TUTORIAL #2 SOLUTIONS

Q 1:

3 POINTS IN THE PLANE ARE:

$P_1(a,0,0)$

$P_2(0,b,0)$

$P_3(0,0,c)$

2 VECTORS IN THE PLANE ARE:

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -a \\ b \\ 0 \end{bmatrix} \quad \overrightarrow{P_1P_3} = \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix}$$

CROSS PRODUCT OF THESE TWO VECTORS WILL PRODUCE A NORMAL TO THE PLANE:

$$\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{bmatrix} -a \\ b \\ 0 \end{bmatrix} \times \begin{bmatrix} -a \\ 0 \\ c \end{bmatrix} = \begin{bmatrix} bc \\ ac \\ ab \end{bmatrix}$$

SCALAR EQUATION OF THE PLANE:

$$bcx + acy + abz + d = 0$$

SUBSTITUTE ONE POINT TO FIND d:

$P_1 \rightarrow \quad bca + ac(0) + ab(0) + d = 0$

$\therefore \quad d = -abc$

$\therefore \quad bcx + acy + abz = abc$

ASSUMING $a \neq 0,\ b \neq 0,\ c \neq 0$

$\div\, abc \qquad \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = \underline{1}$

Q2:

ANY LINE THAT IS PARALLEL TO TWO PLANES IS ORTHOGONAL TO THEIR TWO NORMALS.

PLANE 1: $2x+y-4z=0$ $\qquad \vec{n}_1=\begin{bmatrix}2\\1\\-4\end{bmatrix}$

PLANE 2: $-x+2y+3z+1=0$ $\qquad \vec{n}_2=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

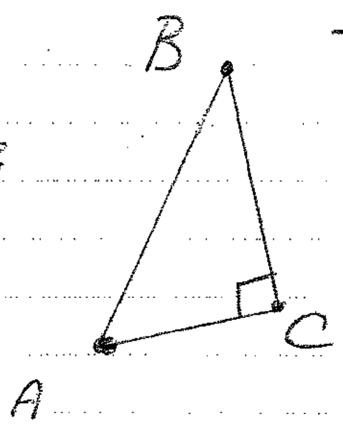
$$\vec{n}_1\times\vec{n}_2=\begin{bmatrix}2\\1\\-4\end{bmatrix}\times\begin{bmatrix}-1\\2\\3\end{bmatrix}=\begin{bmatrix}11\\-2\\5\end{bmatrix}$$

$\therefore$ A DIRECTION VECTOR FOR THE LINE IS $\begin{bmatrix}11\\-2\\5\end{bmatrix}$.

VECTOR EQUATION OF THE LINE:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\5\\0\end{bmatrix}+t\begin{bmatrix}11\\-2\\5\end{bmatrix}$$ t IS A SCALAR

Q3:



GIVEN $\vec{AB} = t\begin{bmatrix}1\\-1\\1\end{bmatrix}$ $\vec{AC} = s\begin{bmatrix}2\\0\\-1\end{bmatrix}$ S AND t SCALARS

BY VECTOR ADDITION: $\vec{AC} + \vec{CB} = \vec{AB}$

$$\vec{CB} = \vec{AB} - \vec{AC} = t\begin{bmatrix}1\\-1\\1\end{bmatrix} - s\begin{bmatrix}2\\0\\-1\end{bmatrix}$$

$$\vec{CB} = \begin{bmatrix}t-2s\\-t\\t+s\end{bmatrix}$$

WE ALSO KNOW THAT $\vec{CB}$ IS ORTHOGONAL TO $\vec{AC}$.

$$\therefore \begin{bmatrix}t-2s\\-t\\t+s\end{bmatrix} \cdot \begin{bmatrix}2\\0\\-1\end{bmatrix} = 0$$

$$2(t-2s) - (t+s) = 0$$

$$2t - 4s - t - s = 0$$

$$t = 5s$$

$$\therefore \quad \overrightarrow{CB} = \begin{bmatrix} 3s \\ -5s \\ 6s \end{bmatrix} = s\begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$$

$\therefore$ THE EQUATION OF THE LINE THROUGH B AND C IS:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$$

S IS A SCALAR

Q4:

A VECTOR PARALLEL TO THE yz-PLANE IS ORTHOGONAL TO A NORMAL OF THIS PLANE.

A NORMAL TO THE yz-PLANE IS $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

NOW WE NEED TO FIND A VECTOR ORTHOGONAL TO BOTH $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ AND $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

USE CROSS PRODUCT OF THESE TWO VECTORS:

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

NOW WE WANT TO FIND ALL UNIT VECTORS PARALLEL TO $\begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$

$$\text{UNIT VECTOR} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$$\text{ANOTHER VALID UNIT VECTOR} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Q5:

AREA OF BASE = AREA OF TRIANGLE DEFINED BY $\vec{b}$ AND $\vec{c}$

$= \frac{1}{2}$ (AREA OF PARALLELOGRAM DEFINED BY $\vec{b}$ AND $\vec{c}$)

$$= \frac{1}{2} \|\vec{b}\times\vec{c}\|$$

HEIGHT CAN BE OBTAINED BY FINDING PROJECTION OF $\vec{a}$ ON A VECTOR ORTHOGONAL TO THE BASE, e.g. $\vec{b}\times\vec{c}$

$$\text{HEIGHT} = \left|\operatorname{proj}_{\vec{b}\times\vec{c}} \vec{a}\right| = \left|\frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\|\vec{b}\times\vec{c}\|^2}\, \vec{b}\times\vec{c}\right|$$

$$= \frac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{\|\vec{b}\times\vec{c}\|}$$

$$\therefore \text{VOLUME} = \frac{1}{3}\cdot\frac{1}{2}\cancel{\|\vec{b}\times\vec{c}\|}\, \frac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{\cancel{\|\vec{b}\times\vec{c}\|}}$$

$$= \frac{1}{6}\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

Q6:

