UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

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PHY293F (Waves and Modern Physics Solutions)
Instructor: Professors N. Grisouard and W. Trischuk

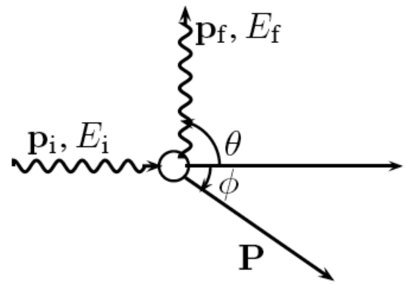
Duration: 2.5 hours

Exam Type C: Non-programmable calculators.

This examination paper consists of **5** pages and 6 questions. Please bring any discrepancy to the attention of an invigilator. Answer all 6 questions.

- Write your name, student number and tutorial group on top of **all** examination booklets and test pages used.
- Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. Not other aid (notes, textbook, dictionary) is allowed. **Communication devices are strictly forbidden. Turn them off and make sure they are in plain sight to the invigilators.**
- Each question is worth 1/6 of your overall grade for this exam. Within each question, a mark breakdown is indicated in square brackets at the beginning of each sub-part. Part marks will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion** any relevant assumptions you are making along the way.
- Do not separate the stapled sheets of the question paper. Hand in the questions with your exam booklet at the end of the test.
- The next two pages include some formulae and constants you may find useful.
- The questions begin on **page 4**. The total number of marks is 60.

1. In a particle physics experiment, a photon with energy E_i collides with an unknown particle that is at rest, as shown in the diagram, opposite. The photon scatters at $90°$ to its original direction retaining 1/4 of its original energy ($E_f = E_i/4$). After the collision, the massive particle has momentum P directed at an angle ϕ relative to the incident photon's momentum direction.



   (a) Derive the relationship between the rest mass of the unknown particle in terms of E_i. We know $E_i = p_i c$ and $E_f = p_f c$ for the incoming and outgoing photon. This is an example of Compton scattering of a photon, not off an electron in this case, but some other unknown charged particle. We can still apply the Compton scattering formula from the formulas given to get:

$$\lambda_f - \lambda_i = \frac{h}{p_f} - \frac{h}{p_i} = \frac{h}{mc}(1 - \cos\theta).$$

   Here θ is the angle of the outgoing photon, 90 degress in this case, so $\cos\theta = 0$. Substituting the photon energies for momenta (and putting in the requisite factor of c) we get:

$$\frac{hc}{E_f} - \frac{hc}{E_i} = \frac{h}{mc} \quad \Rightarrow \quad \frac{1}{E_f} - \frac{1}{E_i} = \frac{1}{mc^2}.$$

   Given that $E_f = 1/4 E_i$ this reduces to $E_i = 3mc^2$ or $m = E_i/(3c^2)$ [3]

   (b) Calculate the angle, ϕ, of the outgoing massive particle after the collision. This is straighforward kinematics: for the massive particle we have $p_\parallel = P\cos(\phi)$, $p_\perp = P\sin(\phi)$ but to conserve momentum $p_\parallel = p_i$ (the incoming photon momentum) and $p_\perp = p_f$ (the outgoing photon momentum). So we can take the ratio $\frac{p_f}{p_i} = \tan(\phi)$ where we can replace the photon momenta by energies (the cs cancel) to get $\frac{E_f}{E_i} = \tan\phi$ or $\tan(\phi) = 1/4$ and $\phi = 14^o$ or 0.245 radians [3]

   (c) If the incident photon energy is 1.5 GeV, determine the rest mass of the target particle. Given the result in part (a), if $E_i = 1.5$ GeV then $m = 0.5$ GeV/c^2 [1].

