The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Final exam - Wednesday April 25, 2018

Instructors: Profs. Micah Stickel and Piero Triverio

Last name: ....

First name: ....

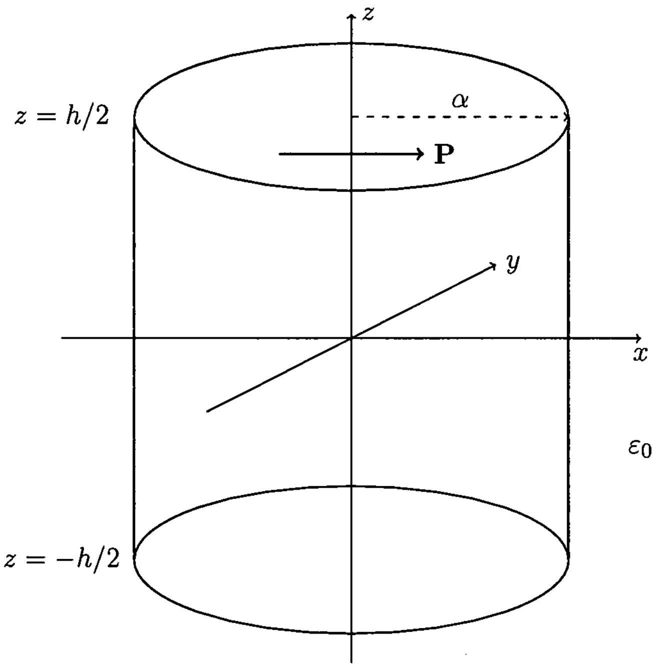
Student number: ....

Instructions

- Duration: 2 hour 30 minutes (14:00 to 16:30)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full credit!**

Marks:	Q1: /20	Q2: /20	Q3: /20	Q4: /20	TOTAL: /80

Question 1



The cylinder in the figure has radius α, height h and lies along the z axis with the origin in the middle. The cylinder is made by a perfect dielectric material and is polarized. The polarization vector is $\mathbf{P} = P_0\mathbf{a}_x$ with $P_0 > 0$.

(a) Find the density of **all** polarization charge distributions that may exist within or on the cylinder. [4 points]

(b) Without doing calculations, determine the direction of the electric field $\mathbf{E}$ at the origin. Briefly justify your answer. [2 points]

(c) Find the electric field $\mathbf{E}$ at the origin. [14 points]
You may find the following integrals useful

$$\int_0^{2\pi} \cos^2 x dx = \pi \qquad \int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

Q1 (c) (continued)

Question 2

A very long wire with radius a lies along the z-axis and has a current density given by $\mathbf{J}_{inner} = J_0 r \mathbf{a}_z$. Coaxial to this wire is situated a very thin cylinder with radius b. The outer cylinder carries a total current that is equal and opposite to the inner conductor. You may assume that for these conductors, $\mu_r = 1$.

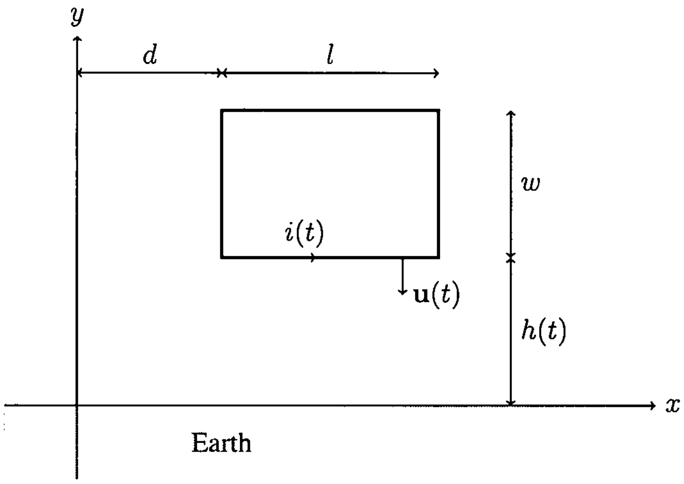
(a) Determine the magnetic field intensity, $\mathbf{H}$, everywhere. [10 points]

