MAT195S CALCULUS II

Midterm Test #1

10 February 2015 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: P. Athavale and J. W. Davis

Family Name: J Davis

Given Name: Sol'ns.

Student #: ____________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	11	
2	10	
3	13	
4	8	
5	5	
6	8	
7	12	
8	8	
TOTAL	75	/70

Tutorial Section: ____________

TA Name: ____________

1) Evaluate the following integrals.

a) $\int \frac{\ln x}{\sqrt{x}} dx$

b) $\int \sec^3 x \, dx$

c) $\int \frac{dx}{\sqrt{x^2+16}}$

(11 marks)

a) $\int \frac{\ln x}{\sqrt{x}} dx$ let $u = \ln x$, $du = \frac{dx}{x}$; $dv = x^{-1/2} dx$, $v = 2x^{1/2}$

$= 2\sqrt{x}\ln x - \int \frac{2\sqrt{x}\,dx}{x} = 2\sqrt{x}\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln x - 4\sqrt{x} + C$

b) $\int \sec^3 x \, dx$ let $u = \sec x$, $du = \sec x \tan x \, dx$; $dv = \sec^2 x \, dx$, $v = \tan x$

$= \sec x \tan x - \int \sec x \tan^2 x \, dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$\rightarrow \int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

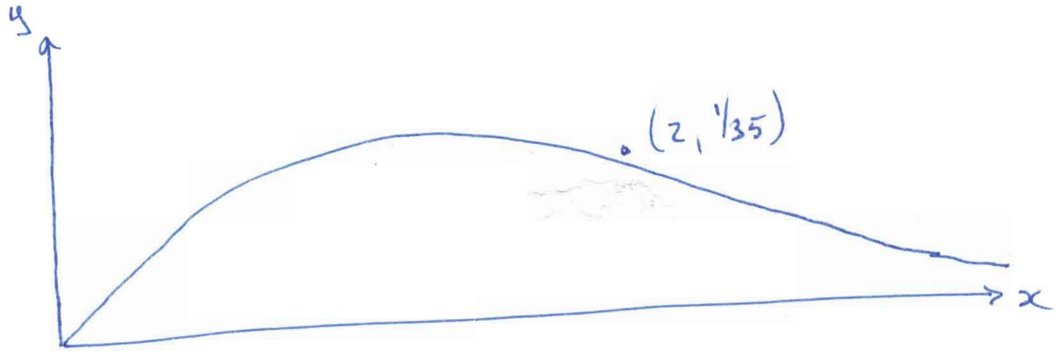
c) $\int \frac{dx}{\sqrt{x^2+16}}$ let $x = 4\tan\theta$, $dx = 4\sec^2\theta$, $\sqrt{x^2+16} = 4\sec\theta$

$= \int \frac{4\sec^2\theta}{4\sec\theta} d\theta = \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$

$= \ln\left|\frac{1}{4}\sqrt{x^2+16} + \frac{x}{4}\right| + C$

2) Find the centroid of the infinitely long region lying between the x-axis and the curve $y = \frac{x}{(x+1)^4}$, and to the right of the y-axis. Provide a sketch of the region.

(10 marks)



$$\text{Area} = \int_0^\infty \frac{x}{(x+1)^4}\,dx \qquad \text{let } u = x+1,\ du = dx \qquad x = u-1$$

$$= \int_1^\infty \frac{(u-1)\,du}{u^4} = \int_1^\infty \frac{du}{u^3} - \int_1^\infty \frac{du}{u^4} = \left[-\frac{1}{2}\frac{1}{u^2}\right]_1^\infty - \left[-\frac{1}{3}\frac{1}{u^3}\right]_1^\infty = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\bar{x}A = \int_0^\infty \frac{x^2}{(x+1)^4}\,dx \qquad \text{let } u = x+1,\ du = dx \qquad x^2 = u^2 - 2u + 1$$

$$= \int_1^\infty \frac{(u^2-2u+1)}{u^4}\,du = \int_1^\infty \frac{du}{u^2} - \int_1^\infty \frac{2du}{u^3} + \int_1^\infty \frac{du}{u^4} = \left[-\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3}\right]_1^\infty$$

$$= 1 - 1 + \frac{1}{3} = \frac{1}{3} \quad \Rightarrow \boxed{\bar{x} = \frac{1/3}{1/6} = 2}$$

