The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Final Exam - Tuesday April 29, 2014

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Last name: ....

First name: ....

Student number: ....

Instructions

- Duration: 2 hours 30 minutes (9:30 to 12:00)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full credit!**

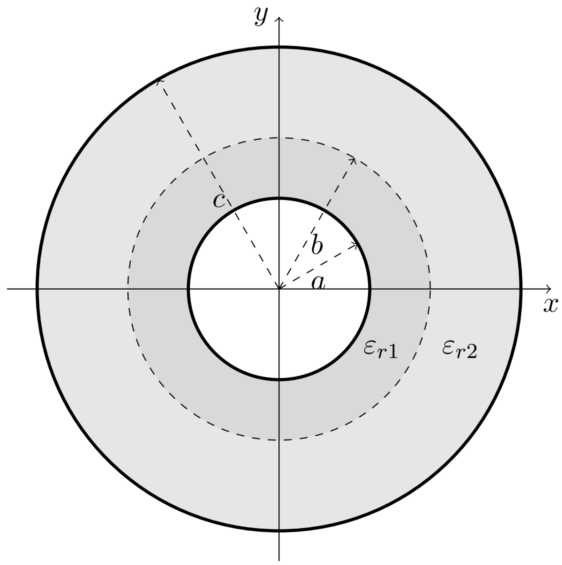
Marks:

Q1: /20	Q2: /20	Q3: /20	Q4: /20	Q5: /20

TOTAL: /100

Question 1

Consider the spherical capacitor shown in the figure below, which is made by two perfectly conducting spheres (shown in black) of radii a and c. The inner dielectric layer is made from mica ($\varepsilon_{r1} = 5.4$) while the outer layer is made from oil ($\varepsilon_{r2} = 2.3$). Remember that $\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{F/m}$.



$a = 10\,\mathrm{mm}$

$b = 20\,\mathrm{mm}$

$c = 30\,\mathrm{mm}$

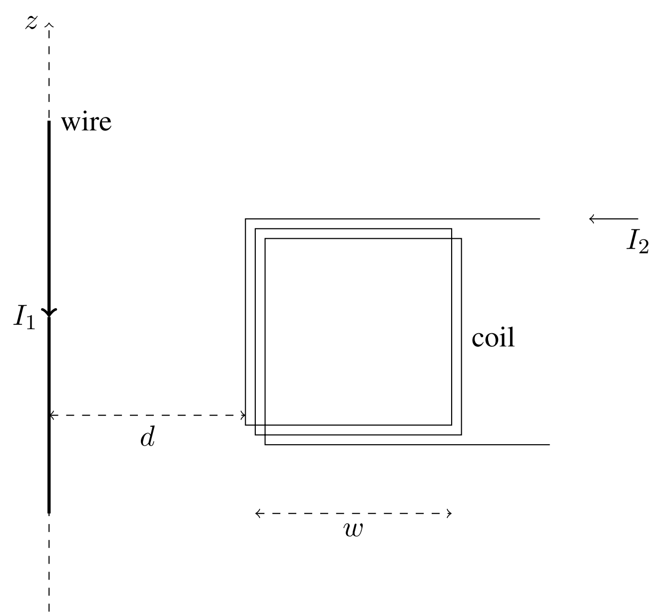
(a) Assume that a charge $+Q$ has been placed on the inner conductor, and charge $-Q$ has been placed on the outer conductor. Calculate the electric field $\mathbf{E}$ inside both dielectric layers. **(8 pts)**

(b) Find the capacitance of the system. Express its value in picoFarad. **(8 pts)**

(c) The capacitor is now connected to a voltage source of value $V = 20\,\mathrm{V}$. The source is then disconnected, and the oil is drained from the capacitor. Find the voltage across the capacitor in the new state. **(4 pts)**

Question 2

An infinitely-long wire is placed along the z-axis near a coil with square cross-section and N turns, as shown in the figure below. The length of the coil edges is w. Constant currents I_1 and I_2 flow in the wire and in the coil, respectively, in the directions shown. Permeability is μ_0 everywhere.



(a) Find the magnetic flux density $\mathbf{B}_1$ caused by the current I_1 of the infinitely-long wire, at an arbitrary point away from the $z-$axis ($r > 0$). **(6 pts)**

