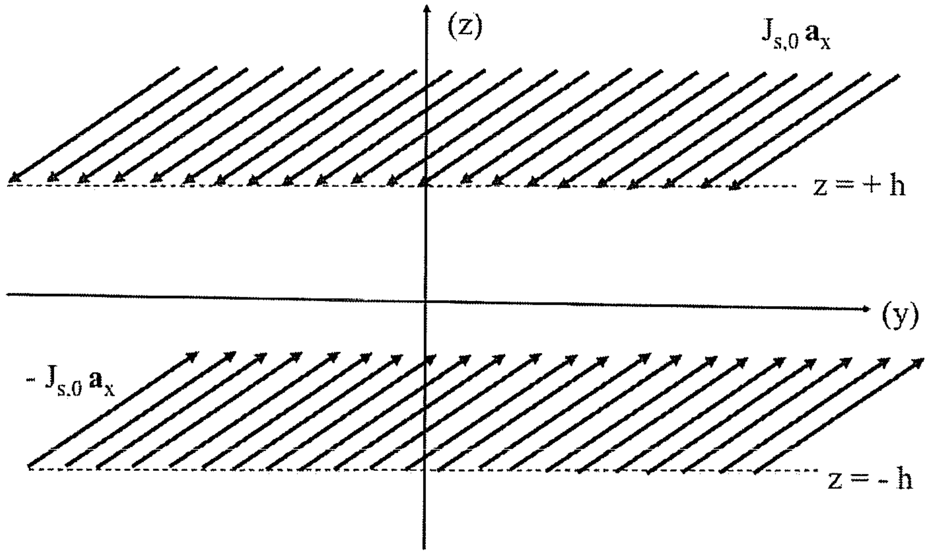
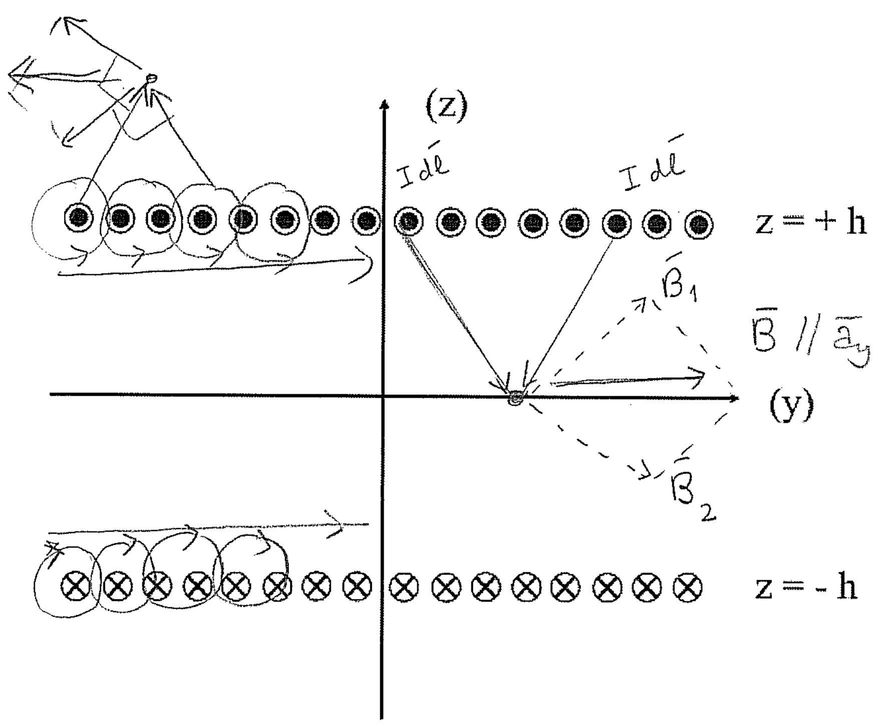
## Question 1

Consider two infinite current-carrying conducting planes at $z = \pm h$, as shown in the figure below. The currents flowing on these conductors have surface current densities $\mathbf{J}_s = \pm \mathbf{a}_x J_{s,0}$, respectively, where $J_{s,0}$ is a positive constant. The space inside and outside the conducting plates has $\epsilon = \epsilon_0$, $\mu = \mu_0$.



