The Edward S. Rogers Sr. Department of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 1 - Thursday February 12, 2015

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Last name: ....

First name: ....

Student number: ....

Tutorial section number: ....

Section	Day	Time	Room
1	Monday	15:00-16:00	WB144
2	Monday	15:00-16:00	BA2159
3	Monday	15:00-16:00	BA3008
4	Monday	15:00-16:00	BA3012
5	Wednesday	13:00-14:00	BA2159
6	Wednesday	13:00-14:00	BA3008
7	Wednesday	13:00-14:00	BA3012
8	Wednesday	13:00-14:00	BA3116

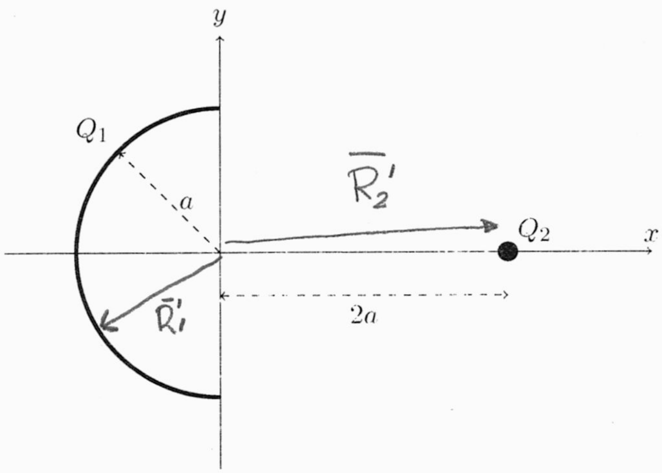
Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full credit!**

Marks:	Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1

1. Consider the charge distribution shown in the figure below. A total charge Q_1 is uniformly distributed along a semicircle with radius a. A point charge of value Q_2 is placed along the x axis at a distance $2a$ from the origin.



Find Q_2 in terms of Q_1 in order to have no electric field at the origin. (**12 pts**)

Electric field produced by Q_2 at the origin

$$\bar{E}_2 = \frac{Q_2}{4\pi\varepsilon_0} \cdot \frac{\bar{R} - \bar{R}_2'}{|\bar{R} - \bar{R}_2'|^3}$$

[1pt] $\bar{R}_2' = 2a\,\bar{a}_x$

[1] $\bar{R} - \bar{R}_2' = -2a\,\bar{a}_x$

[1] $\bar{E}_2 = \frac{Q_2}{4\pi\varepsilon_0} \frac{-\cancel{2}\cancel{a}\,\bar{a}_x}{\cancel{8}_{4}\, a^{\cancel{3}2}} = -\frac{Q_2}{16\pi\varepsilon_0 a^2}\,\bar{a}_x$

Electric field produced by Q_1 at origin

→ we use superposition

$$\bar{E}_1 = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_\ell \cdot d\ell}{|\bar{R}-\bar{R}_1'|^3}(\bar{R}-\bar{R}_1')$$

cylindrical coordinates

[1] $r' = a \quad \varphi' \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \quad z' = 0$

[1pt] $\rho_\ell = \frac{Q_1}{\pi a}$

[1] $\bar{R} = 0$

[1] $\bar{R}_1' = a\,\bar{a}_r' = a\cos\varphi'\,\bar{a}_x + a\sin\varphi'\,\bar{a}_y$

[1] $|\bar{R}-\bar{R}_1'| = a$

2pt

$$\bar{E}_1 = \frac{Q_1}{4\pi\varepsilon_0\,\pi a}\int_{\varphi'=\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cancel{a}\,d\varphi'}{a^{\cancel{3}2}}\cdot\left(-\cancel{a}\cos\varphi'\,\bar{a}_x - \cancel{a}\sin\varphi'\,\bar{a}_y\right) =$$

$$= \frac{Q_1}{4\pi^2\varepsilon_0 a^2}\left[\bar{a}_x\underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}(-\cos\varphi')\,d\varphi'}_{=2} - \bar{a}_y\underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin\varphi'\,d\varphi'}_{=0}\right] = \frac{Q_1}{2\pi^2\varepsilon_0 a^2}\,\bar{a}_x$$

