ECE259H1: Electromagnetism

Term Test 1 – Thursday February 16, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.

- The Term Test is worth 60 marks and has three questions. Each question is work 20 marks.

- Show all of your work, and the final page is left blank which you can use for rough work or for extra space for your answers.

- Take a deep breath and relax 😊.

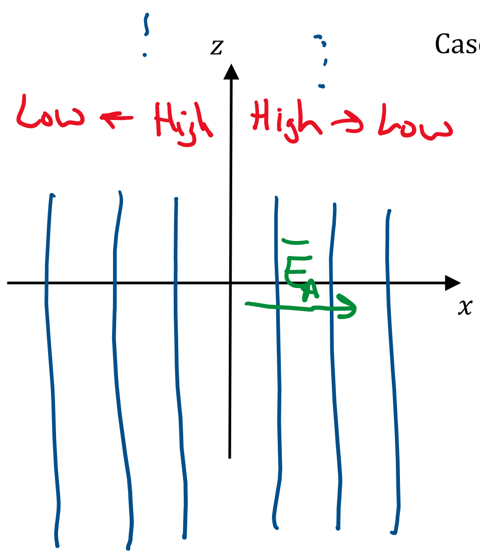
Question #1 (20 marks)

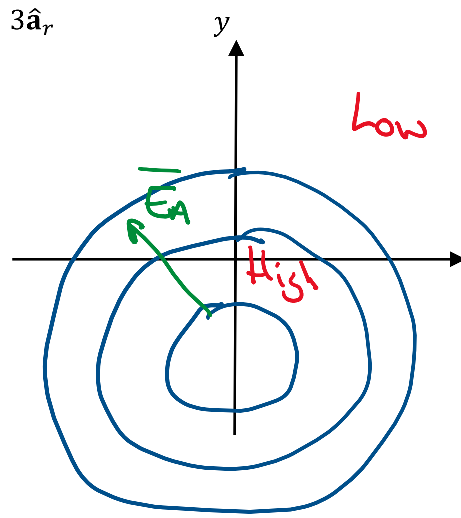
(*7 marks*) 1. (a) Consider the two electric field distributions defined as: $\mathbf{E}_A = 3\hat{\mathbf{a}}_r$ and $\mathbf{E}_B = \frac{2}{R}\hat{\mathbf{a}}_R$. On the figures below sketch the equipotential surfaces associated with each field. Indicate areas of higher and lower electric potential if relevant. Briefly justify in words why you made the sketches that you did. *Note*: For each field, you have to draw two 2D diagrams, one for the xz-plane and one for the xy-plane

$\bar{E}$ must be $\perp$ to the equipotential surfaces.

↳ So for $\bar{E}_A = 3\hat{a}_r$ these form cylinders in (r, ϕ, z) coords.

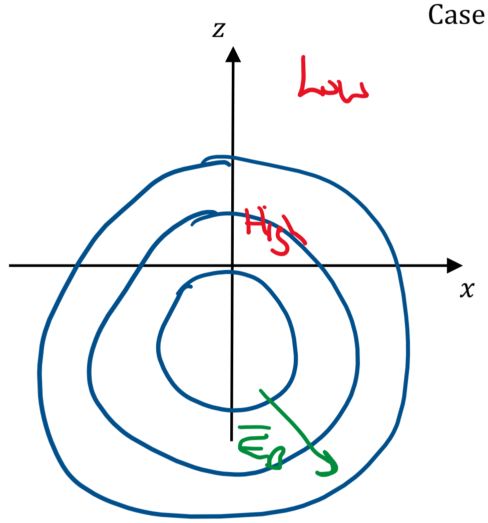
Case A: $\mathbf{E}_A = 3\hat{\mathbf{a}}_r$

