SOLUTIONS

Q1 a) SOLVE FOR $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 7 & 5 & 1 \end{array}\right]$$

$$R2-3R1 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$R1-2R2 \left[\begin{array}{ccc|c} 1 & 0 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

c_1, c_2 LEADING
c_3 FREE

$$c_1 = -2 + 3c_3$$
$$c_2 = 1 - 2c_3$$
$$c_3 = c_3$$

$$\therefore \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

b) TAKE ANY TWO VALUES OF C_3 AND SOLVE FOR C_1 AND C_2.

FOR EXAMPLE: $C_3 = 0 \Rightarrow C_1 = -2 \,;\, C_2 = 1$

$C_3 = 1 \Rightarrow C_1 = 1 \,;\, C_2 = -1$

c) NOT ALWAYS. FOR INSTANCE HAD WE CHOSEN $\vec{u}, \vec{v}$ AND $\vec{w}$ TO BE PARALLEL TO EACH OTHER BUT NOT PARALLEL TO $\vec{b}$, THEN NO COMBINATIONS OF $\vec{u}, \vec{v}$ AND $\vec{w}$ WOULD PRODUCE $\vec{b}$.

Q2 a) $i=1 \qquad -x_2 + 2x_1 - \cancel{x_0}^{\,0} = 1$

$i=2 \qquad -x_3 + 2x_2 - x_1 = 2$

$i=3 \qquad -x_4 + 2x_3 - x_2 = 3$

$i=4 \qquad -\cancel{x_5}^{\,0} + 2x_4 - x_3 = 4$

$\therefore \quad 2x_1 - x_2 = 1$

$-x_1 + 2x_2 - x_3 = 2$

$-x_2 + 2x_3 - x_4 = 3$

$-x_3 + 2x_4 = 4$

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{B}$$

$$\begin{array}{r} b) \\ 2R2+R1 \\ \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ 3R3+R2 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 1 & 2 & 3 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ 4R4+R3 \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ R4\div 5 \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & -3 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R3+3R4 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 & 8/5 & 16/5 & 24/5 & 12/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R3\div 4 \\ \\ \end{array} \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$R_2+2R_3 \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 9/5 & 18/5 & 12/5 & 6/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$R2 \div 3 \left[\begin{array}{cccc|cccc} 2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$R1+R2 \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 8/5 & 6/5 & 4/5 & 2/5 \\ 0 & 1 & 0 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$R1 \div 2 \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4/5 & 3/5 & 2/5 & 1/5 \\ 0 & 1 & 0 & 0 & 3/5 & 6/5 & 4/5 & 2/5 \\ 0 & 0 & 1 & 0 & 2/5 & 4/5 & 6/5 & 3/5 \\ 0 & 0 & 0 & 1 & 1/5 & 2/5 & 3/5 & 4/5 \end{array}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 4/5 & 3/5 & 2/5 & 1/5 \\ 3/5 & 6/5 & 4/5 & 2/5 \\ 2/5 & 4/5 & 6/5 & 3/5 \\ 1/5 & 2/5 & 3/5 & 4/5 \end{bmatrix}$$

c)

$$X = A^{-1}B$$

$$= \begin{bmatrix} 4/5 & 3/5 & 2/5 & 1/5 \\ 3/5 & 6/5 & 4/5 & 2/5 \\ 2/5 & 4/5 & 6/5 & 3/5 \\ 1/5 & 2/5 & 3/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20/5 \\ 35/5 \\ 40/5 \\ 30/5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$$

d) POINT IN R^4

Q3 a) $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$ TAKE $\vec{d} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$\text{proj}_{\vec{d}}\, \vec{u} = \frac{\vec{u}\cdot\vec{d}}{\|\vec{d}\|^2}\,\vec{d}$$

$$= \frac{(6+8+4)}{2^2+2^2+1^2} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{18}{9} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$$

b)
$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

COLUMN VECTORS LIE IN R

c) $X = (A^TA)^{-1} A^T \vec{u}$ (NORMAL SYSTEM ASSOCIATED WITH $AX = \vec{u}$)

PROJECTION OF $\vec{u}$ ON R

$$AX = A(A^TA)^{-1}A^T\vec{u}$$

$$\therefore \quad P = A(A^TA)^{-1}A^T$$

d) $$A^TA = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \frac{1}{5}$$

$$= \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 9/5 \end{bmatrix}$$

$$A(A^TA)^{-1} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/5 & -2/5 \\ -2/5 & 9/5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 2/5 & -4/5 \\ 1/5 & -2/5 \end{bmatrix}$$

$$A(A^TA)^{-1}A^T = \begin{bmatrix} 0 & 1 \\ 2/5 & -4/5 \\ 1/5 & -2/5 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$\therefore \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 2/5 \\ 0 & 2/5 & 1/5 \end{bmatrix}$

$\therefore \quad AX = P\vec{u} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 2/5 \\ 0 & 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 24/5 \\ 12/5 \end{bmatrix}$

