Last name: First name: ID number:

ECE 286
Final exam

April 30, 2022
2:00 pm

Guidelines:

- Write your answer in the space provided for each question. Show your work and use the back side of sheets as needed.
- The exam is Type D. You may use a non-programmable calculator and a two-sided, 8.5 by 11 handwritten note sheet (prepared by you).
- **You will find tables for the normal and t distributions in the back of the exam. You may remove them.**

Problem	Score
1	/13
2	/10
3	/8
4	/6
5	/8
Total	/45

1. Suppose X is a random variable for the roll of a biased die. The probability of a particular roll is given by $P(X = x) = x/21$ for $x = 1, ..., 6$. Answer parts (a)-(c) below. Justify your answers.

   (a) (2 points) What is the probability that $X \leq 3$? *Solution:* $P(X \leq 3) = 6/21$.

(b) You roll the die four times. Answer parts i-iii below.

i. (2 points) What is the probability of rolling 6, 5, 4, 3, in this order? *Solution: It is* $6 \cdot 5 \cdot 4 \cdot 3/21^4 = 0.00185$.

ii. (2 points) What is the probability of rolling 6, 5, 4, 3, in any order? *Solution: There are 4! orderings, each with probability* 0.00185. *Therefore the probability of rolling 6, 5, 4, 3 in any order is* $4! \cdot 0.00185 = 0.044$.

iii. (2 points) What is the probability of rolling two 6's and two 3's, in any order? *Solution: The probability of two 6's and two 3's in a given order is* $6 \cdot 6 \cdot 3 \cdot 3/21^4 = 0.00167$. *This is a permutation with identical items. The number of orderings is*

$$\binom{4}{2,2} = \frac{4!}{2!2!} = 6.$$

The probability is therefore $6 \cdot 0.00167 = 0.01$.

(c) Now suppose Y is a random variable for the roll of a fair die, i.e., $P(Y = y) = 1/6$ for $y = 1, ..., 6$. Let $Z = X + Y$. Note that X and Y are independent. Answer parts i-ii below.

i. (2 points) Find $P(Z = 8)$. *Solution: The PMF of Z is the convolution of the PMFs of X and Y. For $Z = 8$, we have*

$$\begin{aligned}
P(Z = 4) &= \sum_k P(X = k)P(Y = 4 - k) \\
&= P(X = 1)P(Y = 3) + P(X = 2)P(Y = 2) + P(X = 3)P(Y = 1) \\
&= \frac{1}{21} \cdot \frac{1}{6} + \frac{2}{21} \cdot \frac{1}{6} + \frac{3}{21} \cdot \frac{1}{6} \\
&= 0.0476.
\end{aligned}$$

ii. (3 points) Find the covariance of X and Z, σ_{XZ}. *Solution: Observe that*

$$\begin{aligned}\sigma_{XZ} &= E[XZ] - \mu_X \mu_Z \\ &= E[X(X+Y)] - \mu_X(\mu_X + \mu_Y) \\ &= E[X^2] + E[XY] - \mu_X^2 - \mu_X \mu_Y \\ &= E[X^2] + E[X]E[Y] - \mu_X^2 - \mu_X \mu_Y \\ &= E[X^2] - \mu_X^2.\end{aligned}$$

This is because X and Y are independent and therefore uncorrelated. We have that $\mu_X = 3.5$, $E[X^2] = 91/6 = 15.167$, and therefore

$$\sigma_{XZ} = 91/6 - 3.5^2 = 2.917.$$

2. X is a continuous random variable with PDF

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 3 - x & \text{if } 2 \leq x \leq 3 \\ 0 & \text{otherwise.} \end{cases}$$

Answer parts (a)-(e) below. Justify your answers.

(a) (1 point) What is the probability that X is 1, i.e., $P(X = 1)$? *Solution:* $P(X = 1) = 0$.

(b) (3 points) Find the cumulative distribution function of X, $F(x)$. *Solution: We know* $F(x) = \int_0^x f(t)dt$. *We have*

$$F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ x^2/2 & \textit{if } 0 \leq x \leq 1 \\ 1/2 & \textit{if } 1 < x < 2 \\ 3x - x^2/2 - 7/2 & \textit{if } 2 \leq x \leq 3 \\ 1 & \textit{if } 3 < x. \end{cases}$$

(c) (1 points) Find $P(X < 0.5)$. *Solution: From the CDF, it is* $0.5^2/2 = 0.125$.

