UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test I

MAT185H1S — Linear Algebra

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Student Name:	Copy	Fair
	Last Name	First Names

Student No:

e-Address:

Signature:

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution. The total number of marks available is **50**.
3. Write solutions *only* in the boxed space provided for each question. *Do not* write solutions on the reverse side of pages. These will *not* be scanned and therefore will *not* be marked.
4. Two blank pages are provided at the end for rough work. Work on these back pages will *not* be marked unless clearly indicated; in such cases, clearly indicate on the question page(s) that the solution(s) is continued on a back page(s).
5. *Do not* write over the QR code on the top right-hand corner of each page.
6. *No* aid is permitted.
7. The duration of this test is 90 minutes.
8. There are 6 pages and 5 questions in this test paper.

A Note on Notation:

1. ${}^m\mathbb{R}^n = M_{m\times n}(\mathbb{R})$, the former notation is used in the Notes and the latter in Nicholson.

A. Definitions and Statements

Fill in the blanks.

1(a). The *commutative property for vector addition* states that

$$\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x} \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} \text{ (vector space)}$$

/2

1(b). The *image space* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$ is defined as

$$\operatorname{im} \mathbf{A} = \{\mathbf{y} \in {}^m\mathbb{R} \,|\, \mathbf{y} = \mathbf{A}\mathbf{x} \quad \forall \boldsymbol{x} \in {}^n\mathbb{R}\}$$

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1(c). State the *Fundamental Theorem of Linear Algebra*.

As in Notes.

/2

1(d). The *linear independence* of a set of vectors is defined as

As in Notes.

/2

1(e). A *basis* for a vector space $\mathcal{V}$ is defined as

As in Notes.

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B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. A correct answer earns 2 marks but 2 marks will be deducted for an incorrect answer; the minimum total mark for this section is 0.*

2(a). For $\boldsymbol{v}$ in a vector space, if $\alpha\boldsymbol{v} = \beta\boldsymbol{v}$ then $\alpha = \beta$ or $\boldsymbol{v} = \boldsymbol{0}$. $\boxed{\text{T}}$

2(b). The set $S = \{\mathbf{x} \in {}^n\mathbb{R} \,|\, \mathbf{Ax} = \mathbf{b} \neq \mathbf{0}$ for given $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\mathbf{b} \in {}^m\mathbb{R}\}$ is a subspace of ${}^n\mathbb{R}$. $\boxed{\text{F}}$

2(c). The set of vectors $\{1 + x, 1 - x + 2x^2, 1 + x^2\} \subset \mathbb{P}_2$ is linearly independent. $\boxed{\text{F}}$

2(d). Let $\boldsymbol{w}_1 \cdots \boldsymbol{w}_l$ be vectors in a vector space and let $\boldsymbol{v}_i \in \operatorname{span}\{\boldsymbol{w}_1 \cdots \boldsymbol{w}_l\}$ for $i = 1 \cdots k$. Then $\operatorname{span}\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_k\} = \operatorname{span}\{\boldsymbol{w}_1 \cdots \boldsymbol{w}_l\}$. $\boxed{\text{F}}$

2(e). If $\mathcal{W}$ is a subspace of a vector space $\mathcal{V}$, then $\mathcal{W} = \operatorname{span}\mathcal{W}$. $\boxed{\text{T}}$

C. Problems

3. Let $V = \{\boldsymbol{x}, \boldsymbol{y}\}$ be a set with exactly two vectors, $\boldsymbol{x}$ and $\boldsymbol{y}$. Define vector addition and scalar multiplication as follows:

Vector addition: $\boldsymbol{x} + \boldsymbol{x} = \boldsymbol{x}$, $\boldsymbol{y} + \boldsymbol{y} = \boldsymbol{x}$, $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y}$ and $\boldsymbol{y} + \boldsymbol{x} = \boldsymbol{y}$
Scalar multiplication: $\alpha\boldsymbol{x} = \boldsymbol{x}$ and $\alpha\boldsymbol{y} = \boldsymbol{y}$ for all $\alpha \in \mathbb{R}$

Using only the definition of a vector space, show that V is not a vector space.

