UNIVERSITY OF TORONTO, FACULTY OF APPLIED SCIENCE AND ENGINEERING

MAT292H1F - Ordinary Differential Equations

Final Exam - December 15, 2018

EXAMINERS: A. STINCHCOMBE AND A. KHOVANSKII

Time allotted: 150 minutes

Aids permitted: None

Total marks: 100

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Last First

Student Number: ______________________________

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<u>**Instructions**</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers are required. In the second section, justify your answers fully.
- This test is **double-sided**. Make sure you don't skip any problems.
- This test contains 18 pages, including this title page and a formula sheet.
  Make sure you have all of them.
- You can use pages 14–16 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 14–16.
- No calculators, cellphones, or any other electronic gadgets are allowed.
- You may detach the formula sheet. Work on the formula sheet will NOT be graded.

SECTION I No explanation is necessary. (26 marks)

For questions 1–6, please fill in the blanks.

1. **(2 marks)** Find the stable ($y = a$) and unstable ($y = b$) equilibrium points of $y' = e^{2y} - 4e^y + 3$.

   $a =$ ________ $b =$ ________.

2. **(2 marks)** The solution to the initial value problem $\boldsymbol{x}'(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \boldsymbol{x}(t)$, $\boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

   $\boldsymbol{x}(t) =$ ________________.

3. **(2 marks)** The solution to the initial value problem $\frac{d^4y}{dt^4} + 7\frac{d^3y}{dt^3} = 0$, $y(0) = 1$, $y'(0) = y''(0) = y'''(0) = 0$ is

   $y =$ ________________.

4. **(2 marks)** State a first order autonomous differential equation $y' = f(y)$ for which Euler's method gives exactly correct values (for any stepsize):

   $f(y) =$ ________________.

5. **(2 marks)** Assume that the function $z(t) = \sin(t-1)$ satisfies the equation $y''(t) + p(t)y'(t) + q(t)y(t) = 0$ for $1 \leqslant t \leqslant a$. For which value(s) of a does the function $z(t)$ satisfy the boundary condition $y(1) = y(a)$.

   $a =$ ________________.

6. **(2 marks)** For $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, the matrix exponential $e^{At} =$ ________________________.

For questions 7–13, circle **True** or **False**.

7. **(2 marks)** The initial value problem $\sin(y' - y) = 0$, $y(0) = 0$ has a unique solution. **True** **False**

8. **(2 marks)** The solution to $y'(t) = \exp(y)\cos(y)$, $y(0) = 0$, exists for all t. **True** **False**

9. **(2 marks)** For all differential equations and all stepsizes h, the improved Euler method is more accurate than the Euler method. **True** **False**

10. **(2 marks)** The equilibrium point of $\boldsymbol{x}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \boldsymbol{x}$ is stable. **True** **False**

11. **(2 marks)** $\mathcal{L}\{\exp(t)\cos(t)\sin(t)\}(s)$ is a rational function of s. **True** **False**

12. **(2 marks)** All solutions of $y'' + y = \cos t$ are bounded. **True** **False**

13. **(2 marks)** The Wronskian $W[y_1, y_2](t)$ for solutions y_1, y_2 of $y'' + p(t)y' + q(t)y = 0$ can not take values -1, 0, and 1 at the points $t = 1$, $t = 2$, and $t = 3$ correspondingly. **True** **False**

SECTION II **Justify** your answers. (74 marks)

14. Find a function $F(x, y)$ and a constant C such that $F(x, y(x)) = C$ is an implicit solution to the initial value problem $(2y + x)y' + (2x + y) = 0$, $y(1) = 1$. **(5 marks)**

15. Let $y(t)$ for $-\infty < t < \infty$ be the solution of $ay'' + by' + cy = 0$, where a, b, and c are constants and the initial condition is $y(0) = 0, y'(0) = a^{-1}$. Let $z(t)$ be the impulse response, so that $az'' + bz' + cz = \delta(t)$ and $z(0) = 0 = z'(0)$. **(5 marks)**

a) (2 marks) Show that $\mathcal{L}\{y\} = \mathcal{L}\{z\}$.

b) (3 marks) Is it true that $y(t) = z(t)$ for all real t?

16. Consider the initial value problem $y' + ay = g(t)$, $y(0) = 0$, where a is a constant. **(9 marks)**
Find the solution $y(t)$ for $t > 0$ and express it in the exact same form using the following three methods:

a) **(3 marks)** the integrating factor method

b) **(3 marks)** using the Laplace transform and the convolution theorem

c) **(3 marks)** the method of variation of parameters: suppose that $y(t) = c(t)y_1(t)$ for y_1 a solution to the homogeneous equation and then solve for $c(t)$.

