2.8 Exercises for Chapter 2

2.1) Derive equation 2.4.8 for atmospheric pressure as a function of altitude in the presence of a linearly decreasing temperature. Calculate the altitude at which the atmospheric density is one half its sea level value assuming that the temperature declines from an assumed sea-level value of 15°C at the rate dT/dz = 6.45°K/km. Assume that g equals its sea level value, 9.804 m/s^2. Given that the radius of the Earth R_e = 6440 km, estimate the error involved in assuming that g is constant and equal to its sea-level value by assuming an average g for the range of altitudes obtained in the first part of this problem.

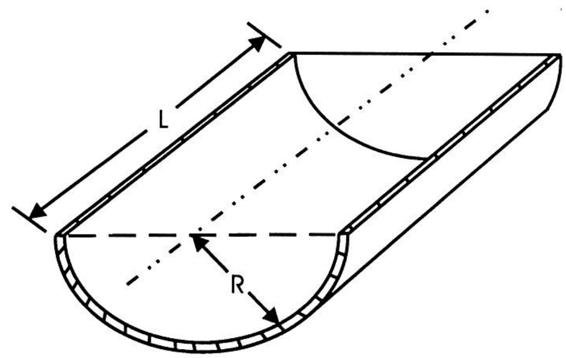
2.2) A vertical mine-shaft extends 3000 m down into the earth and the air temperature, which is determined by that of the surrounding rock, increases linearly from 20 °C at the top to 45°C at the bottom. Find the air pressure at the bottom of the shaft, assuming atmospheric pressure at the surface is 100 kPa. Compare your answer with that obtained by assuming that the air density is constant and equal to its surface value.

2.3) The deepest point in the Earth's oceans is thought to be in the Marinas Trench, about 400 km SW of Guam in the Pacific, with this depth H = 11,033 m. Given that the density ρ_0 of sea water at atmospheric pressure is 1023 kg/m^3, and given that its isothermal bulk modulus $E_T = \rho\,(\mathrm{d}p/\mathrm{d}\rho) = 2.40 \times 10^9$ N/m^2, estimate the density ρ_H and pressure p_H at the bottom of the Marinas Trench. First show that, if g is assumed constant, $1/\rho_H = 1/\rho_0 - gH/E_T$. Find the error incurred in estimating p_H as a result of assuming that $\rho(z) = \rho_0$.

2.4) The theory of inverse square law potentials shows that, within the Earth's interior, matter is attracted to the centre with a force per unit mass that is proportional to the radial distance from the centre. Furthermore, much of the interior is molten rock. Using the values of g at the surface and R_e quoted in Problem 2.1, compute the pressure at the Earth's centre assuming that the material behaves like a liquid with average density ρ = 5600 kg/m^3. For this purpose, think in terms of a tube of constant diameter extending to the centre, or alternatively in terms of a spherical co-ordinate system with a spherical symmetry, $\nabla p = (\partial p/\partial r)\,\boldsymbol{i}_r$. Express your results in atmospheres.

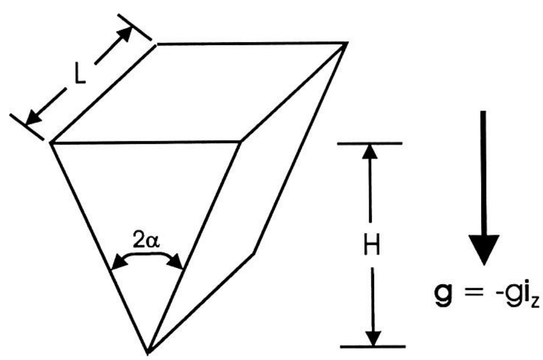
2.5) A truncated cone of base radius R_B, height H and radius $R_T < R_B$ at the top is filled with liquid of density ρ. Thus the pressure of the water on the base is $p = P_a + \rho g H$ and, if the pressure below the base is P_a, the force exerted by the liquid on the base is $\boldsymbol{F}_b = \pi\rho R^2 H \boldsymbol{i}_z$, which is greater than the weight of the liquid. By starting with the pressure acting on the conical sides of the vessel, resolve in detail this apparent paradox.

