MAT195S CALCULUS II

Midterm Test #2

28 March 2019 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: F. Al Faisal and J. W. Davis

Family Name: J Davis.

Given Name: Solutions

Student #: ______

FOR MARKER USE ONLY		
Question	Marks	Earned
1	6	
2	9	
3	10	
4	23	
5	8	
6	8	
7	8	
TOTAL	72	/ 65

Tutorial Section: ______

TA Name: ______

1) Test the series for convergence or divergence:

a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+4}$ b) $\sum_{k=2}^{\infty} \left(\frac{k}{\ln k}\right)^k$

(6 marks)

a) $\lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{1}{n+4/n^2} \to 0$

$\Rightarrow$ for $n \geq 2$, $\frac{4}{n^2} \leq 1$ $\therefore$ $n+1+\frac{4}{(n+1)^2} > n+\frac{4}{n^2}$

$\therefore a_{n+1} = \frac{(n+1)^2}{(n+1)^3+4} = \frac{1}{n+1+\frac{4}{(n+1)^2}} < \frac{1}{n+\frac{4}{n^2}} = \frac{n^2}{n^3+4} = a_n$

$\therefore$ Convergent by Alt. Series test.
(see Stewart Sol'n manual for alternate approach)

b) root test : $\lim_{k\to\infty} (a_k)^{1/k} = \lim_{k\to\infty} \frac{k}{\ln k} \stackrel{*}{=} \lim_{n\to\infty} \frac{1}{1/k} \to \infty$

$\therefore$ divergent

2) a) Using the usual notation $i = \sqrt{-1}$, use the Taylor series expansions for e^x, $\cos x$ and $\sin x$ to show that $e^{i\theta} = \cos\theta + i\sin\theta$. Note that the Taylor series for e^x derived in class works just as well when x is a complex number: it converges to e^x for all complex x.

(5 marks)

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}(-1)^n$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(-1)^n$$

$$\Rightarrow e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$= \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] + i\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \cos x + i \sin x$$

b) Use power series representations of $\cos x$ and e^x to evaluate the limit: $\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}$

(4 marks)

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \Rightarrow 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \Rightarrow 1 + x - e^x = -\frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!} - \cdots$$

$$\therefore \lim_{x\to 0} \frac{1-\cos x}{1+x-e^x} = \lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots}{-\frac{x^2}{2} - \frac{x^3}{3!} - \frac{x^4}{4!} - \cdots}$$

$$= \lim_{x\to 0} \frac{\frac{1}{2} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots}{-\frac{1}{2} - \frac{x}{3!} - \frac{x^2}{4!} - \cdots}$$

$$= \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$

3) a) Prove part (iii) of the Ratio Test: Let $\Sigma\, a_k$ be a series with positive terms, and suppose that:

$$\frac{a_{k+1}}{a_k} \to \lambda \quad as \quad k \to \infty$$

Show that if $\lambda = 1$, the test is inconclusive; the series may either converge or diverge.

Hint: consider $\Sigma(1/k)$ and $\Sigma(1/k^2)$.

(5 marks)

i) let $a_k = \frac{1}{k}$: ratio test : $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{1}{k+1}}{\frac{1}{k}}\right| = \left|\frac{k}{k+1}\right| \longrightarrow 1$ as $k \to \infty$

ii) let $b_k = \frac{1}{k^2}$: ratio test : $\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{k}{k+1}\right|^2 \longrightarrow 1$ as $k \to \infty$

But $\Sigma \frac{1}{k}$ diverges while $\Sigma \frac{1}{k^2}$ converges

$\therefore$ the ratio test is inconclusive when $\lambda = 1$

b) Prove part (ii) of the Root Test: Let $\Sigma\, a_k$ be a series with non-negative terms, and suppose that: $(a_k)^{1/k} \to \rho \quad as \quad k \to \infty$

Show that if $\rho > 1$, then $\Sigma\, a_k$ diverges.

