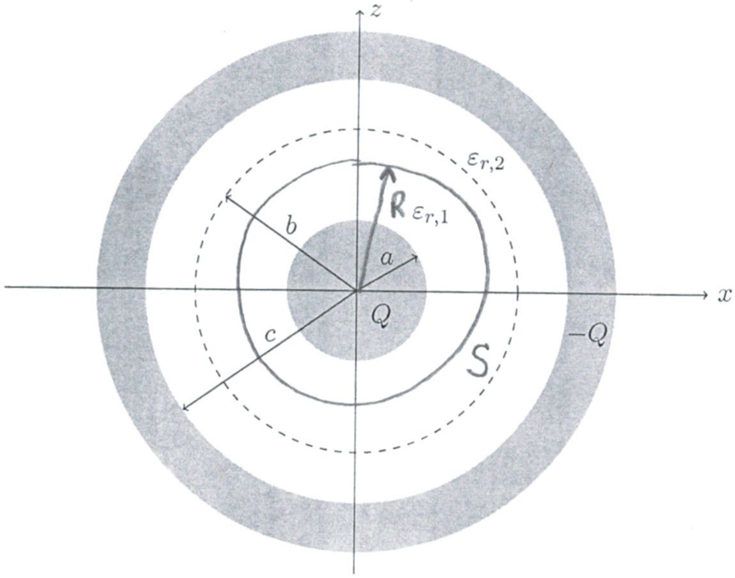
Question 1



1. Consider the **spherical** capacitor shown in the figure above. The capacitor consists of two perfect conductors of radii a and c, separated by two different layers of perfect dielectric. The dielectric in the region $R \in [a, b]$ has relative permittivity $\varepsilon_{r,1}$. The dielectric in the region $R \in [b, c]$ has relative permittivity $\varepsilon_{r,2}$. The charge on the inner conductor is Q. The charge on the outer conductor is $-Q$.

   Use Gauss' law to find the electric field $\mathbf{E}_1(R)$ in the first dielectric and the electric field $\mathbf{E}_2(R)$ in the second dielectric [14 points].

We use generalized Gauss' law

$$\int_S \bar{D} \cdot \bar{dS} = Q$$

[2pt]

Because of spherical symmetry $\bar{D} = D(R)\,\bar{a}_R$ [2pt]

Gaussian surface: spherical surface [2pt]

$$\int_S D(R)\, \bar{a}_R \cdot \bar{a}_R \, dS_R = Q$$ [2pt]

$D(R)\ 4\pi R^2 = Q$

$$\bar{D} = \frac{Q}{4\pi R^2} \bar{a}_R$$ [2pt]

$$\bar{E}_1(R) = \frac{Q}{4\pi \varepsilon_0 \varepsilon_{r,1} R^2} \bar{a}_R$$ [2pt]

$$\bar{E}_2(R) = \frac{Q}{4\pi \varepsilon_0 \varepsilon_{r,2} R^2} \bar{a}_R$$ [2pt]

$\mathbf{E}_1(R) =$
$\mathbf{E}_2(R) =$

2. Find the capacitance C between the two conductors [6 points].

Voltage from "$-Q$" conductor to "$+Q$" conductor

$$V = -\int_{R=c}^{R=a} \bar{E} \cdot \bar{dl} = \int_{R=a}^{R=c} \bar{E} \cdot \bar{dl} = \int_{R=a}^{b} \bar{E}_1 \cdot \bar{dl} + \int_{R=b}^{c} \bar{E}_2 \cdot \bar{dl} =$$

$$= \int_{R=a}^{b} \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,1} (R')^2} dR' + \int_{R=b}^{c} \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,2} (R')^2} dR' =$$

$$= \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,1}} \left[-\frac{1}{R'} \right]_a^b + \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,2}} \left[-\frac{1}{R'} \right]_b^c = \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,1}} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{Q}{4\pi\varepsilon_0 \varepsilon_{r,2}} \left(\frac{1}{b} - \right.$$

[4pt]

