UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test I

First Year — Program 5

MAT185H1S — Linear Algebra

Examiners: G S Scott & G M T D'Eleuterio

26 January 2015

Student Name: Fair Copy

Last Name First Names

Student Number: Tutorial Section: TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 7 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). Give two examples of a *field*.

Among the possibilities... $\mathbb{R}, \mathbb{C}, \mathbb{Q}$

/2

1(b). The *associative property for scalar multiplication* in a vector space is defined as

$$\alpha(\beta \boldsymbol{v}) = (\alpha\beta)\boldsymbol{v} \quad \forall \alpha, \beta \in \Gamma \text{ (field)}, \forall \boldsymbol{v} \in \mathcal{V} \text{ (vector space)}$$

/2

1(c). Let α, β be scalars and $\boldsymbol{v}$ be a vector. Is (*i*) $\alpha - 1/\beta$ a scalar, a vector or neither? What about (*ii*) $\alpha\boldsymbol{v}^{-1}$?

(i) scalar, (ii) neither

/2

1(d). The *span* of a set of vectors $\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\}$ is defined as

$$\text{span}\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_n\} = \left\{ \boldsymbol{v} \;\middle|\; \boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i, \forall \lambda_i \in \Gamma \right\}$$

/2

1(e). Let $\mathbf{A} \in {}^m\mathbb{R}^n$. Describe the *function* defined by $\mathbf{A}$ in mathematical notation.

$\mathbf{A} : {}^n\mathbb{R} \to {}^m\mathbb{R}$ where $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). The set $\mathbb{R}^2$ is a vector space over $\mathbb{C}$ and under the usual definitions of vector addition and scalar multiplication.

F

2(b). The set of matrices $\mathbf{X} \in {}^n\mathbb{R}^n$ satisfying $\mathbf{AX}+\mathbf{XA}^T = -\mathbf{L}$ for given but arbitrary $\mathbf{A}, \mathbf{L} \in {}^n\mathbb{R}^n$ forms a subspace of ${}^n\mathbb{R}^n$ over $\mathbb{R}$ and under the usual definitions of matrix addition and scalar multiplication.

F

2(c). Let $\mathbb{P}_5$ be the vector space of polynomials of degree 5 or less with real coefficients. If $\{p_1, p_2 \cdots p_n\}$ are all polynomials of degree 3, then every polynomial in $\text{span}\{p_1, p_2 \cdots p_n\}$ also has degree 3.

F

2(d). Let $\mathbf{A} \in {}^2\mathbb{R}^7$ and let $\mathbf{v}, \mathbf{w} \in {}^7\mathbb{R}$ be solutions to

$$\mathbf{Av} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \mathbf{Aw} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

T

Then the only linear combination of $\{\mathbf{v}, \mathbf{w}\}$ in $\text{null}\,\mathbf{A}$ is the zero vector.

2(e). $\mathbb{R}^3 = \text{span}\{\begin{bmatrix} 1 & \pi & 0 \end{bmatrix}, \begin{bmatrix} e^2 & -1 & 0 \end{bmatrix}\}$.

F

C. Problems

3. Consider $V = \mathbb{R}^2$ and define vector addition and scalar multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1x_2, x_1y_2 + x_2y_1)$$
$$\lambda(x, y) = (\lambda x, \lambda y)$$

where $\lambda \in \mathbb{R}$.

Check whether the following properties are satisfied:

(i) The associative property for vector addition.

(ii) The existence of a zero vector.

(iii) The existence of a negative for each element in V.

Does V over $\mathbb{R}$, with vector addition and scalar multiplication as defined above, establish a vector space?

(i)
$$\begin{aligned}[(x_1, y_1) + (x_2, y_2)] + (x_3, y_3) &= (x_1x_2, x_1y_2 + x_2y_1) + (x_3, y_3)\\ &= (x_1x_2x_3, x_1x_2y_3 + x_3(x_1y_2 + x_2y_1))\\ &= (x_1x_2x_3, x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3)\end{aligned}$$

and

$$\begin{aligned}(x_1, y_1) + [(x_2, y_2) + (x_3, y_3)] &= (x_1, y_1) + (x_2x_3, x_2y_3 + x_3y_2)\\ &= (x_1x_2x_3, x_1x_2x_3, x_1(x_2y_3 + x_3y_2) + x_2x_3y_1)\\ &= (x_1x_2x_3, x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3)\\ &= [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)\end{aligned}$$

Thus, associativity under vector addition holds.