(DIFFERENT APPROACH THAN THE ONE PRESENTED IN CLASS)

FIND A NORMAL TO THE PLANE: $\begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

FIND THE LINE THROUGH P IN THE DIRECTION OF THE NORMAL:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} \qquad t \text{ SCALAR}$$

FIND WHERE THE LINE INTERSECTS THE PLANE:

$$5(-3+5t) - 3(1-3t) + (3+t) = -4$$

$$35t = 11$$

$$t = \frac{11}{35}$$

$$\therefore \quad x = -3+5t = -\frac{50}{35}$$

$$y = 1-3t = \frac{2}{35}$$

$$z = 3+t = \frac{116}{35}$$

FIND THE DISTANCE BETWEEN THE TWO POINTS:

$$\text{DISTANCE} = \left(\left(-3+\frac{50}{35}\right)^2 + \left(1-\frac{2}{35}\right)^2 + \left(3-\frac{116}{35}\right)^2\right)^{\frac{1}{2}}$$

$$= \frac{11}{\sqrt{35}} \approx 1.859$$

Q7:

THE PLANES ARE PARALLEL IF THEIR NORMALS ARE PARALLEL:

$$(-2)\begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ -12 \end{bmatrix} \quad \therefore \text{ PLANES ARE } \parallel\text{'l.}$$

FIND A VECTOR $\vec{v}$ THAT CONNECTS ONE POINT ON EACH PLANE:

$$3x - y + 6z = 7 \qquad -6x + 2y - 12z = 1$$

TAKE $x = 0$, $y = 0$, $z = 7/6$

TAKE $x = 0$, $y = 0$, $z = -1/12$

$$\vec{v} = \begin{bmatrix} 0-0 \\ 0-0 \\ 7/6 - \left(\frac{-1}{12}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5/4 \end{bmatrix}$$

PROJECT $\vec{v}$ ON THE NORMAL:

TAKE $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}$

MAGNITUDE OF THE PROJECTION:

$$\left| \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{30/4}{\sqrt{3^2+(-1)^2+6^2}} = \frac{7.5}{\sqrt{46}} \approx 1.106$$

GIVES THE DISTANCE BETWEEN THE PLANES.

Q8:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix} \qquad t \text{ SCALAR}$$

Q9:

TO FIND A NORMAL TO THE PLANE, FIND THE CROSS PRODUCT OF THE TWO VECTORS IN THE PLANE:

$$\vec{n} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

PLANE:

$$3x + 6y + 2z + d = 0$$

USE POINT IN THE PLANE TO FIND d:

$$3(-1) + 6(5) + 2(6) + d = 0$$

$$d = -39$$

Q10:

THE EQUATION $ax + by + 0z = 0$ REPRESENTS A PLANE THAT PASSES THROUGH THE LINE $ax + by = 0$ IN THE X-Y PLANE AND IS ORTHOGONAL TO THE X-Y PLANE WITH A NORMAL VECTOR HAVING A ZERO Z-COMPONENT.

Q11: ROW PICTURE CONTAINS 3 PLANES:

→ ONE WITH NORMAL VECTOR $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ THAT PASSES THROUGH $x=2$.

→ ANOTHER WITH NORMAL VECTOR $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ THAT PASSES THROUGH $y=3$

→ A THIRD WITH NORMAL VECTOR $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ THAT PASSES THROUGH $z=4$.

COLUMN PICTURE CONTAINS THE LINEAR COMBINATION OF 3 COLUMN VECTORS THAT ADD UP TO $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$:

$$x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

THE SOLUTION TO THE ROW PICTURE CORRESPONDS TO THE POINT $(2,3,4)$ WHERE THE 3 PLANES INTERSECT.

THE SOLUTION TO THE COLUMN PICTURE CORRESPONDS TO THE ONLY VALUES OF (x,y,z) THAT SATISFY THIS VECTOR EQUATION, NAMELY $(2,3,4)$.

THE SOLUTION TO BOTH PICTURES IS A POINT IN R^3, $(2,3,4)$ WHICH CAN ALSO BE VIEWED AS THE VECTOR $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

Q12:

$$x\begin{bmatrix}1\\1\\2\end{bmatrix} + y\begin{bmatrix}1\\2\\3\end{bmatrix} + z\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}2\\3\\5\end{bmatrix}$$

TWO LINEAR COMBINATIONS THAT WORK:

$x=1,\ y=1,\ z=0$

$x=0\ \ y=1,\ z=1$

$$x\begin{bmatrix}1\\1\\2\end{bmatrix} + y\begin{bmatrix}1\\2\\3\end{bmatrix} + z\begin{bmatrix}1\\1\\2\end{bmatrix} = \begin{bmatrix}4\\6\\c\end{bmatrix}$$

$x+y+z=4 \Rightarrow x+z=4-y$

$x+2y+z=6 \Rightarrow x+z=6-2y$

$2x+3y+2z=c$

$\therefore 4-y=6-2y$

$\therefore y=2$

$\therefore 2(x+z)+3y=c$

$2(4-y)+3y=c$

$8-2y+3y=c$

$8+y=c$

$\therefore c=8+2=10$

LOOKING AT THE ORIGINAL VECTOR EQUATION, ONE CAN SEE THAT WITH $Y=1$, $X+Z=1$. THIS WOULD SUGGEST A SOLUTION THAT CORRESPONDS TO A LINE IN R^3.

LET $P_1 = (1,1,0)$
$P_2 = (0,1,1)$

$$\vec{d} = \overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$