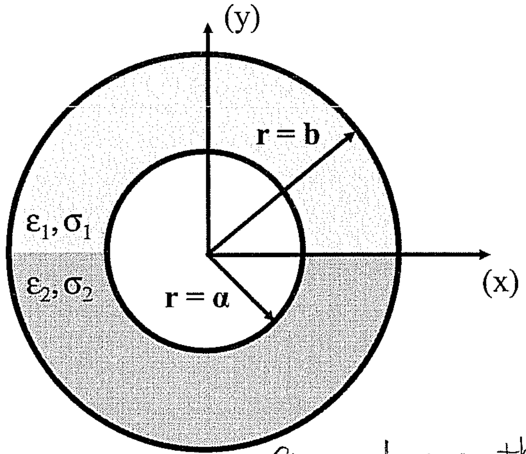
Question 1

A lossy, coaxial cable consists of two cylindrical conductors at $r = \alpha$ and $r = b$. The region between the conductors consists of two regions, filled with lossy dielectrics ϵ_1, σ_1 $(0 \le \phi < \pi)$ and ϵ_2, σ_2 $(\pi \le \phi < 2\pi)$, respectively. The electric potential of the inner cylinder is $V(r = \alpha) = V_0$ and the potential of the outer cylinder is $V(r = b) = 0$.

1. Show that the electric potential in region $\alpha < r < b$ is given by:

$$V(r) = V_0 \frac{\ln \frac{r}{b}}{\ln \frac{\alpha}{b}}$$

and find the electric field $\mathbf{E}$ in the same region. Confirm that the electric field satisfies the boundary condition at the interface between the two lossy dielectrics. **(12 pts)**



Sln #1 : Invoke uniqueness (i.e. show that V satisfies Poisson equation + boundary conditions)

$\nabla^2 V = 0 \Rightarrow \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$ [1pt]

$\frac{dV}{dr} = \frac{V_0}{(\ln\frac{b}{\alpha})\,r} \Rightarrow r\frac{dV}{dr} = \frac{V_0}{\ln\frac{b}{\alpha}} \Rightarrow \frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$ ✓ [3pts]

$V(r=\alpha) = V_0$, $V(r=b) = V_0 \frac{\ln\frac{b}{b}}{\ln\frac{\alpha}{b}} = 0$ ✓ [2pts] (1pt each)

$\bar{E} = -\operatorname{grad} V = \dots = \frac{V_0}{(\ln\frac{b}{\alpha})\,r}\,\bar{a}_r$ [2pts]

$\bar{E}$ tangential to interface [2pts] $\Rightarrow$ needs to be continuous [1pt] and it is continuous as it

does NOT depend on φ [1pt]

Sln # 2: Find $\bar{E}$ $\Rightarrow$ Find V

confirm that it agrees with the expression that is given.

[1pt]

Cylindrical symmetry $\Rightarrow$ $\bar{E} = E_r(r)\,\bar{a}_r$ $\Rightarrow$

$\bar{D}_1 = \epsilon_1 E_r(r)\,\bar{a}_r$, $\bar{D}_2 = \epsilon_2 E_r(r)\,\bar{a}_r$ [2pts]

{ E_r same in two regions due to continuity of tangential $\bar{E}$ at interfaces. [4pts] }

Gauss Law on a cylinder of length L, radius $\alpha < r < b$: [1pts]

$$\oint \bar{D}\cdot d\bar{s} = Q_{enclosed} \Rightarrow \int_{z=0}^{L}\int_{\varphi=0}^{\pi} \epsilon_1 E_r\, r\, d\varphi\, dz + \int_{z=0}^{L}\int_{\varphi=\pi}^{2\pi} \epsilon_2 E_r\, r\, d\varphi\, dz$$

$$= Q_{enclosed} \Rightarrow r.\,E_r\,(\pi\epsilon_1 + \pi\epsilon_2).\,L = Q_{enclosed}$$

$$\Rightarrow \boxed{E_r = \frac{Q_{enclosed}}{\pi r.L\,(\epsilon_1+\epsilon_2)}} \quad \text{or} \quad \boxed{E_r = \frac{C}{r}}, \quad C = \frac{Q_{enclosed}}{\pi\cdot r\cdot L}$$

[2pt]

To find this constant, use $V(b) - V(\alpha) = V_0$ = constant.

$$\Rightarrow \int_a^b \frac{C}{r}\,\bar{a}_r\cdot\bar{a}_r\,dr = V_0 \Rightarrow C = V_0/\ln\frac{b}{\alpha} \quad [1pt]$$

Then,

$$\underbrace{V(r) - \underset{0}{V(r=b)}}_{[2]} = -\int_b^r \bar{E}\cdot d\bar{l} = \int_r^b \frac{V_0}{(\ln\frac{b}{\alpha})r}\,dr = \frac{V_0}{\ln\frac{b}{\alpha}}\ln\frac{b}{r}$$

$$= \frac{V_0}{\ln\frac{\alpha}{b}}\ln\frac{r}{b}$$ as given.

