MAT195S CALCULUS II

Midterm Test #2

24 March 2015 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

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Given Name: Sol'ns

Student #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	6	
2	10	
3	7	
4	10	
5	12	
6	10	
7	11	
8	6	
TOTAL	72	/ 65

Tutorial Section:

TA Name:

1) Test the series for convergence or divergence:

a) $\sum_{k=1}^{\infty} k\left(\frac{2}{3}\right)^k$ b) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

(6 marks)

a) Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k+1}{k} \frac{(2/3)^{k+1}}{(2/3)^k}\right| = \frac{k+1}{k} \cdot \frac{2}{3} \xrightarrow{k \to \infty} \frac{2}{3}$$ converges

b) Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{100^n}{100^{n+1}}\right| = \frac{n+1}{100} \xrightarrow{n \to \infty} \infty$$ diverges

2) Proof of the Limit comparison Test. (You may use the basic comparison test in your proof.)

a) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is convergent. Prove that if $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = 0$, then Σa_n is also convergent.

b) Suppose that Σa_n and Σb_n are series with positive terms and Σb_n is divergent. Prove that if $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \infty$, then Σa_n is also divergent.

(10 marks)

a) Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \to 0$ there is a number, $N > 0$, for which

$\left| \frac{a_n}{b_n} - 0 \right| < 1$ for all $n > N$.

$\therefore$ $a_n < b_n$ for $n > N$ (all $+^{ve}$ terms)

Thus given Σb_n converges, Σa_n converges by the basic comparison test.

b) Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} \to \infty$, there is a number, $M > 0$, for which

$\left| \frac{a_n}{b_n} \right| > 1$ for all $n > M$

$\therefore$ $a_n > b_n$ for $n > M$ (all $+^{ve}$ terms)

Thus given Σb_n diverges, Σa_n diverges by the basic comparison test.

3) a) Find the sum of the series:

$$1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \ldots$$

(2 marks)

$$= \sum_{n=0}^{\infty} \frac{(\ln 2)^n}{n!} = e^{\ln 2} = 2$$

b) For what values of x does the following series converge absolutely? Conditionally? Give the radius and interval of convergence.

$$\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n+3}$$

(5 marks)

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{2^n} \frac{(x-3)^{n+1}}{(x-3)^n} \frac{n+3}{n+4}\right| \xrightarrow{n\to\infty} 2|x-3| \Rightarrow |x-3| < \frac{1}{2}$

$\therefore$ radius of convergence $R = \frac{1}{2}$

$\therefore$ absolutely convergent for $2\frac{1}{2} < x < 3\frac{1}{2}$

test $x = 2\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(-\frac{1}{2}\right)^n}{n+3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+3}$ $\quad a_{n+1} < a_n$ & $a_n \to 0$ $\therefore$ converges by Alt. series test

test $x = 3\frac{1}{2} \Rightarrow \sum_{n=1}^{\infty} \frac{2^n\left(\frac{1}{2}\right)^n}{n+3} = \sum_{n=1}^{\infty} \frac{1}{n+3} = \sum_{n=4}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$\therefore$ interval of convergence: $2\frac{1}{2} \leq x < 3$

conditionally convergent at $x = 2\frac{1}{2}$

4) Determine by directly taking derivatives, the Taylor series for the function $f(x)=\dfrac{1}{\sqrt{x}}$ about $x = 9$. Determine the radius of convergence and show that the series does not converge at $x = 0$.

(10 marks)

$$f(x) = x^{-1/2}$$

$$f'(x) = -\frac{1}{2} x^{-3/2}$$

$$f''(x) = \frac{1}{2}\cdot\frac{3}{2} x^{-5/2}$$

$$f'''(x) = -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2} x^{-7/2}$$

$$f^{iv}(x) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2} x^{-9/2}$$

$$f(9) = \frac{1}{3}$$

$$f'(9) = -\frac{1}{2}\cdot\frac{1}{3^3}$$

$$f''(9) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3^5}$$

$$f'''(9) = -\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{3^7}$$

$$\vdots$$

$$f^{(n)}(9) = \frac{1}{2^n}\, 3\cdot 5\cdot 7\cdots(2n-1)\cdot\frac{1}{3^{2n+1}}(-1)^n$$

$$\therefore \frac{1}{\sqrt{x}} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{2^n}\cdot\frac{1}{3^{2n+1}}\left(1\cdot 3\cdot 5\cdot 7\cdots(2n-1)\right)(-1)^n\,\frac{(x-9)^n}{n!}$$

Ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^n}{2^{n+1}}\,\frac{3^{2n+1}}{3^{2n+3}}\,\frac{n!}{(n+1)!}\,\frac{\left(1\cdot 3\cdot 5\cdot 7\cdots(2n-1)(2n+1)\right)}{\left(1\cdot 3\cdot 5\cdot 7\cdots(2n-1)\right)}\,\frac{(x-9)^{n+1}}{(x-9)^n}\right|$$

$$= \left|\frac{1}{18}\,\frac{2n+1}{n+1}\,(x-a)\right| \longrightarrow \frac{|x-9|}{9} < 1$$

$$\text{or } |x-9| < 9 \Rightarrow 0 < x < 18$$

test $x=0$

$$\Rightarrow \frac{1}{\sqrt{0}} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{3^{2n+1}} \left(1\cdot3\cdot5\cdot7\cdots(2n-1)\right)(-1)^n \frac{(-9)^n}{n!}$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{3^{2n}}{3^{2n+1}} \left(1\cdot3\cdot5\cdot7\cdots(2n-1)\right) \frac{1}{n!}$$

$$= \frac{1}{3} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2^n\cdot n!}$$

$$= \frac{1}{3} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot4\cdot6\cdot8\cdots2n}$$

compare with $\sum \frac{1}{2n+1}$ (diverges by limit comparison test with $\sum \frac{1}{n}$)

$\therefore$ show $\frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot4\cdot6\cdot8\cdots2n} > \frac{1}{2n+1}$

or $1\cdot\underbrace{\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6}\cdot\frac{9}{8}\cdots\frac{2n-1}{2n-2}\cdot\frac{2n+1}{2n}}_{\text{all terms} > 1} > 1$ $\therefore$ true

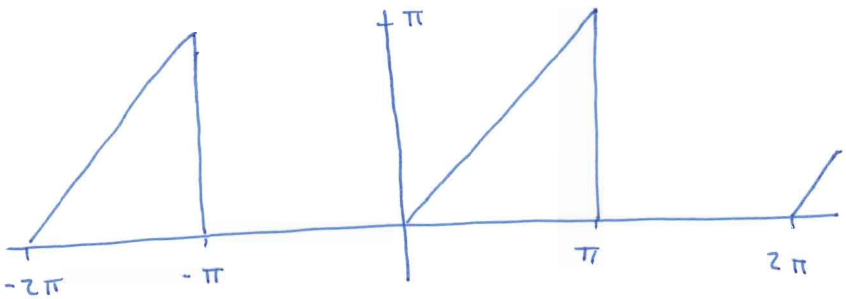
$\therefore \sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdot7\cdots(2n-1)}{2\cdot4\cdot6\cdot8\cdots2n}$ diverges by the comparison test.

5) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(x)=\begin{cases}0 & \text{if } -\pi \le x < 0\\ x & \text{if } 0 \le x < \pi\end{cases}$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

(12 marks)



$T = 2\pi$

$\therefore \omega = \frac{2\pi}{2\pi} = 1$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$$

