UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT292H1F - Calculus III

Final Exam - December 16, 2015

EXAMINERS: B. GALVÃO-SOUSA

Time allotted: 150 minutes. Aids permitted: None.

Solution

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<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 18 pages (including this title page) and a detached formula sheet.

  Make sure you have all of them.
- You can use pages 17–18 to complete a question (**mark clearly**).

GOOD LUCK!

PART I   No explanation is necessary.

1. Find a function $y(t)$ that satisfies the conditions **(3 marks)**

- $y^{(4)} - y = 0$
- $y(0) = y'(0) = 0$
- $y(t)$ is bounded

$y(t) =$ $e^{-t} - \cos(t) + \sin(t) \left(C_2 = C_4 = -C_3\right)$.

2. Find an autonomous differential equation ($y' = f(y)$) with the following properties: **(3 marks)**

- It has equilibrium solutions at $y = 0$, $y = 1$, and $y = 3$;
- The solutions satisfy

$$\begin{cases} y' > 0 & \text{when} \quad 0 < y < 1 \quad \text{or} \quad 1 < y < 3 \\ y' < 0 & \text{otherwise} \end{cases}$$

The differential equation is

$$\frac{dy}{dt} = -y(y-1)^2(y-3).$$

For questions **3.** –**4.** , consider the initial-value problem

$$\vec{x}\,' = \begin{pmatrix} -3 & k \\ k & 1 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \vec{x}_0,$$

where k is a real number.

3. Determine all the values for k for which the matrix has real distinct eigenvalues. **(2 marks)**

$k \in \mathbb{R}$.

4. Determine all the values for k for which the solutions are spirals or centres. **(2 marks)**

Never happens for $k \in \mathbb{R}$

Continued...

5. Consider the following differential equation **(4 marks)**

$$y^{(9)} + 3y^{(8)} + 11y^{(7)} + 25y^{(6)} + 40y^{(5)} + 56y^{(4)} + 48y^{(3)} + 16y'' = te^t + e^{-t} + t\cos(2t)$$

where $x^9 + 3x^8 + 11x^7 + 25x^6 + 40x^5 + 56x^4 + 48x^3 + 16x^2 = x^2(x+1)^3(x^2+4)^2$.

When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* have the form (**select all that apply**):

(a) $A\cos 2t$	**(e)** $E\sin 2t$	**(i)** I	**(m)** Me^t	**(q)** Qe^{-t}
(b) $Bt\cos 2t$	**(f)** $Ft\sin 2t$	**(j)** Jt	**(n)** Nte^t	**(r)** Rte^{-t}
(c) $Ct^2\cos 2t$	**(g)** $Gt^2\sin 2t$	**(k)** Kt^2	**(o)** Ot^2e^t	**(s)** St^2e^{-t}
(d) $Dt^3\cos 2t$	**(h)** $Ht^3\sin 2t$	**(l)** Lt^2	**(p)** Pt^3e^t	**(t)** Tt^3e^{-t}

6. Let $y(t) = 2 - tu_1(t) + (t^2 - 9)u_3(t)$. **(3 marks)**

Then

$$\mathfrak{L}\{y(t)\}(s) = \frac{2}{s} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s} + \left(\frac{2}{s^3} + \frac{6}{s^2}\right)e^{-3s}.$$

7. Let $f(x) = \sin(2x) - \sin\left(\frac{5x}{2}\right) + \frac{1}{2}\cos(1024x)$ for $x \in [-2\pi, 2\pi]$. **(3 marks)**

Then the Fourier Series of $f(x)$ is

$$\sin(2x) - \sin\left(\frac{5x}{2}\right) + \frac{1}{2}\cos(1024x).$$

Continued...

PART II **Justify** your answers.

8. Consider a constant $a > 0$ and the following initial-value problem (IVP). **(16 marks)**

$$\begin{cases} a\dfrac{dy}{dt} = y^2 \\ y(0) = 1 \end{cases}$$

(a) **(5 marks)** Using the Existence and Uniqueness Theorem without solving the IVP, is there a solution? Is it unique? What is its domain?

$$y' = \frac{y^2}{a} \Rightarrow f(t,y) = \frac{y^2}{a} \rightarrow \frac{\partial f}{\partial y} = 2\frac{y}{a}$$

These functions are continuous for $(t,y) \in \mathbb{R}\times\mathbb{R}$ so the Thm states that there is a unique solution defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

(b) **(5 marks)** Solve the IVP and find the solution $y(t)$.