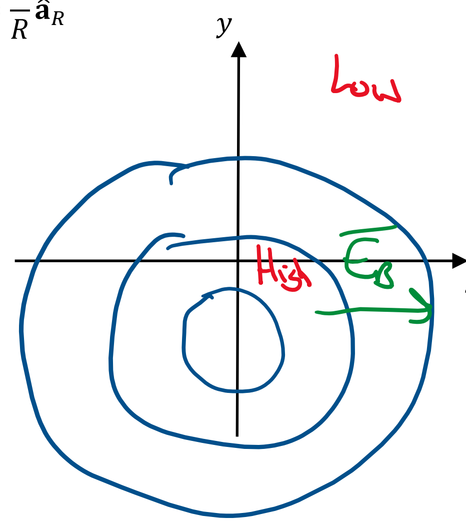




↳ For $\bar{E}_B = \frac{2}{R}\hat{a}_R$ these form spheres in (R, θ, ϕ) coords

Case B: $\mathbf{E}_B = \frac{2}{R}\hat{\mathbf{a}}_R$



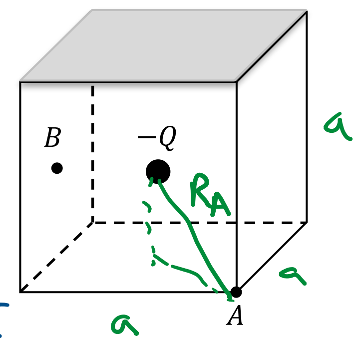


Question #1 (cont'd)

1. (b) A charge $-Q$ sits at the center of a cube with side lengths a. You can assume the charge is placed at the origin of the coordinate system used to describe this situation.

(*3 marks*) i) What is the electric flux, defined as $\Phi_e = \oiint_S \mathbf{D} \cdot d\mathbf{s}$, through the top surface as shown in the figure below? Select the correct answer below and briefly justify your choice.

a. $-Q$
b. $\frac{Q}{4}$
c. $\frac{-Q}{4}$
d. $\frac{-Q}{6}$
e. $\frac{-Q}{8}$
f. $\frac{Q}{16}$
g. Zero
h. None of the above



Total electric flux for cube is $-Q$. Due to the symmetry of the placement of $-Q$ in the cube the electric flux for one face is $-Q/6$.

(*4 marks*) ii) What are the magnitudes of the electric field intensity at point A (at the lower right corner of the cube) and point B (at the center of the left face), as shown in the figure above?

The field magnitude for a point charge is $E = \frac{Q}{4\pi\varepsilon_0 R^2}$

$\therefore$ At A: $R_A^2 = \left(\frac{a}{2}\right)^2 + \left(\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}\right)^2 = \frac{a^2}{4} + \frac{a^2}{2} = \frac{3a^2}{4}$

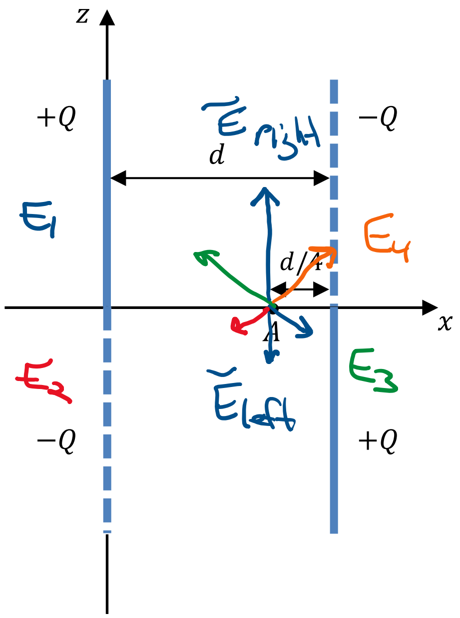
$$E_A = \frac{-Q}{4\pi\varepsilon_0 R_A^2} = \frac{-Q}{4\pi\varepsilon_0 (3a^2/4)} = \frac{-Q}{3\pi\varepsilon_0 a^2}$$

$\therefore$ At B: $R_B^2 = \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$

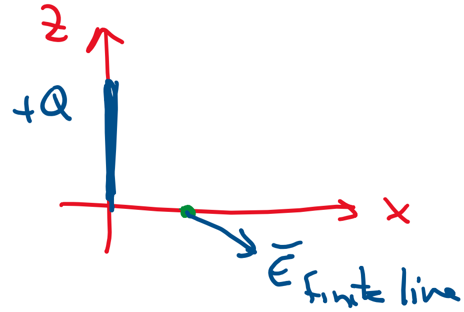
$$E_B = \frac{-Q}{4\pi\varepsilon_0 R_B^2} = \frac{-Q}{4\pi\varepsilon_0 (a^2/4)} = \frac{-Q}{\pi\varepsilon_0 a^2}$$

Question #1 (cont'd)

(*6 marks*) 1. (c) Consider the collection of four charged finite lines as shown below. Each of the solid lines are uniformly charged with a total charge of $+Q$, and each of the dashed lines are uniformly charged with a total charge of $-Q$. All lines have the same length, given by L. What is the direction of the total electric field at point A? Briefly justify your answer with appropriate sketches and descriptions below. No mathematical calculations are needed or expected.