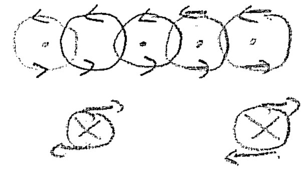

1. Using the Biot-Savart law or otherwise, show that the magnetic flux density $\mathbf{B}$ produced by these currents has only a $y$−component, i.e. $\mathbf{B} = B_y \mathbf{a}_y$. Show your work in the figure below, which depicts the $y - z$ plane. Discuss both $|z| \leq h$ and $|z| > h$. **(5 points)**



From Biot-Savart: $\bar{B} \parallel I d\bar{\ell} \times (\bar{R} - \bar{R}')$. Note that for any $I d\bar{\ell}$ creating $\bar{B}$ with $B_y$ and $B_z$ components there is a symmetrically placed one creating the same $B_y$ and opposite $B_z$ (see figure). So, by superposition, the total field is in the y-direction.

Taking the plane as a superposition of infinite line currents: One line current creates an $\bar{H}$:

. So, all of them:

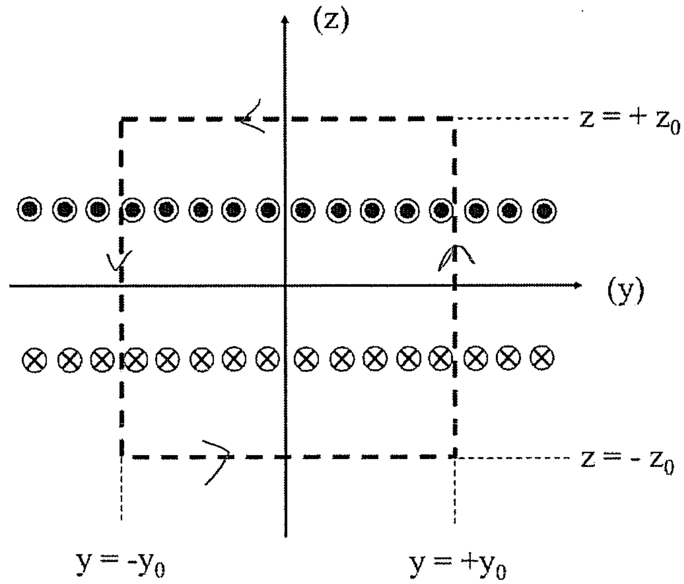


2. Show that $B_y = B_y(z)$, i.e. the only coordinate that the magnetic flux can possibly depend upon is the $z-$coordinate ~~and that $B_y(-z) = -B_y(z)$, i.e. $B_y$ is an odd function of $z$~~. **(3 points)**

Symmetry: $\partial/\partial x = 0 = \partial/\partial y$ because $\bar{J}_s$ is infinite in $x, y$ directions. Hence:

$$B_y = B_y(z).$$

3. By Ampere's law, using the path in dashed lines below, show that $\mathbf{B}(|z| > h) = 0$. You can use the results of 1, 2 even if you have not shown them. **(4 points)**




Assuming $B_y(z=z_0) = -B_y(z=-z_0)$

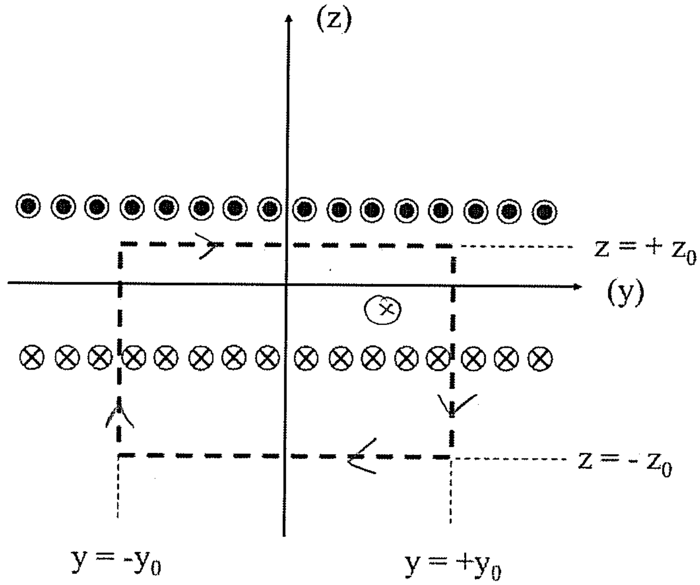
$$\oint \bar{B} \cdot d\bar{\ell} = [-B_y(z=z_0) + B_y(z=-z_0)] \cdot 2y_0$$ (2) pts.

