MAT195S CALCULUS II

Midterm Test #1

4 February 2016 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: Y. Cher and J. W. Davis

Family Name: J W Davis.

Given Name: Solutions

Student #: ______

FOR MARKER USE ONLY		
Question	Marks	Earned
1	12	
2	10	
3	10	
4	7	
5	12	
6	8	
7	9	
8	8	
TOTAL	76	/70

Tutorial Section: ______

TA Name: ______

1) Evaluate the following integrals.

a) $\int \frac{(\ln x)^2}{x^3}\,dx$

b) $\int \sin(8x)\cos(5x)\,dx$

c) $\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$

(12 marks)

a) $\int \frac{(\ln x)^2}{x^3}\,dx$ let $u = (\ln x)^2$ $du = \frac{2\ln x}{x}\,dx$ $dv = x^{-3}\,dx$ $v = -\frac{1}{2}x^{-2}$

$= -\frac{1}{2}\frac{(\ln x)^2}{x^2} + \int \frac{\ln x}{x^3}$ let $u = \ln x$ $du = \frac{dx}{x}$ $dv = x^{-3}dx$ $v = -\frac{1}{2}x^{-2}$

$= -\frac{1}{2}\frac{(\ln x)^2}{x^2} - \frac{\ln x}{2x^2} + \int \frac{dx}{2x^3}$

$= -\frac{1}{2}\frac{(\ln x)^2}{x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2} + C$

b) $\int \sin 8x \cos 5x\,dx = \int \frac{1}{2}\left[\sin(8x-5x) + \sin(8x+5x)\right]dx$

$= \frac{1}{2}\int(\sin 3x + \sin 13x)\,dx = -\frac{1}{2}\cos 3x \cdot \frac{1}{3} - \frac{1}{2}\cos 13x \cdot \frac{1}{13} + C$

$= -\frac{\cos 3x}{6} - \frac{\cos 13x}{26} + C$

c) $\int \frac{x^2}{\sqrt{1+4x^2}}\,dx$ let $2x = \sin\theta$ $2dx = \cos\theta\,d\theta$ $\sqrt{1+4x^2} = \cos\theta$

$= \int \frac{\frac{1}{4}\sin^2\theta \cdot \frac{1}{2}\cos\theta\,d\theta}{\cos\theta} = \frac{1}{8}\int \sin^2\theta\,d\theta = \frac{1}{8}\left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right) + C$

2) Evaluate the integral, or show that it diverges: a) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$

b) $\int_{0}^{\pi} \tan^2 x\,dx$

(10 marks)

a) $\int \frac{e^x}{1+e^{2x}}\,dx$ let $u = e^x$ $du = e^x\,dx$

$= \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(e^x) + C$

$\Rightarrow \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}} = \lim_{a\to -\infty} \left[\tan^{-1}(e^x)\right]_a^0 = \tan^{-1} 1 - \lim_{a\to-\infty} \tan^{-1} e^a$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$

$\Rightarrow \int_{0}^{\infty} \frac{e^x}{1+e^{2x}} = \lim_{b\to\infty} \left[\tan^{-1}(e^x)\right]_0^b = \lim_{b\to\infty} \tan^{-1} e^b - \tan^{-1} 1$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

$\therefore \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$

b) $\int_0^{\pi} \tan^2 x\,dx = \int_0^{\pi/2} \tan^2 x\,dx + \int_{\pi/2}^{\pi} \tan^2 x\,dx$ discontinuity at $x = \pi/2$

consider $\int_0^{\pi/2} \tan^2 x\,dx = \int_0^{\pi/2} (\sec^2 x - 1)\,dx = \left[\tan x - x\right]_0^{\pi/2}$

$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$ $\therefore \int_0^{\pi/2} \tan^2 x\,dx$ diverges

$\therefore \int_0^{\pi} \tan^2 x\,dx$ diverges

Note: $\int_{\pi/2}^{\pi} \tan^2 x\,dx$ also diverges

3) Sketch a graph of the swallowtail catastrophe curve: $x = 12t - 4t^3$ $\quad -\infty < t < \infty$

$y = -6t^2 + 3t^4$

Show all vertical and horizontal tangents, intercepts with the coordinate axes, and identify the asymptotic behaviour. What happens at $t = \pm 1$? The second derivative: $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ may be helpful.