Total $\bar{E}$:

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = \frac{Q_1}{2\pi^2\varepsilon_0 a^2}\,\bar{a}_x - \frac{Q_2}{16\pi\varepsilon_0 a^2}\,\bar{a}_x = \frac{\bar{a}_x}{2\pi\varepsilon_0 a^2}\left(\frac{Q_1}{\pi} - \frac{Q_2}{8}\right)$$

2pt $\bar{E} = 0$ if $\boxed{Q_2 = \frac{8Q_1}{\pi}}$

2. Given the electric displacement vector $\mathbf{D} = 2x^2\mathbf{a}_x + 2xy\mathbf{a}_y + 5x^2z\mathbf{a}_z$, and the cube defined by

$$\begin{aligned} 0 &\leq x \leq 1 \\ 0 &\leq y \leq 1 \\ 0 &\leq z \leq 1 \end{aligned}$$

a) find the volume density of free charge ρ_v everywhere (**2 pts**)

[2pt] Gauss' Law: $\rho_v = \nabla \cdot \bar{D} = 4x + 2x + 5x^2 = \boxed{6x + 5x^2}$

b) find the total outward flux of **D** through the surface of the cube (**4 pts**)

By Gauss' Law (integral form): $\int_S \bar{D} \cdot \bar{dS} = Q$ ← charge enclosed in the cube

charge enclosed [2pt]

$$Q = \iiint_{cube} \rho_v \, dv = \int_{x=0}^{1} \int_{y=0}^{1} \int_{z=0}^{1} (6x + 5x^2)\, dx\, dy\, dz =$$

$$= \int_{x=0}^{1} (6x + 5x^2)\, dx = 6 \left.\frac{x^2}{2}\right|_0^1 + 5 \left.\frac{x^3}{3}\right|_0^1 = 3 + \frac{5}{3} = \frac{14}{3}$$

final answer [2pt]

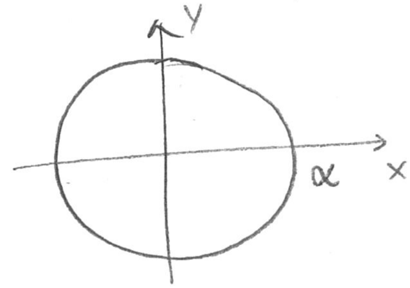
So, $\boxed{\int_{cube} \bar{D} \cdot \bar{dS} = \frac{14}{3}}$

Question 2

1. Consider an infinitely long charged cylinder of radius α and volume charge density

$$\rho_v(r) = \rho_0 \frac{r^2}{\alpha^2} \quad \text{for } 0 \leq r \leq \alpha$$

Find the electric field $\mathbf{E}$ inside and outside the cylinder **(12 pts)**.



The structure has cylindrical symmetry $\Rightarrow$ use Gauss' law

[1pt] $\bar{E}$ must be in the form $\bar{E}(r) = E(r)\,\bar{a}_r$

[1pt] Gaussian surface: cylinder of radius r'

Field inside $(r' < \alpha)$

$$\oint_S \bar{E} \cdot d\bar{S} = \frac{Q}{\varepsilon_0}$$

Q enclosed in cylinder of length L and radius r'

[2pt] $$Q(r') = \int_{z=0}^{L} \int_{\varphi=0}^{2\pi} \int_{r=0}^{r'} \rho_0 \frac{r^2}{\alpha^2}\, r \, d\varphi \, dz \, dr = \frac{\rho_0}{\alpha^2} L \cancel{2}\pi \frac{(r')^4}{\cancel{4}_2} = \frac{\rho_0 L (r')^4}{2\alpha^2}$$

[1pt] $$\oint \bar{E} \cdot d\bar{S} = E(r') \cdot L 2\pi r'$$

Final answ:

[2pt] $$E(r') = \frac{\rho_0 \cancel{L} (r')^{\cancel{4}\,3} \cancel{\pi}}{2\alpha^2 \varepsilon_0 \, 2\cancel{\pi} \cancel{r'} \cancel{L}} = \frac{\rho_0 (r')^3}{4\alpha^2 \varepsilon_0}$$

$$\boxed{\bar{E}(r') = \frac{\rho_0 (r')^3}{4\alpha^2 \varepsilon_0}\, \bar{a}_r}$$

