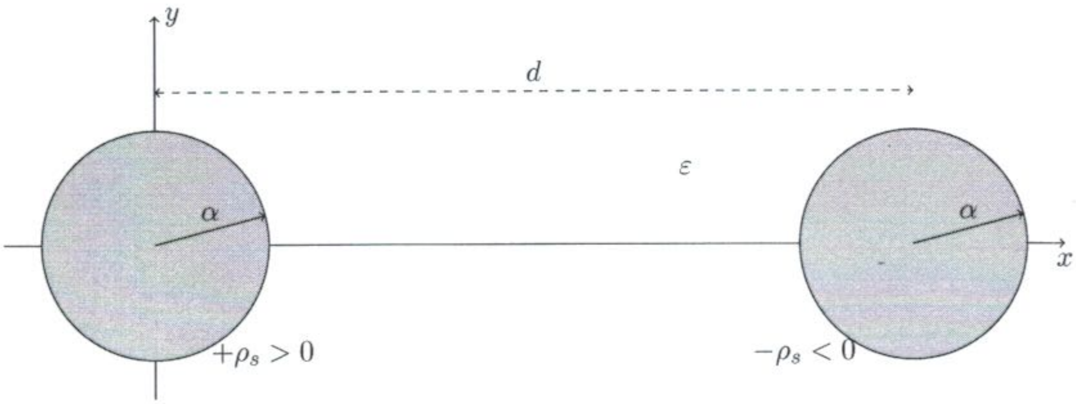
Question 1

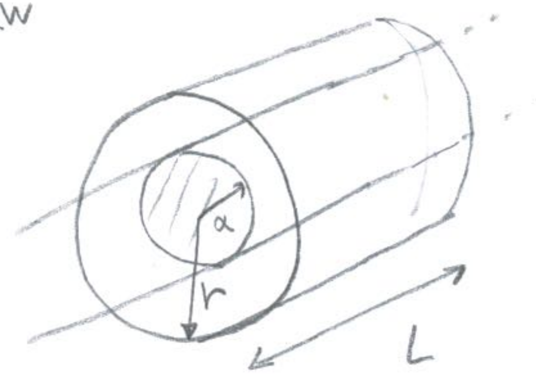


Consider two cylindrical perfect conductors of radius α in free space. The conductors are infinitely long and parallel to the z axis. The conductors are charged with uniform surface charge densities $+\rho_s$ and $-\rho_s$, respectively. The distance between the axes of the conductors is d, as shown in the figure. Conductors are immersed in a dielectric with permittivity ε.

1. Find the electric field $\mathbf{E}_+$ produced by the left conductor alone (the positively charged one). **(5 points)**

We can use generalized Gauss' law

[1pt] Gaussian surface: cylindrical length ℓ, radius $r > \alpha$



$$\int_S \overline{D}_+ \cdot \overline{dS} = Q$$

[1pt] $\overline{D}_+ = D_r(r)\,\overline{a}_r$

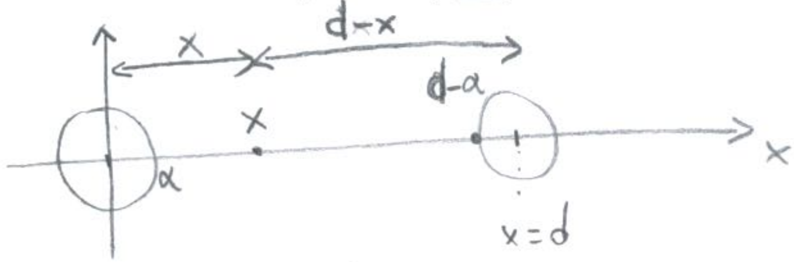
[1pt] $\int D_r(r)\,\overline{a}_r \cdot \overline{a}_r \, dS_r = \rho_s \cdot 2\pi\alpha L$

$D_r(r) \cdot \cancel{2}\pi r \cancel{L} = \rho_s \, \cancel{2\pi}\alpha \cancel{L}$

[1pt] $\overline{D}_+ = \rho_s \dfrac{\alpha}{r}\,\overline{a}_r$

$$\boxed{\mathbf{E}_+ = \frac{\rho_s}{\varepsilon}\frac{\alpha}{r}\,\overline{a}_r}$$ [1pt]