(10 marks)

$\frac{dx}{dt} = 0 \Rightarrow 12 - 12t^2 = 0 \Rightarrow t = \pm 1$

$\frac{dy}{dt} = 0 \Rightarrow -12t + 12t^3 = 0 \Rightarrow t = 0,\ t = \pm 1$

at $t = 0 \Rightarrow (0,0)$: horizontal tangent

$\left.\begin{array}{l} t = 1 \Rightarrow (8,-3) \\ t = -1 \Rightarrow (-8,-3) \end{array}\right\}$ behaviour unknown.

Intercepts: $x = 0 \Rightarrow 12t - 4t^3 = 0 \Rightarrow t = 0$ or $t^2 = 3 \Rightarrow (0, 9)$

$y = 0 \Rightarrow -6t^2 + 3t^4 = 0 \Rightarrow t = 0$ or $t^2 = 2 \Rightarrow (\pm 4\sqrt{2}, 0)$

Asymptotes: for large t: $\left.\begin{array}{l} x \sim -4t^3 \\ y \sim 3t^4 \end{array}\right\}$ $y \sim 3\left(\frac{-x}{4}\right)^{4/3}$

or $y \sim x^{4/3}$

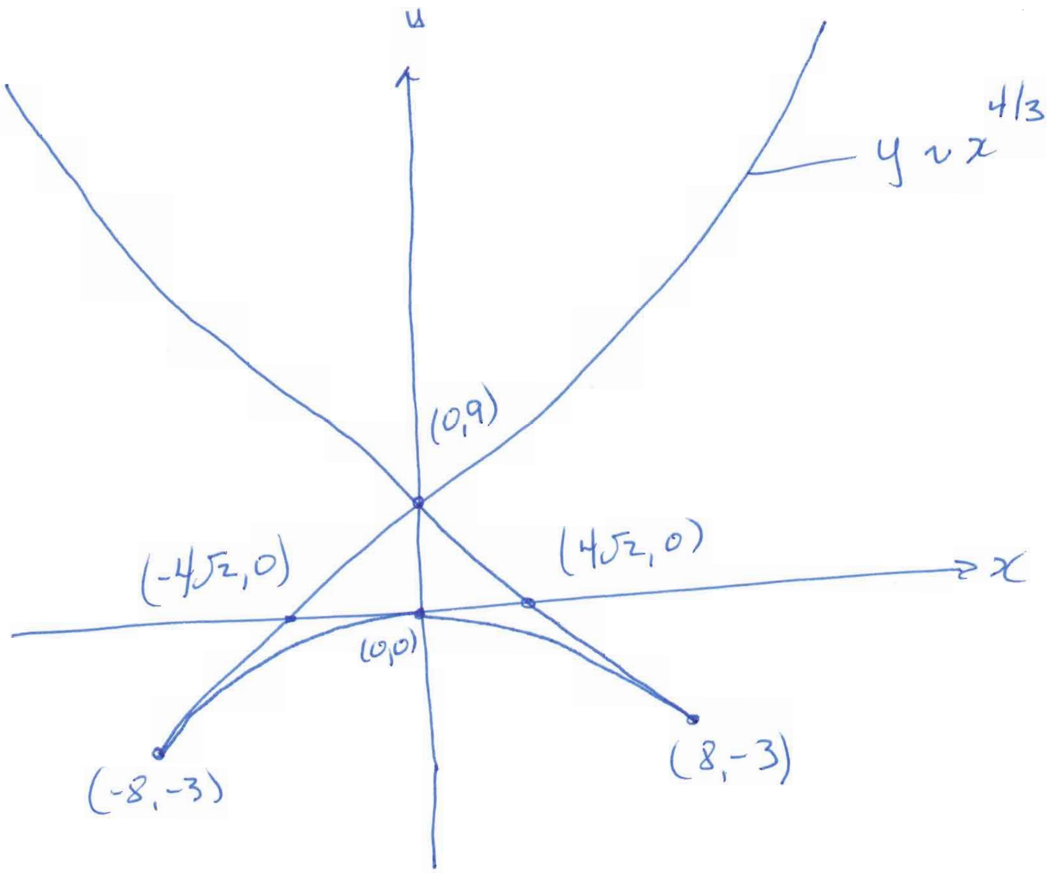
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{-12t(1-t^2)}{12(1-t^2)}\right)}{12(1-t^2)} = \frac{\frac{d}{dt}(-t)}{12(1-t^2)} = \frac{-1}{12(1-t^2)} \quad t \neq \pm 1$$