(2) pts

$$= -4 B_y(z=z_0) \cdot y_0 = I_{enclosed} = 0 \Rightarrow$$

$B_y(z=z_0) = 0$ for any $z_0 > h$.

4. By Ampere's law, using the path in dashed lines below, show that $\mathbf{B}(|z| < h) = \mu_0 J_{s,0}\mathbf{a}_y$. You can use the results of 1, 2 even if you have not shown them. **(4 points)**



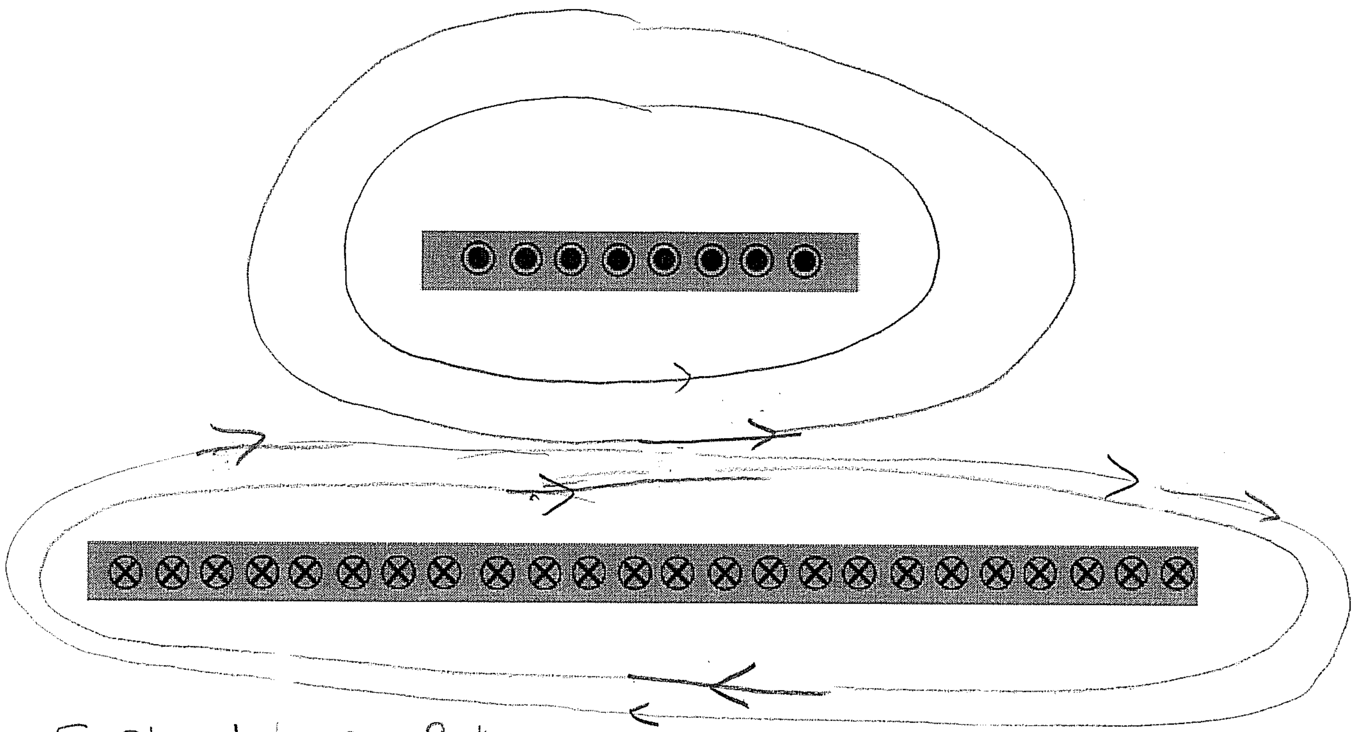

Tracing the path clockwise, the positive current direction is INTO the page.

