MAT195S CALCULUS II

Midterm Test #1

12 February 2013 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: P. Athavale and J. W. Davis

Family Name: J. W. Davis.

Given Name: Solutions.

Student #: ______________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	11	
2	10	
3	12	
4	9	
5	8	
6	10	
7	11	
8	5	
TOTAL	76	/ 70

Tutorial Section: ______________________

TA Name: ______________________

1) Evaluate the following integrals.

a) $\displaystyle\int \frac{4x^2 - 7x - 12}{x(x+2)(x-3)} dx$

b) $\displaystyle\int \cot^5 \theta \sin^4 \theta d\theta$

c) $\displaystyle\int \frac{dx}{(x^2+1)^2}$

(11 marks)

a) $\dfrac{4x^2-7x-12}{x(x+2)(x-3)} = \dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$

$\Rightarrow 4x^2-7x-12 = A(x+2)(x-3) + B(x-3)x + C(x+2)x$

set $x=0 \Rightarrow -12 = -6A \Rightarrow A = 2$

$x=3 \Rightarrow 3 = 15C \Rightarrow C = 1/5$

$x=-2 \Rightarrow 18 = 10B \Rightarrow B = 9/5$

$\therefore \displaystyle\int \frac{4x^2-7x-12}{x(x+2)(x-3)} dx = 2\int \frac{dx}{x} + \frac{9}{5}\int \frac{dx}{x+2} + \frac{1}{5}\int \frac{dx}{x-3}$

$= 2\ln|x| + \frac{9}{5}\ln|x+2| + \frac{1}{5}\ln|x-3| + C$

b) $\displaystyle\int \cot^5\theta \sin^4\theta\, d\theta = \int \frac{\cos^5\theta \sin^4\theta}{\sin^5\theta} d\theta = \int \frac{\cos^5\theta}{\sin\theta} d\theta = \int \frac{(1-\sin^2\theta)^2}{\sin\theta} \cos\theta\, d\theta$

$\displaystyle= \int \frac{1-2u^2+u^4}{u} du = \int \left(\frac{1}{u} - 2u + u^3\right) du = \ln|u| - u^2 + \frac{u^4}{4} + C$

$= \ln|\sin\theta| - \sin^2\theta + \frac{1}{4}\sin^4\theta + C$

c) $\displaystyle\int \frac{dx}{(x^2+1)^2}$

let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$x^2+1 = \sec^2\theta$

$\displaystyle= \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C$

2) a) Show that $\int_0^{\infty} x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^{\infty} e^{-x^2}\,dx$

(5 marks)

$$\int_0^{\infty} x^2 e^{-x^2}\,dx \qquad \text{let } u = x \quad du = dx \qquad dv = x e^{-x^2}\,dx \quad v = -\frac{1}{2}e^{-x^2}$$

$$= \left[-\frac{x}{2}e^{-x^2}\right]_0^{\infty} - \int_0^{\infty} -\frac{1}{2}e^{-x^2}\,dx$$

$$= \underbrace{\lim_{t\to\infty}\left(-\frac{t}{2}e^{-t^2}\right)}_{} + 0 + \int_0^{\infty}\frac{1}{2}e^{-x^2}\,dx = \frac{1}{2}\int_0^{\infty} e^{-x^2}\,dx$$

$$\lim_{t\to\infty}\frac{t}{e^{t^2}} \overset{*}{=} \lim_{t\to\infty}\frac{1}{2te^{t^2}} = 0$$

b) The Gamma function is defined as $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}\,dt$. Show that $\Gamma(4) = 3!$.

(5 marks)

$$\Gamma(4) = \int_0^{\infty} t^3 e^{-t}\,dt \qquad \text{let } u = t^3 \quad du = 3t\,dt \qquad dv = e^{-t}\,dt \quad v = -e^{-t}$$

$$= \left[-t^3 e^{-t}\right]_0^{\infty} + 3\int_0^{\infty} t^2 e^{-t}\,dt$$

using l'Hospital's rule as in (a)

$$= \lim_{t\to\infty}\left(-\frac{t^3}{e^t}\right) + 0 + 3\int_0^{\infty} t^2 e^{-t}\,dt = 0 + 0 + 3\int_0^{\infty} t^2 e^{-t}\,dt$$

$$= 3\left[-t^2 e^{-t}\right]_0^{\infty} + 3\cdot 2\int_0^{\infty} te^{-t}\,dt$$

$$= 3\cdot 2\left[-te^{-t}\right]_0^{\infty} + 3\cdot 2\cdot\int_0^{\infty} e^{-t}\,dt$$

$$= 3\cdot 2\left[-e^{-t}\right]_0^{\infty} = 3\cdot 2\cdot 1 = 3!$$

3) a) Given the formula for the length of a parametric curve, $s = \int \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$, find the formula for the length of a polar curve.

