PHY 294 Thermal Physics Midterm 2
March 29, 2022

Name: ____

Student Number: _________________________________

Five questions worth a total of 20 marks

Possibly Useful Equations:

$$C_V \equiv \left.\frac{\partial U}{\partial T}\right|_V \qquad E = \tfrac{1}{2}mv^2 \qquad U = D\frac{kT}{2}$$

$$\binom{N}{n} \equiv \frac{N!}{n! \cdot (N-n)!}$$

Einstein Solid: $\Omega(N,q) = \binom{q+N-1}{q}$ $\qquad \Omega \approx \left(\frac{eq}{N}\right)^N$ $\quad$ *N>>1* and *q>>N*

Stirling's Approximation: $N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$ $\qquad \ln(N!) \approx N\ln(N) - N$

Ideal Gas: $\Omega_N \approx \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{(3N/2)!}(\sqrt{2mU})^{3N} = f(m,N)V^N U^{3N/2}$ $\qquad S = Nk\left[\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$

$$S \equiv k\ln(\Omega)$$

$$T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}_{N,V}$$

$$Nk_B = nR$$

No aids. No notes. No calculator. No phone.

Question 1 [6 marks]:

Consider a system of two Einstein Solids, A and B, each containing N oscillators, sharing a total of q units of energy. Assume the solids are weakly coupled, and that the total energy is fixed. For this question, express your answers in terms of binomial coefficients, $\binom{n}{r}$ where appropriate, so we don't have to bother with large number approximations. Do not assume $N>>1$, and do not assume $q>>N$.

a) How many different macrostates are available to the combined system, if we define a macrostate by the energy in each of the two solids? (1 pt)

Temperature is a measure of the energy in each solid. Solid A can have between 0 and q units of energy, and q is an integer, so **there are q different macrostates.**

Correction: there are q+1 different macrostates. (Forgot to count 0.) So, count either as correct.

b) How many different microstates are available to the combined system? (1 pt)

$$\Omega_{tot}=\binom{q+2N-1}{q}$$ (There are a total of 2N oscillators – N in A and N in B)

c) Assuming the system is in thermal equilibrium, what is the probability of finding all the energy in solid A? (2 pts)

$$\Omega_A=\binom{q+N-1}{q}$$
$$\Omega_B=1$$
$$P=\frac{\Omega_A\Omega_B}{\Omega_{tot}}=\binom{q+N-1}{q}/\binom{q+2N-1}{q}$$

d) Assuming the system is in thermal equilibrium, what is the probability of finding one third of the energy in solid A? (2 pts)

$$\Omega_A=\binom{q/3+N-1}{q/3}$$
$$\Omega_B=\binom{2q/3+N-1}{2q/3}$$
$$P=\frac{\Omega_A\Omega_B}{\Omega_{tot}}=\binom{q/3+N-1}{q/3}\binom{2q/3+N-1}{2q/3}/\binom{q+2N-1}{q}$$

Question 2 [4 marks]:

a) Suppose you flip 800 coins. What is the probability of getting exactly 400 heads and 400 tails? Expand any binomial coefficients, $\binom{n}{r}$, and use Stirling's approximation to get the get this answer into a form that doesn't need a calculator that can work with numbers larger than 10^{99}. (You don't need to evaluate the expression with a calculator, however). (4 pts)

How many ways can we get 400 heads?

N = 800, n = 400.

$$\Omega_{400} = \binom{N}{n} \equiv \frac{N!}{n! \cdot (N-n)!} = \frac{800!}{(400!)^2}$$

Apply $N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N$

$$\Omega_{400} = \binom{800}{400} \equiv \frac{\sqrt{2\pi 800}e^{-800}800^{800}}{(2\pi 400)e^{-800}400^{800}} = \frac{2^{800}}{\sqrt{400\pi}}$$

What is the total multiplicity?

$$\Omega_{tot} = 2^{800}$$

What is the probability?

$P = \frac{\Omega_{400}}{\Omega_{tot}} = \frac{1}{\sqrt{400\pi}}$ (They can also reduce this to $\frac{1}{20\sqrt{\pi}}$ of course)

Question 3 [6 marks]:

Consider a totally contrived system where the multiplicity is given by $\Omega = \epsilon U^4$ at constant volume and constant number of particles.

a) What is the expression for entropy, S, as a function of ϵ and U? (2 pt)

$$S = k_B \ln \Omega = k_B \ln \epsilon + 4 k_B \ln U$$

b) What is the expression temperature, T as a function of ϵ and U? (2 pt)

$$T \equiv \left(\frac{\partial S}{\partial U} \right)^{-1}_{N,V} = U/4k_B$$

c) What is the expression for the heat capacity at constant volume, as a function of ϵ and T ? (2 pt)

$$U = 4 k_B T \qquad C_V \equiv \left. \frac{\partial U}{\partial T} \right|_V = 4 k_B$$

Question 4 [2 marks]:

According to kinetic theory and the equipartition theorem, what is the expression for the average speed of a molecule of an ideal gas. (2 pts)

$$E = \frac{3k_BT}{2} = \frac{mV^2}{2}$$

$$\frac{3k_B}{m} = V^2 \quad \rightarrow \quad V = \sqrt{\frac{3k_B}{m}}$$

Question 5 [2 marks]:

According to the equipartition theorem, what is the expression for the heat capacity per molecule of a hot diatomic gas with 2 active rotational degrees of freedom and 2 active vibrational degrees of freedom? (2 pts)

There are 7 degrees of freedom (3 translation + 2 + 2):

$$U = D\frac{kT}{2} = \frac{7Nk_BT}{2}$$

$$C_V \equiv \left.\frac{\partial U}{\partial T}\right|_V = \frac{7Nk_b}{2}$$

Per molecule (N=1): $C_V = \frac{7k_b}{2}$