Axiom **MIII**(a) fails. Consider that, by scalar multiplication,

$$(\alpha + \beta)\boldsymbol{y} = \gamma\boldsymbol{y} = \boldsymbol{y}$$

but, by distribution over scalar addition,

$$(\alpha + \beta)\boldsymbol{y} = \alpha\boldsymbol{y} + \beta\boldsymbol{y} = \boldsymbol{y} + \boldsymbol{y} = \boldsymbol{x} \neq \boldsymbol{y}$$

Hence V is not a vector space.

/10

4. Suppose $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are linearly independent vectors in a vector space $\mathcal{V}$ and let

$$\boldsymbol{y}_1 = \boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3, \quad \boldsymbol{y}_2 = \boldsymbol{x}_1 + a\boldsymbol{x}_2, \quad \boldsymbol{y}_3 = \boldsymbol{x}_2 + b\boldsymbol{x}_3$$

Find the condition that must be satisfied by the scalars a and b to make the vectors $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ linearly independent.

The condition for linear independence is that

$$\lambda_1 \boldsymbol{y}_1 + \lambda_2 \boldsymbol{y}_2 + \lambda_3 \boldsymbol{y}_3 = \boldsymbol{0}$$

implies all $\lambda_i = 0$. Substituting for $\boldsymbol{y}_i$, we obtain

$$(\lambda_1 + \lambda_2)\boldsymbol{x}_1 + (\lambda_1 + a\lambda_2 + \lambda_3)\boldsymbol{x}_2 + (\lambda_1 + b\lambda_3)\boldsymbol{x}_3 = \boldsymbol{0}$$

As $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are linearly independent, we must have

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ \lambda_1 + a\lambda_2 + \lambda_3 &= 0 \\ \lambda_1 + b\lambda_3 &= 0 \end{aligned}$$

or

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & a & 1 \\ 1 & 0 & b \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \mathbf{0}$$

To ensure that all $\lambda_i = 0$, the matrix must be invertible, i.e., the determinant of the matrix must be nonzero. Thus

$$(a-1)b + 1 \neq 0$$

ensures that $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3\}$ is linearly independent

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5. Let $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ be vectors in a vector space $\mathcal{V}$. Suppose that $\boldsymbol{x}_3 = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{x}_4 = 2\boldsymbol{x}_1 + 3\boldsymbol{x}_2 - \boldsymbol{x}_3$ and that $\mathcal{W} = \operatorname{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\}$.

(a) What are the possible dimensions of $\mathcal{W}$?

(b) Suppose $\dim \mathcal{W} = 2$. Must $\{\boldsymbol{x}_3, \boldsymbol{x}_4\}$ be linearly independent? Explain.

(a) Given that $\boldsymbol{x}_3$ is a linear combination of $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$, $\boldsymbol{x}_4 = \boldsymbol{x}_1 + 4\boldsymbol{x}_2$, upon substituting for $\boldsymbol{x}_3$, is also a linear combination of $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$. Hence

$$\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4\} \subset \operatorname{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$$

Accordingly,

$$\mathcal{W} = \operatorname{span}\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$$

The maximum dimension of $\mathcal{W}$ is 2. If $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are linearly dependent and not zero, the dimension is 1. If $\boldsymbol{x}_1 = \boldsymbol{x}_2 = \boldsymbol{0}$, then the dimension is 0.

(b) If $\dim \mathcal{W} = 2$, $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ must be a basis and thus linearly independent. Testing the linear independence of $\{\boldsymbol{x}_3, \boldsymbol{x}_4\}$, we consider

$$\lambda_1 \boldsymbol{x}_3 + \lambda_2 \boldsymbol{x}_4 = \boldsymbol{0}$$

which implies that

$$(\lambda_1 + \lambda_2)\boldsymbol{x}_1 + (-\lambda_1 + 4\lambda_2)\boldsymbol{x}_2 = \boldsymbol{0}$$

As $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ is linearly independent, we must have

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0 \\ -\lambda_1 + 4\lambda_2 &= 0 \end{aligned}$$

The only possible solution is $\lambda_1 = \lambda_2 = 0$, from which we conclude that $\{\boldsymbol{x}_3, \boldsymbol{x}_4\}$ must be linearly independent.

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