TUTORIAL 4 – SOLUTIONS

Q1:

$$\vec{AB} = \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ B_z - A_z \end{bmatrix}$$

$$\vec{BC} = \begin{bmatrix} C_x - B_x \\ C_y - B_y \\ C_z - B_z \end{bmatrix}$$

ABC IS A RIGHT ANGLE.

$$\therefore \quad \vec{AB} \cdot \vec{BC} = (B_x - A_x)(C_x - B_x) + (B_y - A_y)(C_y - B_y) + (B_z - A_z)(C_z - B_z)$$

$$= 0$$

$$\left.\begin{aligned} \vec{A'B'} &= \begin{bmatrix} B_x - A_x \\ B_y - A_y \\ 0 \end{bmatrix} \\ \vec{B'C'} &= \begin{bmatrix} C_x - B_x \\ C_y - B_y \\ 0 \end{bmatrix} \end{aligned}\right\} \text{ORTHOGONAL PROJECTIONS ON } \pi_1$$

$$\vec{A'B'} \cdot \vec{B'C'} = (B_x - A_x)(C_x - B_x) + (B_y - A_y)(C_y - B_y)$$

THIS ANGLE A'B'C' IS RIGHT IFF

$$(B_x - A_x)(C_x - B_x) + (B_y - A_y)(C_y - B_y) = 0$$

∴ $A'B'C'$ WILL BE RIGHT IFF

$$(B_z - A_z)(C_z - B_z) = 0$$

∴ $A'B'C'$ WILL BE RIGHT IFF $B_z = A_z$ OR $C_z = B_z$.

∴ $A'B'C'$ WILL BE RIGHT IFF EITHER $\vec{AB}$ OR $\vec{BC}$ OR BOTH $\vec{AB}$ AND $\vec{BC}$ ARE PARALLEL TO Π_1.

Q2:

a) FIND $\vec{BA}$ AND $\vec{BC}$

$$\vec{BA} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}2\\2\\3\end{bmatrix} = \begin{bmatrix}-1\\-1\\-2\end{bmatrix} \qquad \|\vec{BA}\| = \sqrt{6}$$

$$\vec{BC} = \begin{bmatrix}6\\1\\10\end{bmatrix} - \begin{bmatrix}2\\2\\3\end{bmatrix} = \begin{bmatrix}4\\-1\\7\end{bmatrix} \qquad \|\vec{BC}\| = \sqrt{66}$$

$$\cos\theta = \frac{\vec{BA}\cdot\vec{BC}}{\|\vec{BA}\|\,\|\vec{BC}\|} = \frac{\begin{bmatrix}-1\\-1\\-2\end{bmatrix}\cdot\begin{bmatrix}4\\-1\\7\end{bmatrix}}{\sqrt{6}\ \sqrt{66}} = \frac{-17}{\sqrt{396}}$$

$$\theta = 2.59 \text{ rad } (149 \text{ degrees})$$

b) ORTHOGONAL PROJECTION OF $\vec{BA}$ ONTO THE X-Y PLANE CAN BE FOUND USING:

$$\vec{B'A'} = \vec{BA} - \text{proj}_{\vec{n}_1}\vec{BA}$$

WHERE $\vec{n}_1$ IS A NORMAL OF THE X-Y PLANE.

TAKE $\vec{n}_1 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$\operatorname{proj}_{\vec{n}_1} \overrightarrow{BA} = \frac{\overrightarrow{BA} \cdot \vec{n}_1}{\|\vec{n}_1\|^2} \vec{n}_1 = \frac{\begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

$$\therefore \overrightarrow{B'A'} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

NOTE THIS CORRESPONDS TO SETTING THE Z- COMPONENT IN $\overrightarrow{BA}$ EQUAL TO ZERO.

$\therefore$ ORTHOGONAL PROJECTION OF $\overrightarrow{BC}$ ONTO THE X-Y PLANE IS GIVEN BY:

$$\overrightarrow{B'C'} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

$$\cos\theta' = \frac{\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}}{\sqrt{2}\,\sqrt{17}} = \frac{-3}{\sqrt{34}}$$

$$\theta' = 2.11 \text{ rad } (121 \text{ degrees})$$

c) SET Y-COMPONENT EQUAL TO ZERO TO GET ORTHOGONAL PROJECTIONS OF $\vec{BA}$ AND $\vec{BC}$ ONTO THE X-Z PLANE:

$$\overrightarrow{B''A''} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

$$\overrightarrow{B''C''} = \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}$$

$$\cos\theta'' = \frac{\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \circ \begin{bmatrix} 4 \\ 0 \\ 7 \end{bmatrix}}{\sqrt{5}\,\sqrt{65}} = \frac{-18}{\sqrt{325}}$$

$$\theta'' = 3.09 \text{ rad} \quad (177 \text{ degrees})$$

Q3:

a) $c_1\vec{0} + c_2\vec{v} + c_3\vec{w} = \vec{0}$

WE CAN CHOOSE $c_1 \neq 0$ AND $c_2 = c_3 = 0$

$\therefore$ $\{\vec{0}, \vec{v}, \vec{w}\}$ IS NOT INDEPENDENT.

