ECE159H1: Fundamentals of Electric Circuits

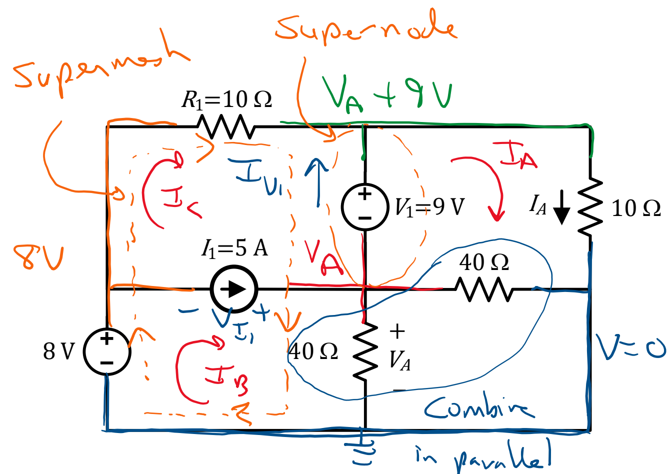
Midterm – Tuesday March 5, 2024
9 AM – 10:50 AM

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The Midterm is worth 80 marks and has four questions. Each question is worth 20 marks.
- Show all of your work, and the final page is left blank which you can use for rough work or for extra space for your answers.
- Marks are allocated for neatness and clarity of communication in your answers.
- Take a deep breath, relax, and enjoy 😊.

Question #1 (20 marks)

1. Consider the circuit shown to the right. Use nodal ***or*** mesh analysis to find the powers of the sources V_1 and I_1, and the power of the resistor R_1. If you want, you can make any reasonable simplifications before you apply nodal or mesh analysis.



(*5 marks*) (a) Take a few minutes to consider how you plan to analyze this circuit to determine the quantities you are asked to find.

Briefly describe your proposed plan below. Your plan should be specific in terms of
(i) Any simplifications you plan to make before your analysis,
(ii) How you plan to set up your nodal or mesh analysis to solve the circuit (i.e., where is the ground node, how are your mesh currents defined, do you need to use any supernodes or supermeshes?), and
(iii) How you plan to use the results of your analysis to calculate the quantities you need to find.
This should be mostly in words, with some equations as needed. Make sure you clearly label all of the unknown quantities you plan to use on the circuit above. The use of bullet points is preferred.

Possible Answers

Nodal Analysis

- Combine 40Ω // 40Ω between node A & ground
- Putting the ground at bottom node results in one unknown, V_A
- Plan: → Do KCL @ supernode to find V_A
  → Once V_A is known, find I_{V_1} from KCL @ node A.
  → Once V_A & I_{V_1} are known:

$$P_{R_1} = \frac{[8-(V_A+9)]^2}{10}, \quad P_{I_1} = 5(8-V_A)^2$$

$$P_{V_1} = I_{V_1} V_1 = I_{V_1}(9)$$

Mesh Analysis

- Combine 40Ω // 40Ω to eliminate one of the meshes
- Define mesh currents as shown on the circuit → three unknowns
- Plan: → Do KVL for supermesh, and mesh A to solve for three unknowns
  → Do KVL in mesh B to find V_{I_1}
  → Once all currents and V_{I_1} are known:

$$P_{R_1} = I_C^2 R_1 = 10 I_C^2$$

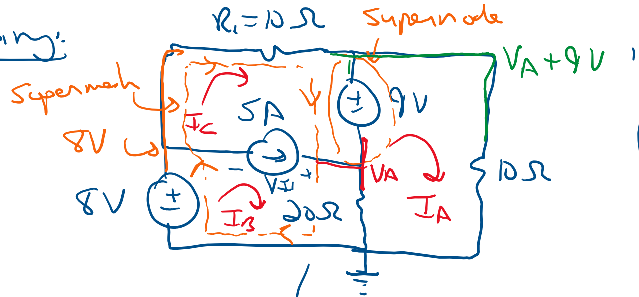
$$P_{I_1} = I_{V_1} V_{I_1}, \quad P_{V_1} = V_1(I_A - I_C)$$

Question #1 (continued)

(*9 marks*) 1. (b) Follow through with the first two steps of the plan you developed in part (a), by making any appropriate simplifications (if necessary), and then completing nodal or mesh analysis.

