University of Toronto – Faculty of Applied Science & Engineering

MAT292 – Final Examination – December 14, 2022

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Exam Type: C | Time alotted: 150 minutes | Calculator Type: none

1 page handwritten "cheatsheet" allowed

Full Name ______________________

Student Number ______________________

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Signature ______________________

DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This exam contains 19 pages (including this title page). Once the exam starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages 15–18 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

! ! ! ! THERE IS A TABLE ON THE LAST PAGE ! ! ! !

GOOD LUCK! YOU GOT THIS!

SECTION I Provide the final answer. No justification necessary.

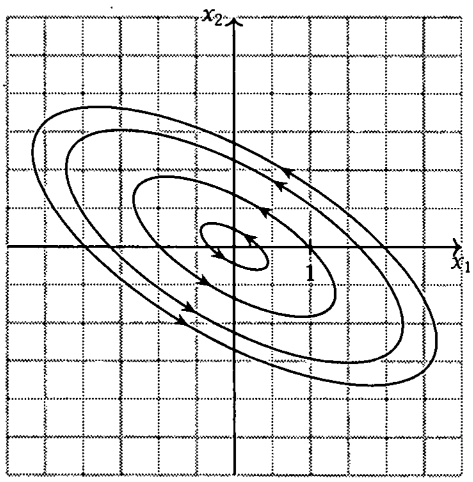
1. **(2 marks)** Consider the initial value problem:

$$\begin{cases} y'(t) = e^{-y^2} y(y-1)(y-10) \\ y(0) = \beta \end{cases}$$

$(a, b) =$ $\boxed{}$

Determine an interval $I = (a, b)$ such that if $\beta \in (a, b)$ then $\lim_{t\to\infty} y(t) = 1$.

2. **(2 marks)** The phase portrait of the system $\frac{d\vec{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}(t)$ is given below. Is $a < 0$ or $a > 0$ or is there not enough information (N.E.I.) to conclude anything?



$\boxed{}$

3. **(2 marks)** You are asked to model the population of squirrels on UofT campus. We have witnessed that once there are over 5,000 squirrels, their population decreases due to territorial disputes caused by severe space limitations at UofT. Furthermore, it is estimated that if the population falls below 10, the harsh winter conditions will not allow the squirrels to survive anymore. Propose a first order autonomous ODE that models the population that captures these three facts. You may include a parameter in your answer.

$\boxed{}$

SECTION II

For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

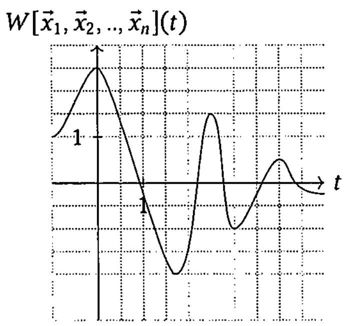
4. **(2 marks)** The initial value problem

$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$$

has a unique solution.

Choose true or false, then justify: ○ TRUE ○ FALSE

5. **(2 marks)** There exists a linear system $\frac{d\vec{x}}{dt} = A\vec{x}$ such that the graph of the Wronskian of its solutions looks like:



Choose true or false, then justify: ○ TRUE ○ FALSE

6. **(2 marks)** You can apply the Laplace Transform on any ODE to get an expression for the solution in the s-domain

Choose true or false, then justify: ○ TRUE ○ FALSE

7. **(2 marks)** Consider the ODEs:

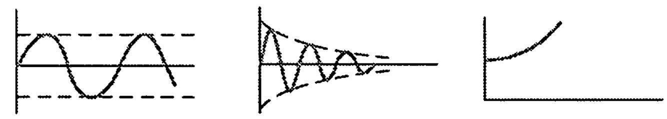
$$a_1 y'' + b_1 y' + c_1 y = g(t)$$

$$a_2 y'' + b_2 y' + c_2 y = g(t)$$

Assuming the ODEs have equivalent impulse responses, then for any forcing function they must have equivalent forced responses.

Choose true or false, then justify: ◯ TRUE ◯ FALSE

8. **(3 marks)** Recall that Re(s) and Im(s) stand for the real and imaginary part of a complex number s. The Laplace transform of each of the following functions has singularities (points in which the function diverges). Denoting by s such a singularity, explain what do we know about Re(s) and Im(s).