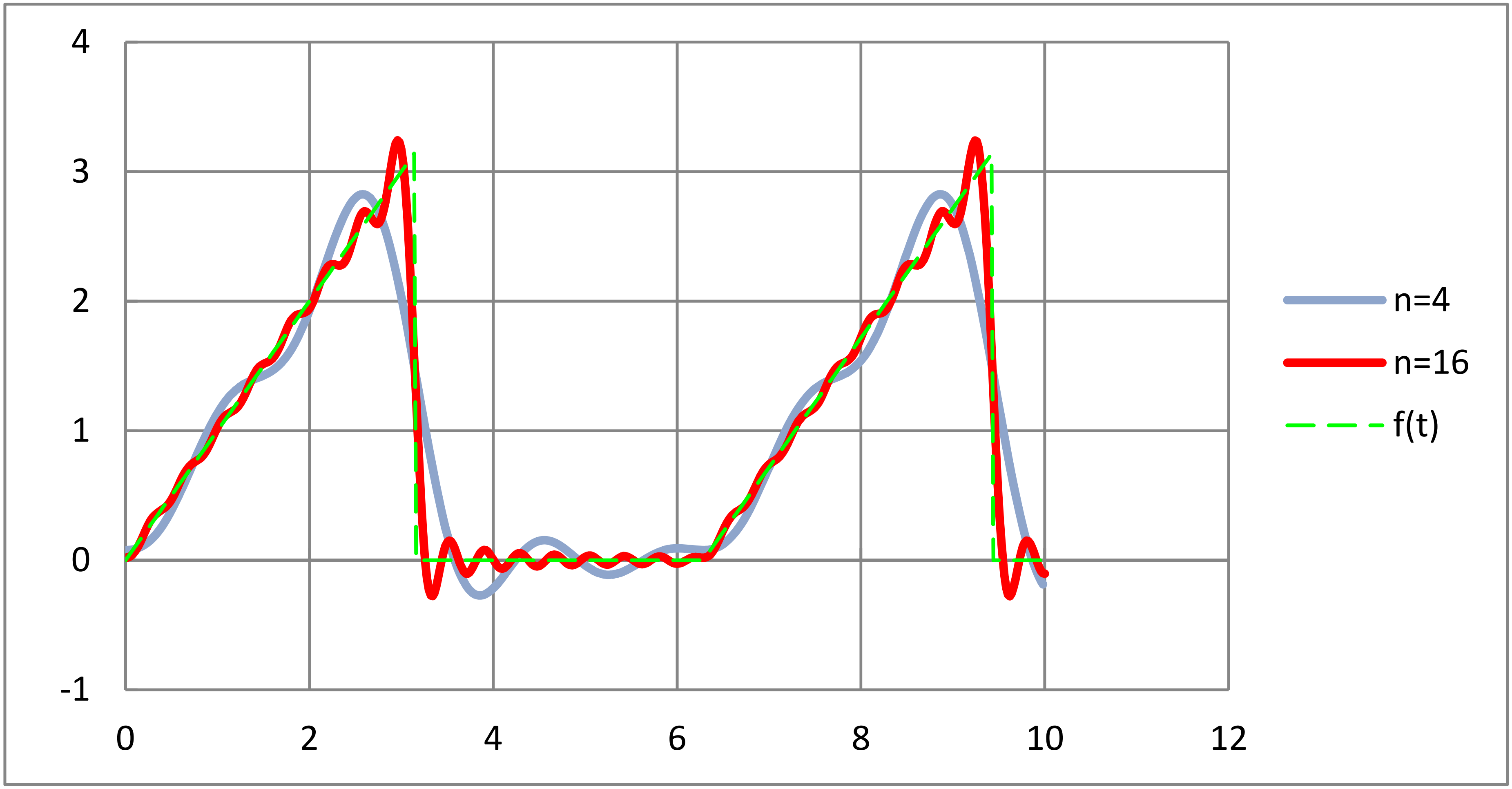
$$a_n = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt \qquad \text{let } u = t \quad dv = \cos nt\,dt$$
$$du = dt \quad v = \frac{1}{n}\sin nt$$

$$= \frac{1}{\pi}\left[\frac{t}{n}\underbrace{\sin nt}_{0}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{1}{n}\sin nt\,dt = \frac{1}{\pi n^2}\left[\cos nt\right]_0^{\pi} = \begin{cases}0 & n \text{ even}\\ \frac{-2}{\pi n^2} & n \text{ odd}\end{cases}$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt \qquad \text{let } u = t \quad dv = \sin nt\,dt$$
$$du = dt \quad v = -\frac{1}{n}\cos nt$$

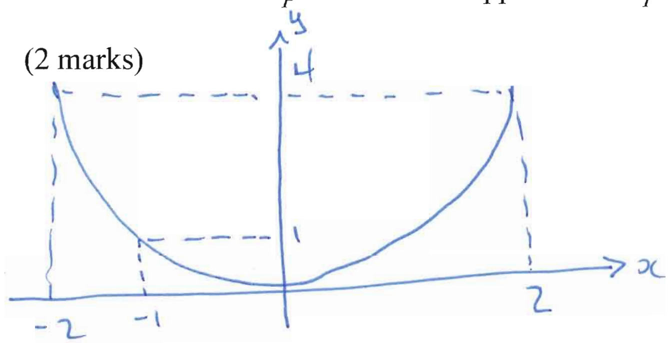
$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos nt\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{1}{n}\underbrace{\cos nt\,dt}_{0} = \begin{cases}-\frac{1}{n} & n \text{ even}\\ \frac{1}{n} & n \text{ odd}\end{cases}$$

$$\therefore f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{-2}{\pi(2n-1)^2}\cos(2n-1)t + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nt$$


4
3
2
1
0
-1
0
2
4
6
8
10
12
n=4
n=16
f(t)

6) a) Let $f(x) = x^2$ for $x \in [-2,2]$. Give an example of a partition of $[-2,2]$ such that the lower sum $L_P = 1$ and the upper sum $U_P = 16$.

(2 marks)



choose $P: [-2,-1,2]$

$L_P = 1 \cdot 1 + 0 \cdot 3 = 1$

$U_P = 4 \cdot 1 + 4 \cdot 3 = 16$

b) Let f be a continuous function on a closed and bounded interval [a,b]. Let $\{P_{2^n}\}_{n=0}^{\infty}$ be a dyadic sequence of partitions of [a,b]. Show that the sequence of upper sums $\{U_{2^n}\}_{n=0}^{\infty}$ is bounded both below and above.