If you do nodal analysis, enter your answer for V_A in the box below. If you do mesh analysis, enter your answer for I_A in the box below.

Redrawing:



$40\Omega // 40\Omega = 20\Omega$ Between node A & ground

Nodal Analysis

KCL @ Supernode:

$$40\,(5) = \left(\frac{V_A}{40} + \frac{V_A}{40} + \frac{(V_A+9)-8}{10} + \frac{V_A+9}{10}\right)40$$

$$V_A + V_A + 4V_A + 36 - 32 + 4V_A + 36 = 200$$

$$V_A = \frac{160}{10} = \underline{16\ V}$$

KCL @ A:

$$I_{V_1} = 5 - \frac{V_A}{40} - \frac{V_A}{40}$$

$$= 5 - \frac{16}{40} - \frac{16}{40} = \underline{4.2\,A}$$

Mesh Analysis

KVL for mesh A:

$$10I_A + 20(I_A - I_B) = 9$$

$$30I_A - 20I_B = 9 \quad (1)$$

KVL for Supermesh:

$$20(I_B - I_A) - 8 + 10I_C + 9 = 0$$

$$-20I_A + 20I_B + 10I_C = -1 \quad (2)$$

Constraint Eqn: $I_B - I_C = 5 \quad (3)$

Solving system of three equations

$I_A = \underline{2.5A}$, $I_B = \underline{3.3A}$, $I_C = \underline{-1.7A}$

KVL for Mesh B:

$$8 + V_{I_1} = 20(I_B - I_A) \rightarrow V_{I_1} = \underline{8V}$$

For those who did nodal analysis:
$V_A =$ 16V

For those who did mesh analysis:
$I_A =$ 2.5A

Question #1 (continued)

(*6 marks*) 1. (c) Determine the quantities you were asked to find at the beginning of this question. Enter your final answers in the boxes below. Use the standard convention that a negative value for power indicates supplied power.

Using equations from part (a):

Nodal

$$P_{R_1} = \frac{[8-(V_A+9)]^2}{10}$$

$$= \frac{[8-(16+9)]^2}{10} = \underline{\underline{28.9\ W}}$$

$$P_{I_1} = 5(8-V_A)^2$$

$$= 5(8-16)^2 = \underline{\underline{40\ W}}$$

(supplying)

$$P_{V_1} = 9 I_{V_1} = 9(4.2)$$

$$= \underline{\underline{37.8\ W}}$$

(supplying)

Mesh

$$P_{R_1} = I_C^2 R_1 = 10 I_C^2 = 10(-1.7)^2 = \underline{\underline{28.9\ W}}$$

$$P_{I_1} = I_{V_1} V_{I_1} = (5)(8) = \underline{\underline{40\ W}}$$ (supplying)

$$P_{V_1} = V_1(I_A - I_C) = (9)[2.5-(-1.7)]$$

$$= \underline{\underline{37.8\ W}}$$ (supplying)

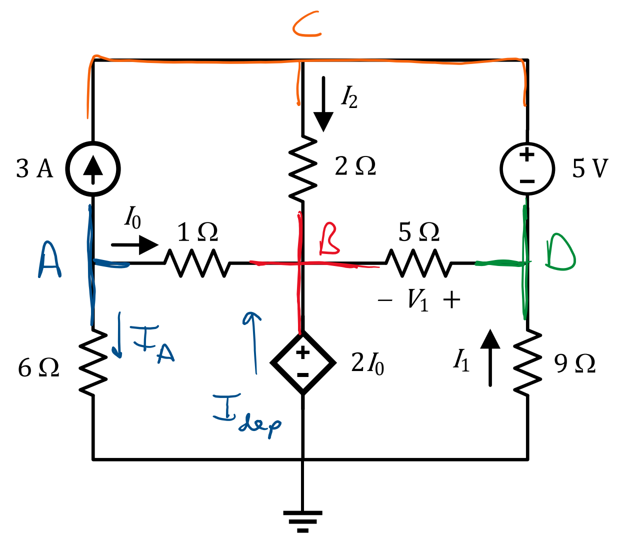
Power of the source I_1	Power of the source V_1	Power of the resistor R_1
$P_{I1} =$ −40 W	$P_{V1} =$ −37.8 W	$P_{R1} =$ +28.9 W

Question #2 (20 marks)

(*8 marks*) 2. (a) In the circuit shown to the right, it is known that $I_0 = -2$ A and $I_1 = \frac{23}{73}$ A.

