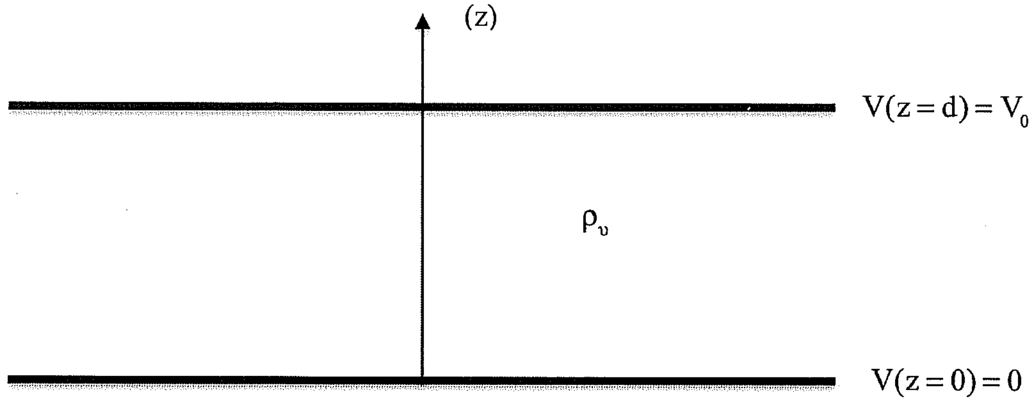
Question 1 [25 pts]

A. Two parallel perfectly conducting plates are separated by a distance d and maintained at potentials 0 and V_0, as shown in the figure. The region between the plates has $\varepsilon=\varepsilon_0$, and it is filled with a distribution of electrons having a volume charge density $\rho_\upsilon = -\rho_0 z/d \, (\rho_0 \geq 0)$.



1. Derive the Poisson equation with respect to the electric potential V(z) between the plates. **[5 pts]**

$$\overset{(2pts)}{\nabla^2 V = -\rho_\upsilon/\varepsilon_0} \quad \underbrace{\Longrightarrow}_{} \quad \overset{(3pts)}{\frac{d^2V}{dz^2} = +\rho_0 z/d}$$

by symmetry

$\partial/\partial x = 0 = \partial/\partial y$, only z-dependence in the problem

2. Solving the Poisson equation or otherwise, determine the electric field between the plates. **[10 pts]**

[2] $\dfrac{dV}{dz} = \dfrac{\rho_0 z^2}{2d} + C_1$

$V(z=0)=0 \Rightarrow C_2 = 0$ [1]

$V(z=d) = V_0 \Rightarrow$

[2] $V(z) = \dfrac{\rho_0 z^3}{6d} + C_1 z + C_2$

$C_1 = \dfrac{1}{d}\left(V_0 - \dfrac{\rho_0 d^2}{6}\right)$ [2]

$= \dfrac{V_0}{d} - \dfrac{\rho_0 d}{6}$

[2]

$$\Rightarrow \quad E(z) = -\bar{a}_z \frac{dV}{dz} = -\bar{a}_z \left[\frac{\rho_0 z^2}{2d} + \frac{V_0}{d} - \frac{\rho_0 d}{6} \right]$$

$$= -\bar{a}_z \frac{V_0}{d} - \bar{a}_z \frac{\rho_0}{2d}\left(z^2 - \frac{d^2}{3}\right) \quad [1]$$

3. Express the surface charge density ρ_s at z=0 and z=d as a function of the electric potential V(z). You don't need to use the result of the previous question.