$$\bar{y}A = \int_0^\infty \frac{1}{2}\left(\frac{x}{(x+1)^4}\right)^2 dx = \int_1^\infty \frac{1}{2}\,\frac{u^2-2u+1}{u^8}\,du = \frac{1}{2}\left[-\frac{1}{5u^5} + \frac{2}{6u^6} - \frac{1}{7u^7}\right]_1^\infty$$

$$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) = \frac{1}{2}\left(\frac{21-35+15}{105}\right) = \frac{1}{210}$$

$$\Rightarrow \boxed{\bar{y} = \frac{1/210}{1/6} = \frac{1}{35}}$$

3) Sketch a graph of the Strophoid: $x = \dfrac{1-t^2}{1+t^2}$ $\quad y = \dfrac{t(1-t^2)}{1+t^2}$.
Indicate the locations of vertical and horizontal tangents, and the slope of the tangents at the origin. When evaluating the horizontal tangents you may use the approximation: $\sqrt{5} - 2 \approx 0.25$.

(13 marks)

$$\frac{dx}{dt} = (1-t^2)(-1)(1+t^2)^{-2}\cdot 2t + -2t(1+t^2)^{-1}$$

$$= \frac{-2t(1-t^2) - 2t(1+t^2)}{(1+t^2)^2} = \boxed{\frac{-4t}{(1+t^2)^2}}$$

$$\boxed{\begin{aligned} &\lim_{t\to\pm\infty} x = -1 \\ &\lim_{t\to\pm\infty} y = \mp\infty \end{aligned}}$$

$$\frac{dy}{dt} = t(1-t^2)(-1)(1+t^2)^{-2}\cdot 2t + (1-3t^2)(1+t^2)^{-1}$$

$$= \frac{-2t^2(1-t^2) + (1-3t^2)(1+t^2)}{(1+t^2)^2} = \frac{-2t^2 + 2t^4 + 1 - 3t^2 + t^2 - 3t^4}{(1+t^2)^2}$$

$$= \boxed{\frac{1-4t^2-t^4}{(1+t^2)^2}}$$

Horizontal tangents: $\frac{dy}{dt} = 0 \Rightarrow t^4 + 4t^2 - 1 = 0 \Rightarrow t^2 = \frac{-4 \pm \sqrt{16+4}}{2} = -2 \pm \sqrt{5}$

must have $t^2 > 0 \therefore t^2 = \sqrt{5} - 2 \doteq 0.25 \Rightarrow t \sim \pm 0.5$

$\Rightarrow (0.6, \pm 0.3)$

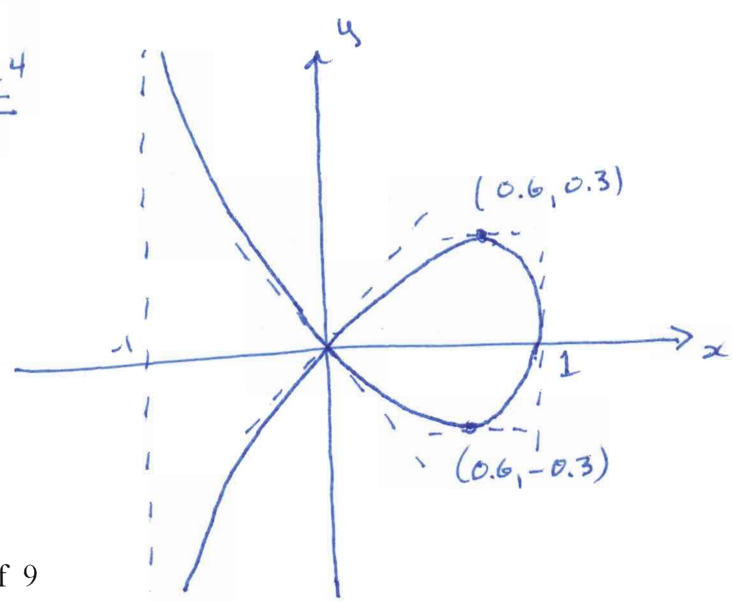
Vertical tangents: $\frac{dx}{dt} = 0 \Rightarrow t = 0 \Rightarrow (1, 0)$