2. Find the resistance R and the dissipated power per unit length. You can use the expression for V provided in the previous question. **(8 pts)**

Take Length $L = 1m$. Then:

$$\bar{E} = \underbrace{-\text{grad } V = -\bar{a}_r \frac{dV}{dr}}_{(2pts)} = \bar{a}_r \frac{V_0}{(\ln\frac{b}{a})r} \quad \text{(using (a))} \quad [2pts]$$

Sln#1

$$\bar{J}_1 = \frac{\sigma_1 V_0}{(\ln\frac{b}{a})r}\bar{a}_r, \quad \bar{J}_2 = \frac{\sigma_2 V_0}{(\ln\frac{b}{a})r}\bar{a}_r \quad \text{in two regions}$$

$$I = \underbrace{\underbrace{\int \bar{J}_1 \cdot d\bar{s}}_{\text{region 1}} + \underbrace{\int \bar{J}_2 \cdot d\bar{s}}_{\text{region 2}}}_{\text{splitting } [2pts]} = \int_{z=0}^{1}\int_{\varphi=0}^{\pi} \frac{\sigma_1 V_0}{(\ln\frac{b}{a})\cancel{r}} \bar{a}_r \cdot \bar{a}_r (\cancel{r}\, d\varphi\, dz) + \ldots$$

$$= \frac{\sigma_1 V_0}{\ln\frac{b}{a}}.\pi + \frac{\sigma_2 V_0}{\ln\frac{b}{a}}\pi = V_0 \frac{\pi}{\ln\frac{b}{a}}(\sigma_1 + \sigma_2) \Rightarrow$$

$$R = \frac{V_0}{I} = \frac{\ln\frac{b}{a}}{\pi(\sigma_1+\sigma_2)}, \quad P = \frac{V_0^2}{R} \quad [2pts]$$

Sln #2:

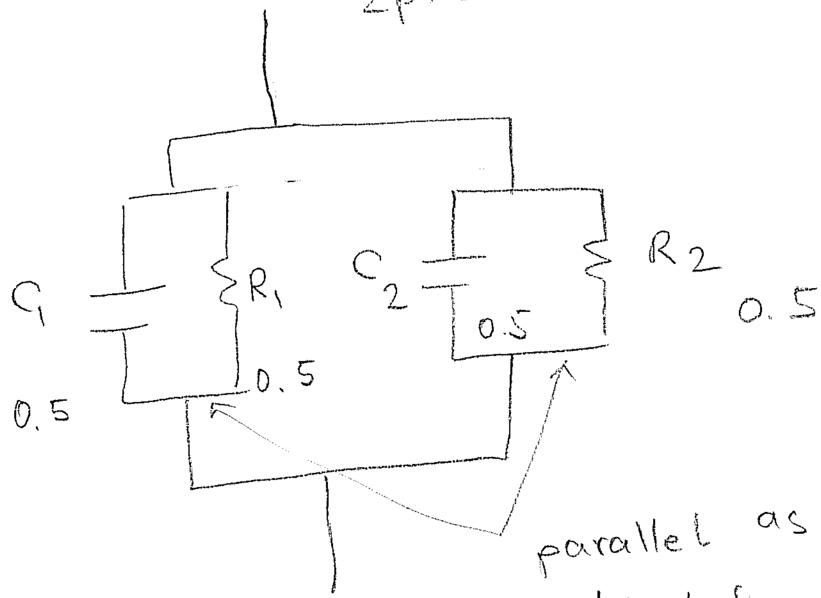
$$\underbrace{dp/dv}_{[1pt]} = \sigma|E|^2 = \begin{cases} \sigma_1 V_0^2 / (\ln\frac{b}{a})^2 r^2 & \text{in reg 1} \\ \frac{\sigma_2 V_0^2}{(\ln\frac{b}{a})^2 r^2} & \text{in reg 2} \end{cases} \quad [2pts]$$