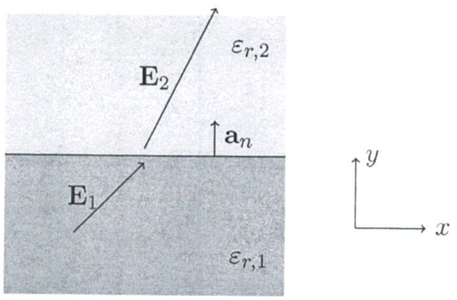
$$C = \frac{Q}{V} = \left[\frac{1}{4\pi\varepsilon_0 \varepsilon_{r,1}} \left(\frac{1}{a} - \frac{1}{b} \right) + \frac{1}{4\pi\varepsilon_0 \varepsilon_{r,2}} \left(\frac{1}{b} - \frac{1}{c} \right) \right]^{-1}$$

[2pt]

$$\boxed{C = 4\pi\varepsilon_0 \left[\frac{b-a}{\varepsilon_{r,1} ab} + \frac{c-b}{\varepsilon_{r,2} bc} \right]^{-1}}$$

Question 2.1

The figure below depicts the electric field just below and just above the interface between two different materials, with permittivity $\varepsilon_{r,1} = 5$ and $\varepsilon_{r,2} = 3$. At the interface, there is some free charge, with density $\rho_s = -3\varepsilon_0$ C/m^2. Below the interface, the electric field is $\mathbf{E}_1 = 3\mathbf{a}_x + 3\mathbf{a}_y$ V/m.



Find the electric field vector $\mathbf{E}_2$ just above the interface [4 points].

$E_{2,t} = E_{1,t} = 3$ V/m } [1pt]

$D_{2,n} - D_{1,n} = \rho_s$ [1pt]

$D_{2,n} = \varepsilon_0 5 \cdot 3 + (-3\varepsilon_0) = 12\,\varepsilon_0$ C/m^2 [1pt]

$E_{2,n} = \dfrac{12\,\cancel{\varepsilon_0}}{3\,\cancel{\varepsilon_0}} = 4$ V/m^2 [1pt]

$\bar{E}_2 = 3\,\bar{a}_x + 4\,\bar{a}_y$ V/m^2

Question 2.2

We have an electrostatic system. Initially, all potentials are defined with reference to infinity. What happens when the origin is adopted as a new reference point for potentials?

1. Both potential V and electric field $\mathbf{E}$ remain the same;
2. Electric field $\mathbf{E}$ changes, potential V remains the same;
3. Potential V changes, electric field $\mathbf{E}$ remains the same; Right answer: 2pt
4. Both potential V and electric field $\mathbf{E}$ change.

Briefly justify your answer [4 points].

All potentials will change by a constant

since

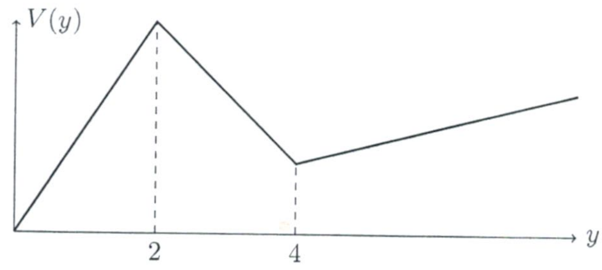
$$V(P) - V(0) = -\int_0^P \bar{E}\cdot d\bar{l} = \underbrace{-\int_\infty^P \bar{E}\cdot d\bar{l}}_{\text{potential with ref to } \infty} - \underbrace{\int_0^\infty \bar{E}\cdot d\bar{l}}_{\text{constant}}$$

