ECE259H1: Electromagnetism

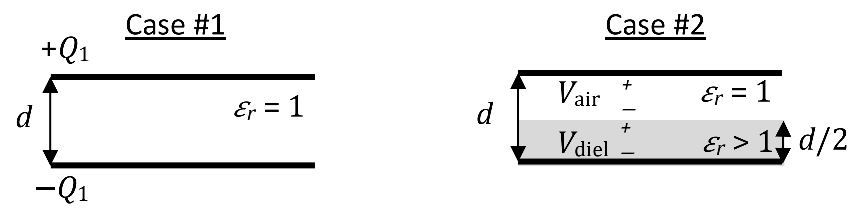
Term Test 2 – Tuesday March 28, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The Term Test is worth 60 marks and has three questions. Each question is work 20 marks. There is a Bonus Question worth 5 additional marks at the end of Question 3.
- Show all of your work, and the final page is left blank which you can use for rough work or for extra space for your answers.
- Take a deep breath and relax 😊.

Question #1 (20 marks)

(*8 marks*) 1. (a) An air-filled parallel-plate capacitor is attached to a battery with voltage V_0 and fully charged up, such that the plates have charges of $\pm Q_1$, and the bottom plate is grounded. This is Case #1. The battery remains connected and then a piece of dielectric material is placed between the plates such that it fills half the volume of the capacitor. This is Case #2. Answer the following questions. You may ignore any effects of fringing fields and you can assume that the charge is uniformly distributed over the area, *S*, of each plate of the capacitor. You may also assume that there is no free surface charge density on the boundary between the air and dielectric.



i) Is there a bound surface charge density on the top layer of the dielectric piece (i.e., at the height of *d*/2 between the plates)? If so, is it positive or negative? If not, why not?

Yes, the dielectric is polarized by the E-field so a ρ_{sb} will exist on this surface.
ρ_{sb} is negative, since $\bar{E}$ is directed downwards.
↳ The downward $\bar{E}$ will push +ve charges downwards, leaving a layer of −ve charges at $d/2$.

ii) In Case #2, is the potential difference across the dielectric region greater or smaller than the potential difference across the air region (i.e., is $V_{diel} > V_{air}$)? If so, why? If not, why not?

↳ Since D is the same in the air & dielectric regions, the electric field intensity E is reduced in the dielectric region by a factor of ε_r : $E_{air} > E_{diel} = \frac{E_{air}}{\varepsilon_r}$

∴ Since both regions are $d/2$ in length,

$V_{diel} = E_{diel}\left(\frac{d}{2}\right) < V_{air} = E_{air}\left(\frac{d}{2}\right)$

Question #1 (continued)

1. (a) iii) Does the capacitance of the capacitor change as you go from Case #1 to Case #2? If so, how does it change? If not, why not?

Sol'n #1: Capacitance

$$C_1 = \frac{\varepsilon_o S}{d}$$

$$C_2 = C_{air} \| C_{diel} = \frac{\left(\frac{\varepsilon_o S}{d/2}\right)\left(\frac{\varepsilon_r \cancel{\varepsilon_o S}}{\cancel{d/2}}\right)}{\frac{\cancel{\varepsilon_o S}}{\cancel{d/2}} + \frac{\varepsilon_r \cancel{\varepsilon_o S}}{\cancel{d/2}}} = \frac{2\varepsilon_r \varepsilon_o S}{(1+\varepsilon_r) d} = \left(\frac{2\varepsilon_r}{1+\varepsilon_r}\right) C_1 > C_1$$

