1. [20 pts.] The position (in metres) of a particle of mass 2.0 kg is

$$\vec{r} = (5.0\,t)\,\hat{\boldsymbol{\imath}} + (2.0\,t^4)\,\hat{\boldsymbol{\jmath}} - 9.5\,\hat{\boldsymbol{k}},$$

where $t$ is time and $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are cartesian unit vectors. (Assume the right units for the numerical constants given here, so that $\vec{r}$ is in metres when $t$ is in seconds. Give your answer as a vector when appropriate, and specify units.)

(a) [5 pts.] What is the **kinetic energy** at time $t$ ?

(b) [5 pts.] What is the **rate of change of momentum** at time $t$ ?

(c) [5 pts.] How much **work** is done between 0 s and time $t$ ?

(d) [5 pts.] What is the **power** of the external force at time $t$?

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2. [20 pts.] Two blocks, of mass $m_1 = 1.0$ kg and $m_2 = 2.0$ kg, are on a collision course. Their initial velocities are $\vec{v}_1 = 2.0\,\hat{\boldsymbol{\imath}} + 1.0\,\hat{\boldsymbol{\jmath}}$ and $\vec{v}_2 = -2.0\,\hat{\boldsymbol{\imath}} + 1.0\,\hat{\boldsymbol{\jmath}}$, where units are m/s. After the collision, they stick together.

(a) [6 pts.] What is the initial **velocity of the centre of mass**?

(b) [7 pts.] What is the **impulse** of block 1 on block 2 due to the collision? Give your answer as a vector, and specify units.

(c) [7 pts.] How much did **mechanical energy change** due to the collision?

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3. [22 pts.] A force $\vec{F} = 2.0x\,\hat{\mathbf{i}}$ N acts on a particle

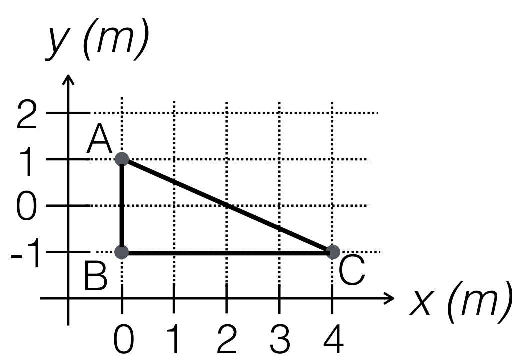
(a) [4 pts.] Calculate the **work** done along **path A-B** (See figure below.)

(b) [4 pts.] Calculate the work done along **path B-C**

(c) [5 pts.] Calculate the work done along **path A-C**

(d) [4 pts.] Is this a **conservative force?** Argue why or why not.

(e) [5 pts.] Give an example of the **potential** $U(x, y)$ that would produce this $\vec{F}$ as an internal force.

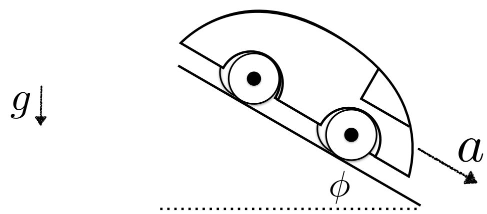



Problem 3.

4. [19 pts.] A car has a total mass $M$, including its **four** wheels; each wheel has radius $R$ and moment of inertia $I$ when rotating about their centre. The car rolls without slipping down a plane inclined by angle $\phi$ from level ground. The coefficient of static friction between the wheels and ground is $\mu_S$.

   (a) [11 pts.] If the car starts at rest, what its **speed** after rolling a **distance** $d$ **down the plane**?

   (b) [8 pts.] What is the **acceleration** of the car?




Problem 4.

