University of Toronto; Division of Engineering Science

STA286S: Probability and Statistics Term Test

Monday, February 27, 2014, 9:10-11am

Examiners: B. Donmez and K. Knight

Name:____

Student Number:___________________________________

Lecture Section: (**circle one**)

- LEC01 (Prof. Knight)
- LEC02 (Prof. Donmez)

Tutorial Section: (**circle one**)

Tutorial	Time	Location	TA
TUT01	Mon 1-2 pm	BA2159	Maryam Merrikhpour
TUT02	Mon 1-2 pm	BA3008	Karen Wong
TUT03	Th 5-6 pm	BA2155	Karen Wong
TUT04	Wed 1-2 pm	WB144	Wayne Giang
TUT05	Mon 2-3 pm	BA3116	Maryam Merrikhpour
TUT06	Fri 10-11 am	BA3008	Zheng Li
TUT07	Tues 1-2 pm	BA3008	Zheng Li
TUT08	Tues 1-2 pm	BA3012	Wayne Giang

Instructions:

- **Time allowed:** 1 hour and 50 minutes.
- **Aids:** a non-programmable calculator and a one-sided A4 size aid sheet.
- There are seven questions. Carefully proportion your time among them. If you do not understand a question, or are having some other difficulty, do not hesitate to ask for clarification.
- There are 12 pages including this page. The last page contains the standard normal table. Please ensure that you are not missing any pages.
- Points for each question are indicated in parenthesis. Total points: 100.

Question	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**Total**
Max	**20**	**24**	**10**	**6**	**13**	**15**	**12**	**100**
Score								

GOOD LUCK!

Question 1: Two teams, the Jets and the Sharks, play a best 4 out of 7 series. (The series ends when one team has won 4 games.) Suppose the outcomes of the games are independent and for each game, the probability that the Jets win is 0.3.

a) (8 points) Suppose you are told that the series has ended in 6 games. What is the probability that the Sharks have won the series given this information?

Partition the sample space into two events: A = Sharks win the series, B = Jets win the series. Define event M: series ends in 6 games.

$$\boldsymbol{P(A|M) = \frac{P(A \cap M)}{P(M)} = \frac{P(A \cap M)}{P(A \cap M) + P(B \cap M)}}$$

$$\boldsymbol{P(A \cap M) = \binom{5}{2}\ \underbrace{0.7^4 0.3^2}}$$

Multiply the probabilities of four game wins by Sharks and two game wins by Jets (independent events).

If 6 games have been played, we know that the last game has to be won by the Sharks. For the first 5 games, we can order the three wins by Sharks and two wins by Jets in $\binom{5}{2}$ =$\binom{5}{3}$ different ways.

$$\boldsymbol{P(B \cap M) = \binom{5}{2}\ 0.7^2 0.3^4}$$

$$\boldsymbol{P(A|M) = \frac{P(A \cap M)}{P(A \cap M) + P(B \cap M)} = \frac{0.21609}{0.21609 + 0.03969} = 0.844828}$$

b) (6 points) What is the probability that the Jets win the series?

Partition the sample space into four events:
K: series ends in 4 games
L: series ends in 5 games
M: series ends in 6 games
N: series ends in 7 games

Define event B: Jets win the series.

$$\boldsymbol{P(B) = P(B \cap K) + P(B \cap L) + P(B \cap M) + P(B \cap N)}$$

$$\boldsymbol{P(B) = 0.3^4 + \binom{4}{1}\, 0.3^4\, 0.7 + \binom{5}{2}\, 0.3^4\, 0.7^2 + \binom{6}{3}\, 0.3^4\, 0.7^3}$$
$$\boldsymbol{= 0.0081 + 0.02268 + 0.03969 + 0.055566 = 0.126036}$$

c) (6 points) Let *X* be the number of games played in the series. Find the probability function of *X*. (The possible values of *X* are 4, 5, 6, and 7.)