Yes, the capacitance increases from Case #1 to #2

Sol'n #2: Fields

Since at the boundary $D_{n1} = D_{n2}$

$$D_{air} = D_{diel} = D_o = \rho_s = Q_2/S$$

And
$$V_o = E_{air}(d/2) + E_{diel}(d/2) = \frac{D_o}{\varepsilon_o}\left(\frac{d}{2}\right) + \frac{D_o}{\varepsilon_r \varepsilon_o}\left(\frac{d}{2}\right) = \frac{Q_2(d/2)}{\varepsilon_o S} + \frac{Q_2(d/2)}{\varepsilon_r \varepsilon_o S}$$

$$\therefore C_2 = \frac{Q_2}{V_o} = \frac{(\cancel{\varepsilon_o} S)(\varepsilon_r \varepsilon_o S)}{\varepsilon_r \cancel{\varepsilon_o S}(d/2) + \cancel{\varepsilon_o S}(d/2)} = \left(\frac{2\varepsilon_r}{1+\varepsilon_r}\right)\left(\frac{\varepsilon_o S}{d}\right) > C_1$$

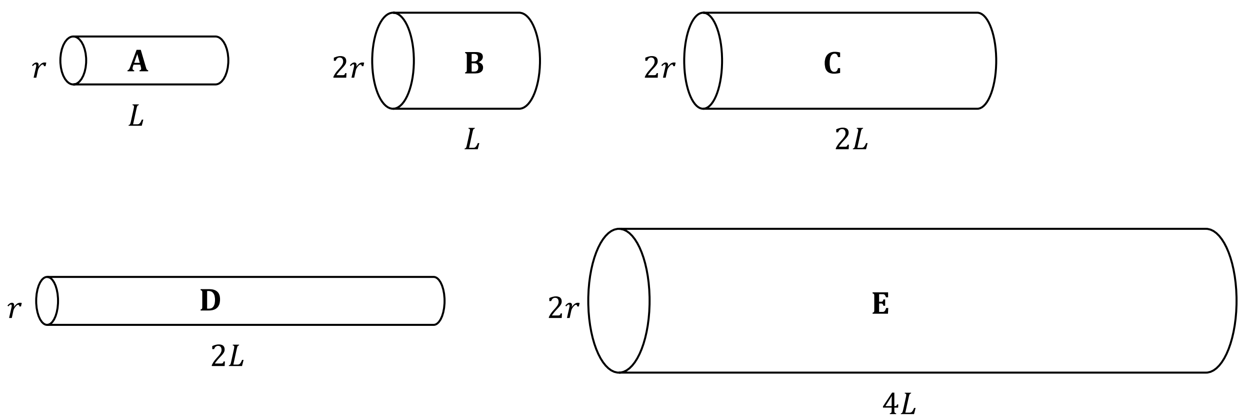
iv) Is the stored energy in Case #2 larger than the stored energy in Case #1? If so, explain where the extra energy comes from. If not, explain where the extra energy goes, or why it would be the same.

Since $W_e = \frac{1}{2} C V_o^2 \rightarrow W_{e2} > W_{e1}$ (since $C_2 > C_1$)

↳ This extra energy comes from the work done by an external agent to force the dielectric into the capacitor. This is needed in order to polarize the dielectric.

Question #1 (cont'd)

(*6 marks*) 1. (b) The wires **A** to **E** shown below are made out of the same material, and have lengths and radii as shown in the diagram. The voltage difference between the ends of each wire is the same. Rank the magnitude of the current densities $J_{\mathbf{A}}$ to $J_{\mathbf{E}}$ within these wires from largest to smallest. If some (or all) are equal clearly indicate that as well. You can assume the current is uniformly distributed throughout each wire.



Largest Current Density Magnitude J_A = J_B > J_C = J_D > J_E *Smallest Current Density Magnitude*

Explain your reasoning or show your work.

