ESC195 - Calculus II
Midterm Test #2
March 26, 2024
9:10 - 10:50 am
Instructor: J. W. Davis

Closed book, no aid sheets, no calculators
There are 7 questions, each worth 10 marks.
Plus a bonus question worth 6 marks.

1. Determine whether the series converges or diverges:

a) $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ b) $\sum_{n=1}^{\infty} \left(\sqrt[n]{2}-1\right)^n$ c) $\sum_{n=1}^{\infty} \frac{n!\,2^n}{(n+1)^n}$

a) $|a_{n+1}| = \frac{1}{\sqrt{n+2}} < \frac{1}{\sqrt{n+1}} = |a_n|$ & $|a_n| = \frac{1}{\sqrt{n+1}} \xrightarrow[n\to\infty]{} 0$

$\therefore \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$ converges by alternating series test

b) root test: $(a_n)^{1/n} = 2^{1/n} - 1 \longrightarrow 1 - 1 = 0 < 1$

$\therefore \sum_{n=1}^{\infty} \left(2^{1/n} - 1\right)^n$ converges

c) ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,2^{n+1}}{(n+2)^{n+1}} \cdot \frac{(n+1)^n}{n!\,2^n}\right| = \frac{2(n+1)}{(n+2)} \cdot \left(\frac{n+1}{n+2}\right)^n = 2\left(\frac{n+1}{n+2}\right)^{n+1}$

let $m = n+1 \Rightarrow \left|\frac{a_m}{a_{m-1}}\right| = 2\left(\frac{m}{m+1}\right)^m = 2\left(\frac{1}{1+\frac{1}{m}}\right)^m = \frac{2}{\left(1+\frac{1}{m}\right)^m}$

$\lim_{m\to\infty} \left(1+\frac{1}{m}\right)^m = e \Rightarrow \left|\frac{a_m}{a_{m-1}}\right| \longrightarrow \frac{2}{e} < 1$

$\therefore \sum_{n=1}^{\infty} \frac{n!\,2^n}{(n+1)^n}$ converges

2. Determine the radius and interval of convergence for the power series:

a) $\displaystyle\sum_{k=0}^{\infty} \frac{(x-2)^k}{k^2+1}$ b) $\displaystyle\sum_{n=1}^{\infty} n^n x^n$ c) $\displaystyle\sum_{k=1}^{\infty} a_k x^k$ where $a_k = \displaystyle\sum_{j=1}^{k} \left(\frac{4}{5}\right)^j$

a) ratio test : $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(x-2)^{k+1}}{k^2+2k+2} \cdot \frac{k^2+1}{(x-2)^k}\right| = |x-2|\left(\frac{k^2+1}{k^2+2k+2}\right) \longrightarrow |x-2|$

$\therefore$ convergent for $|x-2|<1$; $1<x<3$, $R=1$

test $x=1$: $\sum \frac{(-1)^k}{k^2+1}$: $|a_{k+1}| = \frac{1}{k^2+2k+2} < \frac{1}{k^2+1} = |a_k|$ & $\frac{1}{k^2+1} \to 0$

$\therefore$ convergent by alt series test.

test $x=3$: $\sum \frac{1}{k^2+1}$: $a_k = \frac{1}{k^2+1} < \frac{1}{k^2}$ and $\sum \frac{1}{k^2}$ converges (p-series, p=2)

$\therefore$ convergent by comparison test

$\Rightarrow$ Interval of convergence : $[1,3]$

b) root test : $(a_n)^{1/n} = nx \longrightarrow \infty$ $\therefore$ convergent only for $x=0$; $R=0$

c) We first note: $\lim_{k\to\infty} a_k = \sum_{j=1}^{\infty} \left(\frac{4}{5}\right)^j = L = \frac{1}{1-\frac{4}{5}} - \left(\frac{4}{5}\right)^0 = 4$

convergent geometric series.

ratio test : $\left|\frac{b_{k+1}}{b_k}\right| = \left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = |x| \frac{\sum_{j=1}^{k+1}\left(\frac{4}{5}\right)^j}{\sum_{j=1}^{k}\left(\frac{4}{5}\right)^j} \longrightarrow |x| \cdot \frac{L}{L} = |x|$

$\therefore$ convergent for $|x|<1$; $-1<x<1$, $R=1$

test $x=\pm 1$: $a_k = \sum_{j=1}^{k} \left(\frac{4}{5}\right)^j \xrightarrow[k\to\infty]{} 4 \neq 0$

$\therefore \sum (\pm 1)^k a_k$ diverges by test for divergence

$\Rightarrow$ Interval of convergence : $(-1,1)$

3. Determine by directly taking derivatives the Taylor series for the function $f(x) = \dfrac{1}{\sqrt{x}}$ about $x = 9$. Determine the radius of convergence.