$$\Rightarrow P = \int \frac{\sigma_1 V_0^2}{(\ln\frac{b}{a})^2 r^2}.\, r\, d\varphi\, dz\, dr + \int \frac{\sigma_2 V_0^2}{(\ln\frac{b}{a})^2 r^2} r\, d\varphi\, dz\, dr \underset{\text{for } L=1m}{=}$$

$$= \frac{V_0^2}{\ln\frac{b}{a}}.\pi(\sigma_1+\sigma_2) \quad [1pt], \quad R = \frac{V_0^2}{P} = \frac{\ln\frac{b}{a}}{\pi(\sigma_1+\sigma_2)} \quad [2pts]$$

3. Draw an equivalent electric circuit for this lossy capacitor. You do not need to find the values of the circuit elements, however, clarify their physical meaning. **(5 pts)**

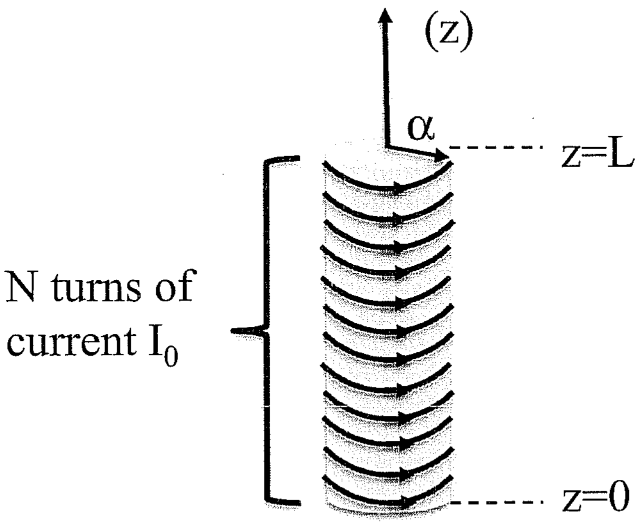
$\underbrace{\text{Parallel connection}}_{\text{2pts}}$ of lossy capacitors



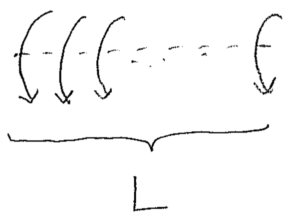
parallel as subject to same voltage [1pt]

Question 2

A finite solenoid of length L and radius α consists of N turns of wire carrying current I_0, as shown in the figure below.



1. Assuming that the N turns of the wire are tightly wound around the cylindrical surface $r = \alpha$, the current distribution on $r = \alpha$ can be approximated by a surface current density $\mathbf{J}_s = J_{s,0}\mathbf{a}_\phi$. Find $J_{s,0}$. **(5 pts)**



We have current $N I$ going through length L $\Rightarrow$ } justification 2 pts

$$\bar{J}_s = \frac{N \cdot I}{L} \bar{a}_\varphi$$

$$\Rightarrow \boxed{J_{s,0} = \frac{N \cdot I}{L}}$$

result 1 pt for correct expression

2 pts for correct units (A/m) even in an incorrect expression.

2. Using the Biot-Savart law, find the magnetic flux density $\mathbf{B}(0,0,z)$, $z > L$, at any point on the positive z–axis outside the solenoid. **(15 pts)**

Coordinates: cylindrical [1pt]

$$d\bar{B} = \frac{\mu_0}{4\pi} \frac{I\, d\bar{l}' \times (\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3}$$

$\bar{R} = z\,\bar{a}_z$ [1pt] $\quad \bar{R}' = \alpha\,\bar{a}_{r'} + z'\,\bar{a}_z$ [1pt]

[2pts] = 1pt $\alpha\, d\varphi'$, 1pt $\bar{a}_{\varphi'}$

$$I\, d\bar{l}' = \underbrace{J_{s,0}\, dz'}_{[2pts]} \underbrace{(\alpha\, d\varphi'}_{[2pts]}\, \bar{a}_{\varphi'})$$

$$\bar{R}-\bar{R}' = (z-z')\,\bar{a}_z - \alpha\,\bar{a}_{r'} \qquad |\bar{R}-\bar{R}'| = \sqrt{\alpha^2 + (z-z')^2} \ \Big\}[1pt]$$

$$I\, d\bar{l}' \times (\bar{R}-\bar{R}') = J_{s,0}\, dz' (\alpha\, d\varphi')\, \bar{a}_{\varphi'} \times \left[(z-z')\,\bar{a}_z - \alpha\,\bar{a}_{r'}\right]$$