(b) Determine the stored energy per unit length of this coaxial system. [5 points]

(c) Determine the inductance per unit length of this coaxial system. [5 points]

Q2 (c) (continued)

Question 3



Consider the rectangular metallic frame shown in the figure. The frame is falling under the effect of gravity with velocity $\mathbf{u}(t) = -u_y(t)\mathbf{a}_y$ where $u_y(t) > 0$. The frame is rigid, has total resistance R and negligible inductance. A magnetic field $\mathbf{B} = B_0 y \mathbf{a}_z$ with $B_0 > 0$ is present in the region $y > 0$.

(a) Using Lenz's law, determine the sign of current $i(t)$. [2 points]

(b) Sketch the direction of the magnetic force acting on each edge of the frame. Briefly justify your answer. [2 points]

(c) Do magnetic forces increase the frame velocity, decrease it, or leave it unchanged? Briefly justify your answer. [2 points]

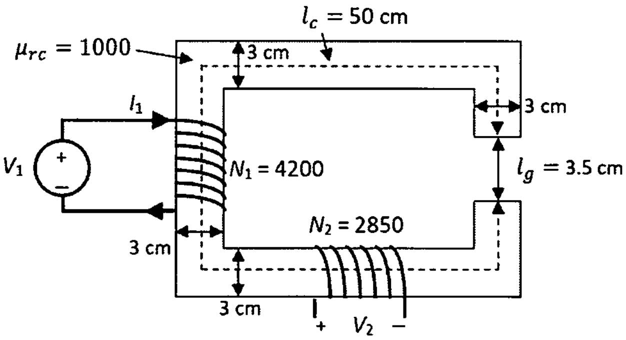
(d) Determine $i(t)$ using Faraday's law in the form $V_{emf} = -\frac{\partial}{\partial t}\Phi(t)$. Express $i(t)$ in terms of $u_y(t)$. [4 points]

(e) Determine $i(t)$ using the alternative form of Faraday's law $V_{emf} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS} + \int_c (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl}$. Express $i(t)$ in terms of $u_y(t)$. [4 points]

(f) Find the net magnetic force $\mathbf{F}_m$ acting on the frame, and express it in terms of $u_y(t)$. [6 points]

Question 4.1

For the magnetic circuit shown below, answer the following True/False questions. For this problem you can ignore the effects of fringing fields, and you can assume the core has a square cross-section (i.e., it extends 3 cm into the page). Both coils are tightly wound around the core. Briefly justify each of your answers with appropriate descriptions and/or calculations. [5 points]



