MAT292 – Fall 2021

Term Test – November 1, 2021

Time alotted: 110 minutes

Full Name ______________________________

Student Number ______________________________

Email ______________________________@mail.utoronto.ca

Signature ______________________________

DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This test contains 8 pages (including this title page). Once the test starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages **??**–8 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

	Short answer	**True/False**	**Long answer**				
Question	Q1-Q5	Q6-Q9	Q10	Q11	Q12	Q13	**Total**
Marks	12	8	14	9	9	9	**61**

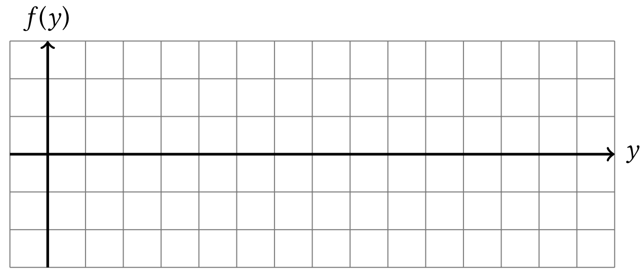
GOOD LUCK! YOU GOT THIS!

SECTION I

Provide the final answer. No justification necessary.

1. **(2 marks)** On the right, draw a phase plot for an autonomous ODE $y' = f(y)$ such that:

   - The ODE has three equilibria.
   - At least one of the equilibria is stable.
   - **NONE** of the equilibria are unstable.

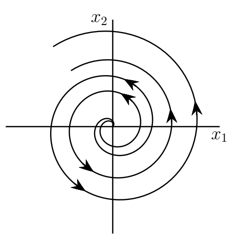


Solution: The graph needs to cross once from above to below the y-axis (stable equilibrium) and "touch" the y-axis another three times (semistable equilibria).

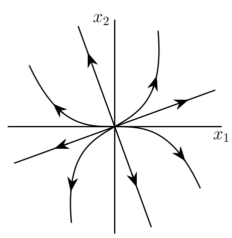
2. **(4 marks)** Consider two-dim. systems of the form $\vec{x}\,' = A\,\vec{x}$ and the following eigenvalue setups for the matrix A.

P: $\lambda_1 = \lambda_2 = -5$ **Q:** $\lambda_1 = \lambda_2 = 5$ **R:** $\lambda_1 = 5 + \pi i$, $\lambda_2 = 5 - \pi i$ **S:** $\lambda_1 = 6$, $\lambda_2 = e$

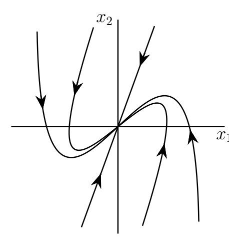
Below each phase plot below, **write the letter** of the matching setup.



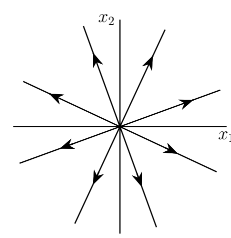
R



S



P



Q

3. **(2 marks)** Consider the IVP $y' = t + y^2$, $y(1) = 1$. Approximate $y(2)$ using the **Improved** Euler method with a single step.

$y(2) \approx$ 7.5

Solution: If we write $f(t, y) = t + y^2$, then the initial slope is $f(1, 1) = 2$. Going along that slope, we get to $(t, y) = (2, 3)$. At that point, we sample the slope $f(2, 3) = 11$. Improved Euler requires us to take the average between the two slopes: $(2 + 11)/2 = 6.5$. We start again at $(t, y) = (1, 1)$ but now using this average slope of 6.5 and get $(t, y) = (2, 7.5)$.

4. **(2 marks)** Consider the IVP $y' = f(t, y)$, $y(0) = y_0$. Approximating $y(1)$ using the Runge-Kutta Method with a fixed step size and 15 steps results in a global truncation error of approximately $\frac{1}{10}$. Give a plausible estimate for the global truncation error if we used 30 steps instead.