$\Rightarrow \frac{d^2y}{dx^2} < 0$ for $t \in (-1, 1)$ $\therefore$ concave down

$\frac{d^2y}{dx^2} > 0$ for $t < -1$ & $t > 1$ $\therefore$ concave up

Consider $t=-1$: $(-8,-3)$

Both x and y have minimum values here, and both 1st and second derivatives are undefined. Thus the curve must have a cusp at this point.



One can also use the 1st derivative: $\frac{dy}{dx} = \frac{-12t(1-t^2)}{12(1-t^2)} = -t$

$(t \neq \pm 1)$

$\Rightarrow$ y has $-$ve slope for $t>0$

y has $+$ve slope for $t<0$

4) Beginning with the arclength formula for parametric curves, find the arclength formula for polar curves. Find the length of the logarithmic spiral $r = e^{\theta}$, $\theta \in [0, 2\pi]$. Provide a sketch of the curve.

(7 marks)

$x = r(\theta)\cos\theta \qquad dx/d\theta = -r\sin\theta + r'\cos\theta$

$y = r(\theta)\sin\theta \qquad dy/d\theta = r\cos\theta + r'\sin\theta$

$$\Rightarrow \quad s = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int \sqrt{(-r\sin\theta + r'\cos\theta)^2 + (r\cos\theta + r'\sin\theta)^2}\, d\theta$$

$$= \int \sqrt{r^2\sin^2\theta - \cancel{rr'\sin\theta\cos\theta} + r'^2\cos^2\theta + r^2\cos^2\theta + \cancel{rr'\sin\theta\cos\theta} + r'^2\sin^2\theta}\, d\theta$$

$$= \int \sqrt{r^2 + (r')^2}\, d\theta$$

$r = e^{\theta} \qquad \theta \in [0, 2\pi]$

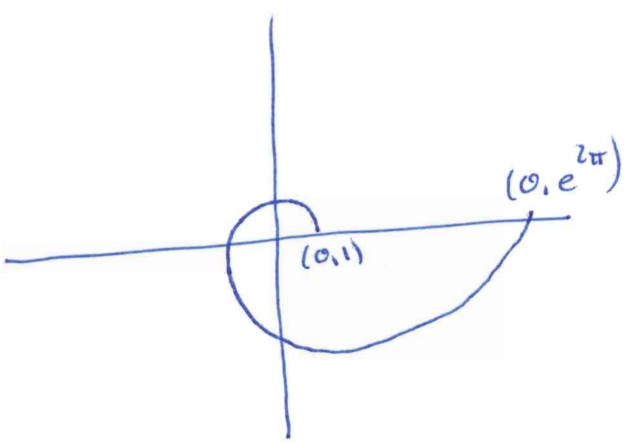
$r' = e^{\theta}$

$$s = \int_0^{2\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} e^{\theta}\, d\theta$$

$$= \sqrt{2}\left[e^{\theta}\right]_0^{2\pi}$$

$$= \sqrt{2}\left(e^{2\pi} - 1\right)$$



5) Sketch the region indicated, and find an integral representing the area of the region. Do not evaluate the integrals.

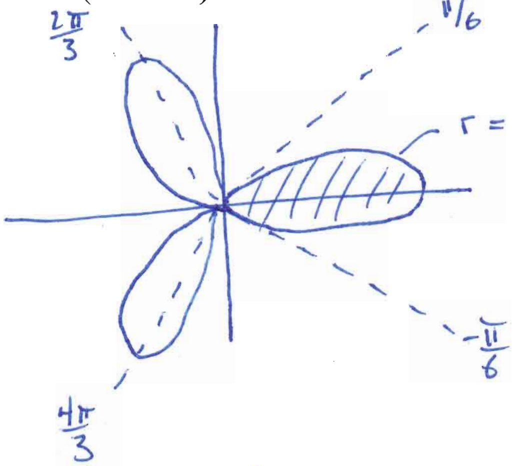
a) The region enclosed by one petal of the curve $r = \cos 3\theta$.

b) The region that lies inside $r = 1 - \sin\theta$ but outside $r = 1$.

c) The region that lies inside the larger loop, but outside the inner loop of $r = 1 + 2\cos\theta$.

