The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

# ECE259: Electromagnetism

## Term Test 1 - Monday February 3, 2020
## Instructors: Profs. Micah Stickel and Piero Triverio

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**Last name**: ..........................................................................................

**First name**: ..........................................................................................

**Student number**: ..........................................................................................

**Tutorial section number**: ..........................................................................................

| **Section** | **Day** | **Time** | **Room** |
|---|---|---|---|
| TUT0101 | Monday | 14:00-15:00 | BA1230 |
| TUT0102 | Monday | 14:00-15:00 | BA2175 |
| TUT0103 | Monday | 14:00-15:00 | BA2135 |
| TUT0104 | Monday | 14:00-15:00 | BA2159 |
| TUT0105 | Wednesday | 13:00-14:00 | BA2165 |
| TUT0106 | Wednesday | 13:00-14:00 | BA2195 |
| TUT0107 | Wednesday | 13:00-14:00 | BA1230 |
| TUT0108 | Wednesday | 13:00-14:00 | BAB024 |

### Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

**Marks**:

| Q1: /20 | Q2: /20 | Q3: /15 | TOTAL: /55 |
|---|---|---|---|

## Question 1

A uniform surface charge density, $\rho_{S0}$ lies in the $xy$-plane, centered about the origin, and exists between $r = a$ and $r = b$ ($b > a$).

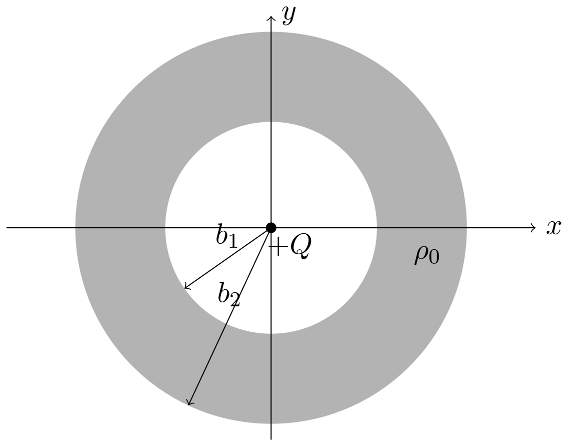
(a) Draw a picture of this situation. Clearly indicate on the figure and state separately the expressions for $dQ'$, **R**, and **R**$'$. [4 points]

(b) For this charge distribution, determine the expression for the electric field intensity at any point on the positive $z$-axis. [10 points]

(c) Use the results of part (b) above to show that the electric field intensity for an infinitely large plate charged with a uniform charge density $\rho_{S0}$ is given by $\mathbf{E}(0,0,z) = \frac{\rho_{S0}}{2\varepsilon_0}\mathbf{a}_z$ for $z > 0$. Make sure to clearly describe your process and reasoning. [2 points]

(d) For the charge distribution geometry described above, identify a non-uniform $\rho_S$ that would result in the electric field intensity along the positive $z$-axis to only have a $y$-component. Briefly justify your answer. [4 points]

## Question 2




Consider the system made by

- a point charge $+Q$ located at the origin;
- a uniform charge distribution in the **spherical** shell $b_1 < R < b_2$, where volume charge density is equal to $\rho_0$.

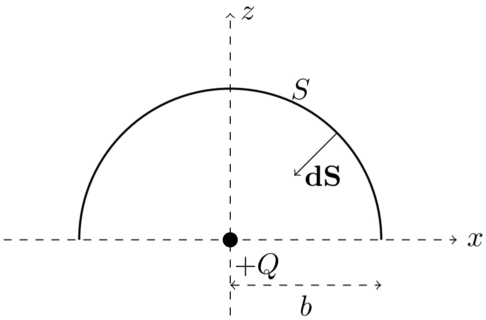
A cross section of the system is shown in the figure above.

i) Use Gauss' law to find the electric field **E** in the region $R < b_1$ [4 points].

ii) Use Gauss' law to find the electric field **E** in the region $b_1 < R < b_2$ [10 points].

iii) Use Gauss' law to find the electric field **E** in the region $R > b_2$ [6 points].

