UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

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PHY293F (Waves and Modern Physics)
Instructor: Professors N. Grisouard and W. Trischuk

Duration: 2.5 hours

Exam Type A: Closed Book. Only non-programmable calculators allowed.

This examination paper consists of **6** pages and 6 questions. Please bring any discrepancy to the attention of an invigilator. Answer all 6 questions.

- Write your name, student number and tutorial group on top of **all** examination booklets and test pages used.
- Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. Not other aid (notes, textbook, dictionary) is allowed. **Communication devices are strictly forbidden. Turn them off and make sure they are in plain sight to the invigilators.**
- Each question is worth 1/6 of your overall grade for this exam. Within each question, a mark breakdown is indicated in square brackets at the end of each sub-part. Part marks will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion** any relevant assumptions you are making along the way.
- Do not separate the stapled sheets of the question paper. Hand in the questions with your exam booklet at the end of the test.
- The next two pages include some formulae and constants you may find useful.
- The questions begin on **page 4**. The total number of marks is 60.

Oscillations

	Amplitude	Velocity	Dissipated Power
Peak freq.	$\omega_{max} = \omega_0\sqrt{1 - 1/(2Q^2)}$	$\omega_{max} = \omega_0$	$\omega_{max} = \omega_0$
Peak value	$A_{max} = \dfrac{QA_f}{\sqrt{1 - 1/(4Q^2)}}$	$V_{max} = \omega_0 Q A_f$	$P_{max} = \dfrac{mA_f^2\omega_0^3 Q}{2}$
Misc.	$A(\omega) = \dfrac{\omega_0^2 A_f}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$	$V(\omega) = \omega A(\omega)$	$\overline{P}(\omega) = \dfrac{m\gamma V^2(\omega)}{2}$
	$\tan\delta = \dfrac{\omega\gamma}{\omega_0^2-\omega^2}$		$\approx \dfrac{P_{max}}{1+\frac{4(\omega_0-\omega)^2}{\gamma^2}}\ (Q \gg 1)$

$$\mathsf{M}\ddot{\vec{X}} + \mathsf{K}\vec{X} = 0; \qquad \det\left(\mathsf{K} - \omega^2\mathsf{M}\right) = 0.$$

$$\mathsf{M}^{-1}\mathsf{K} \quad \text{symmetric and} \quad |\vec{Y}_i| = 1 \quad \Rightarrow \quad \vec{Y}_i \cdot \vec{Y}_j = \delta_{ij}$$

$$\vec{X}(t) = \sum_{n=1}^{N} C_n \vec{Y}_n \cos(\omega_n t + \phi_n), \quad \text{with} \quad C_n\cos\phi_n = \vec{X}_0 \cdot \vec{Y}_n \quad \text{and} \quad C_n \sin\phi_n = -\frac{\vec{V}_0\cdot\vec{Y}_n}{\omega_n}.$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

$$\frac{\partial^2}{\partial t^2} y(x,t) - v^2 \frac{\partial^2}{\partial x^2} y(x,t) = 0 \quad \text{with} \quad v = \sqrt{\frac{T}{\mu}}$$

$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos(\omega_n t + \phi_n)\sin(k_n x) = \sum_{n=1}^{\infty} [\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t)]\sin(k_n x),$$

$$\text{with} \quad \alpha_n = \frac{2}{L}\int_0^L y(0,x)\sin(k_n x)\mathrm{d}x \quad \text{and} \quad \beta_n = \frac{2}{L\omega_n}\int_0^L \dot{y}(0,x)\sin(k_n x)\mathrm{d}x.$$

$$y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right) = A\sin\left(k(x - vt)\right) = A\sin\left(kx - \omega t\right) = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right].$$

$$\omega = 2\pi\nu, \quad \nu = 1/T, \quad k = 2\pi/\lambda, \quad v = \omega/k = \lambda/T = \lambda\nu.$$

$$\text{Energy Flux} = \frac{1}{2}\mu_i v\omega^2 A^2 = \frac{1}{2}\sqrt{T\mu_i}\,\omega^2 A^2.$$

$$\rho = \frac{A_R}{A_I} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}; \qquad \tau = \frac{A_T}{A_I} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

