University of Toronto PHY 293 Fall 2021 Midterm #1

NAME: S(ln)s

STUDENT NUMBER:

TUTORIAL GROUP:

Time: 60 minutes

This is a closed-book exam worth a total of 60 points. Please answer all questions.

ONLY THE FOLLOWING ITEMS ARE ALLOWED ON YOUR DESK DURING THE EXAM:

1. **THIS EXAM BOOK –** It contains this cover page, three question pages and a formula sheet (which may be torn off). Make sure you start by putting your **NAME, ID NUMBER,** and **TUTORIAL GROUP** on the front (cover) page of the exam. The entire exam book (minus the formula sheet) **will be handed in** at the end of the exam and marked.
   a. **FORMULA SHEET**, annotated on the printed page. The formula sheet has to be printed from Quercus.
2. **A CALCULATOR**, which can be anything that does NOT connect to wifi and does NOT communicate with other devices. ACCEPTABLE calculators include programmable and graphing calculators, scientific calculators, etc. UNACCEPTABLE calculators include: cell phones, tablets, laptops, etc.
3. **A PEN OR PENCIL.**
4. **YOUR STUDENT ID CARD**, used to check your identity.

If you are missing anything from the above items, raise your hand and ask an exam supervisor to supply what is missing. If you are missing an item that should have been brought by you (e.g., calculator, pen/pencil) there is a limited supply of extras and are on a first come, first served basis.

COMPLETE SOLUTION INCLUDES:

a) A sketch with a coordinate system, where appropriate. Where a diagram is provided, additional markings can be added to it.
b) Clear and logical workings, including quoting the equations that were used, proper substitutions of numbers (when necessary) with all steps clearly indicated.
c) The final answer with units and three significant figures.

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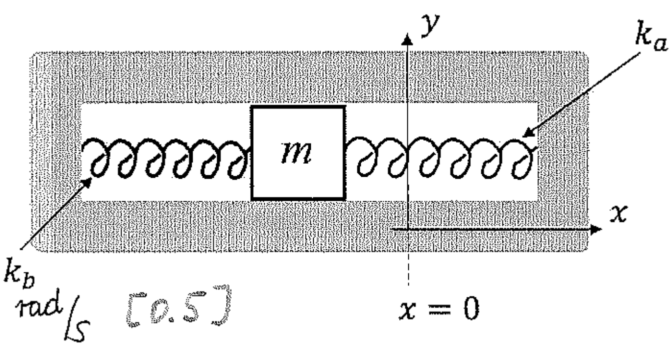
QUESTION	I	II	III	TOTAL
MARK				
MAXIMUM	20	20	20	60

Question I

A mass $m = 0.250$ kg, attached to two horizontal springs (one on each side) with spring constants $k_a = 137\frac{\text{N}}{\text{m}}$ a and $k_b = 59.0\frac{\text{N}}{\text{m}}$, is placed in a frictionless container so that it can oscillate without any losses, as shown in the picture. The spring is oscillating with an amplitude $A = 0.0529$ m. At time $t = 0$ s the spring is moving with velocity $\vec{v}(0) = -0.367\frac{\text{m}}{\text{s}}$ and is **slowing down.**



a. Determine the angular frequency of the motion, ω_0. [2 points]

$\omega_0 = \sqrt{\frac{k}{m}}$ (0.5)

$k_{eff} = k_a + k_b$ [1]

$\omega_0 = \sqrt{\frac{k_a + k_b}{m}} = \sqrt{\frac{137 + 59}{0.25}} =$

$\omega_0 = 28$ rad/s [0.5]

b. Determine the initial phase constant of the motion if the position of the oscillator is described with equation $x(t) = A\cos(\omega t + \phi_0)$. [8 points]

$x(t) = A\cos(\omega t + \phi_0) \rightarrow x(0) = A\cos\phi_0 \leftarrow$ not neccessary

$v(t) = -A\omega\sin(\omega t + \phi_0) \quad v_0(0) = -A\omega\sin\phi_0$ [1]

$-0.367 = -(0.0529)(28)\sin\phi_0$ [2]

NOT CONSIDERING $\rightarrow$ 2 SOLUTIONS [1] 4/8 MAX

$\sin\phi_0 = 0.248 \quad \phi_0 = 0.25$ rad or $\phi_0 = 2.89$ rad [1]

slowing down means $a > 0$ for $v < 0$ (also in the fig $x(0) < 0$) [2]

$a(0) = -A\omega^2\cos\phi_0 \quad \therefore \cos\phi_0 < 0 \quad \therefore \phi_0 = 2.89$ rad [1]

Considering 2 values but arriving to wrong solution: 6.5/8

c. The container is filled with a thick liquid, causing the mass to undergo critical osclations. What is the drag coefficient of the liquid? [4 points]