## Question 3.1




We have a point charge $Q$ at the origin. Let $S$ be the hemispherical surface of radius $b$ shown in the figure ($S : R = b, \varphi \in [0, 2\pi], \theta \in [0, \pi/2]$), with normal oriented inwards. The flux

$$\int_S \mathbf{E} \cdot \mathbf{dS}$$

is equal to:

(a) $-\frac{Q}{2\varepsilon_0}$;

(b) $-\frac{Q}{\varepsilon_0}$;

(c) $0$

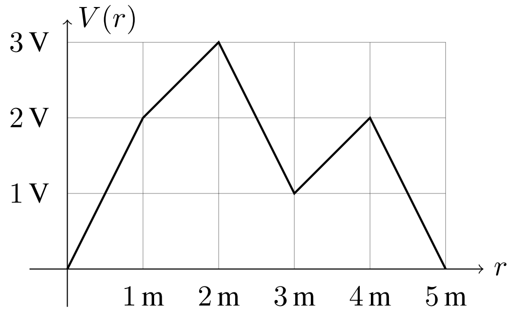
(d) $+\frac{Q}{\varepsilon_0}$;

(e) $+\frac{Q}{2\varepsilon_0}$;

Briefly justify your answer. [5 points]

## Question 3.2

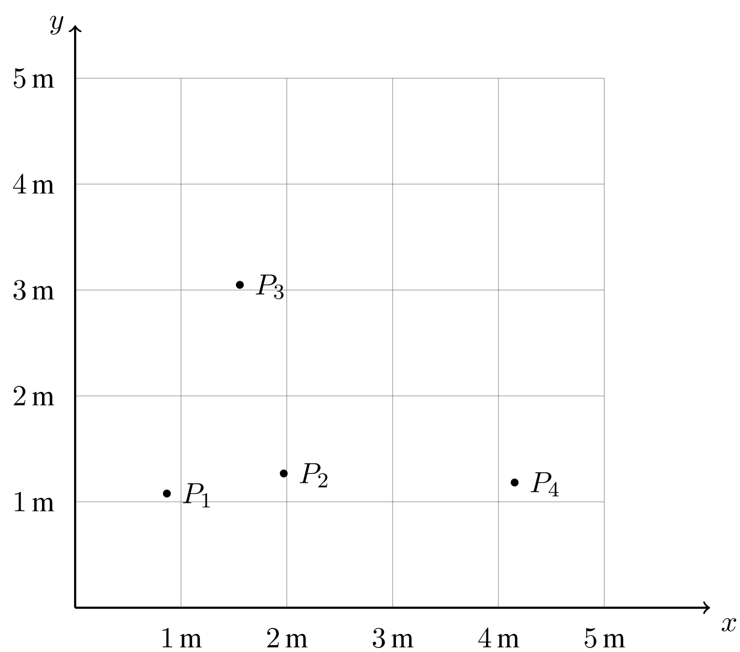
An electrostatic potential $V(r)$ is present in the $xy$ plane and is function of $r$ only. The following graph depicts $V(r)$ as a function of $r$.




Using the axis at the bottom of this page:

- sketch the equipotential lines that pass through the points $(1\,\mathrm{m}, 0)$, $(2\,\mathrm{m}, 0)$ , $(3\,\mathrm{m}, 0)$, $(4\,\mathrm{m}, 0)$;
- Indicate with an arrow the direction of the electric field $\mathbf{E}$ at points $P_1$, $P_2$, $P_3$ and $P_4$.

Briefly justify your answer. [5 points]



## Question 3.3

A very large plate has a constant charge density of $\rho_S = \rho_{SA}$ and is located at $z = -a$, where $a > 0$. A sphere of radius $R = b = \frac{a}{2}$ is centered about the origin and is uniformly charged with a charge density of $\rho_S = \rho_{SB}$. Which of the statements below represents the total electric field intensity at $P(x, y, z) = (0, a, 0)$. Note: You may assume the electric field from the very large plate is the same as it would be for an infinitely-large plate.

