AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 2

26 October 2015 8:50 am - 9:50 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: ________________

Given Name: Solutions

Student #: ________________

Tutorial/TA: ________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	10	
3	10	
4	10	
5	14	
TOTAL	54	/ 50

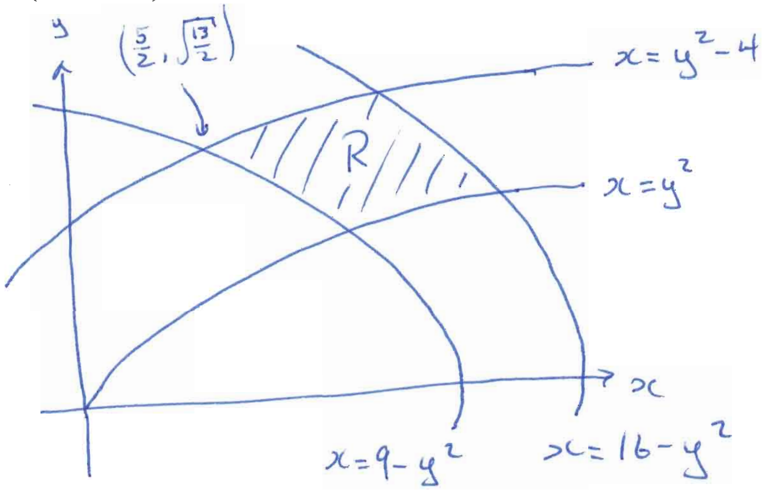
Note: The following integrals may be useful.

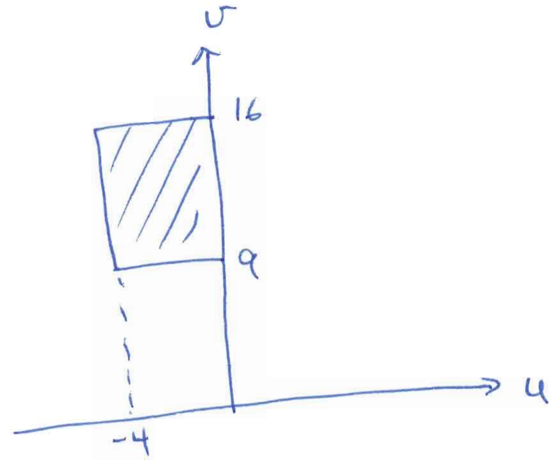
$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \qquad \oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV; \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

1) Let R be the region in the first quadrant bounded by the parabolas: $x - y^2 = 0$, $x - y^2 = -4$, $x + y^2 = 9$, and $x + y^2 = 16$. Use a coordinate transformation to evaluate $\iint_R xy\,dR$. Provide a sketch of the original region in the x-y plane, and the new region in the u-v plane.

(10 marks)





let $u = x - y^2$, $v = x + y^2$

$$\left.\begin{array}{l} -4 \le u \le 0 \\ 9 \le v \le 16 \end{array}\right\} \quad x = \frac{u+v}{2}, \quad y = \sqrt{\frac{v-u}{2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2y \\ 1 & +2y \end{vmatrix} = 4y = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1}$$

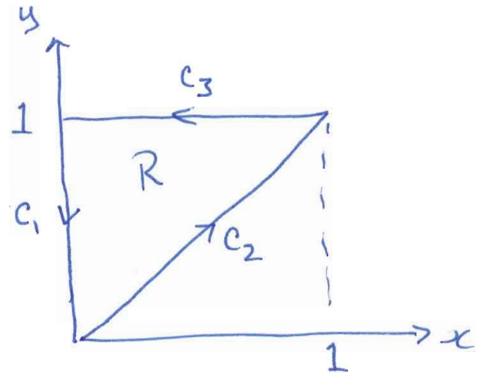
$$\therefore \int_R xy\,dR = \int_{-4}^{0} du \int_{9}^{16} dv\; xy\left(\frac{1}{4y}\right) = \frac{1}{4}\int_{-4}^{0} du \int_{9}^{16} \left(\frac{u+v}{2}\right) dv$$

$$= \frac{1}{8}\int_{-4}^{0} du \left[uv + \frac{v^2}{2}\right]_9^{16} = \frac{1}{8}\int_{-4}^{0}\left(5u + \frac{256-81}{2}\right) du$$

$$= \frac{1}{8}\left[\frac{5u^2}{2} + \frac{175}{2}u\right]_{-4}^{0} = \frac{1}{8}\left(-\frac{80}{2} + \frac{700}{2}\right) = \frac{310}{8}$$

