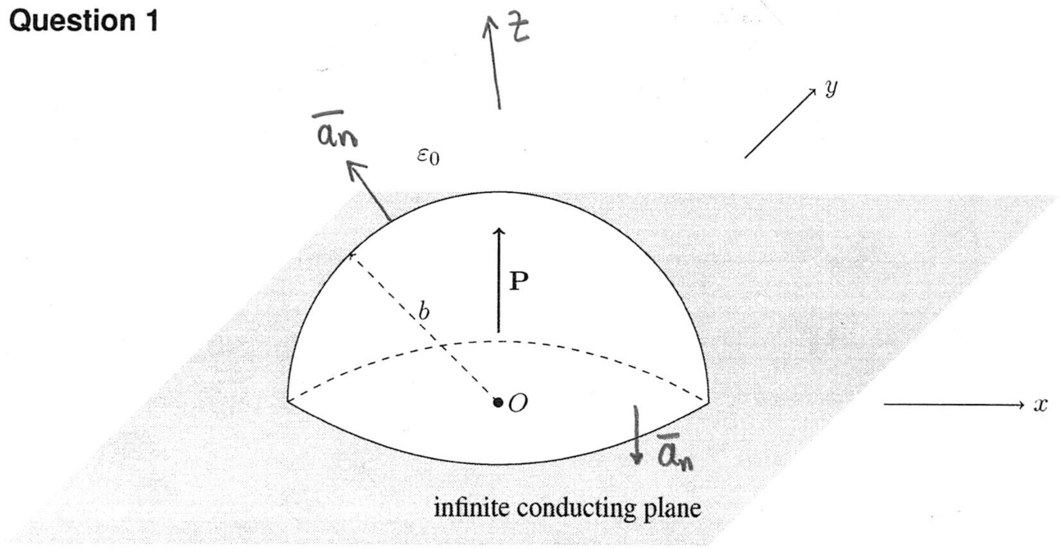
Question 1



A dielectric half-sphere is place just above a perfectly-conducting plane, infinitely-wide. The half-sphere has radius b and is polarized with polarization vector $\mathbf{P} = P_0\mathbf{a}_z$, with $P_0 > 0$. Permittivity is ε_0 everywhere.

1. Find the density of bound charge inside and on the boundaries of the half-sphere (4 points).

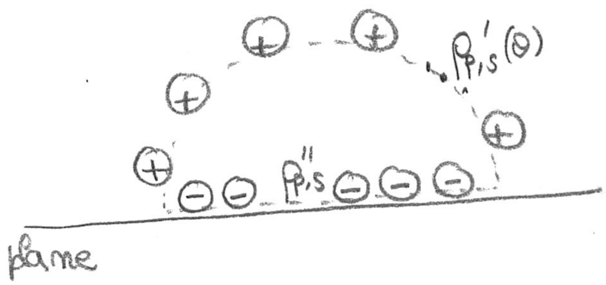
Inside $\rho_{P,v} = -\nabla \cdot \bar{P} = 0$ [1pts]

Top suface $\rho'_{P,S} = \bar{a}_n \cdot \bar{P} = \bar{a}_R \cdot P_0 \bar{a}_z = P_0 \cos\theta$ [1.5 pts]

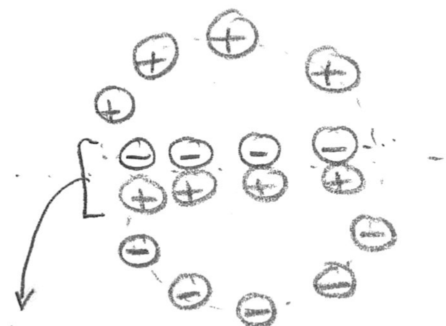
Bottom surface $\rho''_{P,S} = \bar{a}_n \cdot \bar{P} = -\bar{a}_z \cdot P_0 \bar{a}_z = -P_0$ [1.5 pts]

2. Calculate the electric field **E** at a point located just above the center O of the half-sphere (16 points).

Replace dielectric with bound charges
Apply image theory



$\Rightarrow$



[2pt]

these two charge distributions cancel out

[2pt]

Resulting charge distribution:

$$\rho_{P_s}(\theta,\varphi) = P_0 \cos\theta \qquad \theta \in [0, 2\pi]$$

We find $\bar{E}$ just above the origin by superposition

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int_S \frac{dq'}{|\bar{R}-\bar{R}'|^3}(\bar{R}-\bar{R}')$$

$$dq' = \rho_{p,s}(\theta',\varphi')\, dS_R = P_0 \cos\theta'\, b^2 \sin\theta'\, d\theta'\, d\varphi' \quad \Big]\ [2pt]$$

