ECE259: Electromagnetism

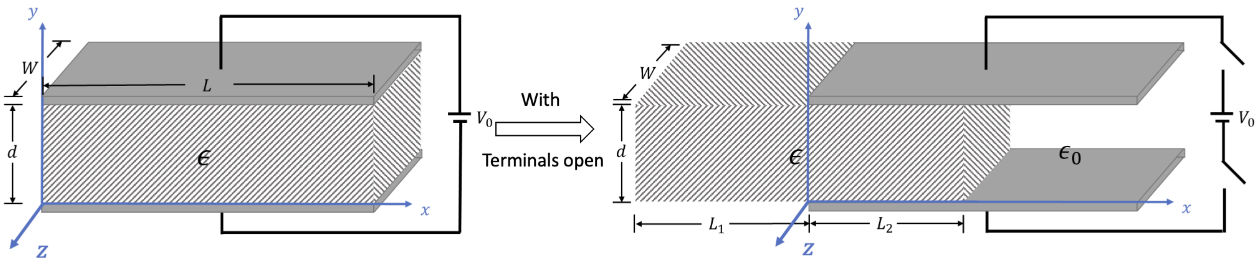
Term Test 2 - March 24th, 2022

Instructors: Profs. Li Qian and Piero Triverio

Instructions

- Duration: 1 hour 30 minutes (18:10 to 19:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**
- Please write with a **dark** pen or pencil.

Question 1



A parallel-plate capacitor with length L, width W, plate separation d, and a dielectric material of permittivity ϵ, is charged to V_0, as shown on the left figure. Afterwards, with the capacitor's terminals open, the dielectric material is partially withdrawn as shown on the right figure.

Ignoring fringing effects, answer the following questions:

a) What is the charge of the capacitor before the withdrawal? Express it in terms of the given parameters [2 points].

$$Q = CV_0 = \frac{\epsilon W L}{d} V_0$$

b) Use generalized Gauss's Law, find the $\mathbf{D}$ field vector inside the capacitor before the withdrawal. (Your expression should contain both magnitude and direction.) [2 points]

$$\iint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enclosed}$$

$$\mathbf{a}_n \cdot \mathbf{D} = \rho_S$$

$$\mathbf{D} = \frac{\epsilon V_0}{d}(-\mathbf{a}_y)$$

c) After the withdrawal, find the $\mathbf{D}$ field vector in

- Region 1: $-L_1 \leq x \leq 0, 0 \leq y \leq d$
- Region 2: $0 \leq x \leq L_2, 0 \leq y \leq d$
- Region 3: $L_2 \leq x \leq L, 0 \leq y \leq d$ [8 points]

- Region 1: $-L_1 \le x \le 0, 0 \le y \le d$

  There is no charge in the domain, and $\mathbf{D}_1 = 0$ [2 points]

- Region 2: $0 \le x \le L_2, 0 \le y \le d$

  The potential of the left plate and the right plate are same because they are in the same perfect conduct plane. According to $-\nabla V = \mathbf{E}$, we have

$$\mathbf{E}_2 = \mathbf{E}_3 = \frac{\mathbf{D}_2}{\epsilon} = \frac{\mathbf{D}_3}{\epsilon_0}$$

  Applying the boundary conditions for the surface charge density on the PEC surface, we have

$$\frac{\rho_{s2}}{\epsilon} = \frac{\rho_{s3}}{\epsilon_0} \qquad (1)$$

  Additionally, the total charge Q has not been changed before and after, which gives

$$Q = CV_0 = \frac{\epsilon W L}{d} V_0 = \rho_{s2} L_2 W + \rho_{s3} L_1 W \qquad (2)$$

  Combining (1) and (2), we have

$$\rho_{s2} = \frac{\epsilon^2 V_0 L}{d(\epsilon L_2 + \epsilon_0 L_1)}$$
$$\vec{D}_2 = \rho_{s2} \cdot \hat{n} = \frac{\epsilon^2 V_0 L}{d(\epsilon L_2 + \epsilon_0 L_1)} (\mathbf{a}_y)$$

  [4 points]

- Region 3: $L_2 \le x \le L, 0 \le y \le d$

$$\rho_{s3} = \frac{\epsilon_0 \epsilon V_0 L}{d(\epsilon L_2 + \epsilon_0 L_1)}$$
$$\vec{D}_3 = \rho_{s2} \cdot \hat{n} = \frac{\epsilon_0 \epsilon V_0 L}{d(\epsilon L_2 + \epsilon_0 L_1)} (\mathbf{a}_y)$$