Left:

Middle:

Right:

SECTION III

Justify all your answers, unless it **specifically** says that you do not need to justify.

9. Consider the following linear system:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

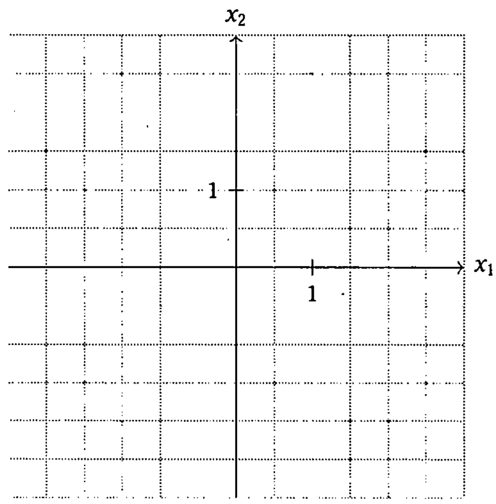
(a) **(2 marks)** Find the general solution to the system for the case where $b_1 = b_2 = 0$.

$y(t) =$

(b) **(2 marks)** Let $\vec{x}_1$ and $\vec{x}_2$ be two linearly independent solutions you found in part a). Compute their Wronskian $W[\vec{x}_1, \vec{x}_2](t)$.

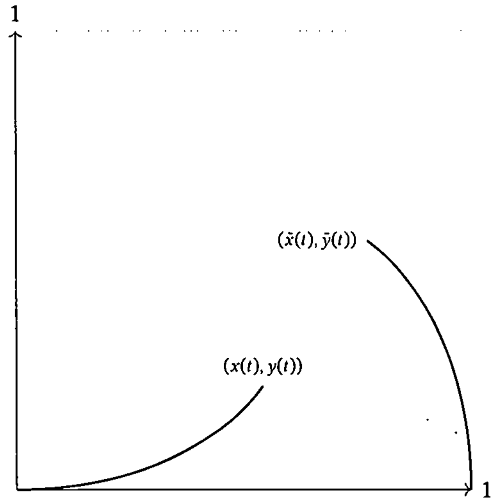
$W[\vec{x}_1, \vec{x}_2](t) =$

(c) (3 marks) Find the general solution to the system for the case where $b_1 = 6$ and $b_2 = 2$ and sketch the phase portrait. Find an expression for the equilibrium solutions.



(d) (3 marks) Find the general solution to the system for the case where $b_1 = 1$ and $b_2 = 3$. Does the system admit an equilibrium solution? If yes, find an expression for the equilibrium solutions.

10. Four kittens are located in the four corners of a square room with a side length of one meter. The bottom left corner is the point $(0,0)$. Each kitten runs towards the kitten to its right and all kittens are running at the same speed.



(a) **(3 marks)** Denote by $(x(t), y(t))$ the coordinates of the kitten starting in the bottom left at time t. What are the coordinates of the kitten starting in the bottom right at time t, i.e., $(\tilde{x}(t), \tilde{y}(t))$? **Hint**: symmetry in image above. **DO NOT ATTEMPT TO FIND AN EQUATION FOR THESE PATHS.**

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} =$$

(b) **(3 marks)** **Explain in words** why the following ODE models the trajectory of the bottom left kitten.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - x - y$$

(c) **(2 marks)** Find the Laplace transform of each of the ODEs above.

(d) **(2 marks)** Solve for $X(s)$ and $Y(s)$ (solutions should be in the s domain).

(e) **(2 marks)** Show that the solution of the system of ODEs is

$$y(t) = \frac{1}{2}\left[1 - e^{-t}\cos t - e^{-t}\sin t)\right]$$
$$x(t) = \frac{1}{2}\left[1 - e^{-t}\cos t + e^{-t}\sin t)\right]$$

(f) **(1 mark)** What can be said about the position of all four kittens after a very long time?