[5 pts]

[1pt] B.C.: $\bar{a}_n \cdot \bar{E} = \rho_s / \epsilon_0$

[2pts] $z=0$: $\bar{a}_z \cdot \bar{a}_z E_z(z=0) = \frac{\rho_s}{\epsilon_0} \Rightarrow \rho_s = \left(-\frac{dV}{dz}\Big|_{z=0}\right) \cdot \epsilon_0$

[2pts] $z=d$: $\bar{a}_n = -\bar{a}_z \quad (-\bar{a}_z) \cdot (\bar{a}_z E_z(z=d)) = \frac{\rho_s}{\epsilon_0}$

$$\Rightarrow \quad \rho_s = -E_z(z=d) \cdot \epsilon_0 = \epsilon_0 \left(+\frac{dV}{dz}\Big|_{z=d}\right)$$

B. Industry Canada Radio Standards Specification 102 states that the maximum electric field amplitude for uncontrolled exposure is 280 V/m. Modeling a power line 5 m above the ground as an infinitely long line charge distribution, determine the maximum allowed ρ_ℓ (C/m), if residential areas exist 40m away (see the figure below). The ground can be considered as a perfect electric conductor for the purposes of this study.

$\varepsilon_0 = 8.8542\times10^{-12} \approx \dfrac{10^{-9}}{36\pi}\,\mathrm{F/m}$ **[5 pts]**

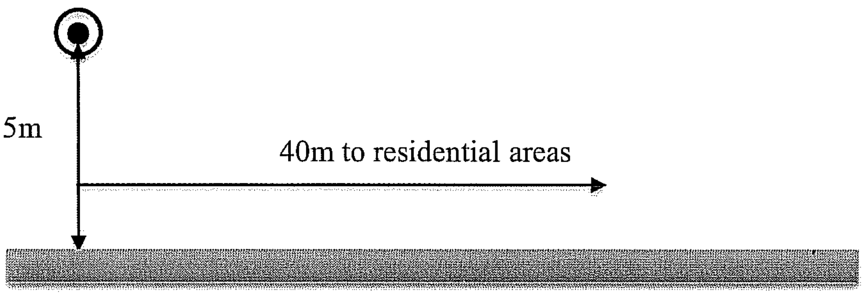
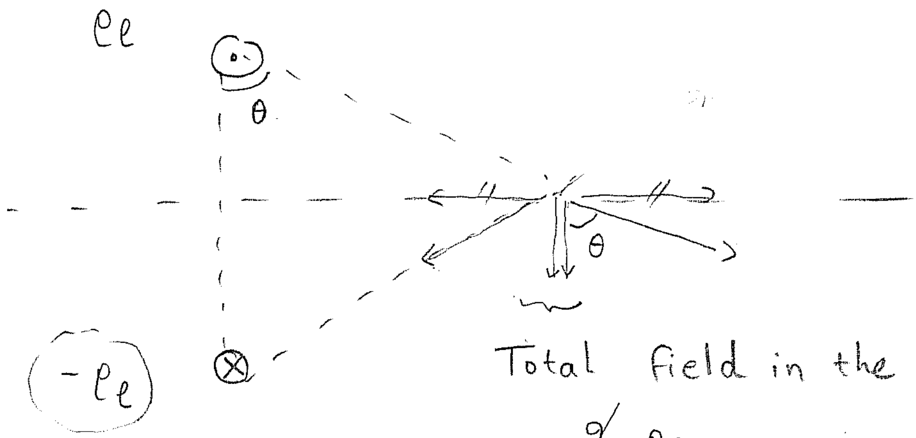


Image theory applies:



correct application of image theory: 2pts

Total field in the $(-z)$ direction

$$\underbrace{E = \frac{\cancel{2}\cdot\rho_\ell}{\cancel{2}\pi\varepsilon_0 r}\cos\theta}_{2\text{pts}}, \quad \cos\theta = \frac{5\,\text{m}}{\sqrt{40^2+5^2}} \approx \frac{5}{40} = 0.125 \Rightarrow$$

$$\frac{\rho_\ell\cdot 0.125}{\frac{\cancel{\pi}\times10^{-9}}{36\cancel{\pi}}\times\sqrt{40^2+5^2}} \leq 280$$