$$\oint \bar{H} \cdot d\bar{l} = I \Rightarrow$$

$$\overbrace{H_y \cdot (2y_0)}^{2pts} = \overbrace{J_{s,0} \cdot 2w_0}^{2pts} \Rightarrow H_y = J_{s,0} \Rightarrow$$

$$B_y = \mu_0 J_{s,0}$$

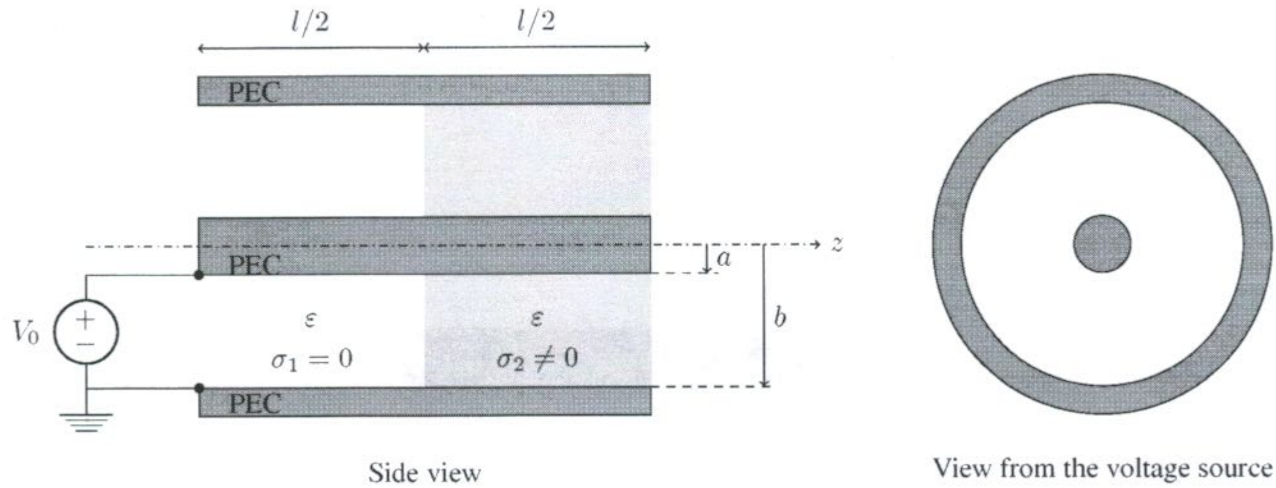
5. A practical situation where these magnetic fields arise is the "microstrip line", a fundamental component of high frequency circuits, where the two conductors are finite. On the figure below, which shows a cross-section of the microstrip line and the direction of the currents on the two conductors, sketch the magnetic flux lines that you expect. Briefly explain. **(4 points)**



Closed lines 2pts.

Correct direction around currents : 2pts (1+1)

## Question 2




Side view

View from the voltage source

1. We have the coaxial cable shown in the figure above. The cable conductors are perfect (PEC). The first half of the cable is filled with a perfect dielectric (permittivity $\varepsilon$, conductivity $\sigma_1 = 0$). The second half of the cable is filled with a lossy dielectric (permittivity $\varepsilon$, conductivity $\sigma_2$). The inner conductor is held at potential $V = V_0$, while the outer conductor is held at $V = 0$. Using the Poisson equation, find the electric field $\mathbf{E}$ inside the cable. You can assume that there is no net free charge $\rho_v$ inside both dielectrics. Neglect any edge effects (i.e. calculate the electric field for an infinite cylinder). **(10 points)**

We first solve for V.