(5 marks)

Given $(a_k)^{1/k} \longrightarrow \rho > 1$ as $k \to \infty$

$\therefore$ for $k > K$ $(a_k)^{1/k} > 1$

$\therefore \left((a_k)^{1/k}\right)^k > 1^k \Rightarrow a_k > 1$

for $k > K$

$\therefore a_k \not\to 0$

$\therefore \Sigma a_k$ diverges by the test for divergence

4) a) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(t) = \begin{cases} 0 & -1 \le t \le 0 \\ 4(t-t^2) & 0 < t \le 1 \end{cases}$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

b) Using (a), show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. [Hint: look at $f(t)$ at $t = 0$]

c) Using (b), show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$.

d) Finally, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$. [Use (a) with $t = 1/2$, and (c)]

Helpful integrals:

$$\int u \sin u \, du = \sin u - u\cos u + C$$

$$\int u \cos u \, du = \cos u + u \sin u + C$$

$$\int u^2 \sin u \, du = 2u \sin u - (u^2 - 2)\cos u + C$$

$$\int u^2 \cos u \, du = 2u \cos u + (u^2 - 2)\sin u + C$$

(23 marks)

$$a_0 = \int_0^1 4(t-t^2)\,dt = 4\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{4}{6} = \frac{2}{3} \Rightarrow \frac{a_0}{2} = \frac{1}{3}$$

$$a_n = \int_0^1 4(t-t^2)\cos(n\pi t)\,dt \qquad \text{where } \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$= \frac{4}{n^2\pi^2}\int_0^1 (n\pi t)\cos(n\pi t)\,d(n\pi t) - \frac{4}{n^3\pi^3}\int_0^1 (n\pi t)^2 \cos(n\pi t)\,d(n\pi t)$$

$$= \frac{4}{n^2\pi^2}\left[\cos(n\pi t) + (n\pi t)\sin(n\pi t)\right]_0^1 - \frac{4}{n^3\pi^3}\left[2(n\pi t)\cos(n\pi t) + ((n\pi t)^2 - 2)\sin(n\pi t)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left(-1 + (-1)^n\right) - \frac{8}{n^2\pi^2}(-1)^n$$

$$= \frac{4}{n^2\pi^2}\left(-1 + (-1)^n - 2(-1)^n\right) = \frac{-4}{n^2\pi^2}\left(1 + (-1)^n\right)$$

4) continued

$$b_n = \int_0^1 4(t-t^2)\sin(n\pi t)\,dt$$

$$= \frac{4}{n^2\pi^2}\int_0^1 (n\pi t)\sin(n\pi t)\,d(n\pi t) - \frac{4}{n^3\pi^3}\int_0^1 (n\pi t)^2\sin(n\pi t)\,d(n\pi t)$$

$$= \frac{4}{n^2\pi^2}\Big[\underbrace{\sin(n\pi t)}_{0} - (n\pi t)\cos(n\pi t)\Big]_0^1 - \frac{4}{n^3\pi^3}\Big[\underbrace{2(n\pi t)\sin(n\pi t)}_{0} - \left((n\pi t)^2 - 2\right)\cos(n\pi t)\Big]_0^1$$

$$= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n\pi}(-1)^n - \frac{4}{n^3\pi^3}\left(-2\left(-1+(-1)^n\right)\right)$$

$$= \frac{8}{n^3\pi^3}\left(1-(-1)^n\right)$$

$$\Rightarrow f(t) = \frac{1}{3} + \sum_{n=1}^{\infty}\left[\frac{-4}{n^2\pi^2}\left(1+(-1)^n\right)\cos n\pi t + \frac{8}{n^3\pi^3}\left(1-(-1)^n\right)\sin n\pi t\right]$$

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b) for $t=0$: $f(t)=0$, $\sin n\pi t = 0$

$$\Rightarrow f(0) = \frac{1}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(1+(-1)^n\right)(1)$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{2}{n^2}\underbrace{\left(1+(-1)^n\right)}_{=\begin{cases}2 & n \text{ even}\\ 0 & n \text{ odd}\end{cases}} = \sum_{n=1}^{\infty}\frac{2}{(2n)^2}\cdot 2 = \sum_{n=1}^{\infty}\frac{2^2}{(2n)^2}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

c) from (b) : $\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right) = \underbrace{\sum_{n=1}^{\infty}\frac{1}{n^2}}_{=\frac{\pi^2}{6}} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$$\therefore -\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

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d) for $t = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2} - \frac{1}{4}\right) = 1$

