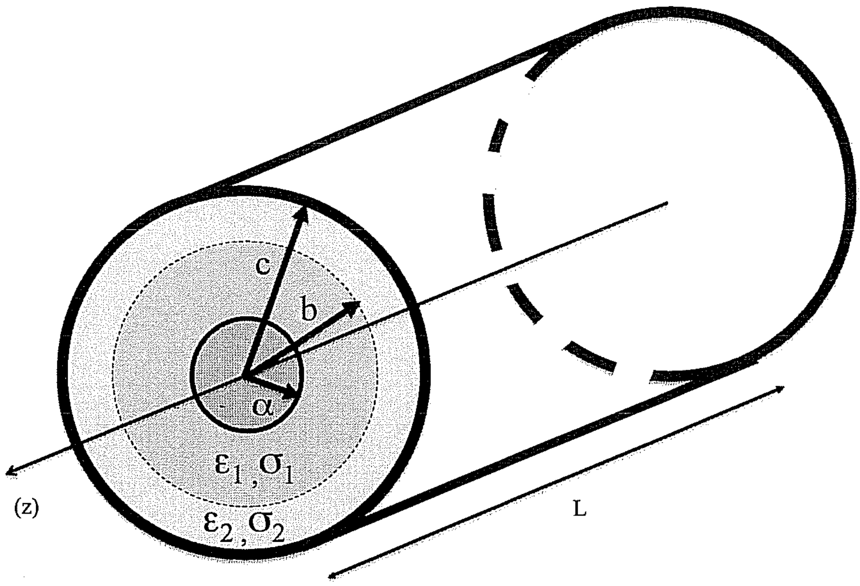
## Question 1

Consider the lossy coaxial capacitor shown in the figure below. The capacitor consists of two perfect conductors at $r = \alpha$ and $r = c$ and two lossy dielectric media with dielectric permittivities and conductivities $\varepsilon_i$, $\sigma_i$, $i = 1, 2$. The interface between the two media is at $r = b$. The capacitor has finite length $L$ in the $z$–direction, however, its electric field can be approximated by the field of a capacitor with $L \to \infty$. The voltage difference between the inner and outer perfect conductor is $V(r = \alpha) - V(r = c) = V_0$.




1. Using the Laplace equation, show that the general form of the electric field in the two regions is $\mathbf{E}_i = \dfrac{A_i}{r}\mathbf{a}_r$, $i = 1, 2$. **(4 pts)**

Laplace applies to both media. In cyl. coordinates

$\nabla^2 V = 0 \Rightarrow \dfrac{1}{r}\dfrac{d}{dr}\left[r\dfrac{dV}{dr}\right] = 0$ (1pt), where use was made of (1pt)

the cylindrical symmetry, which translates to $V = V(r)$. $-C/r$

Then, $r\,dV/dr = C \Rightarrow dV/dr = C$. But, $\underbrace{\bar{E} = -\mathrm{grad}\,\bar{V}}_{(1pt)} = -\dfrac{dV}{dr}\bar{a}_r$

Since these constants (C) are different, in general, the form of $\bar{E}$ is $\underbrace{\dfrac{A_i}{r}\bar{a}_r}_{(1\ pt)}$, $i = 1, 2$.

2. Using boundary conditions for the volume current density **J**, show that $\frac{A_1}{A_2} = \frac{\sigma_2}{\sigma_1}$. **(2 pts)**

B.C. that applies is $\bar{a}_n \cdot (\bar{J}_2 - \bar{J}_1) = -\partial \rho_s / \partial t \to 0$ at steady-state (0.5)

Ohm's Law: $\bar{J} = \sigma \bar{E}$ (0.5 pt) $\Rightarrow$ At $r = b$ $\underbrace{\frac{\sigma_1 A_1}{b} = \frac{\sigma_2 A_2}{b}}_{1\text{ pt}} \Rightarrow$