  [2 points]

d) What's the capacitance of the capacitor shown in the right figure? [4 points]

$$V' = |E_3| d = \frac{\left|\vec{D}_3\right| d}{\epsilon_0} = \frac{\epsilon L}{\epsilon L_2 + \epsilon_0 L_1} V_0$$
$$C' = \frac{Q}{V'} = \frac{\frac{\epsilon W L}{d} V_0}{V'} = \frac{(\epsilon L_2 + \epsilon_0 L_1) W}{d}$$

[2 points for finding V, 2 points for finding C']

e) What's the energy stored in the capacitor shown in the right figure? [2 points]

$$W_e' = 0.5QV' = \frac{\epsilon^2 W L^2 V_0^2}{d(\epsilon L_2 + \epsilon_0 L_1)}$$

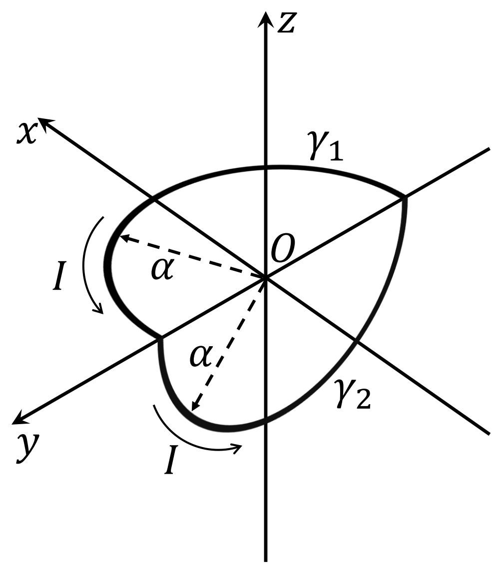
f) Has the energy stored in the capacitor increased or decreased after the partial withdrawal? [2 points]

The energy has increased. [1 point for right answer, 1 point for justification]

$$W_e = 0.5QV_0 = \frac{\epsilon W L V_0^2}{2d}$$

$$\frac{W_e'}{W_e} = \frac{\epsilon L}{\epsilon L_2 + \epsilon_0 L_1} > 1$$

Question 2



A DC current I is flowing in the circuit shown in the figure. The circuit consists of:

- a semicircular path γ_1, centered at the origin, of radius α and laying in the x-y plane. The midpoint of this path is the point $(\alpha, 0, 0)$;
- another semicircular path γ_2, centered at the origin, of radius α and laying in the y-z plane. The midpoint of this path is the point $(0, 0, -\alpha)$;

The circuit is surrounded by vacuum.

a) Find the magnetic flux density vector $\mathbf{B}_1$ produced at the origin O by the current in only the first semicircular part γ_1 of the circuit [12 points].

b) Find the magnetic flux density vector $\mathbf{B}_2$ produced at the origin O by the current in only the second semicircular part γ_2 of the circuit [6 points].

c) Find the total magnetic flux $\mathbf{B}$ at the origin. [2 points]

Solution

a)

$$
\begin{aligned}
\mathbf{R} &= \mathbf{0} \quad [2\,points] \\
\mathbf{R}' &= \alpha(\cos\phi'\mathbf{a}_x + \sin\phi'\mathbf{a}_y) \quad [2\,points] \\
\mathbf{dl}' &= \alpha d\phi'\mathbf{a}_{\phi'} = \alpha d\phi'(-\sin\phi'\mathbf{a}_x + \cos\phi'\mathbf{a}_y) \quad [2\,points] \\
\mathbf{B}_1 &= \frac{\mu_0 I}{4\pi}\int_{\gamma_1}\frac{\mathbf{dl}'\times(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3} \\
&= \frac{\mu_0 I\alpha}{4\pi\alpha^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\alpha\sin^2\phi'\mathbf{a}_z + \alpha\cos^2\phi'\mathbf{a}_z)d\phi' \\
&= \frac{\mu_0 I\alpha^2}{4\pi\alpha^3}\pi\mathbf{a}_z = \frac{\mu_0 I}{4\alpha}\mathbf{a}_z \quad [2\,points for integration limits, 4\,points for correct integration]
\end{aligned}
$$

b) Currents and geometries of paths γ_1 and γ_2 are similar, so one can deduce that the magnitudes of their corresponding magnetic flux density vectors at the origin O are the same. Considering the directions of $\mathbf{B}_1$ and the flow of current in the previous part, one can realize that they obey the right-hand rule. Hence,