This is a separable DE, so the solution satisfies

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{a}\,dt \Leftrightarrow -\frac{1}{y} = \frac{t}{a} + C \Leftrightarrow y = -\frac{1}{t/a + C}$$

Using the initial condition, we get:

$$1 = y(0) = -\frac{1}{C} \Leftrightarrow C = -1$$

So the solution is

$$y = -\frac{1}{\frac{t}{a} - 1} = \frac{a}{a-t}$$

(c) **(2 marks)** What is the domain of the solution?

The solution's domain is

$t < a \iff t \in (-\infty, a)$

(d) **(4 marks)** Compare the results of **(a)** with the results of **(b)** – **(c)** .

Is there a value $\varepsilon^{\star} > 0$ independent of a such that the solution exists and is unique for $t \in (-\varepsilon^{\star}, \varepsilon^{\star})$?

After solving the DE in (b), (c), we can specify $\varepsilon = a$ in (a).

There is no such ε^{*}. If there was, we could choose a value for a satisfying $0 < a < \varepsilon^{*}$ and the solution would not exist for all $t \in (-\varepsilon^{*}, \varepsilon^{*})$.

Continued...

9. Consider the following model for two competing populations. **(16 marks)**

Define the following variables

- $x_1(t)$ = number of lions at time t (in years);
- $x_2(t)$ = number of cheetahs at time t;
- $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Then we have a system of differential equations:

$$\vec{x}\,' = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \vec{x} + \begin{pmatrix} -7 \\ -14 \end{pmatrix}$$

(a) **(4 marks)** Find the equilibrium solution for this system.

We want a solution which is independent of t:

$$\vec{X_E}' = \vec{0} \iff \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \vec{x_E} + \begin{pmatrix} -7 \\ -14 \end{pmatrix}$$

$$\iff \begin{cases} 3x_1 - x_2 = 7 \\ -2x_1 + 2x_2 = 14 \end{cases} \iff \begin{cases} x_2 = 3x_1 - 7 \\ 4x_1 = 28 \end{cases} \iff \begin{cases} x_2 = 14 \\ x_1 = 7 \end{cases}$$

$$\boxed{\vec{x}_{\text{eq}} = \begin{pmatrix} 7 \\ 14 \end{pmatrix}}$$

(b) **(6 marks)** Sketch a phase portrait for this system. (Don't forget the equilibrium point)

$$\vec{x}_h' = \begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix} \vec{x}_h$$

Eigenvalues.

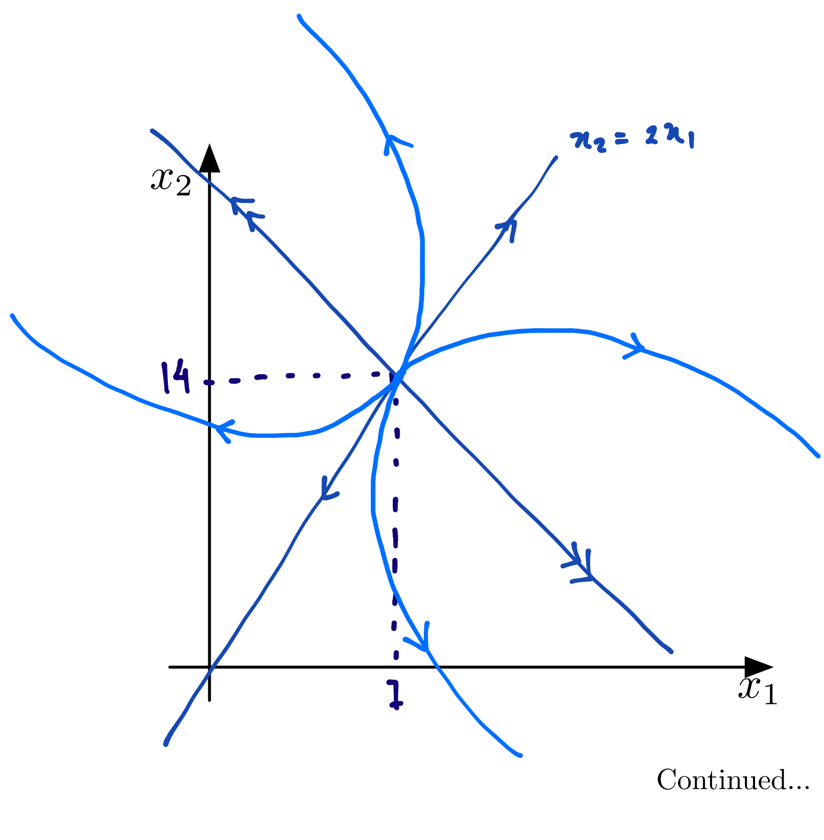
$$(3-\lambda)(2-\lambda) - 2 = 0 \iff \lambda^2 - 5\lambda + 4 = 0 \iff \lambda = 1 \text{ or } \lambda = 4.$$