$\sigma_1 A_1 = \sigma_2 A_2$

3. Find the resistance $R$ of the resistor. **(8 pts)**

$$\boxed{R = \frac{1}{2\pi L} \left\{ \frac{1}{\sigma_1} \ln \frac{b}{a} + \frac{1}{\sigma_2} \ln \frac{c}{b} \right\}}$$

**Derivation:**

$R = \frac{V}{I}$. To find V:

$d\bar{l} = \bar{a}_r\, dr$ (1 pt)

2 pts

$$V_0 = V(r=a) - V(r=c) = \underbrace{\int_a^c \bar{E} \cdot d\bar{l}}_{(1\text{ pt})} = \underbrace{\int_a^b \frac{A_1}{r} \underbrace{\bar{a}_r \cdot \bar{a}_r}_{1} dr + \int_b^c \frac{A_2}{r} \bar{a}_r \cdot \bar{a}_r\, dr}_{\text{breaking integrals (1 pt)}}$$

$$= \underbrace{A_1 \ln \frac{b}{a} + A_2 \ln \frac{c}{b}}_{(1\text{ pt}):\text{ correct result}} = A_1 \sigma_1 \left\{ \frac{1}{\sigma_1} \ln \frac{b}{a} + \frac{1}{\sigma_2} \ln \frac{c}{b} \right\}$$

$\underbrace{I = \int \bar{J} \cdot d\bar{s} = \int \sigma_1 \bar{E}_1 \cdot ds}_{(1\text{ pt})} = \int \sigma_2 \bar{E}_2 \cdot ds$ (can be calculated by integration on a cylinder either in medium 1, or 2. Then:

$$I = \int \sigma_1 \frac{A_1}{\cancel{r}} \underbrace{\bar{a}_r \cdot \bar{a}_r}_{1} \overbrace{(\cancel{r}\, d\varphi\, dz)}^{d\bar{s} = (1\text{ pt})} = \sigma_1 A_1 \int_{\varphi=0}^{2\pi} d\varphi \int_0^L dz = \underbrace{2\pi L \cdot \sigma_1 A_1}_{(\text{result: } 1\text{ pt})}$$

Hence:

$$R = \frac{V}{I} = \frac{\cancel{A_1} \sigma_1}{\cancel{A_1} \sigma_1} \frac{1}{2\pi L} \left\{ \frac{1}{\sigma_1} \ln \frac{b}{a} + \frac{1}{\sigma_2} \ln \frac{c}{b} \right\} \quad \boxed{R: 1\text{ pt}}$$

$$\Rightarrow R = \underbrace{\frac{1}{2\pi L \sigma_1} \ln \frac{b}{a}}_{R_1} + \underbrace{\frac{1}{2\pi L \sigma_2} \ln \frac{c}{b}}_{R_2}$$

(series connection)

4. Is there a surface charge density $\rho_s$ at the interface between the two lossy dielectrics? If yes, calculate it (you can use $A_1$, $A_2$ in this calculation). If not, why not ? **(4 pts)**

Yes (0.5pt)

It can be found from boundary condition at

$r = b$: $\bar{a}_n \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s$ (1 pt)

$\Downarrow$

$$\underbrace{\epsilon_2 E_2 - \epsilon_1 E_1 = \rho_s}_{1.5\text{ pt}} \Rightarrow \overbrace{\boxed{\epsilon_2 \frac{A_2}{b} - \epsilon_1 \frac{A_1}{b} = \rho_s}}^{1\text{ pt (final)}}$$

(correct interpretation of b.c.)