b) Find, but do not solve, the integrals representing the lengths of the following curves:
(i) one loop of the curve $r = \cos 2\theta$
(ii) the lemniscate $r^2 = \cos 2\theta$

Provide sketches of the curves

(12 marks)

a) given $r = r(\theta) \Rightarrow x = r\cos\theta = r(\theta)\cos\theta \Rightarrow x'(\theta) = r'(\theta)\cos\theta - r(\theta)\sin\theta$
$y = r\sin\theta = r(\theta)\sin\theta \rightarrow y'(\theta) = r'(\theta)\sin\theta + r(\theta)\cos\theta$

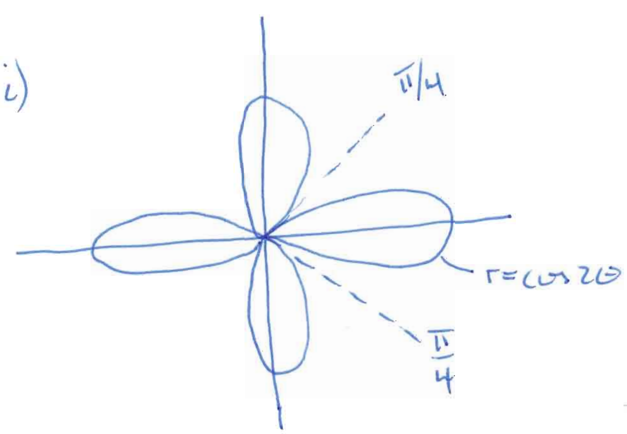
$(x'(\theta))^2 = (r')^2\cos^2\theta - 2rr'\sin\theta\cos\theta + r^2\sin^2\theta$

$(y'(\theta))^2 = (r')^2\sin^2\theta + 2rr'\cos\theta\sin\theta + r^2\cos^2\theta$

$\therefore (x'(\theta))^2 + (y'(\theta))^2 = r^2 + (r')^2$

$\therefore s = \int_\alpha^\beta \sqrt{(r(\theta))^2 + (r'(\theta))^2}\, d\theta$

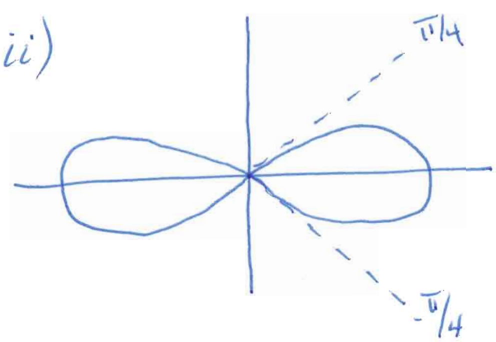
b) i)



$s = \int_{-\pi/4}^{\pi/4} \sqrt{\cos^2 2\theta + 4\sin^2 2\theta}\, d\theta$

$= \int_{-\pi/4}^{\pi/4} \sqrt{1 + 3\sin^2 2\theta}\, d\theta$

ii)



$r^2 = \cos 2\theta$

$2r\,dr = -2\sin\theta\, d\theta$

$\therefore dr = \frac{-2\sin\theta\, d\theta}{2\sqrt{\cos 2\theta}} = -\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$

$\therefore \frac{1}{2}s = \int_{-\pi/4}^{\pi/4} \sqrt{\cos 2\theta + \frac{(\sin 2\theta)^2}{\cos 2\theta}}\, d\theta$

4) Sketch the parametric curve: $x = t^3 - 3t, \quad y = t^2$

(9 marks)

$x = t^3 - 3t$ $\qquad$ $y = t^2$

$x' = 3t^2 - 3$ $\qquad$ $y' = 2t$

$x' = 0 \Rightarrow t = \pm 1 \quad (-2, 1)$ $\qquad$ $y' = 0 \rightarrow t = 0 \qquad (0, 0)$