Explaination 2pt

$\bar{E}$ remains the same since

$$\bar{E} = -\nabla V$$

Question 2.3

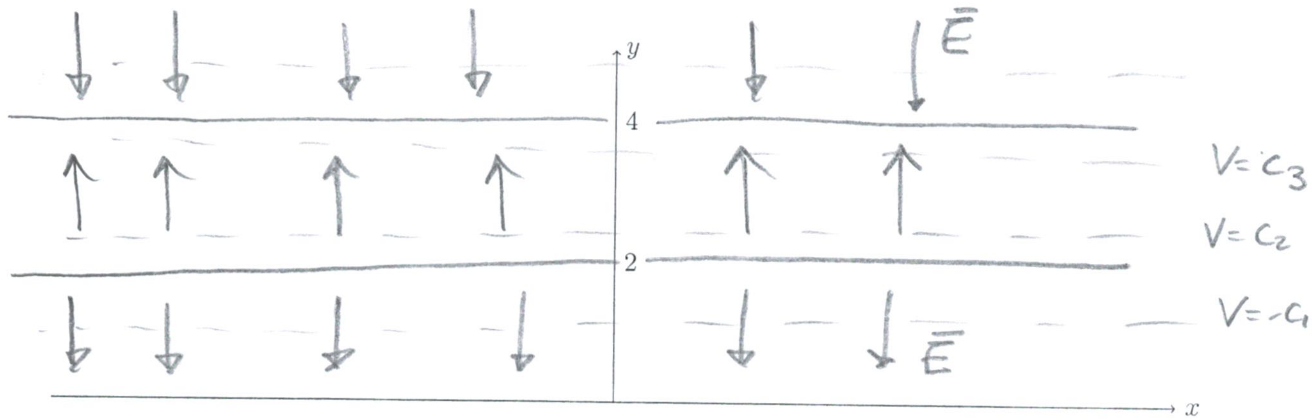


The graph above depicts the potential V in the region $y > 0$ as a function of position. In the graph below, sketch (for $y > 0$):

- some equipotential lines [2 points];
- the direction of the $\mathbf{E}$ field in the various regions [2 points].

Briefly justify your answer.

Graph:
1pt for $\bar{E}$
1pt for V

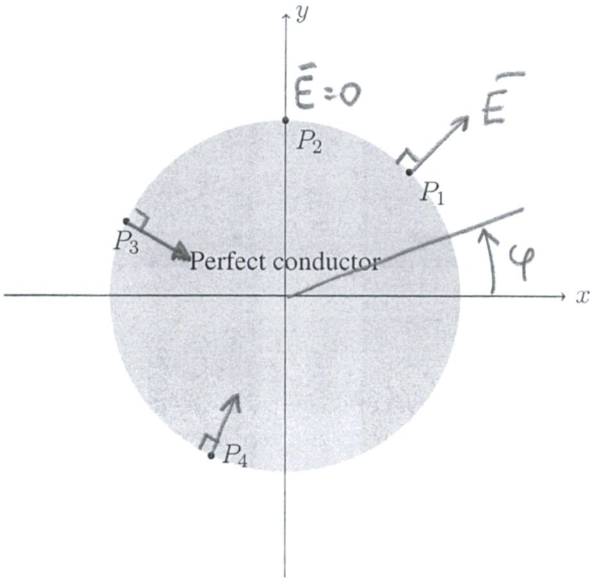


[1pt] equipotential lines are horizontal, since for $y=$ const $\Rightarrow V=$ const

[1pt] $\bar{E} = -\nabla V \Rightarrow \bar{E} \perp$ to equipotential lines and points in the direction of decreasing V

Question 2.4

We have a perfect conductor of cylindrical shape, shown in the figure below. The conductor is surrounded by free space. On the conductor's surface, the density of free charge is $\rho_s = 4\cos(\varphi)\ \mathrm{C/m^2}$. Sketch the direction of the electric field vector $\mathbf{E}$ at the four points shown in the figure, which are on the surface of the conductor. Briefly justify your answer [4 points].



On PEC surface

$E_t = 0$

$E_n = \dfrac{\rho_s}{\varepsilon_0}$

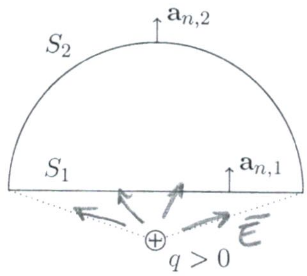
[1pt] for each point

at P_1, $\rho_s > 0 \Rightarrow \bar{E}$ points outward

at P_2, $\rho_s = 0 \Rightarrow \bar{E} = 0$

at P_3, $\rho_s < 0$
at P_4, $\rho_s < 0$ $\Big\} \Rightarrow \bar{E}$ points inward

Question 2.5



Consider the half sphere shown in the figure above, which is made by:

- the circular base S_1, with normal $\mathbf{a}_{n,1}$;
- the surface of the "dome" S_2, with normal $\mathbf{a}_{n,2}$.

