AER210F

VECTOR CALCULUS AND FLUID MECHANICS

Quiz 1

3 October 2016 9:15 am - 10:20 am

Closed Book, No aid sheets, No calculators, No cellphones

Instructor: D. A. Brown

Last Name: Brown

Given Name: David A.

Student #:

Tutorial/TA: Solutions.

FOR MARKER USE ONLY		
Question	Marks	Earned
1	12	
2	4	
3	8	
4	8	
5	8	
6	10	
7	8	
8	2	
TOTAL	60	/55

Note: The following integrals may be useful:

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

1. (a) Evaluate $\displaystyle\int_1^2 \int_0^1 \frac{3y^2}{x\,(y^3+1)}\,dy\,dx.$
(3 marks)

$$\int_1^2 \frac{1}{x}\,dx \int_0^1 \frac{3y^2}{y^3+1}\,dy = \left(\ln x\Big|_{x=1}^{2}\right)\left(\ln(y^3+1)\Big|_{y=0}^{1}\right) = (\ln 2 - 0)(\ln 2 - 0)$$

$$=(\ln 2)^2$$

(b) Write $\displaystyle\int_0^1 \int_0^{2x} y\,dy\,dx + \int_1^3 \int_0^{-x+3} y\,dy\,dx$ as a single integral by changing the order of integration and evaluate. Provide a labeled sketch of the region.
(5 marks)

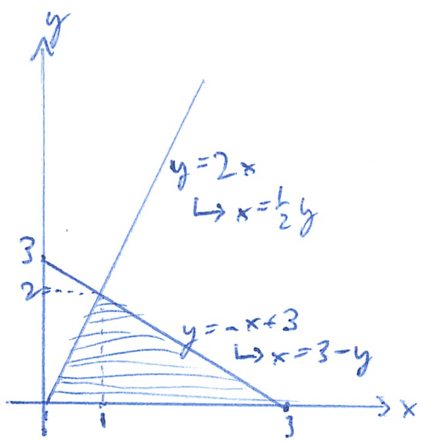
Intersection:

$2x = -x+3$

$\Rightarrow 3x = 3$

$\Rightarrow x = 1$

$\therefore\ y = -1+3 = 2$



$$\int_0^2 \int_{\frac12 y}^{3-y} y\,dx\,dy = \int_0^2 (3-y)y - \tfrac12 y\cdot y\,dy$$

$$=\int_0^2 3y - y^2 - \tfrac12 y^2\,dy$$

$$=\int_0^2 3y - \tfrac32 y^2\,dy = \tfrac32 y^2 - \tfrac12 y^3\Big|_{y=0}^{2}$$

$$=\tfrac32\cdot 2^2 - \tfrac12\cdot 2^3 = 3\cdot 2 - 2^2 = 6-4$$

$$=2$$

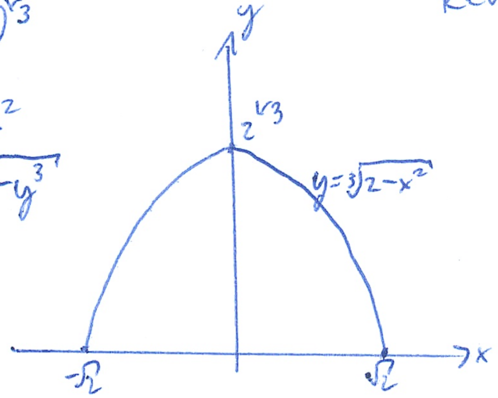
(c) Evaluate $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{0}^{\sqrt[3]{2-x^2}} \sqrt{2-y^3}\, dy\, dx$.

(4 marks)

$y = (2-x^2)^{1/3}$

$\Rightarrow y^3 = 2 - x^2$

$\Rightarrow x = \pm\sqrt{2-y^3}$



Reverse the order of integration.

$$\int_0^{2^{1/3}} \int_{-\sqrt{2-y^3}}^{\sqrt{2-y^3}} \sqrt{2-y^3}\, dx\, dy = \int_0^{2^{1/3}} (2-y^3)+(2-y^3)\, dy$$

$$= 2\int_0^{2^{1/3}} 2-y^3\, dy = 2\left(2y - \tfrac{1}{4}y^4\right)\Big|_{y=0}^{2^{1/3}}$$

$$= 2\left(2\cdot 2^{1/3} - \tfrac{1}{4}2^{4/3}\right) = \left(4\cdot 2^{1/3} - 2^{1/3}\right)$$

$$= 2^{1/3}(4-1) = 3\cdot 2^{1/3}$$

2. Evaluate $f(1)$, where $f(x) = \dfrac{d^2}{dx^2}F(x)$ and $F(x) = \displaystyle\int_x^{x^2+1} \frac{e^y}{y}\, dy$.

(4 marks)