2.6) An open container or trough in the form of a semicircular cylinder of radius R and length L as depicted in the diagram is filled with liquid of density ρ. By direct integration of the pressure forces acting on the surface of the container, verify that the resultant pressure force $\boldsymbol{F}_p$ acting on the container is equal to the weight of the liquid in the container, that is $\boldsymbol{F}_p = -\ \tfrac{1}{2}\pi\rho g R^2 L\, \boldsymbol{i}_z$.



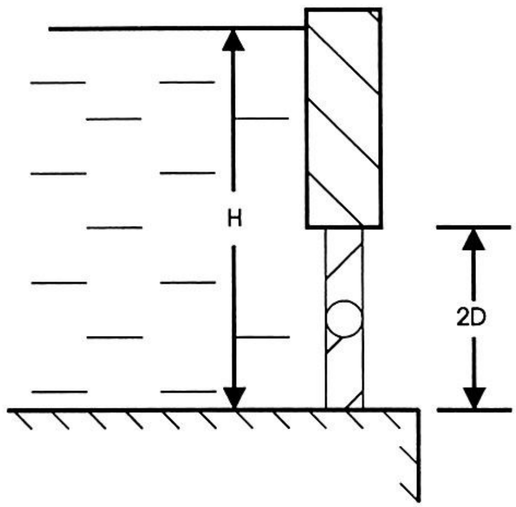
2.7) A vessel containing liquid consists of an inverted cone of apex angle 2α having its axis vertical. If the vessel is filled with liquid of density ρ to a height H above the apex, by integrating the pressure forces acting over the vessel's interior, show that the resultant vertical force exerted on the cone by the liquid is equal to the weight of the liquid.

2.8) Verify Archimedes' Principle for a body immersed in a liquid of constant density ρ subject to $\boldsymbol{g} = -\ g\, \boldsymbol{i}_z$ by determining the forces acting on the inverted isosceles triangular prism of included angle 2α, height H and length L depicted in the diagram. Hint: Use the running coordinate s along an inclined side, and any obvious symmetry properties.



2.9) Verify Archimedes' Principle for a constant density fluid in terrestrial gravity by directly integrating the expression for pressure force on the surface of a sphere of radius A.

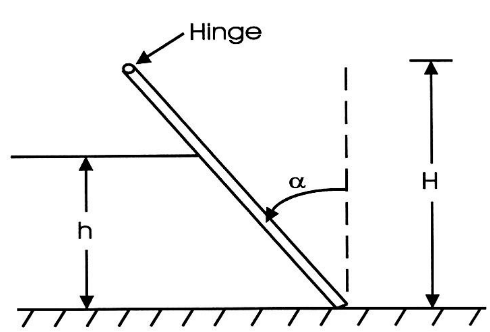
2.10) A rectangular sectioned channel of width W contains water of density ρ to a height H. A barrier mounted transversely contains a control gate consisting of a rectangular panel of width W and height $2D < H$ mounted on a horizontal shaft located at the panel's mid-points depicted in the diagram. Since the pressure at any point on the lower half of the panel is greater than that on the upper half, it will tend to open unless a restraining moment is applied to the shaft. Find this moment, showing that it is independent of H.



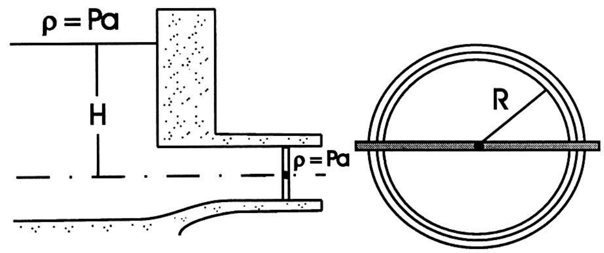
2.11) For Exercise 2.10, the moment is independent of H because the fluid density ρ is assumed constant. To demonstrate this, notwithstanding the fact that ρ depends on z indirectly through its dependence on p, for simplicity assume an explicit dependence on z. That is, with K being a constant, assume $\rho(z) = \rho_0 [1 + K(H - z)]$, and recalculate the moment exerted on the gate.