2) Verify Green's theorem for the line integral $\oint_C x^2 y\,dx + e^y\,dy$, where C is the triangle with vertices (0, 1), (0, 0) and (1, 1).

(10 marks)



Green's Th'm: $\oint_C P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR$

$P = x^2 y \qquad \frac{\partial P}{\partial y} = x^2$

$Q = e^y \qquad \frac{\partial Q}{\partial x} = 0$

a) $\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR = \int_0^1 dx \int_x^1 (-x^2)\,dy = \int_0^1 dx\,(-x^2)\left[y\right]_x^1 = \int_0^1 (-x^2)(1-x)\,dx$

$= \int_0^1 (x^3 - x^2)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = \boxed{-\frac{1}{12}}$

b) C_1: $x = 0$, $dx = 0$ $\quad 1 > y > 0 \Rightarrow y = 1 - t$, $dy = -dt$ $\quad 0 \le t \le 1$

$\int_{C_1} x^2 y\,dx + e^y\,dy = \int_0^1 e^{1-t}(-dt) = -e\int_0^1 e^{-t}\,dt = -e\left[-e^{-t}\right]_0^1 = 1 - e$

C_2: $y = x \Rightarrow x = t$, $y = t$ $\quad dx = dt$, $dy = dt$ $\quad 0 \le t \le 1$

$\int_{C_2} x^2 y\,dx + e^y\,dy = \int_0^1 (t^3 + e^t)\,dt = \left[\frac{t^4}{4} + e^t\right]_0^1 = \frac{1}{4} + e - 1$

C_3: $y = 1$, $dy = 0$ $\quad 1 > x > 0 \Rightarrow x = 1 - t$, $dx = -dt$ $\quad 0 \le t \le 1$

$\int_{C_3} x^2 y\,dx + e^y\,dy = \int_0^1 (1-t)^2 \cdot 1 \cdot (-dt) = -\int_0^1 (1 - 2t + t^2)\,dt$

$= -\left[t - t^2 + \frac{t^3}{3}\right]_0^1 = -1 + 1 - \frac{1}{3} = -\frac{1}{3}$

$\therefore \oint_C x^2 y\,dx + e^y\,dy = 1 - e + \frac{1}{4} + e - 1 - \frac{1}{3} = \frac{1}{4} - \frac{1}{3} = \boxed{-\frac{1}{12}}$

3) The fundamental theorem of calculus as applied to volume integrals gives the following results: for a function $f(x, y, z)$ which is continuous over a volume V enclosed by a surface S, if $\hat{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$ is the unit normal on S pointing to the exterior of V, then

$$\int_V \frac{\partial f}{\partial x} dV = \int_S f\hat{n}\cdot\hat{i}\,dS \ ; \quad \int_V \frac{\partial f}{\partial y} dV = \int_S f\hat{n}\cdot\hat{j}\,dS \ ; \quad \int_V \frac{\partial f}{\partial z} dV = \int_S f\hat{n}\cdot\hat{k}\,dS$$

Use this result to derive the Gradient and Divergence Theorems.