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5. [19 pts.] Sand falls at mass-rate $r$ onto a platform sliding on the ground. The sand has speed $v_S$ downward when it lands on (and instantly sticks to) the platform. At the instant when the platform and accumulated sand is $M_P$, all moving right at speed $v_P$, answer the following questions:

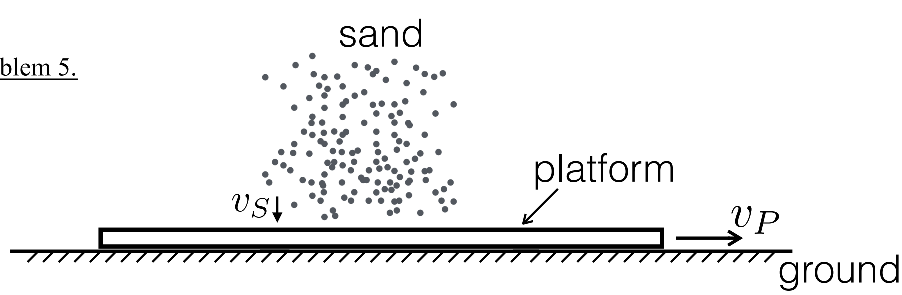
   (a) [10 pts.] What is the **normal force** of the ground on the platform?

   (b) [9 pts.] What is the **instantaneous acceleration** of the platform, if there is no friction between the ground and the platform?

   (c) [4 pts. bonus] What would the instantaneous acceleration of the platform be, if there were furthermore **kinetic friction** with coefficient $\mu_K$ between the platform and the ground?

   For all parts, give a simplified expression in terms of the variables given here and $g$. If you can't solve part (a), then you can also use "$n$" as the normal force, in your answer to parts (b) and (c). *Hint: none of these answers have an explicit dependence on time $t$ in them.*

Problem 5.



(1) $\vec{r} = \langle 5t, 2t^4, -9.5 \rangle$ in mks ; $m = 2$

(a) $KE = \frac{1}{2} m |\vec{v}|^2$

$\vec{v} = \langle 5, 8t^3, 0 \rangle \rightarrow v^2 = 25 + 64t^6$

so $\underline{KE = 25 + 64t^6 \text{ (in J)}}$

(b) $\frac{d\vec{p}}{dt} = \vec{F}$, $\vec{p} = m\vec{v} = \langle 10, 16t^3, 0 \rangle$

$$\boxed{\frac{d\vec{p}}{dt} = \langle 0, 48t^2, 0 \rangle \text{ or } 48t^2 \hat{j}}$$

units are SI here, kg m/s.

(c) Work is change in K for a point particle.

So, $W = \Delta K = K(t) - K(t=0) = \boxed{64t^6 \text{ in J}}$

An alternate solution is to figure out $\vec{F}$ and $d\vec{r}$, multiply & integrate; or find P (see d.) & integrate.

(d) $P = \vec{F} \cdot \vec{v}$

Here $\vec{F} = m\vec{a}$, $\vec{a} = \frac{d}{dt}\vec{v} = \langle 0, 24t^2, 0 \rangle$

So $\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v} = 2 \langle 0, 24t^2, 0 \rangle \cdot \langle 5, 8t^3, 0 \rangle$

$$= \boxed{384t^5 \text{, units in J or } (3.8 \times 10^2) t^5 \text{ for 2 sig figs}}$$

2. (a.) $\vec{V}_{cm} = \dfrac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2}$

$$= \frac{1.0\,\langle 2,1\rangle + 2.0\,\langle -2,1\rangle}{3.0}$$

$$= \boxed{\langle -0.67,\ 1.0\rangle \ m/s \quad \text{or} \quad -0.67\,\hat{i} + 1.0\,\hat{j}}$$

(b.) Impulse = change in momentum

$\vec{I}_{21} = \Delta\vec{p}_2 = m_2\Delta\vec{V}_2 = m_2(\vec{V}_{2f} - \vec{V}_{2i})$

But $\vec{V}_{2f} = \vec{V}_{cm}$ since blocks stuck together.