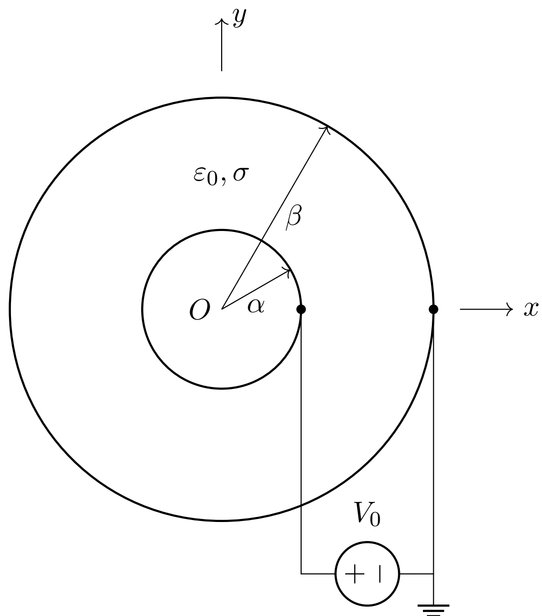
$$\mathbf{B}_2 = -\frac{\mu_0 I}{4\alpha}\mathbf{a}_x$$

[4 points for justification, 2 points for correct B found]

c) [2 points]

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \frac{\mu_0 I}{4\alpha}(-\mathbf{a}_x + \mathbf{a}_z)$$

Question 3.1



The structure shown in the figure above consists of two *spherical* perfect conductors, both very thin, and both center at the origin. The inner conductor has radius α, while the outer conductor has radius $\beta > \alpha$. The volume between the two spherical conductors ($R \in [\alpha, \beta]$) is filled with a conductive material with conductivity σ and permittivity ε_0. There are no free charges in the region between the two conductors ($\rho_v = 0$). A voltage V_0 is applied between the two conductors, and the potential of the outer conductor is taken as reference.

The potential in the region $R \in [\alpha, \beta]$ is given by

$$V(R) = \frac{\alpha}{\alpha - \beta} V_0 \left(1 - \frac{\beta}{R}\right)$$

Find the resistance R_0 between the inner and outer conductors [10 points].

Solution

[finding E: 2 points] The electric field can be obtained from potential as

$$\begin{aligned}\mathbf{E} &= -\nabla V = \\ &= \frac{\alpha\beta}{(\beta - \alpha)R^2} V_0 \mathbf{a}_R\end{aligned}$$

[finding J: 2 points] Current density $\mathbf{J}$ can be found as

$$\mathbf{J} = \sigma\mathbf{E} = \frac{\sigma\alpha\beta}{(\beta-\alpha)R^2} V_0 \mathbf{a}_R$$

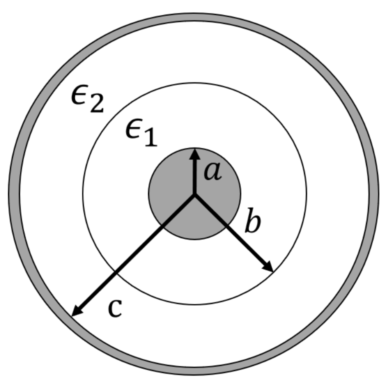
[finding I: 4 points] To find the current I flowing from the inner conductor to the outer conductor. we use a spherical surface (S') centered at the origin with radius R

$$I = \oint_{S'} \mathbf{J}\cdot\mathbf{ds} = \int_0^{2\pi}\int_0^{\pi} \sigma\frac{\alpha\beta}{(\beta-\alpha)R^2} V_0 R^2 \sin\theta d\theta d\phi = \frac{4\pi\sigma\alpha\beta}{\beta-\alpha} V_0$$

[finding R: 2 points] Finally, the resistance between the two conductors is as follows:

$$R_0 = \frac{V_0}{I} = \frac{V_0}{\frac{4\pi\sigma\alpha\beta}{\beta-\alpha} V_0} = \frac{\beta-\alpha}{4\pi\sigma\alpha\beta}$$

Question 3.2



A cylindrical coaxial cable, consisting of an inner conductor of radius a and an outer conductor of radius c, is filled with two dielectrics with permittivity ϵ_1 and ϵ_2, respectively. The related illustration is shown above. The interface of the two perfect dielectrics is at the spherical surface with radius b. The length of the cable is L, which is much larger compared to c. The two conductors carry equal and opposite charge Q, with the positive charge on the inner conductor.

