UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

AER210 Midterm Test #1
Vector Calculus & Fluid Mechanics

Instructor: Alex Bercik October 19, 2023

First name (please write as legibly as possible within the boxes)

S	O	L	U	T	I	O	N	S								

Last name

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Student ID number

This exam contains 20 pages (including this cover page and a 2-page formula sheet at the back) and 8 questions. The total number of points is 65.

This exam will be scanned and graded using Crowdmark. Therefore, please write legibly, within the margins, and **make a note to the grader if you need to continue your work on a separate page** (otherwise, your work will not be seen by the grader). Good luck!

Distribution of Marks

Question:	1	2	3	4	5	6	7	8	Total
Points:	10	15	6	6	4	5	11	8	65
Score:									

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1. (10 points) True or False. Circle the following statements that are **true**. You may circle only the letter before the statement, rather than the entire statement. Remember, in order for a statement to be true, it must *always* be true.

(a) In a conservative force vector field, the work done by the force in moving an object from any point $\boldsymbol{a}$ to point $\boldsymbol{b}$ is the same regardless of the path taken.

TRUE: THM in lecture 13

(b) Let $f(x,y) : \Omega \to \mathbb{R}$ be some function defined on a rectangular domain $\Omega = [a,b] \times [c,d]$. Then

$$\int_\Omega f(x,y)d\Omega = \int_c^d \left(\int_a^b f(x,y)dx\right) dy = \int_a^b \left(\int_c^d f(x,y)dy\right) dx \,.$$

False: No guarantee Fubini's THM holds/applies! e.g. is F integrable?

(c) For two vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$, if $\nabla \times \boldsymbol{F} = \nabla \times \boldsymbol{G}$, then $\boldsymbol{F} = \boldsymbol{G}$.

False: Curl is not unique. e.g. $\vec{\nabla} \times (\vec{F} + \vec{\nabla}\varphi) = \vec{\nabla} \times \vec{F}$

(d) If f and $|f|$ are both Riemann integrable functions on Ω, then $\left|\int_\Omega f \; d\Omega\right| \leq \int_\Omega |f| \; d\Omega$.

TRUE: THM in lecture 3 (Monotonicity)

(e) If the closed line integral $\oint_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell}$ of a vector field $\boldsymbol{F}$ around a specific path γ is found to be zero, then $\boldsymbol{F}$ is conservative.

False: Must be zero for all paths

(f) Given a vector field $\boldsymbol{F} : \Omega \to \mathbb{R}^n$ defined on a domain $\Omega \subseteq \mathbb{R}^n$, if there exists a C^1 scalar field $\phi : \Omega \to \mathbb{R}$ such that $\boldsymbol{F} = \nabla\phi$, then $\boldsymbol{F}$ is conservative.

TRUE: Definition of conservative, lecture 13

(g) Let $f(x,y) : \Omega \to \mathbb{R}$ be a bounded, continuous, but not differentiable function defined on the unit circle $\Omega = \{x^2 + y^2 \leq 1\}$. Any Riemann sum of f with respect to partition P must converge to the same finite value in the limit $||P|| \to 0$. (defn of integral)

TRUE: THM in lecture 3: bounded & C^0 $\Rightarrow$ integrable $\Rightarrow$ all Riemann sums converge

(h) Let $f(x,y,z)$ be some integrable scalar field, and γ be some path in $\mathbb{R}^3$ with its reverse path denoted $-\gamma$. Then $\int_\gamma f(x,y,z)\,ds = -\int_{-\gamma} f(x,y,z)\,ds$.

False: $\int_\gamma f\,ds = \int_{-\gamma} f\,ds$. Standard result stated many times in lecture.

(i) The third-order Taylor polynomial of a function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ about a point (a,b) is unique.

TRUE: Taylor polynomials are unique! Stressed in lecture 9, on P.S., etc.

(j) The following transformation $T : (r,\theta) \to (x,y)$ is *not* a diffeomorphism.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad \text{where} \quad T : [0,\infty) \times [0,2\pi] \to \mathbb{R}^2$$

$\approx$ bijective differentiable invertible

TRUE: not bijective! should be $T:(0,\infty) \times [0,2\pi)$

importance of bounds stressed in lecture 11, on P.S.

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2. Determine the line integrals $\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell}$ for the given vector fields $\boldsymbol{F}$ and paths γ.

