PHY294, Winter 2024, Midterm Exam (Thermal Physics)

Answer all questions on the exam paper. Total number of pages = 11. Duration: 90 minutes.

Name: _____; **Student #**: _____; **Tutorial group**: _____

Please read the questions very carefully. While every bit of information may be relevant, there may be some information thrown in that's not. To maximize your score, do the questions that seem easier first. Some formulae are found at page 10. You can use the space on pages 10 and 11 to scribble on before writing the official solution).

I. Consider a single-atomic ideal classical gas of N atoms. It is somehow engineered to undergo an isochoric quasistatic process to twice the initial pressure (recall that *isochoric* means constant volume).

1. How do you think such a process can be realized? In other words, what do your think is done to the gas so that the pressure increases while the volume is kept fixed.

   (This is not a trick question! A common sense qualitative answer suffices.)

   8 points

   Use the space below for problem **I.1**

   pressure can be increased by compressing, but since $V = $ const, the only choice is heating:

   N, V, T

   ← raise T ⟹ raise p
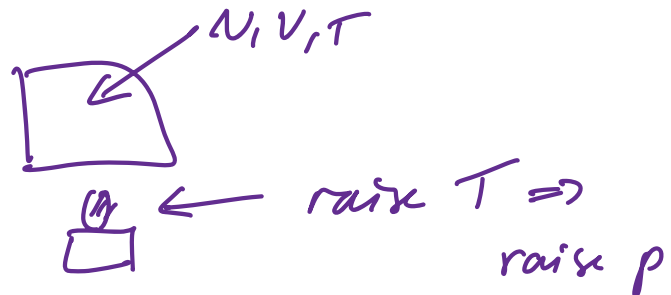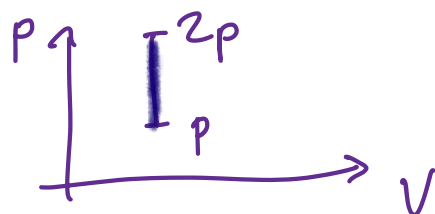
   End of space for problem **I.1**.

2. Is there work being done by (or on) the gas in this process? Is there any heat transferred to or from the gas? If you think that there is heat or work, determine their values, including their signs, in terms of p and V, the initial values of the pressure and volume of the gas.

Use the space below for problem **I.2**

- there's no work, as $V = $ const

$$\left(\begin{array}{c} \text{- or the area} \\ \text{under this} \\ \text{curve} = 0 \end{array} \right.$$

p ↑ ┃ ^{2p}
      ┃ $|_{p}$
  ─────────→ V

$\left. \begin{array}{c} \\ \\ \end{array} \right)$ ideal gas law

- there's heat : $pV = kNT_{initial}$ ⟸ holds

$$2pV = kNT_{final} \swarrow$$

$$\Rightarrow \quad T_{final} = 2T_{initial}$$

So $U_{final} = \frac{3}{2}kNT_{final} = \frac{3}{2} \cdot 2pV$

$V_{initial} = \frac{3}{2}kNT_{init} = \frac{3}{2}pV$

$U_{final} - U_{initial} = Q(\text{absorbed})_{\text{by gas}} = \frac{3}{2}pV(2-1)$

$$= \frac{3}{2}pV \quad \checkmark$$

End of space for problem **I.2**.

2

3. Do you expect the entropy of the gas to change? Determine the change of entropy in this isochoric process.

12 points

Use the space below for problem **I.3**

yes. $S \uparrow$ .. how? use S.-T. formula

since $U_{final} = 2 U_{initial}$ we have

$$S_{fin} - S_{init.} = k N \left(\ln \frac{V}{N} \left(\frac{4 \pi m U_{fin}}{3 N h^2} \right)^{3/2} \right.$$

$$\left. - \ln \frac{V}{N} \left(\frac{4 \pi m U_{in}}{3 N h^2} \right)^{3/2} \right)$$

$$= \frac{3}{2} k N \ln 2$$

$\left(\text{it's also legit to write } S(p) \text{ by subbing} \right.$

$\left. U = \frac{3}{2} p V \ \cdots \ \text{same results} \right).$

End of space for problem **I.3**.

Total number of points for **I.**: *28 points*

II. Consider two ideal monatomic gases. They occupy volumes V_1 and V_2. The atoms in gas 1 are roughly 4 times heavier than the ones in the other, i.e. $m_1 = 4m_2$. In addition, gas 1 consists of N_1 atoms and gas 2 of N_2 atoms, with equal numbers $N_1 = N_2$. Thus, the initial number densities of the gases are equal. The gases are kept in separate vessels but at the same temperature, ~~which is kept fixed throughout~~. We have learned that at sufficiently low density, both gases are classical and obey the ideal gas law and the equipartition theorem. We assume that at the initial values of the volumes (given above) and temperature both gases obey the classical gas laws.