Since V is the same for all, and the material is the same for all,

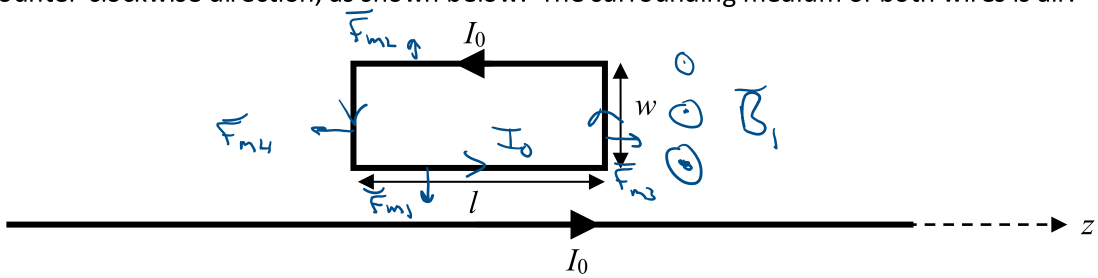
$$|\vec{J}_x| = |\sigma \vec{E}| = \sigma \frac{V}{L_x}$$

$\therefore$ $|\vec{J}_x|$ is inversely proportional to L_x

$\therefore$ $J_A = J_B > J_C = J_d > J_E$

Question #1 (cont'd)

(*6 marks*) 1. (c) An infinitely-long wire carries a DC current of I_0 in the z-direction and is fixed in place. A single-turn rectangular loop is placed near the wire and also carries the same amount of DC current, I_0, in the counter-clockwise direction, as shown below. The surrounding medium of both wires is air.



For the statement below, select the correct response from those options provided. Justify your response.

If the rectangular loop is free to move in any direction, it will initially:

a. Move upwards (away from the wire)
b. Move downwards (towards the wire)
c. Move out of the page (vertically upwards)
d. Move into the page (vertically downwards)
e. Not move at all

* The magnetic field due to I_0 in the infinitely long wire is out of the page where the rectangular current exists. This field decays according to $\frac{1}{r}$ → $\bar{B}_1 = \frac{\mu_0 I_0}{2\pi r}\hat{a}_\phi$

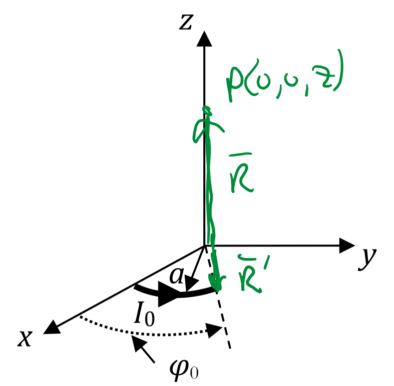
∴ The force on the bottom section of the rectangular wire is stronger than the force on the top section.

⇒ Using the right-hand rule and from $\bar{F} = I_0 \bar{L} \times \bar{B}$, the rectangular loop would move downwards since $|\bar{F}_{m1}| > |\bar{F}_{m2}|$ and since F_{m3} & F_{m4} cancel each other.

Question #2 (20 marks)

(*12 marks*) 2. (a) A short wire carries a current I_0 and is oriented as shown in the figure below. The wire is in the shape of a partial ring of radius a and subtends an angle φ_0. Use the Biot-Savart law or the Vector Magnetic Potential to prove that the expression for the magnetic field intensity for this wire at an arbitrary point on the positive z-axis is:

$$\mathbf{H} = \frac{I_0 a}{4\pi[a^2+z^2]^{3/2}}\left[z\sin\varphi_0\,\hat{\boldsymbol{a}}_x + z\,(1-\cos\varphi_0)\hat{\boldsymbol{a}}_y + a\varphi_0\hat{\boldsymbol{a}}_z\right]$$



Make sure to show all your work.

