Last name: First name: ID number:

ECE 286
Midterm exam

March 23, 2023
9:30 – 10:30 am

Circle your lecture section:

LEC0101 (Monday 9-10)

LEC0102 (Monday 1-2)

Guidelines:

- Write your answer in the space provided for each question. Show your work and use the back side of sheets as needed.
- The exam is Type D. You may use a non-programmable calculator and a one-sided, 8.5 by 11 handwritten note sheet (prepared by you).
- Only exams written in pen will be considered for regrades.

Problem	Score
1	/8
2	/5
3	/5
Total	/18

1. X is a continuous random variable with PDF

$$f(x) = \begin{cases} g(x) & \text{if } 0 \le x \le 1 \\ 1/2 & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise,} \end{cases}$$

where $g(x) > 0$ is some function. Let

$$G(x) = \int_0^x g(t)dt.$$

Answers parts (a)-(d) below. Justify your answers.

(a) (1 point) What is $G(1)$?

Solution: because $f(x)$ is a PDF, we must have $G(1) = 1/2$.

(b) (2 points) Find the CDF, $F(x)$.

Solution: It is

$$F(x) = \begin{cases} 0 & \textit{if } x < 0 \\ G(x) & \textit{if } 0 \leq x \leq 1 \\ 1/2 + (x-1)/2 & \textit{if } 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$

$$= \begin{cases} 0 & \textit{if } x < 0 \\ G(x) & \textit{if } 0 \leq x \leq 1 \\ x/2 & \textit{if } 1 < x \leq 2 \\ 1 & x > 2. \end{cases}$$

(c) (2 points) Let $Y = 3X$. Find the PDF of Y.

Solution: denote PDF of Y $h(y)$. *If* $u(X) = 3X$, *then* $u^{-1}(Y) = Y/3$. *We know that*

$$\begin{aligned} h(y) &= f(u^{-1}(y))\frac{du^{-1}(y)}{dy} \\ &= f(y/3)/3 \\ &= \begin{cases} g(y/3)/3 & \textit{if } 0 \le y \le 3 \\ 1/6 & \textit{if } 3 < y \le 6 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

(d) (3 points) Let μ be the mean of X. Find the minimum and maximum possible values for μ. (Hint: We cannot know μ exactly without knowing $g(x)$, but we do know that $f(x)$ is a PDF.)

Solution: We obtain the minimum by setting $g(x) = 0.5\delta(x)$*, the Dirac* δ*. Then the expectation is*

$$\begin{aligned}
E[X] &= \int_0^2 xf(x) \\
&= \int_0^1 0.5\delta(x)xdx + \int_1^2 x/2dx \\
&= 0.5 \cdot 0 + x^2/4\Big|_1^2 \\
&= 2^2/4 - 1^2/4 \\
&= 1 - 1/4 \\
&= 3/4.
\end{aligned}$$

We obtain the maximum by setting $g(x) = 0.5\delta(x-1)$*. Then*

$$\begin{aligned}
E[X] &= \int_0^2 xf(x) \\
&= \int_0^1 0.5\delta(x-1)xdx + 3/4 \\
&= 0.5 \cdot 1 + 3/4 \\
&= 5/4.
\end{aligned}$$

The minimum and maximum possible means are $3/4$ *and* $5/4$*.*

2. A sample consists of observations $x_1 = 2$ and $x_2 = 4$. The population variance is $\sigma^2 = 1$. Answers parts (a) and (b) below. Justify your answers.

   (a) (3 points) Compute the two-sided 90% confidence interval.

   Solution: The sample mean is $\bar{x} = 3$. Because we know the variance, we can use the CLT. For $\alpha = 0.1$, $z_{\alpha/2} = 1.65$. The CI is thus:

$$\begin{aligned}[\bar{x} - z_{\alpha/2} \cdot \sigma/\sqrt{n}, \bar{x} + z_{\alpha/2} \cdot \sigma/\sqrt{n}] &= [3 - 1.65 \cdot 1/\sqrt{2}, 3 + 1.65 \cdot 1/\sqrt{2}] \\ &= [1.83, 4.17].\end{aligned}$$

(b) (2 points) Compute a one-sided 90% upper confidence interval. (It should be of the form $(\infty, x_U]$.)

Solution: For $\alpha = 0.1$, $z_\alpha = 1.28$ *The upper limit of the CI is* $x_U = 3+1.28 \cdot 1/\sqrt{2} = 3.905$.

3. The distribution of a population is

$$f(x) = \begin{cases} 1/5 & \text{if } 0 \le x \le 5 \\ 0 & \text{otherwise.} \end{cases}$$

We take a sample from the population of size $n = 2$. The sample mean (a random variable) is

$$\bar{X} = \frac{1}{2}(X_1 + X_2).$$

Answer parts (a) and (b) below. Justify your answers.

(a) (3 points) Find the exact value of x_L such that 10% of realizations of $\bar{X}$ are less than x_L. Your answer to part (a) should not involve the Central Limit Theorem or the normal distribution.

Solution: We must first find the distribution of $\bar{X}$. The distribution of $X_1 + X_2$ is

$$\begin{aligned} g(y) &= \int_{-\infty}^{\infty} f(x) f(y - x) dx \\ &= \frac{1}{5} \int_0^5 f(y - x) dx \\ &= \begin{cases} \frac{1}{25} \int_0^y dx & \textit{if } 0 \le y \le 5 \\ \frac{1}{25} \int_{y-5}^5 dx & \textit{if } 5 \le y \le 10 \\ 0 & \textit{otherwise.} \end{cases} \\ &= \begin{cases} y/25 & \textit{if } 0 \le y \le 5 \\ (10 - y)/25 & \textit{if } 5 \le y \le 10 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

The distribution of $\bar{X}$ is

$$\begin{aligned} h(z) &= 2g(2z) \\ &= \begin{cases} 4z/25 & \textit{if } 0 \le 2z \le 5 \\ 2(10 - 2z)/25 & \textit{if } 5 \le 2z \le 10 \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

We want to find the point x_L such that

$$\int_0^{x_L} h(z) dz = 0.1.$$

We have

$$\begin{aligned} \int_0^{x_L} h(z) dz &= \frac{4}{25} \int_0^{x_L} z dz \\ &= \frac{2x_L^2}{25} \\ &= 0.1. \end{aligned}$$

Solving, we have that $x_L = \sqrt{5/4} = \sqrt{5}/2 \approx 1.118$.

(b) (2 points) Consider the sample $X_1, ..., X_n$. The sample mean is

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Let

$$Y = \sqrt{n}\left(\bar{X} - 2.5\right).$$

Find the limiting distribution of Y as $n \to \infty$. (Hint: the variance of a uniform distribution over $[a, b]$ is $\sigma^2 = (b-a)^2/12$.)

Solution: The mean of the population is 2.5. The variance is 25/12. The CLT says the distribution of the mean approaches the normal

$$n(y; 0, 25/12) = \frac{\sqrt{12}}{5\sqrt{2\pi}} e^{-\frac{12y^2}{50}}.$$