(d) (2 points) Find all values of x for which $P(X = x) = 1/2$. *Solution:* $P(X = x) = 0$. *Solution for intended question with* $\leq$*: From the CDF, it is* $x \in [1, 2]$.

(e) (3 points) Let

$$Y = \begin{cases} X & \text{if } X \leq 1.5 \\ 2X & \text{if } X > 1.5. \end{cases}$$

Find the PDF of Y. *Solution: If $Y = u(X)$, then*

$$u^{-1}(Y) = \begin{cases} Y & \textit{if } Y \leq 1.5 \\ Y/2 & \textit{if } Y > 3. \end{cases}$$

It is not defined for other values of Y. The PDF of Y is

$$g(y) = f(u^{-1}(y)) \left| \frac{du^{-1}(y)}{dy} \right|.$$

We have that

$$\left| \frac{du^{-1}(y)}{dy} \right| = \begin{cases} 1 & \textit{if } Y \leq 1.5 \\ 1/2 & \textit{if } Y > 3, \end{cases}$$

and is undefined otherwise. We also have

$$f(u^{-1}(y)) = \begin{cases} Y & \textit{if } 0 \leq Y \leq 1 \\ 3 - Y/2 & \textit{if } 4 \leq Y \leq 6 \\ 0 & \textit{otherwise.} \end{cases}$$

Multiplying, we have

$$g(y) = \begin{cases} Y & \textit{if } 0 \leq Y \leq 1 \\ 3/2 - Y/4 & \textit{if } 4 \leq Y \leq 6 \\ 0 & \textit{otherwise.} \end{cases}$$

3. The random variable X is equal to 1 if a radar detects something. The radar has trouble distinguishing between different types of objects in the airspace. Testing has revealed that the radar will detect something ($X = 1$) with the following probabilities.

- If there is a plane, the radar will detect it with probability 0.9.
- If there is a bird, the radar will detect it with probability 0.2.
- If there is nothing, the radar will (incorrectly) detect something with probability 0.1.

Observation has also shown that

- There are planes in the airspace 20% of the time.
- There are birds in the airspace 30% of the time.
- There is nothing in the airspace 50% of the time.

For simplicity, assume that these events are mutually exclusive. Answer parts (a)-(c) below. Justify your answers.

(a) (2 points) What is the probability the radar detects something? *Solution: Using the law of total probability, it is*

$$\begin{aligned} P(\mathit{detect}) &= P(\mathit{detect} \mid \mathit{plane})P(\mathit{plane}) + P(\mathit{detect} \mid \mathit{bird})P(\mathit{bird}) \\ &\quad + P(\mathit{detect} \mid \mathit{nothing})P(\mathit{nothing}) \\ &= 0.9 \cdot 0.2 + 0.2 \cdot 0.3 + 0.1 \cdot 0.5 \\ &= 0.29. \end{aligned}$$

(b) (2 points) If the radar detects something, what is the probability a plane is in the airspace? *Solution: Bayes' rule gives*

$$\begin{aligned} P(plane \mid detect) &= \frac{P(detect \mid plane)P(plane)}{P(detect)} \\ &= 0.9 \cdot 0.2/0.29 \\ &= 0.62. \end{aligned}$$

(c) (4 points) For this question, note the following definition:

Two events, A and B, are conditionally independent for event C if $P(A, B \mid C) = P(A \mid C)P(B \mid C)$.

To improve reliability, a second radar is also used. It detects with the same probabilities as the first. It is conditionally independent of the first; whatever is or isn't in the airspace, the output of the second radar is independent of the output of the first. If both the first and second radar detect something, what is the probability a plane is in the airspace? *Solution: Bayes' rule gives*

$$\begin{aligned} P(\textit{plane} \mid \textit{both detect}) &= \frac{P(\textit{both detect} \mid \textit{plane})P(\textit{plane})}{P(\textit{both detect})} \\ &= \frac{P(1 \textit{ detects} \mid \textit{plane})P(2 \textit{ detects} \mid \textit{plane})P(\textit{plane})}{P(\textit{both detect})}. \end{aligned}$$

The probability both detect is given by the law of total probability as

$$\begin{aligned} P(\textit{both detect}) &= P(\textit{both detect} \mid \textit{plane})P(\textit{plane}) + P(\textit{both detect} \mid \textit{bird})P(\textit{bird}) \\ &\quad + P(\textit{both detect} \mid \textit{nothing})P(\textit{nothing}) \\ &= 0.9^2 \cdot 0.2 + 0.2^2 \cdot 0.3 + 0.1^2 \cdot 0.5 \\ &= 0.179. \end{aligned}$$