   (a) (5 points) $\boldsymbol{F} = zy\,\hat{\boldsymbol{x}} + x\,\hat{\boldsymbol{y}} + z^2x\,\hat{\boldsymbol{z}}$ along the straight line between $(0,0,0)$ and $(2,3,2)$.

   (b) (5 points) $\boldsymbol{F} = (2xy + \cos(x))\,\hat{\boldsymbol{x}} + x^2\hat{\boldsymbol{y}}$ along the line $\{(x,y) : x^2 + y^2 = 4\}$ in the counterclockwise direction, starting point $\boldsymbol{a} = (2,0)$ and finishing at point $\boldsymbol{b} = (0,2)$.

   (c) (5 points) $\boldsymbol{F} = (2y + \cos(x))\,\hat{\boldsymbol{x}} + \left(x + e^{\sqrt{y}}\right)\hat{\boldsymbol{y}}$ around the region enclosed by $y = x^2$ and $x = y^2$ in the counterclockwise direction.

$\frac{\partial F_x}{\partial y} = z \neq \frac{\partial F_y}{\partial x} = 1$

Note: $\vec{F}$ not conservative $\to$ No G… TH…

& γ not closed $\to$ No Stok… THM

a) let $\vec{r}(t) = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} t$, $t \in [0,1]$

$\Rightarrow \vec{r}\,'(t) = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $\vec{F}(\vec{r}(t)) = \begin{pmatrix} zy \\ x \\ z^2x \end{pmatrix} = \begin{pmatrix} 6t^2 \\ 2t \\ 8t^3 \end{pmatrix}$

$$\Rightarrow \int_\gamma \vec{F} \cdot d\vec{\ell} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \begin{pmatrix} 6t^2 \\ 2t \\ 8t^3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} dt = \int_0^1 12t^2 + 6t + 16t^3\, dt$$

$$= 4t^3 + 3t^2 + 4t^4 \Big|_0^1 = 4 + 3 + 4 = \boxed{11}$$

b) $\frac{\partial F_x}{\partial y} = 2x$, $\frac{\partial F_y}{\partial x} = 2x$, domain Ω of $\vec{F}$ is $\mathbb{R}^2$

$\to$ simply connected

$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$ & Ω simply connected $\overset{\text{THM}}{\Rightarrow}$ $\vec{F}$ conservative

Find the potential φ s.t $\vec{F} = \nabla \varphi$

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b) (cont'd) $\frac{\partial \varphi}{\partial x} = F_x$, $\frac{\partial \varphi}{\partial y} = F_y$

$$\varphi = \int F_y \, dy = \int x^2 \, dy = x^2 y + g(x)$$

$$\frac{\partial \varphi}{\partial x} = 2xy + g'(x) = F_x = 2xy + \cos(x) \Rightarrow g'(x) = \cos(x)$$

$$\Rightarrow g(x) = \sin(x)$$

$$\therefore \varphi(x,y) = x^2 y + \sin(x)$$

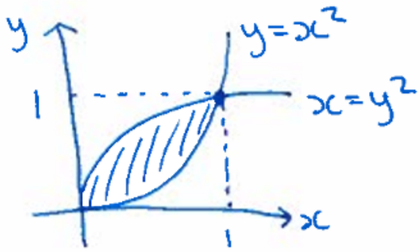
Since $\vec{F}$ is conservative, can use gradient THM

$$\int_\gamma \vec{F} \cdot d\vec{\ell} = \varphi(\vec{b}) - \varphi(\vec{a}) = \varphi\begin{pmatrix}0\\2\end{pmatrix} - \varphi\begin{pmatrix}2\\0\end{pmatrix} = (0 + \sin(0)) - (0 + \sin(2))$$

$$\boxed{= -\sin(2)}$$

c) $\frac{\partial F_x}{\partial y} = 2$ $\frac{\partial F_y}{\partial x} = 1$ $\Rightarrow$ Not conservative, but closed!

i.e. Can use Green's THM



✓ regular region (closed, bounded, no isolated pts)

✓ piecewise-smooth boundary

✓ $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 (at least on $y \geq 0$)

✓ postive orientation (CCW)

$$\oint \vec{F} \cdot d\vec{\ell}$$

$$= \int_\Omega \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) d\Omega = \int_0^1 \int_{x^2}^{\sqrt{x}} -1 \, dy \, dx = + \int_0^1 x^2 - \sqrt{x} \, dx = \left[\frac{x^3}{3} - \frac{2}{3} x^{3/2} \right]_0^1$$

$$= +\frac{1}{3} - \frac{2}{3} = \boxed{\frac{-1}{3}}$$

3. (6 points) Consider the triple integral $\int_0^4 \int_0^{4-z} \int_0^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$.