So $\Delta\vec{p}_2 = 2.0\left(\langle -0.67, 1.0\rangle - \langle -2.0, 1.0\rangle\right)$

$$= \boxed{\langle 2.7, 0\rangle \ kg\ m/s \quad \text{or} \quad +2.7\,\hat{i}}$$

(c.) $\Delta K = K_f - K_i$ , $K = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$

so $K_i = \frac{1}{2}(1.0)(2^2+1^2) + \frac{1}{2}(2.0)(2^2+1^2) = 7.5\ J$

$K_f = \frac{1}{2}(3.0)\left((0.67)^2 + 1^2\right) = 2.2\ J$

$\rightarrow \boxed{\Delta K = -5.3\ J}$

3. $W = \int \vec{F} \cdot d\vec{r}$ , $\vec{F} = \langle 2x, 0 \rangle$

(a) $d\vec{r} = \langle 0, dy \rangle$ , $y: +1 \rightarrow -1$

but $\vec{F} \cdot d\vec{r} = \langle 2x, 0 \rangle \cdot \langle 0, dy \rangle = 0 \rightarrow \boxed{W_{AB} = 0}$

(b) $d\vec{r} = \langle dx, 0 \rangle$, $x: 0 \rightarrow 4$

$$W = \int_0^4 2x \, dx = \left[x^2\right]_0^4 = \boxed{16 \text{ J}}$$

(c) Here the integrand is $\vec{F} \cdot d\vec{r} = \langle 2x, 0 \rangle \cdot \langle dx, dy \rangle$
$= 2x \, dx$ as in (b).

So we can still write this integral as
$W = \int_0^4 2x \, dx = \boxed{16 \text{ J}}$ as in (b).

(d) YES:

argument 1: W doesn't seem to depend on path, because $W_{AC} = W_{AB} + W_{BC}$

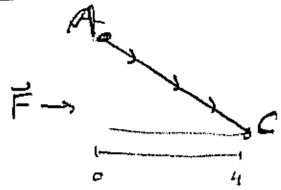
argument 2: $W_{ABCA} = 0$, because $W_{CA} = -W_{AC}$, so

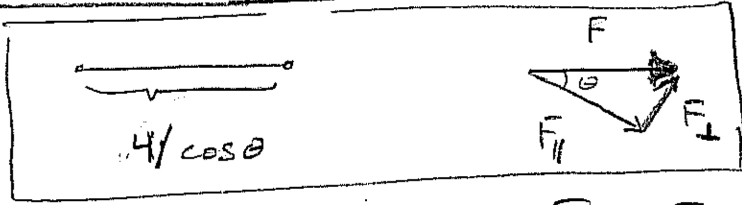
$$W_{ABCA} = W_{AB} + W_{BC} + W_{CA} = 0 + 16 - 16 = 0.$$

(e) We're looking for something that gives
$F_x = 2x$, $F_y = 0$, $F_z = 0$.
Remembering that $F_x = -\frac{dU}{dx}$, etc., then $U = -x^2$ would work! But also, any $U = -x^2 + \text{const}$ is fine also.

Rigorous solutions to ③ c) | THREE OPTIONS: |

(I)

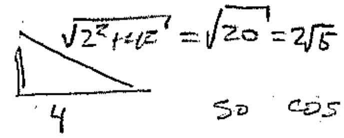



Rotate axes to $\parallel$ and $\perp$ to path AC

$F_{\parallel} = F\cos\theta$

(can calculate $\cos\theta$,




so $\cos\theta = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}}$

but drops out..)

$$W = \int_A^C \vec{F}\cdot d\vec{r} = \int_0^{4/\cos\theta} F_{\parallel}\cdot d\ell = \int_0^{4/\cos\theta} (2x)\cos\theta\, d\ell$$

(Now write $x = \ell\cos\theta$) $\quad = \left(\int_0^{4/\cos\theta} 2\ell\, d\ell\right)\cos^2\theta$

$$= \left[\ell^2\right]_0^{4/\cos\theta} \cos^2\theta = 4^2 = 16$$

---

(II) Write $d\vec{r} = \vec{T}\,ds$

construct a tangent vector $\vec{T}$, here $= \langle +4,-2\rangle / \|\langle +4,-2\rangle\| = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

$= \langle \cos\theta, -\sin\theta\rangle$

$W = \int \vec{F}\cdot\vec{T}\,ds$, where $ds =$ differential of length.

$$= \int_0^{4/\cos\theta} \langle 2x, 0\rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle ds$$

& again $ds = dx/\cos\theta$ or $x = \ell\cos\theta$

= same as above

---

(III) Can use a parameterization "t". Then

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r})\cdot\vec{r}\,'(t)\,dt \quad \text{where} \quad \vec{r} = x(t)\hat{i} + y(t)\hat{j}$$

