Polymer problem

1.)

$$E = -f\ell \sum_{i=1}^{N} s_i$$

$$Z = \sum_{\substack{\{s_1 \dots s_N\} \\ = \pm 1}} e^{\beta f \ell \sum_{i=1}^{N} s_i} = \left(e^{\beta f \ell} + e^{-\beta f \ell}\right)^N$$

$$= \left(2 \cosh \beta f \ell\right)^N$$

2.)

$$\overline{L} = \sum_{\{s_1 \dots s_N\}} \left(\ell \sum_{i=1}^{N} s_i\right) e^{-\beta E} / Z$$

$$= \frac{\partial}{\partial(\beta f)} \log Z = \frac{\partial}{\partial(\beta f)} \left(N \log(2 \cosh \beta f \ell)\right)$$

$$= \frac{N\ell}{\cosh \beta f \ell} (\sinh \beta f \ell) = N \ell \tanh \beta f \ell$$

recall $\beta = \frac{1}{k_B T}$, so $\overline{L} = N\ell \tanh \frac{f\ell}{k_B T}$

for $k_B T \gg f\ell$, $\overline{L} \approx N\ell \frac{f\ell}{k_B T} = f \frac{N\ell^2}{k_B T}$

so we have

$$f = \underbrace{\frac{k_B T}{N \ell^2}}_{k} \bar{L}$$

force

avge length

Hooke's law coefft. $\sim T$

valid for high $k_B T \gg f\ell$

3.)) $F = -k_B T \log Z =$

$$= -k_B T N \log 2 \cosh\left(\frac{f\ell}{k_B T}\right)$$

$$-\left(\frac{\partial F}{\partial f}\right)_{N,T} = \cancel{k_B T} N \frac{\sinh(f\ell/k_B T)}{\cosh(f\ell/k_B T)} \frac{\ell}{\cancel{k_B T}}$$

$$= N \ell \tanh \frac{f\ell}{k_B T} = \bar{L}$$

we also know $-\left(\frac{\partial F}{\partial T}\right)_{f,N} = S$

since F is a fn of (T, N, f)

for fixed N we have

$$dF = \underbrace{\left(\frac{\partial F}{\partial T}\right)_{N,f}}_{-S} dT + \underbrace{\left(\frac{\partial F}{\partial f}\right)_{N,T}}_{-L} df$$

$-L$, by above calculation

So $$dF = -S\,dT - L\,df$$

Einstein solid problem

N oscillators of frequency ω, energies $\hbar\omega k$, $k = 0,1,2$

$$Z = \left(\sum_{k=0}^{\infty} e^{-\beta\hbar\omega k}\right)^N = \left(\frac{1}{1 - e^{-\beta\hbar\omega}}\right)^N$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\log Z = +\frac{\partial}{\partial\beta} N \log\left(1 - e^{-\beta\hbar\omega}\right) =$$

$$= \frac{N}{1-e^{-\beta\hbar\omega}} \left(-e^{-\beta\hbar\omega}\right)(-\hbar\omega)$$

$$\langle E \rangle = \frac{\hbar\omega N}{e^{\beta\hbar\omega}-1}$$ - average energy

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$\langle E^2 \rangle = ?$

$$\langle E^2 \rangle = \frac{\sum_s e^{-\beta E(s)} E(s)^2}{Z}$$

$$= \frac{\sum_s \frac{\partial^2}{\partial\beta^2} e^{-\beta E(s)}}{Z} =$$

it is OK to use w/out derivation (as long as correct)

$$= \frac{1}{Z}\frac{\partial^2}{\partial\beta^2} Z$$

$$\langle E^2 \rangle = \frac{1}{Z}\frac{\partial^2}{\partial\beta^2} Z = \frac{1}{Z}\frac{\partial}{\partial\beta}\frac{\partial}{\partial\beta}\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)^N =$$

$$= \frac{1}{Z}\frac{\partial}{\partial\beta}\left[N\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)^{N-1}(-)\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)^2\left(-e^{-\beta\hbar\omega}\right)(-\hbar\omega\right.$$

$$= -\frac{N}{Z}\frac{\partial}{\partial\beta}\left(\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)^{N+1} e^{-\beta\hbar\omega}\right)\hbar\omega =$$

$$= -\frac{N\hbar\omega}{Z}\left[-\hbar\omega e^{-\beta\hbar\omega}\left(1-e^{-\beta\hbar\omega}\right)^{-(N+1)} + e^{-\beta\hbar\omega}\frac{\partial}{\partial\beta}\left(1-e^{-\beta\hbar\omega}\right)^{-\overbrace{(}^{N+1}}\right.$$

$$= -\frac{N\hbar\omega}{Z}\Big(-\cancel{\hbar\omega}^{\hbar\omega} e^{-\beta\hbar\omega}(1-e^{-\beta\hbar\omega})^{-(N+1)}$$