2. Find the total electric field **E** produced by both conductors on the x axis for $x \in [\alpha, d-\alpha]$. **(5 points)**



we apply superposition

from left cond: $\overline{E}_+ = \overbrace{\frac{\rho_s}{\varepsilon}\frac{\alpha}{x}}^{\text{Right mag [1pt]}}\,\overline{a}_x$ Right dir. [1pt] for $x > \alpha$

right cond: $\overline{E}_- = -\frac{\rho_s}{\varepsilon}\frac{\alpha}{d-x}\left(-\overline{a}_x\right) = \underbrace{\frac{\rho_s}{\varepsilon}\frac{\alpha}{d-x}}_{\text{Right mag [1pt]}}\,\underbrace{\overline{a}_x}_{\text{Right dir [1pt]}}$ for $x < d-\alpha$

$$\boxed{\mathbf{E} = \frac{\rho_s \alpha}{\varepsilon}\left(\frac{1}{x} + \frac{1}{d-x}\right)\bar{a}_x}$$

[1pt]

3. Find the voltage V between the conductors. **(5 points)**

[1pt]

$$V(x=\alpha) - V(x=d-\alpha) = -\int_{x=d-\alpha}^{x=\alpha} \bar{E} \cdot \bar{dl} =$$

[1pt]

$$= \int_{x=\alpha}^{d-\alpha} \frac{\rho_s \alpha}{\varepsilon}\left(\frac{1}{x} + \frac{1}{d-x}\right) \underbrace{\bar{a}_x \cdot \bar{a}_x}_{1} dx$$

$$= \frac{\rho_s \alpha}{\varepsilon} \ln \frac{d-\alpha}{\alpha} + \frac{\rho_s \alpha}{\varepsilon} (-1) \cdot \left[\ln(\alpha) - \ln(d-\alpha)\right] =$$

Integration [2pt]

$$= \frac{\rho_s \alpha}{\varepsilon}\left[\ln\left(\frac{d-\alpha}{\alpha}\right) + \ln\left(\frac{d-\alpha}{\alpha}\right)\right] =$$

$$= 2\frac{\rho_s \alpha}{\varepsilon}\ln\left(\frac{d-\alpha}{\alpha}\right)$$

$$\boxed{V = 2\rho_s\frac{\alpha}{\varepsilon}\ln\left(\frac{d-\alpha}{\alpha}\right)}$$

Answer
[1pt]

4. Find the capacitance per unit length C' between the conductors. **(5 points)**

Correct use of C def [2pt]

$$C' = \frac{Q'}{V}$$ where Q' charge on a 1m long section of conductor

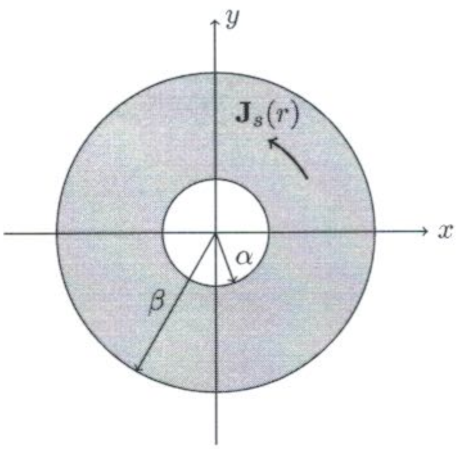
$$Q' = 2\pi\alpha\rho_s$$ [2pt]

$$C' = \frac{\cancel{2}\pi\cancel{\alpha}\cancel{\rho_s}}{\cancel{2}\cancel{\rho_s}\frac{\cancel{\alpha}}{\varepsilon}\ln\left(\frac{d-\alpha}{\alpha}\right)} = \frac{\pi\varepsilon}{\ln\left(\frac{d-\alpha}{\alpha}\right)}$$

$$\boxed{C' = \frac{\pi\varepsilon}{\ln\left(\frac{d-\alpha}{\alpha}\right)}}$$

Answer
[1pt]