Partition the sample space into two disjoint events: A = Sharks win the series, B = Jets win the series.

$$\boldsymbol{P(X = 4) = P((X = 4) \cap A) + P\big((X = 4) \cap B\big) = 0.7^4 + 0.3^4 = 0.2401 + 0.0081 = 0.2482}$$

$$\boldsymbol{P(X = 5) = P\big((X = 5) \cap A\big) + P\big((X = 5) \cap B\big) = \binom{4}{1}\, 0.7^4\, 0.3 + \binom{4}{1}\, 0.3^4\, 0.7}$$
$$\boldsymbol{= 0.28812 + 0.02268 = 0.3108}$$

$$\boldsymbol{P(X = 6) = P\big((X = 6) \cap A\big) + P\big((X = 6) \cap B\big) = \binom{5}{2}\, 0.7^4 0.3^2 + \binom{5}{2}\, 0.3^4\, 0.7^2}$$
$$\boldsymbol{= 0.21609 + 0.03969 = 0.25578}$$

$$\boldsymbol{P(X = 7) = P((X = 7) \cap A) + P\big((X = 7) \cap B\big) = \binom{6}{3}\, 0.7^4\, 0.3^3 + \binom{6}{3}\, 0.3^4\, 0.7^3}$$
$$\boldsymbol{= 0.129654 + 0.055566 = 0.18522}$$

Question 2: A student writes a multiple choice test consisting of 20 questions where each question has 4 options. Unfortunately, the student has spent the past two weeks watching the Winter Olympics and is forced to make a blind guess on each question.

a) (7 points) Find the probability that this student gets 10 or more correct answers on the test. For this question only, it is acceptable if you do not calculate the final probability value as long as you have the correct equation written out.

Let X = the number of correct answers, Y = the number of wrong answers; both follow the binomial distribution. The parameters for Y are: n = 20, p = 0.75.

Then,

$$P(X \geq 10) = P(Y \leq 10) = \sum_{k=0}^{10} P(Y = k) = \sum_{k=0}^{10} \binom{20}{k} (\tfrac{3}{4})^k (\tfrac{1}{4})^{20-k} = 0.01386442 \approx 0.0139$$

b) (7 points) It turns out that 200 students find themselves in the same predicament. Find the **exact** probability that at least 4 students get 10 or more correct answers.

Let V = the number of students getting 10 or more correct answers. V follows the binomial distribution with n = 200 and p = 0.0139.

$$\boldsymbol{P(V \geq 4) = 1 - P(V \leq 3) = 1 - \sum_{k=0}^{k=3} \binom{200}{k} 0.0139^k \times (1 - 0.0139)^{200-k} = 1 - 0.696731}$$

$$\boldsymbol{= 0.30369}$$

c) (5 points) Use an appropriate approximation to estimate the probability you calculated in part a of this question.

We can use the Normal approximation to calculate this probability.

$$\boldsymbol{P(Y \leq 10) = P\left(Z \leq \frac{10 + 0.5 - 20 \times 0.75}{\sqrt{20 \times 0.75 \times 0.25}}\right) = P(Z \leq -2.3238) \cong 0.01}$$

d) (5 points) Use an appropriate approximation to estimate the probability you calculated in part b of this question.

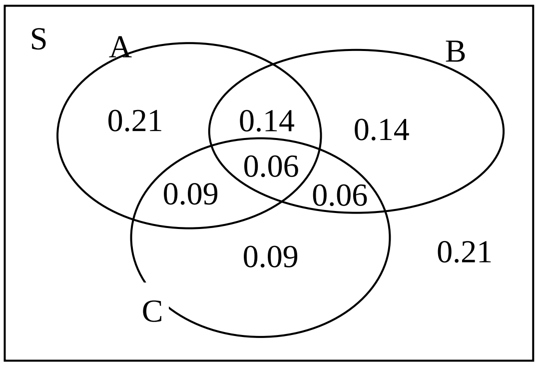
We can use the Poisson approximation to calculate this probability with Poisson mean of np=200x0.0139=2.78.

$$\boldsymbol{P(V \geq 4) = 1 - P(V \leq 3) = 1 - \sum_{k=0}^{k=3} \frac{e^{-2.78} 2.78^{k}}{k!} = 1 - 0.696384 = 0.30361}$$

Question 3: Suppose that A, B, and C are mutually independent events with $P(A) = 0.5$, $P(B) = 0.4$, and $P(C) = 0.3$.

a) (6 points) Find $P(A \cup (B \cap C'))$.

$$\boldsymbol{P(A \cap B) = P(A)P(B) = 0.5 \times 0.4 = 0.2}$$
$$\boldsymbol{P(A \cap C) = P(A)P(C) = 0.5 \times 0.3 = 0.15}$$
$$\boldsymbol{P(B \cap C) = P(C)P(B) = 0.3 \times 0.4 = 0.12}$$
$$\boldsymbol{P(A \cap B \cap C) = P(A)P(B)P(C) = 0.5 \times 0.4 \times 0.3 = 0.06}$$