Suppose now the gases get cooled keeping their volumes fixed (at any point, they have the same temperature). As we learned, upon cooling a gas, classical equipartition and the ideal gas law will fail. As the gases are cooled sufficiently, they will cease to obey classical equipartition. Which gas will stop being classical first? In other words, estimate the ratio of temperatures T_1' (for gas 1) and T_2' (for gas 2) where each of the two gases is expected to cease obeying the classical ideal gas law, i.e. estimate the ratio T_1'/T_2'. Rephrasing the question once more, which gas must be cooled more, and by how much, to make it non-classical?

Total number of points for **II.**: *24 points*

Use the space below for problem **II.**:

gas becomes quantum whenever

$$\lambda \sim \frac{\hbar}{\sqrt{mkT}} > \left(\frac{V}{N}\right)^{1/3} \sim \text{typical distance between particles}$$

Since $\frac{V}{N}$ is same for the two gases,

we have, that for each gas

$$\frac{1}{\sqrt{T}} > \sqrt{m} \quad \text{or} \quad T < \frac{1}{m}$$

thus $T_1' \sim \frac{1}{m_1}$, $T_2' \sim \frac{1}{m_2} \sim \frac{1}{4m_1}$

4

obvious typo here, fix!

(continue)

$$\frac{T_2'}{T_1'} = \frac{1}{4} \quad \therefore \text{ heavier gas has to be cooled more; to } 4 \times \text{ lower } T.$$

End of space for problem **II**.

III. The multiplicity function of a paramagnet of N spins is $\Omega(N, N_{up}) = \frac{N!}{N_{up}!(N-N_{up})!}$, where the number of spins up, N_{up}, can take values $0, 1, ...N$. The number of spins-up determines the macrostate of the system. The energy of the paramagnet placed in an external magnetic field B is $E = -\mu B(2N_{up} - N)$ (where μ is the magnetic moment associated with each spin) and can be written also in terms of the magnetization as $E = -BM$, with $M = \mu(2N_{up} - N)$.

1. Consider a macrostate of the paramagnet with $N_{up} = \frac{N}{2} + x$, with $0 < x \ll N/2$. Notice that we consider the thermodynamic limit where x and N both tend to very large numbers, but their ratio is taken fixed, assumed to obey $x/N \ll 1$. What are the energy E and the magnetization M of the paramagnet in terms of x? What is the ratio M/M_{max}, where M_{max} is the maximum possible magnetization that the paramagnet can ever have?

8 points

Use the space below for problem **III.1**

$$2N_{up} - N = 2\frac{N}{2} + 2x - N = 2x \quad , \text{ so}$$

$$E = -2\mu B x \qquad \& \quad M = 2\mu x$$

$$M_{max} = \mu N, \text{ so} \qquad \frac{M}{M_{max}} = \frac{2x}{N} \ll 1$$

in regime

$$\frac{x}{N/2} \ll 1$$

(continue)

End of space for problem **III.1**.

2. One can show (see the last part of this problem), using Stirling's formula, that with $N_{up} = (N/2) + x$, the multiplicity function, for $x \ll N$, can be written as an expansion

$$\Omega(N, x) \simeq 2^N e^{-2\frac{x^2}{N}} \tag{1}$$

Taking Eqn. (1) for granted, find the entropy of the paramagnet as a function of N and the energy E. Then, find how the energy E depends on the temperature of the paramagnet, i.e. determine the behaviour $E(T)$. What is the temperature range, in terms of the characteristic energy scale associated with the paramagnet, for which the small-x/N assumption is valid? Find $M(T)/M_{max}$ and comment on its T dependence in this regime. (Do you recall if this T-dependence of the magnetization has a name?)

16 points

Use the space below for problem **III.2**

$$S(N, x) = N \ln 2 - 2 \frac{x^2}{N}, \quad \text{use}$$

$$x = -\frac{E}{2\mu B} \quad \text{to} \quad \text{write}$$

● $S(N, E) = k N \ln 2 - k 2 N \left(\frac{E}{2N\mu B} \right)^2$

● $\frac{1}{T} = \frac{\partial S}{\partial E} = -4 N k \frac{E}{(2N\mu B)^2} = -\frac{k}{\mu B} \frac{E}{N \mu B}$

Or $\boxed{\frac{-E}{N\mu B} = \frac{\mu B}{kT}}$ 6 . $\frac{x}{N} \ll 1 \iff \boxed{-\frac{E}{2N\mu B} \ll 1}$

(continue)

since $M = -\dfrac{E}{B}$ we

have $\quad + \dfrac{M}{N\mu} = \dfrac{\mu B}{kT} \Rightarrow \left(\dfrac{M}{M_{max}} = \dfrac{\mu B}{kT} \right)$

$\longrightarrow$ Curie's law.

End of space for problem **III.2**.

3. What limit does the entropy approach in the $x \to 0$ limit? What does it teach us about the T-dependence of the entropy in the corresponding temperature range? Is there a simple way to obtain ~~this result?~~ find an estimate for S in this limit?   *12 points*

Use the space below for problem **III.3**

as $\quad x \to 0 \qquad S \to kN \log 2$

$\longrightarrow$ this is $\quad T \to \infty$

hence

$\quad T \to \infty \qquad S \to S_{max}$

$\quad S_{max} \sim k \log\left(2^N \right)$

$\qquad\qquad\qquad\qquad \uparrow$ total disorder

[this is a special
System where there's
a max. of S!]