Field outside $(r' > \alpha)$

Charge enclosed in cylinder of radius $r' > \alpha$ and height L:

[2pt] $Q(\alpha) = \frac{\rho_0 L \alpha^2}{2}$

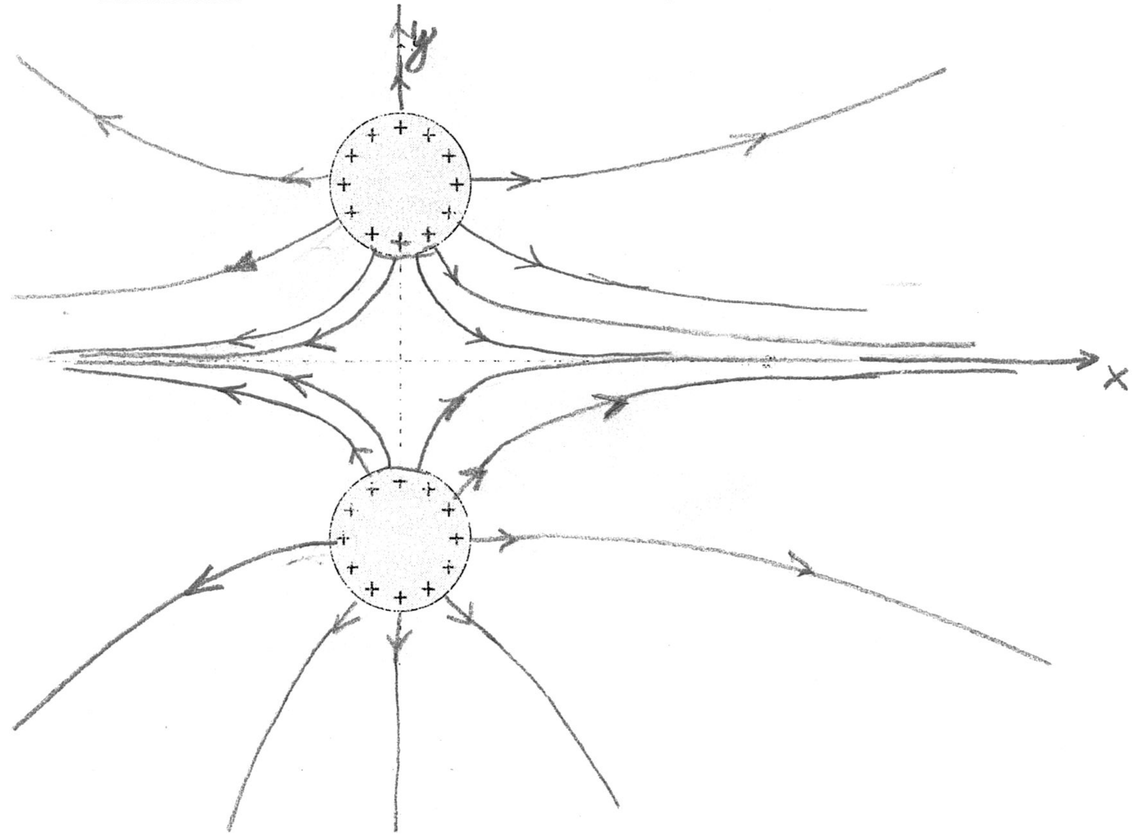
[1pt]

Gauss' law: $E(r')\, 2\cancel{\pi} r' \cancel{L} = \frac{\rho_0 \cancel{L} \alpha^2 \cancel{\pi}}{2\varepsilon_0}$

final answer
[2pt]

$$\boxed{\bar{E}(r') = \frac{\rho_0 \alpha^2}{4\, r' \varepsilon_0}\, \bar{a}_r}$$

2. The cross section of a two-wire cable is shown in the figure below. The two wires can be treated as perfect conductors. The two wires have the same amount of positive charge uniformly distributed along their boundary. Sketch qualitatively the electric field lines in the region. **(6 pts)**