Use this information to determine the current I_2, the voltage V_1, and the powers supplied by the 3 A source and the dependent source. Enter your answers in the boxes below.

Note: You do not need to apply nodal or mesh analysis to solve this problem. But make sure to clearly justify your answers based on fundamental circuit analysis principles.



1) From KCL @ A: $I_A + 3 + I_o = 0 \rightarrow I_A = -3-(-2) = \underline{-1\text{ A}}$

$\therefore V_A = 6(I_A) = \underline{-6\text{ V}}$

2) Observe $V_B = 2I_o = 2(-2) = \underline{-4\text{ V}}$

3) Observe $V_D = -9I_1 = -9\left(\frac{23}{73}\right) = \underline{-\frac{207}{73} \simeq -2.84\text{ V}}$

4) Observe $V_C = 5 + V_D = \underline{\frac{158}{73}} \simeq 2.16\text{ V}$

5) KCL @ B Gives: $I_o + I_{dep} + \frac{V_C - V_B}{2} + \frac{V_D - V_B}{5} = 0 \rightarrow I_{dep} = \underline{-\frac{96}{73} \simeq -1.315\text{A}}$

$$\therefore I_2 = \frac{V_C - V_B}{2} = \frac{\frac{158}{73} - (-4)}{2} = \underline{\underline{\frac{225}{73} \simeq 3.08\text{ A}}}$$

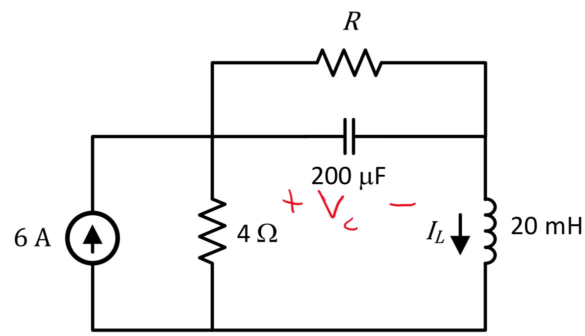
$$V_1 = V_D - V_B = -\frac{207}{73} - (-4) = \underline{\underline{\frac{85}{73} \simeq 1.16\text{ V}}}$$

$P_{3A} = 3(V_C - V_A) = \underline{\underline{24.5\text{W}}}$ (supplying), $P_{dep} = (2I_o)I_{dep} = \frac{384}{73} \simeq \underline{\underline{5.26\text{ W}}}$ (supplying)

Current I_2	Voltage V_2	Power of 3 A Source	Power of Dependent Source
$I_2 = \frac{225}{73} \simeq 3.08\text{A}$	$V_2 = \frac{85}{73} \simeq 1.16\text{V}$	$P_{3A} = -24.5\text{ W}$	$P_{dep} = -5.26\text{ W}$

Question #2 (continued)

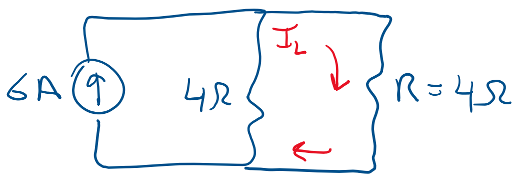
(*6 marks*) 2. (b) The circuit to the right has been in this state for a long time, meaning the circuit has reached its steady-state condition.



Determine if each of the following statements are ***True*** or ***False***, and circle your answer. Provide a brief justification for your answer in words or with appropriate calculations for each choice.

True ***False*** (i) If $R = 4\ \Omega$, then the current through the inductor is $I_L = 3$ A.

At steady-state:



$$I_L = 6\left(\frac{1/4}{1/4 + 1/4}\right) = \underline{\underline{3A}}$$

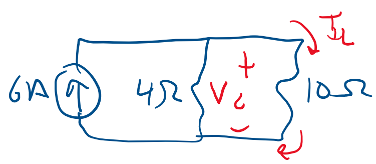
$\underline{\underline{\text{True}}}$

True ***False*** (ii) A resistance of $R = 10\ \Omega$, would ensure that the energy stored in the capacitor is equal to the energy stored in the inductor.

