ESC195S CALCULUS II

Midterm Test #1

11 February 2020 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructor: J. W. Davis

Family Name: JW Davis

Given Name: Solutions

Student #: ____________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	13	
2	7	
3	9	
4	10	
5	8	
6	8	
7	12	
8	9	
TOTAL	76	70

Tutorial Section: ____________________

TA Name: ____________________

1) Evaluate the following integrals.

a) $\int \frac{dx}{\sqrt{x^2+2x+5}}$

b) $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)} dx$

c) $\int x\tan^2 x\, dx$

(13 marks)

a) $x^2+2x+5 = (x+1)^2+4 \Rightarrow$ let $(x+1) = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$(x+1)^2+4 = 4\tan^2\theta+4 = 4\sec^2\theta$

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C$$

b) $\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3} \Rightarrow 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$

$= Ay^2 - Ay - 6A + By^2 - 3By + Cy^2 + 2Cy$

$y^2: 4 = A+B+C$

$y: -7 = -A-3B+2C$

$1: -12 = -6A \Rightarrow A=2$

$2 = B+C$

$-5 = -3B+2C = -3B+2(2-B) = -3B+4-2B$

$\therefore B = \frac{9}{5}$ $\therefore C = \frac{1}{5}$

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}dy = \int_1^2 \frac{2\,dy}{y} + \int_1^2 \frac{9/5\,dy}{y+2} + \int_1^2 \frac{1/5\,dy}{y-3} = \left[2\ln y + \frac{9}{5}\ln(y+2) + \frac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2 + \frac{9}{5}(\ln 4 - \ln 3) + \frac{1}{5}(0-\ln 2) = \frac{9}{5}(\ln + \ln 4 - \ln 3) = \frac{9}{5}\ln\frac{8}{3}$$

c) $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)dx = \int x\sec^2 x\,dx - \int x\,dx$

let $u = x$ $du = dx$

$dv = \sec^2 x\,dx$

$v = \tan x$

$$= x\tan x - \int \tan x\,dx - \frac{x^2}{2}$$

$$= x\tan x - \ln|\sec x| - \frac{x^2}{2} + C$$

2) For what values of p is the integral $\int_0^{\infty} \frac{dx}{x^p + x^{-p}}$ convergent?

(7 marks)

$\Rightarrow$ $\frac{1}{x^p+x^{-p}} = \frac{x^p}{x^{2p}+1}$ is a continuous function for $0 < x < \infty$. $\therefore$ convergence is determined by the behaviour as $x \to 0$ or $x \to \infty$.

$\Rightarrow$ From symmetry we consider $p \geq 0$.

$\Rightarrow$ Special cases: $p = 0 \Rightarrow \int_0^{\infty} \frac{dx}{1+1} = \frac{1}{2}\left[x\right]_0^{\infty} \to \infty$

$p = 1 \Rightarrow \int_0^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\left[\ln(x^2+1)\right]_0^{\infty} \longrightarrow \infty$

$\Rightarrow$ consider $x \to 0$: for $0 < x < 1 \Rightarrow 0 < x^{2p} < 1$

$\therefore 1 > \frac{1}{x^{2p}+1} > \frac{1}{2} \Rightarrow \frac{x^p}{1} > \frac{x^p}{x^{2p}+1} > \frac{x^p}{2}$

$\therefore \int_0^a \frac{x^p}{1}\,dx > \int_0^a \frac{x^p}{x^{2p}+1}\,dx > \int_0^a \frac{x^p}{2}\,dx \Rightarrow \left[\frac{x^{p+1}}{p+1}\right]_0^a > \int_0^a \frac{x^p}{x^{2p}+1}\,dx > \frac{1}{2}\left[\frac{x^{p+1}}{p+1}\right]_0^a$

$\therefore \int_0^a \frac{x^p}{x^{2p}+1}\,dx$ converges for all $p \geq 0$; $a < 1$

