University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test
9:00 – 10:45, 21 November 2019
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

Solutions

JWD

1) Evaluate the integrals:

a) $\int_0^4 (4-t)\sqrt{t}\,dt$ b) $\int \frac{1-\sin^3 x}{\sin^2 x}$ c) $\int \frac{\cos(\ln x)}{x}dx$

d) $\int (x^5 + 5^x)dx$ e) $\int \frac{e^x}{(1-e^x)^2}dx$

a) $\int_0^4 (4-t)\sqrt{t}\,dt = \int_0^4 \left(4t^{1/2} - t^{3/2}\right)dt = \left[4\cdot\frac{2}{3}t^{3/2} - \frac{2}{5}t^{5/2}\right]_0^4 = \frac{64}{3} - \frac{64}{5} = \frac{128}{15}$

b) $\int \frac{1-\sin^3 x}{\sin^2 x}dx = \int \csc^2 x\,dx - \int \sin x\,dx = -\cot x + \cos x + C$

c) $\int \frac{\cos(\ln x)}{x}dx$ let $u = \ln x$ $du = \frac{dx}{x}$

$= \int \cos u\,du = \sin u + C = \sin(\ln x) + C$

d) $\int (x^5 + 5^x)dx = \frac{x^6}{6} + \frac{5^x}{\ln 5} + C$

e) $\int \frac{e^x}{(1-e^x)^2}dx$ let $u = 1-e^x$ $du = -e^x dx$

$= -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{1-e^x} + C$

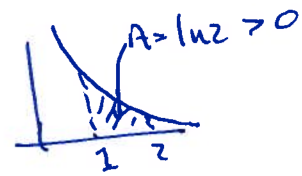
2) **Prove that the logarithm function is unbounded below; that is, prove** $\lim_{x \to 0^+} \ln(x) = -\infty$

Let N be a large $-^{ve}$ number

$\Rightarrow$ show that for $0 < x < x_0$, $\ln x < N$

Consider $x = \frac{1}{2} \Rightarrow \ln \frac{1}{2} = -\ln 2$

where $\ln 2$ is a $+^{ve}$ number :



$\therefore$ there is some $+^{ve}$ integer n, st $n \cdot \ln\frac{1}{2} < N$

$\therefore$ choose $x_0 = \left(\frac{1}{2}\right)^n$

Proof : given $N < 0$, choose $x_0 = \left(\frac{1}{2}\right)^n$

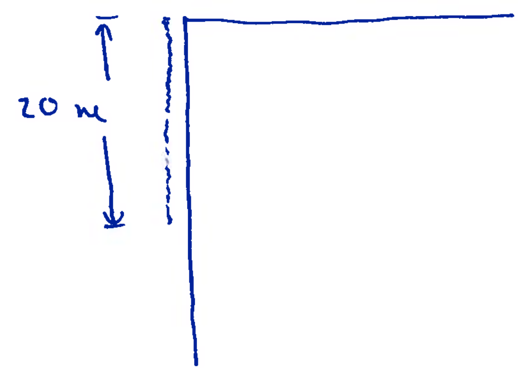
$\therefore \ln x < N$ for $0 < x < x_0 = \left(\frac{1}{2}\right)^n$

$\therefore \lim_{x \to 0^+} \ln x = -\infty$ By the definition of a negative infinite limit.

3) Suppose h is a function such that $h(1) = -2,\ h'(1) = 2,\ h''(1) = 3,\ h(2) = 6,\ h'(2) = 5,$ $h''(2) = 13$, and h'' is continuous everywhere. Evaluate $\int_1^2 h''(u)du$.

$$\int_1^2 h''(u)\,du = \left[h'(u)\right]_1^2 = h'(2) - h'(1) = 5 - 2 = 3$$

4) A heavy rope, 20 m long, weighs 0.5 kg/m and hangs over the edge of a building, 50 m high.
a) How much work is done in pulling the rope to the top of the building?
b) How much work is done in pulling half the rope to the top of the building?