[4pt]

$$\bar{R} = 0 \qquad \bar{R}' = b\,\bar{a}_{R'} = b\left[\bar{a}_x \sin\theta'\cos\varphi' + \bar{a}_y \sin\theta'\sin\varphi' + \bar{a}_z \cos\theta'\right]$$

$$\bar{R} - \bar{R}' = -b\bar{a}_{R'}$$

$$|\bar{R} - \bar{R}'| = b$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \underbrace{\int_{\varphi'=0}^{2\pi}} \int_{\theta'=0}^{\pi} \frac{P_0 \cancel{b^2} \cos\theta' \sin\theta'\, d\theta'\, d\varphi'}{\cancel{b^3}} (-\cancel{b}) \overbrace{\left[\bar{a}_x \sin\theta'\cos\varphi' + \bar{a}_y \sin\theta'\sin\varphi' + \bar{a}_z\cos\theta'\right]}^{\text{integrate to } \varphi}$$

$$= -\frac{P_0}{\underset{2}{\cancel{4\pi}}\varepsilon_0} \cancel{2\pi}\, \bar{a}_z \int_{\theta'=0}^{\pi} \cos^2\theta' \sin\theta'\, d\theta'$$

$$= \frac{P_0}{2\varepsilon_0} \bar{a}_z \int_{\theta'=0}^{\pi} \cos^2\theta' (-\sin\theta')\, d\theta' =$$

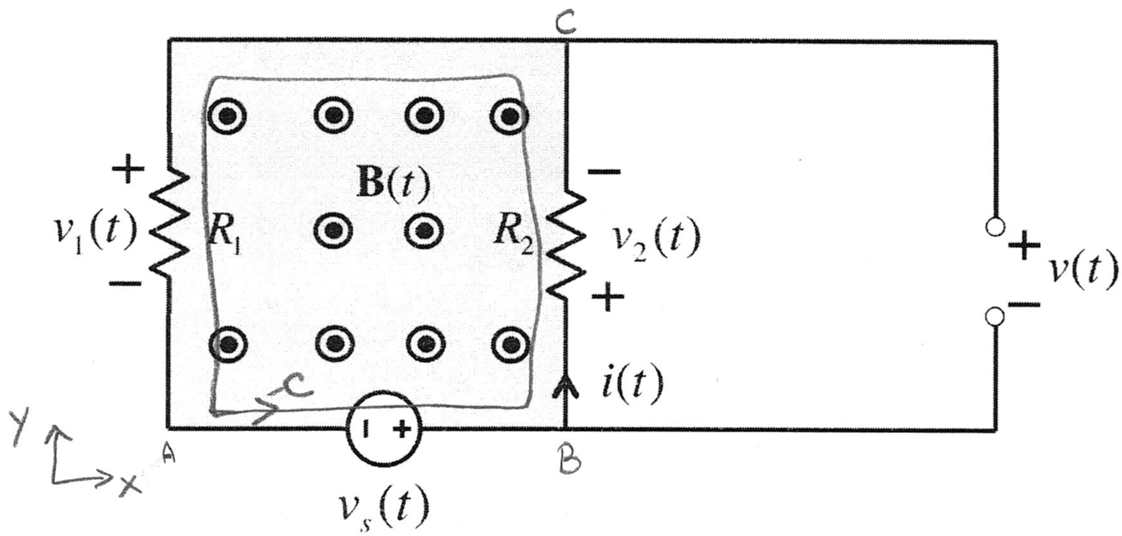
$$= \frac{P_0}{2\varepsilon_0} \bar{a}_z \left. \frac{\cos^3\theta'}{3} \right|_0^{\pi} = \frac{P_0}{2\varepsilon_0} \bar{a}_z \left[-\frac{1}{3} - \frac{1}{3}\right] = \boxed{-\frac{P_0}{3\varepsilon_0} \bar{a}_z}$$

Calculation & final result [6pt]

Question 4

$\bar{a}_z$

Consider the circuit depicted below. The surface enclosed by the left mesh of the circuit (shaded) has area S. The magnetic field $\mathbf{B}(t) = B_0 \sin(\omega t)$ is uniform inside S and is zero elsewhere. The value of the voltage source is $v_s(t) = V_0 \cos(\omega t)$. The self-inductance of the circuit can be neglected.