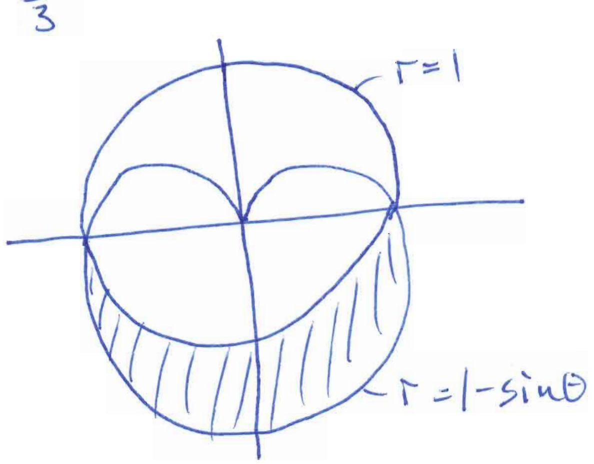
(12 marks)

a)



$$A = \int_{-\pi/6}^{\pi/6} \frac{1}{2}(\cos 3\theta)^2 \, d\theta$$

b)

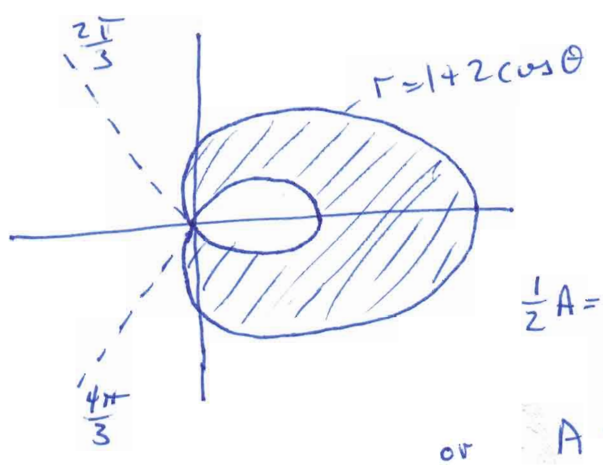


intersection: $1 - \sin\theta = 1$

$\sin\theta = 0$; $\theta = 0, \pi$

$$A = \int_{\pi}^{2\pi} \frac{1}{2}\left[(1-\sin\theta)^2 - 1^2\right] d\theta$$

c)



$r = 0 \Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = \pm \frac{2\pi}{3}$

$$\frac{1}{2}A = \int_0^{2\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2 \, d\theta - \int_{\pi}^{4\pi/3} \frac{1}{2}(1 + 2\cos\theta)^2 \, d\theta$$

or

$$A = \int_{-\frac{2\pi}{3}}^{2\pi/3} \frac{1}{2}(1+2\cos\theta)^2 \, d\theta - \int_{2\pi/3}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos\theta)^2 \, d\theta$$

6) Determine whether the following sequence converges or diverges; if it converges, find the limit:

a) $a_n = n\sin(1/\sqrt{n})$

b) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

c) $a_n = \dfrac{7^n}{(2n)!}$

(8 marks)

a) $a_n = n \sin\frac{1}{\sqrt{n}}$: consider the function $f(x) = x \sin\frac{1}{\sqrt{x}}$

let $u = \frac{1}{\sqrt{x}} \Rightarrow x = \frac{1}{u^2}$

$\therefore f(u) = \frac{\sin u}{u^2}$

$$\lim_{u \to 0} \frac{\sin u}{u^2} \overset{H}{=} \lim_{u\to 0} \frac{\cos u}{2u} \to \infty \qquad \therefore a_n = n\sin\tfrac{1}{\sqrt{n}} \text{ diverges}$$