11. Consider the following second order initial value problem:

$$\begin{cases} ty'' - ty' + y = g(t) \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases}$$

(a) **(2 points)** Apply the Laplace Transform to this IVP and find an ODE that $\mathcal{L}\{y\}(s) = Y(s)$ must satisfy. You may express this ODE in terms of $\mathcal{L}\{g\}(s) = G(s)$

(b) For the remainder of this problem, assume that $y_0 = 3$, $y_1 = 2$ and $g(t) = 3$.

i. **(2 points)** Solve part a) to get an explicit expression for $Y(s)$.

$Y(s) =$

ii. **(2 points)** Find an explicit expression for the solution in the t-domain. Your final solution should have no undefined constants.

$y(t) =$

12. (a) **(2 marks)** Consider the ODE:

$$x' = Bx$$

where

$$B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}.$$

Let t be some non-negative scalar. Compute the even and odd powers of tB:

For $k = 0, 1, 2, \ldots$

$(tB)^{2k+1} =$

$(tB)^{2k} =$

(b) **(2 marks)** Recall the Taylor series $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ and $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$. Using your answer to the previous item, find e^{Bt}.

$e^{Bt} =$

(c) **(3 marks)** Consider the alternative ODE:

$$x' = Ax$$

where

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Find e^{At} using the properties of matrix exponentials. Mention explicitly at each step what property of matrix exponentials you are invoking.

$e^{At} =$

(d) **(3 marks)** Suppose A has a repeated eigenvalue λ and assume $(A - \lambda I)^k = 0$ for $k \geq 2$. Show that

$$e^{At} = e^{\lambda t}(I + (A - \lambda I)t)$$

13. An engineering science student decided to construct from scratch a Positron Emission Tomography (PET) system as a summer project. The student is currently testing a key component of this system: a high energy photon detector. To test the system, the student "shoots" photons at the photon detector and measures its output. Each photon *shocks* the photon detector for an *instantaneous* period of time. The student first shot a single photon at the system and obtained an exponential decay output:

$$\begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The student then shot two photons at the system and noticed that the output is doubled. The student then became more adventurous and shot at times $t = 1, 2, 3, \ldots, n$ a number of photons equal to $c_1, c_2, \ldots, c_n$, respectively. The engineering science student, who previously learned MAT292, decided to model the input-output relation of the photon detector through the following ODE:

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0, \qquad y'(0) = y_1.$$

The following questions concern the very last experiment in which the student shot many photons.

(a) **(2 marks)** What is the forcing function of the photon detector?

(b) **(2 marks)** What is the Laplace transform of the forcing function of the photon detector?

(c) **(3 marks)** What is the Laplace transform of the forcing function assuming $c_1 = c_2 = \cdots = c_n = 1$ and $n \to \infty$? Your answer should not contain an infinite sum.

(d) **(2 marks)** What is the impulse response of the photon detector? The solution should not depend on a, b, c.

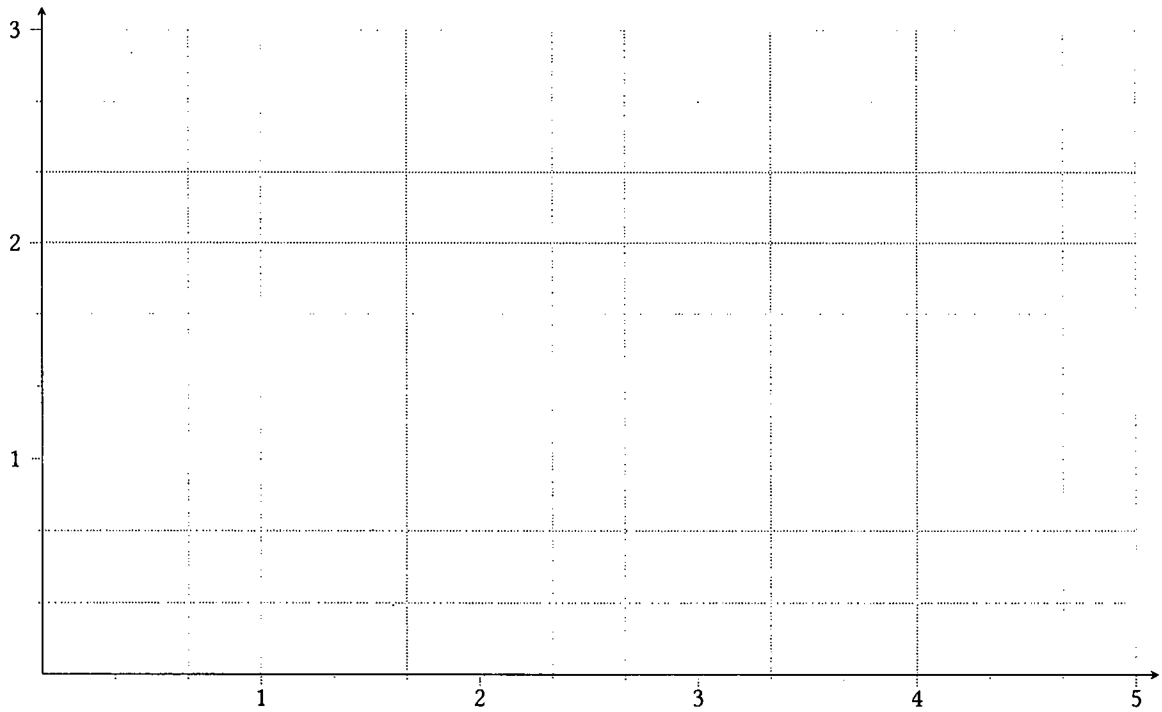
(e) **(2 marks)** What is the transfer function of the photon detector? The solution should not depend on a, b, c.

