University of Toronto, Faculty of Applied Science and Engineering

MAT292H1F - Calculus III

Solution of Final Exam - December 5, 2014

Examiners: B. Galvão-Sousa

Duration: 150 minutes. Aids permitted: None.

Full Name: ______________________________
Last First

Student ID: ______________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 18 pages (including this title page) and a detached formula sheet.
  Make sure you have all of them.
- You can use pages **??**–18 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES **??**–18.

GOOD LUCK!

PART I. No explanation is necessary. **(20 Marks)**

1. Let $y(t)$ be the solution of the initial-value problem

$$\begin{cases} y' = \sin(y)\cos(y), \\ y(0) = a. \end{cases}$$

For which value of a do we have $\lim_{t\to\infty} y(t) = 0$?

(a) $a = -1$.

$\boxed{\textbf{(b)}\ a = 0.}$

(c) $a = 1$.

(d) $a = \frac{\pi}{2}$.

(e) $a = \infty$.

(f) There is no such a.

2. A 3 V battery is connected to an RC–circuit. The circuit has capacitance $C = \frac{1}{25}$ F and the resistance is $R = 5\ \Omega$. The differential equation for an RC-circuit is

$$R\frac{dq}{dt} + \frac{q}{C} = E(t),$$

with $\boldsymbol{i(t) = q'(t)}$. Then

$$\lim_{t\to\infty} i(t) = \underline{\quad 0 \quad}.$$

3. Let $y(x)$ be the unique solution of the initial-value problem

$$\begin{cases} \sin(x)\dfrac{dy}{dx} + \cos(x)y = \dfrac{1}{x+1}, \\ y(1) = -2 \end{cases}$$

Then, the largest interval where there exists a unique solution is:

$$x \in \underline{\ (0,\pi)\ }.$$

4. Let $y(x)$ be a solution to the differential equation

$$\frac{dy}{dx} = e^y \cos(x),$$

which is defined in some interval centered at $x_0 = \frac{\pi}{2}$. Then, the graph of the solution $y(x)$ has what kind of point at $x_0 = \frac{\pi}{2}$?

(**Hint**: Do not try to solve the DE)

$\boxed{\textbf{(a)}\ \text{A local maximum.}}$

(b) A local minimum.

(c) A vertical asymptote.

(d) An inflection point.

(e) None of the above.

5. Consider the differential equation $y''' - 7y'' + 15y' - 9y = te^{3t} + e^t \sin(t)$. To use the method of undetermined coefficients, we assume that the particular solution has the form:

$$y_p(t) = \underline{(At+B)t^2e^{3t} + Ce^t\sin(t) + De^t\cos(t)}.$$

(**Hint.** 1 is a root of $r^3 - 7r^2 + 15r - 9$)

6. Let $f(t) = e^{-st}$ and $g(t) = 1$ with $s > 0$. Then

$$\lim_{t\to\infty} (f * g)(t) = \underline{\frac{1}{s}}.$$

7. The Laplace Transform of the function $f(t) = t^4e^{\pi t}$ exists for:

$$s \in \underline{(\pi, \infty)}.$$

8. Suppose that $f(x)$ has the Fourier sine series expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sin(2nx) \qquad \text{for } 0 < x < \pi.$$

Then

$$\int_0^\pi f(x)\sin(6x)\,dx = \underline{\frac{\pi}{2}\frac{(-1)}{3!} = -\frac{\pi}{12}}.$$

PART II. Answer the following questions. **Justify** your answers.

9. Consider the following initial-value problem: **(16 Marks)**

$$\begin{cases} \sin(y) + x\cos(y)\dfrac{dy}{dx} = 0 \\ y(2) = \dfrac{\pi}{2}. \end{cases}$$

(a) Without solving, what can we say about the existence and uniqueness of solution?