Question 2.1

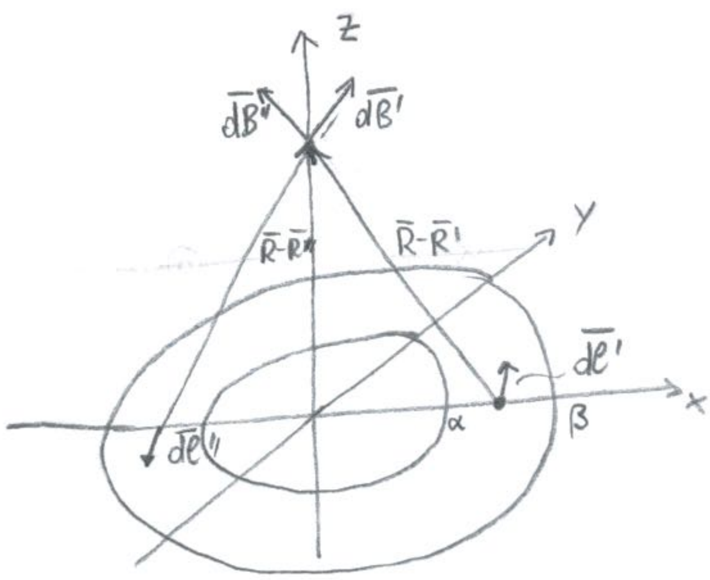


The flat thin ring shown in the figure carries a surface current density

$$\mathbf{J}_s(r) = J_0 \frac{\alpha}{r} \mathbf{a}_\varphi \qquad r \in [\alpha, \beta]$$

The ring is located on the $z = 0$ plane. Permeability is μ_0 everywhere. We want to find $\mathbf{B}$ on the positive z axis.

1. Deduce from geometrical considerations (i.e. no calculations), what is the direction of $\mathbf{B}$ at a point on the positive z axis. **(2 points)**



From Biot-Savart, the direction of $\bar{B}$ is given by $d\bar{\ell}' \times (\bar{R} - \bar{R}')$

Look at the contribution of two symmetric points (current elements on the ring)

$\bar{a}_r$ components cancel

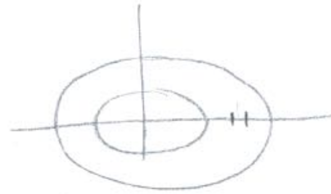
$\bar{a}_z$ add up

Direction: $\bar{a}_z$

2. Calculate **B** at a point $z > 0$ on the positive z axis. **(10 points)**

We use Biot-Savart [1pt] $\bar{B} = \frac{\mu_0 I}{4\pi}\int \frac{\bar{d\ell}' \times (\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3}$

$\bar{R} = z\,\bar{a}_z$ [1pt]



$\bar{R}' = r'\bar{a}_r'$ [1pt]

$\bar{R}-\bar{R}' = z\bar{a}_z - r'\bar{a}_{r'}$ ~~[1pt]~~

$|\bar{R}-\bar{R}'| = \sqrt{z^2+(r')^2}$ [1pt]

$\bar{d\ell}' = r'd\varphi'\bar{a}_\varphi'$ [1pt]

dI: current in a strip of width dr'

$dI = J_0 \frac{\alpha}{r'} dr'$ [1pt]

$$\begin{aligned}\bar{d\ell}' \times (\bar{R}-\bar{R}') &= r'd\varphi'\bar{a}_\varphi' \times (z\bar{a}_z - r'\bar{a}_{r'}) = \\ &= r'd\varphi' z\,\bar{a}_{r'} - (r')^2 d\varphi'(-\bar{a}_z) = \\ &= r'd\varphi' z\,\bar{a}_{r'} + (r')^2 d\varphi'\,\bar{a}_z\end{aligned}$$ [1pt]

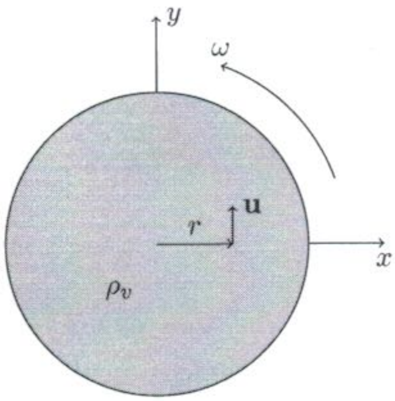
$$\bar{B} = \frac{\mu_0}{4\pi}\underbrace{\int_{r'=\alpha}^{\beta}\int_{\varphi'=0}^{2\pi}}_{[1pt]} J_0\frac{\alpha}{\cancel{r'}}dr' \cdot \frac{\cancel{r'}d\varphi' z\,\bar{a}_{r'} + (r')^{\cancel{2}} d\varphi'\,\bar{a}_z}{\left[z^2+(r')^2\right]^{3/2}} =$$

$$\bar{B} = \underbrace{\frac{\mu_0 J_0 \alpha}{4\pi} \int_{r'=\alpha}^{\beta} \int_{\varphi'=0}^{2\pi} \frac{z\, dr'\, d\varphi'}{\left[z^2+(r')^2\right]^{3/2}}\, \bar{a}_{r'}}_{\text{integrates to zero because of symmetry [1pt]}} + \frac{\mu_0 J_0 \alpha}{4\pi} \int_{r'=\alpha}^{\beta} \int_{\varphi'=0}^{2\pi} \frac{r'\, dr'\, d\varphi'}{\left[z^2+(r')^2\right]^{3/2}}\, \bar{a}_z$$