$\bar{E}$ lines must be: • normal to conductors' boundary [2pt]

• source at positive charges [2pt]

• be symmetric with respect to the x-axis [1pt]

and the y-axis [1pt]

3. Consider again the two-wire cable described at point 2. The electric field **E** is zero:

   a) Only at infinity ($r \to \infty$);

   b) On the x axis and at infinity;

   c) On the y axis and at infinity;

   d) At the origin and at infinity ← [1pt] right answer

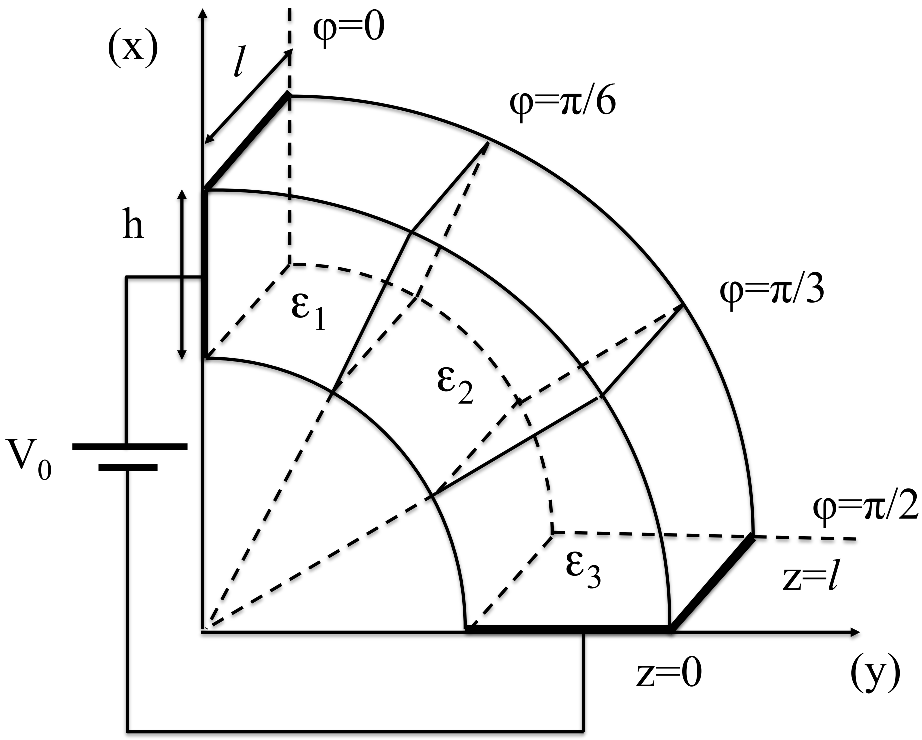
   Briefly justify your answer (**2 pts**)

- $\bar{E}$ will decay like $\frac{1}{r}$ $\Rightarrow$ $\lim_{r\to\infty} E(r) = 0$
- at the origin the field contributions due to the two conductors are equal and opposite $\Rightarrow$ cancel exactly

[1pt]

Question 3

1. Consider the following geometry of a capacitor, consisting of three dielectric sections with: $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = 4\varepsilon_0$, $\varepsilon_3 = 2\varepsilon_0$. The length of the capacitor in the $z-$direction is $l = 5$ cm. The conducting plates, of area $h \times l$ for this capacitor ($h = 2$ cm) lie on $\phi = 0$ and $\phi = \pi/2$ and a voltage source maintains a voltage $V_0 = 1$ V between them.