Modern Physics

Speed of light c	$= 3.00 \times 10^{8}$ m/s	Mass of electron m_e	$= 9.11 \times 10^{-31}$ kg $= 511\,\mathrm{keV/c^2}$
Elementary charge e	$= 1.602 \times 10^{-19}$ C	Mass of proton m_p	$= 1.67 \times 10^{-27}$ kg $= 939\,\mathrm{MeV/c^2}$
Coulomb constant ϵ_0	$= 8.85 \times 10^{-12}\,\mathrm{C^2/(J\,m)}$	Planck's constant h	$= 6.626 \times 10^{-34}$ J s $= 4.14 \times 10^{-15}$ eV s
hc	$= 1.240$ keV nm	1 eV	$= 1.602 \times 10^{-19}$ J

$$L^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \qquad \gamma = (1-\beta^2)^{-1/2} \qquad \beta = v/c$$

$$x' = \gamma(x - \beta ct) \qquad ct' = \gamma(ct - \beta x) \qquad y' = y \qquad z' = z$$

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2}$$

$$\vec{p} = m\vec{u} \qquad \vec{p} = \gamma m \vec{u}$$

$$E = \gamma mc^2 = \sqrt{(pc)^2 + (mc^2)^2} \qquad \left(\frac{p}{mc}\right)^2 = \gamma^2\beta^2$$

$$E = h\nu = \hbar\omega \qquad \lambda = \frac{h}{p} \qquad \vec{p} = \frac{h}{\lambda} = \hbar\vec{k} \qquad \hbar = \frac{h}{2\pi}$$

$$T_{\mathrm{max}} = eV_0 = h\nu - \Phi \qquad \Delta\lambda = \lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta)$$

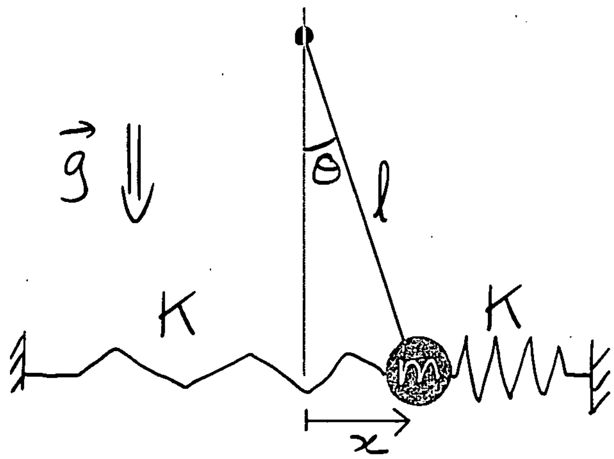
$$E_n = -\frac{1}{2n^2} m_e \left(\frac{e^2}{4\pi\epsilon_0\hbar}\right)^2 = \frac{E_1}{n^2} = -\frac{13.56\,\mathrm{eV}}{n^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x,t)\,\Psi(x,t) = E\,\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

$$\int_{\text{all space}} \left|\Psi(x,t)\right|^2 dx = 1 \qquad \Delta p\,\Delta x \gtrsim \hbar/2 \qquad \Delta E\,\Delta t \gtrsim \hbar/2$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \begin{Bmatrix}\cos^2\theta \\ \sin^2\theta\end{Bmatrix} = \frac{1}{2}(1 \pm \cos 2\theta)$$

1. See figure: the mass m of a simple pendulum of length ℓ is also attached to two identical, horizontal springs of stiffness K, pointing in two opposite directions. Under the small angle approximation, the motion is considered horizontal. The position x is taken to be zero when the pendulum is vertical.



   (a) Write down the differential equation that describes the motion of the mass when friction is neglected. What is the expression for the natural frequency? [5]

   (b) Compute the period of oscillations for $m = 500$ g, $\ell = 40$ cm, $K = 30\ \mathrm{N\,m^{-1}}$ and $g = 9.8\ \mathrm{m\,s^{-2}}$. [2]

   (c) We now introduce damping. The damping coefficient is equal to 10% of that of the critical damping coefficient. What is the value of the quality factor? [3]

2. A long string is connected to an electrically driven oscillator so that a transverse sinusoidal wave is propagated along the string. The string has a mass of 600 g, is 2 m long, and is held under a tension of 150 N.