$\sqrt{40^2+5^2}$ ≈ 40

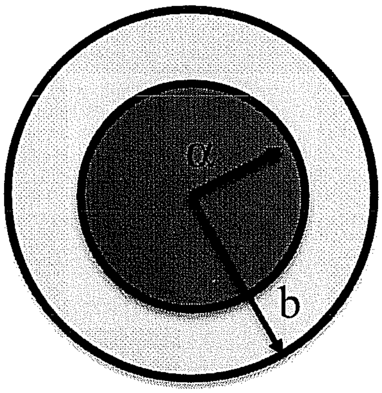
$\rho_\ell \leq 2.48\times10^{-6}$ C/m. } calc 1pt.

--- **End of Question 1**

Question 2 [25 pts]

A. Consider a coaxial (cylindrical) resistor formed by two perfectly conducting cylinders of radii α and b, respectively. The area $\alpha \le r \le b$ between the cylinders is filled by a medium with dielectric permittivity $\varepsilon(r) = \frac{\varepsilon_0 r}{\alpha}$ and conductivity $\sigma(r) = \frac{\sigma_0 r}{\alpha}$.

A voltage source keeps the potential difference between the conductors at $V(r=\alpha)-V(r=b)=V_0$. The cross-section of the resistor is shown below:



1. Determine the per unit length resistance (i.e. for length L=1 m) of this resistor, ignoring edge effects. Intermediate steps: find the electric field $\bar{E}$ [4 pts], express the volume current density $\bar{J}$ in terms of $\bar{E}$ [2 pts]; then, find the total current I [4 pts] and the ratio of voltage V_0 to I [4 pts]. **[14 pts]**

From term test 1:

$$D_r = \frac{\rho_s a}{r} \Rightarrow E_r = \frac{\rho_s a^2}{\epsilon_0 r^2} \quad [2pts]$$

$$\int_a^b \frac{\rho_s a^2}{\epsilon_0 r^2} dr = V_0 \Rightarrow \frac{\rho_s a^2}{\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = V_0$$

$$\Rightarrow \rho_s = \frac{\epsilon_0 V_0}{a^2\left(\frac{1}{a}-\frac{1}{b}\right)} \Rightarrow E_r = \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right) r^2} \quad [2pts]$$

$$J_r = \frac{\epsilon_0 \cancel{r}}{a} \frac{V_0}{\left(\frac{1}{a}-\frac{1}{b}\right) r^{\cancel{2}}} = \frac{\epsilon_0 V_0}{\cancel{a} \frac{b-a}{\cancel{a} b} \cdot r} = \frac{\epsilon_0 \cdot b \cdot V_0}{(b-a) r} \quad [2 pts]$$

$$I = \int J_r \bar{a}_r \cdot \bar{a}_r \underbrace{(r\, d\varphi\, dz)}_{[2pts]}$$

$$= \frac{\sigma_0 b V_0}{b-a} \underbrace{\int_{\varphi=0}^{2\pi}\int_{L=1m} d\varphi\, dz}_{2\pi} = \frac{2\pi \sigma_0 b V_0}{b-a} \quad [1pt] \Rightarrow$$

$$R = \frac{V_0}{I} = \frac{b-a}{2\pi \sigma_0 b} \quad [1pt]$$

Note: $RC = \epsilon_0 / \sigma_0 \quad \left(C = \frac{2\pi \epsilon_0 b}{b-a} \right)$

2. Determine the per unit length (i.e. for length L=1 m) dissipated power P in this resistor. **[6 pts]**

density: $p = \frac{dP}{dv} = \bar{E} \cdot \bar{J} = \sigma |E|^2 = \frac{|J|^2}{\sigma}$

$$= \frac{\cancel{\sigma_0^2} b^2 V_0^2}{(b-a)^2 r^2} \frac{a}{\cancel{\sigma_0} r} = \frac{a b^2 \sigma_0 V_0^2}{(b-a)^2 r^3} \quad [3pts] \Rightarrow$$

$$P = \frac{a b^2}{(b-a)^2} \sigma_0 V_0^2 \underbrace{\int_a^b \frac{\cancel{r}\, dr}{r^{\cancel{3}}}}_{\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}} \underbrace{\int_{L=1m} dz}_{1m} \underbrace{\int_{\varphi=0}^{2\pi} r\, d\varphi}_{2\pi} \quad [2pts]$$