   (d) As measured in the lab frame, the target particle decays 1.7 ps after the collision. How long after the collision does the decay take place in the particle's rest frame? The momentum of the outgoing particle is $P = p_i/\cos(\theta) = 1.55$ GeV/c. From the formula sheet $pc = \gamma\beta m$ so $\gamma\beta = 1.55/0.5 = 3.10$. Now we just have to fiddle with the defintions of γ and β to determine the speed of the outgoing particle. Since $\gamma = 1/\sqrt{1-\beta^2}$ we can derive $\gamma = \sqrt{1 + 9.6} = 3.25$. So the outgoing mystery particle has a boost of more than 3. If the decay in the lab is measured to take 1.7 ps, then time dilation tells us the particle will decay more slowly in the moving (lab) frame than the particle (rest) frame. So in the rest frame the time interval will be $\Delta t' = \Delta t/\gamma$. This formula was not given on the test sheet, but it, along with length contraction should be very familiar to you. So the time interval in the particle rest frame is $\Delta t' = 1.7\ \mathrm{p}s/3.25 = 0.52$ ps. [3].

2. We saw in class that Heisenberg's uncertainty principle can be used to give a reasonable prediction for the size of the hydrogen atom: about 10^{-10} m.

   (a) Without resorting to the specific form of the potential energy of electrons in hydrogen atom use the Heisenberg uncertainty principle to estimate the momentum of the electron when confined to a region of this size. From the formula sheet the Heisenberg uncertainty principle says $\Delta x \Delta p$ can be no smaller than $\hbar/2$. Confining the electron to a region around the proton in a hydrogen atom of size 0.1 fm leads to a momentum uncertainty of $\Delta p = \hbar/(2\Delta x) = \frac{6.63\times10^{-34}/(2\pi)}{2\times10^{-10}} = 5.27 \times 10^{-25}$ kg m/s. While strictly speaking this is the uncertainty on the electron momentum, as we discussed in class, this means the typical electron momentum is going to be the same order of magnitude (not 10x smaller, and not 10x larger) so we take this as our estimate of the electron momentum. [2]. Some students took 10^{-10} m to be the diameter of the atom and then rigorously used the radius 0.5×10^{-10} m as Δx that was ok too.

   (b) Is such an electron likely to be relativistic? As a rough guide, you can use $v \approx 0.1c$ as the boundary between the non-relativistic and relativistic regimes. While we discused the non-relativistice nature of the electron in the hydrogen atom several times in class, you needed to provide some quatitative evidence to get full credit here. The simplest is just to test the assumption that the non-relativistic momentum formula leads to a non-relativsitic speed: $p = mv$ leads to $v = p/m = 0.57 \times 10^6$ m/s or 0.002 c. Non-relativistic OK. [2].

   (c) In higher charge atoms, like lead, the inner-most electrons are confined to a much smaller region of about 10^{-12} m. Are these electrons relativistic ? If the electron is confined to a smaller region, in a lead atom, the Heisenberg uncertainty principle will lead to a larger momentum uncertainty and hence a larger, typical, electron momentum. That turns out to be $p_e^{lead} \approx 5.27 \times 10^{-23}$ kg m/s. If we try to use the non-relativistic momentum formula now we conclude $v_e^{Lead} = 0.57 \times 10^8$ m/s (0.2 c). This is above the border suggested by the problem so these electrons border on relativistic. But you didn't need to go all the way through to determine $\gamma \approx 1.8$ to convince me that the electron was relativistic in this case to get full credit. I hope the relatively low weight for this question would have been enough of a clue not to dive into the full relativistic calculation [2].

   (d) If an electron, in the hydrogen atom, finds itself in the first excited state ($n = 2$) it will decay back to the ground state ($n = 1$).