* Each line will produce a field similar to this since the line is of finite length:



* The contributions due to the +Q & −Q line on the z-axis will be smaller than the lines on the right (due to the point being closer to the right).

* Finally, the +Q & −Q line pairs will result in the cancellation of the x-components.

* The total field thus is in the positive z-direction.

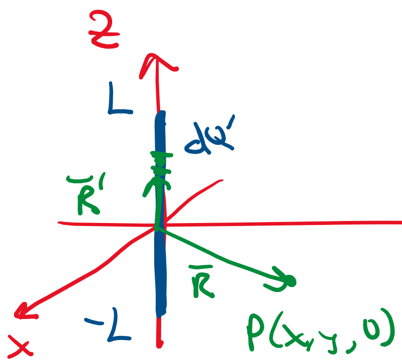
↳ There is no x or y component

Question #2 (20 marks)

(*10 marks*) 2. (a) A uniformly charged length of wire located in free space has a charge density of ρ_l and a total length $2L$. This wire is located along the z-axis and bisected by the xy-plane, meaning it's length ranges from $z = -L$ to $z = +L$. Use Coulomb's law to show that the electric field intensity at any point in the xy-plane is given by:

$$\mathbf{E} = \frac{\rho_l L}{2\pi\varepsilon_0 r\sqrt{r^2 + L^2}}\hat{\mathbf{a}}_r$$

Hint: The following information may be of use $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x/a^2}{\sqrt{x^2+a^2}} + C$ and $\int \frac{xdx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}} + C$ and $e^{i\pi} + 1 = 0$



$$dQ' = \rho_\ell dz'$$

$$\bar{R} = r\hat{a}_r$$

$$\bar{R}' = z'\hat{a}_z$$

$$d\bar{E} = \frac{dQ'(\bar{R} - \bar{R}')}{4\pi\varepsilon_0|\bar{R} - \bar{R}'|^3}$$

$$= \frac{\rho_\ell dz'(r\hat{a}_r - z'\hat{a}_z)}{4\pi\varepsilon_0[r^2 + (z')^2]^{3/2}}$$

$$\bar{E} = \int d\bar{E} = \int_{-L}^{L} \frac{\rho_\ell dz'}{4\pi\varepsilon_0[r^2 + (z')^2]^{3/2}}(r\hat{a}_r - z'\hat{a}_z)$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0}\left[r\hat{a}_r\int_{-L}^{L}\frac{dz'}{[r^2+(z')^2]^{3/2}} - \hat{a}_z\int_{-L}^{L}\frac{z'dz'}{[r^2+(z')^2]^{3/2}}\right]$$

→ integrates to zero (odd fn over symmetric limits)

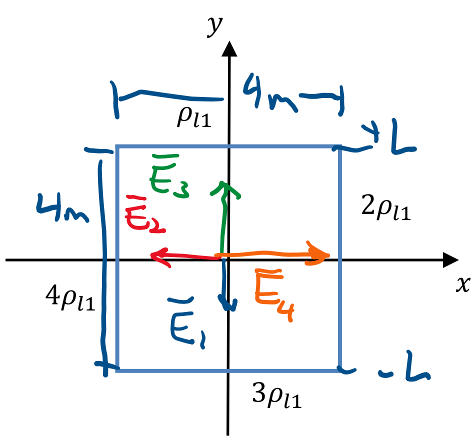
→ Can also argue from symmetry that $E_z \to 0$

$$= \frac{\rho_\ell r\hat{a}_r}{4\pi\varepsilon_0}\left[\frac{z'/r^2}{\sqrt{r^2+(z')^2}}\Bigg|_{-L}^{L}\right]$$

$$= \frac{\rho_\ell\hat{a}_r(\cancel{2}L)}{\cancel{4}_2\pi\varepsilon_0 r\sqrt{r^2+L^2}} = \underline{\underline{\frac{\rho_\ell L\hat{a}_r}{2\pi\varepsilon_0 r\sqrt{r^2+L^2}}}}$$

Question #2 (20 marks)

(*10 marks*) 2. (b) Find the total electric field intensity vector at the origin for the configuration of charged finite lines as shown in the figure below. The configuration is centered about the origin, each line has the same length ($L = 2$ m), and the lines are uniformly charged with different charge densities as indicated. The constant $\rho_{l1} = 4.03\ \mu$C/m.