(a) Rewrite the integral in the form $\int_?^? \int_?^? \int_?^? f(x, y, z)\, dz\, dy\, dx$ with the proper bounds.

(b) Rewrite the integral in the form $\int_?^? \int_?^? \int_?^? f(x, y, z)\, dy\, dx\, dz$ with the proper bounds.

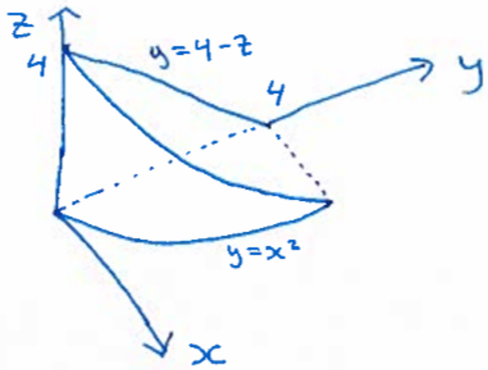
$\int_0^4 \int_0^{4-z} \int_0^{\sqrt{y}} \Rightarrow$

$0 \le z \le 4$

$0 \le y \le 4 - z$

$0 \le x \le \sqrt{y}$

$\Rightarrow 0 \le x^2 \le y$

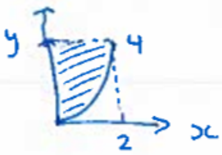


a) $0 \le z \le 4 - y$

$x^2 \le y \le 4$

$0 \le x \le 2$

project to x-y plane



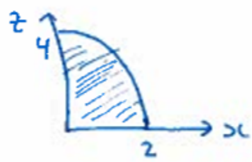
$$\int_0^2 \int_{x^2}^4 \int_0^{4-y} f(x,y,z)\, dz\, dy\, dx$$

b) $x^2 \le y \le 4 - z$

$0 \le x \le \sqrt{4-z}$

$0 \le z \le 4$

project to x-z plane



$$\int_0^4 \int_0^{\sqrt{4-z}} \int_{x^2}^{4-z} f(x,y,z)\, dy\, dx\, dz$$

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4. (a) (3 points) Consider a vector field $\boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n$ with magnitude $f(\boldsymbol{x}) = |\boldsymbol{F}(\boldsymbol{x})|$, and a path γ with parametrization $\boldsymbol{r}(t) : I \subset \mathbb{R} \to \mathbb{R}^n$. Show that if $\boldsymbol{F}$ is everywhere aligned with γ, i.e. it points in the same direction as tangent vectors of $\boldsymbol{r}(t)$ everywhere along the path, then the line integral of $\boldsymbol{F}$ reduces to the line integral of $f(\boldsymbol{x})$. That is,

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell} = \int_\gamma f(\boldsymbol{x})\, ds$$

(b) (3 points) Consider $\mathbb{R}^2$. Show that if $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ points purely in the azimuthal direction, i.e. $\boldsymbol{F} = f(x, y)\,\hat{\boldsymbol{\theta}}$, and we take the line integral along a circle of radius R in the counterclockwise direction from $\theta = \theta_i$ to $\theta = \theta_f$ then the line integral of $\boldsymbol{F}$ reduces to

$$\int_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell} = R \int_{\theta_i}^{\theta_f} f(x(r,\theta), y(r,\theta))\, d\theta$$

a) (done in class, lecture 12, and Review Demo session)