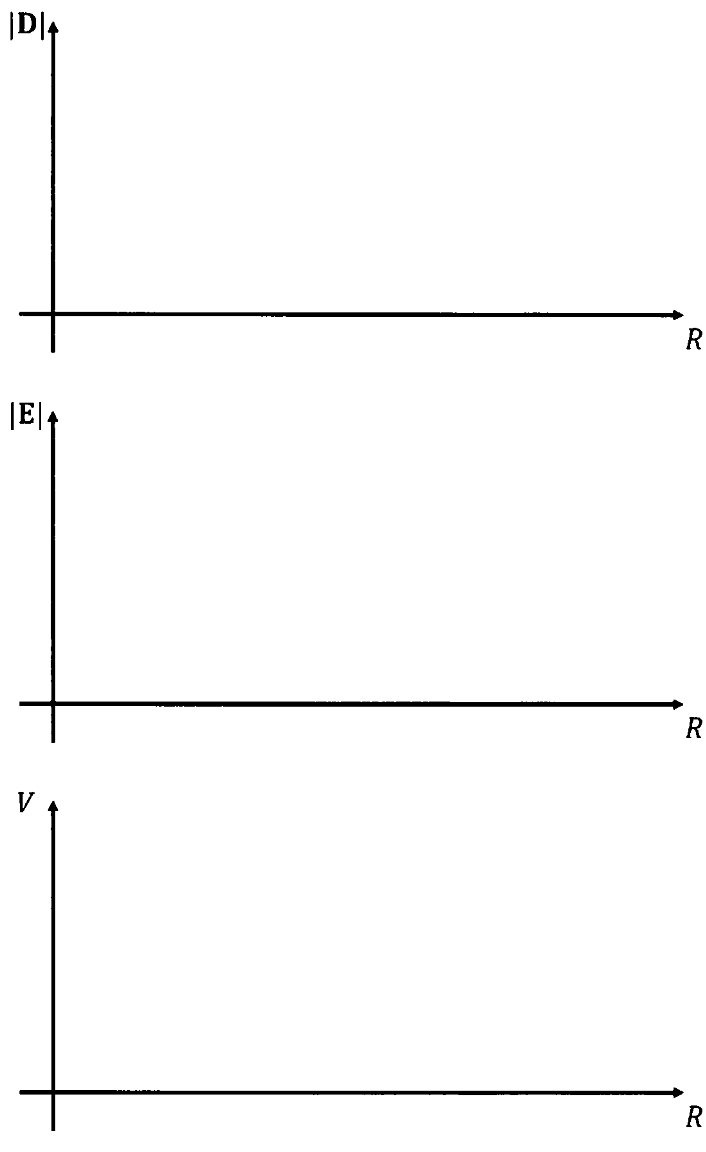
(a) (True / False) The magnitude of the magnetic field intensity, H, is larger in the air gap than in the magnetic core.

(b) (True / False) The reluctance of the air gap is smaller than that of the magnetic core.

(c) (True / False) The self-inductance of the second coil at the bottom is L_{22} = 259 mH (rounded to the nearest mH).

Question 4.2

A solid perfectly conducting sphere of radius $R = 4$ cm is centered on the origin and has a charge density of $\rho_S = 2$ mC/m^2 on its surface. It is surrounded by a spherical dielectric shell $\varepsilon_r = 5$ that extends from $R = 4$ cm to $R = 6$ cm. On the axes below, sketch the variation of the magnitudes of the electric field intensity, electric flux density, and the electric scalar potential (with $V(R = \infty) = 0$). Your plots should include values at key points (i.e., $R = 0$ cm, $R = 4$ cm, and $R = 6$ cm) and should extend from $R = 0$ cm to $R = 10$ cm. [6 points]



Question 4.3

A sphere of radius a that is made of a conductive dielectric ($\sigma = \sigma_0$ and $\varepsilon = \varepsilon_r\varepsilon_0$) is centered about the origin. The sphere is charged at $t = 0$ s with a uniform charge density given by $\rho_v(t = 0) = \rho_0$ for all $R \leq a$, where ρ_0 is a positive constant.

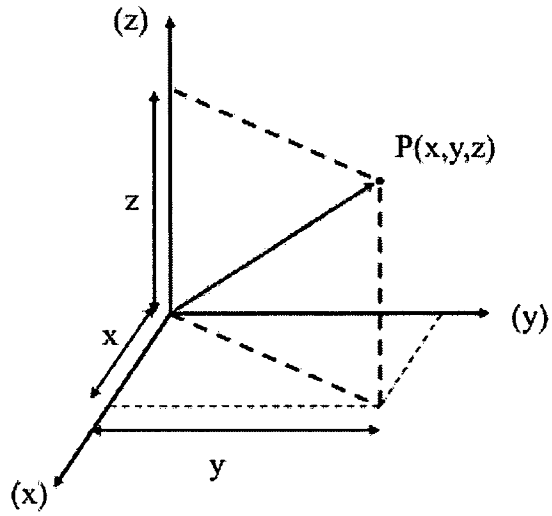
(a) Starting from the continuity equation, $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$, prove that the charge density within the dielectric sphere varies according to $\rho_v(t) = \rho_0 e^{-\frac{\sigma_0 t}{\varepsilon_r \varepsilon_0}}$. [3 points]