Critical $\quad \omega_0^2 - \frac{\gamma^2}{4} = 0$ [2]

$\gamma^2 = 4\omega_0^2, \quad \cancel{\gamma = 4\omega} \quad \gamma = 2\omega_0 = 56\ \frac{\text{rad}}{\text{s}}$

drag coefficient : $\gamma = \frac{b}{m}$ [1] $\rightarrow b = \gamma m = 14$ kg/s [1]

d. After the mass stops, it is displaced from the equilibrium by being given an initial velocity $\vec{v}_i$ in positive x direction. Show that it will take the mass $t = \frac{1}{\omega_0}$ to reach the maximum displacement from the equilibrium position. [6 points]

$x(t) = Ae^{-\gamma t/2} + Bt\,e^{-\gamma t/2} \leftarrow$ [1] recognizing critical damping

$v(t) = -A\frac{\gamma}{2}e^{\gamma t/2} + Be^{-\gamma t/2} + -Bt\frac{\gamma}{2}e^{-\gamma t/2}$ [1]

$x(0) = 0, \quad v = +v_i$ [1]

$x(0) = 0 = A$ [1]

$v(0) = -A\frac{\gamma}{2} + B = v_i \quad$ as $A = 0, \ B = v_i$

max displacement $\frac{dx}{dt} = 0 \rightarrow v_i e^{-\frac{\gamma t}{2}} - v_i t\frac{\gamma}{2}e^{-\frac{\gamma t}{2}} = 0$

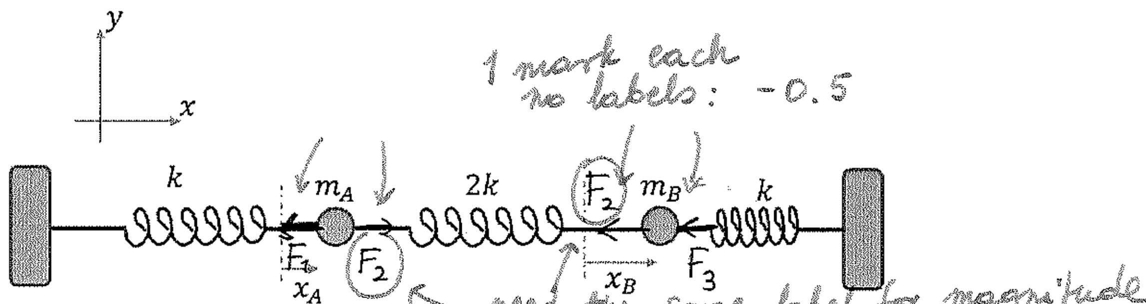
$v_i e^{-\frac{\gamma t}{2}}\left[1 - \frac{\gamma t}{2}\right] = 0$ [1]

$1 - \frac{\gamma t}{2} = 0 \rightarrow \gamma t = 2, \quad t = \frac{2}{\gamma} = \frac{2}{2\omega_0} = \frac{1}{\omega_0}$ [1]

Question II

Two masses, $m_A = m_B = m$ are connected horizontally by three springs of spring constants $k_1 = k, k_2 = 2k$, and $k_3 = k$, in such a way that k_1 connects mass m_A to a rigid support on its left, k_3 connects mass m_B to the rigid support on its right, and k_2 connects two masses together, as shown in the picture below.

a. Assuming both masses are displaced in the $+x$ direction, draw arrows **clearly** indicating the direction of forces on each mass due to each spring attached to it. Label each force. *If it is impossible for you to draw the vectors in the picture, provide labels for the forces and their directions (e.g F_{12} – force on object 1 by agent 2, $-y$ direction).* [4 points]



b. Write the equation of motion for each mass. Clearly label the equations so it is obvious, which mass it is for. [4 points]

$$[2] \quad m_A \frac{d^2x_A}{dt^2} = -k(x_A) + 2k(x_B - x_A) = -kx_A - 2kx_A + 2kx_B = -3kx_A + 2kx_B$$

$$[2] \quad m_B \frac{d^2x_B}{dt^2} = -2k(x_B - x_A) - kx_B = +2kx_A - 3kx_B$$

c. Assuming masses m_A and m_B move according to the equations $x_A(t) = A\cos(\omega t + \phi_A)$ and $x_B = B\cos(\omega t + \phi_B)$, determine the coefficient matrix $\mathbb{M}$ for the system. Express all elements of the matrix in terms of k and m). [6 points]

$$-m_A\omega^2 A = -3kA + 2kB \quad [2]$$

$$-m_B\omega^2 B = +2kA - 3kB \quad [2]$$

$m_A = m_B = m$

$$-\omega^2 \begin{bmatrix} A \\ B \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{3k}{m} & \frac{2k}{m} \\ \frac{2k}{m} & -\frac{3k}{m} \end{bmatrix}}_{\mathbb{M}} \begin{bmatrix} A \\ B \end{bmatrix}$$

