UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test II

First Year — Program 5

MAT185H1S — Linear Algebra

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1 March 2016

Student Name: Fair Copy

Last Name First Names

Student Number:

Tutorial Section: TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 6 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). State the *Fundamental Theorem of Linear Algebra*.

Let $\mathcal{V}$ be a vector space spanned by n vectors. If a set of m vectors from $\mathcal{V}$ is linearly independent, then $m \leq n$.

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1(b). The *dimension* of a vector space is defined as

The dimension of a vector space $\mathcal{V}$, denoted $\dim \mathcal{V}$, is the number of vectors in any of its bases.

/2

1(c). The *row space* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$ is defined as

$\text{row}\,\mathbf{A} = \text{span}\{\mathbf{r}_1, \cdots \mathbf{r}_m\}$, where $\mathbf{r}_i, i = 1 \cdots m$, are the rows of $\mathbf{A}$.

/2

1(d). Give a method by which to determine the *rank* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$.

Row-reduce and count the number of nonzero rows.

/2

1(e). Let E be the standard basis for ${}^n\mathbb{R}$ and let $F = \{\mathbf{f}_1 \cdots \mathbf{f}_n\}$ be another basis for ${}^n\mathbb{R}$. Give an expression for the transformation (transition) matrix from F to E.

$$\mathbf{Q} = \begin{bmatrix} \mathbf{f}_1 & \cdots & \mathbf{f}_n \end{bmatrix}$$

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). Let $\mathcal{Q}$ be $\mathbb{P}_4$ such that $p(1) + p(-1) = 0$ and $p(2) + p(-2) = 0$ for any $p \in \mathcal{Q}$. Then $\mathcal{Q}$ is a three-dimensional subspace of $\mathbb{P}_4$. T

2(b). A basis for ${}^n\mathbb{R}^n$ cannot include any noninvertible matrix. F

2(c). Let $\mathbf{A} \in {}^3\mathbb{R}^3$. Then the set $\{\mathbf{1}, \mathbf{A}, \mathbf{A}^2 \cdots \mathbf{A}^9\}$ is linearly dependent. T

2(d). Let B be a basis for a vector space $\mathcal{V}$ and let $\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_k\}$ be vectors in $\mathcal{V}$ with coordinates $\{\mathbf{v}_1 \cdots \mathbf{v}_k\}$ with respect to B. Then $\dim \operatorname{span}\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_k\} = \dim \operatorname{span}\{\mathbf{v}_1 \cdots \mathbf{v}_k\}$. T

2(e). Let $E = \{\boldsymbol{e}_1 \cdots \boldsymbol{e}_n\}$ and $F = \{\boldsymbol{f}_1 \cdots \boldsymbol{f}_n\}$ be two bases for a vector space $\mathcal{V}$ and let $\mathbf{P}$ be the transformation (transition) matrix from E to F. Then the first row of $\mathbf{P}$ is $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ if and only if $\boldsymbol{e}_1 = \boldsymbol{f}_1$. F

3. Determine a basis for

$$\operatorname{span}\left\{\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 10 \end{bmatrix}, \begin{bmatrix} 3 & 7 \\ -2 & 6 \end{bmatrix}\right\} \sqsubseteq {}^2\mathbb{R}^2$$

Represent each matrix as a row, i.e.,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & c & d \end{bmatrix}$$

(There's a one-to-one correspondence so we can work with the rows rather than the matrices directly.) Putting the resulting rows in a matrix we have

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 1 & 4 \\ -1 & 0 & 1 & 10 \\ 3 & 7 & -2 & 6 \end{bmatrix}$$

In row-reducing, we find at some point (your numbers may vary depending on the arrangement of $\mathbf{A}$) that

$$\mathbf{EA} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & \frac{21}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which informs us that the rows of $\mathbf{A}$ are linearly independent. Thus the original 2×2 matrices are linearly independent and the spanning set itself can serve as a basis for the span.

/10

4. For $\mathbf{A} \in {}^m\mathbb{R}^n$, $\mathbf{B} \in {}^n\mathbb{R}^p$, prove the following:

(a) $\text{rank}\,\mathbf{AB} \leq \min(\text{rank}\,\mathbf{A}, \text{rank}\,\mathbf{B})$

(b) $\dim \text{null}\,\mathbf{A} - \dim \text{null}\,\mathbf{A}^T = n - m$

4(a). Prove $\text{rank}\,\mathbf{AB} \leq \min(\text{rank}\,\mathbf{A}, \text{rank}\,\mathbf{B})$.