2.12) A concrete wall acting as a dam has a rectangular section of height H and width B, and relative density $\sigma_C = \rho_C/\rho_W > 1$, where ρ_W is the water density. Determine the width B required to prevent overturning of the wall if the water can rise to the top of the dam. Assume two cases: (1), water seepage causes the maximum hydrostatic pressure to act on the dam base; (2), as is the characteristic of flow through porous media, the pressure falls linearly under the base.

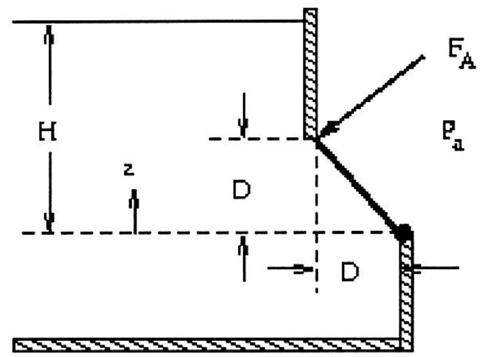
2.13) A hydraulic control gate in a rectangular sectioned channel is a plate hinged at height H above the channel and inclined at an angle α to the vertical. The surface density σ of the plate is to be chosen such that the gate swings open when the water reaches a level $h \le H$. To determine the weight of the plate required, express σ in terms of ρ, h, H, g, and α.



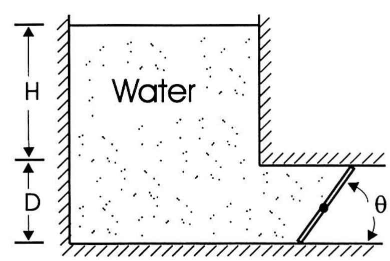
2.14) A reservoir containing water of density ρ to a depth H can be emptied by a horizontal pipe of diameter $D = 2R$ containing a valve in the form of a circular disc of radius R filling the pipe and mounted on a horizontal shaft passing through the middle of the disc as shown in the diagram. Find the moment that has to be applied to the shaft to hold the valve shut. How would you modify the design to avoid the need to apply a moment?



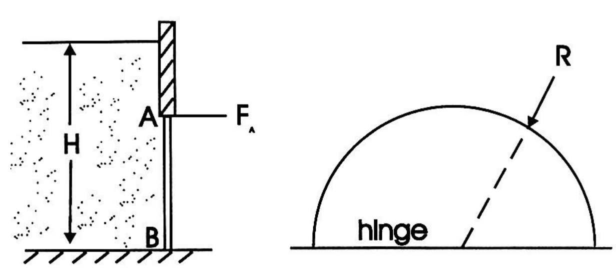
2.15) A hydraulic control gate in a rectangular section channel consists of a hinged plate inclined at 45° to the vertical as shown in the diagram to the right. Determine the force F_A required to keep the gate shut in terms of ρ, g, H, D, and the width of the channel W.



2.16) A rectangular gate attached to a horizontal shaft located at its midpoint prevents the flow of water from a reservoir as shown in the diagram. Calculate the torque that would be required to open the gate, per unit width of gate.

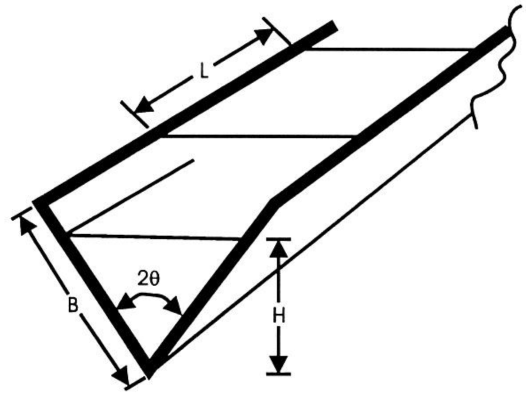


2.17) A semicircular plane gate AB of radius R is hinged along B and held closed by a horizontal force F_A applied at A. The liquid to the left of the gate rises to a height $H > R$ above the gate hinge. Find F_A.



