I

$$\Omega(N,q) = \frac{(N-1+q)!}{(N-1)!\,q!}$$

$$\simeq \frac{(N+q)!}{N!\,q!}$$

I.1 a) – perfectly acceptable

in class we showed $\Omega(N,q)\Big|_{\substack{N\gg1\\q\gg1\\q\gg N}} \approx \left(\frac{qe}{N}\right)^N$

here we want $\Omega(N,q)\Big|_{\substack{N\gg1\\q\gg1\\N\gg q}}$

but $\Omega(N,q) = \Omega(q,N)$, symmetric under $N \leftrightarrow q$

hence $N \gg q$ limit follows from $q \gg N$ limit by $N \leftrightarrow q$:

$$\Omega(N,q)\Big|_{\substack{N\gg1\\q\gg1\\N\gg q}} \simeq \left(\frac{Ne}{q}\right)^q \quad \boxtimes$$

3.1 b) - brute force

$$\ln \Omega(N,q) = \ln (N+q)! - \ln N! - \ln q!$$

(all N, q large)

$$\simeq (N+q)\ln(N+q) - N\ln N - q\ln q$$

$$= (N+q)\ln\left(N\left(1+\frac{q}{N}\right)\right) - N\ln N - q\ln q$$

$q/N \ll 1$

$$\simeq (N+q)\ln N + (N+q)\frac{q}{N} - N\ln N - q\ln q$$

$$= \cancel{N\ln N} + q\ln N + q + \underbrace{q\frac{q}{N}}_{\ll q} - \cancel{N\ln N} - q\ln q$$

$$= \ln N^q + \ln e^q - \ln q^q$$

$$= \ln\left(\frac{Ne}{q}\right)^q \Rightarrow \Omega\Big|_{\substack{N\gg 1\\ q\gg 1\\ N\gg q}} = \left(\frac{Ne}{q}\right)^q$$

I.2 $S = k \ln \Omega =$

$$= kq \ln \frac{Ne}{q} \qquad \Big| \quad q = \frac{E}{\hbar\omega}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = \frac{\partial}{\partial E}\left[\frac{kE}{\hbar\omega} \ln \frac{Ne\hbar\omega}{E}\right]$$

$$= \frac{k}{\hbar\omega} \ln \frac{Ne\hbar\omega}{E} - \frac{k\cancel{E}}{\hbar\omega} \cdot \frac{1}{\cancel{E}}$$

$$= \frac{k}{\hbar\omega}\left(\ln \frac{N\hbar\omega}{E} + \cancel{\ln e}\right) - \cancel{\frac{k}{\hbar\omega}}$$

$$\frac{1}{T} = \frac{k}{\hbar\omega} \ln \frac{N\hbar\omega}{E}$$

so $\dfrac{\hbar\omega}{kT} = \ln \dfrac{N\hbar\omega}{E}$ or ~~ln~~

$$\ln \frac{E}{N\hbar\omega} = -\frac{\hbar\omega}{kT}$$

eqn: $\frac{\hbar\omega}{kT} = \ln \frac{N\hbar\omega}{E} = \ln \frac{N}{q}$

shows that for $N \gg q$

$\frac{\hbar\omega}{kT} \gg 1$ $\left(\ln \frac{N}{q} \gg 1\right)$

or $kT \ll \hbar\omega$: low-T limit

I.3.

$$\frac{E}{N\hbar\omega} = e^{-\frac{\hbar\omega}{kT}}$$

or $E = \hbar\omega N e^{-\hbar\omega/kT}$ at $\hbar\omega/kT \gg 1$

$$c_N = \frac{1}{N}\left(\frac{\partial E}{\partial T}\right)_N =$$

$$= \hbar\omega \frac{\partial}{\partial T} e^{-\frac{\hbar\omega}{kT}} = \frac{(\hbar\omega)^2}{kT^2} e^{-\frac{\hbar\omega}{kT}}$$

$$c_V = k \left(\frac{\hbar\omega}{kT}\right)^2 e^{-\frac{\hbar\omega}{kT}}$$

as $T \to 0$ $\quad \frac{\hbar\omega}{kT} \to \infty$, let $x = \frac{\hbar\omega}{kT}$

we have $\left.\frac{c_V}{k}\right|_{T\to 0} = \lim\limits_{x\to\infty} x^2 e^{-x} = 0$

e^{-x} exponent $\to 0$ faster as $x \to \infty$ than any power of x

11

$$\Omega\left(N, \frac{q}{2} \pm x\right) = \left(\frac{eN}{\frac{q}{2} \pm x}\right)^{\frac{q}{2} \pm x}$$

$$P(x) = \text{const.}\ \Omega\left(N, \frac{q}{2} + x\right) \Omega_N\left(\frac{q}{2} + x\right) =$$