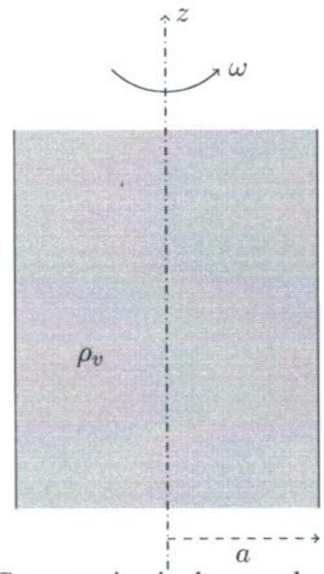
$$= \frac{\mu_0 J_0 \alpha}{2} \int_{r'=\alpha}^{\beta} \frac{r'\, dr'}{\left[z^2+(r')^2\right]^{3/2}}\, \bar{a}_z \qquad [1pt]$$

$$\boxed{\mathbf{B} = \frac{\mu_0 J_0 \alpha}{2} \int_{\alpha}^{\beta} \frac{r'\, dr'}{\left[z^2+(r')^2\right]^{3/2}}\, \bar{a}_z}$$

Question 2.2



Cross section in the x-y plane



Cross section in the x-z plane

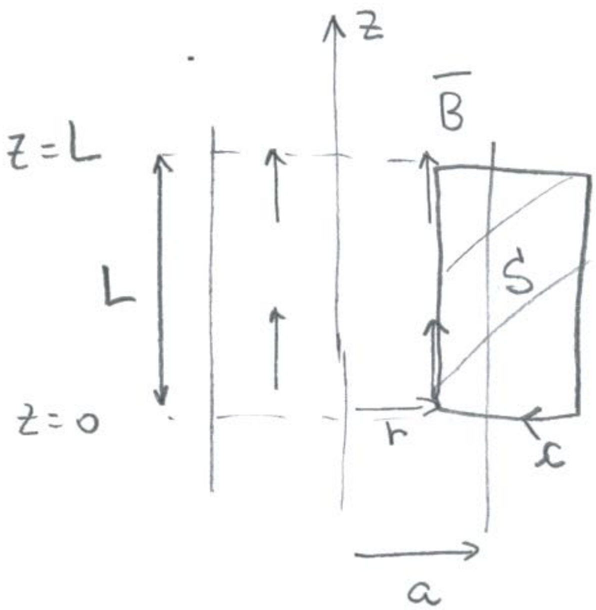
1. An infinitely-long cylinder is uniformly charged with charge density ρ_v, and rotates around its axis with constant angular velocity ω. The cylinder radius is a, and permeability is μ_0 everywhere. Assume that the charge distribution in the cylinder does not change as it rotates. Show that inside the cylinder the current density is in the form $\mathbf{J} = J_0 r \mathbf{a}_\varphi$, and find J_0. **(2 points)**

$$\bar{u} = \omega r \bar{a}_\varphi \quad \Big\} [1pt]$$

$$\bar{J} = \rho_v \bar{u} = \rho_v \omega r \bar{a}_\varphi$$

$$\boxed{J_0 = \rho_v\,\omega}$$ [1pt]

2. Assume that $\mathbf{B} = B(r)\mathbf{a}_z$, and that $B(r) = 0$ for $r > a$. Using Ampere's law, find the magnetic flux density $\mathbf{B}$ inside the cylinder. **(6 points)**



Amperian path: rectangular loop, height L

$$\underbrace{\oint_C \overline{B}\cdot\overline{dl}}_{} = \mu_0 I_{enc}$$

$$L\,B(r)$$

$$I_{enc} = \int_S \overline{J}\cdot\overline{dS} = \int_{r'=r}^{a}\int_{z=0}^{L} \rho_v\omega r'\underbrace{\overline{a}_\varphi\cdot\overline{a}_\varphi}_{1}\,dr'dz =$$

