1.) [10 pts]

Basic idea: angular momentum is conserved, since there is no external torque. This should come out in the math!

<u>Approach 1:</u> $\vec{L} = \vec{r} \times \vec{p}$

$$\vec{p} = m\vec{v} = mv_0\,\hat{\imath}$$

$$\vec{r} = \begin{bmatrix} -c + v_0 t \\ b \\ 0 \end{bmatrix}$$

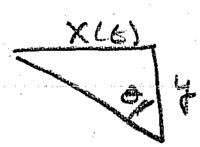
$$\vec{L} = \vec{r} \times \vec{p} = \boxed{-b\,m v_0\,\hat{k}}$$

<u>Approach 2:</u> $\vec{L} = I\vec{\omega}$

Using RHR, $\vec{\omega}$ is in the $-\hat{k}$ direction. So work the rest of the problem with scalars.

$$I = mr^2 = m(x^2+y^2) = m\left((-c+v_0 t)^2 + b^2\right)$$

$\omega = \dfrac{d\theta}{dt}$, θ given by



- If use the fact that L conserved, can evaluate at any t, so choose when $x=0$. In that case, $\omega = v_0/y$, & $I = my^2$, so $L = m y v_0 \longrightarrow$ as above ✓
- Or, just plunge into the math:

$$\tan\theta(t) = x(t)/y$$

Differentiate both sides wrt time:

$$(1+\tan^2\theta)\frac{d\theta}{dt} = \frac{1}{y}\frac{dx}{dt} \rightarrow \frac{d\theta}{dt} = \frac{v_0/y}{1+(x/y)^2}$$

Plug into $L = I\omega = m(x^2+y^2)\cdot\dfrac{v_0 y}{x^2+y^2} = m v_0 y$

and thus $\boxed{\vec{L} = -m v_0 b\,\hat{k}}$

(2) [15 pts.]

Basic idea: collision preserves $\vec{p}$ & $\vec{L}$, because there are no external forces or torques.

(a) [5 pts] Since $\vec{p}=0$ before the collision, $\vec{P}_f=0$.

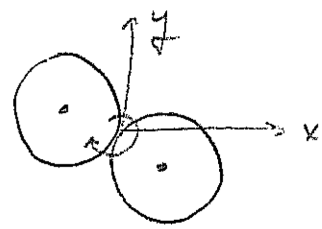
Use the relation $\vec{P}_{tot} = M_{tot}\vec{V}_{cm}$

$\longrightarrow \boxed{\vec{V}_{cm}=0}$

(b) [10 pts.]

Choose axis. The most reasonable is the point of contact, at the instant of collision

Approach: After the collision, $L_f = I_f \omega_f$, and I_f is found for two spheres touching each other:



$I_f = I_A + I_B$ but by symmetry $I_A = I_B$

$I_A = \underbrace{I_{cm}}_{\frac{2}{5}MR^2} + M\underbrace{D^2}_{R}$ using parallel axis eq

$\longrightarrow I_f = 2\left(\frac{2}{5}+1\right)MR^2 = \underline{\frac{14}{5}MR^2}$

By conservation of L, then $\vec{\omega}_f = \dfrac{\vec{L}_i}{I_f} = \dfrac{5L_i}{14MR^2}$

(2.) (b), cont.

The initial $\vec{L}$ cannot be found using $\vec{L} = I\vec{\omega}$, however, because the system before the collision is not a single rigid object. Instead, use

$$\vec{L}_i = \vec{L}_{Ai} + \vec{L}_{Bi},$$

and

$$\vec{L}_{Ai} = \vec{r}_A \times m\vec{V}_A + \underbrace{\sum_j \vec{r}_j \times m_j v_j}_{\text{this } = I_{cm}\omega}$$

... but neither sphere is spinning about its cm, so this second term is zero. The only term left is the cm term:

$$\vec{L}_{Ai} = \vec{r}_A \times m\vec{V}_A$$

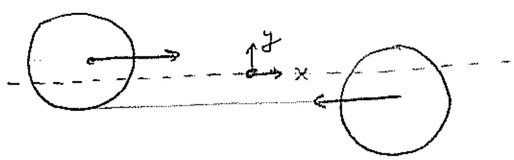
$$= MV_0\left(\frac{R}{2}\right)$$ as shown in problem 1, here in $-\hat{k}$ direction