Because of symmetry, $V = V(r)$ } [1 pt]

$\nabla \cdot (\varepsilon \nabla V) = -\rho_v$

$\rho_v = 0$
$\varepsilon$ uniform inside cable [1 pt]

$\nabla^2 V = 0$

$$\cancel{\frac{1}{r}} \frac{\partial}{\partial r}\left[r \frac{\partial V}{\partial r}\right] = 0$$

$$\frac{\partial V}{\partial r} = \frac{C_1}{r}$$

$V(r) = C_1 \ln r + C_2$ [2pt]

We impose the boundary conditions

$$\begin{cases} V(b) = 0 \\ V(a) = V_0 \end{cases} \rightarrow \begin{cases} C_1 \ln b + C_2 = 0 \\ C_1 \ln a + C_2 = V_0 \end{cases}$$

$$C_2 = -C_1 \ln b$$

$$C_1 \ln a - C_1 \ln b = V_0 \qquad C_1 = \frac{V_0}{\ln \frac{a}{b}} = -\frac{V_0}{\ln \frac{b}{a}}$$

$$C_2 = \frac{V_0}{\ln \frac{b}{a}} \ln b$$

[2pt]

$$V(r) = -\frac{V_0}{\ln b/a} \ln r + \frac{V_0}{\ln \frac{b}{a}} \ln b = \frac{V_0}{\ln \frac{b}{a}} \ln \frac{b}{r} = -V_0 \frac{\ln r/b}{\ln b/a}$$

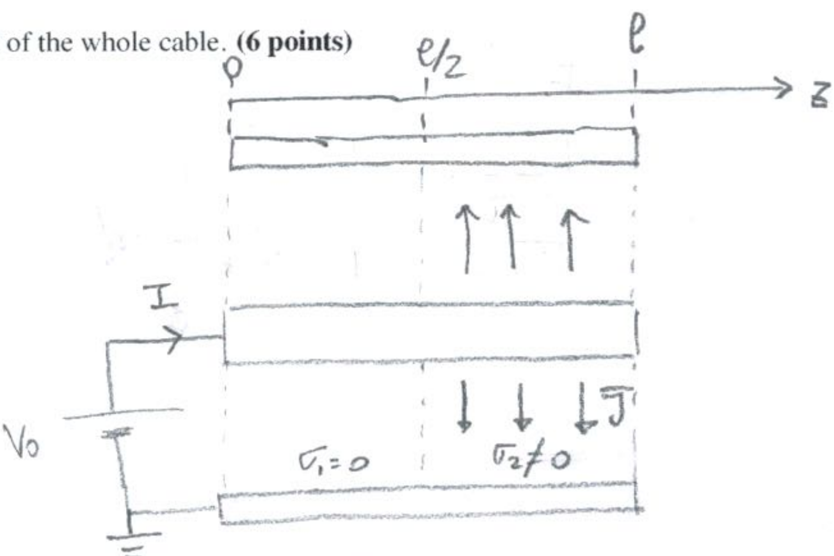
[2pt]

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r} \bar{a}_r = +\frac{V_0}{\ln \frac{b}{a}} \frac{1}{\cancel{b}} \frac{\cancel{b}}{r} \bar{a}_r = \frac{V_0}{r \ln \frac{b}{a}} \bar{a}_r$$

[2pt]

$\mathbf{E} =$

2. Calculate the resistance $R$ of the whole cable. **(6 points)**




$R = \frac{V_0}{I}$

We will have a current density $\bar{J}$ only in the second half of the cable, where $\sigma \neq 0$ [2pt]

$\bar{J} = \sigma_2 \bar{E} = \frac{\sigma_2 V_0}{r \ln \frac{b}{a}} \bar{a}_r$ [1pt]

$$I = \int_S \bar{J} \cdot \overline{dS} =$$

S: cylindrical surface radius r, extending from $z = \ell/2$ to $z = \ell$

$$= \int_{z=\ell/2}^{\ell} \int_{\varphi=0}^{2\pi} \frac{\sigma_2 V_0}{\cancel{r} \ln b/a} \bar{a}_r \cdot \bar{a}_r \cancel{r} \, d\varphi \, dz =$$

[2pt]

