ESC195 - Midterm Test #2
April 1, 2021
9:10 - 10:40 am, EST

The following materials are considered to be acceptable aids during the writing of this test:

- The Stewart textbook and the student solution manuals
- Any course notes or problem solutions prepared by the student
- Any handouts or other materials posed on the ESC195 course website
- Any non-programmable, non-graphing calculator

All questions are worth 10 marks

1. Determine whether the sequence converges or diverges; if it converges, find the limit:

$$\text{a) } a_n = \frac{e^{n/10}}{2^n} \qquad \text{b) } a_n = \frac{\tan^{-1} n}{n} \qquad \text{c) } a_n = n^2 \sin\frac{\pi}{n}$$

a) $a_n = \frac{e^{n/10}}{2^n} = \left(\frac{e^{1/10}}{2}\right)^n = (0.55)^n \longrightarrow 0$ converges

b) $a_n = \frac{\tan^{-1} n}{n}$

$0 \le \tan^{-1} n \le \frac{\pi}{2}$

$\therefore 0 \le \frac{\tan^{-1} n}{n} \le \frac{\pi}{2} \cdot \frac{1}{n}$

$\lim_{n \to \infty} \frac{\pi}{2} \cdot \frac{1}{n} = 0 \quad \therefore a_n = \frac{\tan^{-1} n}{n}$ converges by pinching th'm

c) $a_n = n^2 \sin\frac{\pi}{n}$

since $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$\Rightarrow \lim_{n \to \infty} \frac{n}{\pi} \sin\frac{\pi}{n} = 1$

$\therefore$ for n sufficiently large, $\frac{n}{\pi} \sin\frac{\pi}{n} > \frac{1}{2}$

$\therefore n^2 \sin\frac{\pi}{n} = n\pi \cdot \frac{n}{\pi} \sin\frac{\pi}{n} > n\pi\left(\frac{1}{2}\right) = \frac{n\pi}{2}$

$\lim_{n \to \infty} \frac{n\pi}{2}$ diverges, $\therefore a_n = n^2 \sin\frac{\pi}{n}$ diverges by comparison test

2. Test the series for convergence or divergence:

a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{2k}{k+1}\right)^k$ b) $\frac{1}{1!}+\frac{4}{2!}+\frac{9}{3!}+\frac{16}{4!}+\cdots$ c) $\displaystyle\sum_{n=1}^{\infty}\left(\cos\frac{1}{n}-1\right)$

a) $a_k = \left(\frac{2k}{k+1}\right)^k$ root test: $(a_k)^{1/k} = \frac{2k}{k+1} \longrightarrow 2$ as $k \to \infty$

$\not< 1$ $\therefore$ diverges

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$: ratio test: $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{(n+1)!}\cdot\frac{n!}{n^2} = \left(\frac{n+1}{n}\right)^2\cdot\frac{1}{n+1} \longrightarrow 0$

$\therefore$ convergent.

c) Limit comparison test: $a_n = \cos\frac{1}{n} - 1$ $b_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\left|\frac{1-\frac{1}{n^2 2!}+\frac{1}{n^4 4!}-\frac{1}{n^3 6!}+\cdots -1}{1/n^2}\right| = \left|-\frac{1}{2!}+\frac{1}{n^2 4!}-\cdots\right|$$

$\longrightarrow \frac{1}{2}$

since $\sum b_n = \sum \frac{1}{n^2}$ converges (p-series, $p>1$)

$\therefore \sum\left(\cos\frac{1}{n}-1\right)$ converges

3. a) Use the Taylor series expansions for $\cos x$ and $\sin x$ to verify the identity: $\sin 2x = 2\sin x \cos x$. Consider terms up to the power of x^5.

b) Find the radius of convergence and interval of convergence for the series: $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ $\qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$$\Rightarrow (\sin x)(\cos x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$$

$$= x - \frac{x^3}{2!} - \frac{x^3}{3!} + \frac{x^5}{2!\,3!} + \frac{x^5}{4!} + \frac{x^5}{5!} + \dots$$

$$= x - x^3\left(\frac{1}{2!} + \frac{1}{3!}\right) + x^5\left(\frac{1}{12} + \frac{1}{4!} + \frac{1}{5!}\right) + \dots$$

$$= x - 4\frac{x^3}{3!} + 16\frac{x^5}{5!} + \dots$$

$$\therefore 2\sin x \cos x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots$$

$$= \sin(2x)$$

b) ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} \cdot \frac{(n+1)!\,x^{n+1}}{n!\,x^n}\right| = \frac{n+1}{2n+1}|x| \to \frac{1}{2}|x|$

