UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
FINAL EXAMINATION, April 2020
PHY 294

Maximum number of points possible: 25+25 = 50.
Total number of pages = 4.
You have 48 hrs to finish this exam. As this is a take-home exam, we do not provide a formula sheet. Feel free to use mathematical expressions (for integrals, Taylor expansions, etc.) found in the book without derivation. You may discuss and collaborate with your peers but must submit your own handwritten solutions. Direct copying of someone else's solutions does not count as collaboration.

I. Quantum: *Uncertainty principle* *5 points*

Consider placing a particle of mass m at rest on a flat table of length L (assume one-dimensional world). If you are a classical physicist you can place this particle at any spot on the table, and since there are no horizontal forces acting on this particle, it can stay on the table forever. However, quantum uncertainty messes this up as you know. Assuming you place the ball at the center of the table, and using the uncertainty principle, estimate what is the longest time for which this particle can stay on this table. What is this estimated longest timescale for $L = 1\,\text{meter}$ and $m = 100\,\text{gm}$? What is the estimated timescale to fall off the table if we assume a given position uncertainty $\Delta x = 1\,\text{Å}$? Do these numbers seem consistent with your physical experience?

II. Quantum: *Tunneling* *10 points*

Consider a particle of mass m in one dimension which experiences a potential energy (with $V_0 > 0$):

$$V(x) = V_0\delta(x - L) + V_0\delta(x + L).$$

• Schematically sketch $V(x)$.

• Derive the transmission coefficient T for the particle incident from the far left (region 1: $x < L$) with energy E, and going to the right (i.e., into region 3: $x > L$). Express it in the simplest possible final form.

• Sketch T as a function of $k \equiv (1/\hbar)\sqrt{2mE}$.

III. Quantum: *Two level system* *10 points*

Consider a spin-1/2 particle which at time $t = 0$ starts off in the state $\psi_0 \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Let this particle be subject to a Hamiltonian $H = -B\hat{S}_x$.

(i) Find the state of this particle ψ_1 at time $t = \frac{\pi}{4B}$.
(ii) Find the average values $\langle \hat{S}_x \rangle$ and $\langle \hat{S}_z \rangle$ at time $t = \frac{\pi}{4B}$.

(iii) What is probability that at time $t = \frac{\pi}{4B}$, this particle is in the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

(iv) Is this particle in a stationary state? Why or why not?

Recall the spin operators are given by

$$\hat{S}_x \equiv \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \hat{S}_y \equiv \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \hat{S}_z \equiv \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

IV. Thermal: *Collision rate and pressure from the Maxwell distribution* *10 points*

Consider a classical ideal monatomic gas of N atoms, all of mass m. Imagine that the gas is occupying a large cubic box of volume $V = L^3$, with faces perpendicular to the Cartesian coordinate axes (x, y, z). Let $dN(v_x, v_y, v_z)$ denote the number of atoms whose components of the velocity lie in the infinitesimal intevals $v_x \in [v_x, v_x + dv_x]$, $v_y \in [v_y, v_y + dv_y]$, $v_z \in [v_z, v_z + dv_z]$. According to the Maxwell distribution, we have that

$$dN(v_x, v_y, v_z) = Ce^{-\frac{m\vec{v}^2}{2kT}} dv_x dv_y dv_z \ , \tag{1}$$

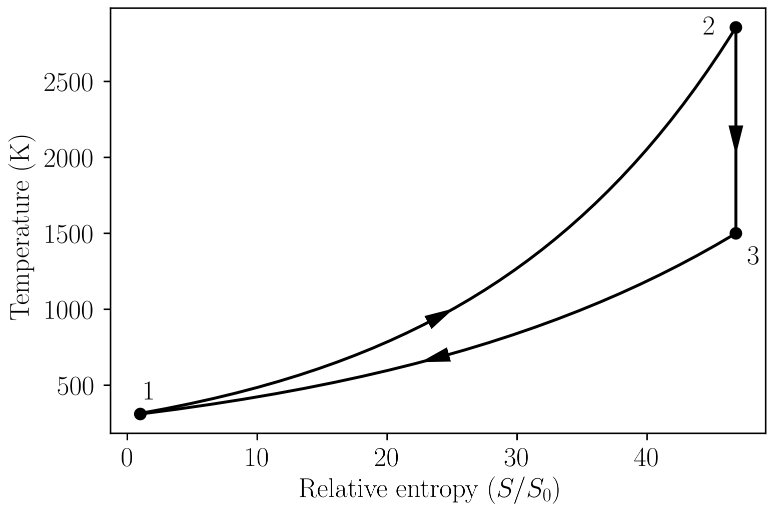
where k is the Boltzmann constant, T-the absolute temperature, and $\vec{v} = (v_x, v_y, v_z)$ is the velocity vector, $\vec{v}^2 = v_x^2 + v_y^2 + v_z^2$, and C is a normalization constant.