(f) **(2 marks)** Using the two previous items, find the constants a, b, c in the ODE.

(g) **(2 marks)** What is the free/transient response of the system? Simplify the answer as much as you can.

(h) **(2 marks)** What is the forced response of the system? Simplify the answer as much as you can.

(i) **(4 marks)** Draw a sketch of the forced response assuming $n = 4$ and $c_1 = c_2 = \cdots = c_4 = 1$. Use the approximation $1/e \approx \frac{1}{3}$.



14. Consider the 2D Laplace Equation with boundary conditions:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 \le x \le L, \quad 0 \le y \le H \\ u(x,0) = 0 = u(x,H) & \\ u(0,y) = f(y), \quad u(L,y) = 0 & \end{cases}$$

Assume the solution factorizes as follows: $u(x,y) = X(x)Y(y)$.

(a) **(2 marks)** Using the PDE, fill in the three blanks.

$$\frac{X''(x)}{\boxed{}} = -\frac{\boxed{}}{\boxed{}} = \lambda \qquad \text{for some constant } \lambda > 0$$

(b) **(2 marks)** Find two ODEs, as well as one boundary condition for $X(x)$ and two boundary conditions for $Y(y)$. These three boundary conditions should be separable (i.e. only depend on X or only depend on Y).

$X''(x) = \boxed{}$ boundary condition: $\boxed{}$

$Y''(y) = \boxed{}$ boundary conditions: $\boxed{}$ $\boxed{}$

(c) **(2 marks)** Show that the solutions for $Y(y)$ and λ must be of the form

$$Y_n(y) = c_1 \sin\left(\frac{n\pi}{H}y\right) \qquad \lambda_n = \frac{n^2\pi^2}{H^2} \qquad \text{where } n = 1, 2, \ldots$$

(d) **(2 marks)** Verify that for $n = 1, 2, \ldots$

$$X_n(x) = \sinh\left(\frac{n\pi(x-L)}{H}\right) = \frac{e^{n\pi(x-L)/H} - e^{-n\pi(x-L)/H}}{2}$$

is a solution to the following ODE:

$$\begin{cases} X'' - \frac{n^2\pi^2}{H^2}X = 0 \\ X(L) = 0 \end{cases}$$

(e) **(2 marks)** Briefly justify why

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi}{H}y\right)$$

solves the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 \le x \le L, \quad 0 \le y \le H \\ u(x,0) = 0 = u(x,H) \\ u(L,y) = 0 \end{cases}$$

(f) **(3 marks)** Using Fourier Series, determine a formula for $c_n \in \mathbb{R}$ such that

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi}{H}y\right)$$

and

$$u(0,y) = f(y).$$

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TABLE 5.3.1 Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
16.	$\delta(t-c)$	e^{-cs}
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18.	$t^nf(t)$	$(-1)^nF^{(n)}(s)$