UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, April 2018

PHY294S - QUANTUM PHYSICS AND THERMAL PHYSICS

Instructor: Professors A. Paramekanti and E. Poppitz

Duration: 150 minutes

Exam Type A: Closed Book
Calculator Type 3: Non-programmable calculators without text storage

- Print in BLOCK LETTERS your name, student number and tutorial group on top of all examination booklets.
- Place your student ID on the desk.
- Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. No other aid (notes, textbook, dictionary) is allowed.
- Turn off any communication device (phone, pager, PDA, iPod, etc.) you may have and place it far from where you are sitting.
- There are two parts of the exam, one for Quantum Physics, and the other one for Thermal Physics. Each part is worth 50 marks in total.
- This examination paper consists of 11 questions. Answer all questions. Show all important steps.
- Within each question, a mark breakdown is indicated in brackets. Part marks will be given for partially correct answers.
- Do not separate the stapled sheets of the question paper. Hand in the questions with your exam booklet at the end of the exam.
- The total number of marks is 100.

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, April 2018
PHY 294

Useful formulae

Units, Constants, Identities

$$\begin{aligned}
1\mathrm{eV} &= 1.6 \times 10^{-19}\mathrm{J} \\
\hbar &= 1.05 \times 10^{-34} kg\ m^2/s \\
k &= 1.38 \times 10^{-23} kg\ m^2\ s^{-2} K^{-1} \\
\mathrm{e}^{ix} &= \cos x + i \sin x
\end{aligned} \tag{1}$$

Oscillator frequency

$$\omega = \sqrt{\frac{k}{m}}$$

Time-independent Schrodinger equation in 1D

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{2}$$

Harmonic oscillator

$$\begin{aligned}
E_n &= (n + \frac{1}{2})\hbar\omega \qquad (n = 0, 1, 2 \ldots) \\
\psi_0(x) &= (\frac{m\omega}{\pi\hbar})^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \\
\psi_1(x) &= (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{m\omega}{2\hbar}}\ 2x \exp\left[-\frac{m\omega x^2}{2\hbar}\right]
\end{aligned}$$

Particle in a 1D box

$$\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2} \qquad (n = 1, 2 \ldots) \\
\psi_n(x) &= \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}
\end{aligned}$$

Fourier transforms

$$\begin{aligned}
\tilde{\psi}(k) &= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}}\ \mathrm{e}^{-ikx}\ \psi(x) \\
\psi(x) &= \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}}\ \mathrm{e}^{+ikx}\ \tilde{\psi}(k)
\end{aligned} \tag{3}$$

Integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ \mathrm{e}^{-\alpha x^2} &= \sqrt{\frac{\pi}{\alpha}} \\
\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\ \mathrm{e}^{-\alpha x^2}\ \mathrm{e}^{-ikx} &= \sqrt{\frac{1}{2\alpha}}\ \mathrm{e}^{-\frac{k^2}{4\alpha}} \\
\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\ x\mathrm{e}^{-\alpha x^2}\ \mathrm{e}^{-ikx} &= -\sqrt{\frac{1}{2\alpha}}\ \frac{ik}{2\alpha}\ \mathrm{e}^{-\frac{k^2}{4\alpha}}
\end{aligned}$$

I. Short questions

(i) Consider nucleons (neutrons and protons) inside a nucleus of radius $R_N = 10^{-14}m$. The mass of a nucleon is $m \approx 10^{-27}$kg. Using the uncertainty principle, estimate the typical energy (in eV) associated with the "strong interaction" which keeps nucleons bound inside the nucleus. (5 points)

(ii) Consider a particle in the ground state of a harmonic oscillator potential. If the spring constant of the oscillator is suddenly halved, what is the probability to find the particle in the (a) ground state of the new potential (b) first excited state of the new potential. (5 points)

(iii) Consider a spin-1/2 particle with the Hamiltonian

$$H = \begin{pmatrix} b_z & -ib_y \\ ib_y & -b_z \end{pmatrix}.$$

If you carry out a single measurement of its energy, what possible results could you get? (5 points)

(iv) Consider 2 electrons, both with spin $\uparrow$, in a box of size L. What are the normalized wavefunctions and energies for the ground state and the first excited state of this two-electron problem? (5 points)