$\Rightarrow$ consider $x \to \infty$: for $x > 1 \Rightarrow x^{2p} > 1$

$\therefore x^{2p} < x^{2p}+1 < 2x^{2p} \Rightarrow \frac{1}{x^{2p}} > \frac{1}{x^{2p}+1} > \frac{1}{2x^{2p}} \Rightarrow \frac{x^p}{x^{2p}} > \frac{x^p}{x^{2p}+1} > \frac{x^p}{2x^{2p}}$

$\therefore \int_b^{\infty} \frac{x^p}{x^{2p}}\,dx > \int_b^{\infty} \frac{x^p}{x^{2p}+1}\,dx > \int_b^{\infty} \frac{x^p}{2x^{2p}} \Rightarrow \int_b^{\infty} x^{-p}\,dx > \int_b^{\infty} \frac{x^p\,dx}{x^{2p}+1} > \frac{1}{2}\int_b^{\infty} x^{-p}\,dx$

$\Rightarrow \left[\frac{x^{1-p}}{1-p}\right]_b^{\infty} > \int_b^{\infty} \frac{x^p\,dx}{x^{2p}+1} > \frac{1}{2}\left[\frac{x^{1-p}}{1-p}\right]_b^{\infty}$

$\therefore \int_b^{\infty} \frac{x^p}{x^{2p}+1}\,dx$ converges for $p > 1$; $b > 1$

$\Rightarrow$ From symmetry $\int_0^{\infty} \frac{dx}{x^p+x^{-p}}$ converges for $|p| > 1$

3) Sketch the parametric curve: $x = t^3 - 3t, \quad y = \dfrac{2}{1+t^2}$

Identify all horizontal and vertical asymptotes, and describe the asymptotic behaviour and what happens at all points where $x = 0$

(9 marks)

$x = t^3 - 3t$

$x' = 3t^2 - 3$

$x' = 0 \Rightarrow t = \pm 1$

$\Rightarrow$ vertical tangents : $(-2,1)$, $(2,1)$

$y = 2(1+t^2)^{-1}$

$y' = -2(1+t^2)^{-2} \cdot 2t = -4t(1+t^2)^{-2}$

$y' = 0 \Rightarrow t = 0$

$\Rightarrow$ horizontal tangent $(0,2)$

Asymptotic behaviour: $t \to +\infty$: $x \to +\infty$, $y \to 0$

$t \to -\infty$: $x \to -\infty$, $y \to 0$

Describe what happens at points where $x = 0$

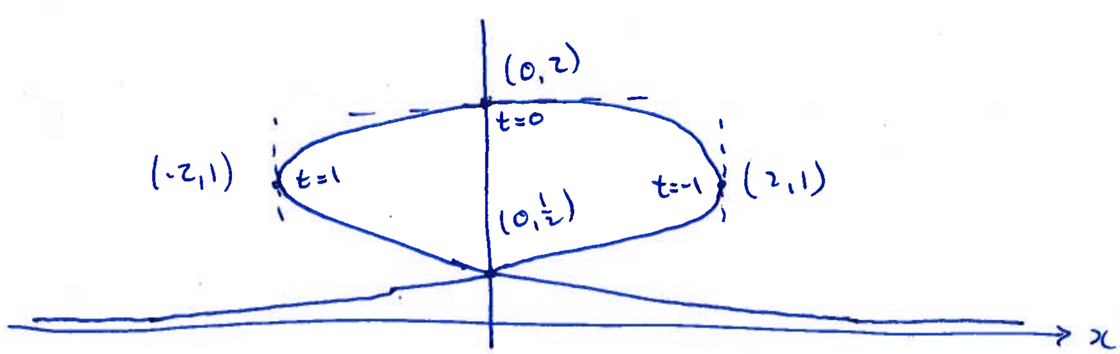
$x = t^3 - 3t = 0 \Rightarrow t = 0, \quad t = \pm\sqrt{3}$

