Q6: (parts (a) and (b) are separate)

a) Using Gaussian elimination, find the 3 values for c so that matrix A is not invertible:

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$$

b) Find the inverse of the $4x4$ matrix A by Gaussian elimination:

$$A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solutions:

a) $\begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{c}{2} & \frac{c}{2} \\ c & c & c \\ 8 & 7 & c \end{bmatrix} 0.5 * R1$

$$\rightarrow \begin{bmatrix} 1 & \frac{c}{2} & \frac{c}{2} \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 8 & 7 & c \end{bmatrix} R2 - c * R1 \; ; 2c - c^2 = c(2-c) \neq 0 \; ; c \neq 0, c \neq 2$$

$$\rightarrow \begin{bmatrix} 1 & \frac{c}{2} & \frac{c}{2} \\ 0 & c - \frac{c^2}{2} & c - \frac{c^2}{2} \\ 0 & 7 - 4c & c - 4c \end{bmatrix} R3 - 8 * R1$$

$$\rightarrow \begin{bmatrix} 1 & \frac{c}{2} & \frac{c}{2} \\ 0 & 1 & 1 \\ 0 & 7 - 4c & c - 4c \end{bmatrix} \frac{2}{2c - c^2} * R2$$

$$\rightarrow \begin{bmatrix} 1 & \frac{c}{2} & \frac{c}{2} \\ 0 & 1 & 1 \\ 0 & 0 & c - 7 \end{bmatrix} R3 - (7 - 4c) * R2 \; ; c - 7 \neq 0 \; ; c \neq 7$$

Therefore, matrix A is not invertible if $c = 0 \; or \; 2 \; or \; 7$.

b) $\left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R3 + c * R4$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & -a & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R2 + b * R3$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & a & ab & abc \\ 0 & 1 & 0 & 0 & 0 & 1 & b & bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R1 + a * R2$$

$$\therefore\ A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$