$t = \pm\infty \Rightarrow (-1, \pm\infty)$

Origin: $t = \pm 1$; $\frac{dy}{dx} = \frac{1-4t^2-t^4}{-4t^2}$

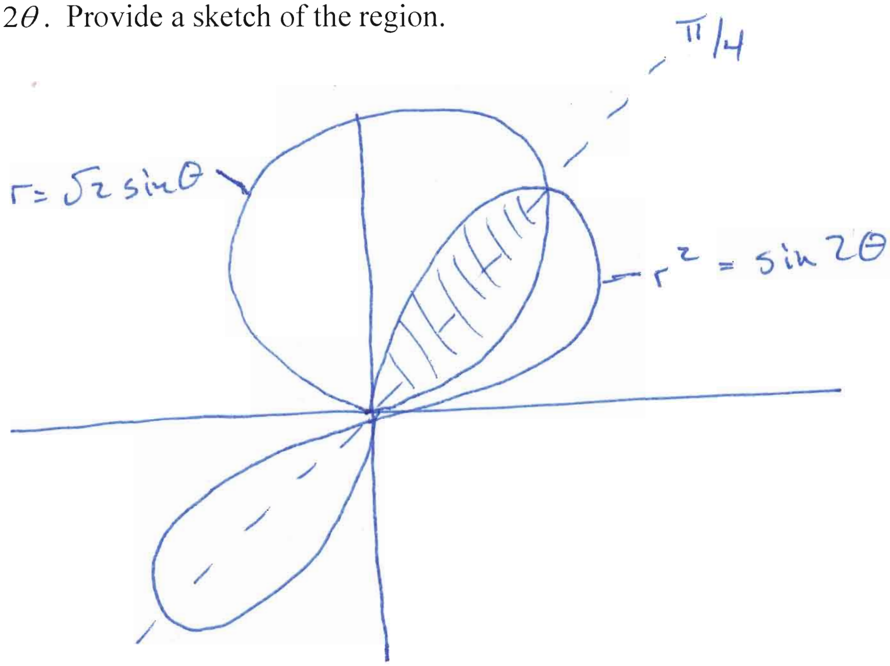
$t = 1 \Rightarrow \frac{dy}{dx} = 1$

$t = -1 \rightarrow \frac{dy}{dx} = -1$



4) Find the area of the region that lies inside both the circle $r = \sqrt{2}\sin\theta$ and inside the lemniscate $r^2 = \sin 2\theta$. Provide a sketch of the region.

(8 marks)



points of intersection: $2\sin^2\theta = \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \pi/4$

$$A = \frac{1}{2}\int_0^{\pi/4} (\sqrt{2}\sin\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin 2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{1-\cos 2\theta}{2} \, d\theta - \frac{1}{4}\Big[\cos 2\theta\Big]_{\pi/4}^{\pi/2}$$

$$= \Big[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\Big]_0^{\pi/4} - 0 + \frac{1}{4}$$

$$= \frac{\pi}{8} - \frac{1}{4} + 0 - 0 + \frac{1}{4} = \frac{\pi}{8}$$

5) Identify and sketch the conic: $r = \dfrac{8}{4+5\sin\theta}$

(5 marks)

$$r = \frac{8}{4+5\sin\theta} = \frac{2}{1+\frac{5}{4}\sin\theta} = \frac{\frac{5}{4}\cdot\frac{8}{5}}{1+\frac{5}{4}\sin\theta}$$