Solution. First, write the differential equation in the form of the Existence and Uniqueness Theorem for Nonlinear First Order DEs:

$$\frac{dy}{dx} = -\frac{\tan(y)}{x}.$$

The function $\frac{\tan(y)}{x}$ is not continuous for $y = \frac{\pi}{2}$, so we cannot deduce anything about the existence and uniqueness of solution without trying to solve it. □

(b) Find $y(x)$ (you can leave the solution in implicit form).

Solution. This equation is exact. Indeed,

$$M(x,y) = \sin y \qquad \text{and} \qquad N(x,y) = x\cos y$$

satisfy

$$M_y = \cos(y) = N_x.$$

Then we can find a function $\Phi(x,y)$ satisfying

$$\Phi_x = M \qquad \text{and} \qquad \Phi_y = N.$$

We have

$$\Phi = \int M\,dx = x\sin y + h(y),$$

and

$$x\cos(y) = N = \Phi_y = x\cos y + h'(y),$$

thus

$$h'(y) = 0 \qquad \Rightarrow \qquad h(y) = C.$$

So we have

$$\Phi(x,y) = x\sin y.$$

The solution satisfies the implicit equation:

$$\Phi(x,y) = C \qquad \Leftrightarrow \qquad x\sin y = C.$$

We now use the initial condition to find the constant C:

$$2\sin\frac{\pi}{2} = C \qquad \Leftrightarrow \qquad 2 = C.$$

The solution is given by

$$x\sin y = 2.$$

□

10. Consider the initial-value problem **(16 Marks)**

$$\begin{cases} y' + p(t)y = g(t)y^2 \\ y(0) = y_0 \end{cases}$$

(a) Consider the new variable $u = \frac{1}{y}$. What initial-value problem does u satisfy?

Solution. Since $u = \dfrac{1}{y}$, we have

$$u' = -\frac{1}{y^2}y' \qquad \Leftrightarrow \qquad y' = -y^2u' \qquad \Leftrightarrow \qquad y' = -\frac{1}{u^2}u'.$$

Thus

$$-y^2u' + p(t)\frac{1}{u} = g(t)\frac{1}{u^2} \qquad \Leftrightarrow \qquad -u' + p(t)u = g(t).$$

The initial condition becomes:

$$u(0) = \frac{1}{y(0)} = \frac{1}{y_0} = u_0.$$

□

(b) Consider $p(t) = -2t$, $g(t) = -t$, and $y_0 = 1$.

Find $u(t)$.

Solution. With these definitions, u satisfies the initial-value problem

$$\begin{cases} u' + 2tu = t \\ u(0) = 1 \end{cases}$$

This equation is linear, so we multiply the DE by the integrating factor $\mu(t)$, which satisfies

$$\mu = e^{\int 2t\,dt} = e^{t^2}.$$

We get

$$\begin{aligned} \left(e^{t^2} u\right)' = te^{t^2} \quad &\Leftrightarrow \quad e^{t^2} u = \int te^{t^2}\,dt \\ &\Leftrightarrow \quad e^{t^2} u = \frac{1}{2} e^{t^2} + C \\ &\Leftrightarrow \quad u = \frac{1}{2} + Ce^{-t^2} \end{aligned}$$

Using the initial condition, we get

$$1 = u(0) = \frac{1}{2} + C \qquad \Leftrightarrow \qquad C = \frac{1}{2}.$$

The solution is

$$u = \frac{1}{2} + \frac{1}{2} e^{-t^2}.$$

$\square$

(c) What is $y(t)$?

Solution. We have

$$y = \frac{1}{u} = \frac{1}{\frac{1}{2}\left(1+e^{-t^2}\right)} = \frac{2}{1+e^{-t^2}}.$$

□

(d) Confirm that the solution $y(t)$ you found solves the original initial-value problem with $p(t), g(t), y_0$ as in **(b)**.