Together, $\vec{L}_i = -MV_0R\hat{k}$

$$\longrightarrow \vec{\omega}_f = -\frac{5\cancel{M}V_0\cancel{R}}{14\cancel{M}R^{\cancel{2}}}\hat{k} = \boxed{-\frac{5}{14}\frac{V_0}{R}\hat{k}}$$

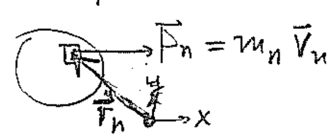
(6) [10pts] Again we use a conservation law: $\underline{\vec{L}_f = \vec{L}_i}$.
$\vec{L}$ is only defined wrt an axis. Here, the most reasonable choice is the point of contact, since after the collision the objects spin about this point.

Initial angular momentum:



Each sphere has only its cm motion to contribute to angular momentum: $\vec{L}_i = 2\cdot(-MV_0(R/2)\,\hat{k})$
or $\boxed{L_{zi} = -MV_0R}$

That only the cm matters can be shown by
$\vec{L} = \sum_n \vec{r}_n \times \vec{p}_n$ for all n pieces of the extended object



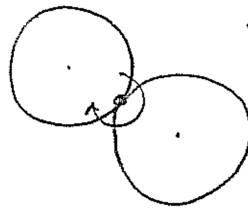
Here, $\vec{V}_n = V_0\,\hat{i}$, so
the cross-product is only $\vec{r}_n \times (m_n V_0 \hat{i}) = -m_n y_n V_0 \hat{k}$.
Now $\vec{L} = \sum_n (-m_n y_n V_0 \hat{k}) = -V_0\hat{k} \underbrace{\sum_n m_n y_n}$
here we recognize My_{cm}

& thus $\vec{L} = -MV_0 y_{cm} \hat{k}$

(More generally, $\vec{L} = M\vec{r}_{cm} \times \vec{V}_{cm}$ for an extended object moving without rotating.)

[(6), cont.]

Now, what is I_f?



Two spheres rotate about point of contact.

$$I_f = 2\left(I_{cm} + MD^2\right), \text{ where } R = D,$$

$$= 2\left(\tfrac{2}{5} + 1\right)MR^2 = \frac{14}{5}MR^2$$

Finally, we apply conservation of angular momentum,

$$L_f = L_i$$

$$\left(\frac{14}{5}MR^2\right)\omega_f = -MV_0R \quad \rightarrow \quad \boxed{\omega_f = \frac{5}{14}V_0/R} \text{ in } -\hat{k} \text{ direction}$$

btw, how much mechanical energy lost? KE initially $2 \times \left(\frac{1}{2}MV_0^2\right)$,

& finally, $KE = \frac{1}{2I}L^2$

$$= \frac{1}{2\left(\frac{14}{5}MR^2\right)} \cdot \left(MV_0R\right)^2 = \frac{5}{28}MV_0^2$$

$\rightarrow$ only 18% of initial KE left! Inelastic collision.

③ [20 pts.]

Basic idea: force needs to stop falling water, as well as support what is in the bucket.

- At $t=0$, $F = \underset{\nearrow\, 2\,kg}{M_b}\, \underset{\nwarrow\, 9.80\ m/s^2}{g}$ upwards. [Don't need to specify direction.]