$2^N = \#$ all microstates

End of space for problem **III.3**.

4. Finally, if you have time, derive Eqn. (1) from the approximate expression for the multiplicity, $\ln \Omega \simeq N \ln N - (N - N_{up}) \ln(N - N_{up}) - N_{up} \ln N_{up}$. This is the usual expression obtained using the leading term in the Stirling formula for the factorials.

   To proceed, you will need the small-a Taylor expansion $\ln(1 + a) = a - \frac{a^2}{2} + ...$, where the dots indicate terms scaling like a^3, which are suppressed, compared to the terms shown, in the $a \to 0$ limit.

   This is straightforward but requires care, hence it is left as the last problem.

   12 points

   Use the space below for problem **III.4**

$$N - N_{up} = \frac{N}{2} - x, \quad N + N_{up} = \frac{N}{2} - x$$

$$\ln \Omega \approx N \ln N - \left(\frac{N}{2} - x\right) \ln\left(\frac{N}{2} - x\right) - \left(\frac{N}{2} + x\right) \ln\left(\frac{N}{2} + x\right) =$$

$$= N \ln N - \left(\frac{N}{2} - x\right) \ln \frac{N}{2} - \left(\frac{N}{2} - x\right) \ln\left(1 - \frac{2x}{N}\right) - \left(\frac{N}{2} + x\right) \ln \frac{N}{2}$$

(cancel)

$$- \left(\frac{N}{2} + x\right) \ln\left(1 + \frac{2x}{N}\right) \approx$$

$$\approx N \ln N - N \ln \frac{N}{2} - \left(\frac{N}{2} - x\right) \ln\left(1 - \frac{2x}{N}\right) - \left(\frac{N}{2} + x\right) \ln\left(1 + \frac{2x}{N}\right)$$

$$= N \ln 2 - \frac{N}{2} \ln\left(1 - \frac{2x}{N}\right)\left(1 + \frac{2x}{N}\right) + x \ln\left(1 - \frac{2x}{N}\right) - x \ln\left(1 + \frac{2x}{N}\right)$$

$$= N \ln 2 - \frac{N}{2} \ln\left(1 - \frac{4x^2}{N^2}\right) + x \left(- \frac{2x}{N}\right) - x \left(\frac{2x}{N}\right)$$

$$= N \ln 2 - \frac{N}{2} \left(- \frac{4x^2}{N^2}\right) - \frac{4x^2}{N}$$

$$= N \ln 2 + \frac{2x^2}{N} - \frac{4x^2}{N} \quad 8 \quad = N \ln 2 - \frac{2x^2}{N}$$

(continue)

End of space for problem **III.4**.

Total number of points for **III.**: *48 points*

Total number of points: 28 + 24 + 48= 100.

Some useful formulae are on the next page.

Some useful mathematical expressions are: $\frac{d \ln x}{dx} = \frac{1}{x}$, $\ln x^a = a \ln x$, $\ln e = 1$, $e^{\ln x} = x$, $4^{1/2} = 2$, $4^4 = 64$, $\frac{d e^{a(x)}}{dx} = e^{a(x)} \frac{da(x)}{dx}$, $\frac{dx^k}{dx} = kx^{k-1}$, the function $e^{-\frac{1}{x}} \to 0$ as $x \to 0$ and is positive for $x > 0$; also $\frac{d^n e^{-\frac{1}{x}}}{dx^n}|_{x \to 0} = 0$, for all $n > 0$. In addition $\ln(ab) = \ln a + \ln b$. The ideal gas law is $pV = NkT$. The energy of a single-atomic classical ideal gas of N atoms is $U = \frac{3}{2}NkT$. The work done by the gas in a quasistatic process along some curve $p(V)$ in the $p - V$ plane is $W = \int_{V_1}^{V_2} p(V)dV$. The entropy and multiplicity function are related by $S(E, N, V) = k \ln \Omega(E, N, V)$, while the temperature is $T = 1/(\partial S/\partial E)_{N,V}$. For a monatomic ideal classical gas, the entropy is given by the Sackur-Tetrode formula $S(E, N, V) = kN \left(\log \left[\frac{V}{N} \left(\frac{4\pi m E}{3Nh^2} \right)^{\frac{3}{2}} \right] + \frac{5}{2} \right)$. The thermal de Broglie wavelength of a gas of particles of mass m at temperature T is $\lambda_{th.} \sim \frac{\hbar}{\sqrt{mkT}}$. The ratio between the thermal de Broglie wavelength and the typical distance between particles in the gas (as determined by the number density of the gas) determines, roughly, under what conditions a gas can be considered classical and obeying the ideal gas law. The energy levels of an oscillator, ignoring the zero-point energy, are $\hbar \omega n$, where $n \geq 0$ is the (integer) number of quanta. The first law of TD expresses energy conservation: the energy of a system of fixed number of particles can change because of the heat transfer and work performed.

Use the space below for scribbles, if needed before writing your official answers.

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