2 (needs to be identified)

d. Determine the normal frequencies of this oscillation. [6 points]

$\frac{k}{m} = \alpha$

$$\begin{bmatrix} -3\alpha + \omega^2 & 2\alpha \\ 2\alpha & -3\alpha + \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \quad [2]$$

2 process neccessary

$$(-3\alpha + \omega^2)^2 - 4\alpha^2 = 0$$

$$9\alpha^2 - 6\alpha\omega^2 + \omega^4 - 4\alpha^2 = 0$$

$$\omega^4 - 6\alpha\omega^2 + 5\alpha^2 = 0$$

$$\omega^2 = \frac{+6\alpha \pm \sqrt{36\alpha^2 - 4 \cdot 5\alpha^2}}{2} = \frac{6\alpha \pm \sqrt{16\alpha^2}}{2} = \frac{6\alpha \pm 4\alpha}{2} \begin{cases} 5\alpha \\ \alpha \end{cases}$$

$$\omega = \sqrt{\alpha} = \sqrt{\frac{k}{m}}, \quad \omega = \sqrt{5\alpha} = \sqrt{\frac{5k}{m}}$$

1 1

Question III

1 mark each
needs to be the same value, can be as general as k

A horizontal tube of length L is filled with Argon at temperature $T = -3.0\ ^\circ\mathrm{C}$ ($v_{sound_{Ar}} = 306\ \frac{\mathrm{m}}{\mathrm{s}}$). At time $t = 0$ s, all Ar particles are placed at their respective amplitudes. The open, left handed side of the tube is placed at $x = 0$ m. The characteristic of the other end of the tube is unknown.

a. Fill in the missing elements (numbers, symbols, constants) in the equation so that it correctly describes n^{th} standing **displacement** wave of amplitude A_n in the tube. [5 points]

HAS TO BE ZERO

$$s_n(x,t) = (\underline{0}\ \sin(\tfrac{n\pi}{mL}x) + \underline{A_n}\ \cos(\tfrac{n\pi}{mL}x))\ \underline{\cos}\ (\omega_n t)$$

k k

m is 1 or 2!, but k NOT $\frac{n\pi}{mL}$ is totally correct

b. Two consecutive harmonics heard in the tube have wavelengths of 0.060 m and 0.068 m. What are their mode numbers? [4 points]

longer λ, smaller f

any logical process is ok!

$\lambda_n = 0.060\,m$

$\lambda_m = 0.068\,m$

$m < n$

if not obvious that λ_n is higher freq. -1

$f_n = \frac{v}{\lambda_n}$

$f_m = \frac{v}{\lambda_m}$

$\frac{f_n}{f_m} = \frac{v/\lambda_n}{v/\lambda_m} = \frac{\lambda_m}{\lambda_n} = \frac{6.0\,cm}{6.8\,cm} = \frac{3.0\,cm}{3.4\,cm} = \frac{15}{17}$

0.068 m ~~0.060 m~~

0.060 m ~~0.068 m~~

c. What can you deduce about the other end of the tube? Briefly (1-2 sentences) justify your answer. [4 points]

As the ratio of consecutive harmonics is a ratio of two odd numbers, tube is closed at one end & open on the other

- Correct conclusion [2 marks]
- correct / reasonable justification [2]

d. What is the length of the tube? [3 points]

$$\lambda_n = \frac{4L}{n} \rightarrow L = \frac{n\lambda_n}{4} = \frac{17 \cdot 0.060}{4} = 0.255\,m$$

1 point for consistent λ_n with c, 2 points for correct n & λ_n

e. In the space below sketch a tube and the first three possible harmonics of a displacement wave in it as carefully as possible. The outline of the top and the bottom of the tube is provided, boundary conditions at the ends need to be identified. It should be clear from the sketch how various characteristics of the standing waves compare. (*If you are unable to sketch, describe the first three possible harmonics*). [4 points]