   (a) Calculate the power that must be supplied to the oscillator to sustain the propagation of the wave if it has a frequency of 12 Hz and an amplitude of 3.0 cm. [4]

   (b) What will be the power required (i) if the frequency is tripled and (ii) if the amplitude is divided by three? [2]

   (c) Now a second uniform wire, of length 0.50 m, and having a mass of 200 g, is spliced (joined seamlessly) to the first wire, producing a single wire. The tension remains the same. A pulse of amplitude 0.50 cm is introduced at the top of the new wire and propagates downward.

      i. What is the amplitude of the wave pulse which is reflected back from the joint between the two wires? [2]

      ii. What is the amplitude of the wave pulse which continues downward beyond the joint between the two wires? [2]

3. Cauchy's formula is an empirical relationship that relates the refractive index $n = c/v$ of a transparent medium to wavelength λ, where $c \approx 3 \times 10^8$ m s^{-1} is the phase speed of light in vacuum, v is the phase speed of light in the transparent medium, and λ is the wavelength of the light in vacuum. The formula is

$$n = A + \frac{B}{\lambda^2},$$

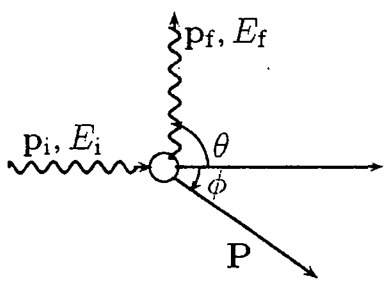
where A and B are constants for the particular medium.

(a) Show that the ratio of group and phase velocities at wavelength λ is given by

$$\frac{v_g}{v} = \frac{A - B/\lambda^2}{A + B/\lambda^2}. \qquad [7]$$

(b) Evaluate this ratio at a wavelength of 500 nm for a particular type of glass for which $A = 1.85$ and $B = 4.2 \times 10^{-14}$ m^2. [3]

4. In a particle physics experiment, a photon with energy E_i collides with an unknown particle that is at rest, as shown in the diagram, opposite. The photon scatters at 90° to its original direction retaining 1/4 of its original energy ($E_f = E_i/4$). After the collision, the massive particle has momentum P directed at an angle ϕ relative to the incident photon's momentum direction.



(a) Derive an expression for the rest mass of the unknown particle in terms of E_i. [3]

(b) Calculate the angle, ϕ, of the outgoing massive particle after the collision. [3]

(c) If the incident photon energy is 1.5 GeV, determine the rest mass of the target particle. [1]

(d) As measured in the lab frame, the target particle decays 1.7 ps after the collision. How long after the collision does the decay take place in the particle's rest frame? [3]

5. We saw in class that Heisenberg's uncertainty principle can be used to give a reasonable prediction for the size of the hydrogen atom: about 10^{-10} m.

   (a) Without resorting to the specific form of the potential energy of electrons in hydrogen atom use the Heisenberg uncertainty principle to estimate the momentum of the electron when confined to a region of this size. [2]

   (b) Is such an electron likely to be relativistic? As a rough guide, you can use $v \approx 0.1c$ as the boundary between the non-relativistic and relativistic regimes. [2]

   (c) In higher charge atoms, like lead, the inner-most electrons are confined to a much smaller region of about 10^{-12} m. Are these electrons relativistic ? [2]

   (d) If an electron, in the hydrogen atom, finds itself in the first excited state ($n = 2$) it will decay back to the ground state ($n = 1$.).

      i. What is the energy that will be released in this transition ? [2]

      ii. If this decay has a lifetime of 2 ns, what limit does the Heisenberg uncertainty principle place on the precision with which we can measure that energy? [2]

6. Suppose the wavefunction for a particle, confined to lie between $0 < x < 1$ is given by $\psi(x) = A(x - x^2)$. Outside the allowed region the wave-function vanishes.

   (a) Sketch what the wave-function looks like, making sure to indicate clearly how it obeys the continuity conditions required by the Schrodinger equation [2].

   (b) Find the probability to find the particle in the right half ($\frac{1}{2} < x < 1$) of the well. [3]

   (c) What is the expectation value of x? [2]

   (d) What is the expectation value of x^2? [2]

   (e) Compute the uncertainty on x: $\Delta x = \sqrt{< x^2 > - < x >^2}$ [1]

End of examination
Total pages: 6