$\boldsymbol{P(A \cup (B \cap C'))}$ **= 0.21+0.14+0.14 + 0.09+0.06= 0.64**

b) (4 points) Find the conditional probability $P(A|(C' \cup A))$.

$$\boldsymbol{P\left(A\middle|(C' \cup A)\right) = \frac{P(A\cap(C'\cup A))}{P(C'\cup A)} = \frac{0.5}{1-0.09-0.06} = \frac{0.5}{0.85} \cong 0.5882}$$

Question 4: (6 points) A class of size 20 consists of 16 right-handed and 4 left-handed people. Suppose that 10 people are selected at random (without replacement) from the class. What is the probability that we have 2 or fewer left-handed people in our sample of 10 people.

Let X = the number of left-handed people selected at random
X follows a hypergeometric distribution.

$$P(X \le 2) = \sum_{k=0}^{2} P(X = k) = \sum_{k=0}^{2} \frac{\binom{16}{10-k}\binom{4}{k}}{\binom{20}{10}} = 0.7089783 \approx 0.71$$

Question 5: Suppose that X and Y are independent random variables with Var(X) = σ_1^2 and Var(Y) = σ_2^2.

a) (5 points) Suppose Z = 5X + 3Y. What is Var(Z) in terms of σ_1^2 and σ_2^2?

$$Var(Z) = Var(5X + 3Y) = Var(5X) + Var(3Y) = 5^2 Var(X) + 3^2 Var(Y) = 25\sigma_1^2 + 9\sigma_2^2$$

b) (8 points) Let U = X + Y and V = X − Y. Under what condition on σ_1^2 and σ_2^2 is Cov (U, V) = 0?

Firstly, let's calculate Var(U) and Var(V):

$$Var(U) = Var(X + Y) = Var(X) + Var(Y) = \sigma_1^2 + \sigma_2^2$$
$$Var(V) = Var(X - Y) = Var(X) + Var(Y) = \sigma_1^2 + \sigma_2^2$$

Therefore,

$$Var(U + V) = Var(X + Y + X - Y) = Var(2X) = 2^2 Var(X) = 4Var(X) = 4\sigma_1^2$$
$$= Var(U) + Var(V) + 2Cov(U,V) = 2\sigma_1^2 + 2\sigma_2^2 + 2Cov(U,V)$$

Then we can conclude Cov(U,V) = 0 only if $\sigma_1^2 = \sigma_2^2$

Question 6: Suppose X and Y are continuous random variables with joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} \frac{6}{7}(x+y)^2, & for\ 0 \le x \le 1\ and\ 0 \le y \le 1 \\ 0, & otherwise \end{cases}$$

a) (5 points) Find the marginal pdf of X.

- when $0 \le x \le 1$ **,**

$$f_x(x) = \int_0^1 f_{x,y}(x,y)dy = \int_0^1 (\frac{6}{7})\ (\text{x} + \text{y})^2\ dy$$

$$= \left[(\frac{6}{7}).(\frac{1}{3})(x+y)^3 \right]_{y=0}^{y=1} = (\frac{2}{7})(x+1)^3 - (\frac{2}{7})x^3$$

- when x is o.w., $f_x(x) = 0$

b) (5 points) Are X and Y independent? **Mathematically** prove or disprove.

Firstly, let's calculated the marginal pdf of Y:

- **when** $0 \le y \le 1$

$$f_y(y) = \int_0^1 f_{x,y}(x,y)dx = \int_0^1 (\frac{6}{7})\ (\text{x} + \text{y})^2\ dy$$

$$= \left[(\frac{6}{7}).(\frac{1}{3})(x+y)^3 \right]_{x=0}^{x=1} = (\frac{2}{7})(y+1)^3 - (\frac{2}{7})y^3$$

- **when y is o.w.,** $f_y(y) = 0$

But obviously, when $0 \le x \le 1$ **and** $0 \le y \le 1$**,** $f_x(x) \bullet f_y(y) \ne f_{x,y}(x,y)$

As a conclusion, X and Y are <u>NOT</u> independent.