From Biot-Savart: $\bar{H} = \int d\bar{H} = \int \frac{I d\bar{l} \times (\bar{R}-\bar{R}')}{4\pi\,|\bar{R}-\bar{R}'|^3}$

$I d\bar{l} = I_0 (a\,d\phi'\,\hat{a}_\phi)$, $\bar{R} = z\hat{a}_z$, $\bar{R}' = a\hat{a}_r$

$$\therefore d\bar{H} = \frac{I_0 (a\,d\phi'\,\hat{a}_\phi) \times (-a\hat{a}_r + z\hat{a}_z)}{4\pi\,(a^2+z^2)^{3/2}}$$

using for cross product: $\hat{a}_z \to \hat{a}_r \to \hat{a}_\phi \to \hat{a}_z$

$\hookrightarrow \hat{a}_\phi \times (-\hat{a}_r) = \hat{a}_z$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_r$

$$= \frac{I_0 a}{4\pi (a^2+z^2)^{3/2}} \left(a\hat{a}_z + z\hat{a}_r\right) d\phi'$$

Since as we integrate $\hat{a}_r$ changes we need to write:

$$d\bar{H} = \frac{I_0 a}{4\pi (a^2+z^2)^{3/2}} \left(z\cos\phi'\,\hat{a}_x + z\sin\phi'\,\hat{a}_y + a\hat{a}_z\right) d\phi'$$

$$\therefore \bar{H} = \int d\bar{H} = \frac{I_0 a}{4\pi (a^2+z^2)^{3/2}} \int_0^{\phi_0} \left(z\cos\phi'\,\hat{a}_x + z\sin\phi'\,\hat{a}_y + a\hat{a}_z\right) d\phi'$$

$\int \cos\phi\, d\phi = \sin\phi + C$

$\int \sin\phi\, d\phi = -\cos\phi + C$

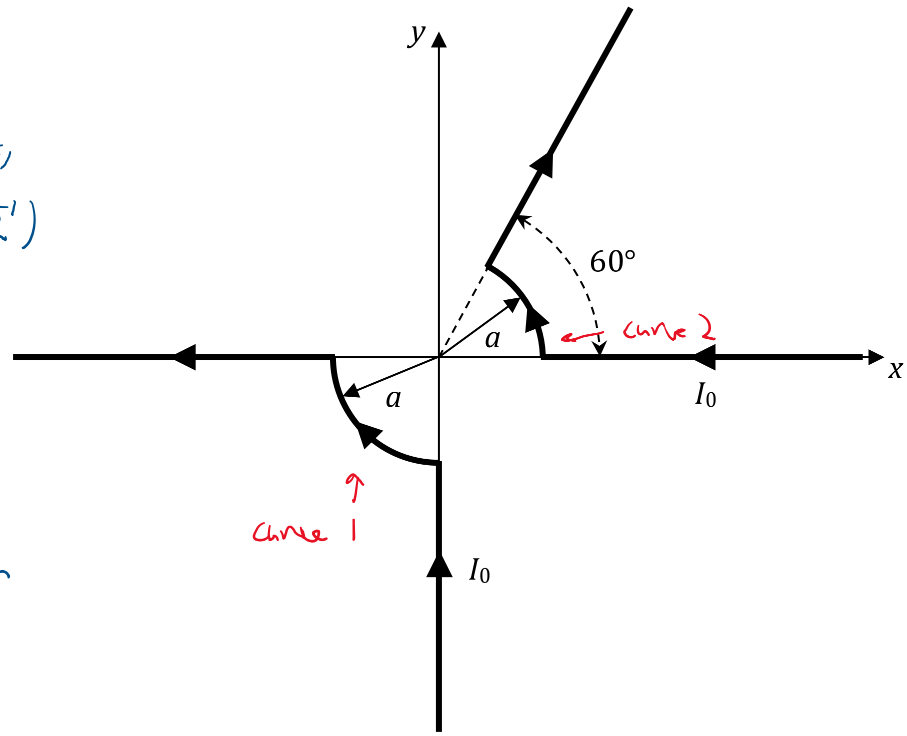
$$= \frac{I_0 a}{4\pi (a^2+z^2)^{3/2}} \left[z\sin\phi_0\,\hat{a}_x + z(1-\cos\phi_0)\,\hat{a}_y + a\phi_0\,\hat{a}_z\right]$$

Question #2 (continued)

2. (a) (continued)

Question #2 (continued)

(*8 marks*) 2. (b) Two wires each carry a current $I_0 = 2$ mA and are oriented as shown in the figure below. The radius of the curved segments is $a = 1$ cm. For this situation, determine the total magnetic field intensity at the origin. You can assume that the straight segments of the wires are very long (i.e., they extend off to infinity).