Eigenvectors.

$\boxed{\lambda_1 = 1}$ $\begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_2 = 2\xi_1 \Rightarrow \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{\lambda_2 = 4}$ $\begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \xi_1 = -\xi_2 \Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

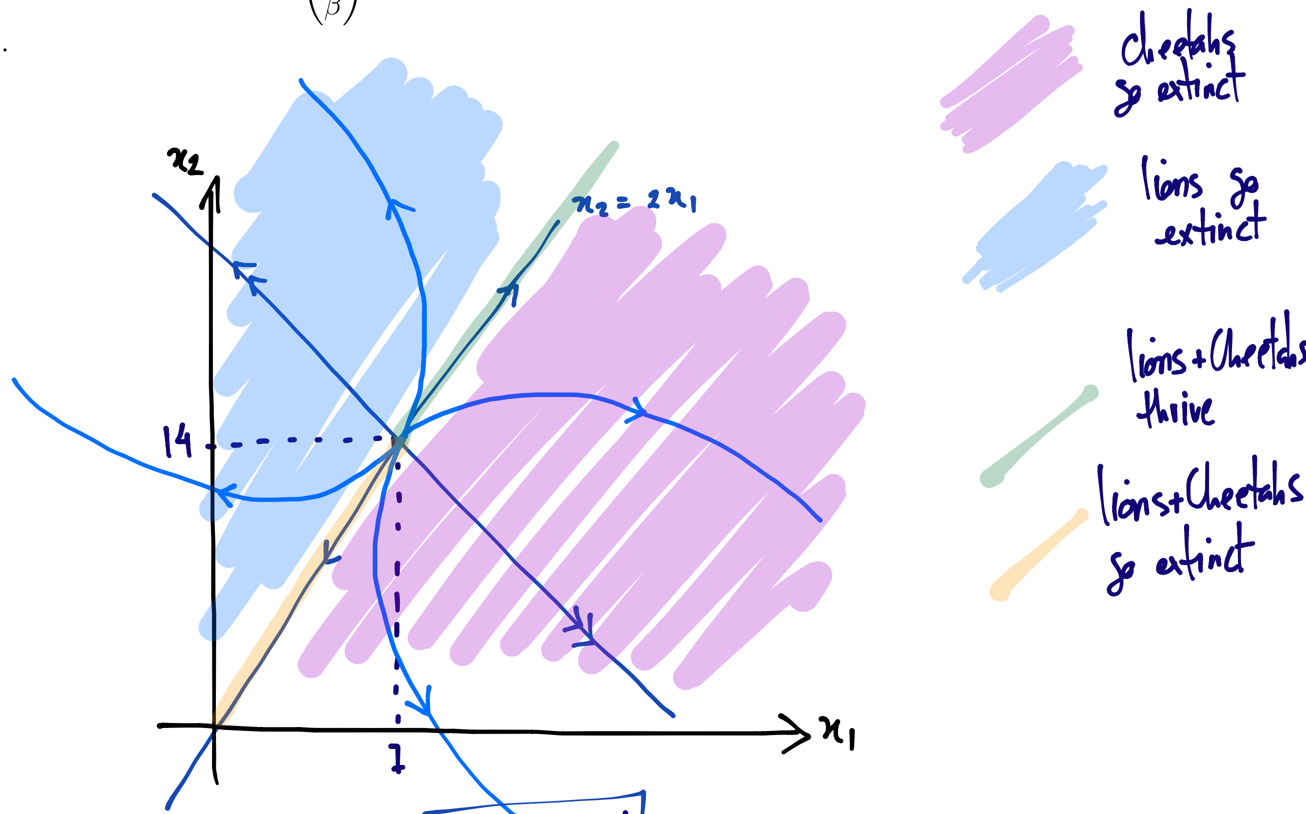
$$\vec{x}_E = \begin{pmatrix} 7 \\ 14 \end{pmatrix}$$



Continued...