$\therefore$ the series converges for $|x| < 2$ or $R = 2$

test $x = \pm 2$: $|a_n| = \frac{n!\,2^n}{1\cdot 3\cdot 5\cdots(2n-1)} = \frac{(1\cdot 2\cdot 3\cdot 4\cdots n)\,2^n}{1\cdot 3\cdot 5\cdots(2n-1)}$

$$= \frac{2\cdot 4\cdot 6\cdot 8\cdots 2n}{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)} > 1$$

$\therefore |a_n| \not\to 0$ $\therefore$ diverges by test for divergence

$\Rightarrow$ Interval of convergence : $x \in (-2, 2)$

4. a) Explain the fallacy in the following argument:

let $x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$

$y = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \cdots$

It is easily shown that $x + y = 2y = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$, which implies that $x = y$.

On the other hand, $x - y = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$, is the sum of all positive terms; we thus conclude that $x > y$.

Thus we have both $x = y$ and $x > y$.

b) Prove that if a power series $\sum a_n x^n$ converges for a non-zero number $x = c$, then it is absolutely convergent whenever $|x| < |c|$.

a) Both x & y are divergent series (ie, they are NOT numbers), thus the normal rules for addition and subtraction do not hold.

b) Given $\sum a_n c^n$ converges $\rightarrow \lim_{n \to \infty} a_n c^n = 0$

$\therefore |a_n c^n| < 1$ for $n > N$

$\therefore |a_n x^n| = \left| \frac{a_n c^n x^n}{c^n} \right| = |a_n c^n| \left| \frac{x}{c} \right|^n < \left| \frac{x}{c} \right|^n$; $n > N$

$\therefore$ If $|x| < |c|$ then $\left| \frac{x}{c} \right| < 1$ and $\sum \left| \frac{x}{c} \right|^n$ is a convergent geometric series.

$\therefore \sum |a_n x^n|$ converges by the comparison test

Note: convergence is determined by the terms as $n \to \infty$; thus the first $N-1$ terms of the series do not affect this result.

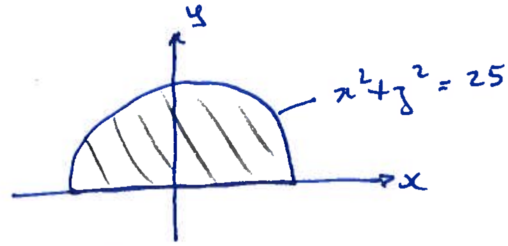
5. a) Find and sketch the domain of the functions:

i) $f(x,y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$ ii) $g(x,y) = \dfrac{\ln(2-x)}{1 - x^2 - y^2}$

b) Find the partial derivatives of the functions:

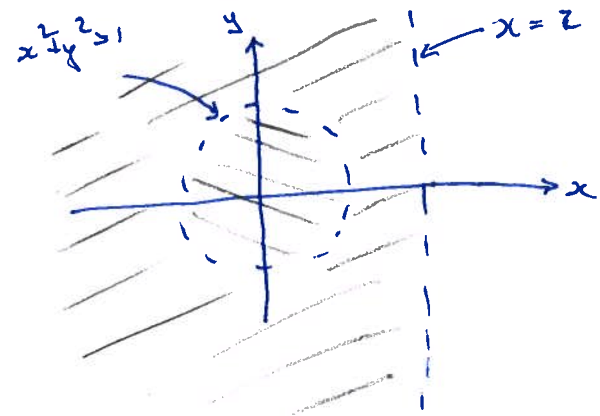
i) $g(x,y) = x^2 e^{-y}$ ii) $f(x,y,z) = z^{xy^2}$

a) i) $y \geq 0$ & $x^2 + y^2 \leq 25$



ii) $2 - x > 0 \Rightarrow x < 2$

$x^2 + y^2 \neq 1$



b) i) $g(x,y) = x^2 e^{-y} \Rightarrow \dfrac{\partial g}{\partial x} = 2xe^{-y}$