$$f(x) = x^{-1/2} \qquad f(9) = \frac{1}{3}$$

$$f'(x) = -\frac{1}{2}x^{-3/2} \qquad f'(9) = -\frac{1}{2}\cdot\frac{1}{3^3}$$

$$f''(x) = \frac{1}{2}\cdot\frac{3}{2}x^{-5/2} \qquad f''(9) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3^5}$$

$$f'''(x) = -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}x^{-7/2} \qquad f'''(9) = -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{3^7}$$

$$\vdots$$

$$f^{(n)}(9) = \frac{(-1)^n}{2^n}\cdot\frac{1}{3^{2n+1}}\cdot(1\cdot 3\cdot 5\cdots(2n-1))$$

$$n \geq 1$$

$$\therefore \frac{1}{\sqrt{x}} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}\,\frac{1}{3^{2n+1}}\,(1\cdot 3\cdot 5\cdots(2n-1))\cdot\frac{(x-9)^n}{n!}$$

ratio test
$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^n}{2^{n+1}}\cdot\frac{3^{2n+1}}{3^{2n+3}}\cdot\frac{n!}{(n+1)!}\cdot\frac{(x-9)^{n+1}}{(x-9)^n}\cdot\frac{(1\cdot 3\cdot 5\cdots(2n-1)\cdot(2n+1))}{(1\cdot 3\cdot 5\cdots(2n-1))}\right|$$

$$= \left|\frac{1}{2}\cdot\frac{1}{9}\cdot\frac{1}{n+1}\cdot(x-9)\cdot(2n+1)\right| = \frac{1}{18}|x-9|\left(\frac{2n+1}{n+1}\right)$$

$$\to \frac{1}{9}|x-9|$$

$\therefore$ convergent for $|x-9| < 9$, $0 < x < 18$, $R = 9$

4. (a) Prove part (ii) of the Ratio Test: Let $\sum a_k$ be a series with positive terms, and suppose that:

$$\frac{a_{k+1}}{a_k} \to \lambda \quad \text{as} \quad k \to \infty$$

Show that if $\lambda > 1$, then $\sum a_k$ diverges.

Given $\frac{a_{k+1}}{a_k} \to \lambda > 1$ as $k \to \infty$

$\therefore \frac{a_{k+1}}{a_k} > 1$ for $k > K$

$\therefore a_{k+1} > a_k$ $\quad \therefore \sum a_k$ diverges by the test for divergence

(b) Prove part (iii) of the Root Test: Let $\sum a_k$ be a series with non-negative terms, and suppose that:

$$(a_k)^{1/k} \to \rho \quad \text{as} \quad k \to \infty$$

Show that if $\rho = 1$, the test is inconclusive: the series may converge or diverge.

Hint: consider $\sum \frac{1}{k}$ and $\sum \frac{1}{k^2}$

$\left(\frac{1}{k}\right)^{1/k} = \frac{1}{k^{1/k}} \to \frac{1}{1} = 1$ as $k \to \infty$

$\left(\frac{1}{k^2}\right)^{1/k} = \left(\frac{1}{k^{1/k}}\right)^2 \to \left(\frac{1}{1}\right)^2 = 1$ as $k \to \infty$

but $\sum \frac{1}{k}$ diverges while $\sum \frac{1}{k^2}$ converges

$\therefore (a_k)^{1/k} \to 1$ can hold for both converging and diverging series

5. Find the Fourier series; i.e., evaluate the Fourier coefficients, for the function:

$$f(t) = |\sin t|, \quad -\pi \le t \le \pi$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

Hint: $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$; $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$; $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin t\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin t\,dt = \frac{1}{\pi}[\cos t]_{-\pi}^{0} - \frac{1}{\pi}[\cos t]_{0}^{\pi}$$

$$= \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi} \Rightarrow \boxed{\frac{a_0}{2} = \frac{2}{\pi}}$$

even function: $b_n = 0$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos t\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin t\cos t\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin t\cos t\,dt$$

$$= \frac{1}{\pi}\left[\frac{\cos^2 t}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{\sin^2 t}{2}\right]_{0}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(nt)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} -\sin t\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}\sin t\cos nt\,dt$$

$(n \ge 2)$

$$= \frac{1}{\pi}\int_{-\pi}^{0}\frac{1}{2}\left[-\sin(1-n)t - \sin(1+n)t\right]dt + \frac{1}{\pi}\int_{0}^{\pi}\frac{1}{2}\left[\sin(1-n)t + \sin(1+n)t\right]dt$$