Error≈ $\frac{1}{160}$

Solution: Doubling the number of steps is equivalent to halving the step size. Since Runge-Kutta is a fourth order method, this approximately reduces the global truncation error by a factor of $2^4 = 16$, so the answer is $\frac{1}{10} \cdot \frac{1}{16}$

5. **(2 marks)** Consider the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $\alpha \in \mathbb{R}$ is a parameter.

Make **exactly one choice** in each box.

◯ The equilibrium is stable	◯ The equilibrium is unstable	◯ The answer depends on the value of α.

◯ $\lim_{t\to\infty} \lvert x(t)\rvert = 0$ for all solutions.	◯ We must know the initial value to determine $\lim_{t\to\infty} \lvert x(t)\rvert$.
◯ $\lim_{t\to\infty} \lvert x(t)\rvert = \infty$ for all solutions.	◯ We must know α to determine $\lim_{t\to\infty} \lvert x(t)\rvert$.
	◯ We must know the initial value **AND** α to determine $\lim_{t\to\infty} \lvert x(t)\rvert$.

Solution:

$$\begin{vmatrix} \alpha - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda - 1$$

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$$

Notice that $\sqrt{\alpha^2 + 4} > |\alpha|$. If $\alpha > 0$, one eigenvalue is positive and one is negative. If $\alpha < 0$, one eigenvalue is positive and one is negative. If $\alpha = 0$, one eigenvalue is positive and one is negative. So the only possibility is having one positive eigenvalue and one negative, i.e., an unstable node. The limit of the magnitude depends on the initial value AND α. A solution that starts on the "eigenvector line" for a negative eigenvalue goes to zero. All other solutions go to infinity.

SECTION II

For each of the following statements, decide if it is true or false. **Then justify your choice.**

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

6. **(2 marks)** Consider the solution $y(t)$ to the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)y, \qquad y(0) = y_0.$

   Given any y_1, there exists a time t_1 at which $y(t_1) = y_1$.

   Solution: There is an equilibrium at $y = T$. Solutions can't cross this equilibrium. That means if $y_0 < T$, the solution to the IVP can never get to a value $y_1 > T$. And if $y_0 > T$, the solution to the IVP can never get to a value $y_1 < T$. If $y_0 = T$ the solution is constant and can't reach any other value.

7. **(2 marks)** Consider a two-dimensional system $\vec{x}\,' = A\vec{x}$ where A has two complex eigenvalues. If there is at least one nonzero solution such that $\lim\limits_{x\longrightarrow\infty} \vec{x}(t) = [0, 0]$, then $\lim\limits_{x\longrightarrow\infty} \vec{x}(t) = [0, 0]$ is true for all solutions.

   Solution: This statement is true. Complex eigenvalues are conjugates of the form $\lambda = \mu \pm i\nu$. The real part can be positive, negative or zero. Since there is a nonzero solution with zero limit, the real part must be negative, giving a decreasing factor of $e^{\mu t}$. This means all solutions are going to zero.

8. **(2 marks)** Euler's Method (**not** improved Euler) is used to approximate the two IVPs below.

   $$\text{(A)} \qquad y' = \sin y, \qquad y(0) = 100 \qquad\qquad \text{(B)} \qquad y' = 5y, \qquad y(0) = 100$$

   The local truncation error should be larger when approximating the solution of (A).

   Solution: The second derivative of the first ODE is $y'' = \cos y y' = \cos y \sin y$ which is bounded by 1. The second derivative of the second ODE is $y'' = 5y' = 25y$ which is unbounded. The difference between the derivatives is already striking at the initial step.

   The error of Euler's Method is bounded in terms of the second derivative. We should expect a much more significant error when approximating the solution of (B).

   Therefore the statement is false.

9. **(2 marks)** Assume $\vec{\phi}(t)$ and $\vec{\psi}(t)$ solve the system $\vec{x}\,' = A\vec{x} + \vec{b}$ where $A \in \mathbb{R}^{n\times n}$ and $\vec{b} \in \mathbb{R}^n$.
   For any $a \in \mathbb{R}$, $\vec{\phi}(t) + a\vec{\psi}(t)$ also solves the system.

