UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

December 17, 2018

PHY293F (Waves and Modern Physics)
Instructor: Professors N. Grisouard and P. Krieger

Duration: 2.5 hours

Exam Type A: Closed Book. Only non-programmable calculators allowed.

This examination paper consists of **7** pages and 6 questions. Please bring any discrepancy to the attention of an invigilator. Answer all 6 questions.

- Write your name, student number and tutorial group on top of **all** examination booklets and test pages used.
- Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. Not other aid (notes, textbook, dictionary) is allowed. **Communication devices are strictly forbidden. Turn them off and make sure they are in plain sight to the invigilators.**
- Each question is worth 1/6 of your overall grade for this exam. Within each question, a mark breakdown is indicated in square brackets at the end of each sub-part. Part marks will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion** any relevant assumptions you are making along the way.
- Do not separate the stapled sheets of the question paper. Hand in the questions with your exam booklet at the end of the test.
- The next two pages include some formulae and constants you may find useful.
- The questions begin on **page 4**. The total number of marks is 60.

Oscillations

	Amplitude	Velocity	Dissipated Power
Peak freq.	$\omega_{max} = \omega_0\sqrt{1-1/(2Q^2)}$	$\omega_{max} = \omega_0$	$\omega_{max} = \omega_0$
Peak value	$A_{max} = \dfrac{QA_f}{\sqrt{1-1/(4Q^2)}}$	$V_{max} = \omega_0 Q A_f$	$P_{max} = \dfrac{mA_f^2\omega_0^3 Q}{2}$
Misc.	$A(\omega) = \dfrac{A_f}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + \frac{1}{Q^2}\frac{\omega^2}{\omega_0^2}}}$ $\tan\delta = \dfrac{\omega\gamma}{\omega_0^2-\omega^2}$	$V(\omega) = \omega A(\omega)$	$\overline{P}(\omega) = \dfrac{m\gamma V^2(\omega)}{2}$ $\approx \dfrac{P_{max}}{1+\frac{4(\omega_0-\omega)^2}{\gamma^2}}\ (Q \gg 1)$

$$\mathsf{M}\ddot{\vec{X}} + \mathsf{K}\vec{X} = 0; \qquad \det\left(\mathsf{K} - \omega^2\mathsf{M}\right) = 0.$$

$$\mathsf{M}^{-1}\mathsf{K} \quad \text{symmetric and} \quad |\vec{Y}_i| = 1 \quad \Rightarrow \quad \vec{Y}_i \cdot \vec{Y}_j = \delta_{ij}$$

$$\vec{X}(t) = \sum_{n=1}^{N} C_n \vec{Y}_n \cos(\omega_n t + \phi_n), \quad \text{with} \quad C_n\cos\phi_n = \vec{X}_0 \cdot \vec{Y}_n \quad \text{and} \quad C_n \sin\phi_n = -\frac{\vec{V}_0 \cdot \vec{Y}_n}{\omega_n}.$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

$$\frac{\partial^2}{\partial t^2}y(x,t) - v^2\frac{\partial^2}{\partial x^2}y(x,t) = 0 \quad \text{with} \quad v = \sqrt{\frac{T}{\mu}}$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \phi_n)\sin(k_n x) = \sum_{n=1}^{\infty}[\alpha_n\cos(\omega_n t) + \beta_n \sin(\omega_n t)]\sin(k_n x),$$

$$\text{with} \quad \alpha_n = \frac{2}{L}\int_0^L y(0,x)\sin(k_n x)\mathrm{d}x \quad \text{and} \quad \beta_n = \frac{2}{L\omega_n}\int_0^L \dot{y}(0,x)\sin(k_n x)\mathrm{d}x.$$

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = A\sin\left(k(x-vt)\right) = A\sin\left(kx - \omega t\right) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right].$$

$$\omega = 2\pi\nu, \quad \nu = 1/T, \quad k = 2\pi/\lambda, \quad v = \omega/k = \lambda/T = \lambda\nu.$$

$$\text{Energy Flux} = \frac{1}{2}\mu_i v\omega^2 A^2 = \frac{1}{2}\sqrt{T\mu_i}\,\omega^2 A^2.$$

$$\rho = \frac{A_R}{A_I} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}; \qquad \tau = \frac{A_T}{A_I} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