$t = 0 \to (0,2)$ horizontal asymptote

$t = \pm\sqrt{3} \to (0, \tfrac{1}{2})$

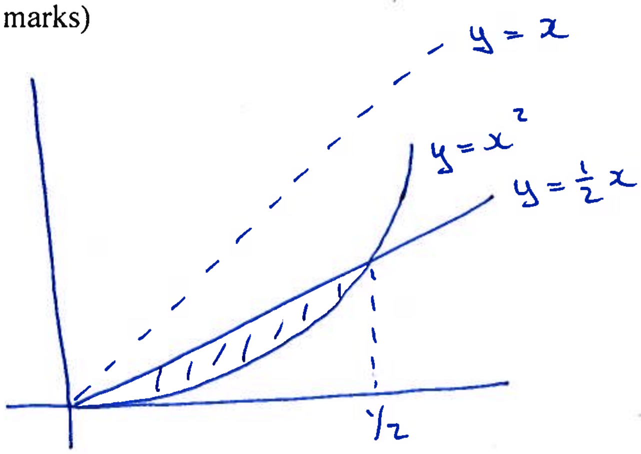
$\dfrac{dy}{dx}(t=\sqrt{3}) = \dfrac{y'(\sqrt{3})}{x'(\sqrt{3})} = \dfrac{-4\sqrt{3}/(1+3)^2}{3 \cdot 3 - 3} = \dfrac{-\sqrt{3}/4}{6} = \dfrac{-\sqrt{3}}{24}$

$\dfrac{dy}{dx}(t=-\sqrt{3}) = \dfrac{4\sqrt{3}/(1+3)^2}{3 \cdot 3 - 3} = \dfrac{+\sqrt{3}}{24}$



4) Find the centroid of the region trapped between the curves $y = x^2$ and $y = \frac{1}{2}x$. Use Pappus's theorem to find the volume formed by rotating this region about the line $y = x$.

(10 marks)



Intersection:

$x^2 = \frac{1}{2}x$

$\Rightarrow x = \frac{1}{2}$

$A = \int_0^{1/2} \left(\frac{1}{2}x - x^2\right) dx = \left[\frac{x^2}{4} - \frac{x^3}{3}\right]_0^{1/2} = \frac{1}{16} - \frac{1}{24} = \frac{1}{48}$

$\bar{x} A = \int_0^{1/2} x\left(\frac{1}{2}x - x^2\right) dx = \left[\frac{x^3}{6} - \frac{x^4}{4}\right]_0^{1/2} = \frac{1}{48} - \frac{1}{64} = \frac{1}{192}$ $\therefore \bar{x} = \frac{48}{192} = \frac{1}{4}$

$\bar{y} A = \int_0^{1/2} \frac{1}{2}\left(\left(\frac{1}{2}x\right)^2 - (x^2)^2\right) dx = \int_0^{1/2} \left(\frac{x^2}{8} - \frac{x^4}{2}\right) dx = \left[\frac{x^3}{24} - \frac{x^5}{10}\right]_0^{1/2} = \frac{1}{8\cdot 24} - \frac{1}{32\cdot 10}$

$= \frac{1}{3\cdot 64} - \frac{1}{5\cdot 64} = \frac{2}{3\cdot 5\cdot 64} = \frac{1}{2\cdot 5\cdot 48}$ $\therefore \bar{y} = \frac{48}{10\cdot 48} = \frac{1}{10}$

Distance to line $y = x$:

1) find $\perp$ line through centroid: slope $= -1$ $\Rightarrow \frac{1}{10} = (-1)\left(\frac{1}{4}\right) + b$

$\therefore b = \frac{1}{10} + \frac{1}{4} = \frac{7}{20}$

$\therefore y = -x + \frac{7}{20}$

2) find intersection with line $y = x$: $y = -y + \frac{7}{20} \Rightarrow y = \frac{7}{40}, \ x = \frac{7}{40}$