$$= J_{s,0}\alpha\, dz'\, d\varphi' \left[\overbrace{(z-z')\,\bar{a}_{r'}}^{(1pt)} + \overbrace{\alpha\,\bar{a}_z}^{(1pt)}\right]$$

$$\Rightarrow d\bar{B} = \frac{\mu_0 J_{s,0}\alpha}{4\pi} \frac{(z-z')\,\bar{a}_{r'} + \alpha\,\bar{a}_z}{\left[\alpha^2 + (z-z')^2\right]^{3/2}}\, d\varphi'\, dz' \ \Big\}[2pts]$$

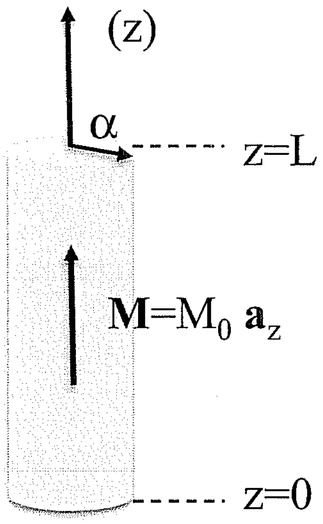
[2pts]

Note that $\bar{a}_{r'}$ integrated over φ' gives zero. $\Rightarrow$

$$\bar{B} = \frac{\mu_0 J_{s,0}\alpha^2}{4\pi}\,\bar{a}_z \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^L \frac{1}{\left[\alpha^2 + (z-z')^2\right]^{3/2}}\, dz' \ \Big\}[1pt]$$

For calculation of integral 2 bonus points (if mark was ≤ 13)

3. This solenoid is designed to produce the same magnetic flux density as a cylindrical magnet of the same length L and radius α, magnetized with constant magnetization $\mathbf{M} = M_0\mathbf{a}_z$. What is the relation between M_0 and $J_{s,0}$? Explain. **(5 pts)**



$\bar{J}_{s,m} = J_{s,0}\,\bar{a}_\varphi$ (2pts)

$\bar{J}_{s,m} = M_0\,\bar{a}_z \times \bar{a}_r = M_0\,\bar{a}_\varphi$ (2pts)

$\Rightarrow \boxed{M_0 = J_{s,0}}$ (1pt)

Question 3

1. Consider the coaxial cable shown in figure 1. A steady current I flows in the positive z direction in the inner conductor, and is uniformly distributed across the conductor's cross section. In the outer conductor, a steady current of intensity I flows in the opposite direction, and is also uniformly distributed across the cross section. Permeability is μ_0 everywhere.

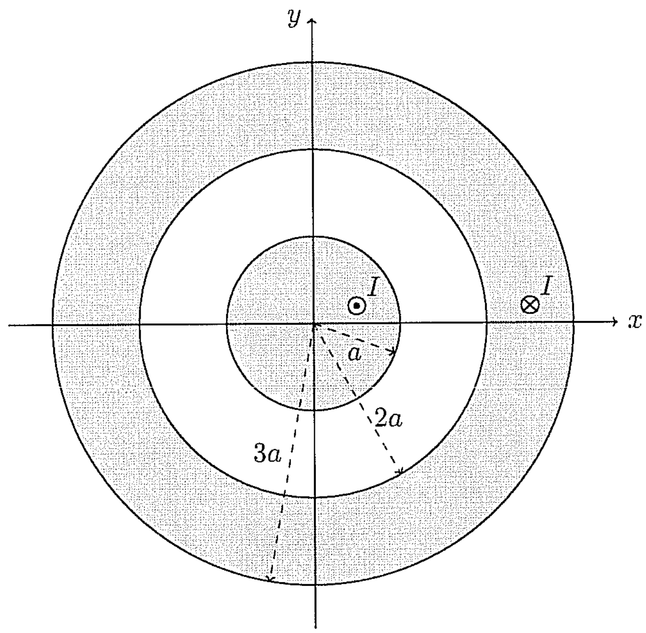


Figure 1

a) Using Ampere's law, find the magnetic flux density vector **B** in the region between the conductors ($a < r < 2a$) **(8 pts)**.