At steady state:

$$V_c = 6(4 \| 10) = 6(2.857\Omega) = 17.143\text{ V}$$

$$I_L = 6\left(\frac{4}{4+10}\right) = 1.714\text{ A}$$



$$W_c = \frac{1}{2}CV_c^2 = \frac{1}{2}(200\times10^{-6})(17.143)^2 = \underline{29.4\text{mJ}}$$

$$W_L = \frac{1}{2}LI_L^2 = \frac{1}{2}(20\times10^{-3})(1.714)^2 = \underline{29.4\text{ mJ}}$$

$\therefore$ With $R = 10\Omega$, $W_c = W_L \rightarrow$ $\underline{\underline{\text{True}}}$

True ***False*** (iii) The 200 μF capacitor is a parallel-plate capacitor. To double the value of the capacitance, one should double the spacing between the parallel plates.

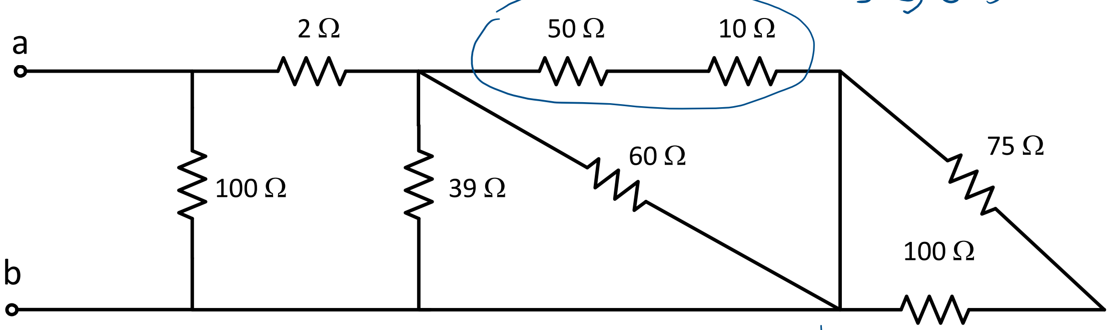
$C = \frac{\varepsilon A}{d} \rightarrow$ To double C, one must halve d.

$\hookrightarrow$ $\underline{\underline{\text{False}}}$

Question #2 (continued)

(*6 marks*) 2. (c) Find the equivalent resistance as seen from terminals a-b for the circuit shown below. Be sure to provide a clear description of your steps to find the equivalent resistance. Enter your answer (to one decimal place) in the box below.

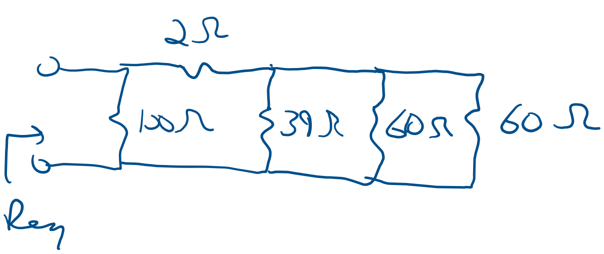
Series ⇒ 60Ω



R_{eq}

Since the series combination of the 100Ω and 75Ω resistor is shorted out, they don't contribute to R_{eq}.

Re drawing



$$\therefore R_{eq} = 100 \,//\, (2 + 39 \,//\, 60 \,//\, 60)$$

$$= \left[\frac{1}{100} + \frac{1}{2 + \left[\frac{1}{39} + \frac{1}{60} + \frac{1}{60}\right]^{-1}}\right]$$