Q4 a) IF A IS SQUARE AND INVERTIBLE, THEN ITS INVERSE CAN BE EXPRESSED AS A PRODUCT OF ELEMENTARY MATRICES

$$E_k E_{k-1} \cdots E_2 E_1 A = I$$

AND $A^{-1} = E_k E_{k-1} \cdots E_2 E_1$

IF B IS OBTAINED BY EXCHANGING THE FIRST TWO ROWS OF A, THEN A IN TURN CAN BE OBTAINED BY EXCHANGING THE FIRST TWO ROWS OF B. THEREFORE THERE EXISTS AN ELEMENTARY MATRIX OF THE FORM

$$E_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

SUCH THAT $A = E_0 B$

$\therefore E_k E_{k-1} \cdots E_2 E_1 E_0 B = I$

HENCE B IS INVERTIBLE AND

$$B^{-1} = A^{-1} E_0$$

b) IF B IS THE INVERSE OF A^2, THEN

$$BA^2 = A^2B = I$$

$$\therefore \quad B(AA) = (AA)B = I$$

$$(BA)A = A(AB) = I$$

$\therefore$ BA IS AN INVERSE OF A AND AB IS AN INVERSE OF A BUT SINCE AN INVERTIBLE MATRIX ONLY HAS ONE INVERSE, THEN

$$AB = BA$$

Q5 a) $\frac{dy}{dt} = (-y+z)e^{(1-t)} + 0.5y$

$\frac{dz}{dt} = y - z^2$

$t_0 = 0 \quad y_0 = 3 \quad z_0 = 0.2$

$t_1 = 0.25$

$y_1 = y_0 + 0.25\left[(-y_0+z_0)e^{(1-t_0)} + 0.5y_0\right]$

$= 3 + 0.25\left[(-3+0.2)e^{(1-0)} + (0.5)(3)\right]$

$= 3 + (0.25)(-6.1112)$

$= 1.4722$

$z_1 = z_0 + 0.25(y_0 - z_0^2)$

$= 0.2 + (0.25)(3 - 0.2^2)$

$= 0.2 + (0.25)(2.96)$

$= 0.94$

$t_2 = 0.5$

$$y_2 = y_1 + 0.25\left[(-y_1 + z_1)e^{(1-t_1)} + 0.5\, y_1\right]$$

$$= 1.4722 + 0.25\left[(-1.4722 + .94)e^{(1-.25)} + (0.5)(1.4722)\right]$$

$$= 1.4722 + (0.25)(-.3906)$$

$$= 1.3746$$

$$z_2 = z_1 + 0.25(y_1 - z_1^2)$$

$$= .94 + (0.25)(1.4722 - .94^2)$$

$$= .94 + (0.25)(.5886)$$

$$= 1.0872$$

b)

$$t_m = t_n + \frac{\Delta t}{2}$$

$$y_m = y_n + \frac{\Delta t}{2} f(t_n, y_n)$$

$$t_{n+1} = t_n + \Delta t$$

$$y_{n+1} = y_n + \Delta t f(t_m, y_m)$$

Q6 a)

$$-2\left(\frac{y(x+\Delta x)-2y(x)+y(x-\Delta x)}{(\Delta x)^2}\right)+y(x)=e^{-0.2x}$$

$$-2y(x+\Delta x)+4y(x)+(\Delta x)^2y(x)-2y(x-\Delta x)=(\Delta x)^2e^{-0.2x}$$

$$-2y(x+\Delta x)+(4+(\Delta x)^2)\,y(x)-2y(x-\Delta x)=(\Delta x)^2e^{-0.2x}$$

b) $\Delta x=0.25$

$x=.25$

1 (B.C.)

$$-2y(0)+4.0625\,y(.25)-2y(0.5)=.05945$$

$x=.5$

$$-2y(.25)+4.0625\,y(.5)-2y(.75)=.05655$$

$x=.75$

$$-2y(.5)+4.0625\,y(.75)-2y(1)=.05379$$

c) RECOMMEND BACKWARD DIFFERENCE BECAUSE IT USES VALUES CONTAINED IN THE INTERVAL, WHEREAS FORWARD AND CENTRAL DIFFERENCES WOULD REQUIRE VALUES OUTSIDE THE INTERVAL. THE INTERVAL BEING REFERRED TO IS $0 \leq X \leq 1$.

d) GIVEN $\frac{dy}{dx} = -y$

USING BACKWARD DIFFERENCE

$$\frac{y(x) - y(x-\Delta x)}{\Delta x} = -y(x)$$

AT $X=1$

$$\frac{y(1) - y(.75)}{.25} = -y(1)$$

$$y(1) + .25\,y(1) = y(.75)$$

$$1.25\,y(1) = y(.75)$$

$$\underbrace{\begin{bmatrix} 4.0625 & -2 & 0 & 0 \\ -2 & 4.0625 & -2 & 0 \\ 0 & -2 & 4.0625 & -2 \\ 0 & 0 & 1 & -1.25 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(0.25) \\ y(0.5) \\ y(0.75) \\ y(1) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 2.05945 \\ .05655 \\ .05379 \\ 0 \end{bmatrix}}_{B}$$