b) $c_1\vec{v} + c_2\vec{w} + c_3(3\vec{v} - 4\vec{w}) = \vec{0}$

WE CAN CHOOSE $c_1 = -3$, $c_2 = 4$, $c_3 = 1$

$\therefore$ $\{\vec{v}, \vec{w}, 3\vec{v} - 4\vec{w}\}$ IS NOT INDEPENDENT.

c) $c_1\vec{v} + c_2\vec{w} + c_3(\vec{v} \times \vec{w}) = \vec{0}$

SINCE $c_1\vec{v} + c_2\vec{w}$ IS ALWAYS IN THE PLANE DEFINED BY THESE TWO VECTORS $\vec{v}$ AND $\vec{w}$, THEN THE ONLY SOLUTION IS $c_1 = c_2 = c_3 = 0$ BECAUSE $\vec{v} \times \vec{w}$ IS ORTHOGONAL TO THIS PLANE.

$\therefore$ $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ IS INDEPENDENT.

THIS IS NOT A FORMAL PROOF THOUGH.

PROOF OF 1 c) AS PROVIDED BY AN ESC103F STUDENT IN 2012-13:

$$(\vec{v}\times\vec{w})\cdot\left(c_1\vec{v}+c_2\vec{w}+c_3\vec{v}\times\vec{w}\right)=(\vec{v}\times\vec{w})\cdot\vec{0}=0$$

SINCE $\vec{v}$ AND $\vec{w}$ ARE ORTHOGONAL TO $\vec{v}\times\vec{w}$:

$$0+0+c_3\|\vec{v}\times\vec{w}\|^2=0$$

SINCE $\vec{v}$ AND $\vec{w}$ ARE NON-ZERO, NON-PARALLEL VECTORS:

$$\|\vec{v}\times\vec{w}\|^2\neq 0$$

$$\therefore\quad c_3=0$$

$$\therefore\quad c_1\vec{v}+c_2\vec{w}=\vec{0}$$

HOWEVER, SINCE $\vec{v}$ AND $\vec{w}$ ARE NOT PARALLEL:

$$c_1=c_2=0$$

HENCE, THE 3 VECTORS ARE INDEPENDENT.

Q4:

a) Is $T(k\vec{u}) = kT(\vec{u})$? $\quad$ K IS A SCALAR

$$T(k\vec{u}) = \begin{bmatrix} kx \\ kx+ky \\ kx+ky+kz \end{bmatrix} = k\begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix} = kT(\vec{u}) \checkmark$$

Let $\vec{v} = \begin{bmatrix} x_1 \\ x_1 \\ z_1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

Is $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$?

$$T(\vec{v}+\vec{w}) = \begin{bmatrix} x_1+x_2 \\ x_1+x_2+y_1+y_2 \\ x_1+x_2+y_1+y_2+z_1+z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1+y_1 \\ x_1+y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2+y_2 \\ x_2+y_2+z_2 \end{bmatrix}$$

$$= T(\vec{v}) + T(\vec{w}) \checkmark$$

$\therefore$ T IS A LINEAR TRANSFORMATION.

b) $$M_T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$T(\vec{u}) = M_T\,\vec{u}$$