5. Is there a volume charge density $\rho_v$ within the capacitor? If yes, why; if not, why not ? **(2 pts)**

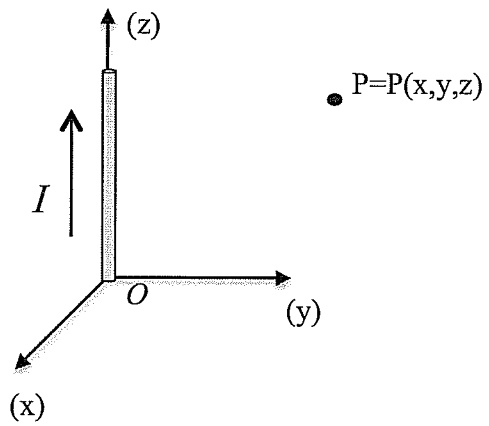
0,5pt $\equiv$ No relaxation effect applies to both media since they are uniform, so any volume charge distributions become surface charge distributions (1.5pt)

## Question 2

1. A thin wire of length $L$ carries current $I$ along the $z-$axis for $0 \leq z \leq L$. Using the Biot-Savart law, the magnetic field that this wire produces at an arbitrary observation point $P(x, y, z)$ can be expressed as follows:

$$\mathbf{B}(x,y,z) = \frac{\mu_0 I}{4\pi} \int_{z'=0}^{z'=L} \frac{?}{(x^2+y^2+(z-z')^2)^{3/2}}$$

Derive the term that is missing in this expression. **(10 pts)**




**Answer:**

$$\boxed{? = I\,dz'\left(x\,\bar{a}_y - y\,\bar{a}_x\right) = I\,dz'\,r\,\bar{a}_\varphi}$$

**Derivation:**

Biot-Savart: $\bar{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I\,d\bar{\ell}' \times (\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3}$

(1pt) (1pt)

$I\,d\bar{\ell}' = I\,\bar{a}_z\,dz'$

(1pt) $\bar{R} = r\,\bar{a}_r + z\,\bar{a}_z = x\,\bar{a}_x + y\,\bar{a}_y + z\,\bar{a}_z$

(1pt) $\bar{R}' = z'\,\bar{a}_z$

$\bar{R}-\bar{R}' = x\,\bar{a}_x + y\,\bar{a}_y + (z-z')\,\bar{a}_z$

$I\,d\bar{\ell}' \times (\bar{R}-\bar{R}') = I\,dz'\,\bar{a}_z \times \left(x\,\bar{a}_x + y\,\bar{a}_y + (z-z')\,\bar{a}_z\right)$

$= I\,dz'\underbrace{\left(x\,\bar{a}_y - y\,\bar{a}_x\right)}_{(2\,\text{pts})}$, $|\bar{R}-\bar{R}'| = \sqrt{x^2+y^2+(z-z')^2}$ 1pt

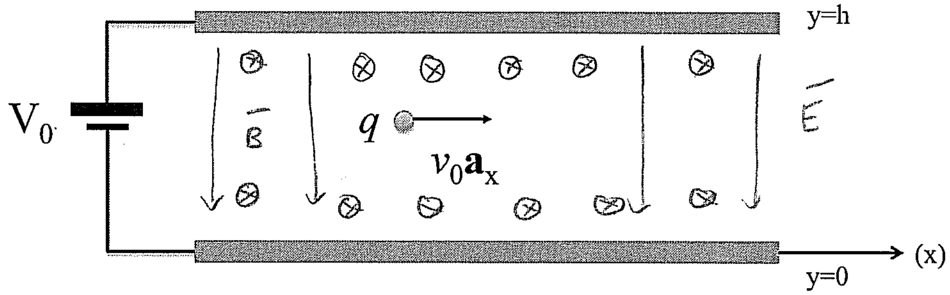
(1 per component)

$$\Rightarrow \quad d\bar{B} = \frac{\mu_0 I}{4\pi} \int_0^L \frac{I dz' \overbrace{(x\bar{a}_y - y\bar{a}_x)}^{r\bar{a}_\varphi}}{\left[x^2 + y^2 + (z-z')^2\right]^{3/2}} \quad \Big\} \; (2pts)$$

$$\Downarrow$$

$$\text{Missing} = I dz' (x\bar{a}_y - y\bar{a}_x) \qquad (1pt).$$

2. The charge $q$ shown in the figure moves with constant velocity within the electric field of a parallel plate capacitor with voltage $V_o$ and plate separation $h$, due to a constant magnetic field within the capacitor. Find the magnitude and direction of the magnetic flux density **B** of this magnetic field. **(4 pts)**




