MAT185 Linear Algebra
Term Test 2

Full Name: ______________________ **SOLUTIONS** ______________________

Student number: ____

Email : ____@mail.utoronto.ca

Signature: ____

Instructions:

1. This test contains a total of 9 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. Consider the basis $\alpha = 1+x, \frac{1}{2}x, -\frac{3}{2}x^2+\frac{1}{2}$ for $P_2(\mathbb{R})$. Then the coordinate vector of x^2 with respect to $\alpha =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
- ○ $\begin{bmatrix} 0 \\ 0 \\ -\frac{2}{3} \end{bmatrix}$.
- ○ $\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$.
- ● $\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$.
- ○ $\begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$.

2. Let α and β be two bases for $P_2(\mathbb{R})$ where $\alpha = x^2, 1+x, x+x^2$. Given that $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ is the change of basis matrix from α to β and that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the coordinate vector of $p(x)$ with respect to β, then $p(x) =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $-1+x+x^2$.
- ○ $1+2x+2x^2$.
- ● $-1+x+3x^2$.
- ○ $2+2x+x^2$.
- ○ $1-x-3x^2$.

Multiple Choice: No justification is required. Only your final answer will be graded.

3. Suppose that A is a 3×4 matrix such that each of the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

belong to the nullspace of A. Which of the following statements are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ● The rows of A are linearly dependent.
- ○ The equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in {}^3\mathbb{R}$.
- ○ The solution to $A^{\mathrm{T}}\mathbf{x} = \mathbf{y}$, when it exists, is unique.
- ● The vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ belongs to the nullspace of A.
- ○ $\operatorname{null} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

4. $\dim \operatorname{span}\{e^x, \cos^2 x, \cos 2x, 1, \sin^2 x\} =$ ______? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ 1
- ○ 2
- ● 3
- ○ 4
- ○ 5

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

5. Let V and W be 3-dimensional vector spaces, and let $T : V \to W$ be a linear transformation.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●). Each part is worth 3 marks: 1 mark for a correct final answer; 2 marks for a correct explanation.

(a) If B is a subspace of W, then the set $U = \{\mathbf{x} \mid T\mathbf{x} \in B\}$ is a subspace of V.

● True.

○ False.

Apply the Subspace Test:

SI. B is a subspace of W; so $\mathbf{0} \in B$. Therefore, $\mathbf{0} \in U$ because $T\mathbf{0} = \mathbf{0}$ owing to T being a linear transformation.

SII. Let $\mathbf{x}_1, \mathbf{x}_2 \in U$: $T(\mathbf{x}_1 + \mathbf{x}_2) = T\mathbf{x}_1 + T\mathbf{x}_2 \in B$ because $T\mathbf{x}_1, T\mathbf{x}_2 \in B$ and B is a subspace. Thus U is closed under vector addition.

SIII. Let $\mathbf{x} \in U$ and $\alpha \in \mathbb{R}$: $T(\alpha\mathbf{x}) = \alpha T\mathbf{x} \in B$ because again $T\mathbf{x} \in B$ and B is a subspace. Thus U is closed under scalar multiplication.

Therefore U is a subspace of V.

(b) If T is bijective, then there exist bases α for V and β for W such that the matrix of T with respect to α and β is the 3×3 identity matrix.

● True.

○ False.

If $\alpha = \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is any basis for V, then, since T is bijective, $\beta = T\mathbf{x}_1, T\mathbf{x}_2, T\mathbf{x}_3$ is a basis for W (cf. Assignment 3, Q3(a)). The matrix of T with respect to this α and β is

$$\begin{bmatrix} [T\mathbf{x}_1]_\beta & [T\mathbf{x}_2]_\beta & [T\mathbf{x}_2]_\beta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

Let α and β' be arbitrary bases for V and W, respectively, and let the matrix representation of T be $[T]_{\alpha\beta'}$ in these bases. In general, $[\mathbf{w}]_\alpha = [T]_{\alpha\beta'}[\mathbf{v}]_{\beta'}$, where $[\mathbf{w}]_\alpha$ and $[\mathbf{v}]_{\beta'}$ are coordinates.