Solution. First verify that $y(t)$ satisfies the initial condition

$$y(0) = \frac{2}{1+1} = 1.$$

We now want to show that

$$y' = 2ty - ty^2 = ty(2-y).$$

We have,

$$\begin{aligned} y' &= -2\big(1+e^{-t^2}\big)^{-2}\big(-2te^{-t^2}\big) = \frac{4te^{-t^2}}{\big(1+e^{-t^2}\big)^2} \\ &= t\frac{2}{1+e^{-t^2}}\frac{2e^{-t^2}}{1+e^{-t^2}} = ty\frac{2e^{-t^2}}{1+e^{-t^2}}. \end{aligned}$$

Now check that

$$2 - y = 2 - \frac{2}{1+e^{-t^2}} = 2\frac{1+e^{-t^2}-1}{1+e^{-t^2}} = 2\frac{e^{-t^2}}{1+e^{-t^2}}.$$

So,

$$y' = ty(2-y).$$

□

11. Consider the system **(16 Marks)**

$$\vec{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \vec{x}$$

(a) Solve the system for $\alpha = 1/2$. What are the eigenvalues of the coefficient matrix?

Classify the equilibrium point at the origin as to type (node / saddle-point / spiral) and asymptotical stability.

Solution. We compute the characteristic equation.

$$P(\lambda) = \det(\mathbb{I}\lambda - A) = \begin{vmatrix} \lambda + 1 & 1 \\ \alpha & \lambda + 1 \end{vmatrix} = \lambda^2 + 2\lambda + 1 - \alpha = 0 \implies \lambda_\pm = -1 \pm \sqrt{\alpha}$$

So in this case of $\alpha = 1/2$, we have

$$\lambda_1 = -1 + \frac{1}{\sqrt{2}} \quad \& \quad \lambda_2 = -1 - \frac{1}{\sqrt{2}}$$

Since both eigenvalues are negative and different, this is a node which is asymptotically stable.

□

(b) Solve the system for $\alpha = 2$. What are the eigenvalues of the coefficient matrix?

Classify the equilibrium point at the origin as to type (node / saddle-point / spiral) and asymptotical stability.

Solution. Using the above, we clearly see

$$\lambda_1 = -1 + \sqrt{2} \quad \& \quad \lambda_2 = -1 - \sqrt{2}$$

In this case since the eigenvalues are of different signs, we have a saddle point, which is unstable.

□

(c) Notice the change of solutions from **(a)** to **(b)**. What α is the critical point when solutions begin to change. Justify your answer.

Solution. By the above, it is when

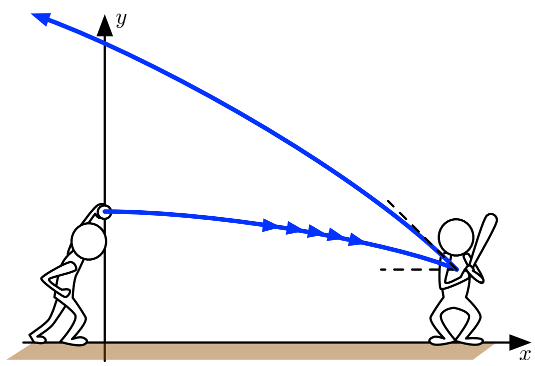
$$-1 + \sqrt{\alpha} = 0 \implies \alpha = 1$$

□

12. A baseball pitcher (at position $(0, 2)$ in meters) throws a ball horizontally to the batter (at position $x = 18$ m) at an initial velocity of 36 m/s. **(16 Marks)**

The batter hits the ball with an impulse of $56\sqrt{2}$ Ns at an angle of $\frac{\pi}{4}$.

Do not consider drag and do not give a numeric value for g. (it could be Martian baseball!)



(a) How much time does the ball take to reach the batter?

Solution. Since the horizontal velocity is 36 m/s, it takes $\frac{1}{2}$ s to cover the 18 m distance between the pitcher and the batter.

□

(b) What is the height of the ball when the batter hits it?