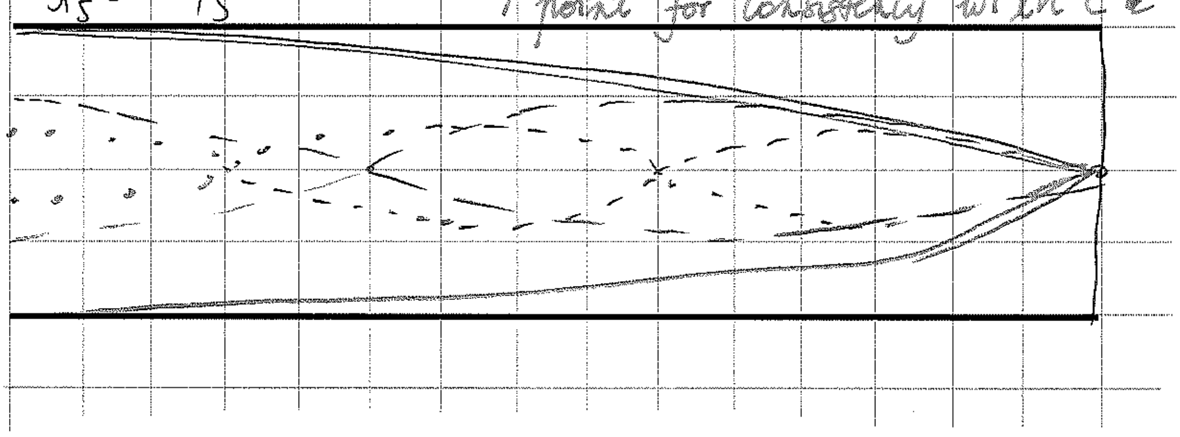
$\lambda_1 = 4L$ —

$\lambda_3 = 4L/3$ – – –

$\lambda_5 = 4L/5$ • • •

1 point for each wave (HAS TO HAVE CORRECT λ) ~~AND CONSISTENT~~

1 point for consistency with c.



OSCILLATIONS

$\omega = 2\pi f = \frac{2\pi}{T}$	$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{mgd}{I}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
$x(t) = A\cos(\omega t + \phi_i)$	$x(t) = A_0 \exp\left(-\frac{\gamma t}{2}\right)\cos(\omega t + \phi_i))$		$x(t) = A(\omega)\cos(\omega t - \delta)$
	$x(t) = A\exp\left(-\frac{\gamma t}{2}\right) + Bt\exp\left(-\frac{\gamma t}{2}\right)$		
$x(t) = A\exp\left(\left(-\frac{\gamma}{2} + \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right) + B\exp\left(\left(-\frac{\gamma}{2} - \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right)$			
$q_0(\omega) = \frac{\varepsilon_0}{\omega Z}$	$q(t) = q_0(\omega)\cos(\omega t - \delta)$	$Z = \sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$	$i = \frac{dq}{dt}$
$V_R = i(t)R$	$V_C = \frac{q}{C}$	$V_L = L\frac{di}{dt}$	
$K = \frac{1}{2}mv^2$	$U = \frac{1}{2}kx^2$	$E(t) = E_0\exp(-\gamma t)$	$P = \frac{dE}{dt} = Fv$
$Q = \frac{\omega_0}{\gamma}$	$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$		
$A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$		$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$	
$\bar{P}(\omega) = \frac{b[v_0(\omega)]^2}{2}$	$\bar{P}_{max} = \frac{F_0^2}{2m\gamma}$	$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma\left[\frac{4(\Delta\omega)^2}{\gamma^2} + 1\right]}$	

WAVES

$v = \lambda f$	$y(x,t) = f(x \pm vt)$	$y(x,t) = A\cos(kx \pm \omega t + \phi_i)$	
$k = \frac{2\pi}{\lambda}$	$y(x,t) = (A\sin(kx) + B\cos(kx))\cos(\omega t)$		
$v = \sqrt{\frac{F_T}{\mu}}$	$v = \sqrt{\frac{B}{\rho}}$	$v = \sqrt{\frac{Y}{\rho}}$	$v = \sqrt{\frac{\gamma RT}{M}}$
$\frac{\partial y^2}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$	$\omega = \frac{2\pi}{T}$	$f = \frac{1}{T}$	
$\omega_n = \frac{n\pi v}{L}$	$\omega_n = \frac{n\pi v}{2L}$	$f_n = nf_1$	$E = \frac{1}{4}\mu\omega_n^2 A_n^2 L$

MATHEMATICAL FORMULAE

$\cos\alpha + \cos\beta = 2\cos\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]$	$\cos\alpha - \cos\beta = -2\sin\left[\frac{\alpha+\beta}{2}\right]\sin\left[\frac{\alpha-\beta}{2}\right]$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$	$\tan^{-1}(x) = \{\theta, \theta + \pi\} + 2\pi n$ $\cos^{-1}(x) = \pm\theta + 2\pi n$
$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$	$\sin^{-1}(x) = \{\theta, \pi - \theta\} + 2\pi n$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$	$\tilde{A} = Ae^{j\theta} = A(\cos\theta + j\sin\theta)$

CONSTANTS

$c = 3.00 \times 10^8 \frac{\text{m}}{\text{s}}$			$g = 9.81\frac{\text{m}}{\text{s}}$
$v_{sound\ at\ 20°C} = 343\frac{\text{m}}{\text{s}}$	$T_K = T_{°C} + 273.15°\text{C}$		