$\dfrac{\partial g}{\partial y} = -x^2 e^{-y}$

ii) $f(x,y,z) = z^{xy^2} \Rightarrow \dfrac{\partial f}{\partial x} = y^2 \ln z \; z^{xy^2}$

$\dfrac{\partial f}{\partial y} = 2xy \ln z \; z^{xy^2}$

$\dfrac{\partial f}{\partial z} = xy^2 \, z^{(xy^2-1)}$

6. a) Determine whether the function $f(x,y) = \dfrac{x - y^4}{x^3 - y^4}$ has a limit at $(x,y) = (1,1)$.

b) Consider the function: $f(x,y) = \begin{cases} \dfrac{xy(y^2 - x^2)}{x^2 + y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

Show that $f(x,y)$ is continuous at $(x,y) = (0,0)$.

a) consider $x=1 \Rightarrow f(1,y) = \dfrac{1-y^4}{1-y^4} = 1 \Rightarrow \lim_{y\to 1} 1 = 1$

consider $y=1 \Rightarrow f(x,1) = \dfrac{x-1}{x^3-1} = \dfrac{1}{x^2+x+1} \Rightarrow \lim_{x\to 1} \dfrac{1}{x^2+x+1} = \dfrac{1}{3}$

$\therefore \lim_{(x,y)\to(1,1)} \dfrac{x-y^4}{x^3-x^4}$ DNE

b) $\Rightarrow$ show $\lim_{(x,y)\to(0,0)} \dfrac{xy(y^2-x^2)}{x^2+y^2} = 0$

$|x| \leq \sqrt{x^2+y^2}$

$|y| \leq \sqrt{x^2+y^2}$

$\Rightarrow 0 \leq \left| \dfrac{xy(y^2-x^2)}{x^2+y^2} \right| = \dfrac{|x||y||y^2-x^2|}{x^2+y^2} \leq \dfrac{\sqrt{x^2+y^2}\cdot\sqrt{x^2+y^2}\,|y^2-x^2|}{x^2+y^2}$

$= |y^2-x^2|$

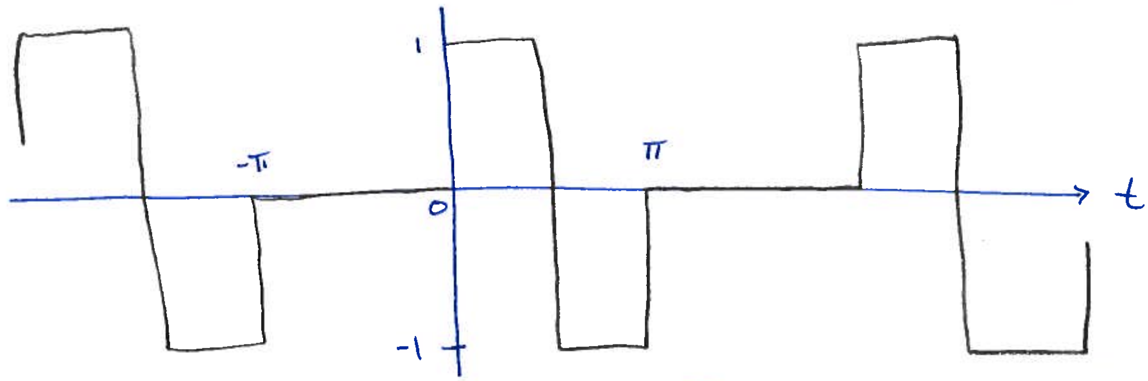
$\lim_{(x,y)\to(0,0)} |y^2-x^2| = 0$ by squeeze th'm.

$\therefore \lim_{(x,y)\to(0,0)} \dfrac{xy(y^2-x^2)}{x^2+y^2} = 0 = f(0,0)$

7. Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(t) = \begin{cases} 0 & -\pi \le t \le 0 \\ 1 & 0 < t \le \pi/2 \\ -1 & \pi/2 < t \le \pi \end{cases}$$

Provide a sketch of the function, and a sketch of what you imagine the sum of the first few terms of the series would look like.