a) $$W = \int_0^{20} \underbrace{\lambda g \, dx}_{\text{weight}} \cdot \underbrace{x}_{\text{distance}} = \left[\frac{\lambda g x^2}{2} \right]_0^{20} = 0.5 \frac{kg}{m} \cdot 9.8 \, m/s^2 \cdot \frac{400 \, m^2}{2}$$

$$= 980 \, kg \, m^2/s^2$$

$$= 980 \, J$$

b) $$W = \int_0^{10} \lambda g x \, dx + \underbrace{10 \lambda g}_{\text{weight}} \cdot \underbrace{10}_{\text{distance}}$$

$$= \left[\frac{\lambda g x^2}{2} \right]_0^{10} + 100 \cdot 0.5 \cdot 9.8$$

$$= 0.5 \cdot 9.8 \cdot \frac{100}{2} + 0.5 \cdot 9.8 \cdot 100 = 735 \, J$$

5) Consider the region bounded by the curves: $y = (x-1)^2$ and $y = x + 1$. Find, but do NOT solve, integrals which represent:

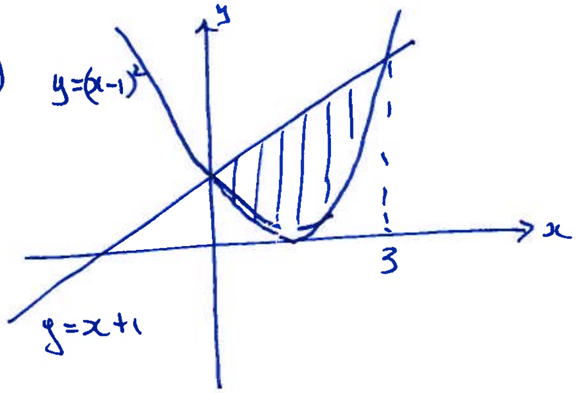
a) The area of this region.

b) The volume formed when the region is rotated about the *x*-axis, formulated using the washer method.

c) The volume formed when the region is rotated about the *y*-axis, formulated using the shell method.

Provide a sketch of the regions or volumes in each part of this question.

a)



Intersection: $(x-1)^2 = x+1$

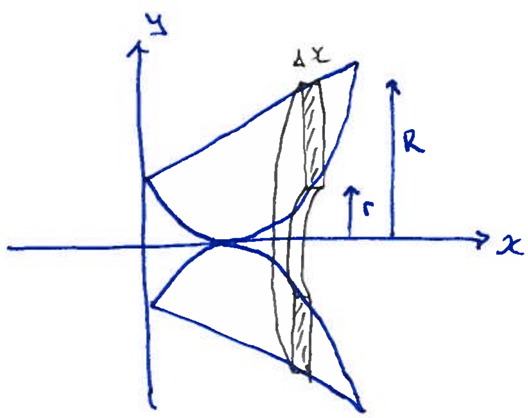
$x^2-2x+1 = x+1$

$x^2 = 3x \Rightarrow x=0$

$x=3$

$$A = \int_0^3 \left((x+1) - (x-1)^2\right) dx$$

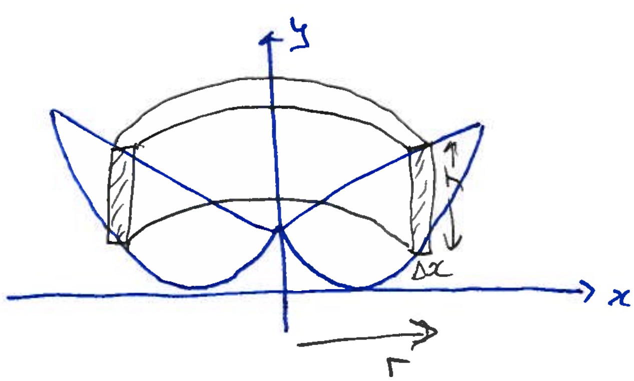
b)



$$V = \int \pi \left(R^2 - r^2\right) dx$$

$$= \int_0^3 \pi \left((x+1)^2 - (x-1)^4\right) dx$$

c)



$$V = \int 2\pi r \cdot h \Delta x$$

$$= \int_0^3 2\pi x \left((x+1) - (x-1)^2\right) dx$$

6) Directly calculate the limit of a Riemann sum to evaluate the areas of the following regions:

a) The area between the line $y = 1 - x$, $x \in [0,1]$ and the x-axis

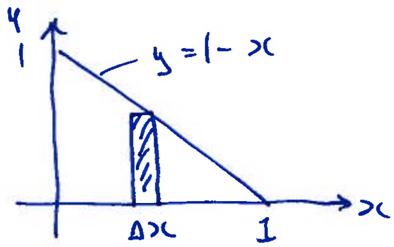
b) The area between the x-axis and the curve $y = 1 - x^2$, $x \in [2,3]$