Modern Physics

$$\begin{aligned}
\text{Speed of light } c &= 3.00\times10^{8}\,\text{m/s} & \text{Mass of electron } m_e &= 9.11\times10^{-31}\,\text{kg} = 511\,\text{keV/c}^2\\
\text{Elementary charge } e &= 1.60\times10^{-19}\,\text{C} & \text{Mass of proton } m_p &= 1.67\times10^{-27}\,\text{kg} = 939\,\text{MeV/c}^2\\
\text{Coulomb constant } \epsilon_0 &= 8.85\times10^{-12}\,\text{C}^2/(\text{J m}) & \text{Planck's constant } h &= 6.63\times10^{-34}\,\text{J s} = 4.14\times10^{-15}\,\text{eV s}\\
hc &= 1240\,\text{eV nm} & 1\,\text{eV} &= 1.60\times10^{-19}\,\text{J}
\end{aligned}$$

$$x' = \gamma\,(x-\beta\,ct) \qquad t' = \gamma\,(t-\frac{\beta}{c}\,x) \qquad y' = y \qquad z' = z \qquad \gamma = (1-\beta^2)^{-1/2} \qquad \beta = v/c$$

$$u'_x = \frac{u_x - v}{1-u_x v/c^2} \qquad u'_y = \frac{u_y}{\gamma(1-u_x v/c^2)} \qquad f_{obs} = f_{source}\cdot\frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta}$$

$$\text{Classically} \qquad \vec{p} = m\,\vec{u} \qquad E = \frac{1}{2}mv^2$$

$$\text{Relatvistically} \qquad \vec{p} = \gamma\,m\,\vec{u} \qquad E = \gamma\,m\,c^2$$

$$\text{Position-time} \qquad (x,y,z,ct) \qquad \text{Energy-momentum} \qquad (p_x,p_y,p_z,\frac{E}{c})$$

$$\text{Invariants} \qquad (\Delta s)^2 = (c\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2 \qquad \frac{E^2}{c^2}-p_x^2-p_y^2-p_z^2 = \frac{E^2}{c^2}-p^2 = m^2c^2$$

$$E = \frac{hc}{\lambda} = h\nu = \hbar\omega \qquad \lambda = \frac{h}{p} \qquad p = \frac{h}{\lambda} = \hbar k \qquad \hbar = \frac{h}{2\pi}$$

$$KE_{\text{max}} = h\nu - \Phi \qquad \Delta\lambda = \lambda' - \lambda = \frac{h}{mc}(1-\cos\theta)$$

$$\text{Hydrogen atom} \qquad E_n = -\frac{1}{2n^2}m_e\Big(\frac{e^2}{4\pi\epsilon_0\hbar}\Big)^2 = \frac{E_1}{n^2} = -\frac{13.6\,\text{eV}}{n^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + U(x,t)\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t) \qquad \Psi_n(x,t) = \psi_n(x)e^{-i\frac{E_n}{\hbar}t}$$

$$\int_{\text{all space}} \left|\Psi(x,t)\right|^2 dx = 1 \qquad \Delta p\,\Delta x \gtrsim \hbar/2 \qquad \Delta E\,\Delta t \gtrsim \hbar/2$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \qquad \begin{Bmatrix}\cos^2\theta\\ \sin^2\theta\end{Bmatrix} = \frac{1}{2}(1\pm\cos 2\theta)$$

1. Note: Q1a and Q1b are independent.

   (a) The phase velocity v of transverse waves in a crystal of atomic separation a is given by

   $$v = c\frac{\sin(ka/2)}{ka/2}. \tag{1}$$

   i. Show that the value of the group velocity is $v_g = c\cos(ka/2)$. [2.5]

   ii. What is the limiting value of the group velocity for long wavelengths? [2.5]

   (b) A rectangular dish containing mercury is connected to the cone of a loudspeaker so that when the loudspeaker is driven by an oscillating voltage a standing wave is set up on the surface of the mercury. When a beam of light is shone on the surface, the standing wave acts like a diffraction grating and the observed diffraction pattern enables the spacings of the antinodes of the standing wave to be determined. It is found that the spacing of the antinodes for a standing wave of frequency 1.35 kHz is 0.25 mm.