* Using the result from part(a) we can adapt this for each of the four lines, with total length of $2L = 4$m for each line

$$|E| = \frac{\rho_l L}{2\pi\varepsilon_0 r\sqrt{r^2+L^2}}$$

$$|\bar{E}_1| = \frac{\rho_{l1} \cancel{L}}{2\pi\varepsilon_0(\cancel{L})\sqrt{L^2+L^2}} = \frac{\rho_{l1}}{2\sqrt{2}\pi\varepsilon_0 L} \quad \therefore \bar{E}_1 = \frac{-\rho_{l1}}{2\sqrt{2}\pi\varepsilon_0 L}\hat{a}_y$$

Similarly, $\bar{E}_2 = \frac{-\cancel{2}\rho_{l1}}{\cancel{2}\sqrt{2}\pi\varepsilon_0 L}\hat{a}_x$, $\bar{E}_3 = \frac{3\rho_{l1}}{2\sqrt{2}\pi\varepsilon_0 L}\hat{a}_y$, $\bar{E}_4 = \frac{\overset{2}{\cancel{4}}\rho_{l1}}{\cancel{2}\sqrt{2}\pi\varepsilon_0 L}\hat{a}_x$

$$\therefore \bar{E}_{TOT} = \bar{E}_1 + \bar{E}_2 + \bar{E}_3 + \bar{E}_4$$

$$= \frac{\rho_{l1}}{\sqrt{2}\pi\varepsilon_0 L}\hat{a}_y + \frac{\rho_{l1}}{\sqrt{2}\pi\varepsilon_0 L}\hat{a}_x$$

$$= \frac{\rho_{l1}}{\sqrt{2}\pi\varepsilon_0 L}(\hat{a}_x + \hat{a}_y) = \underline{\underline{5.12\times10^4(\hat{a}_x+\hat{a}_y)\left[\tfrac{V}{m}\right]}}$$

* Note, we also accept answers that used total lengths of 2m as correct. In this case, $\bar{E}_{TOT} = \frac{\sqrt{2}\rho_{l1}}{\pi\varepsilon_0 L}(\hat{a}_x+\hat{a}_y)$

$$= 10.2\times10^4(\hat{a}_x+\hat{a}_y)\left[\tfrac{V}{m}\right]$$

Question #3 (20 marks)

3. Consider the situation of two spherical shells that are both centered about the origin, as shown in the figure below. The inner shell ($R = a$) contains a volume charge distribution given by

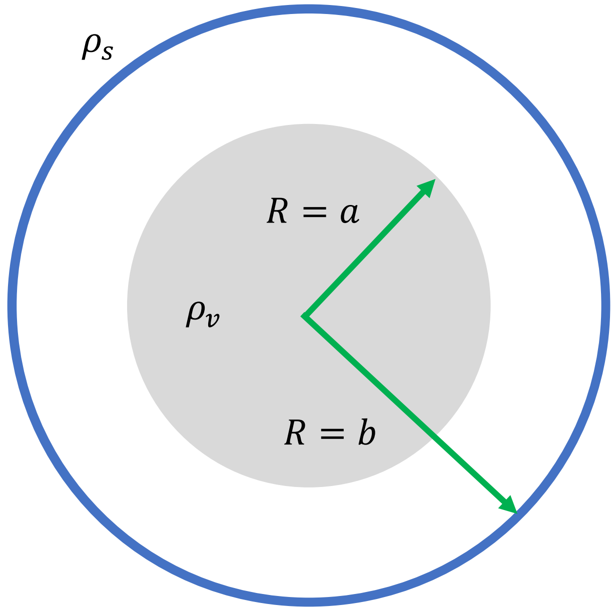
$$\rho_v = \rho_0 \frac{R^2}{a} \quad 0 \le R \le a,\ \rho_0 > 0$$

The outer shell is infinitely thin and has a radius $R = b$ and has a uniform surface charge distribution, ρ_s. The shells are separated by a region of free space ($\varepsilon = \varepsilon_0$).

(*5 marks*) (a) Show that the surface charge density on the outer shell is $\rho_s = -\frac{\rho_0 a^4}{5b^2}$, if it is known that the electric field intensity is zero for $R > b$.

(*7 marks*) (b) Use Gauss's law to find the electric field intensity field within the regions $0 \le R \le a$ and $a < R \le b$.

(*8 marks*) (c) Find the scalar electric potential field *everywhere*, taking the potential of the outer shell as the reference (i.e., $V(R = b) = 0$).