$\vec{F}$ aligned with $\gamma \Rightarrow \vec{F}(\vec{r}(t)) = F(\vec{r}(t))\, \hat{T}(\vec{r}(t))$ ← tangent vector to γ

$$= F(\vec{r}(t)) \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

$$\Rightarrow \int_\gamma \vec{F} \cdot d\vec{\ell} = \int_{t_i}^{t_F} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_{t_i}^{t_F} F(\vec{r}(t)) \frac{\vec{r}\,'(t) \cdot \vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_{t_i}^{t_F} F(\vec{r}(t)) \frac{\|\vec{r}\,'(t)\|^2}{\|\vec{r}\,'(t)\|}\, dt$$

$$= \int_{t_i}^{t_F} F(\vec{r}(t)) \underbrace{\|\vec{r}\,'(t)\|\, dt}_{ds} = \int_\gamma F(\vec{x})\, ds \quad \square$$

b) on next page

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b) Posted as note on Piazza (@26)

take polar coordinates $\Rightarrow \vec{F} = F\hat{\theta}$, $d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$\Rightarrow \vec{F}\cdot d\vec{\ell} = F\,r\,d\theta$ since $\hat{\theta}\cdot\hat{r} = 0$

$$\therefore \int_\gamma \vec{F}\cdot d\vec{\ell} = \int_\gamma F(\vec{x})\,r(\vec{x})\,d\theta = \int_{\theta_i}^{\theta_f} F(\vec{x})\,R\,d\theta$$ since $r = R$ along γ

$$= R\int_{\theta_i}^{\theta_f} F(x(R,\theta), y(R,\theta))\,d\theta \qquad \square$$

OR

Compute directly using $\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$

$$\Rightarrow \vec{F} = \begin{bmatrix} -F\sin\theta \\ F\cos\theta \end{bmatrix}, \quad \vec{r}(\theta) = \begin{bmatrix} R\cos\theta \\ R\sin\theta \end{bmatrix}, \quad \vec{r}\,'(\theta) = \begin{bmatrix} -R\sin\theta \\ R\cos\theta \end{bmatrix}$$

$$\therefore \int_\gamma \vec{F}\cdot d\vec{\ell} = \int_{\theta_i}^{\theta_f} \vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta)\,d\theta = \int_{\theta_i}^{\theta_f} \begin{bmatrix} -F\sin\theta \\ F\cos\theta \end{bmatrix}\cdot\begin{bmatrix} -R\sin\theta \\ R\cos\theta \end{bmatrix} d\theta$$

$$= \int_{\theta_i}^{\theta_f} RF(\sin^2\theta + \cos^2\theta)\,d\theta, \quad \sin^2\theta + \cos^2\theta = 1$$

$$= R\int_{\theta_i}^{\theta_f} F(x(R,\theta), y(R,\theta))\,d\theta \qquad \square$$

5. (4 points) Given the vector field $\boldsymbol{F} = y\,\hat{\boldsymbol{x}} - x\,\hat{\boldsymbol{y}}$, show that the line integral around a piecewise smooth, simple, closed, and bounded curve γ in the x-y plane in the clockwise direction satisfies

$$\oint_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell} = 2A$$

where A is the area enclosed by the curve γ.

- $\vec{F}$ is C^1 on $\mathbb{R}^2$
- Simple, closed, bounded $\gamma \Rightarrow$ regular region Ω
- piecewise-smooth boundary $\gamma = \partial\Omega$

$\Rightarrow$ Green's THM applies

$\frac{\partial F_x}{\partial y} = 1$, $\frac{\partial F_y}{\partial x} = -1$, Note: CW direction is <u>negative</u> orientation!

$$\therefore \oint_\gamma \vec{F} \cdot d\ell = -\int_\Omega \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) d\Omega$$

$$= -\int_\Omega -2 \; d\Omega$$

$$= 2\int_\Omega d\Omega$$

$$= 2A \quad \text{since } A = \int_\Omega d\Omega \qquad \square$$

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6. (5 points) Let $\boldsymbol{F} : \Omega \to \mathbb{R}^2$ and $\boldsymbol{G} : \Omega \to \mathbb{R}^2$ be vector fields over a domain $\Omega \subset \mathbb{R}^2$. Prove or disprove (e.g. find a counterexample to) the following statement.

If $\boldsymbol{F} + \boldsymbol{G}$ is conservative, then $\boldsymbol{F}$ and $\boldsymbol{G}$ are also conservative.

(similar to HW problems Lec 13)

The statement is <u>False</u>.