NOTE: One could tell that the two random variables are dependent as the joint pdf cannot be written as a product of two separate functions of X and Y. This response however only would identify that the two are independent. The question asks for a mathematical proof.

c) (5 points) Define V = max(X, Y) to be the maximum of X and Y. Find the pdf of V.

Because $V = Max\{X,Y\} \le v \Leftrightarrow Min\{X,Y\} \le Max\{X,Y\} \le v \Leftrightarrow X \le v, Y \le v$ **;**

$$P(V \le v) = P(X \le v, Y \le v) = \int_{y=0}^{v}\int_{x=0}^{v} f_{x,y}(x,y)dxdy = \int_{y=0}^{v}\int_{x=0}^{v}(\frac{6}{7})\,(\mathrm{x}+\mathrm{y})^2 dxdy$$

$$= (\frac{2}{7})\left[v^3 y + (\tfrac{3}{2})v^2 y^2 + v y^3\right]_{y=0}^{y=v} = v^4$$

$$\boldsymbol{f(v)} = \begin{cases} \mathbf{4}\boldsymbol{v^3}, & \boldsymbol{for}\ \mathbf{0} \le \boldsymbol{v} \le \mathbf{1} \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$$

Question 7: The amount of liquid a soft-drink company puts into its 2 litre bottles is normally distributed with mean 2.03 litres and variance σ^2.

a) (6 points) Suppose $\sigma^2 = 2.5 \times 10^{-4}$. What proportion of bottles will contain less than 2 litres of liquid?

Let X = amount of liquid in a 2 litre bottle. Then, X ~ N(2.03, σ^2=2.5 x 10^{-4}).

We want to find k such that: $P(X < 2) = k$ **.**

Using the fact that $Z = (X - 2.03)/\sigma$ **is a N(0,1) random variable. We have that,**

$$\boldsymbol{P}(\boldsymbol{X} < 2) = \boldsymbol{P}\left(\boldsymbol{Z} < \frac{\mathbf{2 - 2.03}}{\sqrt{\mathbf{2.5}} \times \mathbf{10^{-2}}}\right) \cong \boldsymbol{P}(\boldsymbol{Z} < -1.8974) \cong \boldsymbol{P}(\boldsymbol{Z} < -1.90) = \mathbf{0.0287}$$

b) (6 points) To avoid legal action, the company would like to adjust the filling mechanism (and hence σ^2) so that fewer than 0.1 % of bottles contain less than 2 litres. How small must σ (or σ^2) be for this to be true? (Assume that the mean is still 2.03 litres).

Let X = amount of liquid in a 2 litre bottle. Then, X ~ N(2.03, σ^2).

We want to find σ such that: $P(X < 2) = 0.001$**.**

Using the fact that $Z = (X - 2.03)/\sigma$ **is a N(0,1) random variable. We have that,**

$$\boldsymbol{P}(\boldsymbol{X} < 2) = \boldsymbol{P}\left(\boldsymbol{Z} < \frac{\mathbf{2 - 2.03}}{\boldsymbol{\sigma}}\right) = \mathbf{0.001}$$

From Table A.3 we find that $\frac{\mathbf{2-2.03}}{\boldsymbol{\sigma}}$ **has to equal -3.1 (-3.08 and -3.09 are also true).**

$$\boldsymbol{If}\ \frac{\mathbf{2 - 2.03}}{\boldsymbol{\sigma}} = -\mathbf{3.1}\ \boldsymbol{then}\ \boldsymbol{\sigma} \cong \mathbf{9.677} \times \mathbf{10^{-3}}$$

$$\boldsymbol{If}\ \frac{\mathbf{2 - 2.03}}{\boldsymbol{\sigma}} = -\mathbf{3.08}\ \boldsymbol{then}\ \boldsymbol{\sigma} \cong \mathbf{9.704} \times \mathbf{10^{-3}}$$

$$\boldsymbol{If}\ \frac{\mathbf{2 - 2.03}}{\boldsymbol{\sigma}} = -\mathbf{3.09}\ \boldsymbol{then}\ \boldsymbol{\sigma} \cong \mathbf{9.709} \times \mathbf{10^{-3}}$$

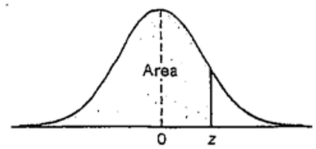


Table **A.3** Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
−1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
−1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
−1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
−1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
−1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
−0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
−0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
−0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
−0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
−0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

END!