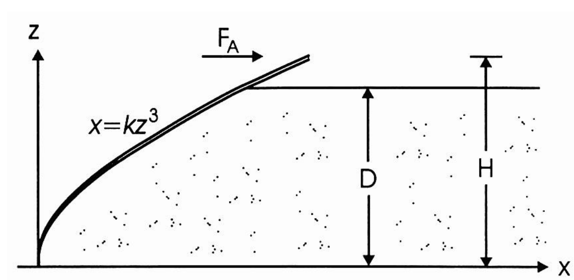
---

density ρ is made from plates of width B hinged together and constrained by cables spaced L apart so that they form a vee of included angle 2θ. If the water fills the channel to height H, find the tension T in the cables in terms of ρ, B, H, L and the acceleration due to gravity g.

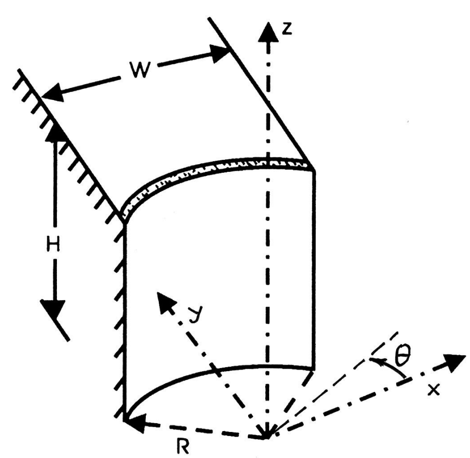


2.19) A liquid having density ρ is confined by a wall in the form of a cylindrical parabola having the shape $z = x^2$, and by the plane $z = 0$. With $\boldsymbol{g} = -g\boldsymbol{i}_z$, if the liquid lies to the left of the parabola, and if its free surface is at $z = H$, calculate in magnitude and direction the force per unit length in the x-direction exerted by the liquid on the wall. Do this two ways: first by direct integration along the parabola, and second by considering the forces acting on a control volume consisting of the parabola and the planes $x = 0$ and $z = H$.

2.20) The cylindrical gate OA in a channel of width W has the shape $x = Kz^3$ for $0 \le z \le H$. If the channel is filled with water of density ρ to a depth $D < H$, find the horizontal force F_A that must be applied at the height H to hold the gate shut.

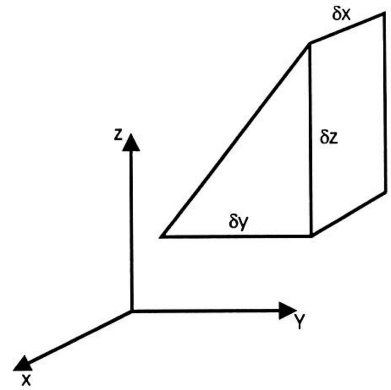


2.21) A barrier in a rectangular channel of width W and depth H takes the form of part of a surface of a right circular cylinder of radius R with its axis vertical as depicted in the diagram. Assuming that the channel contains water of density ρ to the full depth of the channel, using cylindrical coordinates, by carrying through the surfaceintegral, calculate the force acting on the barrier, showing that it is independent of R. Hint: Use z and the angle θ (or any other suitable angle) to parametrize the surface, thus



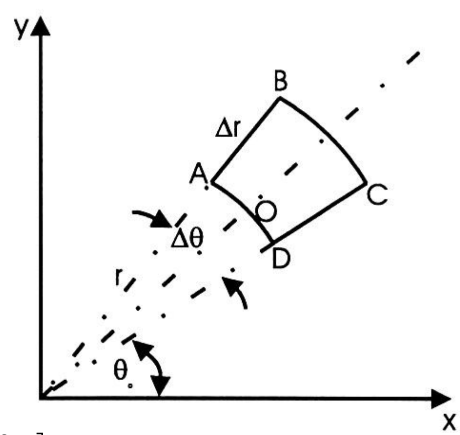
obtaining the area of a surface element δS by a simple geometrical argument.