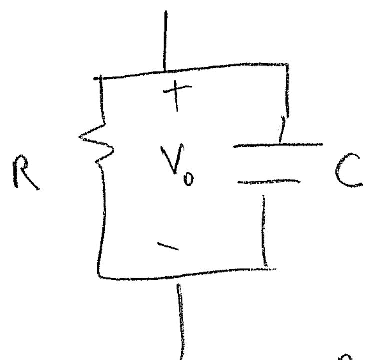
$$= \frac{\cancel{a} b^{\cancel{2}}}{(b-a)^{\cancel{2}}} \sigma_0 V_0^2 \frac{\cancel{b-a}}{\cancel{a}\cancel{b}} \cdot 2\pi = \frac{2\pi \sigma_0 b}{b-a} V_0^2 = \frac{V_0^2}{R}$$

[1pt]

Full marks if R had been found in # 1 and $P = V_0^2/R$ was written (or $I^2 R$, where I had been found before)

Part marks will be given as shown above.

3. Derive a circuit model for this structure. You do not need to specify the values for each element of this circuit. **[5 pts]**



In parallel because they are subject to the same voltage.

R-C : 2.5 pts

"parallel" : 2.5 pts

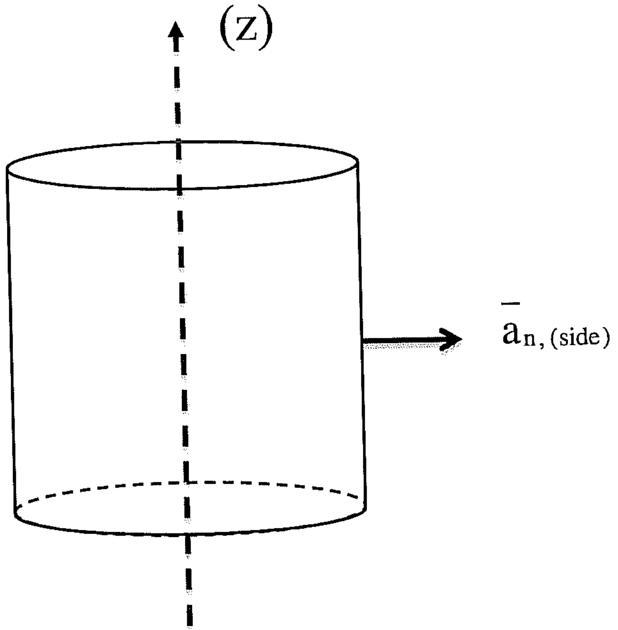
--- End of Question 2

Question 3 [25 pts]

A. Consider the cylindrical surface of radius α and length L shown in the figure. The axis of the cylinder is the z-axis. The surface is immersed in a magnetic field with flux density $\overline{B} = 2y\overline{a}_x + 3x\overline{a}_y$ (T). Find the magnetic flux:

$$\Phi_{side} = \int_{S_{side}} \overline{B} \cdot \overline{ds} = \int_{S_{side}} \overline{B} \cdot \overline{a}_{n,(side)}\, ds$$

through the side surface of the cylinder (r=α, 0<z<L). Hint: You do not need to perform an integration. **[5 points]**



$\oint \overline{B} \cdot d\overline{s} = 0 \Rightarrow \Phi_{top} + \Phi_{bottom} + \Phi_{side} = 0$ [2pts]