$$= \rho_v\omega\cdot L\int_{r'=r}^{a} r'dr'$$

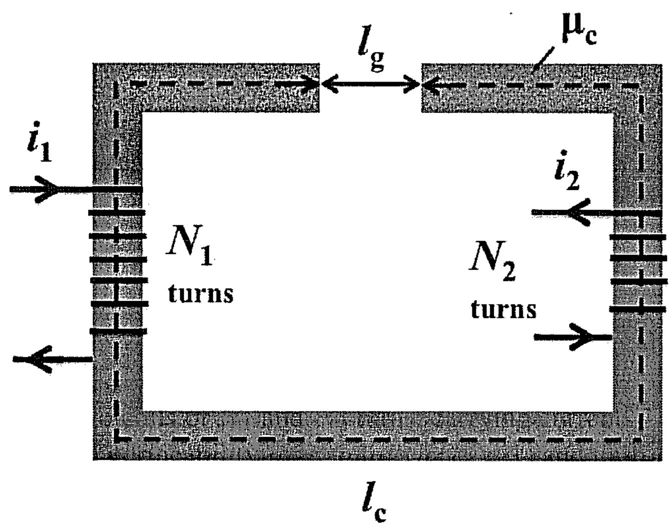
$$= \rho_v \omega L \left(\frac{a^2}{2} - \frac{r^2}{2}\right)$$

$$\cancel{L} B(r) = \mu_0 \rho_v \omega \cancel{L} \frac{a^2 - r^2}{2}$$

$$\boxed{\mathbf{B} = \mu_0 \rho_v \omega \frac{a^2 - r^2}{2} \bar{a}_z}$$

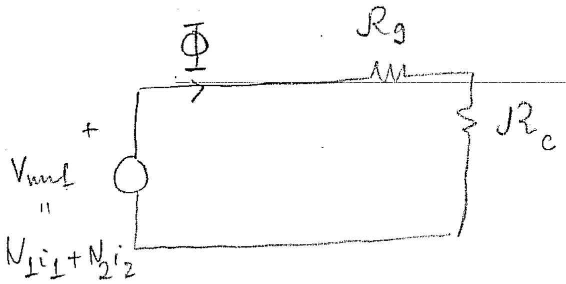
Question 3

A magnetic circuit consists of a two coils of N_1 and N_2 turns, carrying currents i_1, i_2, respectively. These coils are wound around a core with high permeability $\mu_c = \mu_0\mu_r$ ($\mu_r >> 1$) and uniform cross-section S_c. The path length of the magnetic flux within the high permeability core of this circuit is l_c and within the air gap is l_g. The geometry is shown in the figure below.



1. Draw the equivalent magnetic circuit for this geometry, clearly indicating the magnetomotive forces and reluctances involved. Determine the total magnetomotive force $\mathcal{V}_{mmf}$ and the total reluctance of the circuit $\mathcal{R}$. **(4 points)**

Equivalent circuit:



$$\mathcal{R}_c = \frac{l_c}{\mu_c S_c}$$

$$\mathcal{R}_g = \frac{l_g}{\mu_0 S_c}$$

$$V_{mmf} = N_1 i_1 + N_2 i_2$$

$$\Phi = \frac{N_1 i_1 + N_2 i_2}{\mathcal{R}_g + \mathcal{R}_c}$$

Schematic : 2 pts (1 for correct flux direction
1 for correct elements)

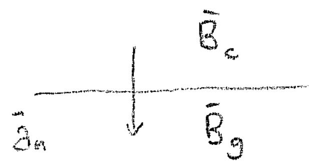
V_{mmf} : 1 pt $\mathcal{R}_c$, $\mathcal{R}_g$: 1 pt.

$$\boxed{\mathcal{V}_{mmf} = N_1 i_1 + N_2 i_2}$$

$$\boxed{\mathcal{R} = \frac{l_c}{\mu_c S_c} + \frac{l_g}{\mu_0 S_c}}$$

2. What is the relation between the magnetic flux Φ_c within the core and the magnetic flux Φ_g within the air gap ? **(2 points)**

$\bar{\Phi}_c = \bar{\Phi}_g = \bar{\Phi}$ from boundary condition:



$$\underbrace{\bar{a}_n \cdot \left(\bar{B}_g - \bar{B}_c\right) = 0}_{1} \Rightarrow \frac{\bar{\Phi}_c}{\cancel{S_c}} = \frac{\bar{\Phi}_g}{\cancel{S_c}} \Rightarrow$$

$$\bar{\Phi}_c = \bar{\Phi}_g \Big\} 1$$

3. What is the relation between the magnetic field intensity H_c within the core and the magnetic field intensity H_g within the air gap ? **(2 points)**

$$\underbrace{B_c = B_g}_{1} \Rightarrow \mu_c H_c = \mu_0 H_g \quad 1$$

$$\Rightarrow \boxed{\frac{H_c}{H_g} = \frac{\mu_0}{\mu_c} = \frac{1}{\mu_r}}$$

4. Show that the self-inductance of coil 1 is $L_{11} = \frac{N_1^2}{\mathcal{R}}$ and the self-inductance of coil 2 is $L_{22} = \frac{N_2^2}{\mathcal{R}}$, where $\mathcal{R}$ is the total reluctance of the circuit (see part 1 of this question). **(5 points)**

Total flux $\Phi = \frac{N_1 i_1 + N_2 i_2}{R} = \Phi_1 + \Phi_2$

1 1

with $\boxed{\Phi_1 = \frac{N_1 i_1}{R}}$, $\boxed{\Phi_2 = \frac{N_2 i_2}{R}}$.