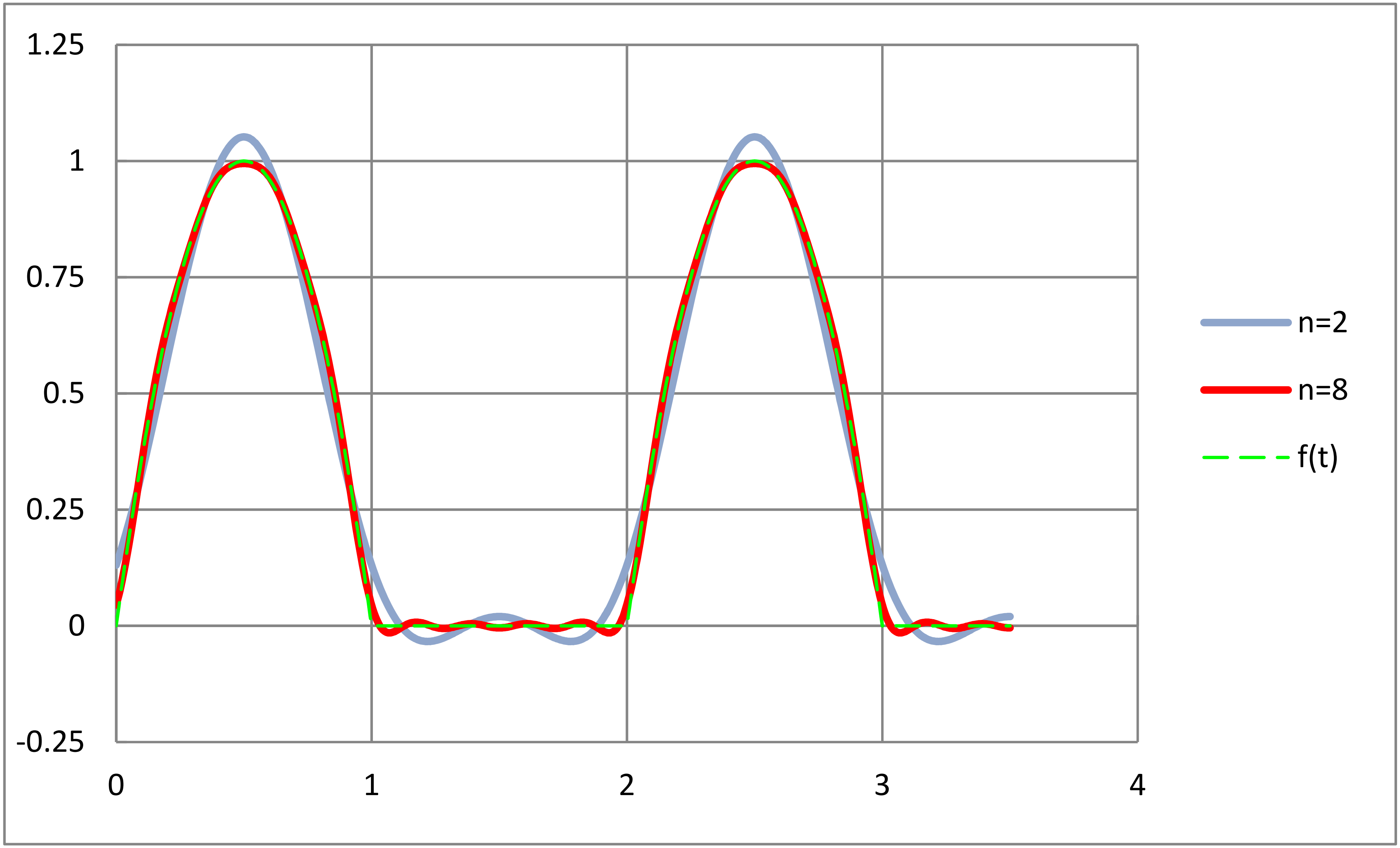
$$\therefore 1 = \frac{1}{3} + \underbrace{\sum_{n=1}^{\infty}\frac{-4}{n^2\pi^2}\left(1+(-1)^n\right)\cos\frac{n\pi}{2}}_{\text{all odd terms} = 0} + \underbrace{\sum_{n=1}^{\infty}\frac{8}{n^3\pi^3}\left(1-(-1)^n\right)\sin\frac{n\pi}{2}}_{\text{all even terms} = 0}$$

