Duration: 150 minutes **Saturday April 22, 2022**

Faculty of Applied Science & Engineering
University of Toronto

MAT185 Linear Algebra
Final Exam

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Instructions:

1. This test contains a total of 11 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this exam, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
8. You must use the methods learned in this course to solve all of the problems.
9. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. Let V be a five-dimensional vector space. What are the possible dimensions of the intersection of any two three-dimensional subspaces of V? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ 0.
- ○ 1.
- ○ 2.
- ○ 3.
- ○ 4.

2. Consider the linear transformations $S : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $S(p(x)) = p(x+1)$, and $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ defined by $T(p(x)) = (p(0), p'(0))$. Which of the following statements is true? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ Both S and T are injective.
- ○ T is injective but not surjective.
- ○ Both S and T are surjective.
- ○ S is surjective but not injective.
- ○ Both S and T are bijective.

Multiple Choice: No justification is required. Only your final answer will be graded.

3. If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = 3$, then $\det \begin{bmatrix} 4a-2c & 6b & 2c \\ 2d-f & 3e & f \\ 2g-k & 3h & k \end{bmatrix} =$ _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ 3.

○ 6.

○ 12.

○ 24.

○ 36.

4. Let A be a 4×4 matrix with $\det A = 2$. Let $a = \det(A^{-1} + \operatorname{adj} A)$. Then $a =$ _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

○ 16.

○ $\frac{81}{2}$.

○ $\frac{9}{2}$.

○ $\frac{5}{2}$.

○ $\frac{27}{2}$.

Multiple Choice: No justification is required. Only your final answer will be graded.

5. If $A = \begin{bmatrix} 4 & -3 & 1 \\ -1 & 2 & -2 \\ -6 & 6 & -4 \end{bmatrix}$ is similar to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then the first column of an invertible matrix S such that $A = SDS^{-1}$ is _____? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
- ○ $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.
- ○ $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.
- ○ $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.
- ○ $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

6. The matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 2 \\ -1 & 0 & 4 \end{bmatrix}$ has $\lambda = 3$ as an eigenvalue. Which of the following statements are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $E_3(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.
- ○ The algebraic multiplicity of $\lambda = 3$ is 2.
- ○ $E_3(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.
- ○ $\lambda = 3$ is the only eigenvalue of A.
- ○ A is not diagonalizable.

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●). Each part is worth 3 marks: 1 mark for a correct final answer; 2 marks for a correct justification.

7(a) If $A \in {}^n\mathbb{R}^n$ is non-zero, then the set of all $n \times n$ matrices that are similar to A is a subspace of ${}^n\mathbb{R}$.

○ True.

○ False.

7(b) Let $A, B, C \in {}^n\mathbb{R}^n$. Suppose C is invertible and $C = AB$. Then the list (i.e. ordered set) of columns of A, B and C are each bases for ${}^n\mathbb{R}$ (call them α, β, and γ respectively), and B is the change of basis matrix (i.e. matrix of transition) such that $[\mathbf{x}]_\alpha = B\,[\mathbf{x}]_\gamma$ for every $\mathbf{x} \in {}^n\mathbb{R}$.

○ True.

○ False.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

8. Consider the basis $\alpha = 1 + x, 2x, 1 - x^2$ for $P_2(\mathbb{R})$, and let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the derivative transformation. That is $T(p(x)) = \dfrac{d}{dx}(p(x))$ for every $p(x) \in P_2(\mathbb{R})$.

(a) Determine $[x^2]_\alpha$, the coordinate vector of x^2 with respect to α, and $[T]_{\alpha\alpha}$, the matrix of T with respect to α. [4 marks]

(b) Use your answer from part (a) to compute $\dfrac{d}{dx}(x^2)$. [2 marks]

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

9. Let $A \in {}^n\mathbb{R}^n$.

(a) Define what it means for a vector $\mathbf{x} \in {}^n\mathbb{R}$ to be an eigenvector of A. Be sure to give a precise statement. No partial credit will be given for statements that are "close" to the definition. [2 marks]

(b) Prove that if $\mathbf{x}_1, \mathbf{x}_2$ are eigenvectors of A corresponding to unequal eigenvalues λ_1, λ_2 respectively, then the vector $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is never an eigenvector of A for any choices of non-zero $c_1, c_2 \in \mathbb{R}$ [4 marks]

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

10. Let $A \in {}^n\mathbb{R}^n$.

(a) Define what it means for A to be diagonalizable. Be sure to give a precise statement. No partial credit will be given for statements that are "close" to the definition. [2 marks]

(b) Let A be diagonalizable with only two unequal (i.e. distinct) eigenvalues λ_1 and λ_2. Prove that $\mathrm{col}\,(\lambda_1 I - A) = E_{\lambda_2}(A)$. [4 marks]

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

11. Let A be an 3×3 matrix and consider the system

$$\mathbf{x}(t+1) = A\mathbf{x}(t), \quad \mathbf{x}(1) = \mathbf{b}$$

for $t = 1, 2, 3, \ldots$.

(You may think of $\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3) \ldots$, as a sequence of measured values of some quantity $\mathbf{x}$ every hour, say. The system $\mathbf{x}(t+1) = A\mathbf{x}(t)$ relates the value of $\mathbf{x}$ at each hour to the value at the next hour).

Suppose A is diagonalizable, and let $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ be eigenvectors of A with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively. Let S be the 3×3 matrix

$$S = \begin{bmatrix} \mathbf{s}_1 & \mathbf{s}_2 & \mathbf{s}_3 \end{bmatrix}$$

Show that for $t = 1, 2, 3, \ldots$, the general solution $\mathbf{x}(t)$ to the system may be expressed as

$$\mathbf{x}(t) = c_1\lambda_1^{t-1}\mathbf{s}_1 + c_2\lambda_2^{t-1}\mathbf{s}_2 + c_3\lambda_3^{t-1}\mathbf{s}_3$$

where c_1, c_2, c_3 are the solution to $S \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{b}$. [4 marks]

IF NEEDED, USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

If you wish to have this page marked, make sure to refer to it in your original solution.

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