b) $$a_n = n - \sqrt{n+1}\sqrt{n+3} = \frac{n^2 - (n+1)(n+3)}{n+\sqrt{n+1}\sqrt{n+3}} = \frac{n^2-(n^2+4n+3)}{n+\sqrt{n+1}\sqrt{n+3}}$$

$$= \frac{-4n-3}{n+\sqrt{n^2+4n+3}} = \frac{-4-3/n}{1+\sqrt{1+4/n+3/n^2}} \longrightarrow \frac{-4}{2} = -2$$

c) $$a_n = \frac{7^n}{(2n)!} = \frac{7}{2n(2n-1)} \cdot \underbrace{\frac{7}{(2n-2)(2n-3)} \cdots \frac{7}{4\cdot 3}}_{\text{all terms} < 1} \cdot \frac{7}{2\cdot 1}$$

$$< \frac{7}{2n(2n-1)} \cdot \frac{7}{2} \longrightarrow 0$$

7) Determine whether the series converges or diverges:

a) $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$

b) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{8^n}$

(9 marks)

a) $\sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n}$ $\Rightarrow$ Limit comparison test with $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3n^2-4n}{n^2-2n} = \lim_{n\to\infty} \frac{3-4/n}{1-2/n} = 3$

since $\sum b_n = \sum \frac{1}{n}$ diverges $\therefore \sum a_n = \sum \frac{3n-4}{n^2-2n}$ diverges

b) $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ $\Rightarrow$ $\frac{k^{1/3}}{(k^3+4k+3)^{1/2}} < \frac{k^{1/3}}{k^{3/2}} = \frac{1}{k^{7/6}}$

$\sum \frac{1}{k^{7/6}}$ $\Rightarrow$ p-series, $p > 1$ $\therefore$ convergent

$\therefore \sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$ converges by comparison test

c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{8^n}$ $\Rightarrow$ Alternating series test

$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \frac{8^n}{8^{n+1}} = \frac{1}{8} \frac{n+1}{n} \longrightarrow \frac{1}{8} < 1$ $\therefore$ decreasing

show $a_n \to 0$: work with $f(x) = \frac{x}{8^x}$

$\lim_{x\to\infty} \frac{x}{8^x} \stackrel{*}{=} \lim_{x\to\infty} \frac{1}{8^x \ln 8} \longrightarrow 0$ $\therefore$ convergent

8) a) For what values of p does the series $\sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converge?

(4 marks)

① $p < 0$: $\lim_{n\to\infty} \frac{1}{n^p \ln n} = \lim_{n\to\infty} \frac{n^{-p}}{\ln n} \longrightarrow \infty$ $\therefore$ series diverges

② $0 \le p \le 1$: $n^p \ln n \le n \ln n \Rightarrow \frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$; $\sum \frac{1}{n \ln n}$ diverges

$$\Rightarrow \int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \left[\ln u\right]_{\ln 2}^{\infty} = \left[\ln(\ln x)\right]_2^{\infty} \longrightarrow \infty$$

③ $p > 1$: use limit comparison test with $a_n = \frac{1}{n^p \ln n}$ $b_n = \frac{1}{n^p}$

$\Rightarrow \sum b_n$ converges : p-series, $p > 1$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\ln n} = 0$ $\therefore \sum \frac{1}{n^p \ln n}$ converges

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^p \ln n}$ converges for $p > 1$

b) Given $a_n > 0$, show that if $\sum a_n$ diverges, then so does $\sum \sqrt{a_n}$.

(4 marks)

case ① $\lim_{n\to\infty} a_n \not\to 0$

If $a_n \not\to 0$, then $\sqrt{a_n} \not\to 0$

$\therefore \sum \sqrt{a_n}$ diverges by the test for divergence

case ② $\lim_{n\to\infty} a_n \to 0$

Given $a_n \to 0$, there must be same number k such that for $n > k$, $a_n < 1$.

$\therefore$ For $n > k$, $a_n < 1$ $\therefore \sqrt{a_n} > a_n$

$\Rightarrow$ Since $\sum a_n$ diverges, $\sum \sqrt{a_n}$ diverges by the comparison test.