$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dt + \int_{0}^{\pi/2} 1\,dt + \int_{\pi/2}^{\pi}(-1)\,dt\right] = \frac{1}{\pi}\left(0+\frac{\pi}{2}-\frac{\pi}{2}\right) = 0$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cos nt\,dt + \int_{0}^{\pi/2}\cos nt\,dt - \int_{\pi/2}^{\pi}\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_0^{\pi/2} - \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left(\sin\frac{n\pi}{2} - \sin 0 - \sin n\pi + \sin\frac{n\pi}{2}\right)$$

$$= \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{0}^{\pi/2}\sin nt\,dt - \int_{\pi/2}^{\pi}\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi/2} - \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

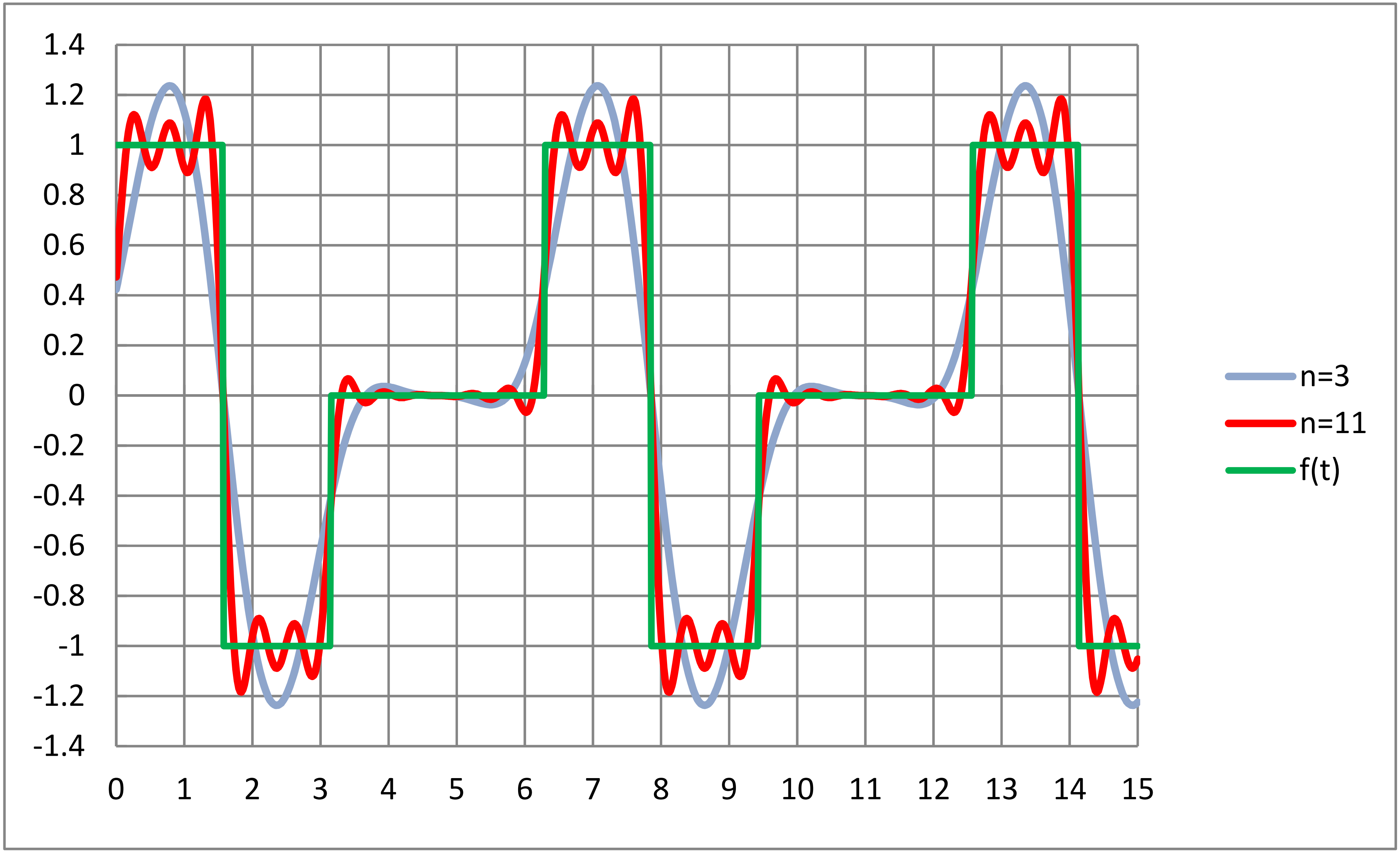
$$= \frac{1}{n\pi}\left(-\cos\frac{n\pi}{2} + \cos 0 + \cos n\pi - \cos n\frac{\pi}{2}\right)$$

