UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test I

First Year — Program 5

MAT185H1S — Linear Algebra

Examiners: G S Scott & G M T D'Eleuterio

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Student Name: Fair Copy

Last Name First Names

Student Number:

Tutorial Section: TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 8 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). *Closure under vector addition* in a vector space is defined as

$u + v \in \mathcal{V}$, the vector space, for all $u, v \in \mathcal{V}$

/2

1(b). The *unitary property* in a vector space is defined as

$1u = u$ for all $u \in \mathcal{V}$

/2

1(c). The *subtraction of vectors* means

$u - v = u + (-v)$

/2

1(d). State the *Cancellation Theorem*.

If $u + w = v + w$ (where $u, v, w \in \mathcal{V}$) then $u = v$.

/2

1(e). A *linear combination* of $\{v_1, v_2 \dots v_n\}$ is defined as

$\lambda_1 v_1 + \cdots + \lambda_n v_n$ for any $\lambda_i \in \Gamma$, the scalar field.

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). Let $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\mathbf{v} \in {}^n\mathbb{R}$. If $\mathbf{Av} = \mathbf{0}$ then either $\mathbf{A} = \mathbf{O}$ or $\mathbf{v} = \mathbf{0}$. $\boxed{\mathcal{F}}$

2(b). The negative of $\boldsymbol{0}$ in any vector space is $\boldsymbol{0}$. $\boxed{\mathcal{T}}$

2(c). Let $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\mathbf{B} \in {}^n\mathbb{R}^p$. If null $\mathbf{B}$ contains a nonzero vector then null $\mathbf{AB}$ must also contain a nonzero vector. $\boxed{\mathcal{T}}$

2(d). The set $U = \{f \in \mathcal{F}[\mathbb{R}] \mid f'' + f^2 = 0\}$ is a subspace of $\mathcal{F}[\mathbb{R}]$. $\boxed{\mathcal{F}}$

2(e). Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, a vector space, then

$$\operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}\} = \operatorname{span}\{\boldsymbol{u}, \boldsymbol{v}\} \cup \operatorname{span}\{\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}\}$$

$\boxed{\mathcal{F}}$

C. Problems

3. Show that $(-\alpha)\boldsymbol{v} = \alpha(-\boldsymbol{v})$ where $\boldsymbol{v}$ is an arbitrary vector in a vector space and α is any scalar. (You may use any property of vector spaces except the property that you are being asked to prove.)

As in Chapter 4.

/10

4. Consider $V = \mathbb{R}^2$ and define vector addition and scalar multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$\lambda(x, y) = (\lambda x, \lambda y)$$

where $\lambda \in \mathbb{R}$. (Geometrically, $\boldsymbol{u} + \boldsymbol{v}$ is the midpoint of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.)

Check whether the following properties are satisfied:

(i) The distributive property over vector addition, *i.e.*, $\lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v}$.

(ii) The existence of a zero vector.

(iii) The associative property for vector addition.

Does V over $\mathbb{R}$, with vector addition and scalar multiplication as defined above, establish a vector space?

(i) Note

$$\lambda[(x_1, y_1) + (x_2, y_2)] = \left(\lambda\frac{x_1 + x_2}{2}, \lambda\frac{y_1 + y_2}{2}\right)$$

and

$$\begin{aligned}\lambda(x_1, y_1) + \lambda(x_2, y_2) &= (\lambda x_1, \lambda y_1) + (\lambda y_1 + \lambda y_2)\\ &= \left(\frac{\lambda x_1 + \lambda x_2}{2}, \frac{\lambda y_1 + \lambda y_2}{2}\right)\\ &= \left(\lambda\frac{x_1 + x_2}{2}, \lambda\frac{y_1 + y_2}{2}\right)\end{aligned}$$

Therefore, distribution holds.

(ii) We seek a fixed (a, b) such that $(x, y) + (a, b) = (x, y)$ for all $x, y \in \mathbb{R}$ according to $AIII$. Now

$$(x, y) + (a, b) = \left(\frac{x + a}{2}, \frac{y + b}{2}\right)$$

which requires $\frac{1}{2}(x + a) = x$ or $a = x$. A fixed value of a cannot satisfy AIII for all x. Therefore, a zero does not exist.

. . . cont'd

4. . . . cont'd

(iii) Note

$$[(x_1,y_1)+(x_2,y_2)]+(x_3,y_3) = \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)+(x_3,y_3)$$
$$= \left(\frac{x_1+x_2+2x_3}{4},\frac{y_1+y_2+2y_3}{4}\right)$$

but

$$(x_1,y_1)+[(x_2,y_2)+(x_3,y_3)] = (x_1,y_1)+\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right)$$
$$= \left(\frac{2x_1+x_2+x_3}{4},\frac{2y_1+y_2+y_3}{4}\right)$$

Therefore, associativity for vector addition does not hold.

We conclude that, because at least one of the axioms of a vector space is violated (namely, at least AII and AIII), V does not constitute a vector space.

/10

5. Let $\mathcal{V}$ be a vector space over a field Γ.

(a) State the Subspace Test.

(b) If $\mathcal{V} = {}^3\mathbb{R}$ and

$$\mathcal{S}_t = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \; 2x - y + 3z = t \right\}$$

for a real scalar t, prove that $\mathcal{S}_t$ is a subspace of ${}^3\mathbb{R}$ if and only if $t = 0$.

5(a). State the Subspace Test.

As in Chapter 5.

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5(b). If $\mathcal{V} = {}^3\mathbb{R}$ and

$$\mathcal{S}_t = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, 2x - y + 3z = t \right\}$$

for a real scalar t, prove that $\mathcal{S}_t$ is a subspace of ${}^3\mathbb{R}$ if and only if $t = 0$.

[$\Rightarrow$] If $\mathcal{S}_t$ is a subspace, then SI must hold and the zero of $\mathcal{S}_t$ must be the zero of $\mathcal{V}$. This requires that $x = y = z = 0$ and hence $t = 0$.

[$\Leftarrow$] If $t = 0$, we check that $\mathcal{S}_0$ is a subspace of $\mathcal{V}$ by the Subspace Test:

SI. $[0 \;\; 0 \;\; 0]^T \in \mathcal{S}_0$; hence the zero exists.

SII. Let $\mathbf{u} = [x_1 \;\; y_1 \;\; z_1]^T$ and $\mathbf{v} = [x_2 \;\; y_2 \;\; z_2]^T$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$$

Now

$$\begin{aligned} 2(x_1 + x_2) - (y_1 + y_2) + 3(z_1 + z_2) &= (2x_1 - y_1 + 3z_1) + (2x_2 - y_2 + 3z_2) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

which means that $\mathbf{u} + \mathbf{v} \in \mathcal{S}_0$.

SIII. Let $\mathbf{u} = [x \;\; y \;\; z]^T$. Then

$$\lambda\mathbf{u} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

and

$$\begin{aligned} 2(\lambda x) - (\lambda y) + 3(\lambda z) &= \lambda(2x - y + 3z) \\ &= \lambda(0) \\ &= 0 \end{aligned}$$

which means the $\lambda\mathbf{u} \in \mathcal{S}_0$.

According to the Subspace Test then, $\mathcal{S}_t$ is a subspace of $\mathcal{V}$ when $t = 0$.

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