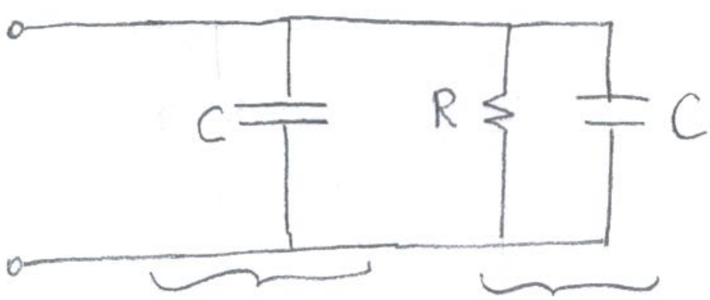
$$= \frac{\sigma_2 V_0}{\ln b/a} \frac{\ell}{\cancel{2}} \cancel{2}\pi =$$

$$\boxed{R = \frac{V_0}{I} = \frac{\ln b/a}{\sigma_2 \pi \ell}}$$

[1pt]

3. Propose an equivalent circuit model for the cable. Indicate *separately* the component(s) that model the first half of the cable and the component(s) that model the second half. You do not have to provide the component values. **(4 points)**

parallel connection [1pt]
two capacitors [1pt]
one resistor [1pt]




model for 1st half of cable

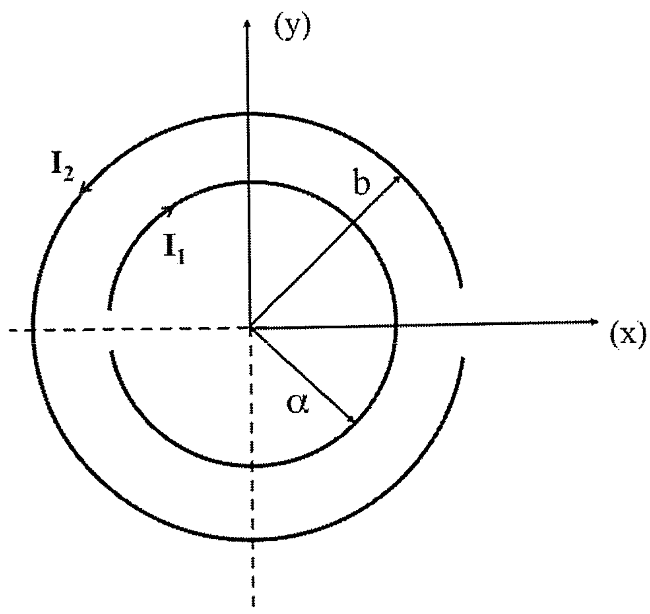
model for 2nd half

explanation of what models each part [1pt]

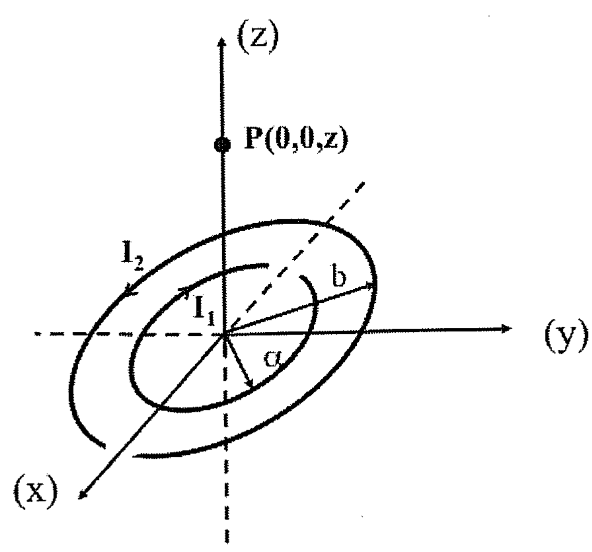
## Question 3

A split-ring resonator (SRR) consists of two "split" rings, as shown in the figure below. The two rings support currents:

- $I_1$ flowing in the $-\mathbf{a}_\phi$ direction, at $r = \alpha$, $(0 \leq \phi \leq \pi - \phi_0$ and $\pi + \phi_0 \leq \phi < 2\pi)$, $z = 0$.
- $I_2$ flowing in the $\mathbf{a}_\phi$ direction, at $r = b$, $\phi_0 \leq \phi \leq 2\pi - \phi_0$, $z = 0$.