   Solution: One way is to show that this is false by giving a counterexample.

   Another way is to explain how the superposition principle doesn't apply to solutions of non-homogeneous systems.

   Alternatively, a more general approach works as follows: If we write $\vec{p} = \vec{\phi} + a\vec{\psi}$, we get

   $$A\vec{p} + \vec{b} = A(\vec{\phi} + a\vec{\psi}) + \vec{b} = A\vec{\phi} + aA\vec{\psi} + \vec{b}$$

   Using that $\vec{\phi}$ and $\vec{\psi}$ themselves are solutions, we also get

   $$\vec{p}\,' = \vec{\phi}\,' + a\vec{\psi}\,' = A\vec{\phi} + \vec{b} + a(A\vec{\psi} + \vec{b})$$

   For $\vec{p}$ to be a solution, we need that both of the previous lines equal:

   $$A\vec{\phi} + aA\vec{\psi} + \vec{b} = A\vec{\phi} + \vec{b} + a(A\vec{\psi} + \vec{b}) \Longleftrightarrow \vec{b} = \vec{b} + a\vec{b}$$

   Unless $a = 0$ or $\vec{b} = 0$, this is not true. So in general the statement is false.

SECTION III **Justify** all your answers.

10. Consider the temperature of two adjacent rooms in a house.

Denote by $A(t)$ the temperature in room A and by $B(t)$ the temperature in room B.

We measure temperature in degrees Celsius and time in hours.

There are two effects, coming from the fact that the air in one room heats/cools the air in the other room.

- The temperature in room A changes at a rate proportional to the difference in temperature between room A and room B. The proportionality constant is 2.
- The temperature in room B changes at a rate proportional to the difference in temperature between room B and room A. The proportionality constant is 3.

(a) **(3 marks)** Find two ODEs involving $A(t)$ and $B(t)$. **Explain.**

Solution: The difference between the room temperatures is given by $B - A$ or $A - B$. Using the proportionality constant, the first law translates to either $A' = 2(B - A)$ or $A' = 2(A - B)$. The latter can't be correct since it would mean that if A is hotter than B, A heats up even more. Therefore $A' = 2(B - A)$ is the correct equation. Similarly, we get $B' = 3(A - B)$

(b) **(1 mark)** The ODEs that you found in the previous part produce a system of two ODEs. Fill in the matrix on the right.

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

(c) **(2 marks)** Explain **from a physical perspective** why it's **impossible** for the matrix in part (b) to have positive eigenvalues.

Solution: If the matrix had a positive eigenvalue, at least for certain initial values, the room temperatures would increase to infinity. Energy would be produced "out of nowhere" with no external influence. That defies the basic laws of thermodynamics.

(d) **(6 marks)** Now find the general solution to the system and draw a phase portrait on the next page.
Make sure to label the phase portrait appropriately.

Put your final answer in the box on the next page.

Solution: Find the eigenvalues

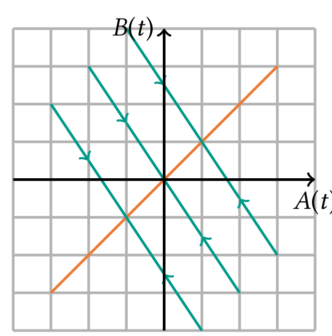
$$\begin{vmatrix} -2-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (2+\lambda)(3+\lambda) - 6 = \lambda^2 + 5\lambda = \lambda(\lambda+5)$$

So we have eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -5$.

Either by solving a linear system or by inspection, matching eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

Therefore, the general solution is

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$



And the phase portrait looks like the one on the right. Note that the lines are not orthogonal due to the different slopes.

(e) **(2 marks)** Find the equlibrium/equlibria of the system. Explain the result from a physical perspective.