2.22) According to $\mathfrak{T}$4.1 the resultant pressure force acting on a fluid particle of volume δV is $\delta \boldsymbol{F}_p = -\nabla p \delta V$. In Section 2.4 we use a Taylor's series approach to obtain this result for rectangular volume. Use this method for the triangular volume in the diagram, and thus verify the theorem. For simplicity assume $p = p(y, z)$ alone.



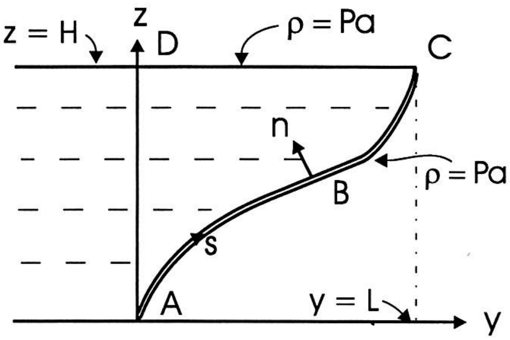
2.23) Demonstrate $\mathfrak{T}$4.1 for a right circular cylinder of radius R and length L having its axis located on the x-axis by assuming a pressure field of the form $p(x, y, z) = p_0 + ay + bz$ and then by integrating the pressure forces around the periphery of the cylinder.

2.24) Given a pressure field in cylindrical polar coordinates $p(r, \theta, z)$, by determining the pressure forces acting on the four sides AB, BC, CD, and DA of the sectorial volume depicted below show that, in the limit $\delta V \to 0$, the resultant pressure force acting in the $\boldsymbol{i}_r$ direction is $\delta F_r = -(\partial p/\partial r)\delta V$. Begin by assuming that the pressure field has no dependence on z, and can be expressed as

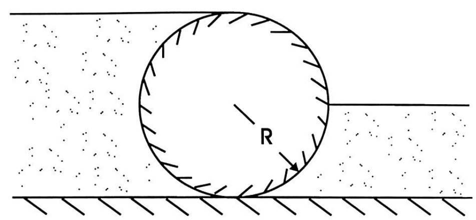


$$p - p_O = \left[\frac{\partial p}{\partial r}\right]_O (r - r_O) + \left[\frac{\partial p}{\partial \theta}\right]_O (\theta - \theta_O)$$

2.25) A cylindrical wall in the form of a thin rigid sheet of width W has the shape $y = y_w(s)$ for $0 \le y \le L$ and $z = z_w(s)$ for $0 \le z \le H$, where s is the distance along the wall as shown in the diagram. It is acted upon by liquid of density ρ from above and by atmospheric pressure P_a from below as depicted. By direct integration of the pressure distribution along the wall, show that the vertical component of the force acting on the wall is equal to the weight of the liquid in the volume ABCD directly above it, and that the horizontal component is equal to the force exerted on the fluid in ABCD lying to the left of DA. Hint: First show that the unit normal on the wall pointing to the interior of ABCD is given by $\boldsymbol{n} = -(dz_w/ds)\,\boldsymbol{i}_y + (dy_w/ds)\,\boldsymbol{i}_z$



2.26) A right circular cylinder of radius R is used as a barrier in a channel of width B. If water, of density ρ, rises to a height R on one side and to a height $2R$ on the other, find the mass M that the cylinder must have so that it is not lifted by the water.

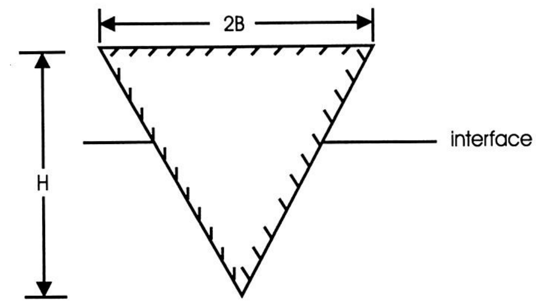


2.27) A closed thin-walled circular cylindrical metal drum of height H and diameter $D = 3H$ floats with its axis vertical in water of density ρ to a height $h \le H$ because the interior contains only air. Owing to corrosion a small hole opens in the base of the drum, so that water leaks slowly in. Assuming that none of the air escapes, find the depth to which the drum must sink before the air pressure buildup prevents any further leakage into the drum. Assume that the walls of the drum are so thin that its interior volume is effectively the same as its exterior, that the air inside is initially at atmospheric pressure P_a, and that both P_a and the atmospheric temperature remain constant throughout the process.

