MAT292 – Fall 2022

Quiz – October 3, 2022

Time alotted: 50 minutes

Full Name ______________________________

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Signature ______________________________

DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This quiz contains 7 pages (including this title page). Once the quiz starts, make sure you have all of them.

In the first section, only answers are required.

In the second section, justify your answers fully.

You can use pages 6–7 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

GOOD LUCK!

SECTION I   Provide only the final answer.

1. **(1 mark)** Newton's law of cooling is a homogeneous ODE.
   ○ True   ○ False
   Solution: False. Newton's Law of Cooling can be written as

$$u' + ku = kT_0$$

   Since the right hand side is not zero, the ODE is nonhomogeneous.

2. **(2 marks)** Consider a first-order autonomous ODE $y' = f(y)$. Suppose y_1 is an equilibrium solution.

   If $f'(y_1) < 0$ then y_1 is a... ○ stable equilibrium ○ unstable equilibrium ○ more information is needed

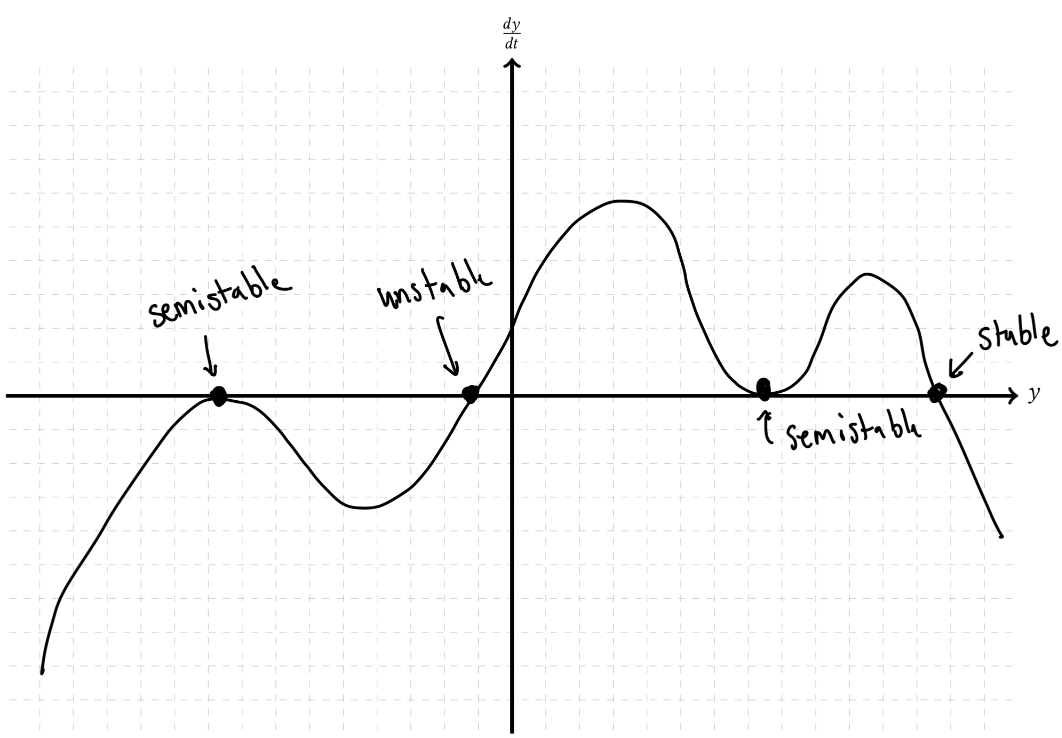
   If $f'(y_1) = 0$ then y_1 is a... ○ stable equilibrium ○ unstable equilibrium ○ more information is needed

   Solution: If $f'(y_1) < 0$, then y_1 is a stable equilibrium. If $f'(y_1) = 0$, then we need more information as y_1 could be either stable, unstable, or semistable.