17. Consider the initial value problem $y' + ay = \exp(bt)$, $y(0) = 0$, where $b \neq a$ are constants. Find the solution $y(t)$ for $t > 0$ using the following methods: **(9 marks)**

a) **(3 marks)** the method of undetermined coefficients

b) **(3 marks)** using the Laplace transform

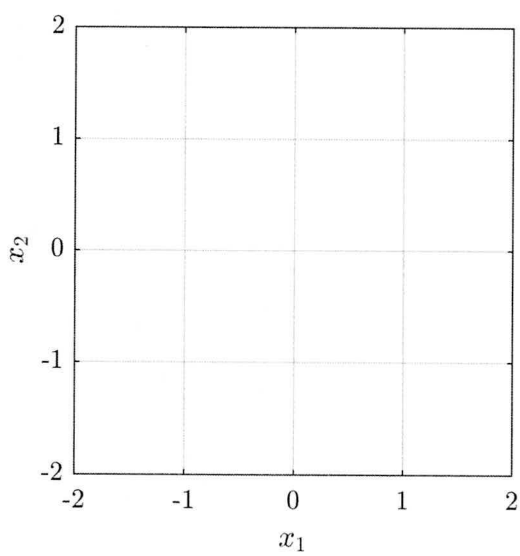
c) **(3 marks)** evaluating the integral in your answer to problem **16** with $g(t) = \exp(bt)$.

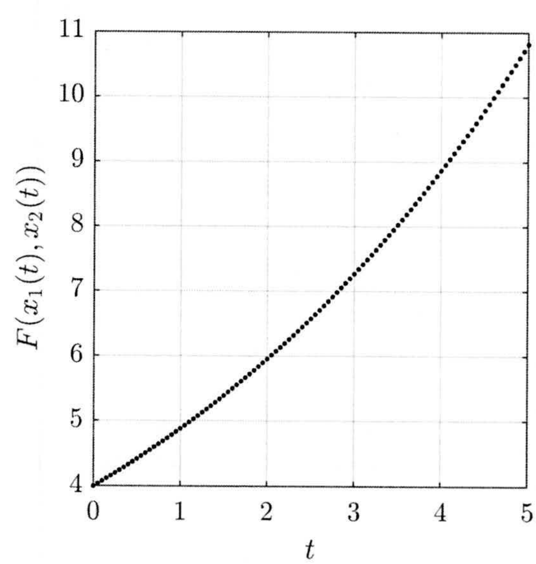
18. Consider the autonomous system, $x_1'(t) = -x_2(t), \quad x_2'(t) = 4x_1(t)$. **(10 marks)**

a) **(2 marks)** What are the eigenvalues of the coefficient matrix?

b) **(2 marks)** Does the trajectory starting at $x_1(0) = 1, x_2(0) = 0$ return to its initial value? To decide, calculate the time derivative of $F(x_1, x_2) = 4x_1^2 + x_2^2$.

c) **(4 marks)** Sketch the phase plane on axes below, to the left. Include the contours (curves of constant value) of F and the trajectory passing through $(1, 0)$.





d) **(2 marks)** The plot above, on the right, shows the value of F evaluated from an Euler's method numerical solution with stepsize $h = 0.1$. Why does F increase instead of remaining constant?

19. Consider the system of equations $\mathbf{x}'(t) = \begin{pmatrix} 2 & 1 \\ \alpha & 0 \end{pmatrix} \mathbf{x}(t)$ with real parameter α. **(8 marks)**

a) **(3 marks)** For which values of α is $\mathbf{0}$ the unique unstable critical point of the system?

b) **(5 marks)** Find the general (real) solution for $\alpha = -2$.

20. An igloo is heated by an oil lamp called a qulliq. Let $y(t)$ represent the temperature **(10 marks)** of the igloo in degrees Celsius at time t in hours, which is modelled by the initial value problem

$$y'(t) = -0.1(y(t) + 50) + 5u_a(t), \quad y(0) = -50.$$

a) **(3 marks)** Describe the assumptions that resulted in this initial value problem.

b) **(5 marks)** Find $Y(s) = \mathcal{L}\{y(t)\}(s)$ and invert the Laplace transform to find $y(t)$.

c) **(2 marks)** When should the lamp be lit so that the temperature in the igloo will be -25 degrees Celsius at time $t = 24$ hours?

21. A qualitative model of the human circadian clock **(8 marks)**
(the body's light-driven, 24-hour time-keeping mechanism) is given by the differential equation

$$y'' + \frac{\pi}{30} y' + \left(\frac{2\pi}{24}\right)^2 y = L(t),$$

in which t is the time in hours since sunrise, $L(t)$ is the light input that drives the clock, and y is the circadian output variable which typically oscillates with a period of 24-hours. The variable y corresponds directly to body temperature, which rhythmically varies by 1 degree Celsius each day.

(a) **(2 marks)** Is the system undamped, underdamped, critically damped, or overdamped?

(b) **(2 marks)** For constant light input, what is the long-run behaviour of the body temperature?

(c) **(2 marks)** Does the light input $L(t) = \frac{1}{2}\left[1 + \sin\left(\frac{2\pi}{24} t\right)\right]$ result in unbounded solutions? Explain.

(d) **(2 marks)** How would the body temperature behave on Mars with 25-hour long days?