Proof by counter-example:

let $\vec{F} = y\hat{x} - x\hat{y}$

$\vec{G} = -y\hat{x} + x\hat{y} + \nabla\varphi$ for some C^1 φ on Ω

by previous Q5, $\vec{F}$ and $\vec{G}$ are not conservative

$$\left[\frac{\partial F_x}{\partial y} = 1 \neq -1 = \frac{\partial F_y}{\partial x}, \quad \frac{\partial G_x}{\partial y} = -1 + \frac{\partial^2\varphi}{\partial x \partial y} \neq 1 + \frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial G_y}{\partial x}\right]$$

but $\vec{F} + \vec{G} = (y-y)\hat{x} + (-x+x)\hat{y} + \nabla\varphi$

$= \nabla\varphi$

i.e. $\vec{F}+\vec{G}$ is conservative by definition $\square$

i.e. we showed,

($\vec{F}+\vec{G}$ is conservative, but neither $\vec{F}$ nor $\vec{G}$ are conservative)

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7. Consider the volume Ω in $\mathbb{R}^3$ bounded by $y = x^2 + z^2$ and the plane $y = 4$.

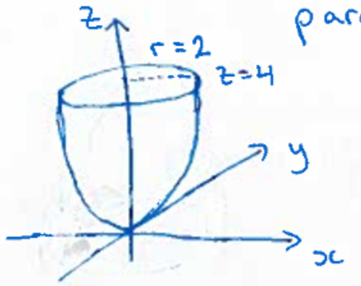
(a) (5 points) Evaluate the integral $\int_\Omega \sqrt{x^2 + z^2}\, d\Omega$.

(b) (6 points) Compute the total surface area $\Gamma = \partial\Omega$ of the volume Ω

a) Without loss of generality, swap y & z

i.e. $\underbrace{z = x^2 + y^2, \; z = 4}_{\text{paraboloid}}$, $\int_\Omega \sqrt{x^2+y^2}\, d\Omega$

$= \int_\Omega r\, d\Omega$ cylindrical coordinates!



$$\therefore \int_\Omega \sqrt{x^2+y^2}\, d\Omega = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} r \cdot r\, dr\, d\theta\, dz$$

$$= 2\pi \int_0^4 \frac{1}{3} r^3 \Big|_0^{\sqrt{z}} dz$$

$$= \frac{2\pi}{3} \int_0^4 z^{3/2}\, dz = \frac{2\pi}{3} \cdot \frac{2}{5} z^{5/2} \Big|_0^4$$

2, 4, 8, 16, 32, 64, 128

$$= \frac{4\pi}{15} 2^5 = \boxed{\frac{128\pi}{15}}$$

b) on next page

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b) Surface Area $= \Gamma_{\text{paraboloid}} + \Gamma_{\text{top cap}}$

$$\Gamma_{\text{paraboloid}} = \int_\Omega \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; d\Omega \quad , \quad f(x,y) = z = x^2+y^2$$

$$= \int_\Omega \sqrt{(2x)^2+(2y)^2+1}\; d\Omega = \int_\Omega \sqrt{4(x^2+y^2)+1}\; d\Omega$$

$$= \int_\Omega \sqrt{4r^2+1}\; d\Omega$$ use polar coordinates! $r^2 = x^2+y^2$

$(x,y) \to (r,\theta)$

$$= \int_0^{2\pi} \int_0^{2} \sqrt{4r^2+1}\; r\,dr\,d\theta$$

$$= 2\pi \int_1^{17} \sqrt{u} \cdot \frac{1}{8}\, du$$

let $u = 4r^2+1$ $\quad r=0 \Rightarrow u=1$

$du = 8r\,dr$ $\quad r=2 \Rightarrow u = 17$

$$= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2} \Big|_1^{17}$$

$$\boxed{= \frac{\pi}{6}\left(17^{3/2}-1\right)} \approx 5.76$$

$$\Gamma_{\text{top cap}} = \pi r^2 = \pi (2)^2 = \boxed{4\pi}$$

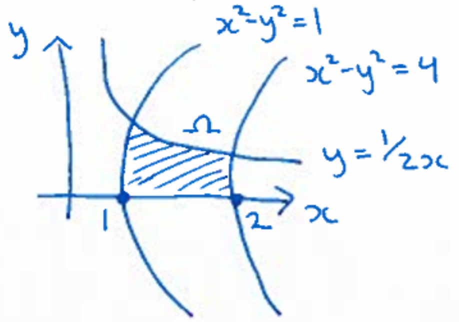
$\therefore$ Surface Area $\boxed{\Gamma = 4\pi + \frac{\pi}{6}\left(17^{3/2}-1\right)}$