Disregarding "edge effects," the electric field in the three dielectrics is given as:

$$\mathbf{E} = \mathbf{a}_\phi \begin{cases} \dfrac{A_1}{r}, & 0 \le \phi < \pi/6 \\ \dfrac{A_2}{r}, & \pi/6 \le \phi < \pi/3 \\ \dfrac{A_3}{r}, & \pi/3 \le \phi < \pi/2 \end{cases}$$

where A_1, A_2 and A_3 are constants.

a) Show that $\varepsilon_1 A_1 = \varepsilon_2 A_2 = \varepsilon_3 A_3$, by applying electric field boundary conditions at $\phi = \pi/6$ and $\phi = \pi/3$. **(4 pts)** Electric field is normal to interface, hence the condition $\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$ applies (2 pts), with $\rho_s = 0$ (since the interface consists of two dielectric materials) (1 pt) . At $\phi = \pi/6$, the condition gives: $\varepsilon_1 A_1 = \varepsilon_2 A_2$ (0.5 pt); at $\phi = \pi/3$, the condition gives: $\varepsilon_2 A_2 = \varepsilon_3 A_3$ (0.5 pt).

b) Find the capacitance C by expressing the charge Q on the positively charged plate and the voltage V_0 between the plates as a function of A_1. You can use the result of (a). **(10 pts)**

Solution based on C=Q/V:

Find Q from boundary condition at the positive plate on $\phi = 0$:

$$\rho_s = \mathbf{a}_\phi \cdot \mathbf{a}_\phi \varepsilon_1 \frac{A_1}{r}$$

(2 pts)

Hence,

$$Q = \int_{r=\alpha}^{r=\alpha+h} \int_{z=0}^{z=l} \varepsilon_1 \frac{A_1}{r} dz dr = A_1 \varepsilon_1 \ln \frac{\alpha + h}{\alpha} l.$$

(2 pts = 1 pt for correct integral, 1 pt for correct dS)

Express V in terms of A_1:

$$V = V(\phi = 0) - V(\phi = \pi/2) = -\int_{\phi=\pi/2}^{\phi=0} \mathbf{E} \cdot d\mathbf{l} = \int_{\phi=0}^{\phi=\pi/2} \mathbf{E} \cdot \mathbf{a}_\phi r d\phi$$

(2 pts for correct definition of V and 1 pt for $d\mathbf{l}$). Then,

$$V = \int_{\phi=0}^{\phi=\pi/6} \frac{A_1}{r} \mathbf{a}_\phi \cdot \mathbf{a}_\phi r d\phi + \int_{\phi=\pi/6}^{\phi=\pi/3} \frac{A_2}{r} \mathbf{a}_\phi \cdot \mathbf{a}_\phi r d\phi + \int_{\phi=\pi/3}^{\phi=\pi/2} \frac{A_3}{r} \mathbf{a}_\phi \cdot \mathbf{a}_\phi r d\phi$$

(2 pts) Hence:

$$V = (A_1 + A_2 + A_3) \frac{\pi}{6} = A_1 \left(1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_3}\right)$$

Dividing:

$$C = \frac{\ln \frac{\alpha + h}{\alpha} l}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}}$$

(1 pt)

Solution based on energy

$$W_e = \frac{1}{2} \int_{z=0}^{z=l} \int_{r=\alpha}^{\alpha+h} \int_{\phi=0}^{\phi=\pi/2} \varepsilon E^2 r d\phi dz dr$$

Hence:

$$W_e = \frac{1}{2} \int_{z=0}^{z=l} \int_{r=\alpha}^{\alpha+h} \int_{\phi=0}^{\phi=\pi/6} \left(\varepsilon_1 \frac{A_1^2}{r^2} + \varepsilon_2 \frac{A_2^2}{r^2} + \varepsilon_3 \frac{A_3^2}{r^2}\right) r d\phi dz dr$$

(correct formulation of energy integral = 2 pts, dv=1 pt). Then:

$$W_e = \frac{1}{2}\ln\frac{\alpha+h}{\alpha}l\frac{\pi}{6}\left(\varepsilon_1 A_1^2 + \varepsilon_2 A_2^2 + \varepsilon_3 A_3^2\right)$$

(1 pt) From (a):

$$W_e = \frac{1}{2}\ln\frac{\alpha+h}{\alpha}l\frac{\pi}{6}A_1^2\left(\varepsilon_1 + \frac{\varepsilon_1}{\varepsilon_2} + \frac{\varepsilon_1}{\varepsilon_3}\right)$$

and finally,

$$W_e = \frac{1}{2}\ln\frac{\alpha+h}{\alpha}l\frac{\pi}{6}\varepsilon_1^2 A_1^2\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) = \frac{1}{2}CV^2$$