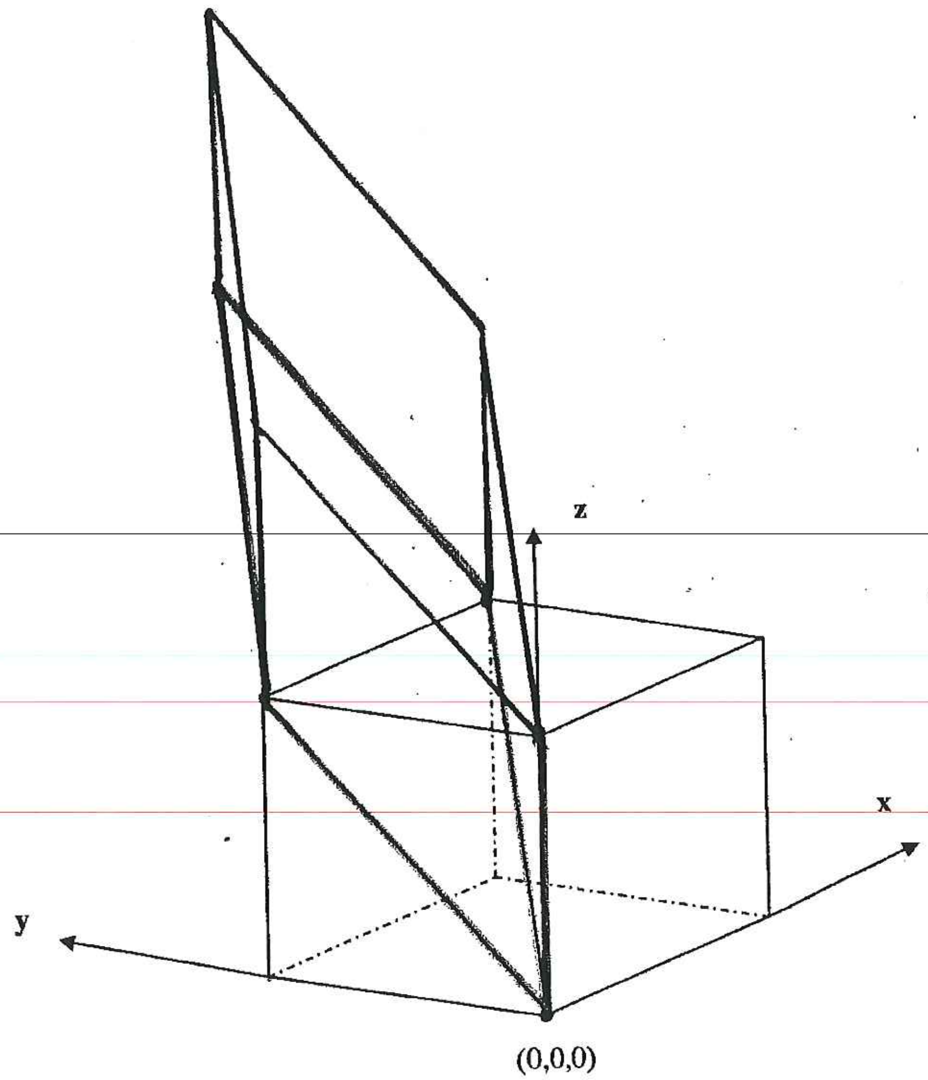
(C) FIND WHERE T TAKES THE EIGHT CORNERS OF THE UNIT CUBE

$$\begin{bmatrix}0\\0\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}1\\1\\1\end{bmatrix} \quad \begin{bmatrix}1\\1\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}1\\2\\2\end{bmatrix} \quad \begin{bmatrix}0\\1\\0\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\0\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}1\\1\\2\end{bmatrix} \quad \begin{bmatrix}1\\1\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}1\\2\\3\end{bmatrix} \quad \begin{bmatrix}0\\1\\1\end{bmatrix} \xrightarrow{T} \begin{bmatrix}0\\1\\2\end{bmatrix}$$

(c)

What does the transformation T do to the unit cube shown in the figure below? Provide a sketch of your answer directly on the figure.



Q5:

(a)

$$A^2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$= \begin{bmatrix} 4 & 1 & 2 \\ 2 & 7 & -4 \\ 1 & -5 & 7 \end{bmatrix}$$

$$A^3 = \begin{pmatrix} 4 & 1 & 2 \\ 2 & 7 & -4 \\ 1 & -5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{pmatrix}$$

$$= \begin{bmatrix} 10 & 9 & 0 \\ 0 & -17 & 18 \\ 9 & 27 & -17 \end{bmatrix}$$

(b) $$9A - 8I = 9\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 3 & -1 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{pmatrix} 18 & 9 & 0 \\ 0 & -9 & 18 \\ 9 & 27 & -9 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{bmatrix} 10 & 9 & 0 \\ 0 & -17 & 18 \\ 9 & 27 & -17 \end{bmatrix} = A^3$$

(c) $A^6 = (A^3)^2 = (9A - 8I)^2$

$$= 81A^2 - (2)(72)A + 64I$$

$$= 81A^2 - 144A + 64I$$

$$= aA^2 + bA + cI$$

$\therefore$ $a = 81$
$b = -144$
$c = 64$

Q6:

a) $K\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ -z \end{bmatrix}$

+1

$$\therefore \quad M_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ -z \end{bmatrix} = M_k \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$J\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \\ z \end{bmatrix}$$

$$\therefore \quad M_J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -x \\ y \\ z \end{bmatrix} = M_J \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

b) K FOLLOWED BY J

$$M_{COMPOSITION} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

c) J FOLLOWED BY K

$$M_{COMPOSITION} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

GIVEN THAT THE COMPOSITION MATRICES ABOVE ARE THE SAME THEN THE COMPOSITION OF THESE TWO TRANSFORMATIONS IS THE SAME REGARDLESS OF THE ORDER IN WHICH THE TRANSFORMATIONS ARE PERFORMED.