8. (8 points) Consider a 2D plate defined in the x-y plane with mass density

$$\rho(x,y) = \left(x^2+y^2\right) e^{|xy|}\ \frac{\text{kg}}{\text{m}^2}\ ,$$

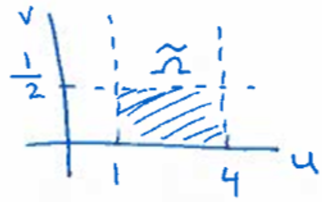
where x and y have units of m. Suppose we are only interested in the region of this plate Ω found in the 1st quadrant, bounded between the curves $x^2-y^2=1$, $y^2=x^2-4$, $y=0$, and $y=\dfrac{1}{2x}$. Compute the total mass of the plate contained in the region Ω.

$yx=\frac{1}{2}$ $\quad$ $yx=0$



let $u=x^2-y^2$

$v=yx$ $\longrightarrow$



$\Rightarrow \rho(x,y) = (x^2+y^2)\,e^{|v|}$

$= (x^2+y^2)\,e^{v}$ since $v \geq 0$

$\vec{T}:(u,v)\rightarrow(x,y)$ and $\vec{T}^{-1}:(x,y)\rightarrow(u,v)$

$D_x\vec{T}^{-1} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}$ $\rightarrow$ $\vec{T}^{-1}$ is differentiable (C^1) & continuous

$\det(D_u\vec{T}) = \dfrac{1}{\det(D_x\vec{T}^{-1})} = \dfrac{1}{\begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix}} = \dfrac{1}{2x^2+2y^2} = \dfrac{1}{2(x^2+y^2)} > 0$ since $x>0,\ y\geq 0$

$\therefore$ $\vec{T}^{-1}$ is C^1, determinants >0, $\vec{T}^{-1}$ bijective (no proof required) $\Rightarrow$ invertible

$\Rightarrow$ $\vec{T}$ & $\vec{T}^{-1}$ are diffeomorphisms $\Rightarrow$ use change of variables

$$\int_\Omega \rho(x,y)\,d\Omega = \int_0^{1/2}\int_1^4 \cancel{(x^2+y^2)}\,e^v \cdot \frac{1}{2\cancel{(x^2+y^2)}}\,du\,dv$$

$$= \int_1^4 du \cdot \frac{1}{2}\int_0^{1/2} e^v = \frac{3}{2}\,e^v\Big|_0^{1/2} = \boxed{\frac{3}{2}\left(\sqrt{e}-1\right)}$$

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Useful Formulas

Vector Identities

$$
\begin{aligned}
\nabla (fg) &= f(\nabla g) + g(\nabla f) \\
\nabla (\boldsymbol{f} \cdot \boldsymbol{g}) &= \boldsymbol{f} \times (\nabla \times \boldsymbol{g}) + \boldsymbol{g} \times (\nabla \times \boldsymbol{f}) + (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} + (\boldsymbol{g} \cdot \nabla)\boldsymbol{f} \\
\nabla (f\boldsymbol{g}) &= f(\nabla \cdot \boldsymbol{g}) + \boldsymbol{g} \cdot (\nabla f) \\
\nabla \times (f\boldsymbol{g}) &= f(\nabla \times \boldsymbol{g}) - \boldsymbol{g} \times (\nabla f) \\
\nabla \times (\boldsymbol{f} \times \boldsymbol{g}) &= (\boldsymbol{g} \cdot \nabla)\boldsymbol{f} - (\boldsymbol{f} \cdot \nabla)\boldsymbol{g} + \boldsymbol{f}(\nabla \cdot \boldsymbol{g}) - \boldsymbol{g}(\nabla \cdot \boldsymbol{f}) \\
\nabla \cdot (\nabla \times \boldsymbol{f}) &= 0 \\
\nabla \times (\nabla f) &= 0 \,, \quad \text{where } f \text{ is the \textit{scalar potential} of the \textit{conservative} vector field } \nabla f
\end{aligned}
$$

(Note: the following are *not* rigorous mathematical statements!)