Solution. The height y of the ball satisfies:

$$y'' = -g,$$

and

$$y(0) = 2 \qquad \text{and} \qquad y'(0) = 0.$$

So it satisfies

$$y(t) = -\frac{g}{2}t^2 + 2,$$

so when the batter hits the ball, the height of the ball is

$$y\left(\tfrac{1}{2}\right) = -\frac{g}{8} + 2.$$

□

(c) Write a system of differential equations that gives the position of the ball after the ball is thrown by the pitcher.

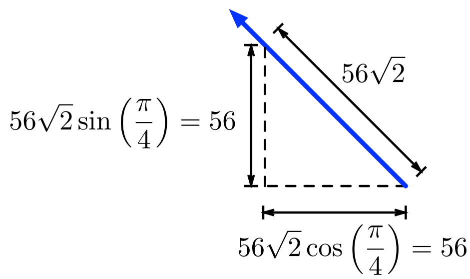
(**Hint.** Dirac delta "function")

Solution. The x and y components satisfy

$$\begin{cases} x'' = 0 \\ y'' = -g \end{cases}$$

for all $t > 0$ except when the ball is struck at $t = \frac{1}{2}$, where it receives an impulse of $I = 56\sqrt{2}$ Ns in a diagonal direction (with angle $\frac{\pi}{4}$).

The components of the impulse are



This means that these components satisfy

$$\begin{cases} x'' = -56\,\delta\left(t - \frac{1}{2}\right) \\ y'' = -g + 56\,\delta\left(t - \frac{1}{2}\right) \end{cases}$$

with the initial conditions

$$\begin{cases} x(0) = 0 \\ y(0) = 2 \end{cases}$$

$\square$

(d) Find the position of the ball $x(t)$ and $y(t)$ for all $t \geqslant 0$.

Solution. Let $X(s) = \mathfrak{L}\{x(t)\}(s)$ and $Y(s) = \mathfrak{L}\{y(t)\}(s)$.
Then

$$\begin{cases} s^2X(s) - 36 = -56e^{-\frac{s}{2}} \\ s^2Y(s) - 2s = -\frac{g}{s} + 56e^{-\frac{s}{2}} \end{cases}$$

This means that

$$\begin{cases} X(s) = \frac{36}{s} - \frac{56}{s^2}e^{-\frac{s}{2}} \\ Y(s) = \frac{2}{s} - \frac{g}{s^3} + \frac{56}{s^2}e^{-\frac{s}{2}} \end{cases}$$

Apply the inverse Laplace transform:

$$\begin{cases} x(t) = 36t - 56(t - \frac{1}{2})u_{\frac{1}{2}}(t) \\ y(t) = 2 - \frac{1}{2}gt^2 + 56(t - \frac{1}{2})u_{\frac{1}{2}}(t) \end{cases}$$

This means that

$$x(t) = \begin{cases} 36t & \text{if } t < \frac{1}{2} \\ 28 - 20t & \text{if } t \geqslant \frac{1}{2} \end{cases} \qquad \text{and} \qquad y(t) = \begin{cases} -\frac{g}{2}t^2 + 2 & \text{if } t < \frac{1}{2} \\ -\frac{g}{2}t^2 + 56t - 26 & \text{if } t \geqslant \frac{1}{2} \end{cases}$$

$\square$

Bonus. Assume that the "home run distance" is 150 m and they are in playing on Earth. **(3 Marks)** Did the batter hit a homerun?