   i. Use these data to obtain a value for the surface tension σ of mercury. Assume the dispersion relation

   $$\omega^2 = gk + \frac{\sigma k^3}{\rho}, \tag{2}$$

   where ρ is the density and assume also that the wavelength is sufficiently small that the wave properties are determined by surface tension and not by gravity. [2.5]

   ii. What is the numerical value of the group velocity? (use $\rho = 13.6 \times 10^3$ kg m^{-3} and $g = 10$ m s^{-2}) [2.5]

2. See figure 1: a channel connecting a river to the ocean is modeled as a one-dimensional layer of water which, when it is at rest, has depth h_0 that is independent of x, the along-channel coordinate. The channel starts at the location $x = 0$ (river mouth) and ends at $x = L$ (where the open ocean starts).

   In this question, we assume that damping is negligible and that the water surface elevation y (with $y(x,t) = 0$ when the water surface is at rest) is well described by the following equation:

   $$\frac{\partial^2 y}{\partial t^2} - gh_0\frac{\partial^2 y}{\partial x^2} = 0. \tag{3}$$

   (we do not ask you to derive it!)

   We also assume that the tidal forcing from the open ocean imparts a boundary condition on the ocean side

   $$y(x = L, t) = y_0\cos(\omega t). \tag{4}$$

   Whenever needed, you will use the following numbers:

   - $g = 10$ m s^{-2} (gravitational acceleration),
   - $h_0 = 20$ m (depth of the channel),
   - $L = 150$ km (length of the channel),
   - $y_0 = 0.3$ m (tidal amplitude on the ocean side of the channel),

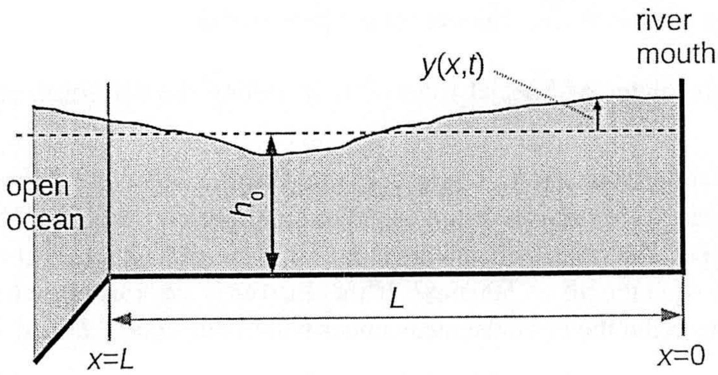

Figure 1. Sketch of the channel of question 2.

- $\omega = 1.45 \times 10^{-4}$ rad s^{-1} (tidal angular frequency).

(a) We seek a solution of the form

$$y(x,t) = y_M \cos(\omega t)\cos(kx), \tag{5}$$

which we do not ask you to justify. Express k as a function of ω, g and h_0 (*note: make sure to obtain* $k > 0$). What is the speed of propagation of the surface waves? [2.5]

(b) Find the expression of y_M as a function of y_0, ω, g, h_0 and L. [2.5]

(c) Deduce that for certain angular frequencies $\omega = \omega_n$, the amplitude in the channel becomes infinite ($\lim_{\omega \to \omega_n} y_M = \infty$). [2.5]

(d) Compute y_M for the numerical values of ω, g, h_0 and L provided earlier: is the system resonant or near-resonant? [2.5]

3. Three equal masses m are connected to fixed walls at either end by four springs of varying stiffness (k or $2k$) as shown in fig. 2. Ignore gravity.

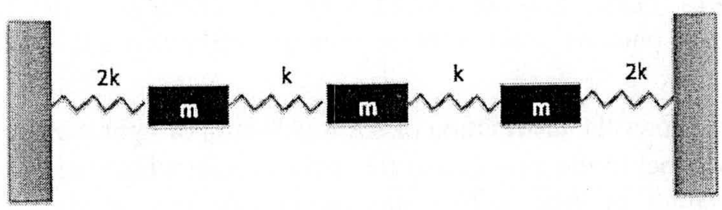

Figure 2. Masses+springs system.