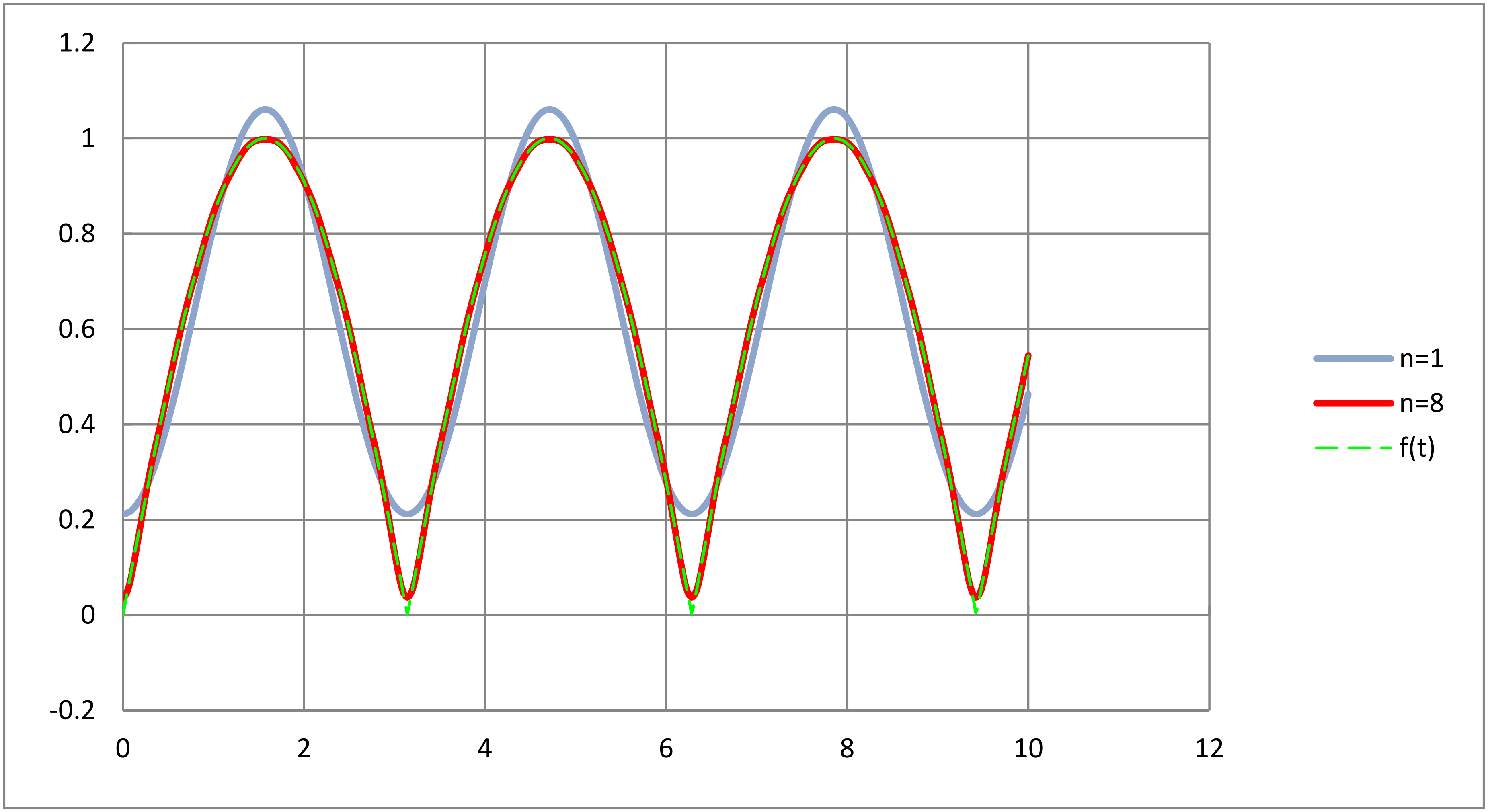
$$= \frac{1}{2\pi}\left[\frac{\cos(1-n)t}{1-n} + \frac{\cos(1+n)t}{1+n}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[-\frac{\cos(1-n)t}{1-n} - \frac{\cos(1+n)t}{1+n}\right]_{0}^{\pi}$$

$$\begin{cases} = 0 \text{ for } n \text{ odd} \\ = \frac{1}{2\pi}\left(\frac{1+1}{1-n} + \frac{1+1}{1+n}\right) + \frac{1}{2\pi}\left(-\frac{-1-1}{1-n} - \frac{-1-1}{1+n}\right) \end{cases}$$

$$= \frac{1}{\pi}\left(\frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n}\right) = \frac{2}{\pi}\left(\frac{1}{1-n} + \frac{1}{1+n}\right) = \frac{2}{\pi}\left(\frac{2}{1-n^2}\right) \quad n \text{ even}$$

let $n = 2k \Rightarrow a_k = \frac{4}{\pi}\left(\frac{1}{1-4k^2}\right) \qquad k = 1, 2, 3, \ldots$

$$\therefore f(t) = \frac{2}{\pi} + \sum_{k=1}^{\infty}\frac{4}{\pi}\left(\frac{1}{1-4k^2}\right)\cos 2kt$$