Ignoring the fringing effect, please find:

a) $\mathbf{P}(r)$ inside the outer dielectric material with permittivity ϵ_2. [3 points]

Take a Gaussian surface at r, where $b < r < c$. We have

$$\iint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

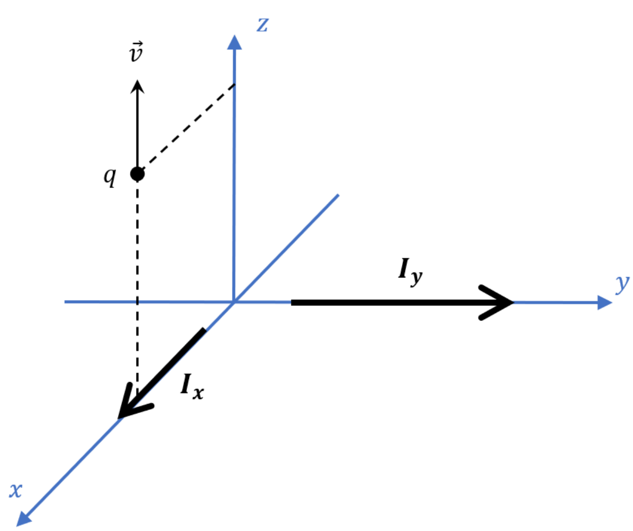
$$\mathbf{D} = \frac{Q}{2\pi |r| L}\mathbf{a}_r = \epsilon_2 \mathbf{E}$$

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \frac{Q}{2\pi |r| L}(1 - \frac{\epsilon_0}{\epsilon_2})\mathbf{a}_r$$

b) The bound surface charge density on the *outer* dielectric material at the interface, i.e. $r = b$. [2 points]

$$\begin{aligned}\rho_s(r = b) &= \mathbf{P} \cdot \mathbf{a}_n \\ &= \frac{Q}{2\pi bL}(1 - \frac{\epsilon_0}{\epsilon_2})\mathbf{a}_n \cdot (-\mathbf{a}_n) \\ &= -\frac{Q}{2\pi bL}(1 - \frac{\epsilon_0}{\epsilon_2})\end{aligned}$$

Question 3.3



Two very long wires carry currents I_x and I_y locating at the x and y axis, respectively. I_x flows at $+\mathbf{a}_x$ direction and I_y flows at $+\mathbf{a}_y$ direction. A charged particle carrying charge q is traveling in the system. When it passes the point $(x_0, 0, z_0)$, its velocity is v in the $+\mathbf{a}_z$ direction.

Find the magnetic force acting on the charged particle at the point $(x_0, 0, z_0)$. [5 points]

Hint: The magnitude of the $\mathbf{B}$ field due to a long, straight wire carrying a current I and r distance away is

$$B = \frac{\mu_0 I}{2\pi r}$$

[finding B1: 1 point] $\mathbf{B}_1$ due to I_x at $(x_0, 0, z_0)$ is

$$\mathbf{B}_1 = \frac{\mu_0 I_x}{2\pi z_0}(-\mathbf{a}_y)$$

[finding B2: 3 point] $\mathbf{B}_2$ due to I_y at $(x_0, 0, z_0)$ is

$$\mathbf{B}_2 = \frac{\mu_0 I_y}{2\pi\sqrt{x_0^2 + z_0^2}}(\mathbf{a}_{B_2}) \qquad (1)$$

where

$$\mathbf{a}_{B_2} = \frac{z_0}{\sqrt{x_0^2 + z_0^2}}\mathbf{a}_x - \frac{x_0}{\sqrt{x_0^2 + z_0^2}}\mathbf{a}_z \qquad (2)$$

Combining $(1)(2)$, we have

$$\mathbf{B}_2 = \frac{\mu_0 I_y z_0}{2\pi(x_0^2 + z_0^2)}\mathbf{a}_x - \frac{\mu_0 I_y x_0}{2\pi(x_0^2 + z_0^2)}\mathbf{a}_z$$

[finding force: 1 point] The magnetic force can be then calculated, yielding

$$\begin{aligned}
\mathbf{F}_m &= q\mathbf{v} \times \mathbf{B} = q\mathbf{v} \times (\mathbf{B}_1 + \mathbf{B}_2) \\
&= qv\mathbf{a}_z \times \left(\frac{\mu_0 I_x}{2\pi z_0}(-\mathbf{a}_y) + \frac{\mu_0 I_y z_0}{2\pi(x_0^2 + z_0^2)}\mathbf{a}_x - \frac{\mu_0 I_y x_0}{2\pi(x_0^2 + z_0^2)}\mathbf{a}_z\right) \\
&= \frac{\mu_0 q v}{2\pi}\left(\frac{I_x}{z_0}\mathbf{a}_x + \frac{I_y z_0}{x_0^2 + z_0^2}\mathbf{a}_y\right)
\end{aligned}$$