(10 marks)

Gradient Theorem: consider $f = f(x, y, z)$

$$I = \int_S f\hat{n}\,dS = \int_S f(n_x\hat{i} + n_y\hat{j} + n_z\hat{k})\,dS$$

$$= \int_S f\hat{n}\cdot\hat{i}\,dS\,\hat{i} + \int_S f\hat{n}\cdot\hat{j}\,dS\,\hat{j} + \int_S f\hat{n}\cdot\hat{k}\,dS\,\hat{k}$$

$$= \int_V \frac{\partial f}{\partial x} dV\,\hat{i} + \int_V \frac{\partial f}{\partial y} dV\,\hat{j} + \int_V \frac{\partial f}{\partial z} dV\,\hat{k}$$

$$= \int_V \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) dV = \int_V \nabla f\,dV$$

$$\Rightarrow \boxed{\int_S f\hat{n}\,dS = \int_V \nabla f\,dV}$$

Divergence Theorem: consider $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$I = \int_S \vec{F}\cdot\hat{n}\,dS = \int_S (Pn_x + Qn_y + Rn_z)\,dS$$

$$= \int_S P\hat{n}\cdot\hat{i}\,dS + \int_S Q\hat{n}\cdot\hat{j}\,dS + \int_S R\hat{n}\cdot\hat{k}\,dS$$

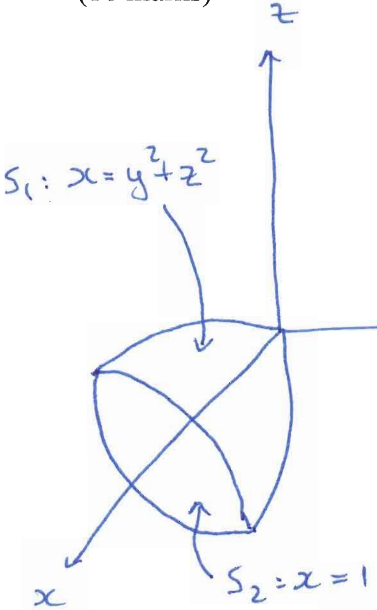
$$= \int_V \frac{\partial P}{\partial x} dV + \int_V \frac{\partial Q}{\partial y} dV + \int_V \frac{\partial R}{\partial z} dV$$

$$= \int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \int_V \nabla\cdot\vec{F}\,dV$$

$$\Rightarrow \boxed{\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{F}\,dV}$$

4) Evaluate the surface integral $\int_S \vec{F}\cdot d\vec{S}$ for the vector field $\vec{F} = x\hat{i} - y\hat{j} + z\hat{k}$, where S consists of the paraboloid $x = y^2 + z^2$, $0 \le x \le 1$, and the disk $y^2 + z^2 \le 1$, $x = 1$.

(10 marks)



S_1: let $x = u^2+v^2$, $y = u$, $z = v$

$\Rightarrow \vec{r}(u,v) = (u^2+v^2, u, v)$

$u^2+v^2 \le 1$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2u & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = (1, -2u, -2v)$$

Take $-ve$ to get outward normal:

$\vec{N} = (-1, 2u, 2v)$

$$\int_{S_1} \vec{F}\cdot\vec{N}\,du\,dv = \int_{u^2+v^2\le 1} (u^2+v^2, -u, v)\cdot(-1, 2u, 2v)\,du\,dv$$

let $u = r\cos\theta$
$v = r\sin\theta$
$0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$= \int_{u^2+v^2\le 1} (-u^2 - v^2 - 2u^2 + 2v^2)\,du\,dv = \int_{u^2+v^2\le 1} (v^2 - 3u^2)\,du\,dv$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr\,(r^2\sin^2\theta - 3r^2\cos^2\theta) = \int_0^{2\pi} (\sin^2\theta - 3\cos^2\theta)\,d\theta \int_0^1 r^3\,dr$$

$$= \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta - \frac{3\theta}{2} - \frac{3}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 = \frac{1}{4}(\pi - 3\pi) = -\frac{\pi}{2}$$