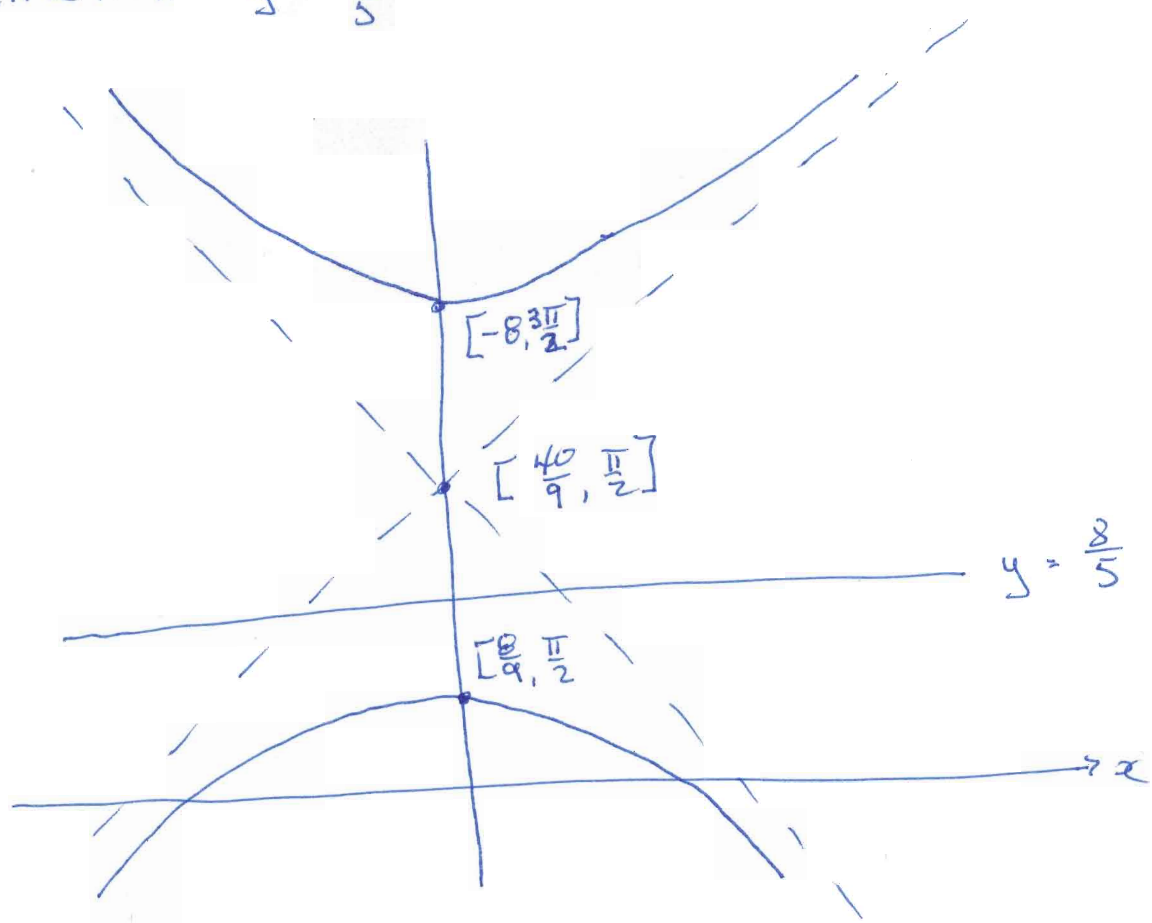
$\Rightarrow e = \frac{5}{4} > 1 \quad \therefore$ hyperbola

$\Rightarrow +e\sin\theta$ appears in the denominator

$\therefore$ oriented along y-axis

$\therefore$ directrix above x-axis

$\Rightarrow$ directrix $y = \frac{8}{5}$



6) Determine whether the following sequence converges or diverges; if it converges, find the limit:

a) $a_n = \dfrac{n^2}{\sqrt{n^3 + 4n}}$

b) $a_n = \ln(n+1) - \ln(n)$

c) $\left\{\sqrt{\alpha}, \sqrt{\alpha\sqrt{\alpha}}, \sqrt{\alpha\sqrt{\alpha\sqrt{\alpha}}}, \ldots\right\}$ $\qquad \alpha > 0$

(8 marks)

a) $a_n = \dfrac{n^2}{\sqrt{n^3+4n}} = \dfrac{1}{\sqrt{1/n + 4/n^3}} \xrightarrow{n \to \infty} \dfrac{1}{0}$ diverges

b) $\ln(n+1) - \ln(n) = \ln\left(\dfrac{n+1}{n}\right)$

$\lim\limits_{n\to\infty} \ln\left(\dfrac{n+1}{n}\right) = \ln\left(\lim\limits_{n\to\infty} \dfrac{n+1}{n}\right) = \ln 1 = 0$

$\ln x$ is continuous at $x = 1$

c) $\left\{\sqrt{\alpha}, \sqrt{\alpha\sqrt{\alpha}}, \sqrt{\alpha\sqrt{\alpha\sqrt{\alpha}}}, \ldots\right\}$

$= \left\{\alpha^{1/2}, \alpha^{1/2}\alpha^{1/4}, \alpha^{1/2}\alpha^{1/4}\alpha^{1/8}, \ldots\right\}$

$= \left\{\alpha^{1/2}, \alpha^{\frac{1}{2}+\frac{1}{4}}, \alpha^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}, \ldots\right\}$

now $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1$

$\therefore$ the limit of the sequence is $\alpha^1 = \alpha$

Note: Stewart gives two alternate solutions.