1

$\boxed{L_{11} = \frac{N_1 \cdot \Phi_1}{i_1}} = \frac{N_1^2 \, i_1}{R \cdot i_1} = \frac{N_1^2}{R}$ 0,5

Similarly, $\boxed{L_{22} = \frac{N_2 \cdot \Phi_2}{i_2}} = \frac{N_2^2 \cdot i_2}{R \cdot i_2} = \frac{N_2^2}{R}$. 0,5

1

5. Show that the mutual inductance between the coils is $L_{12} = \dfrac{N_1 N_2}{\mathcal{R}}$, where $\mathcal{R}$ is the total reluctance of the circuit (see part 1 of this question). **(5 points)**

2 2

$$\boxed{L_{12} = \frac{N_2 (\Phi_1)}{i_1}} = \frac{N_1 \Phi_2}{i_2}, \quad \boxed{\Phi_1 = \frac{N_1 i_1}{\mathcal{R}}}, \quad \Phi_2 = \frac{N_2 i_2}{\mathcal{R}}$$

$$= \boxed{N_1 N_2 / \mathcal{R}} \quad \text{or} \quad \frac{N_1 \Phi_2}{i_2} \;\ldots$$

1

6. The magnetic flux in this circuit is caused by:
   (a) Free currents.
   (b) Magnetization currents.
   (c) Free and magnetization currents. 0.5

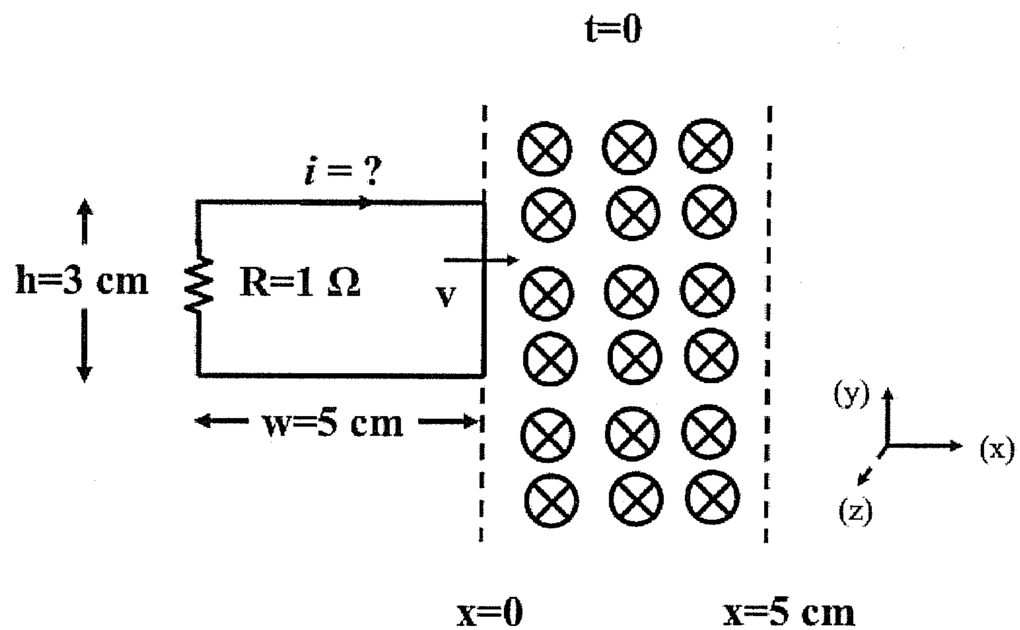
   Choose the correct answer and briefly explain. **(2 points)**