$(2, 1)$

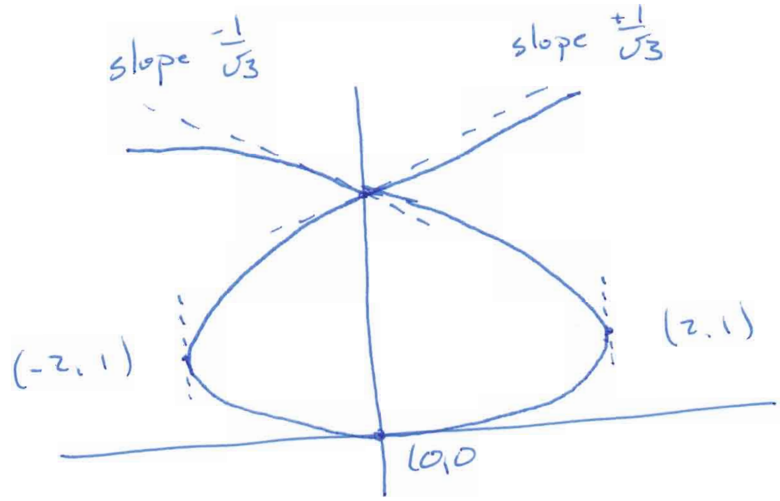
Intercepts: $y = 0 \Rightarrow t = 0 \Rightarrow (0, 0)$

$x = 0 \Rightarrow t = 0$

$t = \pm\sqrt{3} \Rightarrow (0, 3)$

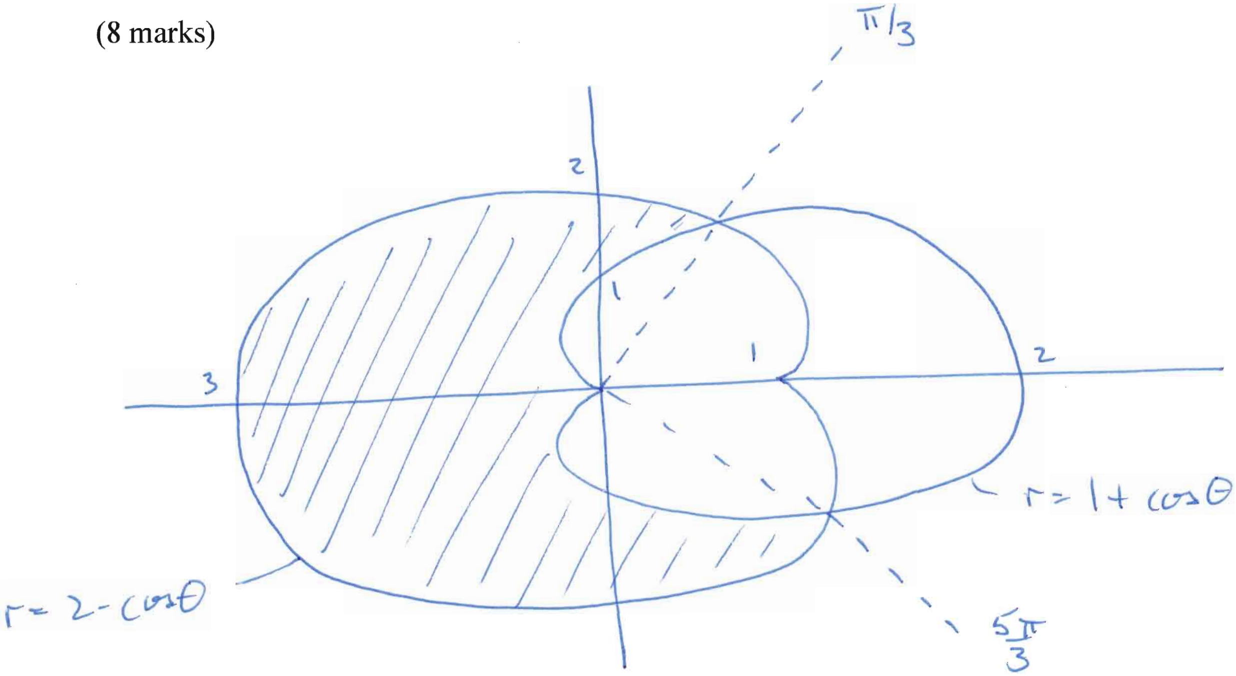
Slope at $t = \pm\sqrt{3}$: $\dfrac{y'}{x'} = \dfrac{\pm 2\sqrt{3}}{6} = \pm \dfrac{1}{\sqrt{3}}$

Asymptotic behavior : As $t \to \pm\infty$: $x \to t^3 \quad \therefore y = x^{2/3}$



5) Find the area of the region that lies inside $r = 2 - \cos\theta$ but outside $r = 1 + \cos\theta$. Provide a sketch of the region.