7) a) Find the sums of the following series:

i) $\displaystyle\sum_{k=1}^{\infty} \alpha(1-\alpha)^{k-1}$ $\quad 0 < \alpha < 1$

ii) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k+2)}$

(6 marks)

i) let $x = 1-\alpha$: $\displaystyle\sum_{k=1}^{\infty} \alpha(1-\alpha)^{k-1} = \alpha \sum_{k=1}^{\infty} x^{k-1} = \alpha \sum_{k=0}^{\infty} x^{k} - \frac{\alpha}{1-x}$

$\Rightarrow \dfrac{\alpha}{1-x} = \dfrac{\alpha}{1-1-\alpha} = 1$

ii) $\dfrac{1}{k(k+2)} = \dfrac{A}{k} + \dfrac{B}{k+2} \Rightarrow A(k+2) + B(k) = 1 \Rightarrow A = \frac{1}{2}, \; B = -\frac{1}{2}$

$\therefore \dfrac{1}{k(k+2)} = \dfrac{1}{2}\left(\dfrac{1}{k} - \dfrac{1}{k+2}\right)$

$\displaystyle\sum_{k=2}^{n} \frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{8} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\right)$

$= \dfrac{1}{2}\left(\dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{n+1} - \dfrac{1}{n+2}\right) \xrightarrow{n \to \infty} \dfrac{5}{12}$

b) Use the integral test to determine whether the series is convergent or divergent.

i) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

ii) $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$

(6 marks)

i) $f(x) = \dfrac{1}{\sqrt{x+1}}$ is continuous, positive and decreasing

on $[1, \infty)$, $\therefore$ integral test applies.

$\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x+1}} = \left[2(x+1)^{1/2}\right]_1^{\infty} \longrightarrow$ diverges

$\therefore \sum \dfrac{1}{\sqrt{n+1}}$ diverges

ii) $f(x) = x^2 e^{-x} \Rightarrow f'(x) = \dfrac{2xe^x - x^2e^x}{(e^x)^2} = \dfrac{x(2-x)}{e^x}$

< 0 for $x > 2$

$\therefore$ decreasing

$\therefore f(x)$ is continuous, positive and decreasing on $[3, \infty)$, $\therefore$ integral test applies.

$$\int x^2e^{-x}dx = -x^2e^{-x} + 2\int xe^{-x}dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}dx$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= -\frac{(x^2+2x+2)}{e^x} + C$$

$$\therefore \int_3^\infty x^2e^{-x}dx = -\left[\frac{x^2+2x+2}{e^x}\right]_3^\infty$$

$$= \frac{17}{e^3} - \lim_{t\to\infty} \frac{t^2+2t+2}{e^t}$$

$$\overset{*}{=} \frac{17}{e^3} - \lim_{t\to\infty} \frac{2t+2}{e^t}$$

$$\overset{*}{=} \frac{17}{e^3} - \lim_{t\to\infty} \frac{2}{e^t}$$

$$= \frac{17}{e^3} \rightarrow \text{converges.}$$

$\therefore \sum_{n=3}^{\infty} \dfrac{n^2}{e^n}$ converges

8) If the series $\sum_{n=1}^{\infty} a_n$ is convergent, does it necessarily imply that $a_n \to 0$? Prove, or provide a counterexample. Is the converse true. Again, either prove or provide a counterexample.

(8 marks)

a) Yes.

Let $S_n = \sum_{n=1}^{n} a_n = \{a_n, a_1 + a_2, a_1 + a_2 + a_3, \ldots\}$

If $\sum a_n$ is convergent, then $\lim_{n \to \infty} S_n = L$

Similarly, $\lim_{n \to \infty} S_{n-1} = L$

But $S_n - S_{n-1} = a_n$

Thus we conclude that $a_n \xrightarrow{n \to \infty} 0$

b) No

Let $a_n = \frac{1}{n}$: $a_n \to 0$ but $\sum \frac{1}{n}$ diverges