II. Momentum

Consider a particle in the first excited state of a harmonic oscillator, with the normalized wavefunction

$$\psi(x) = (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{m\omega}{2\hbar}}\ 2x\ \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

Determine the most likely outcomes when you measure the momentum of this particle. (15 points)

III. Tunneling

Consider a particle moving in a potential

$$V(x) = \begin{cases} V_0/2 & \text{for } x < 0 \\ V_0 & \text{for } 0 < x < L \\ V_0/2 & \text{for } x > L \end{cases}$$

Derive the transmission coefficient through this potential for a particle of mass m for the special energy value $E = V_0$. Show all steps. Express result in terms of $m, V_0, L, \hbar$. (15 points)

(*Hint*: For the middle region, you need to use the **most** general solution to the Schrodinger equation which has **two** arbitrary constants).

IV. Paramagnet

Consider a spin-1/2 paramagnet of N spins. The spins take two values, $s = \pm 1$. The paramagnet is placed in magnetic field B so that the energy of a single spin is $E_s = -\mu B s$, where μ is the magnetic moment of each spin. It is in thermal equilibrium with a reservoir at temperature T. The magnetization of the magnet is $M = \mu S = \mu(s_1 + s_2 + \ldots + s_N)$, where S is the total spin.

For this problem, you may need the equations: $\cosh x = \frac{e^x+e^{-x}}{2}$, $\frac{d\cosh x}{dx} = \sinh x = \frac{e^x-e^{-x}}{2}$ and $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$. Recall also that $Z = \sum_{\textit{all microstates}} e^{-\frac{E}{kT}}$, and that the average energy is $\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z$, with $\beta = \frac{1}{kT}$.

1. Find the Boltzmann partition function of the paramagnet as a function of T, N, B (and μ, of course).

(7 points)

2. Find the average energy $\langle E\rangle$ of the paramagnet and use it to deduce the average magnetization $\langle M\rangle$. Briefly describe the behaviour of $\frac{\langle M\rangle}{|M_{max}|}$ as a function of the temperature, in the limits of low, $T\to 0$, and high $T\to\infty$, temperatures. Here, $|M_{max}|$ is the absolute value of the maximal possible value that the magnetization can take.

(7 points)

3. Explain qualitatively, using the T, B dependence of $\langle M\rangle$, how the paramagnet can be used for "magnetic cooling".

(6 points)

V. Gases mixing

Two identical ideal monatomic gases, each containing the same number of particles N, are initially kept in separate vessels of volumes V_1 and V_2 at the same pressure p, but at different temperatures T_1 and T_2, respectively. The two vessels, now isolated from their respective thermostats and anything else, are then connected to each other (e.g. by quickly removing the partition between them). The gases are left to fill the entire volume and to, after some time, reach thermal equilibrium.

In this problem, you may need the ideal gas law, $pV = NkT$, the energy of the ideal monatomic gas $U=\frac{3}{2}NkT$, and its entropy $S = kN\left(\ln\left[\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{\frac{3}{2}}\right]+\frac{5}{2}\right)$.

Answer the following questions making sure to explain your reasoning:

1. What is the final temperature of the gas?

(4 points)

2. What is the final pressure of the gas?

(4 points)

3. Find an expression for the change of entropy in this process. To get maximum credit for this part, be sure to express the result as a function *only* of the initial temperatures T_1, T_2 and the number of particles N (plus, of course, the relevant fundamental constants). There is more than one way to go about finding the entropy change. So long as it is correct, any derivation will get full credit. Bonus points—not included in the score below—will be awarded to those who present, or even correctly indicate, more than one point of view!

(7 points)

VI. Short questions

Please give brief answers to the following questions.

1. What is the importance of chemical potential in thermal physics?

(5 points)

2. When we study the thermodynamics and statistical mechanics of the ideal gas at room temperature, we did not consider the structure of the nucleus, or the quark structure of the proton. Can you briefly explain why?

(5 points)

3. Consider two gases, one made of potassium atoms (atomic weight ≈ 40) and another one of rubidium atoms (atomic weight ≈ 85). Suppose that both gases are at the same density, and are sufficiently dilute so that at room temperature they obey the classical ideal gas law. The gases, keeping the density fixed, are cooled to lower temperatures. Which of the two gases will first cease to obey the classical ideal gas law as the temperature is lowered? (Equivalently, which gas will exhibit quantum properties up to a higher temperature?)

(5 points)