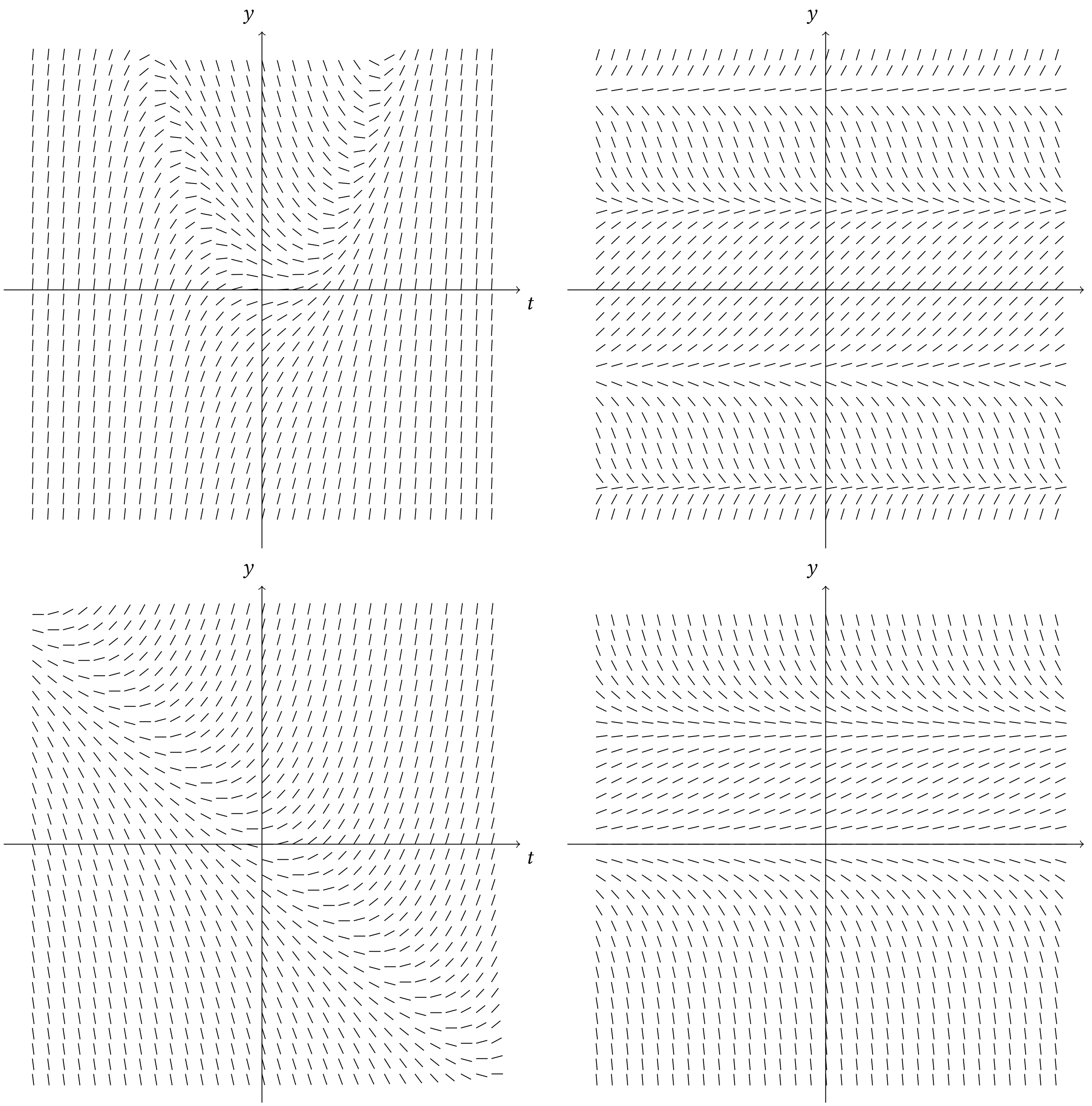
3. **(4 marks)** Consider an autonomous ODE of the form

$$\frac{dy}{dt} = f(y)$$

   - Graph a function f such that there exists an equilibrium solution of each of the four types. **You do not have to find an expression for f.**
   - In the plot of f, label the equilibrium points with the type of equilibrium that they are (you do not need to distinguish between the types of semistable equilibrium).



4. (4 marks) All four direction fields correspond to first order ODEs. Circle all those that correspond to an **autonomous** ODE. Circling one corresponding to a non-autonomous ODE will deduct your score for this question by 2 points (0 is the lowest score you can get).



Solution: The two phase portraits on the right are those corresponding to autonomous ODE. This can be determined by noticing that the slope of the solutions do *not* depend on t.

SECTION II Justify all your answers.

5. For this question, consider the following system of ODEs:

$$\begin{cases} \frac{dx}{dt} = \frac{xe^{t^2}}{t^2-4} + \frac{1}{t^2-1} \\ \frac{dy}{dt} = txy^2 \\ \frac{dz}{dt} = xz + y \end{cases}$$

with the following initial conditions:

$$x(0) = 1, \quad y(0) = 2, \quad z(0) = 1$$

Answer the following questions using the Existence and Uniqueness Theorems and give the strongest possible answer that can be guaranteed by the two theorems.

NOTE: For this question you should **not** use the existence and uniqueness theorems for **systems of ODEs**.

(a) **(3 marks)** Explain why there exists a unique solution for x. Describe what we know about its domain.