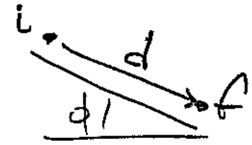
$$\& \quad \vec{r}\,' = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

here, can choose $x = t$, $y = 1 - t/2$, $a = 0$, $b = 4$.

then $\begin{cases} \vec{F} = 2x\hat{i} = 2t\hat{i} \\ \frac{d\vec{r}}{dt} = 1\hat{i} - \frac{1}{2}\hat{j} \end{cases}$ so $W = \int_0^4 \langle 2t, 0\rangle \cdot \langle 1, -\tfrac{1}{2}\rangle \cdot dt = \left[t^2\right]_0^4$

$= 16\text{ J}.$

(4.) (a) Even though the problem mentions friction, it's static friction, and the wheels roll without slip, so no kinetic friction — ENERGY CONSERVED.

"Distance d" refers to  on the plane,

so drop in height of cm is $\underline{d \sin\phi = -\Delta y}$

Now for the system = car + earth (+ other perhaps), energy conserved: $\underline{\Delta K + \Delta U = 0}$, where $\underline{\Delta U = -Mgd\sin\phi}$

$\Delta K = K_f - K_i$, but $\underline{K_i = 0}$ starting at rest.

$$K_f = 4(K_{wheel}) + K_{\text{rest of car}}$$

$$K_{wheel} = (K_R)_{wheel} + (K_{trans})_{wheel}$$

$$\text{But } K_{rest} + 4(K_{trans})_{wheel} = \frac{1}{2}(M_{total})V_{cm}^2$$

Knowing $I_{wheel} = I$, $(K_R)_{wheel} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{I}{R^2}V_{cm}^2$ if rolls without slip

Thus $K_{tot} = \frac{1}{2}MV_{cm}^2 + 2\frac{I}{R^2}V_{cm}^2 = \left(\frac{M}{2} + \frac{2I}{R^2}\right)V_{cm}^2$

Setting $K_{tot} = Mgh$, we find

$$\boxed{V_{cm} = \sqrt{\frac{Mgd\sin\phi}{\frac{M}{2} + \frac{2I}{R^2}}} \text{ or } \sqrt{\frac{2gd\sin\phi}{1 + 4I/MR^2}}}$$

(4.) (6.) Now, even though the system has multiple parts, $\vec{a}_{cm}$ = constant, because the external force is constant.

Now several ways to solve this:

(I) Use $V_f^2 - V_i^2 = 2ad \rightarrow a = V_f^2/2d$

(II) Take the time derivative of the energy equation

$$(Mg\sin\phi)\,d = \frac{1}{2}(M + 4I/R^2)\,V^2$$

$$\frac{d}{dt}: \quad (\quad)\,V = \frac{1}{2}(\quad)\,2V\frac{dV}{dt}$$

cancel the V from each side, & solve for $\frac{dV}{dt} = a$

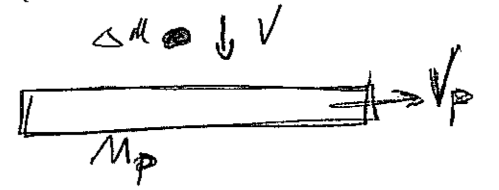
(III) Notice that the car acquired kinetic energy as $\frac{1}{2}M_{eff}V_{cm}^2$, where $M_{eff} = M + 4I/R^2$, & now use $F = M_{eff}\,a \rightarrow a = \frac{Mg\sin\phi}{M_{eff}}$

In all methods, one finds

$$\boxed{a_{cm} = \frac{g\sin\phi}{1 + 4I/MR^2}}$$

Direction of $\vec{a}$ is along the ramp, downward (or $\vec{a} = a_{car}\langle\cos\phi, -\sin\phi\rangle$, but don't need to write this explicitly.)