Confirm both results using the Fundamental Theorem of Calculus.

Hint: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

a) use uniform partition, RH end point. $\Rightarrow$ $\Delta x = \frac{1}{n}$

$x_i^* = x_i = \frac{i}{n}$



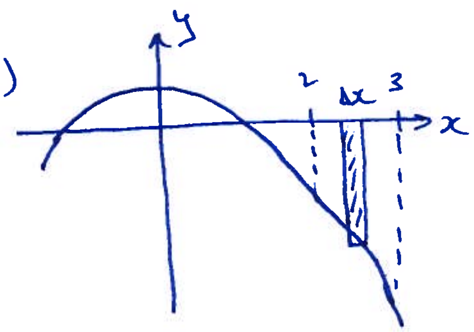
$A_i = (1 - x_i)\,\Delta x_i$

$A \simeq \sum A_i = \sum_{i=1}^{n} (1 - x_i)\,\Delta x_i$

$\therefore A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} 1 - \frac{1}{n^2}\sum_{i=1}^{n} i$

$= \lim_{n\to\infty} \frac{n}{n} - \lim_{n\to\infty} \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = 1 - \frac{1}{2} = \frac{1}{2}$

b)



$\Delta x_i = \frac{3-2}{n} = \frac{1}{n}$

$x_i = 2 + \frac{i}{n}$

$A_i = -\left(1 - x_i^2\right)\Delta x_i$

$= \left(x_i^2 - 1\right)\Delta x_i$

$A \simeq \sum_{i=1}^{n} \left(\left(2 + \frac{i}{n}\right)^2 - 1\right)\cdot\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} + \frac{i^2}{n^2} - 1\right)$

$\Rightarrow A = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{4i}{n} + \frac{i^2}{n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{3n}{n} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right)$

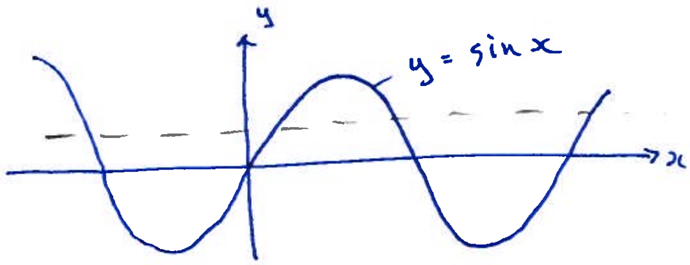
$= 3 + \frac{4}{2} + \frac{2}{6} = \frac{16}{3}$

a) $\int_0^1 (1-x)\,dx = \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$

b) $\int_2^3 -(1-x^2)\,dx = \left[\frac{x^3}{3} - x\right]_2^3 = 9 - \frac{8}{3} - 3 + 2 = \frac{16}{3}$

7) Show that $h(x) = sinx, x \in \mathbb{R}$, is not one-to-one, but its restriction $f(x) = sinx, -\pi/2 \le x \le \pi/2$, is one-to-one. <u>Use implicit differentiation</u> to compute the derivative of $f^{-1} = sin^{-1}$.

a)

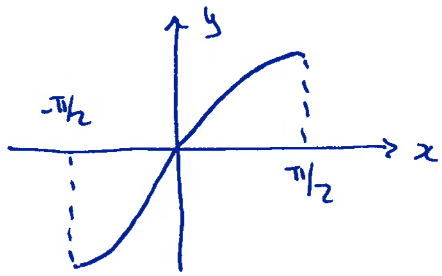


$\Rightarrow$ horizontal line test fails

$\Rightarrow$ or $\sin x = \sin(x + 2n\pi)$

$\therefore$ not 1-1

b) $\sin x, \ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$



$\Rightarrow$ horizontal line test works

$\Rightarrow$ or : $(\sin x)' = \cos x > 0$ $\quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore$ increasing, $\therefore$ 1-1

c) inverse function: let $y = \sin^{-1} x \Rightarrow \sin y = x$

implicit differentiation: $\cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