For the straight line segments, $\bar{H} = 0$ since from Biot-Savart,

$\bar{H} \propto I d\bar{l} \times (\bar{R} - \bar{R}') = I d\bar{l} \times (-\bar{R}')$

$(\bar{R} = 0)$, and so for these straight line segments $d\bar{l}$ is parallel or anti-parallel to $(-\bar{R}')$. Therefore, the cross-product (and thus field) is zero at the origin for each of these four straight line segments.

$\therefore \quad \bar{H}_{TOT} = \bar{H}_{curve1} + \bar{H}_{curve2}$

Using the expression in part 2a), with $z = 0$:

$$\bar{H}_{TOT} = \frac{I_0 a}{4\cancel{\pi} (a^2)^{3/2}} \left[-a\left(\frac{\cancel{\pi}}{2}\right)\hat{a}_z + a\left(\frac{\cancel{\pi}}{3}\right)\hat{a}_z \right]$$

$$= \frac{I_0 \cancel{a^2}\, \hat{a}_z}{4a^{\cancel{3}1}} \left(-\frac{1}{6}\right) = \frac{-I_0}{24a} = \frac{-1}{120}\hat{a}_z = -8.3\times10^{-3}\,\hat{a}_z \left[A/m\right]$$

Question #3 (20 marks)

A spherical capacitor consists of two concentric perfectly conducting spheres of radii $R = a$ and $R = b$. Between the conducting spheres exists a dielectric material with a relative permittivity given by $\varepsilon_r = k/R^2$, where k is a positive constant. It is known that the potential difference between the two spheres is V_0, with the outer sphere being grounded.

(*10 marks*) 3. (a) Determine the expression for the variation of the electric scalar potential between the spheres of the capacitor, i.e., $V(R)$ for $a \le R \le b$.

* Since no information is given, assume $\rho_v = 0$ between spheres.

$$\nabla \cdot (\varepsilon_r \varepsilon_0 \nabla V) = 0 \rightarrow \boxed{\frac{1}{R^2} \frac{d}{dR}}\left[\left(\frac{k}{\cancel{R^2}}\right)\varepsilon_0 \cancel{R^2} \frac{dV}{dR}\right] = 0$$

Integration #1:

$$\int d\left(k\varepsilon_0 \frac{dV}{dR}\right) = \int 0 \,(R^2 dR)$$

$$\boxed{k\varepsilon_0 \frac{dV}{dR}} = C_1$$

Integration #2:

$$\int dV = \int \frac{C_1}{k\varepsilon_0} dR$$

$$V(R) = \frac{C_1}{k\varepsilon_0} R + C_2$$

Using boundary conditions:

$$V(a) = V_0 = \frac{C_1}{k\varepsilon_0} a + C_2 \quad (1)$$

$$V(b) = 0 = \frac{C_1}{k\varepsilon_0} b + C_2 \quad (2)$$

(1) − (2): $V_0 = \frac{C_1}{k\varepsilon_0}(a-b) \rightarrow C_1 = -\frac{V_0 k \varepsilon_0}{b-a} \quad (3)$

(3) ⇒ (2): $C_2 = -\frac{C_1}{k\varepsilon_0} b = \frac{V_0 b}{b-a}$

$$\therefore V(R) = \left(\frac{V_0}{b-a}\right)R + \left(\frac{V_0}{b-a}\right)b = \underline{\underline{\left(\frac{V_0}{b-a}\right)(b-R)}}$$

Question #3 (continued)

(*4 marks*) 3. (b) Determine the expression for the electric field intensity $\mathbf{E}(R)$, between the spheres.