$$= \left[\frac{1}{100} + \frac{1}{2 + 16.9565}\right]$$

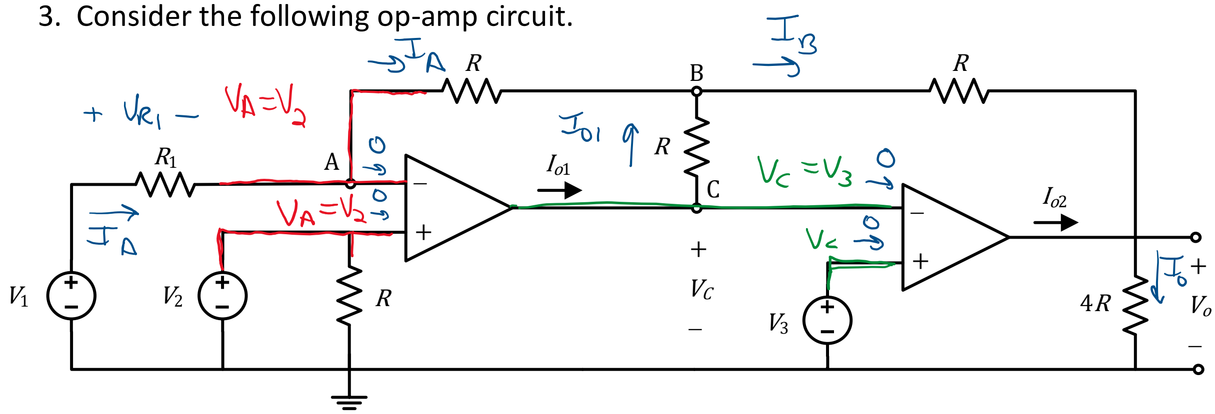
$$= \underline{\underline{15.9\ \Omega}}$$

Equivalent Resistance

$R_{eq} =$ 15.9 Ω

Question #3 (20 marks)

3. Consider the following op-amp circuit.



(*6 marks*) (a) Apply the ideal op-amp model to this circuit, and describe three conclusions about the operation of the circuit you can make as a result of using this ideal model. For example, consider what this model tells you about the voltages V_A and V_C. You do not need to do any calculations for this part, and your answer can be in words or simple equations.

Possible Answers (should have 3)

- Observe $V_A = V_2$
- Observe $V_c = V_3$
- Observe $I_A = \frac{V_{R_1}}{R_1} = \frac{V_1 - V_A}{R_1} = \frac{V_1 - V_2}{R_1}$
- Observe the current through R_1, I_A, is the same as that through the first feedback resistor, R
- The above point means: $I_A = \frac{V_1 - V_2}{R_1} = \frac{V_2 - V_B}{R} \rightarrow V_B = V_2 - \frac{R}{R_1}(V_1 - V_2)$
  $= (1 + \frac{R}{R_1})V_2 - \frac{R}{R_1}V_1$
- Observe $I_B = I_A + I_{o1}$
- Observe $I_{o1} = \frac{V_C - V_B}{R} = \frac{V_3 - V_B}{R} = \frac{V_3}{R} + \frac{V_1}{R_1} - (\frac{1}{R} + \frac{1}{R_1})V_2$

Question #3 (continued)

(*10 marks*) (b) Determine the expression for the output voltage V_0 in terms of V_1, V_2, V_3, R_1 and R.

<u>Applying KCL @ B:</u> $I_B = I_A + I_{o1}$

But $I_B = \dfrac{V_B - V_o}{R}$ and using
$$\left.\begin{aligned} V_B &= \left(1+\frac{R}{R_1}\right)V_2 - \frac{R}{R_1}V_1 \\ I_A &= \frac{V_1 - V_2}{R_1} \\ I_{o1} &= \frac{V_3}{R} + \frac{V_1}{R_1} - \left(\frac{1}{R}+\frac{1}{R_1}\right)V_2 \end{aligned}\right] \text{ from above}$$

$$\left(\frac{1}{R}+\frac{1}{R_1}\right)V_2 - \frac{1}{R_1}V_1 - \frac{V_o}{R} = \frac{V_1}{R_1} - \frac{V_2}{R_1} + \frac{V_3}{R} + \frac{V_1}{R_1} - \left(\frac{1}{R}+\frac{1}{R_1}\right)V_2$$

$$\therefore\ V_o = R\left[2\left(\frac{1}{R}+\frac{1}{R_1}\right)V_2 + \frac{V_2}{R_1} - \frac{V_3}{R} - \frac{3V_1}{R_1}\right]$$

$$= \underline{\underline{-\frac{3R}{R_1}V_1 + \left(2+3\frac{R}{R_1}\right)V_2 - V_3}}$$

Question #3 (continued)

(*4 marks*) (c) Given the values of $V_1 = 4$ V, $V_2 = 2$ V, $V_3 = 1$ V, $R_1 = 2$ kΩ and $R = 1$ kΩ, determine the op-amp output currents I_{o1} and I_{o2}. Enter your results in the boxes below.