1. Use the Biot-Savart law to find **only** the $z$−component of the magnetic flux density $B_z$ at an observation point $P(0, 0, z)$ on the positive $z$−axis. Clearly indicate all your steps to identify the terms $I\mathbf{dl} \times (\mathbf{R} - \mathbf{R}')$, $|\mathbf{R} - \mathbf{R}'|$ in Biot-Savart law and derive the final result. **(16 points)**



For any of the two rings:

$$I\,d\bar{l} = \underbrace{I\,r_o\,d\varphi'}_{(2)}\ \underbrace{\bar{a}_{\varphi'}}_{(1)} \qquad (r_o = a, b,\ I = I_1, I_2)$$

$\bar{R} = z\,\bar{a}_z$ (1)

$\bar{R}' = r_o\,\bar{a}_{r'} = r_o\left[\bar{a}_x \cos\varphi' + \bar{a}_y \sin\varphi'\right]$ (2)

$|\bar{R} - \bar{R}'| = \sqrt{r_o^2 + z^2}$ (1)

$I\,d\bar{l} \times (\bar{R} - \bar{R}') = I\,r_o\,d\varphi'\ \bar{a}_{\varphi'} \times \left(z\,\bar{a}_z - r_o\,\bar{a}_{r'}\right)$ (2)

$= I r_o\,d\varphi'\left[z\,\bar{a}_{r'} + r_o\,\bar{a}_z\right]$ (1)

$$dB_z = \frac{\mu_o}{4\pi}\,\frac{I r_o^2\,d\varphi'}{\left[r_o^2 + z^2\right]^{3/2}}$$

For the first:

(1) $dB_{1,z} = \dfrac{\mu_o}{4\pi}\,\dfrac{I_1 a^2\,d\varphi'}{\left[a^2 + z^2\right]^{3/2}}$, $\quad dB_{2,z} = \dfrac{\mu_o}{4\pi}\,\dfrac{I_2 b^2\,d\varphi'}{\left[b^2 + z^2\right]^{3/2}}$ (1)

Integrate with respect to $\varphi'$:

(2) $B_{1,z} = \dfrac{\mu_o}{4\pi}\,\dfrac{I_1 a^2}{\left[a^2 + z^2\right]^{3/2}}\,2(\pi - \varphi_o)$, $\quad B_{2,z} = \dfrac{\mu_o}{4\pi}\,\dfrac{I_2 b^2}{\left[b^2 + z^2\right]^{3/2}}\,2(\pi - \varphi_o)$ (2)

Superposition:

$B_z = B_{1,z} + B_{2,z}$

$$\boxed{B_z = \frac{\mu_0}{2\pi}(\pi - \varphi_0)\left\{\frac{I_1 a^2}{(a^2+z^2)^{3/2}} + \frac{I_2 b^2}{(b^2+z^2)^{3/2}}\right\}.}$$

2. The following question is independent of the previous one: For the magnetic flux over a closed surface $S$, $\oiint_S \mathbf{B} \cdot \mathbf{dS} = 0$. This means that (check all that apply and briefly explain): **(4 points)**

   a) The magnetic field is conservative.

   b) There is no magnetic monopole. 0.5

   c) The magnetic flux lines terminate at infinity.

   d) The magnetic flux lines are closed. 0.5

   e) $\nabla \times \mathbf{B} = 0$.

   f) $\nabla \cdot \mathbf{B} = 0$. 0.5

   g) $\nabla \cdot \mathbf{B} \neq 0$.

+ 1.5 explanation

+ 1 for rejecting a, c, e, g.

(a): No, there is no relation to $\bar{B}$ being "conservative" $\bar{F} = q\bar{u} \times \bar{B} \Rightarrow \int \bar{F} \cdot d\bar{\ell} = 0$ for $d\bar{\ell}$ in the direction of $\bar{u}$.

(b) Just comparing to Gauss Law for electricity, we conclude there is no "magnetic monopole".

(c). No. See next

(d) Indeed, $\nabla \cdot \bar{B} = 0$ means there are no "sinks"/"sources" of lines $\Rightarrow$ they have to be closed.

(e) No, see next.

(f) Divergence theorem $\Rightarrow$ $\nabla \cdot \bar{B} = 0$.

(g) See above.