2.28) A vessel containing water stands on a balance which records a total weight of vessel and water of 5.0 kg. An iron block of mass 2.7 kg and relative density 7.5 is suspended by a fine wire from a spring balance and then lowered into the water until it is completely immersed. What are the readings on the two balances?

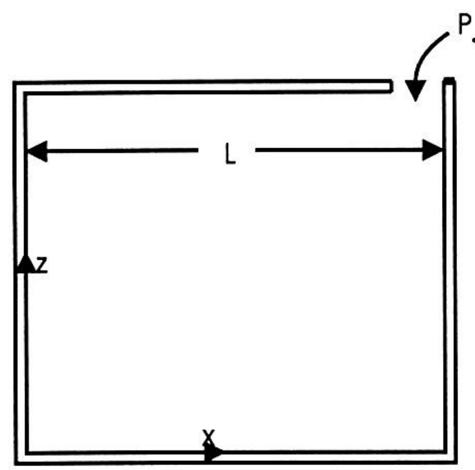
2.29) A long right circular cylinder of radius R and relative density $\sigma_B = 0.9$ floats with its axis horizontal at the interface between two immiscible liquids, the lower having $\sigma_L = 1.5$, and the upper, $\sigma_U = 0.8$. The cylinder axis subtends an angle 2θ with respect to the interface between the two liquids. Show that $\theta - \sin\theta\cos\theta = \pi/7$. Hint: For this exercise it is simpler to develop a version of Archimedes' principle.

2.30) A triangular beam with width $2B$, height H, and relative density 0.9 floats apex down at the interface of two immiscible liquids having relative densities 1.0 and 0.8 respectively. By integrating the pressure distributions along the sides to determine the forces acting there, find the distance the apex penetrates the lower liquid. Express your result as a fraction of H.



2.31) Given that measurements of the density of icebergs show that they are somewhat porous, having σ_U = 0.92 approximately, estimate the fraction of an iceberg that floats under water. For this purpose, assume that sea water has σ_L = 1.023. First compute your result assuming that, for air at 0 degC, σ_{air} = 0, and then estimate the error incurred by incorporating the effect of air density at the conditions of 101.3 kPa and 0°K.

2.32) A square box has interior lengths L and is closed except for a small slot in the upper right-hand corner of the diagram. It is filled with a liquid having density ρ, and then subject to positive accelerations a_x and a_z in the x and z-directions respectively. No liquid spills from the box, so that it remains completely full, and the slot remains subject to the atmospheric pressure P_a. Assuming that each particle is subject to both gravity and the accelerations imposed on the box, find the pressure distribution in the box.

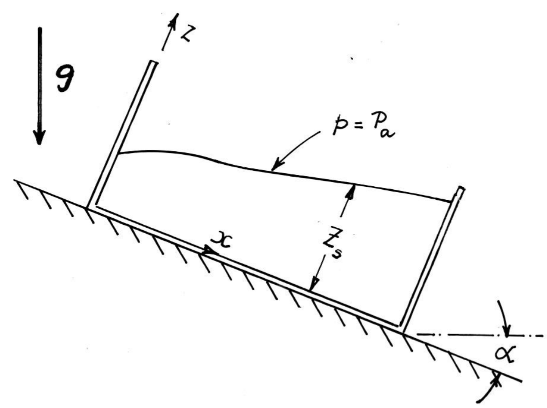


2.33) A cubical box having an internal side length L as depicted in the diagram for Exercise 2.32 is filled with water to a height h. A slot along one side is open to the atmosphere. It is subject to an acceleration $a_x = g/2$ in the x-direction. Assume that no liquid is spilled during start-up. Determine the shape of the surface and the pressure at the two lower corners. Verify that the force exerted on the bottom of the box remains equal to the weight of the liquid. Assume that the fluid does not contact the upper surface of the box.