For these values: $V_0 = \frac{-3R}{R_1} V_1 + \left(2 + 3\frac{R}{R_1}\right) V_2 - V_3 = 0\text{V}$

$$I_A = \frac{V_1 - V_2}{R_1} = 1\text{mA}$$

$$V_B = \left(1 + \frac{R}{R_1}\right) V_2 - \frac{R}{R_1} V_1 = 1\text{V}$$

$$I_B = \frac{V_B - V_0}{R} = 1\text{mA}$$

$$\therefore I_{o1} = I_B - I_A = \underline{\underline{0\text{ mA}}}$$

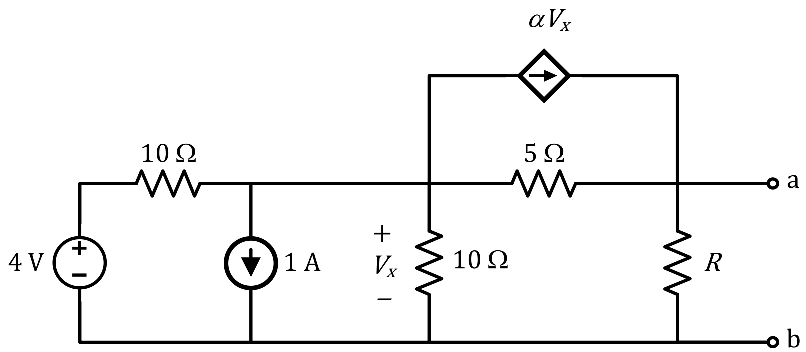
$$I_{o2} = I_0 - I_B = \frac{V_0}{4R} - I_B = \underline{\underline{-1\text{mA}}}$$

Output current for op-amp #1:	Output current for op-amp #2:
$I_{o1} =$ 0mA	$I_{o2} =$ −1mA

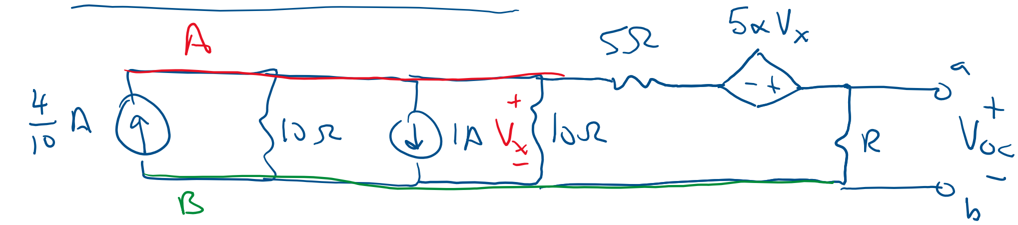
Question #4 (20 marks)

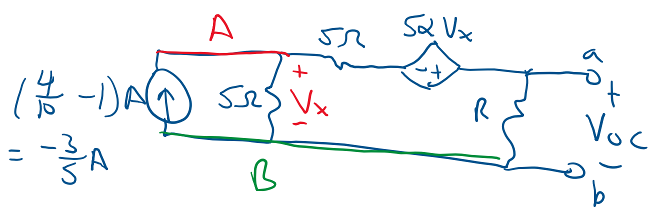
4. In the circuit given below,

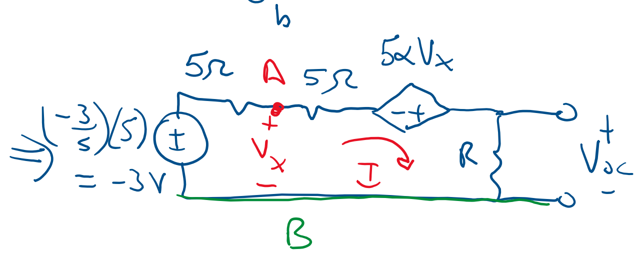
(*12 marks*) (a) Find the Thévenin equivalent circuit for the complete circuit, as seen at the terminals a-b. Enter your final answers for V_{Th} and R_{Th} in the boxes on the next page. Your answers should be in terms of R and α. You may assume that α is a positive constant.