(b) If it is known that at $t = 0$ s the conduction current density within the sphere is given by $\mathbf{J}(R, t = 0) = \frac{\rho_0 \sigma_0}{3\varepsilon_r\varepsilon_0} R\mathbf{a}_R$, determine the expression for the conduction current density for $t \geq 0$ s. Hint: Assume this current density is only a function of R. [3 points]

(c) Find the ratio of the magnitude of the conduction current density relative to the magnitude of the displacement current density for $t \geq 0$ s. [3 points]

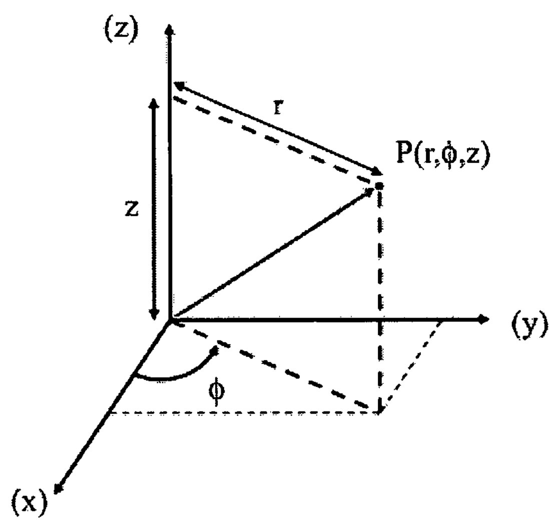
1 Coordinate Systems

1.1 Cartesian coordinates



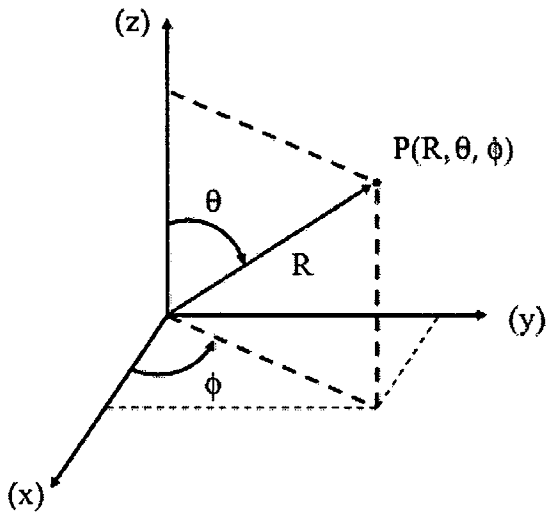
Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates



Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates



Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2 Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3 Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4 Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

5 Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

5.4 Curl

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

6 Electromagnetic formulas

Table 1 Electrostatics

$$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3}dQ'$$

$$\nabla \times \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{D} = \rho_v$$

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0 \qquad \oint_S \mathbf{D}\cdot d\mathbf{S} = Q$$

$$\mathbf{E} = -\nabla V \qquad V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{|\mathbf{R}-\mathbf{R}'|}$$

$$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2}\mathbf{E}\cdot d\mathbf{l}$$

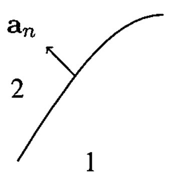
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E} \qquad \mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$$

$$\rho_{p,v} = -\nabla\cdot\mathbf{P}$$

$$\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{a}_n\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

$$E_{1,t} = E_{2,t}$$



$$Q = CV \qquad W_e = \frac{1}{2}QV$$

$$W_e = \frac{1}{2}\int_{vol}\rho_v V\,dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}\,dv$$

$$\nabla\cdot(\varepsilon\nabla V) = -\rho_v \qquad \nabla\cdot(\varepsilon\nabla V) = 0$$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$