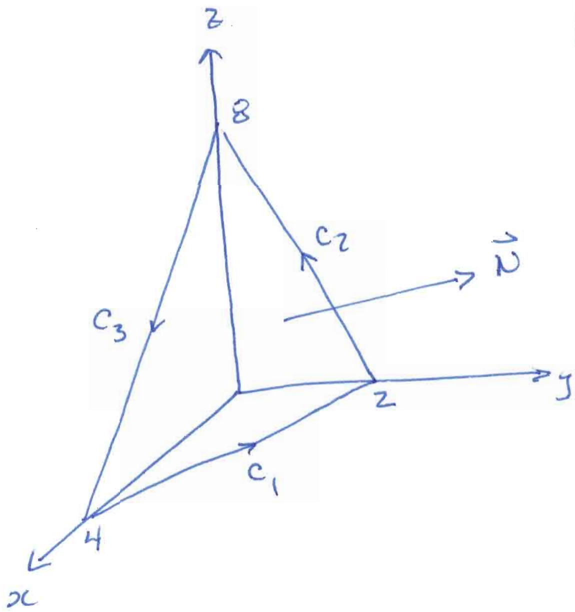
$$\int_{S_2} \vec{F}\cdot\hat{n}\,dS = \int_{S_2} (x, -y, z)\cdot\hat{i}\,dS = \int_{S_2} (\overset{x=1}{1}, -y, z)\cdot(1,0,0)\,dS$$

$$= \int_{S_2} dS = \int_{y^2+z^2\le 1} dA = \pi$$

$$\Rightarrow \oint_S \vec{F}\cdot d\vec{S} = \pi - \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

5) Verify the Stokes' theorem for the vector field $\bar{F}(x,y,z) = z\hat{i} + 2xz\,\hat{j} + xy\,\hat{k}$ over the part of the plane $2x+4y+z=8$ in the first octant. Provide a sketch of the surface and the boundary curve.

(14 marks)



Stokes' Th'm: $\displaystyle\int_S \nabla\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

$$\vec{F} = (z,\ 2xz,\ xy)$$

$$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & 2xz & xy \end{vmatrix}$$

$$= (x-2x,\ 1-y,\ 2z)$$

$$= (-x,\ 1-y,\ 2z)$$

parameterize surface: $\bar{r}(u,v) = (u,\ v,\ 8-2u-4v)$

$$0 \le u \le 4 \qquad 0 \le v \le 2-\frac{u}{2}$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{vmatrix} = (2,4,1)$$

$$\Rightarrow \int_S \nabla\times\vec{F}\cdot\vec{N}\,du\,dv = \int_0^4 du \int_0^{2-\frac{u}{2}} (-u,\ 1-v,\ 16-4u-8v)\cdot(2,4,1)$$

$$= \int_0^4 du \int_0^{2-\frac{u}{2}} dv\,(-2u+4-4v+16-4u-8v) = \int_0^4 du \int_0^{2-\frac{u}{2}} (20-6u-12v)\,dv$$

$$= \int_0^4 du \left[20v-6uv-6v^2\right]_0^{2-u/2} = \int_0^4 \left(40-10u-12u+3u^2-24+12u-\frac{3u^2}{2}\right)du$$

$$= \int_0^4 \left(16-10u+\frac{3}{2}u^2\right)du = \left[16u-5u^2+\frac{1}{2}u^3\right]_0^4 = 64-80+32$$

$$= \boxed{16}$$

$$\vec{F} = (z, 2xz, xy)$$

$C_1: \vec{r}(t) = (4-4t, 2t, 0) \quad 0 \le t \le 1$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 dt (0, 0, 8t - 8t^2) \cdot (-4, 2, 0) = \int_0^1 0\, dt = 0$$

$C_2: \vec{r}(t) = (0, 2-2t, 8t) \quad 0 \le t \le 1$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 dt (8t, 0, 0) \cdot (0, -2, 8) = \int_0^1 0\, dt = 0$$

$C_3: \vec{r}(t) = (4t, 0, 8-8t) \quad 0 \le t \le 1$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 dt (8-8t, 64t - 64t^2, 0) \cdot (4, 0, -8)$$

$$= \int_0^1 (32 - 32t)\, dt = \left[32t - 16t^2 \right]_0^1 = 32 - 16 = 16$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0 + 0 + 16 = \boxed{16}$$