$\Rightarrow \cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y} = \pm\sqrt{1-x^2}$

but $\cos y \ge 0$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ $\quad \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\therefore (\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$

8) Consider the function: $f(x) = \ln\left(\frac{x^4}{x-1}\right)$

a) Determine the domain of f.

b) Find the intervals in which f increases or decreases.

c) Find the extreme values.

d) Determine the concavity of the graph, and find the inflection points.

e) Sketch the graph specifying the asymptotes, if any.

a) Domain: $\frac{x^4}{x-1} > 0 \Rightarrow x-1 > 0 \Rightarrow x > 1$ or $x \in (1, \infty)$

b) $f'(x) = \frac{4x^3}{x^4} - \frac{1}{x-1} = \frac{4}{x} - \frac{1}{x-1} = \frac{4x-4-x}{x(x-1)} = \frac{3x-4}{x(x-1)}$

$f''(x) = \frac{-4}{x^2} + \frac{1}{(x-1)^2} = \frac{x^2 - 4(x-1)^2}{x^2(x-1)^2} = \frac{x^2-4x^2+8x-4}{x^2(x-1)^2} = \frac{-3x^2+8x-4}{x^2(x-1)^2}$

$= -\frac{(3x-2)(x-2)}{x^2(x-1)^2}$

$\Rightarrow f'(x) = 0 \Rightarrow x = \frac{4}{3} \Rightarrow x \in (1, \frac{4}{3}) : f'(x) < 0 \therefore$ decr

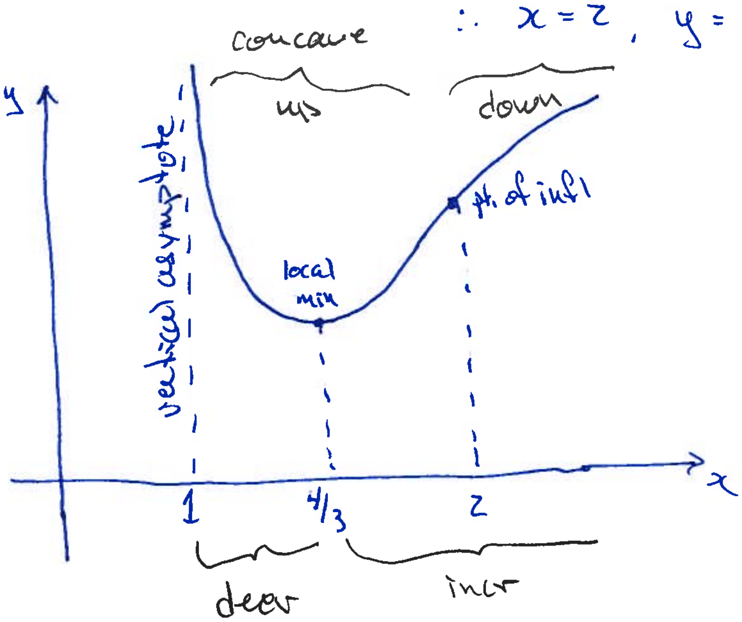
$x \in (\frac{4}{3}, \infty) : f'(x) > 0 \therefore$ incr

c) $f(\frac{4}{3}) = \ln\left(\frac{4^4}{3^4} \cdot \frac{3}{1}\right) = 4\ln 4 - 3\ln 3$ $(\simeq 2.25)$ $\Rightarrow$ local minimum

d) $f''(x) = 0 \Rightarrow \underbrace{x = \frac{2}{3}}_{\text{not in Domain}}, x = 2 \Rightarrow x \in (1,2) : f''(x) > 0 \therefore$ concave up

$x \in (2, \infty) : f''(x) < 0 \therefore$ concave down

$\therefore x = 2, y = \ln 2^4$ is a pt. of inflection

e) 