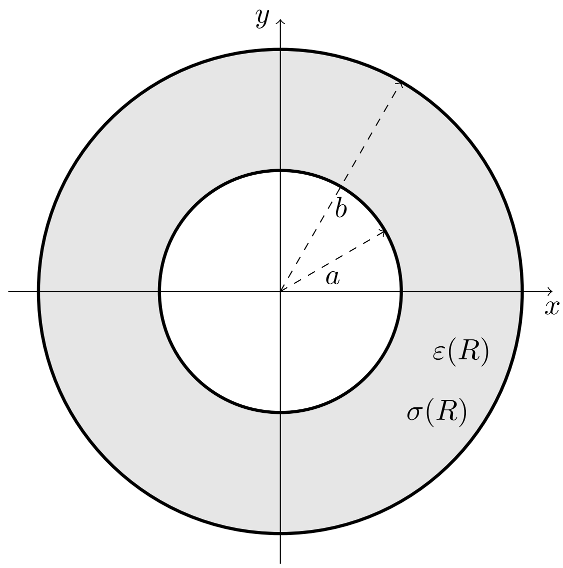
(b) Find the mutual inductance L_{12} between the wire and the N-turn coil. **(6 pts)**

(c) Find the net force **F** acting on the N-turn coil due to the current of the infinitely-long wire. **(8 pts)**

Question 3

Consider the spherical capacitor shown in the figure below. Two perfectly conducting spheres $R = \alpha$ and $R = b$ enclose a lossy and inhomogeneous dielectric. The permittivity and conductivity of the dielectric are given by:

$$\varepsilon(R) = \varepsilon_0 \frac{b}{R}$$
$$\sigma(R) = \sigma_0 \frac{b^2}{R^2}$$



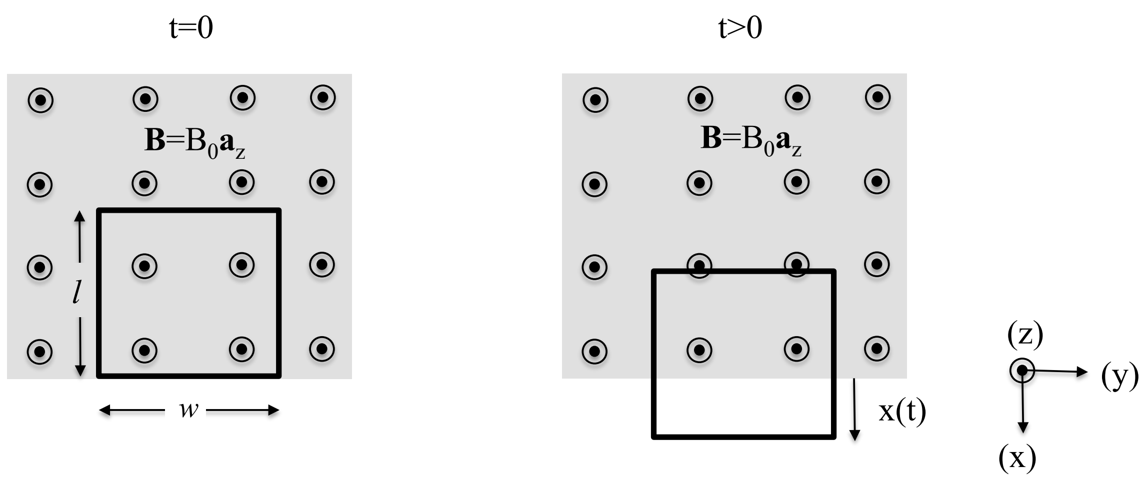
(a) Determine the resistance of the capacitor (hint: assume that a current I flows from the inner to the outer conductor, and determine the volume current density in the dielectric) **(10 pts)**

(b) Determine the density of free charge inside the dielectric **(5 pts)**

(c) Determine the density of bound charge inside the dielectric **(5 pts)**

Question 4

A thin, conducting, rectangular frame of resistance R is released from rest and allowed to fall from a region of constant magnetic field $\mathbf{B} = B_0\mathbf{a}_z$ into a region with no magnetic field. The frame has length l and width w. For the following questions, assume that part of the frame is inside the magnetic field, as shown in the figure.



(a) Use Lenz's law to find the direction of the induced current I flowing around the frame and show that the magnetic field will apply a magnetic braking force on the falling frame, counteracting the force of gravity.

(4 pts)

(b) For frame velocity $\mathbf{v}(t) = v_x(t)\,\mathbf{a}_x$, find the induced electromotive force (emf) and the current I, using Faraday's law $\mathrm{V}_{\text{emf}} = -\dfrac{d\Phi}{dt}$. Clearly indicate the polarity of the emf and the direction of the current.