$$+ e^{-\beta\hbar\omega}(-(N+1))(1-e^{-\beta\hbar\omega})^{-N-1-1} \times$$

$$\times(-e^{-\beta\hbar\omega})(-\hbar\omega)\Big) =$$

$$= N\hbar\omega(1-e^{-\beta\hbar\omega})^{+N} \times \Big[e^{-\beta\hbar\omega}(1-e^{-\beta\hbar\omega})^{-N-1}\hbar\omega$$

$$+ (N+1)\hbar\omega\, e^{-2\beta\hbar\omega}(1-e^{-\beta\hbar\omega})^{-N-2}\Big]$$

$$\langle E^2\rangle = N(\hbar\omega)^2\left[\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} + (N+1)\frac{e^{-2\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}\right]$$

while $\langle E\rangle^2 = \dfrac{N^2(\hbar\omega)^2 e^{-2\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2}$

$$\sigma_E^2 = \langle E^2\rangle - \langle E\rangle^2 = (\hbar\omega)^2\left[N(N+1)\frac{e^{-2\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2} + N\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\right.$$

$$\left. - \frac{e^{-2\beta\hbar\omega}N^2}{(1-e^{-\beta\hbar\omega})^2}\right] = \longrightarrow$$

$$\sigma_E^2 = (\hbar\omega)^2 N \left(\frac{e^{-2\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2} + \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} \right) \quad (1)$$

on the other hand, the heat capacity is

$$C = \frac{d\langle E\rangle}{dT} = \frac{d\beta}{dT}\frac{d\langle E\rangle}{d\beta} = -\frac{1}{kT^2}\frac{d}{d\beta}\langle E\rangle =$$

$$= -k\beta^2 \frac{d}{d\beta}\langle E\rangle$$

so

$$\frac{C}{k} = -\beta^2\frac{d}{d\beta}\langle E\rangle = -\beta^2\frac{d}{d\beta}\left(\frac{N\hbar\omega}{e^{\beta\hbar\omega}-1}\right) =$$

$$= +\beta^2 \frac{N\hbar\omega}{(e^{\beta\hbar\omega}-1)^2} e^{\beta\hbar\omega}\hbar\omega =$$

$$\frac{C}{k} = \beta^2 \frac{N(\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2} \quad (2)$$

from (1)

$$\sigma_E^2 = N(\hbar\omega)^2 \frac{e^{-2\beta\hbar\omega} + e^{-\beta\hbar\omega}(1-e^{-\beta\hbar\omega})}{(1-e^{-\beta\hbar\omega})^2} = \rightarrow$$

$$\sigma_E^2 = N(\hbar\omega)^2 \frac{e^{-\beta\hbar\omega}}{(1-e^{-\beta\hbar\omega})^2} = \underbrace{N(\hbar\omega)^2 \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}}$$

$=$ from (2) $\longrightarrow$ $\frac{1}{\beta^2}\frac{C}{k}$

$$\sigma_E^2 = \frac{1}{\beta^2}\frac{C}{k} \Rightarrow \sigma_E = \sqrt{\frac{1}{\beta^2}\frac{C}{k}} = kT\sqrt{\frac{C}{k}}$$

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part 3:

also: $\sigma_E = \sqrt{N}\, \frac{\hbar\omega\, e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega}-1}$, from above

$$\langle E\rangle = N \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$, from part 1.

$$\frac{\sigma_E}{\langle E\rangle} = \frac{1}{\sqrt{N}} e^{\beta\frac{\hbar\omega}{2}} \sim \frac{1}{\sqrt{N}}$$, fluctuations are small in equilibrium. for $N\to\infty$

The short question:

1.) since densities are same, we habe that in order that the thermal de Broglie wavelengths be equal, since $l_Q^2 \sim \frac{h^2}{m k T}$ the heavier gas has to be cooled more: ^{40}K has mass $\sim \frac{40}{3} \times$ bigger than that of ^{3}H; so roughly ^{40}K has to be cooled $13 \times$ more.

2) various options $\longrightarrow$

2. A. an AC takes heat Q_c from room and expels heat Q_h to outside (both Q's are > 0);

total entropy non-decreasing means

$\frac{Q_c}{T_c} \le \frac{Q_h}{T_h}$, since outside is $T_h > T_c$,

we have $Q_h > Q_c$; if we place it in room, we'd have

$T_c = T_h$ and heat $Q_h \ge Q_c$

will be expelled into room –
– which defeats purpose, as there's no heat outflow from room.

or

2. B. AC operates as fridge, moving heat from system to reservoir. If you place it in window, walls are a barrier between system & reservoir, if you place it in room, reservoir & system are same & heat is returned to reservoir –
– but fewer points! (must connect to increase of total S, as in 2A above).