Last name: First name: ID number:

ECE 286
Midterm exam

February 9, 2022
9:30 – 10:30 am

Circle your lecture section:

LEC0101 (Monday 9-10)

LEC0102 (Monday 1-2)

Guidelines:

- Write your answer in the space provided for each question. Show your work and use the back side of sheets as needed.
- The exam is Type D. You may use a non-programmable calculator and a one-sided, 8.5 by 11 handwritten note sheet (prepared by you).
- Only exams written in pen will be considered for regrades.

Problem	Score
1	/10
2	/5
3	/7
Total	/22

1. Consider a coin with $P(H) = 0.7$ and $P(T) = 0.3$, where H denotes 'Heads' and T denotes 'Tails'. Answer parts (a)-(d) below. Justify your answers.

   (a) You flip the coin three times. Let the random variable X be the number of Heads. Answer parts (i)-(iii) below.

      i. (2 points) Find the probability mass function of X.

      Solution:

$$\begin{aligned} f(0) &= 0.3^3 = 0.027 \\ f(1) &= 3 \times 0.7 \times 0.3^2 = 0.189 \\ f(2) &= 3 \times 0.3 \times 0.7^2 = 0.441 \\ f(3) &= 0.7^3 = 0.343. \end{aligned}$$

      ii. (2 points) Find the cumulative distribution of X.

      Solution:

$$\begin{aligned} F(0) &= 0.027 \\ F(1) &= 0.216 \\ F(2) &= 0.657 \\ F(3) &= 1. \end{aligned}$$

      iii. (1 point) Find the probability that $1 \leq X \leq 2$.

      Solution:

$$F(2) - F(0) = 0.63.$$

   (b) (2 points) Suppose you flip the coin six times. What is the probability of the sequence $TTTTTH$?

   Solution:

$$0.3^5 \times 0.7 = 0.001701$$

   (c) (3 points) Suppose you flip the coin six times. What is the probability that the sequence has exactly three heads in a row? (For example, $THHHTT$ has exactly three heads in a row. $HTHHTT$ and $HHHHHH$ do not.)

   Solution: We must compute how many sequences have three heads in a row. Let $*$ denote H or T. We can have:

   - $HHHT**$. *There are 4 such sequences, because each $*$ can be H or T. We have:*

$$0.7^3 \times 0.3 \times \left(0.3^2 + 2 \times 0.3 \times 0.7 + 0.7^2\right) = 0.1029.$$

- $THHHT*$. *There are 2 such sequences.We have:*

$$0.7^3 \times 0.3^2 \times (0.3 + 0.7) = 0.03087.$$

- $*THHHT$. *There are 2 such sequences.We have:*

$$0.7^3 \times 0.3^2 \times (0.3 + 0.7) = 0.03087.$$

- $**THHH$. *There are 4 such sequences.We have:*

$$0.7^3 \times 0.3 \times \left(0.3^2 + 2 \times 0.3 \times 0.7 + 0.7^2\right) = 0.1029.$$

Summing the probabilities, we have 0.26754

2. A bag contains 2 red, 3 blue, and 5 green balls. You remove three balls from the bag. Answer parts (a) and (b) below.

   (a) (2 points) What is the probability of drawing first a green ball, second a red ball, and third a green ball, in this order?

   Solution: It is

$$P(G, R, G) = \frac{5}{10} \cdot \frac{2}{9} \cdot \frac{4}{8} = 1/18 = 0.0555...$$

   (b) (3 points) What is the probability of drawing one red ball and two green balls, in any order?

   Solution: There are

$$\binom{2}{1} = \frac{2!}{1!1!} = 2$$

   ways to draw one red ball. There are

$$\binom{5}{2} = \frac{5!}{2!3!} = 10$$

   ways to draw 2 green balls. There are thus $2 \cdot 10 = 20$ *ways to draw one red and two green balls. There are*

$$\binom{10}{3} = \frac{10!}{3!7!} = 120$$

   ways to draw 3 balls of any color. Therefore, the chance of drawing 1 red and two green balls is

$$20/120 = 1/6.$$

3. A factory produces cans of blue and green paint. The events are:

   - B: a can has blue paint inside.
   - G: a can has green paint inside.
   - C_B: the can is labeled blue.
   - C_G: the can is labeled green.

   We know the following probabilities:

   - 30% of cans are labeled blue, i.e., $P(C_B) = 0.3$.
   - The probability a can is labeled green and has green paint inside is $P(G \cap C_G) = 0.6$.

   Answer parts (a)-(c) below.

   The probability that a can contains the correct paint color is 0.9.

   (a) (2 points) If you buy a can that is labeled green, what is the probability it has green paint inside?

   Solution: We want to know $P(G \mid C_G)$. From the definition of conditional probability,

   $$P(G \mid C_G) = P(G \cap C_G)/P(C_G).$$

   We know $P(C_G) = 1 - P(C_B) = 0.7$, and therefore $P(G \mid C_G) = 0.6/0.7 = 6/7$.

   (b) (2 points) If you buy a can that is labeled green, what is the probability it has blue paint inside?

   Solution: We want to know $P(B \mid C_G)$. From the definition of conditional probability,

   $$P(B \mid C_G) = P(B \cap C_G)/P(C_G).$$

   We know that $P(C_G) = 1 - P(C_B) = 0.7$. The law of total probability says

   $$P(C_G) = P(B \cap C_G) + P(G \cap C_G).$$

   Because $P(G \cap C_G) = 0.6$, we know $P(B \cap C_G) = 0.1$. Therefore

   $$P(B \mid C_G) = 0.1/0.7 = 1/7.$$

   Alternatively, we can see that $P(B \mid C_G) = 1 - P(G \mid C_G) = 1/7$.

   (c) (3 points) Suppose also that the probability a can is labeled blue and has green paint inside is $P(G \cap C_B) = 0.1$. If a can has blue paint inside, what is the probability that the can is labeled green?

   Solution: We want to know $P(C_G \mid B)$. Bayes' Rule says

   $$P(C_G \mid B) = P(B \mid C_G)P(C_G)/P(B).$$

We know that $P(C_G) = 0.7$ *and* $P(B \mid C_G) = 1/7$*. We must find* $P(B)$*. We know that*

$$\begin{aligned} 1 &= P(G \cap C_G) + P(G \cap C_B) + P(B \cap C_G) + P(B \cap C_B) \\ &= 0.6 + 0.1 + 0.1 + P(B \cap C_B). \end{aligned}$$

Therefore, $P(B \cap C_B) = 0.2$*. We know that*

$$\begin{aligned} P(B) &= P(B \cap C_G) + P(B \cap C_B) \\ &= 0.1 + 0.2 \\ &= 0.3. \end{aligned}$$

Therefore,

$$P(C_G \mid B) = 0.7 \cdot (1/7)/0.3 = 1/3.$$