(8 pts)

(c) Repeat the previous calculation using the form of Faraday's law:

$$\mathrm{V}_{\mathrm{emf}} = \oint_C (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{dl} - \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$$

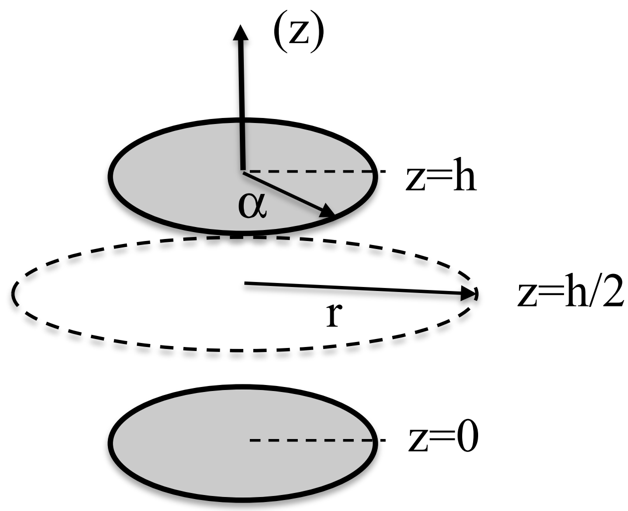
(8 pts)

Question 5

The following questions are independent from each other.

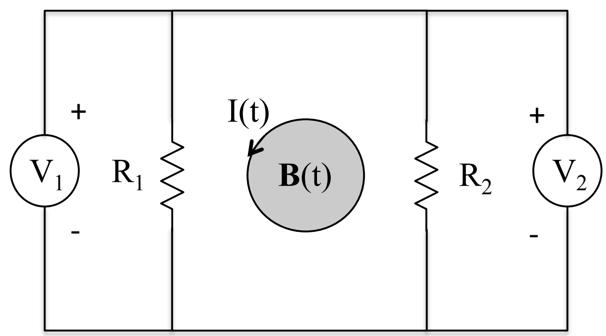
1. A capacitor with circular parallel plates of radius α at $z = 0$ and $z = h$, has a time-varying voltage $V(t)$. Find the magnetic field $\mathbf{H}$ (magnitude and direction) at $r > \alpha, z = h/2$ (i.e. along the dashed line shown in the figure). State your assumptions.

(5 pts)



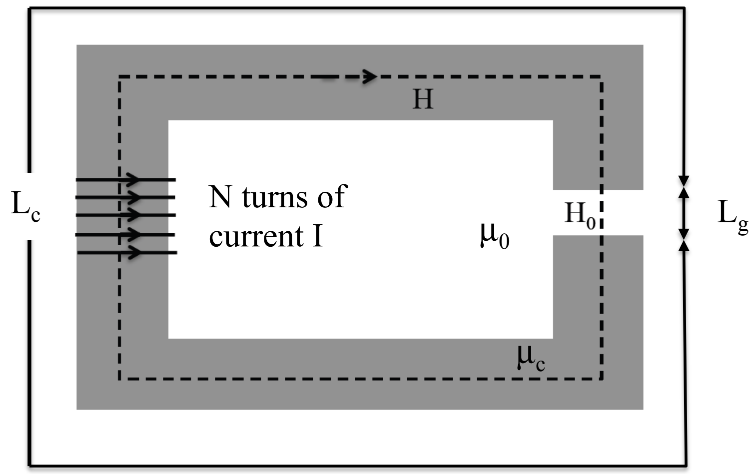
2. Two voltmeters, with infinite internal resistance, are connected to a circuit consisting of two resistors, as shown in the figure. A long solenoid of radius α, with time-varying magnetic field $\mathbf{B}(t)$ passes through the center of the circuit. The cross-section of the solenoid and the current of one of its turns are shown in the figure below. Find the voltages V_1 and V_2 shown by the voltmeters, with the polarity indicated in the figure. Are they the same or different?

(5 pts)



3. Draw the equivalent magnetic circuit of the following structure, providing expressions for all magnetomotive forces and reluctances of the circuit. What is the ratio H/H_0 of the magnetic field intensities inside the magnetic core and the gap ? Assume that all parts of this circuit have uniform cross-section S_c.

(5 pts)



4. In Gauss' law stated as:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$$

the charge Q on the right hand-side represents:

a) free charge

b) the sum of free and bound (polarization) charge

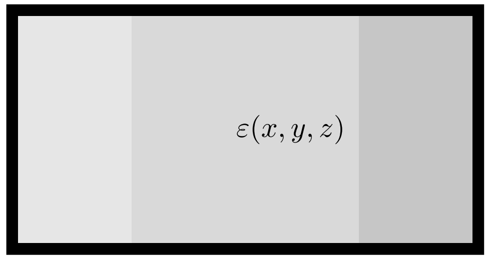
c) bound (polarization) charge

enclosed by the closed surface S.

Choose the correct answer and briefly explain.

(2.5 pts)

5. Consider a rectangular box made by a perfect conductor (shown in black in the figure below). Inside the box we have a non-uniform dielectric without losses ($\sigma = 0$). We want to find the electric field $\mathbf{E}$ inside the box using Poisson equation.



Which statement is correct?

a) We cannot use Poisson equation to find $\mathbf{E}$, since the dielectric is not uniform.

b) We can use Poisson equation. The electric field inside the box is $\mathbf{E} = 0$.

c) We can use Poisson equation. The electric field inside the box will be small but not exactly zero.