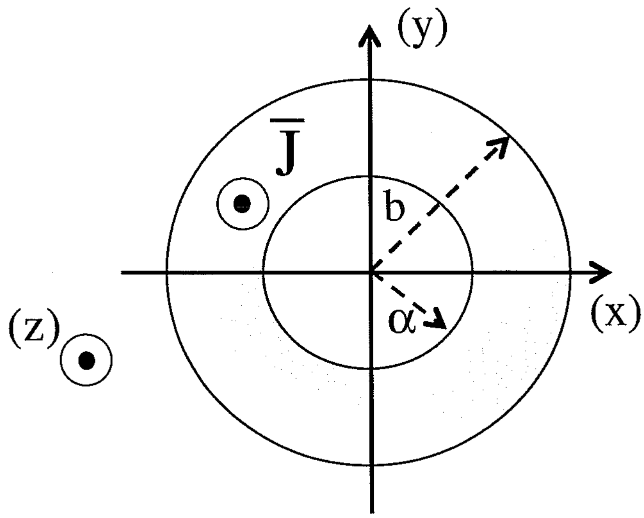
$\Phi_{top} = \Phi_{bottom} = 0$ because $\overline{a}_n = \pm \overline{a}_z \Rightarrow$
$\overline{B} \cdot d\overline{s} = 0$ [2pts]

$\Rightarrow \Phi_{side} = 0$ [1pt]

B. A steady current flows through an infinitely long hollow cylindrical conductor of inner and outer radius α and b, respectively. The cross section of the conductor is shown in the figure below. The current density vector is equal to:

$$\bar{J}(r) = \begin{cases} 0 & \text{for } r < \alpha \\ J_0 \bar{a}_z & \text{for } a < r < b \\ 0 & \text{for } r > b \end{cases}$$

The conductor is surrounded by free space. Find the magnetic flux density everywhere. Hint: as the current distribution is cylindrically symmetric, $\bar{B}$ will be in the form $\bar{B} = B_\varphi(r)\bar{a}_\varphi$. **[15 points]**



$$\oint \bar{B} \cdot d\bar{l} = \mu_0 I_{enclosed} \Rightarrow 2\pi r B_\varphi = \mu_0 I_{enclosed}$$ [3pts]

$$I_{enclosed} = \begin{cases} 0, & r < a & [1pt] \\ J_0 \pi (r^2 - a^2), & a < r < b & [2pts] \\ J_0 \pi (b^2 - a^2), & r > b & [2pts] \end{cases}$$

$$\Rightarrow B_\varphi = \begin{cases} 0, & r < a & [1pt] \\ \frac{J_0 \pi (r^2 - a^2)}{2\pi r}, & a < r < b & [3 pts] \\ \frac{J_0 \pi (b^2 - a^2)}{2\pi r}, & r > b & [3 pts] \end{cases}$$

C. A point charge q=1 nC is travelling in a region of free space where there is a uniform electrostatic field $\bar{E} = 2\bar{a}_x$ V/m and a uniform magnetostatic field $\bar{B} = 3\bar{a}_y$ A/m. The charge is moving on the x-z plane with constant velocity. What is the velocity of the charge (magnitude [2 points] and direction [3 points])? **[5 points]**

$\bar{F}_E + \bar{F}_B = 0$ for constant velocity: [1 pt]

direction (3pts)

$\bar{F}_E = q\bar{E} // \bar{a}_x$ [1pt]

$\bar{F}_B = q\underbrace{\bar{v}}_{-?} \times \underbrace{\bar{B}}_{\bar{a}_y} = -\bar{a}_x$ needs to be in (-x) direction $\Rightarrow \bar{v} // \bar{a}_z$ [2pts]

$qE = qvB \Rightarrow v = \frac{E}{B} = \frac{2}{3}$ [1pt]

$$\boxed{\bar{v} = \frac{2}{3}\bar{a}_z}$$

--- End of Question 3