$$\Rightarrow \frac{2}{3} = \sum_{n=1}^{\infty}\frac{-4}{(2n)^2\pi^2}\cdot 2(-1)^n + \sum_{n=1}^{\infty}\frac{8}{(2n-1)^3\pi^3}\cdot 2\cdot(-1)^{n+1}$$

$$= \sum_{n=1}^{\infty} -\frac{2^2}{(2n)^2\pi^2}\cdot 2(-1)^n + \sum_{n=1}^{\infty}\frac{16}{(2n-1)^3\pi^3}(-1)^{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{-2}{n^2\pi^2}(-1)^n + \sum_{n=1}^{\infty}\frac{16(-1)^{n+1}}{(2n-1)^3\pi^3} = \underbrace{\frac{2}{\pi^2}\sum\frac{(-1)^{n+1}}{n^2}}_{=\frac{\pi^2}{12}\cdot\frac{2}{\pi^2}} + \sum_{n=1}^{\infty}\frac{16(-1)^{n+1}}{(2n-1)^3\pi^3}$$

$$\Rightarrow \frac{2}{3} - \frac{1}{6} = \frac{1}{2} = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3} \Rightarrow \frac{\pi^3}{32} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}$$


1.25
1
0.75
0.5
0.25
0
-0.25
0
1
2
3
4
n=2
n=8
f(t)

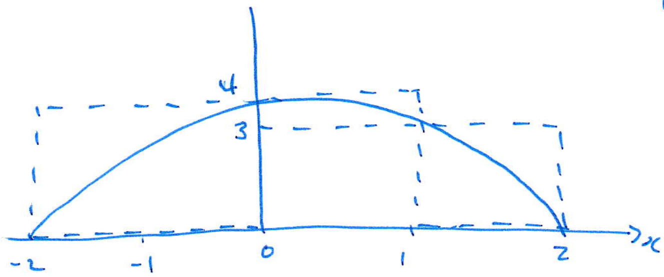
5) a) Let $f(x) = 4 - x^2$ for $x \in [-2,2]$. Give an example of a partition of $[-2,2]$ such that the lower sum $L_P = 3$ and the upper sum $U_P = 15$.

(2 marks)

choose partition $[-2, 0, 1, 2]$

$L_P = 0 \cdot 2 + 3 \cdot 1 + 0 \cdot 1 = 3$

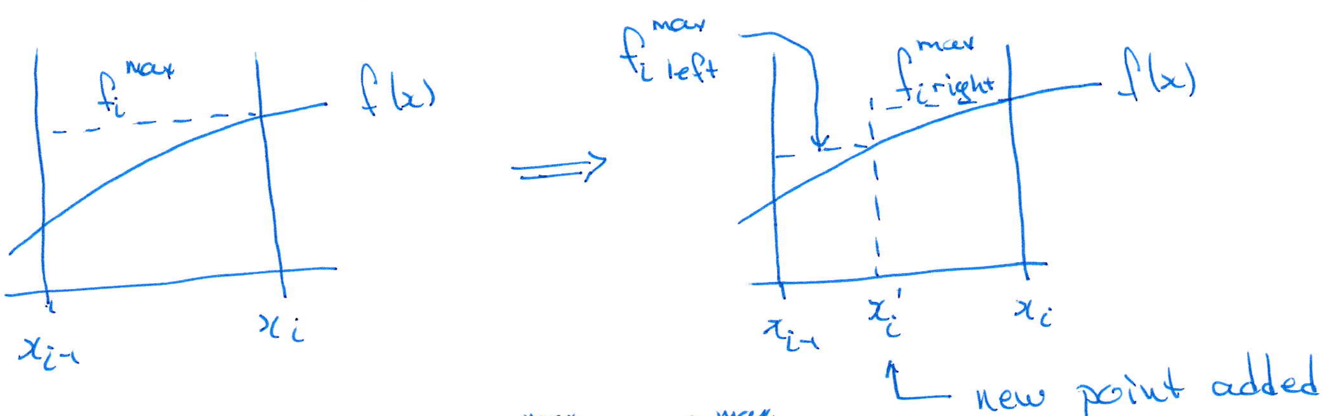
$U_P = 4 \cdot 2 + 4 \cdot 1 + 3 \cdot 1 = 15$



b) Given f, a continuous function on $[a,b]$, and given the partition, P, of the x-axis, $x \in [a,b]$, we create a refinement of P, P', by adding more points of subdivision to P. Show that $U_{P'} \le U_P$, where, $U_P \equiv \sum_{i=1}^{n} f_i^{\max} \Delta x_i$ and $f_i^{\max}$ is the maximum value of f in the interval $[x_{i-1}, x_i]$. Support your arguments with a sketch.