(5.) initial




final


But momentum not conserved, because there is a normal force upwards, $n$, and a frictional force backwards $\mu_K n$, plus gravity $(M_p+\Delta M)g$

So: $\vec{P}_f = \vec{P}_i + \vec{F}_{net}\,\Delta t$

x: $(M+\Delta M)(V_p+\Delta V) = M_p V_p - \mu_K n\,\Delta t$

y: $0 = -(\Delta M)V_s + (n - M_p g - \Delta M g)\Delta t$

Solve the y equation first. In one $\Delta t$, $\Delta M = r\,\Delta t$, so

$0 = -rV_s\,\Delta t + n\Delta t - M_p g\,\Delta t - rg(\Delta t)^2$

Drop $\Delta t^2$, & get $\boxed{n = M_p g + rV_s}$ (a)

---

Now x equation. Use $\Delta m = r\Delta t$, and $\Delta V = a\,\Delta t$.

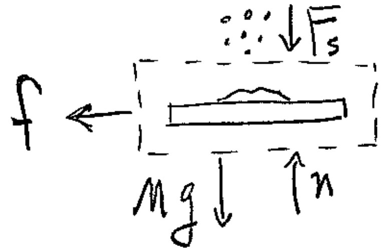
$\cancel{M_pV_p} + rV_p\,\Delta t + M_p a\,\Delta t = \cancel{M_pV_p} - \mu_K n\,\Delta t$ {After neglecting terms with $(\Delta t)^2$

$m_p a = -rV_p - \mu_K n$

$= -rV_p - \mu_K m_p g - \mu_K r V_s$

$$\boxed{a = -\left(\frac{rV_p}{m_p}\right) - \mu_K\left(g + \frac{rV_s}{m_p}\right)}$$ (b, c)

(5.) Alternative soln using system = platform + accumulated sand




$$\vec{p} = M\vec{V}_p$$

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(M\vec{V}_p) = \underbrace{\frac{dM}{dt}}_{r}\vec{V}_p + M\underbrace{\frac{d\vec{V}_p}{dt}}_{\vec{a}}$$

y direction: $\frac{dp_y}{dt} = \sum_{ext} F_y = n - Mg - F_s$ , but $\frac{dp_y}{dt} = 0$, so

$n = Mg + F_s$ where $F_s$ is force of falling sand on the platform.

What is $F_s$? $\left.\frac{dp_y}{dt}\right|_{sand} = -V_s r$, so by Newton 3, negative of same force is felt on platform by the falling sand.

$\rightarrow$ $\boxed{n = M_p g + V_s r}$

x direction: $p_x = MV_p$, and $\frac{dp_x}{dt} = \sum_{ext} F_x = -\mu_K n$

Thus $\frac{d}{dt}(M_p V_p) = -\mu_K n$

$$\underbrace{\frac{dM}{dt}}_{r} V_p + M_p \underbrace{\frac{dV_p}{dt}}_{a} = -\mu_K n$$

solve for a: $\boxed{a = -\frac{rV_p}{M_p} - \frac{\mu_K n}{M_p}}$

substituting answer from part (a), $a = -\frac{rV_p}{M_p} - \mu_K\left(g + \frac{V_s r}{M_p}\right)$

5. $\frac{dm}{dt} = r$ , $\mu_k$ , $v_s$

$M_p$ , $v_p$

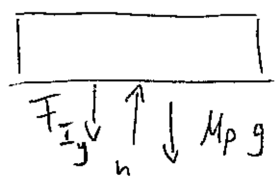
(a) $I_x = \Delta p = \Delta(M_p v_p) = \left(\frac{dM_p}{dt} v_p + M_p \cancel{\frac{dv_p}{dt}}^{\to 0}\right) dt$

$= r v_p \, dt$

$I_y = \frac{dm}{dt} v_s \, dt = r v_s \, dt$

normal force needs to balance gravitational force & force due change in impulse $F_I$

$F_{I_y} = r v_s$




$\Sigma F_y = 0$

$n - F_I - M_p g = 0$

$n = M_p g + r v_s$

(b) $\Sigma F_x = M_p a$

$-F_{I_x} - \mu_k n = M_p a$

$-r v_p - \mu_k (M_p g + r v_s) = M_p a$

$$a = -\frac{r(v_p + \mu_k v_s) + \mu_k M_p g}{M_p}$$