(ii) We seek $\boldsymbol{0} = (x_0, y_0)$ such that $(x, y) + (x_0, y_0) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$. Now

$$(x, y) + (x_0, y_0) = (xx_0, xy_0 + x_0y)$$

So we must have

$$xx_0 = x, \qquad xy_0 + x_0y = y, \qquad \forall (x, y) \in \mathbb{R}^2$$

which are satisfied by $x_0 = 1$ and $y_0 = 0$. Thus the zero exists and is $\boldsymbol{0} = (1, 0)$.

. . . cont'd

3. *...cont'd*

(iii) We seek $-(x, y) = (x_{-1}, y_{-1})$ such that $(x, y) + (x_{-1}, y_{-1}) = (1, 0)$ for all $(x, y) \in \mathbb{R}^2$. Now

$$(x, y) + (x_{-1}, y_{-1}) = (xx_{-1}, xy_{-1} + x_{-1}y)$$

This means that

$$xx_{-1} = 1, \qquad xy_{-1} + x_{-1}y = 0, \qquad \forall (x, y) \in \mathbb{R}^2$$

which requires that $x_{-1} = 1/x$ but this cannot hold if $x = 0$. Hence, a negative does not exist for every $(x, y) \in \mathbb{R}^2$.

Accordingly, we conclude that $V = \mathbb{R}^2$ is not a vector space as de□ned above.

/10

4. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be elements in a vector space and α a scalar in the corresponding field. Show that if $\alpha\boldsymbol{u} = \alpha\boldsymbol{v}$ and $\alpha \neq 0$ then $\boldsymbol{u} = \boldsymbol{v}$.

$$
\begin{aligned}
\boldsymbol{u} &= 1\boldsymbol{u} && \text{[MIV]} \\
&= \left(\frac{1}{\alpha}\alpha\right)\boldsymbol{u} && [\alpha \neq 0] \\
&= \frac{1}{\alpha}(\alpha\boldsymbol{u}) && \text{[MII]} \\
&= \frac{1}{\alpha}(\alpha\boldsymbol{v}) && \text{[given]} \\
&= \left(\frac{1}{\alpha}\alpha\right)\boldsymbol{v} && \text{[MII]} \\
&= 1\boldsymbol{v} \\
&= \boldsymbol{v} && \text{[MIV]}
\end{aligned}
$$

. . . cont'd

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to have an *integer period* if there exists a positive integer a such that $f(x) = f(x+a)$ for all $x \in \mathbb{R}$. Show that the set $\mathcal{F}$ of all functions with integer period is a subspace of the vector space of all functions. (Note that $\mathcal{F}$ admits functions with *any* integer period; for example, functions f and g such that $f(x) = f(x+2)$ and $g(x) = g(x+3)$ are elements in $\mathcal{F}$.)

Use the Subspace Test:

SI. The zero function $z : \mathbb{R} \to \{0\}$ is contained in $\mathcal{F}$ because it has period of every integer, i.e., $z(x+a) = z(x)$ for any positive integer a. (Note that for any f, $f + z = f$.)

SII. We begin by noting that if f has period a then it also has period ab where b is any positive integer. That is,

$$f(x+ab) = f(x+(b-1)a+a) = f(x+(b-1)a) = \\ = f(x+(b-2)a+a) = f(x+(b-2)a) = \cdots = f(x)$$

Now consider arbitrary $f, g \in \mathcal{F}$ where f has period a and g has period b. Then we assert that $f+g$ has period ab. Using the definition of function (vector) addition,

$$(f+g)(x+ab) = f(x+ab) + g(x+ab) = f(x) + g(x) = (f+g)(x)$$

So $f + g \in \mathcal{F}$.

SIII. For arbitrary $f \in \mathcal{F}$ with period a and $\lambda \in \mathbb{R}$, using the definition of scalar multiplication for functions,

$$(\alpha f)(x+a) = \alpha f(x+a) = \alpha f(x) = (\alpha f)(x)$$

implying that $\alpha f \in \mathcal{F}$.

Thus $\mathcal{F}$ is a subspace of the vector space of all functions.

/10