- For water accumulated, $M_w = (4\,kg/s)\,t$, also need normal force: $F = M_w g = 4tg$ in SI units

- For water falling, need to stop it. This is a changing mass problem.

before: $\boxed{\Delta m}\downarrow 10\,m/s$ after: $\boxed{\Delta m}$ $(v=0)$

$$\Delta p = v_i\,\Delta m$$
$$= v_i\,(4\,kg/s)\,\Delta t$$

Thus force is $\lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t} = \frac{v_i\,(4\,kg/s)\,\Delta t}{\Delta t} = v_i\,(4\,kg/s)$

{check units: $m/s \cdot kg/s = kg\,m/s^2 = N$ ✓} $= 40\,N$

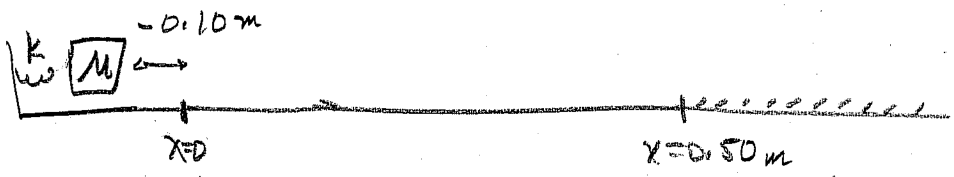
Direction of this force is up, to stop water falling down.

- In sum, $F = F_{bucket} + F_{stopping} + F_{water}$

$$= (2.0)(9.80) + 40 + (4.0)(9.80)\,t$$

$$\boxed{F = 60.\,N + (39.\,N/s)\,t}$$

4. [20 pts.]



(a) Quarter oscillation period: $\omega = \sqrt{k/m}$, $= 4.47\ s^{-1}$

so $\frac{1}{4}T = \frac{\pi}{2\omega} = \frac{\pi}{2}\sqrt{\frac{m}{k}} = \boxed{0.35\ s}$

(b) Conservation of energy:

$\frac{1}{2}kA^2 = \frac{1}{2}mv^2$

$v^2 = \frac{k}{m}A^2 \longrightarrow |v| = \sqrt{\frac{k}{m}}|A| = \boxed{0.45\ m/s \text{ in } +\hat{i} \text{ direction}}$

(c) Given constant acceleration,

$v_f = v_i - at = 0$, & here $a = \mu g$

$\longrightarrow t = \frac{v_i}{a} = \frac{0.447\ m/s}{0.30 \cdot 9.80} = 0.152\ s$

Now, how far does a block go in that time?

$\Delta x = v_i t - \frac{1}{2}at^2 = 0.034\ m$

In total, $x = 0.50\ m + \Delta x = \boxed{0.53\ m}$

(d) Conservation of energy: spring energy is converted to internal energy. So, final

$E_{int} = \frac{1}{2}kA^2 = \boxed{2.0\ J}$

5. [15pts.]

(a) [5pts.] Centrifugal force to keep mass in circular motion is inwards, so string must stretch to provide this: $R > \ell$.

(b) [10pts.] Idea: from length of spring, we know what velocity must be, so we can find $L = I\omega$.

- Force balance: $K(R-\ell) = mR\omega^2$ or $= m\frac{v^2}{R}$

  thus $\omega = \sqrt{\frac{k}{m}\left(1 - \frac{\ell}{R}\right)}$

- Moment of inertia: $I = mR^2$

- Product: $L = I\omega = mR^2\sqrt{\frac{k}{m}(1 - \ell/R)}$

  simplify: $\boxed{L = \sqrt{mR^3 k (R-\ell)}}$

[check units! $kg\ m^4\ N/m = kg\ m^3\ kg\ m/s^2 = \frac{kg^2 m^4}{s^2}$

$\longrightarrow \frac{kg\ m^2}{s}$, which is correct for $L = I\omega$ ✓]

⑥ [20 pts.]

(a) [10 pts.]

Idea is to treat the slider + sphere as one system, & note that its momentum is changed by external impulse, F.T.

How long was force applied? Consider position of cm, X_{cm}, under constant acceleration, $(F/2m) = a_{cm}$.

Then $(X_{cm})_f = \frac{1}{2}(a_{cm})T^2$, & we know $(X_{cm})_f = \frac{X_1 + X_2}{2}$.