**Answer:**

$$\boxed{\mathbf{B} = (-\bar{a}_z)\,\frac{V_0}{v_0 h}}$$

**Derivation:**

$$\text{Constant velocity} \Rightarrow \quad q\bar{E} + q\bar{v}\times\bar{B} = 0 \qquad (1pt)$$

$$\Rightarrow \quad \bar{E} = -\bar{v}\times\bar{B} \quad \Rightarrow \quad \boxed{\frac{V_0}{h}(-\bar{a}_y) = -v_0\bar{a}_x \times \bar{B}} \quad (1pt)$$

$$\frac{V_0}{h}\bar{a}_y = v_0\,\bar{a}_x \times \bar{B} \quad \Rightarrow \quad \bar{B} = \frac{V_0}{h\cdot v_0}(-\bar{a}_z)$$

1pt      1pt.

3. The two circular coils shown in the figure, centered at $A_1$ and $A_2$, support co-directional currents $I_1 = I_2 = I$. Let the magnetic field densities generated by each of the two coils alone be $\mathbf{B_1}$, $\mathbf{B_2}$. The total magnetic field density $\mathbf{B}$ at point P on the axis is:

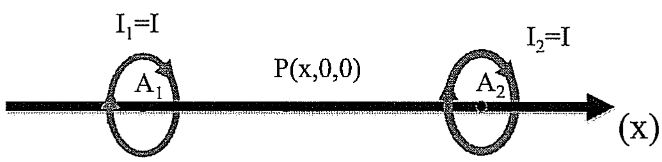
   a) In the positive $x$ direction and has magnitude smaller than the magnitude of $\mathbf{B_1}$ at P.

   b) In the positive $x$ direction and has magnitude smaller than the magnitude of $\mathbf{B_2}$ at P.

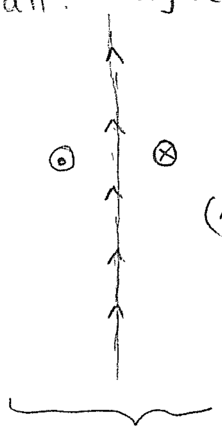
   c) In the negative $x$ direction and has magnitude greater than the magnitude of $\mathbf{B_2}$ at P. (0.5 pt)

   d) In the negative $x$ direction and has magnitude greater than the magnitude of $\mathbf{B_1}$ at P. (0.5 pt)

   Choose all answers that apply and briefly explain. **(4 pts)**




Recall: Magnetic dipole



(1 pt)

⇒ Both $\bar{B}_1$ and $\bar{B}_2$ are pointing in the same, negative x direction for any P(x,0,0). Hence, $\bar{B}$ has magnitude that is equal to the sum of $|\bar{B}_1| + |\bar{B}_2|$ (1 pt)

(1 pt)

⇒ $|\bar{B}| > |\bar{B}_1|, |\bar{B}_2|$ and $\bar{B}$ points in $-x$.

4. Which of the following expressions can represent a magnetic flux density $\mathbf{B}$? Choose all answers that apply and briefly explain. **(2 pts)**

   a) $B_0\mathbf{a}_x$, where $B_0$ is a constant. 0.5

   b) x y $\mathbf{a}_x$.

   c) $\dfrac{B_0}{r}\mathbf{a}_r$, where $B_0$ is a constant.

   d) $\dfrac{B_0}{r}\mathbf{a}_\phi$, where $B_0$ is a constant. 0.5

$\underbrace{\mathrm{div}\,\bar{B} = 0}_{1\,pt}$ is satisfied only by (a), (d).