(**Hint.** it requires difficult inequalities – try only after you solved the other questions)

Solution. First we estimate the value of T when $y(T) = 0$:

$$\begin{aligned}
y = 0 \quad &\Rightarrow \quad 2 - \frac{1}{2}gT^2 + 56(T - \frac{1}{2}) = 0 \\
&\Rightarrow \quad \frac{g}{2}T^2 - 56T + 26 = 0 \\
&\Rightarrow \quad T = \frac{56 + \sqrt{56^2 - 52g}}{g} \\
&\Rightarrow \quad T > \frac{56 + \sqrt{56(56 - 10)}}{g} \\
&\Rightarrow \quad T > \frac{56 + \sqrt{56 \cdot 46}}{g} \\
&\Rightarrow \quad T > \frac{56 + \sqrt{46^2}}{g} \\
&\Rightarrow \quad T > \frac{56 + 46}{g} = \frac{102}{g} > 10
\end{aligned}$$

This means that

$$x(T) < x(10) = 360 - 56(10 - \tfrac{1}{2}) = -200 + 28 < -150.$$

So the answer is: Yes, he did hit a home run (with a 1 kg ball !) □

13. Consider an completely insulated rod, which is modelled by the problem **(16 Marks)**

$$\begin{cases} \dfrac{\partial u}{\partial t} = 25\dfrac{\partial^2 u}{\partial x^2} & \text{for } 0 \leqslant x \leqslant \pi, \quad t \geqslant 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 \\ \frac{\partial u}{\partial x}(\pi,t) = 0 \\ u(x,0) = 4\sin^2(3x). \end{cases}$$

Find the solution $u(x,t)$.

Hint.

(a) Write $u(x,t) = \phi(x)G(t)$ and find differential equations for G and ϕ and boundary conditions.

(b) Find $G(t)$.

(c) Find eigenfunctions $\phi(x)$.

(d) Write down the general solution $u(x,t)$.

(e) Write the initial condition as a Fourier series of the same form as ϕ. Recall that $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$.

(f) Conclude with the final formula for $u(x,t)$.

Solution. Following the hint, I will split the solution in these 6 steps:

(a) Using separation of variables, I write $u(x,t) = \phi(x)G(t)$, then using this formula into the PDE, I get

$$\phi(x)G'(t) = 25\phi''(x)G(t) \qquad \Leftrightarrow \qquad \underbrace{\frac{G'(t)}{25G(t)}}_{\substack{\text{depends}\\ \text{only on } t}} = \underbrace{\frac{\phi''(x)}{\phi(x)}}_{\substack{\text{depends}\\ \text{only on } x}}$$

So both sides must be constant:

$$G'(t) = -25\lambda G(t) \qquad \Leftrightarrow \qquad \phi''(x) = -\lambda\phi(x).$$

Moreover the boundary conditions imply that

$$\phi'(0)G(t) = 0 \qquad \Leftrightarrow \qquad \phi'(0) = 0 \text{ or } G(t) = 0.$$

Since $G(t) = 0$ implies that we obtain a trivial solution $u(x,t) = 0$, we impose $\phi'(0) = 0$.

Similarly, we impose $\phi'(\pi) = 0$.

(Continuation of solution to 14)

(b) Now, solve the DE for $G(t)$:

$$G(t) = Ce^{-25\lambda t}.$$

(c) From the formula sheet, we know that

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2$$
$$\phi_n(x) = \cos(nx)$$

for $n = 0, 1, 2, \ldots$.

(d) We just found a sequence of solutions:

$$u_0(x,t) = 1$$
$$u_n(x,t) = \cos(nx)e^{-25n^2 t}.$$

Using the principle of superposition, we obtain the general solution

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) e^{-25n^2 t}.$$

(e) To simplify the process of finding the constants a_n, we write

$$u(x,0) \quad = \quad 4\sin^2(3x) \quad = \quad 4\frac{1-\cos(6x)}{2} \quad = \quad 2 - 2\cos(6x).$$

Since this function is already in the form of a cosine Fourier series, we obtain

$$\frac{a_0}{2} = 2 \quad , \quad a_6 = -2 \quad , \quad a_n = 0 \text{ for } n \neq 0, 6.$$

(f) The solution is

$$u(x,t) \quad = \quad 2 - 2\cos(6x)e^{-30^2 t} \quad = \quad 2 - 2\cos(6x)e^{-900t}.$$

$\square$