$$= \frac{1}{n\pi}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)$$

n	$\sin\frac{n\pi}{2}$	$1+\cos n\pi - 2\cos\frac{n\pi}{2}$
1	1	$1 - 1 - 0 = 0$
2	0	$1 + 1 + 2 = 4$
3	-1	$1 - 1 - 0 = 0$
4	0	$1 + 1 - 2 = 0$
5	1	0
6	0	4
7	-1	0
8	0	0

$$\Rightarrow f(t) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos(2k-1)t$$

$$+ \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k-2}\sin(4k-2)t$$



1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
n=3
n=11
f(t)

8. Find the unit tangent vector and the principle unit normal vector at $t = 1$ on the curve: $\vec{r}(t) = t\,\hat{\imath} + \frac{1}{t}\,\hat{\jmath} + \sqrt{2}\ln(t)\,\hat{k}$. Find an equation in x, y, z for the osculating plane at the point corresponding to $t = 1$.

$\vec{r}(t) = \left(t, \frac{1}{t}, \sqrt{2}\ln t\right)$ $\qquad\qquad$ $\vec{r}(1) = (1,1,0)$

$\vec{r}\,'(t) = \left(1, -1/t^2, \sqrt{2}/t\right)$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1 + \frac{1}{t^4} + \frac{2}{t^2}} = \frac{1}{t^2}\sqrt{t^4+2t^2+1} = \frac{t^2+1}{t^2}$

$\vec{T}(t) = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \frac{t^2}{t^2+1}\left(1, \frac{-1}{t^2}, \frac{\sqrt{2}}{t}\right)$ $\qquad\qquad$ $\vec{T}(1) = \frac{1}{2}(1,-1,\sqrt{2})$

$\Rightarrow \vec{T}\,'(t) = \left(\frac{2t(t^2+1) - t^2(2t)}{(t^2+1)^2}\right)\left(1, -\frac{1}{t^2}, \frac{\sqrt{2}}{t}\right) + \frac{t^2}{t^2+1}\left(0, \frac{2}{t^3}, -\frac{\sqrt{2}}{t^2}\right)$

$= \frac{2t}{(t^2+1)^2}\left(1, -\frac{1}{t^2}, \frac{\sqrt{2}}{t}\right) + \frac{t^2}{t^2+1}\left(0, \frac{2}{t^3}, \frac{-\sqrt{2}}{t^2}\right)$

$\vec{T}\,'(1) = \frac{1}{2}(1,-1,\sqrt{2}) + \frac{1}{2}(0,2,-\sqrt{2}) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

$\|\vec{T}(1)\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 0} = \frac{1}{\sqrt{2}}$

$\Rightarrow \vec{N}(t=1) = \frac{\vec{T}\,'(1)}{\|\vec{T}\,'(1)\|} = \sqrt{2}\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{1}{\sqrt{2}}(1,1,0)$

$\vec{B}(1) = \vec{T}(1) \times \vec{N}(1) = \frac{1}{2}(1,-1,\sqrt{2}) \times \frac{1}{\sqrt{2}}(1,1,0)$

$= \frac{1}{2\sqrt{2}}\begin{vmatrix} i & j & k \\ 1 & -1 & \sqrt{2} \\ 1 & 1 & 0 \end{vmatrix} = \frac{1}{2\sqrt{2}}(-\sqrt{2}, \sqrt{2}, 2) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

point on osculating plane : $\vec{r}(1) = (1,1,0)$

$\therefore$ eq'n of osculating plane: $-\frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{\sqrt{2}}(z-0) = 0$

$$\boxed{z = \frac{1}{\sqrt{2}}(x-y)}$$