(c) **(6 marks)** Let $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Fill in the blanks below with expressions involving only α and β.

Solution #1.



Solution #2. $\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + \underbrace{\boxed{c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}}}_{\text{dominant term}} + \begin{pmatrix} 7 \\ 14 \end{pmatrix}$

- $c_2 > 0 \Rightarrow$ Cheetahs go extinct
- $c_2 < 0 \Rightarrow$ lions go extinct
- $c_2 = 0$ $<$ $c_1 < 0 \Rightarrow$ both go extinct; $c_1 > 0 \Rightarrow$ both thrive

and $\vec{x}(0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\Downarrow$

$\begin{cases} c_1 = \dfrac{\alpha+\beta}{3} - 7 \\ c_2 = \dfrac{2\alpha - \beta}{3} \end{cases}$

- Lions will become extinct if $\beta > 2\alpha$
- Cheetahs will become extinct if $\beta < 2\alpha$
- Lions and Cheetas will both become extinct if $\beta = 2\alpha$ and $\beta < 14$, $\alpha < 7$
- Lions and Cheetahs will thrive if $\beta = 2\alpha$ and $\beta \geq 14$, $\alpha \geq 7$

10. The goal of this exercise is to show that we can solve this problem with or without using the Laplace transform to obtain the same solution. **(16 marks)**

Consider the initial-value problem

$$\begin{cases} y' + y = \delta(t-3) \\ y(0) = 0 \end{cases}$$

(a) **(6 marks) Without using** the Laplace transform, we approximate the Dirac delta with a unit impulse function:

$$y' + y = d_a(t-3) = \begin{cases} 0 & \text{if } t < 3-a \\ \frac{1}{2a} & \text{if } 3-a < t < 3+a \\ 0 & \text{if } t > 3+a \end{cases}.$$

Find the (continuous) solution to this problem.

$\boxed{t < 3-a}$ $\quad y' + y = 0 \iff y = C_1 e^{-t}$

$y(0) = 0 \implies C_1 = 0$

$\implies \boxed{y = 0}$

$\boxed{3-a < t < 3+a}$ $\quad \begin{cases} y' + y = \frac{1}{2a} \\ y(3-a) = 0 \end{cases}$

Using integrating factor: $\quad (e^t y)' = \frac{1}{2a} e^t \iff e^t y = \frac{e^t}{2a} + C_2$

$\iff y = \frac{1}{2a} + C_2 e^{-t}$

Use the conditions

$0 = y(3-a) = \frac{1}{2a} + C_2 e^{-(3-a)} \iff C_2 = -\frac{e^{3-a}}{2a}$

So $\boxed{y = \frac{1}{2a}\left(1 - e^{3-a-t}\right)}$

Continued...

$\boxed{t > 3+a}$

$$\begin{cases} y' + y = 0 \\ y(3+a) = \frac{1}{2a}\left(1 - e^{-2a}\right) \end{cases} \Rightarrow y = C_3 e^{-t}$$

Same DE as for $t < 3-a$

$$\frac{1}{2a}\left(1-e^{-2a}\right) = y(3+a) = C_3 e^{-(3+a)} \iff C_3 = \frac{1}{2a}\left(1-e^{-2a}\right)e^{3+a}$$

So $\boxed{y = \frac{1}{2a}\left(1-e^{-2a}\right)e^{3+a-t}}$

$y_a(t) =$	0	if $t < 3-a$
	$\frac{1}{2a}\left(1 - e^{3-a-t}\right)$	if $3-a < t < 3+a$
	$\frac{1}{2a}\left(1-e^{-2a}\right)e^{3+a-t}$	if $t > 3+a$

(b) **(4 marks)** Take your solution to **(a)** and find the limit as $a \to 0^+$.