(a) Find that the eigenfrequencies of this system are ω_0, $\sqrt{3}\omega_0$ and $2\omega_0$, with $\omega_0 = \sqrt{k/m}$. [8]

(b) Assume that all masses are held steady at $t = 0$ (their velocities are all initially zero). Find a set of initial displacements $X(t=0)$ which would excite only the eigenmode with frequency $\sqrt{3}\omega$. [2]

4. Answer the following questions: (a), (b) and (c) are independent.

   (a) State the two postulates of Special Relativity, including the definition any necessary concepts. [3]

   (b) A distant light source travels (in some direction) with a velocity of magnitude $0.4c$ relative to an observer on Earth. If the source emits light of wavelength 500 nm in its rest frame, is it possible for on observer on the earth to also see the light emitted from this source at a wavelength of 500 nm? If the answer is no, explain why not. If the answer is yes, describe the circumstances under which this could happen. [3.5]

   (c) According to Anna, on Earth, Bob is on a spaceship moving at a speed of $0.6c$ towards the Earth, while Carl, who is further out (see figure 3 below) is approaching Earth at $0.8c$. According to Bob, how fast is Carl approaching the Earth? [3.5]

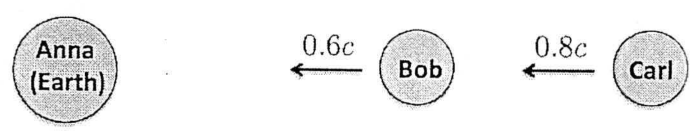

Figure 3. Figure for Problem 4(c)

5. Answer the following questions: (a), (b) and (c) are independent.

   (a) The Davisson-Germer experiment involved shooting a beam of electrons of fixed energy at a nickel crystal.

      i. Briefly describe the purpose of the experiment and state what physical concept it was testing. [2]

      ii. Briefly describe on what basis one would decide what energy to use for the electrons in this experiment. [2]

   (b) A neutral pion (π^0) has a mass of 140 MeV/c^2 and a lifetime of 10^{-16} s. If a π^0 at rest decays to two photons, what is the relative uncertainty on the wavelength of the photons? [3]

   (c) Figure 4 below, shows the distribution of the wavelength of light scattered at an angle of 135° (with respect to the direction of the incident light which has $\lambda = 0.0708$ nm) from a carbon target, as observed by Compton in his famous experiment. Explain the fact that there are two peaks observed, and determine the wavelengths at which these appear. [3]

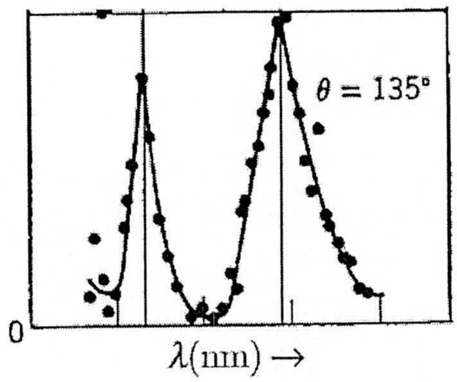

Figure 4. Figure for Problem 5(c)

6. We showed in the lectures that the solutions of the Time Independent Schrödinger Equation for an infinite square well potential, in which the potential is infinite outside the region $0 \leq x \leq L$ and zero within it, are of the form:

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \tag{6}$$

in the region $0 \leq x \leq L$ and zero elsewhere, corresponding to energies

$$E_n = \frac{n^2\hbar^2\pi^2}{2mL^2}. \tag{7}$$

(a) Provide an argument based on standing waves that yields the same set of discrete energies. [3]

(b) If, at $t = 0$, a particle in this potential well has a spatial wavefunction which is a linear combination formed of equal parts of $\psi_1(x)$ and $\psi_2(x)$, write an expression for the probability density at later time t_1. (For full marks, please simplify your result as much as possible.) [4]

(c) If a measurement performed on this system at some time $t_2 > t_1$ were to find the particle at the centre of the well, briefly explain what can you say about the probability density at a later time $t_3 >> t_2$. [3]

End of examination
Total pages: 7