Free currents: i_1, i_2 on wires 0.5

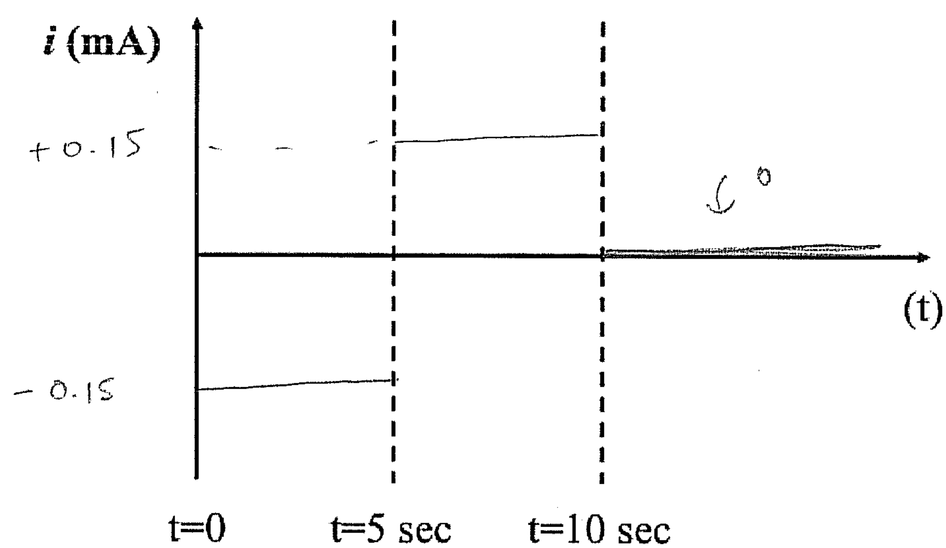
Magnetization currents in the core. 1

Question 4

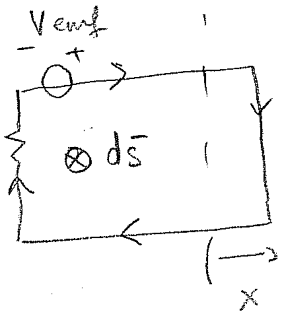
1. The figure below shows a rectangular frame (at $t = 0$), moving with constant velocity $\mathbf{v} = v_0\mathbf{a}_x$, where v_0=1 cm/sec. At $t = 0$, the frame enters a uniform and constant magnetic field of magnetic flux density $\mathbf{B} = -\mathbf{a}_z 0.5$ T, which extends in $0 < x < 5$ cm. The dimensions of the frame are shown ($w = 5$ cm, $h = 3$ cm). The frame has total resistance $R = 1\,\Omega$.



(a) Plot the current i induced in the circuit in the direction shown in the figure (note: i can be positive or negative), as a function of time. Explain the procedure you applied for this calculation. **(8 points)**



4(a) - cont-d $\quad 0 < t < 5\,\text{sec}$



3.5

$$\boxed{\begin{aligned}\Phi &= B \cdot x(t) \cdot h = \int (-\bar{a}_z B_0) \cdot \underbrace{(-\bar{a}_z)\, dx\, dy}_{d\bar{s}} \\ \Rightarrow \frac{d\Phi}{dt} &= B \cdot h \frac{dx}{dt} = B \cdot h \cdot v_0\end{aligned}}$$

EMF: Value: 1 Polarity 1

$V_{emf} = -\, B \cdot h\, v_0$ and from KVL

$$iR - V_{emf} = 0 \Rightarrow i = \frac{V_{emf}}{R} = -\frac{B \cdot h\, v_0}{R}$$

value 1

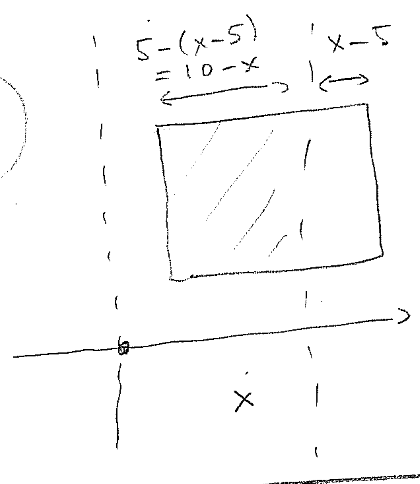
$$= \frac{-0.5 \times 3\times 10^{-2} \times 10^{-2}}{1\ \Omega} = -1.5 \times 10^{-4} = \boxed{-}\ \boxed{0.15\ mA}$$

0.5 polarity

For $5 < t < 10\,\text{sec}$ $\Rightarrow$ $\boxed{\Phi = B \cdot (10 - x(t)) \cdot h \Rightarrow \quad d\Phi/dt = -B v_0 h}$

