First name (please write as legibly as possible within the boxes)

Last name

Student ID number

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

December 19, 2019

Instructor – Professor W.R. Cluett

Closed book.

All questions are of equal value.

Permitted calculators (with ANY suffixes):

- Sharp EL-520
- Casio FX-991

This test contains 20 pages including this page and the cover page, printed two-sided. Do not tear any pages from this test. Present complete solutions in the space provided.

Given information:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\,\vec{u} = \frac{\vec{u}\cdot\vec{d}}{||\vec{d}\,||^2}\vec{d}$$

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The normal system of equations corresponding to $A\vec{x} = \vec{b}$ is given by:

$$A^T A\vec{x}_{LS} = A^T\vec{b}$$

Euler's method for solving a first order differential equation $y'(t) = f(t, y(t))$ is given by:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

Q1:

a) One result that we did not discuss in class is that for square matrices, an inverse on one side is automatically an inverse on the other side. In other words, if $CD = I$ then automatically $DC = I$ and D is C^{-1}. Using this result, if matrix B is the inverse of matrix A^2, show that $AB = BA$.

b) If the product $M = ABC$ of three square matrices is invertible, then A, B and C are invertible. Find a formula for B^{-1} that involves M^{-1}, A and C.

c) There are sixteen different 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Q2:

Consider the equation of a plane in R^3:

$$ax + by + cz = d$$

For the purposes of this question, this equation can be rearranged as follows:

$$z = \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y = d' + a'x + b'y \text{ assuming } c \neq 0$$

a) Find the plane in R^3 that gives the least squares fit in the z-direction at the four points corresponding to the corners of a square in the x-y plane, namely:

$$\{(1,0,0), (0,1,1), (-1,0,3), (0,-1,4)\}$$

b) Show that at the centre of the square (0,0) the value of z at this point on the plane found in part (a) is equal to the average of the z values at the four points corresponding to the corners of the square.

Q3:
Consider matrix A:

$$A = \begin{bmatrix} +4 & -1 & -1 & -1 \\ -1 & +4 & -1 & -1 \\ -1 & -1 & +4 & -1 \\ -1 & -1 & -1 & +4 \end{bmatrix}$$

The inverse of matrix A is known to be of the form:

$$A^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}$$

a) Write two algebraic equations in terms of the unknowns a and b that must hold true for $AA^{-1} = I$.

b) Solve for the unknowns a and b in part (a).

c) By determining its rank, show that matrix B is **NOT** invertible, where:

$$B = \begin{bmatrix} +3 & -1 & -1 & -1 \\ -1 & +3 & -1 & -1 \\ -1 & -1 & +3 & -1 \\ -1 & -1 & -1 & +3 \end{bmatrix}$$

Q4:

Compute L and U for the matrix A given by

$$A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}$$

To receive full marks for this part you must show the steps of your decomposition.

Near the end of your decomposition, you will have one entry in L that is undefined because the corresponding entry of A is equal to zero. Derive the value for this undefined entry of L and then show that $A = LU$.

Q5:

a) Determine the elementary row operations that reduce matrix A to matrix B, where:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

For instance, R2-R1 reduces matrix A to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

and so on.

b) By making use of the elementary matrices corresponding to the above elementary row operations, find the matrix M such that $MA = B$. To receive full marks for this part, you must show that $MA = B$.

Q6:

a) Consider the matrix A where:

$$A = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

One of the eigenvalues (λ) of this matrix A is equal to 0. Find a corresponding eigenvector $\vec{x}$. To receive full marks for this part, you must show that $A\vec{x} = \lambda\vec{x}$.

b) The matrix A in part (a) has another eigenvalue equal to 1 and this eigenvalue has an eigenvector that is a linear combination of two non-parallel vectors. Find an eigenvector $\vec{x}$ associated with $\lambda = 1$ that has no zero components. To receive full marks for this part, you must show that $A\vec{x} = \lambda\vec{x}$.