Thus: $T^2 = \frac{2X_{cm}}{a_{cm}} = \frac{X_1 + X_2}{F/2m} = 2m\frac{(X_1+X_2)}{F}$

$\hookrightarrow T = \sqrt{\frac{2m}{F}(X_1 + X_2)}$

Impulse is then $FT = \sqrt{2mF(X_1+X_2)}$

This is equal to $p_f = M_{TOT}(V_{cm})_f = 2m(V_{cm})_f$

$\longrightarrow \boxed{(V_{cm})_f = \sqrt{\frac{F}{2m}(X_1 + X_2)}}$

(6.) (6.) [10 pts.]

This must be solved with conservation of energy.

$$E = E_{cm} + E_{int}$$

where $E_{cm} = \frac{1}{2} M_{TOT} V_{cm}^2$ and E_{int} is the oscillatory motion of the glider + sphere. In general, this oscillation energy is kinetic + potential, but when the pendulum is @ its max height, all $E_{int} = U_g$, so that $mgh = E_{int}$, & $h = \ell(1-\cos\theta)$.

- First, find total energy E: given by work of external force. $W = Fx_1$ in this case, since force is constant. $\rightarrow$ $E = Fx_1$.
- Next, find E_{int} by subtracting off E_{cm}.

$$E_{cm} = \frac{1}{2}(M_{TOT})V_{cm}^2 = \frac{1}{2}(2m)\left(\frac{F}{2m}(x_1+x_2)\right)$$

$$= \frac{F}{2}(x_1+x_2) \rightarrow E_{int} = \frac{F}{2}(x_1 - x_2)$$

- Finally, we see how high sphere could be for this E_{int} = gravitational energy.

$$U = mgh = mg\ell(1-\cos\theta) = \frac{F}{2}(x_1 - x_2)$$

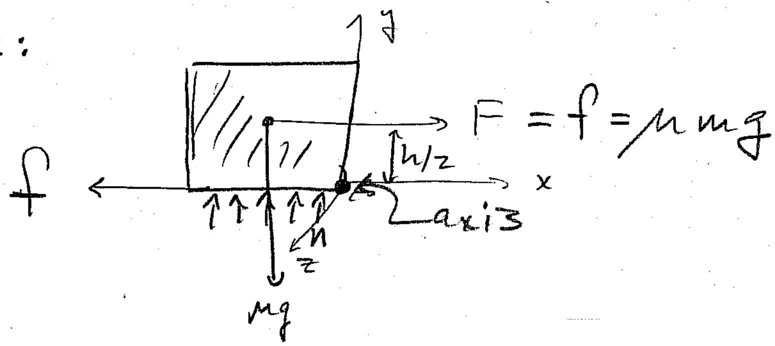
$$\cos\theta = 1 - F(x_1 - x_2)/2mg\ell$$

$$\boxed{\theta = \cos^{-1}\left[1 - F(x_1-x_2)/2mg\ell\right]}$$

7i) [5pts.]

Idea is to require $\sum \vec{\tau} = 0$, & see when there is a contradiction.

Calculate $\sum \vec{\tau}$ about the front corner of the block:



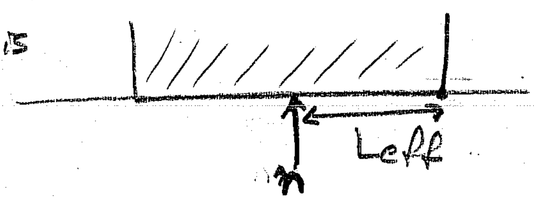
$$\sum \tau_z = +mg\, w/2 - (\mu mg)\, h/2 + \tau_n = 0$$

& thus τ_n, torque from distributed normal force, is

$$\tau_n = -mg\left(\frac{W}{2} - \mu \frac{h}{2}\right)$$

Of course, $mg = n$, magnitude of normal force. So the effective lever arm is

$$L_{eff} = \frac{W}{2} - \mu \frac{h}{2}$$



This must be positive, because n cannot act in front of the block! Thus

$$\frac{W}{2} - \mu\frac{h}{2} > 0 \rightarrow \mu\frac{h}{2} < \frac{W}{2} \quad \text{or} \quad \boxed{\frac{h}{W} < \frac{1}{\mu}}$$

{check units: ✓}

For low friction (small μ), can have a taller block. Makes sense!