Solution: Looking at the first equation in the system, we see that it does not depend on y or z so we can use this equation to solve for x. Furthermore, this is a linear ODE so we can apply the Existence and Uniqueness theorem for Linear ODE. The largest interval that contains $t = 0$ where $\frac{xe^{t^2}}{t^2-4}$ and $\frac{1}{t^2-1}$ are both continuous is $(-1, 1)$. Thus we can conclude that a unique solution for $x(t)$ satisfying $x(0) = 1$ exists on the interval $(-1, 1)$.

(b) **(1 marks)** Explain why the unique solution to $x(t)$ is differentiable (and therefore continuous).

Solution: By part a), we know $x(t)$ solves the first equation. Thus, it has to be differentiable on its domain $(-1, 1)$. Since differentiability implies continuity, we know it is also continuous.

(c) **(3 marks)** Explain why there exists a unique solution for y. Describe what we know about its domain.

Solution: Looking at the second equation of the system, we see it does not depend on z. So we can solve the second equation for y (notice that x has already been *uniquely* determined by the first equation). This is a non-linear first order ODE so we have to use E&U for nonlinear ODEs. In this case:

$$f(t, y) = tx(t)y^2$$

and

$$\frac{\partial f}{\partial y} = 2yt \cdot x(t)$$

Notice that since $x(t)$ is only continuous on the interval $(-1, 1)$, the rectangle where f and $\frac{\partial f}{\partial y}$ are continuous is

$$(-1, 1) \times (-\infty, \infty)$$

Since this rectangle contains the initial condition point $(0, 2)$, we can conclude that there exists a unique solution $y(t)$ on some subinterval of $(-1, 1)$ that contains $t = 0$ such that $y(0) = 2$.

(d) **(2 marks)** Does there exist a unique solution for z? If yes, describe what we know about its domain. If not, justify why.

Solution: Yes. Looking at the third equation, we can see that the equation is linear with respect to z. Using the existence and uniqueness theorem for linear ODEs, by part a) and part c), we can conclude that there will be a unique solution for z that is defined on the *same* subinterval as the domain of y from part c).

(e) **(1 marks)** Suppose we change the initial conditions to the following:

$$x(0) = 1, \quad y(0) = 2, \quad z(1) = 3$$

Is there anything that needs to be changed in regards to your answers in parts (a), (b), and (c)? Justify your answers.

Solution: We would need to change our answer for part d). Since $x(t)$ is not necessarily continuous at $t = 1$, we can no longer use the existence and uniqueness theorem for linear ODEs to guarantee the existence or uniqueness of a solution for z that satisfies the initial condition $z(1) = 3$. Thus, the system might not necessarily have a solution anymore (including x and y when viewed as a system).

<u>**Note:**</u> If you **clearly justified** that the first two equations and initial conditions of x and y are unchanged and independent of z, then you will also be given the point.

6. A colony of y_0 people inhibited planet Mars. The population reproduces in a seasonal manner. A student in MAT292 proposed the following mathematical model for the size of the human population:

$$y' = y - 2000\sin(t), \qquad y(0) = y_0.$$

(a) **(3 marks)** Verify the solution to the initial value problem is given by

$$y(t) = (y_0 - 1000)e^t + 1000\sin(t) + 1000\cos(t).$$

Solution: We plug the solution into both sides of the ODE. The left hand side becomes

$$LHS = (y_0 - 1000)e^t + 1000\cos(t) - 1000\sin(t)$$

The right hand side becomes

$$\begin{aligned} RHS &= (y_0 - 1000)e^t + 1000\sin(t) + 1000\cos(t) - 2000\sin(t) \\ &= (y_0 - 1000)e^t - 1000\sin(t) + 1000\cos(t) \\ &= LHS \end{aligned}$$

We need to check that the initial condition is satisfied. If we substitute $t = 0$ into the given solution we get:

$$y(0) = y_0 - 1000 + 1000 = y_0$$

Thus, we have verified that $y(t)$ is a solution to the initial value problem.

(b) **(2 marks)** What initial condition y_0 will guarantee $y(t)$ remains bounded as t tends to infinity? Does this initial condition lead to a physically meaningful solution?
Solution: We need $y_0 = 1000$. With this initial condition, the solution will oscillate between negative and positive values. In the context of the model, this is not physically meaningful as populations can not be negative.

(c) **(1 marks)** What interval of initial conditions guarantees the population will reach **over** 10,000 individuals at some point in time (regardless of whether or not the solution is physically meaningful).
Solution: We need $y_0 > 1000$.

(d) **(1 marks)** Does the ODE have equilibrium points? If so, classify them.
Solution: No the ODE does not have any equilibrium solutions. This can be determined by looking at the general solution given in part a) or by observing that there is no value of y such that $y - 2000\sin(t) = 0$ for **all** $t \in \mathbb{R}$.

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