Solution: The equilibria are on the line $A = B$, which can be found from setting $A' = B' = 0$ in the equation in part (b). This makes sense since, if the temperature is the same in both rooms, in the absence of external effects we expect that the temperatures remain unchanged.

11. A MAT292 student is trying to to approximate the solution of the IVP $y' = \sin t \cos y$, $y(0) = 1$ at the point $t = 5$ using a numerical method implemented in MATLAB, having fixed step size $h = 0.5$.

The code they produced has an error and is inefficient.

```
1  t=0;h=0.5;y=zeros(11,1);y(1)=1;
2  for i=1:10
3     slope = eval(t,y(i),1,h)+2*eval(t,y(i),2,h)...
4             +2*eval(t,y(i),3,h)+eval(t,y(i),4,h);
5     slope = slope/6;
6     y(i+1) = y(i)+slope*h;    t=t+h;
7  end
8
9  disp('The value y(5) is approximately:');
10 disp(y(11));
11
12 function a=eval(t,y,n,h)
13    f=@(t,y) sin(t)*cos(y);
14    if n == 1
15       a = f(t,y);
16    else
17       a = f(t+h,y+h*eval(t,y,n-1,h));
18    end
19 end
```

(a) **(1 mark)** Before we talk about errors or inefficiencies: Which numerical method is the student *trying* to implement? No justification necessary.
Solution: This looks like Runge-Kutta. The most telling sign is the weighting of four slopes in lines 3/4 with weights 1/2/2/1.

(b) **(2 marks)** Consider the method you just identified in part (a). In an *ideal* implementation of it, how many times would the function $f(t, y) = \sin t \cos y$ need to be evaluated to get to the desired approximation of $y(5)$?
Solution: Runge-Kutta requires four slope samples per step and there are ten steps. That makes 40 slope samples.

(c) **(3 marks)** Now let's have a look at the MATLAB code above. How many times is the function $f(t, y) = \sin t \cos y$ being evaluated if this MATLAB script is run? Explain. You are encouraged to reference line numbers.

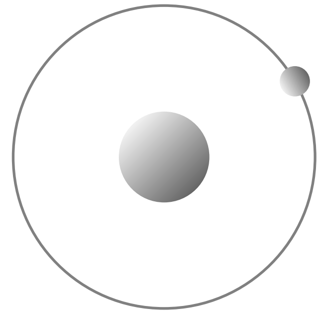
Solution: Consider line 3/4. The first call of eval produces one evaluation of f. The second call of eval produces two evaluations of f (due to the recursive nature of the function `eval`) and so on. In total, line 3/4 produces 1+2+3+4=10 evaluations of f. The for loop is run ten times, so in total there are 100 evaluations of f.

(d) **(3 marks)** The code is not only inefficient, but it is also an *incorrect* implementation of the method you identified in part (a). Explain what is wrong. You are encouraged to reference line numbers.

We are not looking for a clerical error like "they forgot a comma" or "the parentheses don't match".
There is an actual semantic mistake in the code.

Solution: The issue is the recursive nature of the call of `eval`. More specifically, in line 17, the time at which f is evaluated is always a full step of `t+h`. The actual Runge-Kutta Method does evaluations at half step. A way to fix the code would be to use `h` or `h/2` depending on the value of `i` in line 17. That would not solve the problem of efficiency, but at least it would be correct code. Note that fixing line 3/4 alone can't solve the problem since that would still result in wrong step sizes during the recursive calls. It was necessary to fix line 17.

12. A SpaceX aircraft is orbiting around Mars in a perfect circle (see figure). The aircraft's location, represented by three coordinates $[x_1, x_2, x_3]$, satisfies the following linear system of differential equations:



$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x} \qquad \text{with} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

The centre of Mars is located at the origin $[0, 0, 0]$.

(a) **(2 marks)** Looking at the physical model, why should the following be true? Explain briefly.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2 + x_2^2 + x_3^2\right) = 0$$

Solution: The distance of the spaceship from Mars stays fixed. Therefore $\|x\|^2$ doesn't change over time.