      i. What is the energy that will be released in this transition ? [2] From the formula sheet $E_n = -13.56/n^2$ for the hydrogen atom. So $E_2 - E_1 = 13.56 - 13.56/4 = 10.2$ eV. [2]. This is 1.63 $\times 10^{-18}$ J, if the students answered in those units

      ii. If this decay has a lifetime of 2 ns, what limit does the Heisenberg uncertainty principle place on the precision with which we can measure that energy? The energy time version of the Heisenberg uncertainty principle, given in the formula page and discussed (briefly) in class, constrains the precision we can have on an energy measurement (ΔE) given the amount of time we have available to measure the transition. Since the $n = 2$ state decays down to the hydrogen ground state in 2 ns, we only have 2 ns to measure the intial energy. Taking this as Δt (again we might only measure it for half the lifetime, or 1/3 of the lifetime, but we can measure it for 10x the lifetime because it will have decayed so we take the lifetime itself as the typical time unceratinty) we find: $\Delta E \approx \frac{\hbar}{2\Delta t} = \frac{4.14\times10^{-15}}{2\times2\times10^{-9}} = 1.03\times 10^{-6}$ eV (or 1.7×10^{-25} J). So the Heinsenberg limit on the uncertainty of the first excited state of hydrogen is really pretty small (about one part in a million of the actual energy level which is a few eV). [2].

3. Suppose the wavefunction for a particle, confined to lie between $0 < x < 1$ is given by $\psi(x) = A(x - x^2)$. Outside the allowed region the wave-function vanishes.

   (a) Sketch what the wave-function looks like, making sure to indicate clearly how it obeys the continuity conditions required by the Schrodinger equation. Ideally the students will have drawn a straight line (the x term in the wave-function) and then shown how the x^2 term 'subtracts away' from the linear term in a parabolic fashion. But I guess anything that peaks at 1/2 (that is where the derivative of this function goes to 0) and passes through 0 at $x = 0$ and $x = 1$ (the edges of the well) is good enough to get 1.5/2 for this part of the question. To get the last half point they **must** indicate atht he wave-function is 0 below $x = 0$ and above $x = 1$. It's written right in the question. But I wouldn't be surprised if many of them don't draw this part of the wavefunction. [2]

   (b) Find the probability to find the particle in the right half ($\frac{1}{2} < x < 1$) of the well. To find this, we need to integrate the wave-function^2 from $x = 1/2$ to $x = 1$. I expected you to be able to integrate polynomials without including them on the equation sheet. In fact, first you need to determine the normalisation constant: $A^2\int_0^1 |\Psi|^2 dx = 1$ gives $A^2(x^3/3 - x^4/2 + x^5/5)|_0^1 = 1$ or $A^2 = 30$. Then do the integral for only the right half of the well. It's the same integral so get: $A^2(x^3/3 - x^4/2 + x^5/5)|_{0.5}^1 = 30[(1/3 - 1/2 + 1/5) - (1/24 - 1/32 + 1/160)] = 1/2$ when you work it out. Only one point awarded for just 'guessing' the answer was 1/2. One point for setting up the proper intergrals. But you only lost 1 point if you didn't find your way through the limits of integration. [3]

   (c) What is the expectation value of x? [2] More integration. One point for setting up the integral to be done: $A^2\int_0^1 |\Psi|^2 x dx = 30\int_0^1 |\Psi|^2 x dx = 30[1/60] = 1/2$ again. Again, only lost one point if you didn't manage to complete the integration, even if you pointed out the physical conclusion that $< x >= 1/2$ **had to be** given the shape of the wavefunction. [2].

   (d) What is the expectation value of x^2? [2] One last calculation: $A^2\int_0^1 |\Psi|^2 x^2 dx = 30\int_0^1 |\Psi|^2 x^2 dx = 30[1/105] = 0.286$. There was no physical expectation (at least I didn't have one) for this number. I hope it is right. If the majority of the class disagrees with me, please let me know and/or double check their answers ... and then I'll change this solution before posting it for the students. [2]

   (e) Compute the uncertainty on x: $\Delta x = \sqrt{< x^2 > - < x >^2}$ [1] This was almost a free point. Even if you got one of the two answers above wrong (I hope not (c)), you would get full credit for putting your wrong answer there into the correct formula here and getting a consistent 'wrong answer' here. For the record the right answer is $\Delta x = \sqrt{0.286 - 0.5^5} = 0.19$. [1]

End of examination
Total pages: 5