A positive point charge q is located below the half-sphere, as shown. Let Φ_1 and Φ_2 be the flux of the electric field through the surfaces S_1 and S_2

$$\Phi_1 = \int_{S_1} \mathbf{E} \cdot \mathbf{a}_{n,1} dS_1 \qquad \Phi_2 = \int_{S_2} \mathbf{E} \cdot \mathbf{a}_{n,2} dS_2$$

Which statement is correct? Briefly justify your answer [4 points].

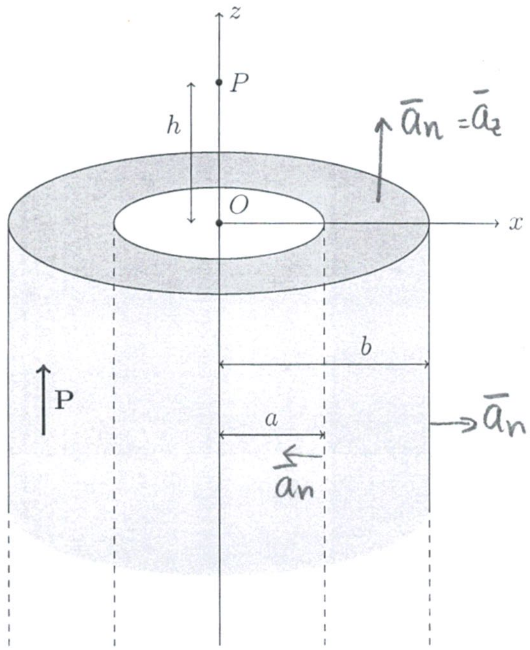
1. Φ_1 and Φ_2 are both positive, and $\Phi_1 = \Phi_2$; [1pt] for right answer
2. Φ_1 and Φ_2 are both positive, and $\Phi_1 > \Phi_2$;
3. Φ_1 and Φ_2 are both positive, and $\Phi_1 < \Phi_2$;
4. $\Phi_1 < 0$ and $\Phi_2 > 0$;
5. $\Phi_1 > 0$ and $\Phi_2 < 0$;

$\bar{E} \cdot \bar{a}_{n,1} > 0$ on any point of S_1 $\Rightarrow \Phi_1 > 0$ [1pt]

$\bar{E} \cdot \bar{a}_{n,2} > 0$ " " " " S_2 $\Rightarrow \Phi_2 > 0$ [1pt]

From Gauss' law, $\Phi_2 - \Phi_1 = 0 \Rightarrow \Phi_1 = \Phi_2$ [1pt]

Question 3



A hollow cylinder of radii a and b extends from $z = 0$ to $z = -\infty$. The cylinder is made by a perfect dielectric material. The cylinder is uniformly polarized with polarization vector $\mathbf{P} = P_0\mathbf{a}_z$.

1. Find the volume density of polarization charge $\rho_{p,v}$ in the hollow cylinder [1 points].

$\rho_{p,v} = -\nabla \cdot \bar{P} = 0$ since $\bar{P}$ constant [1 pt]

2. Find the surface density of polarization charge $\rho_{p,S}$ on the three surfaces of the cylinder (inner lateral surface, outer lateral surface, top surface) [3 points].

$\rho_{p,s} = \bar{P} \cdot \bar{a}_n$

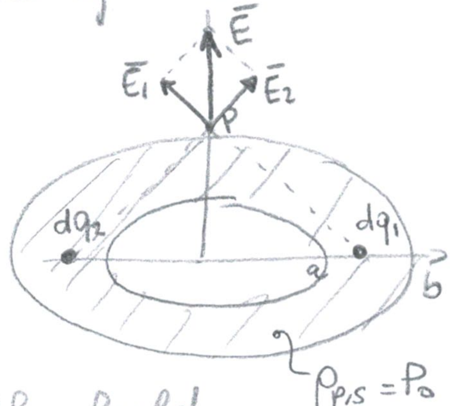
on the inner lateral surface, $\bar{P} \perp \bar{a}_n \Rightarrow \rho_{p,s} = 0$ [1 pt]

on the outer lateral surface, $\bar{P} \perp \bar{a}_n \Rightarrow \rho_{p,s} = 0$ [1 pt]

on the top surface $\rho_{p,s} = \bar{P} \cdot \bar{a}_z = P_0\, \bar{a}_z \cdot \bar{a}_z = P_0$ [1 pt]

3. Find the electric field **E** at a point $P(x = 0, y = 0, z = h > 0)$ on the positive z axis. Hint: the electric field has only a z-component. Justify why and use it to expedite your calculations. [16 points].