$$\lim_{a\to 0^+} \underbrace{\frac{1-e^{-2a}}{2a}}_{\frac{0}{0}} \underbrace{e^{3+a-t}}_{\to e^{3-t}} = e^{3-t} \lim_{a\to 0^+} \frac{1-e^{-2a}}{2a} \overset{\text{L'Hôpital}}{=} e^{3-t} \lim_{a\to 0^+} \frac{\cancel{2}e^{-2a}}{\cancel{2}} = e^{3-t}$$

$$\lim_{a\to 0^+} y_a(t) = \begin{cases} 0 & \text{if } t < 3 \\ e^{3-t} & \text{if } t > 3 \end{cases}$$

(c) **(4 marks) Using** the Laplace transform, find the solution of the original problem

$$\begin{cases} y' + y = \delta(t-3) \\ y(0) = 0 \end{cases}$$

Let $Y(s) = \mathcal{L}\{y(t)\}(s)$, which satisfies

$$sY(s) - \underbrace{\cancel{y(0)}}_{=0} + Y(s) = e^{-3s}$$

$$(s+1)Y(s) = e^{-3s}$$

$$Y(s) = \frac{1}{s+1}e^{-3s}$$

Then $y(t) = \mathcal{L}^{-1}\{Y(s)\}(t) = e^{-(t-3)}u_3(t)$

$y(t) =$ $e^{-(t-3)}u_3(t)$

(d) **(2 marks)** Compare the results of **(b)** and **(c)** .

The solutions of (b) and (c) are the same.

Continued...

11. The following problem is a new problem you haven't seen before. **(16 marks)**

To solve it, you only need to know the material studied in this course.

Consider a vibrating string (like a guitar string) of length π, which is attached at both ends and consider the function

$$u(x,t) = \text{height of the string at the position } x \text{ and time } t.$$

Then $u(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < \pi \text{ and } t > 0, \\ u(x,0) = 0 & \text{for } 0 < x < \pi, \\ \dfrac{\partial u}{\partial t}(x,0) = 1 & \text{for } 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0 & \text{for } t > 0. \end{cases}$$

Find a solution $u(x,t)$ for this problem.

Hint.

(a) Write $u(x,t) = \phi(x)G(t)$ and find differential equations for G and ϕ and boundary conditions.

(b) Find eigenvalues λ and eigenfunctions $\phi(x)$.

(c) Find $G(t)$. Keep c with this term to simplify the notation.

(d) Write down the general solution $u(x,t)$.

(e) Write the initial conditions as Fourier series of the same form as ϕ and find the constants.

(f) Conclude with the final formula for $u(x,t)$.

(a) Using $u(x,t) = \phi(x)G(t)$ implies:

PDE $\Leftrightarrow$ $G''(t)\phi(x) = c^2 G(t)\phi''(x)$ $\Leftrightarrow$ $\underbrace{\dfrac{G''(t)}{c^2 G(t)}}_{\text{depends only on } t} = \underbrace{\dfrac{\phi''(x)}{\phi(x)}}_{\text{depends only on } x}$

So both sides must be constant: $\dfrac{G''(t)}{c^2 G(t)} = \dfrac{\phi''(x)}{\phi(x)} = \text{constant} = -\lambda$

Continued...

(Continuation of solution to 11.)

$$\Rightarrow \boxed{\begin{cases} G''(t) = -\lambda c^2 G(t) \\ \phi''(x) = -\lambda \phi(x) \end{cases}}$$

$$\text{BC} \Rightarrow \begin{cases} \phi(0)G(t) = 0 \\ \phi(\pi)G(t) = 0 \end{cases} \Rightarrow \boxed{\begin{cases} \phi(0) = 0 \\ \phi(\pi) = 0 \end{cases}}$$

(b) From the formula sheet (1st column): $\boxed{\begin{cases} \lambda_n = n^2 \\ \phi_n(x) = \sin(nx) \end{cases}}$, $n = 1, 2, 3, \ldots$

(c) $G''(t) = -(nc)^2 G(t)$

$G''(t) + (nc)^2 G(t) = 0 \quad \Rightarrow$ Assume $G(t) = e^{rt}$.