To find $V_{Th} = V_{oc}$:







KVL: $-3 = 5I + 5I - 5\alpha V_x + RI$ (1)

But $V_x = -5I - 3$ (2)

(2) → (1) $-3 = (10+R)I - 5\alpha(-5I-3) = (10+R+25\alpha)I + 15\alpha$

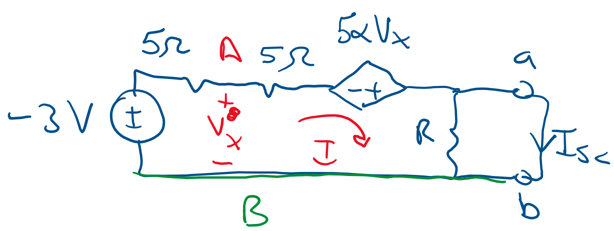
$\therefore I = \dfrac{-3-15\alpha}{10+R+25\alpha} \rightarrow V_{oc} = V_{Th} = RI = \dfrac{-3(1+5\alpha)R}{10+R+25\alpha}$

Question #4 (continued)

4. (a) (continued)

For R_{Th}, find I_{sc}:

Using the above simplified circuit



$\Rightarrow$

$\therefore$ KVL: $-3 = (5+5)I_{sc} - 5\alpha V_x$ but $V_x = -5I_{sc} - 3$

$$-3 = 10I_{sc} - 5\alpha(-5I_{sc} - 3)$$

$$I_{sc} = \frac{-3 - 15\alpha}{10 + 25\alpha}$$

$$\therefore R_{Th} = \frac{V_{oc}}{I_{sc}} = \frac{\frac{\cancel{-3(1+5\alpha)}R}{10 + R + 25\alpha}}{\frac{\cancel{-3(1+5\alpha)}}{10+25\alpha}} = \frac{R(10+25\alpha)}{10 + 25\alpha + R}$$

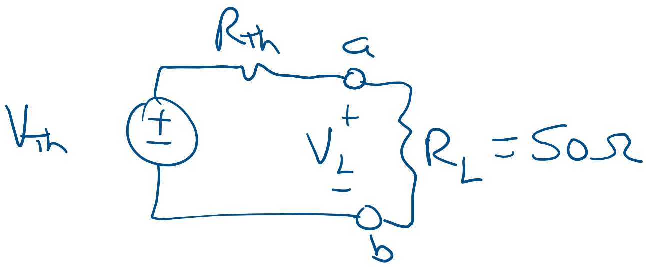
$$= \frac{R}{1 + \frac{R}{10+25\alpha}}$$

Thévenin equivalent source	Thévenin equivalent resistance
$V_{Th} = \frac{-3(1+5\alpha)R}{10 + R + 25\alpha}$	$R_{Th} = \frac{R}{1 + \frac{R}{10+25\alpha}}$

Question #4 (cont'd)

(*8 marks*) 4. (b) Determine the required values of R and α so that a maximum power of $P_{max} = 4.39$ W is delivered by the circuit to a $50\ \Omega$ load resistor (i.e., a resistor that is placed across the terminals a-b).

For maximum power:



For maximum power:

$R_{Th} = R_L = 50\Omega$

$V_L = \frac{V_{Th}}{2}$

For $P_{max} = 4.39\ \omega = \frac{V_{Th}^2}{4R_{Th}} = \frac{V_{Th}^2}{4(50)} \rightarrow V_{Th} = \pm 29.63\ V$

$\therefore R_{Th} = \frac{R}{1 + \frac{R}{10+25\alpha}} = 50 \rightarrow R = 50 + \frac{50R}{10+25\alpha}$

$\therefore 10R + 25R\alpha = 500 + 1250\alpha + 50R \rightarrow 40R + (1250 - 25R)\alpha = -500$

$\hookrightarrow \alpha = \frac{-500 - 40R}{1250 - 25R} = \frac{-100 - 8R}{250 - 5R}$

Observing that α & R is positive and that $V_{Th} = \frac{-3(1+5\alpha)R}{10 + R + 25\alpha}$

we choose $V_{Th} = -29.63V = \frac{-3(1+5\alpha)R}{10+R+25\alpha}$

$-296.3 - 29.63R - 740.8\alpha = -3R - 15\alpha R$

$-296.3 - 26.63R - 740.8\left(\frac{-100-8R}{250-5R}\right) + 15R\left(\frac{-100-8R}{250-5R}\right) = 0$

$-74078 + 1481.6R - 6657.8R + 133.2R^2 + 74078 + 5926.2R - 1500R - 120R^2 = 0$

$13.2R^2 - 750R = 0 \rightarrow R = 0$ (not relevant)

$R = 57\Omega \quad \alpha = 15.9$

Required value for R:

$R =$ 57Ω

Required value for α:

$\alpha =$ 15.9

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