UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test III

First Year — Program 5

MAT185H1S — Linear Algebra

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St Patrick's Day 2016

Student Name:	Fair Copy	
	Last Name	First Names
Student Number:		Tutorial Section: TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 8 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). By definition, how does the *determinant* differ from a *determinant function*?

The determinant is the determinant function Δ_n for which $\Delta_n(\mathbf{1}) = 1$.

/2

1(b). What is the *Laplace expansion* along column i for the determinant of $\mathbf{A} \in {}^n\mathbb{R}^n$?

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ji} c_{ji}$$

where c_{ji} is the (j, i)-cofactor.

/2

1(c). What is the determinant of each kind of the elementary matrix?

$$det\mathbf{E}(i, j) = -1, \det \mathbf{E}(\lambda; 1) = \lambda, \det \mathbf{E}(\lambda; i, j) = 1$$

/2

1(d). State the *transpose rule* for determinants.

$$\det \mathbf{A}^T = \det \mathbf{A}$$

/2

1(e). The *adjoint* of $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

$$\text{adj}\, \mathbf{A} = \mathbf{C}^T$$

where $\mathbf{C}$ is the cofactor matrix.

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). Let $\mathbf{A} \in {}^n\mathbb{R}^n$ have the property that for every k, $k = 1 \cdots n$, each number in row k is an integer multiple of k. Then $\det \mathbf{A}$ is an integer multiple of $n!$. (Recall that $n! = n(n-1)(n-2)\cdots 1$.)

T

2(b). For the $n \times n$ matrix,

$$\mathbf{T} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

F

$\det \mathbf{T} = 1 + (-1)^n$.

2(c). The columns of the minor $\mathbf{M}_{ij}(\mathbf{A})$ span ${}^{n-1}\mathbb{R}$ if and only if the (i, j)-cofactor of $\mathbf{A}$ is nonzero.

T

2(d). If $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonal, then $\operatorname{adj} \mathbf{A}$ is also diagonal.

T

2(e). If $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{A} \in {}^m\mathbb{R}^n$, has only the trivial solution, then $\det \mathbf{A}\mathbf{A}^T \neq 0$.

F

C. Problems

3. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ -4 & 1 & 2 & 0 \\ -3 & -3 & 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \lambda \\ 1 \end{bmatrix}$$

(a) Calculate $\det \mathbf{A}_4$ (the determinant of $\mathbf{A}$ with the 4th column replaced by $\mathbf{b}$).

(b) Let $\mathbf{x} = [\ x_1 \quad x_2 \quad x_3 \quad x_4\ ]^T$ be the solution to $\mathbf{Ax} = \mathbf{b}$. For what values of λ is $x_4 \geq 0$?

3(a). Calculate $\det \mathbf{A}_4$ (the determinant of $\mathbf{A}$ with the 4th column replaced by $\mathbf{b}$).

$$\det \mathbf{A}_4 = \det \begin{bmatrix} 3 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ -4 & 1 & 2 & \lambda \\ -3 & -3 & 1 & 1 \end{bmatrix} = 24 + 8\lambda$$

/4

3(b). Let $\mathbf{x} = [\ x_1 \quad x_2 \quad x_3 \quad x_4\ ]^T$ be the solution to $\mathbf{Ax} = \mathbf{b}$. For what values of λ is $x_4 \geq 0$?

$$\det \mathbf{A} = 3$$

By the Maclaurin-Cramer rule,

$$x_4 = \frac{\det \mathbf{A}_4}{\mathbf{A}} = \frac{24 + 8\lambda}{3}$$

Thus for $x_4 \geq 0$, $\lambda \geq -3$.

/6

4. Given that $\mathcal{F} = \{f \mid f : {}^n\mathbb{R}^n \to \mathbb{R}\}$ is a vector space over $\mathbb{R}$ where

(i) $(f+g)(\mathbf{A}) = f(\mathbf{A}) + g(\mathbf{A})$, for all $f, g \in \mathcal{F}$
(ii) $(\lambda f)(\mathbf{A}) = \lambda f(\mathbf{A})$, for all $f \in \mathcal{F}, \lambda \in \mathbb{R}$

show that the set of all possible determinant functions is a subspace of $\mathcal{F}$.