1. Starting from Faraday's law

$$\oint_c \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathbf{dS}\,.$$

derive the KVL for the left mesh of the circuit (4 points).

$$-\oint_c \bar{E} \cdot \bar{dl} = \frac{\partial}{\partial t} \int_S \bar{B} \cdot \bar{dS}$$

$$-\int_A^B \bar{E} \cdot \bar{dl} - \int_B^C \bar{E} \cdot \bar{dl} - \int_C^A \bar{E} \cdot \bar{dl} = \frac{\partial}{\partial t} B_0 S \sin(\omega t)$$

$$\underbrace{v_s(t) - v_2(t) - v_1(t)}_{\text{LHS: [2pt]}} = \underbrace{B_0 S \omega \cos(\omega t)}_{\text{RHS: [2pt]}}$$

2. Find $i(t)$ (2 points).

$$v_2 = R_2\, i(t) \qquad v_1 = R_1\, i(t)$$

$$-(R_2+R_1)\, i(t) = B_0 S\omega \cos\omega t - V_0 \cos\omega t$$

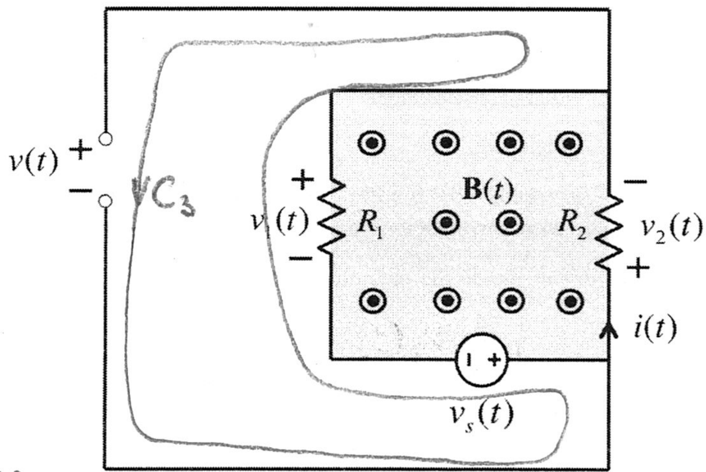
$$i(t) = \frac{V_0 - B_0 S\omega}{R_1+R_2} \cos\omega t$$

3. Find $v(t)$ (2 points).

KVL on right mesh. There is no emf

$$v(t) = -v_2(t) = -R_2\, i(t) = -\frac{R_2}{R_1+R_2}(V_0 - B_0 S\omega)\cos\omega t$$

4. The circuit layout is now changed, as shown in the figure below. Everything remained the same, except for the position of the two wires that end on $v(t)$. Find $v(t)$ in the new configuration (4 points).



i(t) remains the same

KVL on loop C_3. There is no induced emf on this loop.

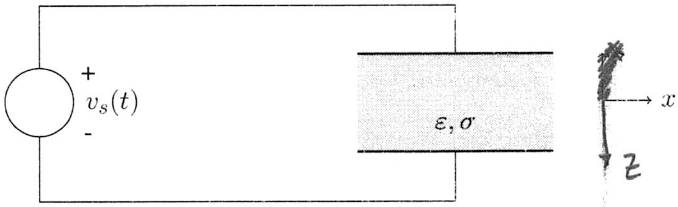
$$v(t) = v_1(t) - v_s(t) = R_1\, i(t) - v_s(t) =$$

$$= \frac{R_1(V_0 - B_0 S\omega)}{R_1+R_2}\cos\omega t - V_0 \cos\omega t =$$

$$= \frac{\cancel{R_1 V_0} - R_1 B_0 S\omega - \cancel{V_0 R_1} - V_0 R_2}{R_1+R_2}\cos\omega t =$$

$$= -\frac{R_1 B_0 S\omega + V_0 R_2}{R_1+R_2}\cos\omega t$$

Question 5



A voltage $v_s(t) = V_0 \cos(\omega t)$ is applied to a parallel plates capacitor with plates area A and plates distance d. The capacitor dielectric is lossy with permittivity ε and conductivity σ. Edge effects can be neglected.

1. Is there a conduction current inside the capacitor? If yes, calculate the density of conduction current **J** (4 points).

Yes, since $\sigma \neq 0$ [2 pts]

$$\bar{J} = \sigma \bar{E}$$

$$\bar{E} = \frac{v_s(t)}{d} = \frac{V_0}{d} \cos(\omega t)\, \bar{a}_z$$

$$\bar{J} = \frac{\sigma V_0}{d} \cos(\omega t)\, \bar{a}_z$$

[2pt]

2. Is there a displacement current inside the capacitor? If yes, calculate the density of displacement current $\mathbf{J}_d$ (4 points).

Yes, because $\bar{E}$ changes over time] [2pt]

$$\bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \varepsilon \frac{\partial \bar{E}}{\partial t} = \varepsilon \frac{V_0}{d} \omega (-\sin \omega t) \bar{a}_z$$] [2pt]