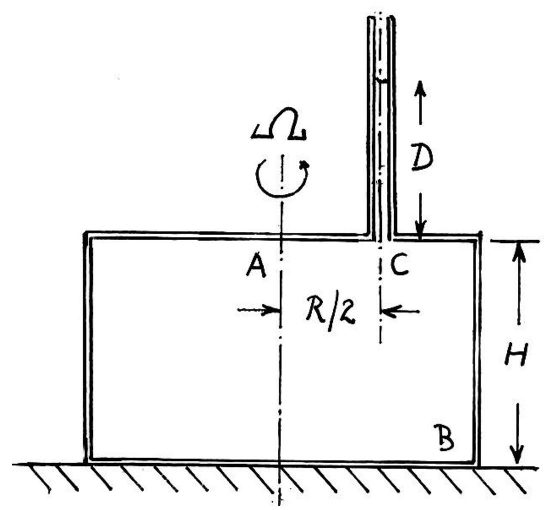
2.34) A rectangular box of length L in the x-direction, width W and height H is open at the top and is filled with a liquid. It is gradually accelerated horizontally, in the direction L, and vertically. The accelerations are built up gradually from zero to $a_x = a_z = g/2$, and maintained at this level for an extended period. If exactly half the liquid spills out of the box, what is the length L?

2.35) A cubic block of density ρ_B and having edges of length B floats in a vessel containing liquid of density $\rho_L > \rho_B$. If the vessel moves upwards with uniform acceleration a, find the change in the position of the block relative to the liquid surface.

2.36) A rectangular box containing liquid slides down an inclined plane under the action of gravity as shown in the diagram. If the plane is inclined at an angle α to the horizontal, and if the coefficient of sliding friction between the box and the plane is k_f find the shape of the surface relative to the coordinate system attached to the box.

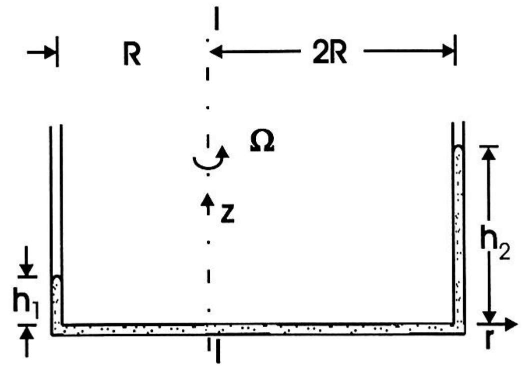


2.37) A right circular cylinder of internal radius R and height H is mounted on a turntable with its axis vertical. The cylinder is closed at the top and a small diameter vertical tube is mounted on the top surface at a distance $R/2$ from the cylinder axis as shown in the diagram. The cylinder and the tube are filled with a liquid of density ρ such that this liquid is at the height D above the top interior surface of the cylinder. The turntable rotates the cylinder about its axis at angular speed Ω and, after a suitable interval, all particles rotate at Ω.



Find the pressure distribution, $p(r, \theta, z)$, within the apparatus and determine the pressure at the points A, B, and C indicated on the diagram. Determine the value of Ω at which liquid begins to move up the tube.

2.38) Liquid in the U-tube shown in the diagram below fills both legs to a height H. The tube then rotates about the offset vertical axis I-I at angular speed Ω, and the liquid eventually reaches different heights h_1 and h_2 in the two legs. Find these heights by assuming that no spillage occurs during start-up.



2.39) A vessel in the form of a right-circular cylinder of internal diameter $D = 2R$ has its axis vertical and is open at the top. Two immiscible liquids are placed in it, one on top of the other, each to a height H. The lower liquid, of density ρ_L, rotates as a rigid body around the cylinder axis at angular speed Ω_L, and the upper liquid, of density $\rho_U < \rho_L$, also rotates at angular speed Ω_U. Find the shape of the interface.