$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \frac{3x-4}{x(x-1)} \to -\infty$

$\therefore$ vertical asymptote

for large x: $f(x) \to 3 \ln x$

$f(\frac{4}{3})$ = Abs minimum

No Abs maximum

9) a) Let f be differentiable on (a, b) and continuous on $[a, b]$.

i) Prove that if there is a constant M such that $f'(x) \leq M$ for all $x \in (a, b)$ then
$$f(b) \leq f(a) + M(b - a)$$

ii) Prove that if there is a constant m such that $f'(x) \geq m$ for all $x \in (a, b)$ then
$$f(b) \geq f(a) + m(b - a)$$

iii) Parts (a) and (b) imply that if there exists a constant L such that $|f'(x)| \leq L$ on (a, b), then
$$f(a) - L(b - a) \leq f(b) \leq f(a) + L(b - a)$$
Verify this result.

b) Consider the function $f(x) = \begin{cases} x + 1, x \geq 1 \\ x - 1, x < 1 \end{cases}$.
Does $f(x)$ have an inverse? If not, why not? If it does have an inverse, find $f^{-1}(x)$.

a i) $f'(x) \leq M \quad , \quad x \in (a, b)$

MVT : $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad ; \quad c \in (a, b)$

$\therefore f'(c) \leq M \Rightarrow \dfrac{f(b) - f(a)}{b - a} \leq M \Rightarrow f(b) \leq f(a) + M(b - a)$

ii) $f'(x) \geq m \quad , \quad x \in (a, b)$

$\therefore f'(c) \geq m \Rightarrow \dfrac{f(b) - f(a)}{b - a} \geq m \Rightarrow f(b) \geq f(a) + m(b - a)$

iii) $|f'(x)| \leq L \Rightarrow -L \leq f'(x) \leq L \qquad x \in (a, b)$

RHS $\Rightarrow f(b) \leq f(a) + L(b - a)$

LHS $\Rightarrow f(b) \geq f(a) - L(b - a)$

$\therefore f(a) - L(b - a) \leq f(b) \leq f(a) + L(b - a)$

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b) $f(x) = \begin{cases} x+1 & x \geq 1 \\ x-1 & x < 1 \end{cases} \Rightarrow f'(x) = 1 \quad \therefore$ increasing $\therefore$ 1-1
$\therefore f(x)$ has an inverse

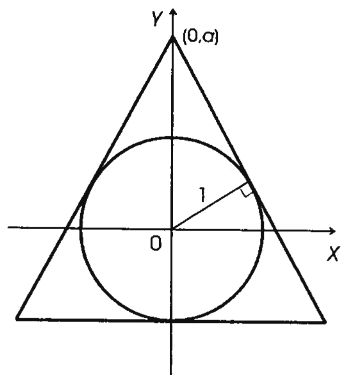
$\Rightarrow x \geq 1$ (Range $y \geq 2$) : $x = y + 1 \Rightarrow y = f^{-1}(x) = x - 1 \quad x \geq 2$

$x < 1$ (Range $y < 0$) : $x = y - 1 \Rightarrow y = f^{-1}(x) = x + 1 \quad x < 0$

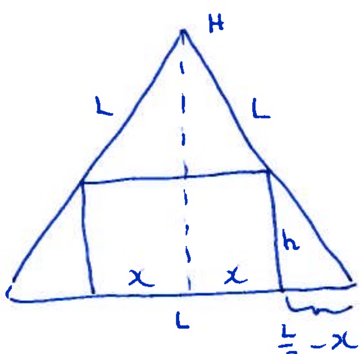
$$\Rightarrow f^{-1}(x) = \begin{cases} x - 1 & x \geq 2 \\ x + 1 & x < 0 \end{cases}$$

10) a) (less difficult) Find the dimensions of the rectangle of largest area that can be inscribed in an equilateral triangle of side L if one side of the rectangle lies on the base of the triangle.

b) (more difficult) An isosceles triangle is circumscribed about the unit circle so that the equal sides meet at the point $(a, 0)$ on the y-axis (see the figure). Find the value of a that minimizes the lengths of the equal sides.