(a) $\mathbf{E}_{total} = \left(\frac{\rho_{SA}}{2\varepsilon_0} + \frac{\rho_{SB}}{2\varepsilon_0}\right)\mathbf{a}_z$

(b) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z$

(c) $0$

(d) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z + \frac{\rho_{SB}}{4\varepsilon_0}\mathbf{a}_y$

(e) $\mathbf{E}_{total} = \frac{\rho_{SA}}{2\varepsilon_0}\mathbf{a}_z + \frac{\rho_{SB}}{2b\varepsilon_0}\mathbf{a}_R$

(f) None of the above.

Briefly justify your answer. [5 points]

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**1. Coordinate Systems**

*1.1 Cartesian coordinates*

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

*1.2 Cylindrical coordinates*

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

*1.3 Spherical coordinates*

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

**2. Relationship between coordinate variables**

| - | Cartesian | Cylindrical | Spherical |
|---|---|---|---|
| $x$ | $x$ | $r\cos\phi$ | $R\sin\theta\cos\phi$ |
| $y$ | $y$ | $r\sin\phi$ | $R\sin\theta\sin\phi$ |
| $z$ | $z$ | $z$ | $R\cos\theta$ |
| $r$ | $\sqrt{x^2+y^2}$ | $r$ | $R\sin\theta$ |
| $\phi$ | $\tan^{-1}\frac{y}{x}$ | $\phi$ | $\phi$ |
| $z$ | $z$ | $z$ | $R\cos\theta$ |
| $R$ | $\sqrt{x^2+y^2+z^2}$ | $\sqrt{r^2+z^2}$ | $R$ |
| $\theta$ | $\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$ | $\tan^{-1}\frac{r}{z}$ | $\theta$ |
| $\phi$ | $\tan^{-1}\frac{y}{x}$ | $\phi$ | $\phi$ |

### 3. Dot products of unit vectors

| $\cdot$ | $\mathbf{a}_x$ | $\mathbf{a}_y$ | $\mathbf{a}_z$ | $\mathbf{a}_r$ | $\mathbf{a}_\phi$ | $\mathbf{a}_z$ | $\mathbf{a}_R$ | $\mathbf{a}_\theta$ | $\mathbf{a}_\phi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{a}_x$ | 1 | 0 | 0 | $\cos\phi$ | $-\sin\phi$ | 0 | $\sin\theta\cos\phi$ | $\cos\theta\cos\phi$ | $-\sin\phi$ |
| $\mathbf{a}_y$ | 0 | 1 | 0 | $\sin\phi$ | $\cos\phi$ | 0 | $\sin\theta\sin\phi$ | $\cos\theta\sin\phi$ | $\cos\phi$ |
| $\mathbf{a}_z$ | 0 | 0 | 1 | 0 | 0 | 1 | $\cos\theta$ | $-\sin\theta$ | 0 |
| $\mathbf{a}_r$ | $\cos\phi$ | $\sin\phi$ | 0 | 1 | 0 | 0 | $\sin\theta$ | $\cos\theta$ | 0 |
| $\mathbf{a}_\phi$ | - $\sin\phi$ | $\cos\phi$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\mathbf{a}_z$ | 0 | 0 | 1 | 0 | 0 | 1 | $\cos\theta$ | $-\sin\theta$ | 0 |
| $\mathbf{a}_R$ | $\sin\theta\cos\phi$ | $\sin\theta\sin\phi$ | $\cos\theta$ | $\sin\theta$ | 0 | $\cos\theta$ | 1 | 0 | 0 |
| $\mathbf{a}_\theta$ | $\cos\theta\cos\phi$ | $\cos\theta\sin\phi$ | $-\sin\theta$ | $\cos\theta$ | 0 | $-\sin\theta$ | 0 | 1 | 0 |
| $\mathbf{a}_\phi$ | - $\sin\phi$ | $\cos\phi$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