EMF value: 1 pol. 1

3.5



$\Rightarrow V_{emf} = +B v_0 h \Rightarrow$

1

$$i = \frac{V_{emf}}{R} = +\ 0.15\ mA$$

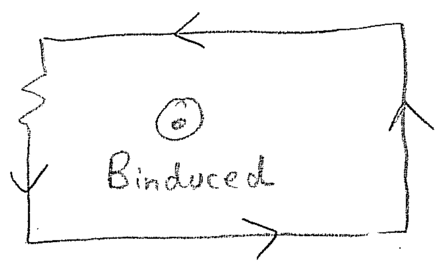
0.5

1 $\quad \boxed{t > 10: \quad V_{emf} = 0 \Rightarrow i = 0.}$ $\quad$ 1 pt

(b) Explain the direction of the induced current i in terms of Lenz' law. **(4 points)**

$0 < t < 5$ ⇒ flux increases, i in a direction that produces a magnetic field OPPOSITE to the external (i.e. in $+\bar{a}_z$ direction)

Hence:



⊗ Bexternal

(2 pts)

0.5 correct $\bar{B}$ direction

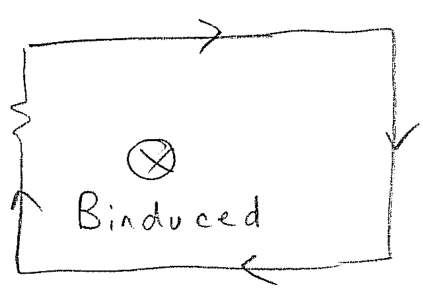
1.5 correct justif.

$5 < t < 10$: ⇒ flux decreases ⇒ i in a direction that produces a magnetic field REINFORCING the external (i.e. in $-\bar{a}_z$):

2 pts

0.5 correct $\bar{B}$ direction

1.5 justification



⊗ Bext.

(2pts)

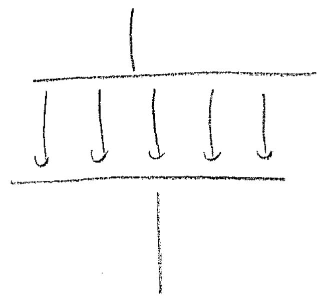
These indeed are consistent with the results shown before.

2. The following three questions are independent from the previous one and from each other.

(a) Can the displacement current exist in vacuum? Briefly explain. **(2 points)**

Yes. Just consider a capacitor with vacuum between the plates

0.5



$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}\left[\frac{\epsilon V}{h}\right](-\bar{a}_z) = \frac{\epsilon}{h}\frac{\partial V}{\partial t}(-\bar{a}_z)$$

1.5

(b) Consider a medium with dielectric permittivity ϵ, magnetic permeability μ. The conductivity $\sigma = 0$. If the magnetic field in this medium is $\mathbf{H} = H_0 \sin(\omega t - kz)\mathbf{a}_y$, where $\frac{\omega}{k} = \frac{1}{\sqrt{\epsilon\mu}}$, find the electric field $\mathbf{E}$. You can assume that at $t = 0$, $\mathbf{E} = 0$ at $z = 0$. **(4 points)**

$$\nabla \times \bar{H} = \underbrace{\bar{J}}_{= \sigma \bar{E},\ \sigma = 0} \xrightarrow{} 0 + \frac{\partial \bar{D}}{\partial t} = \epsilon \frac{\partial \bar{E}}{\partial t}$$

1

$$\nabla \times \bar{H} = \bar{a}_z \frac{\partial}{\partial z} \times \bar{a}_y H_0 \sin(\omega t - kz)$$

$$= -\bar{a}_x H_0 \cdot (-k) \cos(\omega t - kz)$$

1

$\bar{a}_z \times \bar{a}_y = -\bar{a}_x$

$$= +\bar{a}_x k \cdot H_0 \cdot \cos(\omega t - kz) = \epsilon \frac{\partial \bar{E}}{\partial t}$$

2

So, $\bar{E} = \bar{a}_x E_x$ and

$$\epsilon \frac{\partial E_x}{\partial t} = k \cdot H_0 \cos(\omega t - kz) \Rightarrow E_x = \frac{k H_0}{\epsilon} \frac{\sin(\omega t - kz)}{\omega} + \cancel{C}$$

$$= \sqrt{\frac{\mu}{\epsilon}} \cdot H_0 \sin(\omega t - kz)$$

(c) The displacement current causes magnetic field in the same way conduction current does. True or false? Choose the correct answer and briefly explain. **(2 points)**

True. Just look at

0.5

$$\nabla \times \bar{H} = \underbrace{\bar{J}_c + \bar{J}_d}_{1.5}$$

conduction current density.

displacement current density