The electric field $\bar{E}$ is produced by the polarization charge on the top surface of the hollow cylinder [2pt]

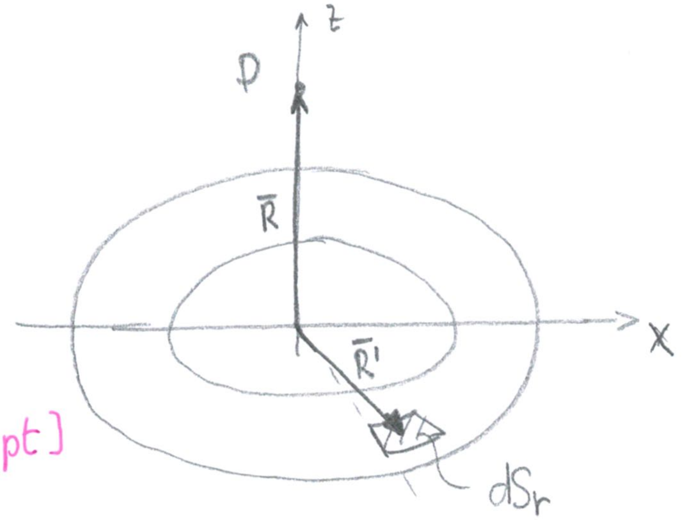


$\bar{E}$ can be found by superposition

Because of the cylindrical symmetry, the $\bar{a}_r$ component of the field cancel

$\Rightarrow \bar{E} \parallel \bar{a}_z$

Justification of $\bar{E} \parallel \bar{a}_z$: [2pt]



$$\bar{E} = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\overbrace{\rho_{P,S}\, dS}^{dq'}\,(\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3}$$

- use cylindrical coordinates [1pt]

  Surface S: $r' \in [a,b]$, $\varphi' \in [0, 2\pi]$ $z' = 0$

- $dq' = P_0\, r'\, d\varphi'\, dr'$ [1pt]

- $\bar{R} = h\, \bar{a}_z$ [1pt]

$$\bar{R}' = r'\bar{a}_{r'} = r'(\cos\varphi'\,\bar{a}_x + \sin\varphi'\,\bar{a}_y) \quad \text{[1pt]}$$

$$\bar{R} - \bar{R}' = h\,\bar{a}_z - r'\cos\varphi'\,\bar{a}_x - r'\sin\varphi'\,\bar{a}_y \quad \text{[1pt]}$$

$$|\bar{R} - \bar{R}'| = \sqrt{h^2 + (r')^2} \quad \text{[1pt]}$$

calculations: [4pt]

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int_{r'=a}^{b}\int_{\varphi'=0}^{2\pi}\frac{\rho_0\, r'd\varphi'\,dr'}{[(r')^2+h^2]^{3/2}}\Big[h\,\bar{a}_z \underbrace{-r'\cos\varphi'\,\bar{a}_x}_{\text{will integrate to 0}} \underbrace{-r'\sin\varphi'\,\bar{a}_y}_{\text{will integrate to 0}}\Big]$$

$$= \frac{\rho_0 h}{4\cancel{\pi}\varepsilon_0}\,2\cancel{\pi}\,\bar{a}_z\int_{r'=a}^{b}\frac{r'\,dr'}{[(r')^2+h^2]^{3/2}} = \frac{\rho_0 h}{4\varepsilon_0}\,\bar{a}_z\int_{r'=a}^{b}\frac{2r'}{[(r')^2+h^2]^{3/2}}\,dr' =$$

$$= \frac{\rho_0 h}{\underset{2}{\cancel{4}}\varepsilon_0}\,\bar{a}_z\left[-\frac{\cancel{2}}{\sqrt{(r')^2+h^2}}\right]_a^b = \frac{\rho_0 h}{2\varepsilon_0}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{b^2+h^2}}\right]\bar{a}_z$$

final answer: [2pt]