(8 marks)



Intersection: $2 - \cos\theta = 1 + \cos\theta$

$1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

$$\therefore A = \frac{1}{2}\int_{\pi/3}^{5\pi/3} \left[(2-\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3} \left(4 - 4\cos\theta + \cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_{\pi/3}^{5\pi/3} (3 - 6\cos\theta)\, d\theta = \frac{3}{2}\left[\theta - 2\sin\theta\right]_{\pi/3}^{5\pi/3}$$

$$= \frac{3}{2}\cdot\frac{4\pi}{3} - 3\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 2\pi + 3\sqrt{3}$$

6) Let $a_1 = 1$ and $a_{n+1} = \sqrt{1+2a_n}$ $(n = 1,2,3,\ldots)$. Show that $\{a_n\}$ is increasing and bounded above. (Hint: show that 3 is an upper bound.) Hence conclude that the sequence is convergent and find its limit.

(10 marks)

1) Increasing: $a_1 = 1$

$a_2 = \sqrt{1+2} = \sqrt{3} > a_1$

given $a_{k+1} > a_k$, then $a_{k+2} = \sqrt{1+2a_{k+1}} > \sqrt{1+2a_k} = a_{k+1}$

$a_{k+1} > a_k$

$\therefore \{a_n\}$ is increasing by induction

2) Bounded: $a_1 = 1 < 3$

given $a_k < 3$, then $a_{k+1} < \sqrt{1+2\cdot 3} = \sqrt{7} < 3$

$\therefore a_n < 3$ for all n

3) Sequence is increasing and bounded above

$\therefore$ limit exists.

let $\lim\limits_{n\to\infty} a_n = L$ $\quad \therefore \lim\limits_{n\to\infty} a_{n+1} = L$

$\Rightarrow L = \sqrt{1+2L} \Leftrightarrow L^2 - 2L - 1 = 0$

$L = \dfrac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2} \Rightarrow L = 1 + \sqrt{2}$

$(L > 0)$

7) a) If $\{a_n\}$ is convergent, use the definition of the limit of a sequence (ϵ - N argument) to show that $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$.

b) Determine whether the sequence converges or diverges. If it converges, find the limit:

(i) $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$ (ii) $a_n = \dfrac{3^{n+2}}{5^n}$ (iii) $a_k = \dfrac{\ln k}{\ln 2k}$

(11 marks)

a) given $\{a_n\}$ convergent, then $\lim_{n\to\infty} a_n = L$

This implies, given $n > N$ that $|a_n - L| < \epsilon$ for all $\epsilon > 0$

Now, since $n+1 > n > N$ $\therefore |a_{n+1} - L| < \epsilon$

$\therefore \lim_{n\to\infty} a_{n+1} = L = \lim_{n\to\infty} a_n$

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b) i) $a_n = \dfrac{\sin 2n}{1+\sqrt{n}} \Rightarrow |a_n| \le \dfrac{1}{1+\sqrt{n}} \xrightarrow{n\to\infty} 0$ $\therefore a_n \to 0$ by squeeze Th'm.

ii) $a_n = \dfrac{3^{n+2}}{5^n} = 9 \cdot \left(\dfrac{3}{5}\right)^n$

$\lim_{n\to\infty} a_n = 9 \cdot \lim_{n\to\infty} \left(\dfrac{3}{5}\right)^n = 9 \cdot 0 = 0$

iii) $a_k = \dfrac{\ln k}{\ln 2k} = \dfrac{\ln k}{\ln 2 + \ln k} = \dfrac{1}{\frac{\ln 2}{\ln k} + 1}$

$\lim_{k\to\infty} \dfrac{\ln 2}{\ln k} = 0$ $\therefore a_k \xrightarrow{k\to\infty} 1$

8) Identify and sketch the conic: $r = \dfrac{10}{5 - 6\sin\theta}$

(5 marks)

$$r = \frac{10}{5-6\sin\theta} = \frac{2}{1-\frac{6}{5}\sin\theta} = \frac{\frac{6}{5}\cdot\frac{10}{6}}{1-\frac{6}{5}\sin\theta}$$

$\Rightarrow e = \frac{6}{5} > 1 \quad \therefore$ hyperbola

$\Rightarrow d = \frac{10}{6} \quad \therefore$ directrix $y = -\frac{10}{6}$

$\Rightarrow -e\sin\theta$ appears in the denominator, $\therefore$ oriented along y-axis.

$\Rightarrow$ vertices: $\theta = \frac{\pi}{2} \rightarrow r = -10$

$\theta = \frac{3\pi}{2} \rightarrow r = \frac{10}{11}$