We apply the subspace test to check that $\mathcal{D}$, the set of all possible determinant functions, is a subspace of $\mathcal{F}$.

SI. Clearly, $f(\mathbf{A}) = 0$ for all $\mathbf{A} \in {}^n\mathbb{R}^n$ is a determinant function as it satisfies DI and DII, defining a determinant function. This is the zero function.

SII. Let $f(\mathbf{A})$ and $g(\mathbf{A})$ be two determinant functions. Then

DI.

$$\begin{aligned}(f+g)[\mathbf{E}(1;i,j)\mathbf{A}] &= f[\mathbf{E}(1;i,j)\mathbf{A}] + g[\mathbf{E}(1;i,j)\mathbf{A}] \\ &= f(\mathbf{A}) + g(\mathbf{A}) = (f+g)(\mathbf{A})\end{aligned}$$

DII.

$$\begin{aligned}(f+g)[\mathbf{E}(\alpha;i)\mathbf{A}] &= f[\mathbf{E}(\alpha;i)\mathbf{A}] + g[\mathbf{E}(\alpha;i)\mathbf{A}]\alpha f(\mathbf{A}) + \alpha g(\mathbf{A}) \\ &= \alpha(f+g)(\mathbf{A})\end{aligned}$$

Therefore, $f+g$ is also a determinant function.

SIII. Let $f(\mathbf{A})$ be a determinant function and $\lambda \in \mathbb{R}$. Then

DI.

$$(\lambda f)[\mathbf{E}(1;i,j)\mathbf{A}] = \lambda f[\mathbf{E}(1;i,j)\mathbf{A}] = \lambda f(\mathbf{A}) = \lambda f(\mathbf{A})$$

DII.

$$(\lambda f)[\mathbf{E}(\alpha;i)\mathbf{A}] = \lambda f[\mathbf{E}(\alpha;i)\mathbf{A}] = \lambda[\alpha f(\mathbf{A})] = \alpha[\lambda f(\mathbf{A})] = \alpha(\lambda f)(\mathbf{A})$$

Therefore, λf is also a determinant function.

By the subspace test, then, $\mathcal{D}$ is a subspace of $\mathcal{F}$.

. . . cont'd

4. *...cont'd*

Alternatively, it can be observed that any determinant function $f(\mathbf{A})$ is a scalar multiple of $\det \mathbf{A}$ and thence apply the subspace test.

/10

5. Let $\mathbf{A} \in {}^n\mathbb{R}^n$ be invertible. Show that $\operatorname{adj}(\mathbf{A}^{-1}) = (\operatorname{adj}\mathbf{A})^{-1}$.

Note that

$$\mathbf{A}\operatorname{adj}\mathbf{A} = (\operatorname{adj}\mathbf{A})\mathbf{A} = (\det\mathbf{A})\mathbf{1}$$

and

$$\mathbf{A}^{-1}\operatorname{adj}\mathbf{A}^{-1} = (\det\mathbf{A}^{-1})\mathbf{1}$$

Taking the inverse of the first relation, we have

$$\mathbf{A}^{-1}(\operatorname{adj}\mathbf{A})^{-1} = \frac{1}{\det\mathbf{A}}\mathbf{1}$$

But $1/\det\mathbf{A} = \det\mathbf{A}^{-1}$ and hence

$$\mathbf{A}^{-1}(\operatorname{adj}\mathbf{A})^{-1} = \mathbf{A}^{-1}\operatorname{adj}\mathbf{A}^{-1}$$

or

$$\mathbf{A}^{-1}[(\operatorname{adj}\mathbf{A})^{-1} - \operatorname{adj}\mathbf{A}^{-1}] = \mathbf{O}$$

As $\mathbf{A}^{-1}$ is invertible (the inverse is $\mathbf{A}$), we conclude that

$$\operatorname{adj}\mathbf{A}^{-1} = (\operatorname{adj}\mathbf{A})^{-1}$$

/10