Plugging into the above, we have

$$\begin{aligned} P(\textit{plane} \mid \textit{both detect}) &= \frac{P(1 \textit{ detects} \mid \textit{plane})P(2 \textit{ detects} \mid \textit{plane})P(\textit{plane})}{P(\textit{both detect})} \\ &= \frac{0.9^2 \cdot 0.2}{0.179} \\ &= 0.91. \end{aligned}$$

4. The below sample is from a normally distributed population:

$$x_1 = 1 \; x_2 = 2 \; x_3 = 3.$$

Answer parts (a)-(c) below.

(a) (2 points) Compute the sample variance. *Solution: the sample mean is*

$$\bar{x} = (1 + 2 + 3)/3 = 2.$$

The sample variance is

$$s^2 = \left((1-2)^2 + (2-2)^2 + (3-2)^2\right)/(3-1) = 1.$$

(b) (2 points) Find the 95%, one-sided confidence interval on the mean of the form $\underline{\mu} \leq \mu$. *Solution: We should use the* t*-distribution because the population is normal and the sample size is small. Because* $n = 3$*, the number of degrees of freedom is* $v = n - 1$*. We have* $t_{0.05} = 2.920$*. Because the distribution is symmetric, we have*

$$\begin{aligned}\underline{\mu} &= \bar{x} - t_{0.05}\frac{s}{\sqrt{n}} \\ &= 2 - 2.920\frac{1}{\sqrt{3}} \\ &= 0.314.\end{aligned}$$

(c) (2 points) Find the 95%, two-sided confidence interval on the mean. *Solution: We should again use the t-distribution because the population is normal and the sample size is small. Because $n = 3$, the number of degrees of freedom is $v = n - 1$. We have $t^2_{0.025} = 4.303$. Because the distribution is symmetric, we have*

$$\begin{aligned}\underline{\mu} &= \bar{x} - t^2_{0.025}\frac{s}{\sqrt{n}} \\ &= 2 - 4.303\frac{1}{\sqrt{3}} \\ &= -0.484.\end{aligned}$$

A similar calculation gives $\overline{\mu} = 4.484$ for the upper limit.

5. X and Y are random variables with joint PDF

$$f(x,y) = \begin{cases} 1/\pi, & x^2+y^2 \le 1 \\ 0 & \text{otherwise,} \end{cases} .$$

Answer parts (a)-(c) below. Justify your answers.

(a) (3 points) Find the marginal distributions of X and Y, $g(x)$ and $h(y)$. *Solution: For $|x| \le 1$, marginal of X:*

$$\begin{aligned} g(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1/\pi dy \\ &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \\ &= \frac{2}{\pi}\sqrt{1-x^2}. \end{aligned}$$

By symmetry, $h(y) = g(y)$.

(b) (2 points) Are X and Y independent? *Solution: No. We can see that*

$$g(x)h(y) = \frac{4}{\pi^2}\sqrt{1-x^2}\sqrt{1-y^2} \neq f(x,y).$$

(c) (3 points) Are X and Y correlated? *Solution: The covariance is* $\sigma_{XY} = E[XY] - E[X]E[Y] = 0 - 0$. *We have that*

$$\begin{aligned} E[XY] &= \int_{-1}^{1}\int_{-1}^{1} xyf(x,y)dxdy \\ &= \int_{0}^{1}\int_{0}^{1} xyf(x,y)dxdy + \int_{0}^{1}\int_{-1}^{0} xyf(x,y)dxdy \\ &\quad + \int_{-1}^{0}\int_{0}^{1} xyf(x,y)dxdy + \int_{-1}^{0}\int_{-1}^{0} xyf(x,y)dxdy. \end{aligned}$$

Terms 1 and 4 are positive, 2 and 3 negative, cancel. Therefore $E[XY] = 0$. *Also,*

$$\begin{aligned} E[X] &= \int_{-1}^{1} xg(x)dx \\ &= \int_{-1}^{0} xg(x)dx + \int_{0}^{1} xg(x)dx \\ &= 0, \end{aligned}$$

again by cancellation. Therefore $\sigma_{XY} = E[XY] - E[X]E[Y] = 0 - 0$. X *and* Y *are therefore uncorrelated.*