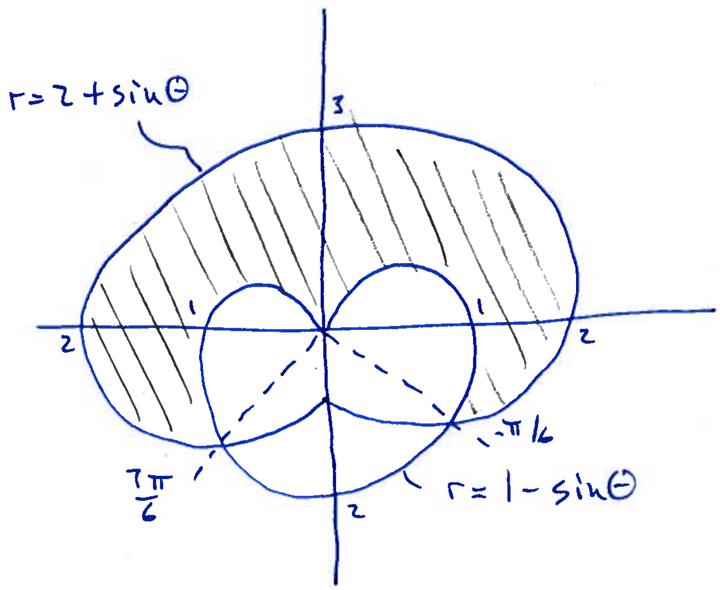
3) distance from centroid to line $y = x$: $\bar{R} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$\bar{R} = \sqrt{\left(\frac{1}{4} - \frac{7}{40}\right)^2 + \left(\frac{1}{10} - \frac{7}{40}\right)^2} = \sqrt{\left(\frac{3}{40}\right)^2 + \left(\frac{3}{40}\right)^2} = \frac{3\sqrt{2}}{40}$

$\therefore$ Pappus's Th'm: $V = 2\pi \bar{R} A = 2\pi \cdot \frac{3\sqrt{2}}{40} \cdot \frac{1}{48} = \frac{\pi\sqrt{2}}{320}$

5) Find the area of the region that lies inside $r = 2 + \sin\theta$ but outside $r = 1 - \sin\theta$. Provide a sketch of the region.

(8 marks)



Intersection:

$2 + \sin\theta = 1 - \sin\theta$

$\sin\theta = -1/2$

$\Rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

$$\Rightarrow A = \int_{-\pi/6}^{\frac{7\pi}{6}} \frac{1}{2}\left[(2+\sin\theta)^2 - (1-\sin\theta)^2\right] d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\frac{7\pi}{6}} (4 + 4\sin\theta + \sin^2\theta - 1 + 2\sin\theta - \sin^2\theta)\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/6}^{\frac{7\pi}{6}} (3 + 6\sin\theta)\, d\theta$$

$$= \frac{1}{2}\left[3\theta - 6\cos\theta\right]_{-\pi/6}^{7\pi/6}$$

$$= \frac{1}{2}\left[3\left(\frac{7\pi}{6} + \frac{\pi}{6}\right) - 6\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= 2\pi + 3\sqrt{3}$$

6) Use the formal definition of the limit of a sequence (ϵ-N argument) to prove: $\lim_{k\to\infty} \frac{k}{k-1} = 1$

(8 marks)

① Find N st. $\left| \frac{k}{k-1} - 1 \right| < \epsilon$ for $k > N$

$\Rightarrow \left| \frac{k}{k-1} - 1 \right| = \left| \frac{1}{k-1} \right| = \frac{1}{k-1}$ for $k > 2$

We therefore require $\frac{1}{k-1} < \epsilon \Rightarrow k-1 > \frac{1}{\epsilon} \Rightarrow k > \frac{1}{\epsilon} + 1$

$\therefore$ let $N = \frac{1}{\epsilon} + 1$

② Given $\epsilon > 0$, let $N = \frac{1}{\epsilon} + 1$ (or $N = 2$ for $\epsilon > 1$)

$\therefore \left| \frac{k}{k-1} - 1 \right| = \frac{1}{k-1} < \frac{1}{\frac{1}{\epsilon}+1-1} = \epsilon$ for $k > N$

$\therefore \lim_{k\to\infty} \frac{k}{k-1} = 1$ by the definition of a limit.