But, from the previous solution:

$$V = (A_1 + A_2 + A_3)\frac{\pi}{6} = A_1\varepsilon_1\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)$$

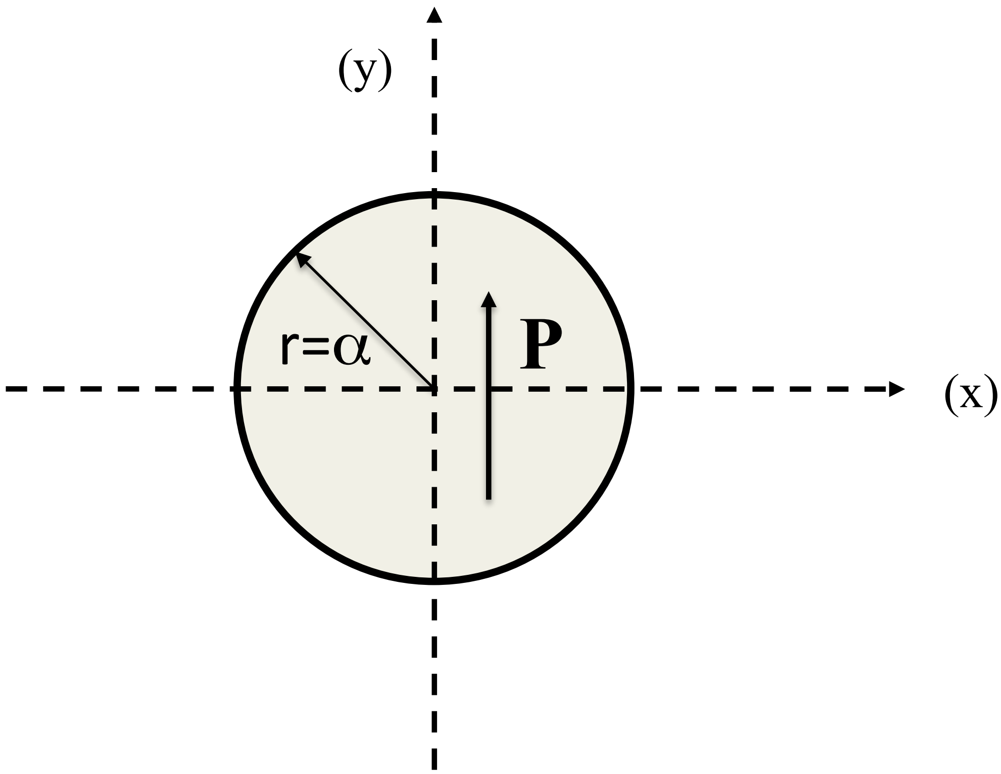
(5 pts, see marking scheme above) Then,

$$C = \frac{\ln\dfrac{\alpha+h}{\alpha}l}{\dfrac{1}{\varepsilon_1} + \dfrac{1}{\varepsilon_2} + \dfrac{1}{\varepsilon_3}}$$

(1 pt) as before.

c) Is this a case of a series or a parallel connection of capacitors? **(2 pts)** Series, since the sections do NOT share the same voltage. (answer=0.5, justification=1.5)

2. Consider a cylinder with constant polarization $\mathbf{P} = P_0\mathbf{a}_y$.



a) Find the surface polarization charge density on the surface of the cylinder $r = \alpha$. Sketch these surface charges. **(2 pts)**

$$\rho_s^P = \alpha_r \cdot P_0\mathbf{a}_y = P_0 \sin\phi$$

(1 pt)

Sketch positive charges for $0 < \phi < \pi/2$ with density becoming larger close to $\pi/2$ and negative charges below. (1 pt)

b) Find the volume polarization charge density within the cylinder $r = \alpha$ and provide a physical explanation of the result. **(2 pts)**

$$\rho_v^P = \nabla \cdot P_0\mathbf{a}_y = 0$$

(1 pt)

Bound charges from positive and negative induced dipoles cancel each other out (1 pt).