1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
2
4
6
8
10
12
n=1
n=8
f(t)

6. Find the unit tangent vector, the principle unit normal vector and an equation in x, y, z for the osculating plane at the point $t = 1$ on the curve: $\vec{r}(t) = t^2\,\hat{\imath} - t^3\,\hat{\jmath} + t\,\hat{k}$.

$$\vec{r}(t) = (t^2, -t^3, t)$$

$$\vec{r}\,'(t) = (2t, -3t^2, 1) \implies \|\vec{r}\,'(t)\| = \sqrt{4t^2+9t^4+1}$$

$$\vec{T}(t) = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \frac{(2t, -3t^2, 1)}{\sqrt{4t^2+9t^4+1}} \implies \vec{T}(1) = \frac{1}{\sqrt{14}}(2,-3,1)$$

$$\vec{T}'(t) = \frac{(2,-6t,0)}{\sqrt{4t^2+9t^4+1}} + \left(-\tfrac{1}{2}\right)(4t^2+9t^4+1)^{-3/2}(8t+36t^3)(2t,-3t^2,1)$$

$$\vec{T}'(1) = \frac{1}{\sqrt{14}}(2,-6,0) - (2,-3,1)\frac{44}{2(14)^{3/2}} = \frac{1}{\sqrt{14}}(2,-6,0) - \frac{1}{\sqrt{14}}\left(\frac{22}{7}, -\frac{33}{7}, \frac{11}{7}\right)$$

$$= \frac{1}{\sqrt{14}}\left(-\frac{8}{7}, -\frac{9}{7}, -\frac{11}{7}\right) = \frac{-1}{7\sqrt{14}}(8, 9, 11)$$

$$\vec{N}(1) = \frac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{-(8,9,11)}{\sqrt{8^2+9^2+11^2}} = \frac{-1}{\sqrt{266}}(8,9,11)$$

$$\vec{B}(1) = \vec{T}(1)\times\vec{N}(1) = \frac{1}{\sqrt{14}}(2,-3,1) \times \frac{-1}{\sqrt{266}}(8,9,11)$$

$$= \frac{-1}{\sqrt{14}}\frac{1}{\sqrt{266}}\begin{vmatrix} i & j & k \\ 2 & -3 & 1 \\ 8 & 9 & 11 \end{vmatrix} = \frac{-1}{\sqrt{14}}\frac{1}{\sqrt{266}}(-33-9,\ 8-22,\ 18+24)$$

$$= \frac{-1}{\sqrt{14}}\frac{1}{\sqrt{266}}(-42, -14, 42)$$

$$= \frac{14}{\sqrt{14}}\frac{1}{\sqrt{266}}(3,1,-3) = \frac{1}{\sqrt{19}}(3,1,-3)$$

point on osculating plane: $\vec{r}(1) = (1,-1,1)$

$\therefore$ eq'n of osculating plane: $3(x-1)+1(y+1)-3(z-1)=0$

or $\boxed{3x+y-3z=-1}$

7. Use the $\epsilon - \delta$ method to prove the limit: $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

1) Find δ s.t. $\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \epsilon$ whenever $\sqrt{(x-0)^2+(y-0)^2} < \delta$

$$\Rightarrow \left.\begin{array}{l} |x| \le \sqrt{x^2+y^2} \\ |y| \le \sqrt{x^2+y^2} \end{array}\right\} \Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \left|\frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right| = \sqrt{x^2+y^2}$$

$\therefore$ choose $\delta = \epsilon$

2) Proof: given $0 < \sqrt{x^2+y^2} < \delta = \epsilon$

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| \le \left|\frac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right| = \sqrt{x^2+y^2} < \epsilon$$

$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$ by the definition of a limit

8. Bonus Question: Match the name of the surface to the equation:

(a) Cone

(b) Ellipsoid

(c) Elliptic Paraboloid

(d) Hyperbolic Paraboloid

(e) Hyperboloid of One Sheet

(f) Hyperboloid of Two Sheets

(i) $\frac{z^2}{c^2} - \frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ Hyperboloid of Two sheets

(ii) $\frac{z^2}{a^2} = \frac{x^2}{c^2} + \frac{y^2}{b^2}$ Cone

(iii) $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Ellipttc Paraboloid

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Hyperboloid of One sheet

(v) $\frac{x^2}{a^2} + \frac{y^2}{c^2} + \frac{z^2}{b^2} = 1$ Ellipsoid

(vi) $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Hyperbolic Paraboloid