UNIVERSITY OF TORONTO
Engineering Science

PHY293, Part A: Waves and Oscillations

Term Test 1, 15 October 2018

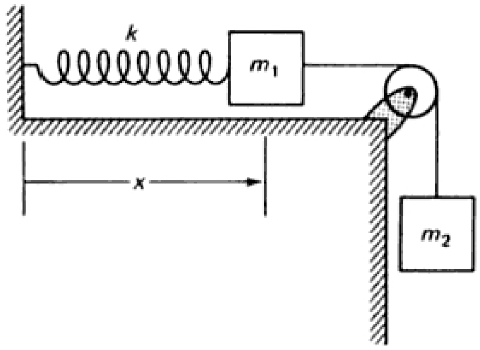
Duration: 60 minutes

- *Write your name, student number and tutorial group on top of **all** examination booklets and test pages used.*
- *Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. Not other aid (notes, textbook, dictionary) is allowed. **Communication devices are strictly forbidden. Turn them off and make sure they are in plain sight to the invigilators.***
- *Answer **all** questions. For each question, the mark breakdown for each subsection is listed in square brackets at the beginning of the question.*
- *There are three questions in this mid-term. Partial credit will be given for partially correct answers. So, please show any intermediate calculations that you do and write down, in a clear fashion, any relevant assumptions you are making along the way.*
- *Do not separate the stapled sheets of the question paper. Hand in the question and rough work sheets together with your exam booklet at the end of the test.*
- *This test has 3 pages, and the total number of marks is 100.*

Some possibly (but not necessarily!) useful equations.

	Amplitude	Velocity	Dissipated Power
Peak freq.	$\omega_{max} = \omega_0\sqrt{1-1/(2Q^2)}$	$\omega_{max} = \omega_0$	$\omega_{max} = \omega_0$
Peak value	$A_{max} = \dfrac{QA_f}{\sqrt{1-1/(4Q^2)}}$	$V_{max} = \omega_0 Q A_f$	$P_{max} = \dfrac{mA_f^2\omega_0^3 Q}{2}$
Miscellaneous	$A(\omega) = \dfrac{\omega_0^2 A_f}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$	$V(\omega) = \omega A(\omega)$	$\overline{P}(\omega) = \dfrac{m\gamma V^2(\omega)}{2}$
	$\tan\delta = \dfrac{\omega\gamma}{\omega_0^2-\omega^2}$		$\approx \dfrac{P_{max}}{1+\frac{4(\omega_0-\omega)^2}{\gamma^2}}\ (Q \gg 1)$

1. [*30 marks*] Consider a mass m_1 attached to an ideal spring (of un-stretched length d) and pulled by constant force F_2, with $F_2 = m_2 g$, as shown in the following figure *(note that the definition of x is different than the one we usually use in class).*



   (a) [*5*] Suppose that the system is in equilibrium when $x = L$. Is $L > d$ or is $L < d$?

   (b) [*10*] If L and d are known, what is the spring constant k?

   (c) [*15*] If the system is at rest in the position $x = L$ and the mass m_2 is suddenly removed (for example, by cutting the string that connects m_1 and m_2). We neglect damping. What is the period and amplitude of the oscillations that m_1 will start to execute? *(use the expression of k from 1(b))*

2. [*40 marks*] The equation of motion of a forced harmonic oscillator with damping is given by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\cos(\omega t). \tag{1}$$

   Assuming a solution $x = A(\omega)\cos(\omega t - \delta)$:

   (a) [*10*] Give the expressions, as functions of mass m, natural frequency ω_0^2, frequency ω, phase shift δ, and amplitude $A(\omega)$, for

      i. the instantaneous kinetic energy $K(t,\omega)$,

      ii. the instantaneous potential energy $U(t,\omega)$,

      iii. the instantaneous total energy $E(t,\omega)$

   of the oscillator.

   (b) [*5*] For what value of ω is the total energy constant with respect to time?

   (c) [*10*] Obtain an expression for the ratio of the time-averaged kinetic energy $\overline{K}$ to the time-averaged $\overline{E}$ of the oscillator in terms of the dimensionless quantity ω_0/ω (*recall that* $\int_0^T \sin^2(\omega t)dt = \int_0^T \cos^2(\omega t)dt = T/2$, *with* $T = 2\pi/\omega$).

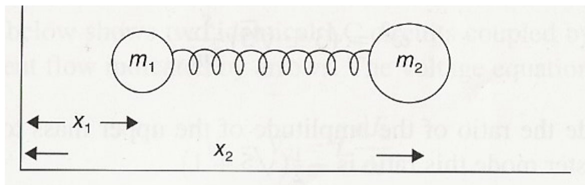
   (d) [*5*] For what value of ω are the average values of the kinetic and potential energies equal?

(e) [*10*] Show that the average total energy of the oscillator varies with angular frequency ω according to

$$\overline{E}(\omega) = \frac{F_0^2\left(\omega_0^2+\omega^2\right)}{4m\left[\left(\omega_0^2-\omega^2\right)^2+\omega^2 b^2/m^2\right]}. \tag{2}$$

Hint: use page 1 of this test, and recall that $\omega_0^2 A_f = F_0/m$ *and* $\gamma = b/m$.

3. [*30 marks*] In the figure below, two masses m_1 and m_2 are coupled by a spring of stiffness k and natural length l (*note again the different definitions of* x_1, x_2).



(a) [*20*] If x is the extension of the spring, show that equations of motion along the x-axis are

$$m_1\ddot{x}_1 = +kx, \qquad \text{and} \tag{3}$$
$$m_2\ddot{x}_2 = -kx, \qquad \text{where} \tag{4}$$
$$x = x_2 - x_1 - l, \tag{5}$$

and combine these to show that the system oscillates with a frequency

$$\omega_0^2 = \frac{k}{\mu}, \qquad \text{where} \qquad \mu = \frac{m_1 m_2}{m_1+m_2} \tag{6}$$

is called the reduced mass.

(b) [*10*] The figure now represents a diatomic molecule as a harmonic oscillator with an effective mass equal to its reduced mass. If a sodium chloride molecule has a natural vibration frequency $\nu_0 = 1.14\times10^{13}$ Hz, show that the interatomic force constant is $k \approx 120\ \mathrm{N\,m^{-1}}$.

- 1 a.m.u. = 1.67×10^{-27} kg,
- Mass of Na atom = 23 a.m.u.,
- Mass of Cl atom = 35 a.m.u.

THIS IS THE END OF THE TEST.