Cylindrical symmetry $\Rightarrow$ $\bar{B} = B_\varphi(r)\,\bar{a}_\varphi$ [2pts] $\Rightarrow$

Ampere's Law should be applied along magnetic flux lines which in this case are circles around z-axis. [2pts]

$\oint_c \bar{B}\cdot d\bar{l} = \mu_0 I_{enclosed}$ [1]

$\underbrace{B_\varphi \cdot 2\pi r}_{[2]} = \mu_0 \underbrace{I}_{[1]} \Rightarrow B_\varphi = \dfrac{\mu_0 I}{2\pi r}$

b) Find the current density vector **J** in the outer conductor ($2a < r < 3a$) **(4 pts)**.

$$\bar{J} = \underbrace{-\bar{a}_z}_{1\,pt} \underbrace{\frac{I}{S}}_{2pts}, \quad S = \text{cross-section of}$$

the outer conductor

$$\left.\begin{aligned} S &= \pi(3\alpha)^2 - \pi(2\alpha)^2 \\ &= \pi[9-4]\alpha^2 = 5\pi\alpha^2 \end{aligned}\right\} 1\,pt$$

$$\boxed{\bar{J} = -\bar{a}_z \frac{I}{5\pi\alpha^2}}$$

c) Using Ampere's law, find the magnetic flux density vector $\mathbf{B}$ inside the outer conductor ($2a < r < 3a$) **(8 pts)**.

See marking scheme for (a) for 4pts
(symmetry > 2pts, circles for Amp. Law > 2pts)

$$I_{enclosed} = I - I \underbrace{\frac{\pi r^2 - \pi (2a)^2}{\pi (3a)^2 - \pi (2a)^2}}_{\text{portion of outer conductor current enclosed}}$$

$$= I - I \frac{r^2 - 4a^2}{5a^2} = I \frac{9a^2 - r^2}{5a^2}$$

(3pts)

$$B_\varphi \cdot 2\pi r = \mu_0 I_{enclosed}$$

$$\Rightarrow B_\varphi = \frac{\mu_0}{2\pi r} I \frac{9a^2 - r^2}{5a^2} \quad (1\text{ pt})$$

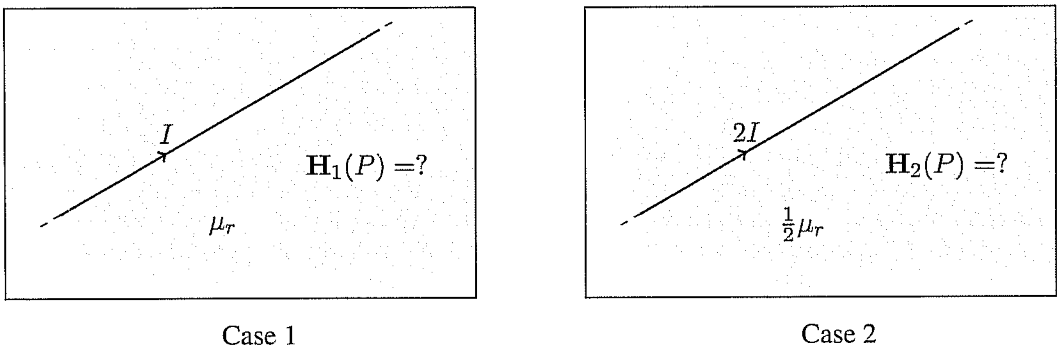


Figure 2

2. In "Case 1" of figure 2, we have a straight line carrying a steady current I. Relative permeability is μ_r everywhere. In "Case 2", the current in the wire is doubled, but the relative permeability at any point is halved. Let $\mathbf{H}_1(P)$ and $\mathbf{H}_2(P)$ be the magnetic field vector at a given point P in case 1 and 2, respectively. The intensity of $\mathbf{H}_2(P)$ is

   a) stronger than the intensity of $\mathbf{H}_1(P)$;

   b) weaker than the intensity of $\mathbf{H}_1(P)$;

   c) equal to the intensity of $\mathbf{H}_1(P)$.

   Briefly justify your answer **(5 pts)**.

$\oint \bar{H} \cdot d\bar{l} = I_{\underline{free}}$ } [3pts] to $\bar{H}$ depending on free currents.

Since case 2 has a stronger current $\Rightarrow \bar{H}_2(P)$ [2pts] will be stronger in magnitude.

Another way to see this is by stating the Biot-Savart law in terms of $\bar{H}$:

$$d\bar{H} = \frac{1}{4\pi} \frac{I\, d\bar{l}' \times (\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|}$$

The two cases have all terms equal except I.

$\Rightarrow d\bar{H}_2(P) = 2\, d\bar{H}_1(P)$.