$$= \text{const}' \left(\frac{1}{\frac{q}{2} + x}\right)^{\frac{q}{2} + x} \left(\frac{1}{\frac{q}{2} - x}\right)^{\frac{q}{2} - x} =$$

$$= c \cdot e^{-\left(\frac{q}{2} + x\right) \ln\left(\frac{q}{2} + x\right) - \left(\frac{q}{2} - x\right) \ln\left(\frac{q}{2} - x\right)} =$$

$$= c \cdot e^{\cancel{-\left(\frac{q}{2} + x\right) \ln \frac{q}{2}} - \left(\frac{q}{2} + x\right) \ln\left(1 + \frac{2x}{q}\right)} \times$$

$$\times e^{\cancel{-\left(\frac{q}{2} - x\right) \ln \frac{q}{2}} - \left(\frac{q}{2} - x\right) \ln\left(1 - \frac{2x}{q}\right)} =$$

$$= c'' e^{-\left(\frac{q}{2} + x\right) \ln\left(1 + \frac{2x}{q}\right) - \left(\frac{q}{2} - x\right) \ln\left(1 - \frac{2x}{q}\right)} =$$

$$= c'' e^{-\frac{q}{2} \ln\left(1 + \frac{2x}{q}\right)\left(1 - \frac{2x}{q}\right) - x \ln\left(1 + \frac{2x}{q}\right) + x \ln\left(1 - \frac{2x}{q}\right)}$$

$$\simeq c'' e^{-\frac{q}{2} \ln\left(1 - \left(\frac{2x}{q}\right)^2\right) - \frac{2x^2}{q} - \frac{2x^2}{q}} \simeq$$

$$\simeq c'' e^{+\cancel{\frac{q}{2}\left(\frac{2x}{q}\right)^2} - \cancel{\frac{2x^2}{q}} - \frac{2x^2}{q}} = c'' e^{-\frac{2x^2}{q}}$$

$$\underline{P(x) = c'' e^{-q \frac{x^2}{q^2} \cdot 2}}$$

/ width of $\left(\frac{x}{q}\right)$-distrib. $\to 0$ as $q \to \infty$

(111)

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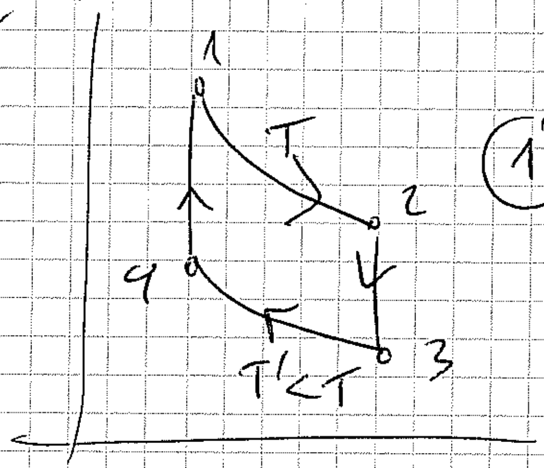

(12) $W_{12}^{\text{by gas}} = kT \ln \frac{V_2}{V_1}$

$Q_{12}^{\text{absorbed}} = kT \ln \frac{V_2}{V_1}$

(isotherm, no change in energy)

(23) no work

$Q_{23}^{\text{absorbed}} = C_V(T' - T)$ (negative : cools gives away heat)

(34) $W_{34}^{\text{by gas}} = -kT' \ln \frac{V_2}{V_1}$

$Q_{34}^{\text{absorbed}} = -kT' \ln \frac{V_2}{V_1}$ (negative : isoth compression, gives heat)

(41) no work

$Q_{41}^{\text{absorbed}} = C_V(T - T')$

benefit: $W_{12}^{\text{by gas}} + W_{34}^{\text{by gas}} =$

benefit $= k(T - T') \ln V_2/V_1$

cost $= Q_{12}^{\text{absorbed}} + Q_{41}^{\text{absorbed}}$

$$= kT \ln \frac{V_2}{V_1} + C_V (T - T')$$

so (efficiency) $= e = \dfrac{k(T - T') \ln V_2/V_1}{kT \ln \frac{V_2}{V_1} + C_V (T - T')} =$

$$= \frac{kT \ln \frac{V_2}{V_1} + C_V (T - T') - kT' \ln \frac{V_2}{V_1} - C_V (T - T')}{kT \ln \frac{V_2}{V_1} + C_V (T - T')}$$

$$e = 1 - \frac{C_V (T - T') + kT' \ln \frac{V_2}{V_1}}{\underbrace{C_V (T - T') + kT \ln \frac{V_2}{V_1}}_{T' < T \text{ so this is } < 1, \; = 1 \text{ if } T = T'}}$$

So, $0 \le e \le 1$

$T = T'$ $T' \Rightarrow 0$ only.

no engine really

$\boxtimes$