1. Determine the value of the normalization constant C.
2. Find the number of atoms per unit volume whose x-th component of the velocity, v_x, lies in the interval $v_x \in [v_x, v_x + dv_x]$, irrespective of the values of v_y and v_z. Denote this quantity by $dn(v_x)$.
3. Use the distribution $dn(v_x)$ to determine the number of collisions, in time Δt, of the atoms of the gas with the wall of the box perpendicular to the positive-x direction. Use the result to show that the number of collisions of atoms of the gas with a unit area of the wall per unit time is
$$j = \frac{1}{4} n \bar{v} \tag{2}$$
where $\bar{v} = \sqrt{\frac{8kT}{\pi m}}$, as per eq. (6.52) of the book, is the average speed of the atoms in the gas, and n is the density (number of atoms per unit volume) of the gas.
4. Next, use $dn(v_x)$ to compute the pressure of the gas, using momentum conservation (or, equivalently, Newton's third law). Have you seen a similar—but more simplistic—derivation and obtained a similar expression for the pressure before?

`Hint:` *You may use, without derivation, expressions for the integrals you encounter along the way (such as the ones, say, from the Appendix of the book). However, you must carefully explain the physical picture and justification for the various steps involved in your calculations, rather than just present a string of formulae. A correct formula without explanation will not get full credit!*

V. Thermal: *Efficiency of Lenoir cycle* *10 points*

Consider the Lenoir cycle, shown on the Figure in $T-S$ coordinates (please disregard the numerical values shown in the Figure). This is an idealized thermodynamic cycle often used to model a pulse jet engine. It is based on the operation of an engine patented by Jean Joseph Etienne Lenoir in 1860. The engine is often thought of as the first commercially produced internal combustion engine. The absence of any compression process in the design leads to lower thermal efficiency than the more well known Otto and Diesel cycles.



The cycle is that of monatomic ideal gas undergoing the following steps:

- $1-2$: constant volume (isochoric) heat addition,
- $2-3$: isentropic (i.e. adiabatic quasistatic) expansion,
- $3-1$: constant pressure (isobaric) heat rejection.

To help your thinking, re-draw the cycle in the more familiar $p-V$ coordinates. For definiteness, take the initial pressure and volume, the $p-V$ coordinates of point 1 to be p_1 and V_1, respectively. Also, let the pressure of point 2 be p_2.

1. Find the efficiency e of the cycle as a function of the ratio of pressures $x = \frac{p_2}{p_1}$ (and no other variables), in a form that makes manifest what values e can take.
2. Find the efficiency in the limits of $x \to 1$ and $x \gg 1$ (formally, the latter limit can be taken as $x \to \infty$).

Hint: Notice that you can argue for the limits in 2. without calculation based on general principles, so this is somewhat independent of the correctness of your answer in 1. above. In fact, a full mark will only be given if you explain how your result about the efficiency in these limits (based on the calculation in 1.) squares with general ideas about efficiencies of heat engines!

VI. Thermal: *Paramagnet and magnetic cooling* *5 points*

Consider a spin-1/2 paramagnet of N spins, labeled by s_i, $i = 1, ..., N$. The spins take two values, $s_i = \pm 1$. The paramagnet is placed in magnetic field B so that the energy of a single spin is $E_{s_i} = -\mu B s_i$ (for $i = 1, ..., N$), where μ is the magnetic moment of each spin. It is in thermal equilibrium with a reservoir at temperature T. The magnetization is $M = \mu S = \mu(s_1 + s_2 + \ldots + s_N)$, where S is the total spin.

1. Find the Boltzmann partition function of the paramagnet as a function of T, N, B (and μ and k—the Boltzmann constant, of course).

2. Find the average energy $\langle E \rangle$ of the paramagnet and use it to deduce the average magnetization $\langle M \rangle$. Sketch the behaviour of $\frac{\langle M \rangle}{|M_{max}|}$ as a function of the temperature, and discuss the behaviour in the limits of low, $T \to 0$, and high, $T \to \infty$, temperatures. Here, $|M_{max}|$ is the absolute value of the maximal possible value that the magnetization can take.

3. Explain qualitatively, using the T, B dependence of $\langle M \rangle$, how the paramagnet can be used for "magnetic cooling".