$$\bar{E}(R) = -\bar{\nabla} V = -\frac{dV}{dR}\hat{a}_R \text{, which from above is}$$

$$= -\frac{d}{dR}\left[\left(\frac{V_0}{b-a}\right)(b-R)\right]\hat{a}_R$$

$$= \underline{\underline{\frac{V_0}{b-a}\hat{a}_R}}$$

(*6 marks*) 3. (c) Prove that the expression for the capacitance of this structure is given by $C = \frac{4\pi k\varepsilon_0}{b-a}$.

$$C = \frac{Q}{V_0} \text{ and } Q = \iint \rho_s \, dS \text{ for } \rho_s \text{ at } R=a$$

From the boundary conditions:

$$\rho_s(R=a) = D(R=a) = \varepsilon_r \varepsilon_0 E(R=a)$$

$$= \frac{k}{a^2}\varepsilon_0\left(\frac{V_0}{b-a}\right)$$

$$\therefore Q = \iint \rho_s \, dS = \rho_s(4\pi a^2)$$

$$= \frac{4\pi k \varepsilon_0 V_0}{b-a}$$

$$\therefore C = \frac{Q}{V_0} = \underline{\underline{\frac{4\pi k \varepsilon_0}{b-a}}}$$

BONUS Question

(*5 marks*) If the dielectric material described in the spherical capacitor of Q3 above, also has a finite conductivity given by $\sigma = \frac{2.59\times10^{-14}}{R^2}$ S/m, and $a = 1$ cm, $b = 4$ cm, $V_0 = 4$ V, and $k = 0.06$, determine the resistance and the power loss of this structure.

We know that $RC = \frac{\varepsilon_r \varepsilon_0}{\sigma} = \left(\frac{k\varepsilon_0}{\cancel{R^2}}\right)\left(\frac{\cancel{R^2}}{2.59\times10^{-14}}\right)$

$$\therefore R = \left(\frac{\cancel{k\varepsilon_0}}{2.59\times10^{-14}}\right)\frac{b-a}{4\pi \cancel{k\varepsilon_0}} = \frac{0.03}{4\pi(2.59\times10^{-14})} = \underline{\underline{9.22\times10^{10}\ \Omega}}$$

$$\text{Power loss} = P = \frac{V_0^2}{R} = \frac{16}{9.22\times10^{10}} = \underline{\underline{0.174\ \text{nW}}}$$

OR From $R = \frac{V_0}{I} = \frac{V_0}{\iint \bar{J}\cdot\bar{ds}} = \frac{V_0}{\iint \sigma\bar{E}\cdot\bar{ds}}$

We have
$$R = \frac{\cancel{V_0}}{\int_0^{2\pi}\int_0^{\pi}\left(\frac{2.59\times10^{-14}}{\cancel{R^2}}\right)\left(\frac{\cancel{V_0}}{b-a}\right)\cancel{R^2}\underbrace{\sin\theta\, d\theta\, d\phi}_{\text{Integration} = 4\pi}} = \frac{b-a}{4\pi(2.59\times10^{-14})}$$
$$= \underline{\underline{9.22\times10^{10}\ \Omega}}$$

$$P = \iiint \bar{J}\cdot\bar{E}\,dv = \iiint \sigma|\bar{E}|^2\,dv$$
$$= \int_a^b\int_0^{2\pi}\int_0^{\pi}\left(\frac{2.59\times10^{-14}}{\cancel{R^2}}\right)\left[\frac{V_0^2}{(b-a)^2}\right]\cancel{R^2}\underbrace{\sin\theta\, d\theta\, d\phi}_{\text{Integration} = 4\pi}\, dR$$
$$= \frac{4\pi V_0^2}{(b-a)^{\cancel{2}}}(2.59\times10^{-14})\overbrace{\cancel{(b-a)}}^{R\ \text{integration}}$$
$$= \frac{V_0^2(4\pi)(2.59\times10^{-14})}{(b-a)} = \underline{\underline{0.174\ \text{nW}}}$$

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