(b) **(2 marks)** Using the linear system, show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2 + x_2^2 + x_3^2\right) = 0$ is in fact true.

Solution: Using chain rule and the equation governing the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2 + x_2^2 + x_3^2\right) = 2x_1x_1' + 2x_2x_2' + 2x_3x_3' = 2x_1(ax_2 + bx_3) + 2x_2(-ax_1 + cx_3) + 2x_3(-bx_1 - cx_2) = 0$$

(c) **(3 marks)** Show that $\left(e^{At}\right)^T = e^{-At}$. Hint: notice that $A^T = -A$.

Solution: Using the definition of the exponential (series expansion) we can check.

$$\left(e^{At}\right)^T = \left(\sum_{i=0}^{\infty} \frac{t^iA^i}{i!}\right)^T = \sum_{i=0}^{\infty} \frac{t^i\left(A^i\right)^T}{i!} = \sum_{i=0}^{\infty} \frac{t^i\left(A^T\right)^i}{i!} = \sum_{i=0}^{\infty} \frac{t^i(-A)^i}{i!} = e^{-At}$$

Note that it was necessary to verify $\left(e^{At}\right)^T = e^{A^Tt}$ using the series expansion. Not all operations can be pulled in like this. For example $\left(e^{At}\right)^2 = e^{A^2t}$ is false.

For the next part, recall: a matrix U is orthogonal if and only if $U^TU = UU^T = I$, where I is the identity matrix.

(d) **(2 marks)** Using part (c), show that the solution of the system is $\vec{x}(t) = Q\,\vec{x}(0)$ for some orthogonal matrix Q.

Solution: The solution of the differential equation is $\vec{x}(t) = e^{At}\vec{x}(0)$ since e^{At} is a special fundamental matrix of the system for $t = 0$ (that was the point of why we defined matrix exponentials). Using part (c) we get $\left(e^{At}\right)^T e^{At} = e^{-At}e^{At} = e^{0t} = I$. As such, $Q = e^{At}$ is an orthogonal matrix that fulfils the requirements.

13. Consider the differential equation $ty' = y^{\alpha}$ where $\alpha \in \mathbb{R}$ is a scalar parameter. We only consider $t \geq 0$.

(a) **(3 marks)** Consider $\alpha = 1$ and $y(1) = 0$. How many solutions does this IVP have? Justify.

Solution: This ODE is linear and can be written as $y' = \frac{1}{t}y$. Pick the interval $I = (0.5, 1.5)$. The initial time $t = 1$ is in this interval and the function $p(t) = \frac{1}{t}$ is continuous on this interval. Therefore, a solution exists and is unique. Of course other choices for I were possible (or a general statement that around $t = 1$, $p(t)$ is continuous). It wasn't necessary to find the solution, but the solution is the constant solution $y(t) = 0$.

(b) **(3 marks)** Consider $\alpha = 1$ and $y(0) = 0$. How many solutions does this IVP have? Justify.

Solution: None of the E-U-Theorems work here, so we need to try other things. The constant solution $y = 0$ works. But also $y = Ct$ is a solution for any constant C. There are many ways to find these solutions. One could even just guess around, shuffle things in what ever way, and finally just verify that $y = Ct$ solves the ODE and the initial value.

Since any C works, there are infinitely many solutions to this IVP. There might even be additional solutions not of the for $y = Ct$, but since we already have infinitely many, we don't need to look further.

(c) **(3 marks)** Consider $\alpha = 2$ and $y(1) = 0$. How many solutions does this IVP have? Justify.

Solution: The conditions of the general E-U-Theorem apply:

$$y' = f(t, y) = \frac{y^2}{t} \implies f_y = \frac{2y}{t}.$$

Both f and f_y are continuous around the point $(t, y) = (1, 0)$. So the general E-U theorem applies and there is only one solution. We don't know for how long it exists, but that is not necessary to make the statement that there is only one solution.

It wasn't necessary to find the solution, but the unique solution is the constant solution $y(t) = 0$.