(8 marks)

Extreme Value Theorem: f has a maximum and minimum value on $[a,b]$

Let $m = \min f(x)$, $x \in [a,b]$; $M = \max f(x)$, $x \in [a,b]$

Let $f_i^{max} = \max f(x)$, $x \in [x_{i-1}, x_i]$ $\equiv$ max value of f on the i^{th} interval

$$m \le f_i^{max} \le M \quad \text{for all } i$$

$$m \Delta x \le f_i^{max} \Delta x \le M \Delta x$$

summing over all intervals:

$$m \sum_{i=1}^{2^n} \Delta x_i \le \sum_{i=1}^{2^n} f_i^{max} \Delta x_i \le M \sum_{i=1}^{2^n} \Delta x_i$$

or $\quad m(b-a) \le U_{2^n} \le M(b-a)$

7) a) Find the length of the curve segment in $\Re^4$ given parametrically by:
$\bar{r}(t) = (t, t, \frac{4}{3}t^{3/2}, \frac{\sqrt{2}}{2}t^2)$ for $0 \le t \le 1$.

(4 marks)

$$\bar{r} = \left(t, t, \frac{4}{3}t^{3/2}, \frac{1}{\sqrt{2}}t^2\right) \to \bar{r}' = \left(1, 1, 2t^{1/2}, \sqrt{2}t\right)$$

$$\therefore s = \int_0^1 \sqrt{1^2 + 1^2 + 4t + 2t^2}\, dt$$

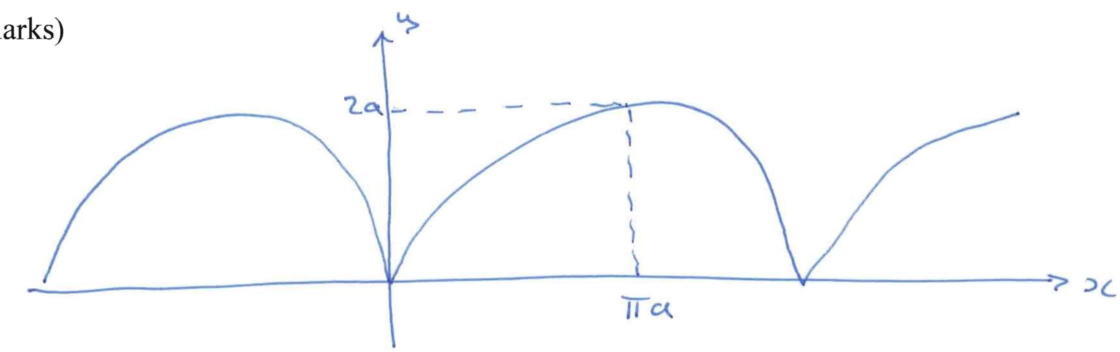
$$= \sqrt{2}\int_0^1 \sqrt{t^2 + 2t + 1}\, dt = \sqrt{2}\int_0^1 (t+1)\, dt$$

$$= \sqrt{2}\left[\frac{t^2}{2} + t\right]_0^1 = \frac{3\sqrt{2}}{2}$$

b) Find the curvature at the highest point of an arch of the cycloid:

$x(t) = a(t - \sin t)$ $\qquad$ $y(t) = a(1 - \cos t)$

(7 marks)



Maximum value of y occurs at $t = \pi$; $y = 2a$

$x(t) = a(t - \sin t)$ $\qquad$ $y(t) = a(1 - \cos t)$

$x'(t) = a(1 - \cos t)$ $\qquad$ $y'(t) = a \sin t$

$x'(t=\pi) = 2a$ $\qquad$ $y'(t=\pi) = 0$

$x''(t) = a \sin t$ $\qquad$ $y''(t) = a \cos t$

$x''(t=\pi) = 0$ $\qquad$ $y''(t=\pi) = -a$

$$K = \frac{|x'y'' - x''y'|}{\left[(x')^2 + (y')^2\right]^{3/2}} = \frac{|2a(-a) - 0(0)|}{\left[(2a)^2 + (0)^2\right]^{3/2}} = \frac{2a^2}{8a^3} = \frac{1}{4a}$$

8) Match the name of the surface to the equations:

a) Ellipsoid
b) Elliptic Paraboloid
c) Hyperbolic Paraboloid
d) Cone
e) Hyperboloid of One Sheet
f) Hyperboloid of Two Sheets

i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of one sheet

ii) $\frac{x^2}{a^2}+\frac{y^2}{c^2}+\frac{z^2}{b^2}=1$ Ellipsoid

iii) $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Hyperbolic paraboloid

iv) $\frac{z^2}{c^2}-\frac{x^2}{b^2}-\frac{y^2}{a^2}=1$ Hyperboloid of two sheets

v) $\frac{z^2}{a^2}=\frac{x^2}{c^2}+\frac{y^2}{b^2}$ Cone

vi) $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Elliptic paraboloid

(6 marks)