Change of Variables

Let $\boldsymbol{T} : \widetilde{\Omega} \to \Omega$ be a diffeomorphism (invertible & differentiable bijection). Then

$$\int_\Omega f(\boldsymbol{x})\, d\Omega = \int_{\widetilde{\Omega} = \boldsymbol{T}^{-1}(\Omega)} f(\boldsymbol{T}(\boldsymbol{u}))\, |\det(D_{\boldsymbol{u}}\boldsymbol{T})|\, d\widetilde{\Omega}$$

where $D_{\boldsymbol{u}}\boldsymbol{T}$ is the Jacobian matrix of $\boldsymbol{T}$ with respect to variables $\boldsymbol{u} = \boldsymbol{T}^{-1}(\boldsymbol{x})$.

Gradient Theorem (Fundamental Theorem of Calculus for line integrals)

Let γ be a continuous curve which starts at $\boldsymbol{a}$ and ends at point $\boldsymbol{b}$. Then

$$\int_\gamma (\nabla f) \cdot d\boldsymbol{\ell} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

Green's Theorem

Let $\Omega \subset \mathbb{R}^2$ be a regular region with boundary $\partial\Omega = \gamma$. Then

$$\oint_\gamma \boldsymbol{F} \cdot d\boldsymbol{\ell} = \int_\Omega \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) d\Omega$$

Cartesian Coordinates (x, y, z)

$$
\begin{aligned}
\text{Line element:} \quad & d\boldsymbol{\ell} = dx\, \hat{\boldsymbol{x}} + dy\, \hat{\boldsymbol{y}} + dz\, \hat{\boldsymbol{z}} \\
\text{Volume element:} \quad & d\Omega = dx\, dy\, dz \\
\text{Gradient:} \quad & \nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}} \\
\text{Divergence:} \quad & \nabla \cdot \boldsymbol{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} \\
\text{Curl:} \quad & \nabla \times \boldsymbol{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \right)\hat{\boldsymbol{x}} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \right)\hat{\boldsymbol{y}} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right)\hat{\boldsymbol{z}}
\end{aligned}
$$

Spherical Coordinates (r, θ, ϕ)

$$x = r\cos\theta\sin\phi \qquad \hat{\boldsymbol{x}} = \cos\theta\sin\phi\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\phi}}$$

$$y = r\sin\theta\sin\phi \qquad \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\cos\phi\,\hat{\boldsymbol{\phi}}$$

$$z = r\cos\phi \qquad \hat{\boldsymbol{z}} = \cos\phi\,\hat{\boldsymbol{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \hat{\boldsymbol{r}} = \cos\theta\sin\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\phi\,\hat{\boldsymbol{z}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}$$

$$\phi = \cos^{-1}\left(\frac{z}{r}\right) \qquad \hat{\boldsymbol{\phi}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\cos\phi\,\hat{\boldsymbol{y}} - \sin\phi\,\hat{\boldsymbol{z}}$$

Line element: $d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\sin\phi\,d\theta\,\hat{\boldsymbol{\theta}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$

Volume element: $d\Omega = r^2\sin\phi\,dr\,d\theta\,d\phi$

Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla\cdot\boldsymbol{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi f_\phi)$

Curl:
$$\nabla\times\boldsymbol{f} = \frac{1}{r\sin\phi}\left(\frac{\partial}{\partial\phi}(\sin\phi f_\theta) - \frac{\partial f_\phi}{\partial\theta}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r f_\phi) - \frac{\partial f_r}{\partial\phi}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{1}{\sin\phi}\frac{\partial f_r}{\partial\theta} - \frac{\partial}{\partial r}(r f_\theta)\right)\hat{\boldsymbol{\phi}}$$

Cylindrical Coordinates (r, θ, z)

$$x = r\cos\theta \qquad \hat{\boldsymbol{x}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$$

$$y = r\sin\theta \qquad \hat{\boldsymbol{y}} = \sin\theta\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$$

$$z = z \qquad \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

$$r = \sqrt{x^2+y^2} \qquad \hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}$$

Line element: $d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\boldsymbol{z}}$

Volume element: $d\Omega = r\,dr\,d\theta\,dz$

Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

Divergence: $\nabla\cdot\boldsymbol{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$

Curl: $\nabla\times\boldsymbol{f} = \left(\frac{1}{r}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r f_\theta) - \frac{\partial f_r}{\partial\theta}\right)\hat{\boldsymbol{z}}$