(6 marks)

Consider the i^{th} interval to be representative of all intervals when a point of subdivision has been added:



$f_{i\,left}^{max} \le f_i^{max}$ & $f_{i\,right}^{max} \le f_i^{max}$

$\therefore f_i^{max} \cdot (x_i - x_{i-1}) \ge f_{i\,left}^{max} \cdot (x_i' - x_{i-1}) + f_{i\,right}^{max} \cdot (x_i - x_i')$

Thus, the terms of U_P can only decrease by adding further points of subdivision; $\therefore U_{P'} \le U_P$

6) Consider the curve given parametrically by $\vec{r}(t) = (x(t), y(t))$, where $x(t) = \int_0^t \cos(\pi\tau^2)d\tau$ and $y(t) = \int_0^t \sin(\pi\tau^2)d\tau$, $t \geq 0$.

a) Find $\vec{r}\,'(t)$ and ds/dt, and hence show that the arclength of this curve is given by $s(t) = t$.

b) Find the unit tangent vector $\hat{T}(s)$.

c) Find the curvature $\kappa(s)$.

(8 marks)

a) $x(t) = \int_0^t \cos(\pi\tau^2)\,d\tau \Rightarrow x'(t) = \cos(\pi t^2)$

$y(t) = \int_0^t \sin(\pi\tau^2)\,d\tau \Rightarrow y'(t) = \sin(\pi t^2)$

$\Rightarrow \vec{r}\,'(t) = (\cos(\pi t^2), \sin(\pi t^2)) \Rightarrow \frac{ds}{dt} = \|\vec{r}\,'(t)\| = \sqrt{(x')^2 + (y')^2}$

$= \sqrt{\cos^2(\pi t^2) + \sin^2(\pi t^2)}$

$= 1$

$\therefore s(t) = \int_0^t \|\vec{r}\,'(\tau)\|\,d\tau = \int_0^t d\tau = t$

b) $\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = (\cos(\pi t^2), \sin(\pi t^2))$

$\Rightarrow s = t \Rightarrow \vec{T}(s) = (\cos(\pi s^2), \sin(\pi s^2))$

c) $\kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d}{ds}(\cos(\pi s^2), \sin(\pi s^2)) \right\|$

$= \|(-2\pi s \sin(\pi s^2), 2\pi s \cos(\pi s^2))\|$

$= 2\pi s \sqrt{\sin^2(\pi s^2) + \cos^2(\pi s^2)}$

$= 2\pi s$

7) Is it possible to define $f(x,y)$ at $(0,0)$ to make the given function continuous?

a) $f(x,y) = \dfrac{xy}{x^2+y^2}$

b) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

(8 marks)

a) $\lim_{x\to 0} f(x,0) = \dfrac{0}{x^2} = 0$

$\lim_{y\to 0} f(0,y) = \dfrac{0}{y^2} = 0$

$\lim_{x\to 0} f(x,x) = \dfrac{x^2}{2x^2} = \dfrac{1}{2}$

$\therefore \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ DNE

$\therefore$ It is not possible to make $f(x,y)$ continuous at $(0,0)$.

b) show $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$

$\Rightarrow \quad |x| \le \sqrt{x^2+y^2} \quad \therefore \dfrac{|x|}{\sqrt{x^2+y^2}} \le 1$

$\therefore \dfrac{|xy|}{\sqrt{x^2+y^2}} \le |y| \longrightarrow 0$ as $(x,y)\to(0,0)$

Squeeze Th'm: $0 \le \left| \dfrac{xy}{\sqrt{x^2+y^2}} \right| = \dfrac{|y|\,|x|}{\sqrt{x^2+y^2}} \le |y| \longrightarrow 0$

$\therefore \dfrac{xy}{\sqrt{x^2+y^2}} \longrightarrow 0$

$\therefore f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ can be made continuous at $(0,0)$

by setting $f(0,0) = 0$.