### 4. Relationship between vector components

| = | Cartesian | Cylindrical | Spherical |
|---|---|---|---|
| $A_x$ | $A_x$ | $A_r\cos\phi - A_\phi\sin\phi$ | $A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ |
| $A_y$ | $A_y$ | $A_r\sin\phi + A_\phi\cos\phi$ | $A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ |
| $A_z$ | $A_z$ | $A_z$ | $A_R\cos\theta - A_\theta\sin\theta$ |
| $A_r$ | $A_x\cos\phi + A_y\sin\phi$ | $A_r$ | $A_R\sin\theta + A_\theta\cos\theta$ |
| $A_\phi$ | -$A_x\sin\phi + A_y\cos\phi$ | $A_\phi$ | $A_\phi$ |
| $A_z$ | $A_z$ | $A_z$ | $A_R\cos\theta - A_\theta\sin\theta$ |
| $A_R$ | $A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ | $A_r\sin\theta + A_z\cos\theta$ | $A_R$ |
| $A_\theta$ | $A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ | $A_r\cos\theta - A_z\sin\theta$ | $A_\theta$ |
| $A_\phi$ | $-A_x\sin\phi + A_y\cos\phi$ | $A_\phi$ | $A_\phi$ |

### 5. Differential operators

*5.1 Gradient*

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

*5.2 Divergence*

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

*5.3 Laplacian*

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

*5.4 Curl*

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

## 6. Electromagnetic formulas

**Table 1** Electrostatics

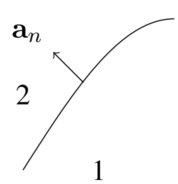
| | |
|---|---|
| $\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{\lvert\mathbf{R}_2-\mathbf{R}_1\rvert^3}(\mathbf{R}_2-\mathbf{R}_1)$ | $\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R}-\mathbf{R}')}{\lvert\mathbf{R}-\mathbf{R}'\rvert^3}dQ'$ |
| $\nabla\times\mathbf{E} = 0$ | $\nabla\cdot\mathbf{D} = \rho_v$ |
| $\oint_C \mathbf{E}\cdot\mathbf{dl} = 0$ | $\oint_S \mathbf{D}\cdot\mathbf{dS} = Q$ |
| $\mathbf{E} = -\nabla V$ | $V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R}-\mathbf{R}'\rvert}$ |
| $V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2}\mathbf{E}\cdot\mathbf{dl}$ | |
| $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$ | $\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$ |
| $\rho_{p,v} = -\nabla\cdot\mathbf{P}$<br>$\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2-\mathbf{P}_1)$<br>$\mathbf{a}_n\cdot(\mathbf{D}_2-\mathbf{D}_1) = \rho_s$<br>$E_{1,t} = E_{2,t}$ |  |
| $Q = CV$ | $W_e = \frac{1}{2}QV$ |
| $W_e = \frac{1}{2}\int_{vol}\rho_v V dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}dv$ | |
| $\nabla\cdot(\varepsilon\nabla V) = -\rho_v$ | $\nabla\cdot(\varepsilon\nabla V) = 0$ |

**Table 2** Magnetostatics

| | |
|---|---|
| $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ | $\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$ |
| $\nabla \times \mathbf{B} = \mu \mathbf{J}$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$ | $\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$ |
| $\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert \mathbf{R} - \mathbf{R}' \rvert^3}$ | $\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$ |
| $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$ | $\mathbf{M} = \chi_m \mathbf{H}$ |
| $\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$ | $\mathbf{J}_m = \nabla \times \mathbf{M}$ |
| $B_{1,n} = B_{2,n}$ | $\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$ |
| $\nabla \times \mathbf{H} = \mathbf{J}$ | $\oint \mathbf{H} \cdot \mathbf{dl} = I$ |
| | $W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$ |
| $L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$ | $L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$ |
| $\mathcal{R} = \frac{l}{\mu S}$ | $V_{mmf} = NI$ |

**Table 3** Faraday's law, Ampere-Maxwell law

| | |
|---|---|
| $V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$ | $V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$ |
| $\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$ | $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ |

**Table 4** Currents

| | |
|---|---|
| $I = \int_S \mathbf{J} \cdot \mathbf{dS}$ | $\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$ |
| $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$ | $\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$ |
| $J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$ | $\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$ |
| $R = \frac{l}{\sigma S}$ | $\sigma = -\rho_e \mu_e = \frac{1}{\rho}$ |