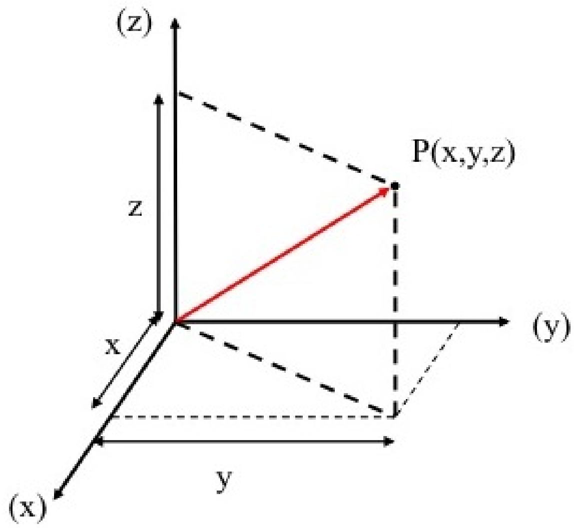
d) We can use Poisson equation. More information on $\varepsilon(x, y, z)$ is needed to calculate $\mathbf{E}$.

<u>Justify</u> your answer.

(2.5 pts)

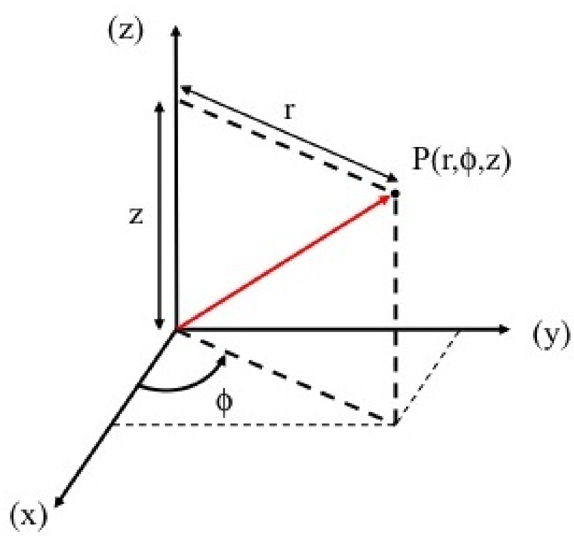
1 Coordinate Systems

1.1 Cartesian coordinates



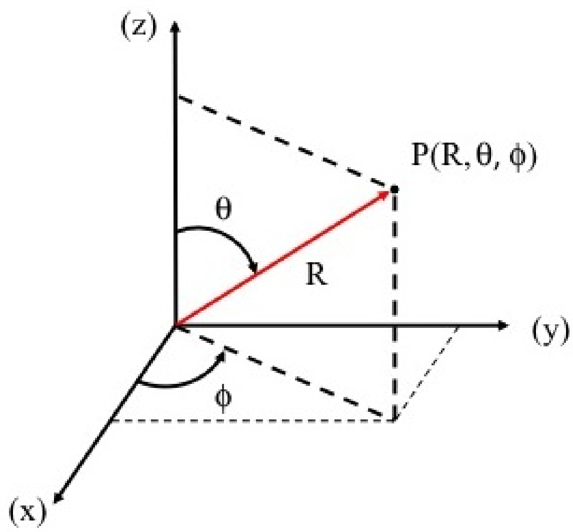
Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates



Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates



Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2 Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3 Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4 Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi - A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_r	-$A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

5 Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_x \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

6 Electromagnetic formulas

Table 1 Electrostatics

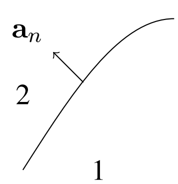
$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{\lvert\mathbf{R}_2-\mathbf{R}_1\rvert^3}(\mathbf{R}_2-\mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R}-\mathbf{R}')}{\lvert\mathbf{R}-\mathbf{R}'\rvert^3}dQ'$
$\nabla\times\mathbf{E} = 0$	$\nabla\cdot\mathbf{D} = \rho_v$
$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0$	$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R}-\mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2}\mathbf{E}\cdot d\mathbf{l}$	
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$	$\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$
$\rho_{p,v} = -\nabla\cdot\mathbf{P}$ $\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2-\mathbf{P}_1)$ $\mathbf{a}_n\cdot(\mathbf{D}_2-\mathbf{D}_1) = \rho_s$ $E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2}QV$
$W_e = \frac{1}{2}\int_{vol}\rho_v V dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}dv$	
$\nabla\cdot(\varepsilon\nabla V) = -\rho_v$	$\nabla\cdot(\varepsilon\nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert \mathbf{R} - \mathbf{R}' \rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} = \mathbf{E} \cdot \mathbf{J} dv$
$J_{1,n} = J_{2,n}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$