22. A ball has mass m and position $x(t)$, a function of time. **(10 marks)**

In a *potential well*, the ball's position is governed by the differential equation $mx'' = -V'(x)$ for potential $V(x) = x^{2p}$ for positive integer p.

a) (2 marks) Find any equilibrium solutions and classify them as stable or unstable.

b) (2 marks) Show that the energy of the ball $E = \frac{1}{2}m(x')^2 + V(x)$ is constant, i.e. $\frac{dE}{dt} = 0$ for $x(t)$ a solution of the differential equation.

c) (2 marks) In the limit $p \to \infty$, $V(x) = 0$ for $x \in [-1, 1]$. Explain why the ball is confined within $[-1, 1]$ and why, in the long-run, it spends an equal amount of time near each position $x \in [-1, 1]$. Use the initial condition $x(0) = 0, x'(0) = 1$.

d) **(3 marks)** If the ball is very small, it will not spend an equal amount of time near each point in $[-1, 1]$ in the limit $p \to \infty$. According to quantum physics, the position of a particle is determined from its wave-function $\psi(x)$ as $\int_a^b |\psi(x)|^2 dx =$ the probability of finding the ball in $[a, b]$. In the case of an infinite square well potential ($p \to \infty$), the steady-state wave-function $\psi(x)$ satisfies the differential equation

$$\frac{d^2\psi}{dx^2} = -k^2\psi,$$

with two boundary conditions $\psi(-1) = 0 = \psi(1)$ for a parameter $k > 0$. Solve for $\psi(x)$ and show that only particular values of k (particle energies) are permitted.

Although the particle spends different amounts of time near different x, it becomes uniform as $k \to \infty$.

e) **(1 mark)** Explain why $f(x) = \frac{\sqrt{15}}{4}(1 - x)(1 + x)$ can be written as a linear combination of solutions $\psi(x)$ from part **d**.

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FORMULA SHEET

First-Order Linear Differential Equations. $y' + p(t)y = g(t)$.

- $\mu(t) = e^{\int p(t)\,dt}$
- $y = \dfrac{1}{\mu(t)} \displaystyle\int \mu(t)g(t)\,dt + \dfrac{C}{\mu(t)}$.

Exact First-Order Differential Equations. $M(x,y) + N(x,y)y' = 0$

- Exact if and only if $M_y = N_x$.
- Solution $\Psi(x,y) = C$ where $\Psi_x = M$ and $\Psi_y = N$.

Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $t_n = t_0 + n \cdot h$
- $y_{n+1} = y_n + f(t_n, y_n)h$ or $y'(t_n) = \dfrac{y_{n+1} - y_n}{h}$
- $E_n \leqslant Ch$

Improved Euler Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + k_{n,2}}{2}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f(t_{n+1}, y_n + k_{n,1}h)$
- $E_n \leqslant Ch^2$

Runge-Kutta Method. $y' = f(t,y)$ $\quad y(t_0) = y_0$.

- $y_{n+1} = y_n + \dfrac{k_{n,1} + 2k_{n,2} + 2kl_{n,3} + k_{n,4}}{6}h$
- $k_{n,1} = f(t_n, y_n)$
- $k_{n,2} = f\left(t_n + \frac{h}{2}, y_n + k_{n,1}\frac{h}{2}\right)$
- $k_{n,3} = f\left(t_n + \frac{h}{2}, y_n + k_{n,2}\frac{h}{2}\right)$
- $k_{n,4} = f(t_{n+1}, y_n + k_{n,3}h)$
- $E_n \leqslant Ch^4$

Euler's Formula. $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Continued...

Limits and Series.

- $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $r < 1$.
- $\exp(At) = e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$.
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}A\right)^n = e^A$.

Variation of Parameters.

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W[y_1,y_2](t)}dt + y_2(t)\int \frac{y_1(t)g(t)}{W[y_1,y_2](t)}dt$$

Laplace Transforms.

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt.$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{a\,t}\} = \frac{1}{s-a},$$

$$\mathcal{L}\{\sin(k\,t)\} = \frac{k}{s^2+k^2}, \quad \mathcal{L}\{\cos(k\,t)\} = \frac{s}{s^2+k^2},$$

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0), \quad \mathcal{L}\{f''(t)\} = s^2\,F(s) - s\,f(0) - f'(0),$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,F(s) - s^{n-1}\,f(0) - s^{n-2}\,f'(0) - \cdots - s\,f^{(n-2)}(0) - f^{(n-1)}(0),$$

$$\mathcal{L}\{e^{a\,t}\,f(t)\} = F(s-a), \quad \mathcal{L}\{u_a(t)\,f(t-a)\} = e^{-s\,a}\,F(s),$$

$$\mathcal{L}\{t^n\,f(t)\} = (-1)^n\,\frac{d^n}{ds^n}F(s),$$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}} \text{ for } T\text{-periodic } f,$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\left\{\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right\} = F(s)\,G(s),$$

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-s\,t_0}.$$

The end.