Then $r^2 e^{rt} + (nc)^2 e^{rt} = 0 \Leftrightarrow r^2 = -(nc)^2 \Leftrightarrow r = \pm nci$

Thus $\boxed{G_n(t) = a_n \cos(nct) + b_n \sin(nct)}$

(d) Using the superposition principle:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(nct) + b_n \sin(nct)\right] \sin(nx)}$$

(e) $u(x,0) = \sum_{n=1}^{\infty} a_n \sin(nx) = 0 \quad \Rightarrow \boxed{a_n = 0}$ for all n.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} ncb_n \cos(nct)\sin(nx)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} ncb_n \sin(nx) = 1$$

From formula sheet:

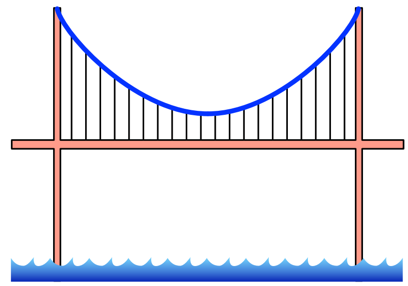
$$ncb_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

$$b_n = \frac{2}{n^2 c\pi}\left[1 - \cos(n\pi)\right]$$

The solution is: $\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2 c\pi}\left[1 - \cos(n\pi)\right] \sin(nct)\sin(nx)}$

Continued...

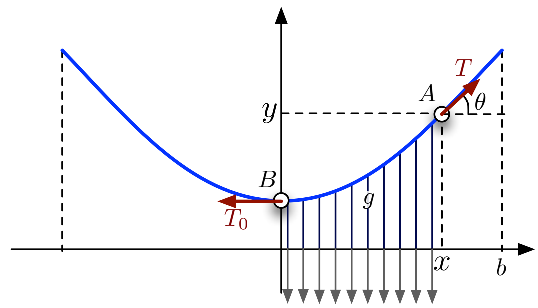
12. Consider a suspension bridge like the one in the figure. **(16 marks)**



We want to figure out the shape of the supporting cable.
To do this, we assume that

The cable supports the weight of the bridge.
The weight of the cables is negligible.

Also, the cable is in equilibrium, so all the forces add up to zero.
Define the variables as in the figure on the right:



- The shape of the cable is given by a function $y(x)$
- Consider the portion of the cable from point B to A
- T_0 is the magnitude of the tension on point B
- At the point A the tension on the cable has angle θ with the horizontal and magnitude T.

(a) **(4 marks)** Obtain an equation for the horizontal components of the forces.

$\vec{F}_{T,B} = (-T_0, 0)$
$\vec{F}_{T,A} = (T\cos\theta, T\sin\theta)$

Horizontal Forces add up to zero:

$$-T_0 + T\cos\theta = 0$$

$$\boxed{T_0 = T\cos\theta}$$

(b) **(4 marks)** Assume that the bridge has mass density ρ (in kg/m). What is the Gravitational force acting on that portion of the cable? Obtain an equation for the vertical components of the forces.

$$\vec{F_G} = (0, -mg)$$

$m =$ mass supported by this portion of the cable

$= \rho \cdot x$

So $\vec{F_G} = (0, -\rho g x)$

$\vec{F}_{T,A} = (T\cos\theta, T\sin\theta)$

Vertical forces add up to zero:

$$-\rho g x + T\sin\theta = 0$$

$$\boxed{\rho g x = T\sin\theta}$$

(c) **(3 marks)** Using **(a)** –**(b)** , find a differential equation for $y(x)$. **Hint.** Observe that $y'(x) = \tan\theta$.

$$y'(x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{T\sin\theta}{T\cos\theta} = \frac{\rho g x}{T_0}$$

$$\Rightarrow \boxed{y' = \frac{\rho g}{T_0} x}$$

Continued...

(d) **(2 marks)** Solve the differential equation you found in **(c)** .

$y(x) =$ $\frac{\rho g}{2T_0}x^2 + C$

(e) **(3 marks)** The Golden Gate bridge has the following design specs:

1 • The support cable touches the deck

2 • The bridge was built with $\rho = 6000$ kg/m in mind.

3 • The two towers are 1300m apart

4 • The towers are 150m high

Find the tension T_0 (in N).

$1 \Leftrightarrow y(0) = 0$ $\qquad 3 + 4 \Leftrightarrow y(650) = 150$

$2 \Leftrightarrow \rho = 6000$

$\Rightarrow \quad 0 = y(0) = C \Rightarrow y = \frac{\rho g}{2T_0}x^2$

$\Rightarrow \quad 150 = \frac{6000 \cdot g}{2T_0}(650)^2 \Leftrightarrow T_0 = \frac{3000 g (650)^2}{150} = 20g(650)^2$

$T_0 =$ $20g(650)^2$

Continued...