a)



$$A = 2x \cdot h$$

$$H = \sqrt{L^2 - (L/2)^2}$$

$$\frac{h}{H} = \frac{L/2 - x}{L/2} \quad \text{(similar triangles)}$$

$$\therefore\ h = \sqrt{L^2 - (L/2)^2}\left(1 - \frac{2x}{L}\right)$$

$$\therefore\ A = H \cdot 2x\left(1 - \frac{2x}{L}\right) = H\left(2x - \frac{4x^2}{L}\right)$$

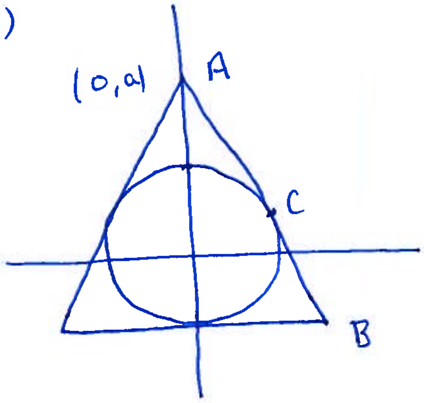
$$\frac{dA}{dx} = H\left(2 - \frac{8x}{L}\right) = \frac{2H}{L}(L - 4x) \Rightarrow \frac{dA}{dx} = 0 \Rightarrow x = \frac{L}{4}$$

Maximum as $A \to 0$ as $x \to 0$ or $x \to \frac{L}{2}$

$$\therefore\ A^{max} = 2 \cdot \frac{L}{4}\sqrt{L^2 - (L/2)^2}\left(1 - \frac{2}{L} \cdot \frac{L}{4}\right) = \frac{L^2}{4}\left(1 - \left(\frac{1}{2}\right)^2\right)^{1/2} = \frac{L^2\sqrt{3}}{8}$$

$$h = \frac{L\sqrt{3}}{4}$$

10 b)



Equation of line AB : $y = a + \alpha x$

where α is the slope of the tangent to the circle at C:

$$y = \sqrt{1-x^2} \Rightarrow y' = \tfrac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1-x^2}} = \alpha$$

Intersection pt. C: $a - \dfrac{x}{\sqrt{1-x^2}} = \sqrt{1-x^2} \Rightarrow a\sqrt{1-x^2} = 1 - x^2 + x^2 = 1$

$$\Rightarrow a^2(1-x^2) = 1 \Rightarrow 1 - x^2 = \frac{1}{a^2} \Rightarrow x = \sqrt{1 - 1/a^2}$$

(x is +ve in 1st quadrant)

$$\therefore \alpha = \frac{-\sqrt{1-1/a^2}}{1/a} = -\sqrt{a^2-1}$$

$\therefore$ equ'n of line AB: $y = a - \sqrt{a^2-1}\, x$

Intersection with line $y = -1 \Rightarrow -1 = a - \sqrt{a^2-1}\, x$

$$a+1 = \sqrt{a^2-1}\, x \Rightarrow x = \frac{a+1}{\sqrt{a^2-1}}$$

length of side AB : $l^2 = (a+1)^2 + \left(\dfrac{a+1}{\sqrt{a^2-1}}\right)^2 = (a+1)^2\left(1 + \dfrac{1}{a^2-1}\right)$

$$= (a+1)^2\left(\frac{a^2}{a^2-1}\right) = (a^3+a^2)(a-1)^{-1}$$

$$\frac{dl^2}{da} = (3a^2+2a)(a-1)^{-1} + (a^3+a^2)(a-1)^{-2}(-1) = \frac{(3a^2+2a)(a-1) - (a^3+a^2)}{(a-1)^2}$$

$$= \frac{a}{(a-1)^2}(3a^2 - 3a + 2a - 2 - a^2 - a) = \frac{2a(a^2-a-1)}{(a-1)^2}$$

$$\frac{dl^2}{da} = 0 \Rightarrow a = 0 \text{ or } a = \frac{1 \pm \sqrt{1+4}}{2}$$

but $a > 1$ $\therefore$ $\boxed{a = \dfrac{1+\sqrt{5}}{2}}$