ESC103F Engineering Mathematics and Computation: Tutorial #5

Question 1: Suppose *A*, *B* and *C* are *nxn* matrices and that both *A* and *B* commute with *C*. That is *AC*=*CA* and *BC*=*CB*. Prove that *AB* commutes with *C*.

Question 2: Suppose *P* and *Q* are *nxn* matrices. Prove that *PQ*=*QP* if and only if $(P-Q)(P+Q)=P^2-Q^2$.

Question 3: Consider the direction vector in R^2 given by $\vec{d}=\begin{bmatrix}1\\2\end{bmatrix}$.

a) From your lecture notes, write the transformation $P(\vec{u})$ which assigns each vector $\vec{u}=\begin{bmatrix}x\\y\end{bmatrix}$ to its projection on the line with direction vector $\vec{d}$.

b) Working with your answer to part a) and by examining the geometric relationship between projection and reflection of vectors, develop the transformation $S(\vec{u})$ which assigns each vector $\vec{u}$ to its reflection in the line with direction vector $\vec{d}$.

c) Develop the matrices associated with both transformations.

d) Determine if these matrices have inverses based on their determinants and, if so, find their inverses.

Question 4: Consider *A* to be the linear transformation with matrix $M_A=\begin{bmatrix}1 & -1\\2 & -2\end{bmatrix}$.

a) Show that if $\vec{u}$ is any vector in R^2 which lies on the line through the origin with direction vector $\begin{bmatrix}1\\1\end{bmatrix}$, then $A(\vec{u})=\vec{0}$.

b) Show that if $\vec{u}$ is any vector in R^2, then $A(\vec{u})$ lies on the line through the origin with direction vector $\begin{bmatrix}1\\2\end{bmatrix}$.

Question 5: Let A be a linear transformation and $M_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ its matrix. Assume λ is an eigenvalue of M_A and $\vec{u}$ is a corresponding eigenvector with $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Therefore $M_A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$.

a) Develop a polynomial equation of the form $\lambda^2 + \alpha\lambda + \beta = 0$ and expressions for the coefficients α and β in terms of a, b, c, d that must be solved for finding the eigenvalues.

b) Using the results from part a), determine the eigenvalues associated with the two matrices in Question 3c) and check to be sure these values match those developed in lecture based on geometric arguments.

Question 6: On two separate figures showing the x-y plane, plot the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the result of transforming this vector under the transformations summarized by the two matrices below. By inspection, determine the eigenvalue(s) and eigenvector(s) of the two matrices. On the same two figures, plot an example of each eigenvector and the result of transforming this eigenvector under the transformations summarized by the two matrices.

a) $\begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Question 7: By generalizing the method developed in lecture for the $2x2$ case, find the inverse N of the matrix M such that $NM = I$,

where $M = \begin{bmatrix} 5 & 5 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.