Let $\mathbf{A} = [a_{ij}]$ and

$$\mathbf{B}' = \mathbf{AB} = \begin{bmatrix} \mathbf{b}'_1 \\ \vdots \\ \mathbf{b}'_m \end{bmatrix} \in {}^m\mathbb{R}^p$$

Each row $\mathbf{b}'_i = \sum_{j=1}^n a_{ij}\mathbf{b}_j \in \text{row}\,\mathbf{B}$. Hence $\text{row}\,\mathbf{B}'$ is a subspace of $\text{row}\,\mathbf{B}$. (This is a generalization to Proposition J in Chapter 7, in which the left-multiplying matrix is not necessarily square.) That is, $\text{row}\,\mathbf{AB} \sqsubseteq \text{row}\,\mathbf{B}$ and accordingly $\dim \text{row}\,\mathbf{AB} \leq \dim \text{row}\,\mathbf{B}$. Thus $\text{rank}\,\mathbf{AB} \leq \text{rank}\,\mathbf{B}$.

In similar fashion, $\text{col}\,\mathbf{AB} \sqsubseteq \text{row}\,\mathbf{A}$ and accordingly $\dim \text{col}\,\mathbf{AB} \leq \dim \text{col}\,\mathbf{A}$. Thus $\text{rank}\,\mathbf{AB} \leq \text{rank}\,\mathbf{A}$.

Therefore $\text{rank}\,\mathbf{AB}$ is less than either the rank of $\mathbf{A}$ or $\mathbf{B}$, i.e., $\text{rank}\,\mathbf{AB} \leq \min(\text{rank}\,\mathbf{A}, \text{rank}\,\mathbf{B})$.

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4(b). Prove $\dim \text{null}\,\mathbf{A} - \dim \text{null}\,\mathbf{A}^T = n - m$.

Using the dimension formula,

$$\begin{aligned} \dim \text{null}\,\mathbf{A} &= n - \text{rank}\,\mathbf{A} \\ \dim \text{null}\,\mathbf{A}^T &= m - \text{rank}\,\mathbf{A}^T \end{aligned}$$

But $\text{rank}\,\mathbf{A} = \text{rank}\,\mathbf{A}^T$. Subtracting then one formula from the other yields

$$\dim \text{null}\,\mathbf{A} - \dim \text{null}\,\mathbf{A}^T = n - m$$

/5

5. Prove that the following statements are equivalent:

1. $\mathbf{Ax} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in {}^m\mathbb{R}$
2. $\operatorname{col} \mathbf{A} = {}^m\mathbb{R}$
3. $\operatorname{rank} \mathbf{A} = m$

where $\mathbf{A} \in {}^m\mathbb{R}^n$.

[1 ⇒ 2] If $\mathbf{Ax} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in {}^m\mathbb{R}$ then the columns of $\mathbf{A}$ must span ${}^m\mathbb{R}$. So $\operatorname{col} \mathbf{A} = {}^m\mathbb{R}$.

[2 ⇒ 3] If $\operatorname{col} \mathbf{A} = {}^m\mathbb{R}$, then $\dim \operatorname{col} \mathbf{A} = m$, i.e., $\operatorname{rank} \mathbf{A} = m$.

[3 ⇒ 1] If $\operatorname{rank} \mathbf{A} = m$, then $\dim \operatorname{col} \mathbf{A} = m$. This means that $\operatorname{col} \mathbf{A} = {}^m\mathbb{R}$ because, clearly, $\operatorname{col} \mathbf{A} \sqsubseteq {}^m\mathbb{R}$ and the dimensions of $\operatorname{col} \mathbf{A}$ and ${}^m\mathbb{R}$ are both m. Hence by Theorem VI in Chapter 6, the two spaces must be equal.

As $\operatorname{col} \mathbf{A} = {}^m\mathbb{R}$, any column in ${}^m\mathbb{R}$ can be expressed as a linear combination of the columns of $\mathbf{A}$. In other words, there exists some $\mathbf{x} \in {}^n\mathbb{R}$ such that $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b} \in {}^m\mathbb{R}$.

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