tan\theta)}$$




In this case, $\Sigma F_x = n\sin\theta + f\cos\theta = m\dfrac{v_{\max}^2}{R}$ , which gives

$$v_{\max} = \sqrt{\frac{Rg(\tan\theta + \mu_s)}{1-\mu_s\tan\theta}} .$$