7) Determine whether the sequence converges or diverges. If it converges, find the limit:

(i) $a_n = \dfrac{4^n}{1+9^n}$ (ii) $a_n = \dfrac{\tan^{-1} n}{n}$

(iii) $a_n = n^{2/n}$ (iv) $a_n = \dfrac{n!}{2^n}$

(12 marks)

i) $a_n = \dfrac{4^n}{1+9^n} = \dfrac{(4/9)^n}{1+(1/9)^n} \longrightarrow \dfrac{0}{1+0} = 0$ since $\left(\frac{4}{9}\right)^n$ & $\left(\frac{1}{9}\right)^n$ converge to zero

ii) $a_n = \dfrac{\tan^{-1} n}{n}$

$\Rightarrow \lim\limits_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ $\therefore \lim\limits_{n \to \infty} \tan^{-1} n = \frac{\pi}{2}$

$\therefore \lim\limits_{n \to \infty} \dfrac{\tan^{-1} n}{n} \longrightarrow 0$

iii) $a_n = n^{2/n} = \left(n^{1/n}\right)^2$

$\Rightarrow \lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{\ln x^{1/x}} = \lim\limits_{x \to \infty} e^{\frac{\ln x}{x}}$

$\Rightarrow \lim\limits_{x \to \infty} \dfrac{\ln x}{x} \overset{H}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$ $\therefore \lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{\frac{\ln x}{x}} = e^0 = 1$

$\therefore \lim\limits_{n \to \infty} n^{1/n} = 1$ $\therefore \lim\limits_{n \to \infty} \left(n^{1/n}\right)^2 = 1^2 = 1$

iv) $a_n = \dfrac{n!}{2^n} = \dfrac{1 \cdot 2 \cdot 3 \cdot 4 \cdots\cdots n}{2 \cdot 2 \cdot 2 \cdot 2 \cdots\cdots 2} > \dfrac{120}{32} \cdot \dfrac{n}{2} \longrightarrow \infty$

$\therefore a_n$ diverges

8) a) For what values of p do the following series converge:

i) $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ ii) $\sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln \ln k)^p}$

b) Which of the following series converges faster? Explain.

$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$ $\sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln \ln k)^2}$

(9 marks)

a i) Consider $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$ let $u = \ln x$ $du = \frac{dx}{x}$

$= \int_{\ln 2}^{\infty} \frac{du}{u^p} = \left[\frac{u^{1-p}}{1-p} \right]_{\ln 2}^{\infty} = \left[\frac{(\ln x)^{1-p}}{1-p} \right]_2^{\infty}$ converges for $p > 1$

$\therefore \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^p}$ converges for $p > 1$ by the integral test

ii) Consider $\int_3^{\infty} \frac{dx}{x \ln x (\ln \ln x)^p}$ let $u = \ln \ln x$ $du = \frac{dx}{x \ln x}$

$= \int_{\ln \ln 3}^{\infty} \frac{du}{u^p} = \left[\frac{u^{1-p}}{1-p} \right]_{\ln \ln 3}^{\infty} = \left[\frac{(\ln \ln x)^{1-p}}{1-p} \right]_3^{\infty}$ converges for $p > 1$

$\therefore \sum_{k=3}^{\infty} \frac{1}{k \ln k (\ln \ln k)^p}$ converges for $p > 1$ by the integral test

b) Consider $\lim_{k \to \infty} \frac{a_k}{b_k} = \lim_{k \to \infty} \frac{k \ln k (\ln \ln k)^2}{k(\ln k)^2} = \lim_{k \to \infty} \frac{(\ln \ln k)^2}{\ln k}$

$\lim_{x \to \infty} \frac{(\ln \ln x)^2}{\ln x} \to \frac{\infty}{\infty} \overset{*}{\Rightarrow} \lim_{x \to \infty} \frac{2 \ln \ln x \cdot \frac{1}{x \ln x}}{1/x} = \lim_{x \to \infty} \frac{2 \ln \ln x}{\ln x} \to \frac{\infty}{\infty}$

$\overset{*}{\Rightarrow} \lim_{x \to \infty} 2 \frac{\frac{1}{x \ln x}}{1/x} = \lim_{x \to \infty} \frac{1}{\ln x} \to 0$

$\therefore$ The terms of $a_k = \frac{1}{k(\ln k)^2}$ thus approach 0 much more quickly than the terms of $b_k = \frac{1}{k \ln k (\ln \ln k)^2}$

$\therefore \sum a_k$ converges faster.