Now imagine another basis β for V and denote the transition matrix from β to β' by $\mathbf{P}_{\beta'\beta}$. Thus $[\mathbf{w}]_\alpha = [T]_{\alpha\beta'}\mathbf{P}_{\beta'\beta}[\mathbf{v}]_{\beta'}$. We desire the new matrix representation to be the identity matrix, i.e., $[T]_{\alpha\beta'}\mathbf{P}_{\beta'\beta} = \mathbf{1}$. As T is bijective, $[T]_{\alpha\beta'}$ must be invertible and therefore we can select β such that $\mathbf{P}_{\beta'\beta} = [T]^{-1}_{\alpha\beta'}$, which shows we can always choose a set of bases such that the matrix representation of T is the identity matrix.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

6. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by

$$T(p(x)) = x^2 p''(x) - p'(x) + p(x)$$

(a) Determine the matrix of T with respect to the standard basis of $P_2(\mathbb{R})$. [3 marks]

Let arbitrary $p \in P_2(\mathbb{R})$ be

$$p(x) = a_0 + a_1 x + a_2 x^2$$

Then

$$T(p(x)) = (a_0 - a_1) + (a_1 - 2a_2)x + 3a_2 x^2$$

Using the standard basis,

$$\mathbf{T} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix}$$

is the matrix representation of T.

(b) Use your answer from part (a) to find *all* solutions $p(x) \in P_2(\mathbb{R})$ to the differential equation

$$x^2 p''(x) - p'(x) + p(x) = 1 + 2x + 3x^2.$$

[3 marks]

We seek all the solutions to $T(p(x)) = 1 + 2x + 3x^2$ but let us convert to coordinates using the standard basis. The coordinates corresponding to the right-hand side are

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in {}^3\mathbb{R}$$

and let $\mathbf{s} \in {}^3\mathbb{R}$ be the coordinates of $p(x)$, a solution to the differential equation. We then seek all solutions $\mathbf{s}$ satisfying

$$\mathbf{Ts} = \mathbf{b}$$

This gives

$$\mathbf{s} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

We observe that $\operatorname{rank} \mathbf{T} = 3$ so $\dim \operatorname{null} \mathbf{T} = 0$ (and $\dim \ker T = 0$), which means that this $\mathbf{s}$ is the only solution. That is, the only solution in $P_2(\mathbb{R})$ to the differential equation is

$$p(x) = 5 + 4x + x^2$$

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

7.

(a) Define the rank of a matrix. Be sure to give a precise statement. No partial credit will be given for statements that are "close" to the definition. [2 marks]

The *rank* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$ is the dimension of the column space of A (or, equivalently, the dimension of the row space of $\mathbf{A}$).

(b) Let $A \in {}^3\mathbb{R}^n$ where $n \neq 3$. If $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \subset {}^3\mathbb{R}$ is linearly independent and $A\mathbf{x} = \mathbf{b}_j$ has at least one solution for each $\mathbf{b}_j$, show that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in {}^3\mathbb{R}$. [4 marks]

We note that as $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is linearly independent and $\dim {}^3\mathbb{R} = 3$, $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ constitutes a basis for ${}^3\mathbb{R}$. If $A\mathbf{x} = \mathbf{b}_j$ has a solution for each $\mathbf{b}_j$ then

$$ {}^3\mathbb{R} = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \subseteq \operatorname{col} A \subseteq {}^3\mathbb{R} $$

So $\operatorname{col} A = {}^3\mathbb{R}$ and $\operatorname{rank} A = \dim \operatorname{col} A = 3$. As such $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b} \in {}^3\mathbb{R}$. It also follows that $n \geq 3$. But if $n \neq 3$, we must have $n > 3$. Accordingly $\dim \operatorname{null} A = n - \operatorname{rank} A > 0$. This means that $A\mathbf{x} = \mathbf{0}$ has an infinite number of solutions and, as a consequence, so does $A\mathbf{x} = \mathbf{b}$.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

8.

(a) State the Dimension Theorem (Formula). Be sure to give a precise statement. No partial credit will be given for statements that are "close" to the statement. [2 marks]

For any $\mathbf{A} \in {}^m\mathbb{R}^n$,

$$\dim \operatorname{null} \mathbf{A} = n - \operatorname{rank} \mathbf{A}$$

or

Let V and W be vector spaces. If V is finite dimensional, then for any linear transformation $T : V \to W$,

$$\dim \ker T + \dim \operatorname{im} T = \dim V$$

(b) Let V and W be finite dimensional vector spaces, and let $T : V \to W$ and $S : W \to V$ be injective linear transformations. Prove that $\dim V = \dim W$. [4 marks]

If $T : V \to W$ is injective then $\ker T = \{\mathbf{0}\}$ and $\dim \ker T = 0$. By the Dimension Theorem,

$$\dim \operatorname{im} T = \dim V$$

But $\operatorname{im} T \subseteq W$ and hence $\dim \operatorname{im} T \leq \dim W$, Thus

$$\dim V \leq \dim W$$

Interchanging the roles of T with S and V with W, we also have

$$\dim W \leq \dim V$$

We therefore conclude that $\dim V = \dim W$ as required.