(a) For $E(R>b) = 0$,

$Q_{TOT}(R=b) = -Q_{TOT}(R \le a)$

$Q_{TOT}(R \le a) = \iiint \rho_v \, dv$

$= \int_a^b \int_0^{2\pi} \int_0^{\pi} \left(\rho_0 \frac{R^2}{a}\right) R^2 \sin\theta \, d\theta \, d\phi \, dR$

$= \frac{\rho_0}{a} \int_a^b R^4 \underbrace{\int_0^{2\pi} \int_0^{\pi} \sin\theta \, d\theta \, d\phi}_{= 4\pi} \, dR$

$= \frac{4\pi \rho_0}{a} \frac{R^5}{5}\Big|_0^a = \frac{4\pi \rho_0 a^4}{5}$

$\therefore Q_{TOT}(R=b) = -\frac{4\pi \rho_0 a^4}{5}$

$\rho_s(R=b) = \frac{Q_{TOT}}{4\pi b^2} = \underline{\underline{-\frac{\rho_0 a^4}{5b^2}}}$

Question #3 (continued)

3(b) Due to spherical symmetry, $\bar{E} = E_R \hat{a}_R$ only.

↳ Use a Gaussian sphere of radius R.

For $R \leq a$: $\oiint_S \bar{E} \cdot d\bar{s} = \frac{Q_{enc}}{\varepsilon_0}$

use a dummy int. variable since upper limit is R.

$$\oiint_S \bar{E} \cdot d\bar{s} = \cancel{4\pi} R^2 E_R = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho_0 \frac{u^2}{a} \underbrace{u^2 \sin\theta \, d\theta \, d\phi}_{\text{result of } d\phi d\theta \text{ integral is } 4\pi} du$$

$$= \frac{1}{\varepsilon_0} (\cancel{4\pi}) \frac{\rho_0}{a} \left(\frac{R^{\cancel{5}\,3}}{5} \right)$$

$$\therefore \bar{E}_1 = \frac{\rho_0 R^3}{5 \varepsilon_0 a} \hat{a}_R \quad (R \leq a)$$

For $a < R \leq b$:

$$\oiint_S \bar{E} \cdot d\bar{s} = \cancel{4\pi} R^2 E_R = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_0^a \int_0^{2\pi} \int_0^{\pi} \rho_0 \frac{R^2}{a} R^2 \sin\theta \, d\theta \, d\phi \, dR$$

$$= \frac{1}{\varepsilon_0} (4\pi) \frac{\rho_0}{a} (a^5/5)$$

$$= \frac{\cancel{4\pi} \rho_0 a^4}{5 \varepsilon_0}$$

$$\therefore \bar{E}_2 = \frac{\rho_0 a^4}{5 \varepsilon_0 R^2} \hat{a}_R \quad (a < R \leq b)$$

Question #3 (continued)

3(k) Using condition from (a), $\bar{E} = 0$ $(R > b)$,

thus $\rho_s = \dfrac{-\rho_0 a^4}{5b^2}$

* With $V(R=b) = 0$, $V(R) = 0$ for $R > b$ since $E = 0$ in this region.

<u>For $a < R \leq b$:</u> $V(R) = -\int_b^R \bar{E}_2 \cdot \bar{dl}$

$$= -\int_b^R \frac{\rho_0 a^4}{5\varepsilon_0 u^2}\, du = -\frac{\rho_0 a^4}{5\varepsilon_0}\left(-\frac{1}{u}\Big|_b^R\right)$$

$$= \underline{\underline{\frac{\rho_0 a^4}{5\varepsilon_0}\left(\frac{1}{R} - \frac{1}{b}\right)}}$$

<u>For $R \leq a$:</u> $V(R) = V(R=a) - \int_a^R \bar{E} \cdot \bar{dl}$

$$V(R) = \frac{\rho_0 a^4}{5\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) - \int_a^R \frac{\rho_0 u^3}{5a\varepsilon_0}\, du$$

$$= \frac{\rho_0 a^3}{5\varepsilon_0}\left(\frac{b-a}{b}\right) - \frac{\rho_0}{5a\varepsilon_0}\left[\frac{u^4}{4}\Big|_a^R\right] = \frac{\rho_0 a^3}{5\varepsilon_0}\left(\frac{b-a}{b}\right) - \frac{\rho_0}{5a\varepsilon_0}\left(\frac{R^4 - a^4}{4}\right)$$

$$= \underline{\underline{\frac{\rho_0 a^3}{5\varepsilon_0}\left(\frac{b-a}{b}\right) + \frac{\rho_0 a^3}{20\varepsilon_0} - \frac{\rho_0 R^4}{20a\varepsilon_0}}}$$

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