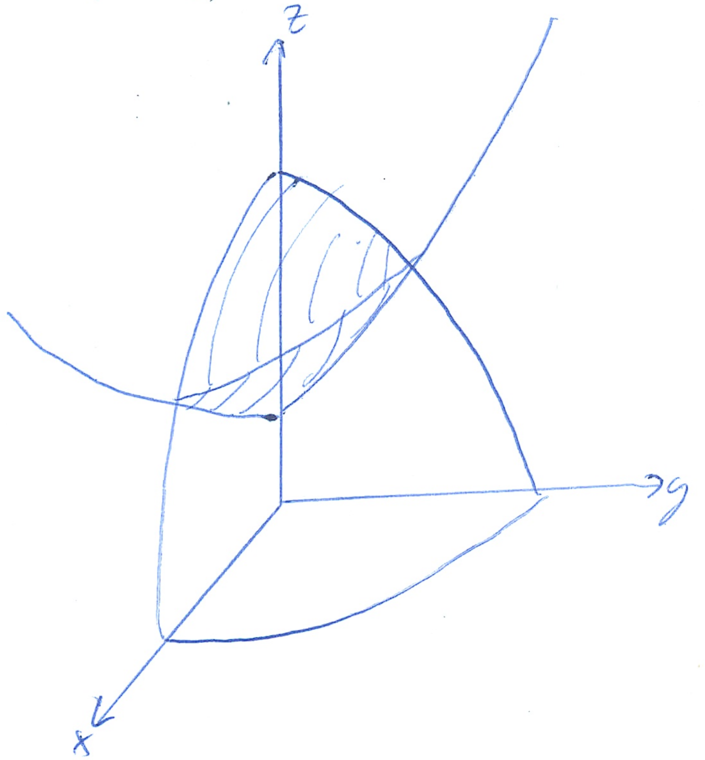
$$f(x) = \frac{d^2}{dx^2}\int_x^{x^2+1} \frac{e^y}{y}\, dy = \frac{d}{dx}\left(2x\frac{e^{x^2+1}}{x^2+1} - \frac{e^x}{x}\right)$$

$$= \frac{\left(2e^{x^2+1} + 4x^2e^{x^2+1}\right)(x^2+1) - 2x\left(2xe^{x^2+1}\right)}{(x^2+1)^2} - \frac{e^x x - e^x}{x^2}$$

$$f(1) = \frac{(2e^2 - 4e^2)2 - 2(2e^2)}{4} - \frac{e-e}{1} = \frac{12e^2 - 4e^2}{4} - 0 = \frac{8e^2}{4} = 2e^2$$

3. Use either polar or cylindrical coordinates to find the volume of the region enclosed by both the paraboloid $z = 1 + 2x^2 + 2y^2$ and the ellipsoid $z^2 + 4x^2 + 4y^2 = 13$. Provide a sketch of the region in the first octant.

(8 marks)



Intersection:

$z = 1 + 2x^2 + 2y^2 = 1 + 2r^2$

$z^2 + 4x^2 + 4y^2 = 13$

$\Rightarrow z^2 + 4r^2 = 13$.

$\therefore (1+2r^2)^2 + 4r^2 = 13$

$\Rightarrow 1 + 4r^2 + 4r^4 + 4r^2 - 13 = 0$

$\Rightarrow 4r^4 + 8r^2 - 12 = 0$.

$\Rightarrow r^4 + 2r^2 - 3 = 0$

$\Rightarrow (r^2+3)(r^2-1) = 0$

$\Rightarrow r^2 = -3, \ r^2 = 1$

$r = 1$ is the only meaningful solution.

$$V = 4\int_0^{\pi/2}\int_0^1 \left[\sqrt{13-4r^2} - (1+2r^2)\right] r\,dr\,d\theta$$

$$= 4\left(\frac{\pi}{2}\right)\int_0^1 r\sqrt{13-4r^2} - r - 2r^3\,dr$$

$$= 2\pi\left[(13-4r^2)^{3/2}\left(\frac{2}{3}\right)\left(-\frac{1}{8}\right) - \frac{1}{2}r^2 - \frac{1}{2}r^4 \Big|_{r=0}^{1}\right]$$

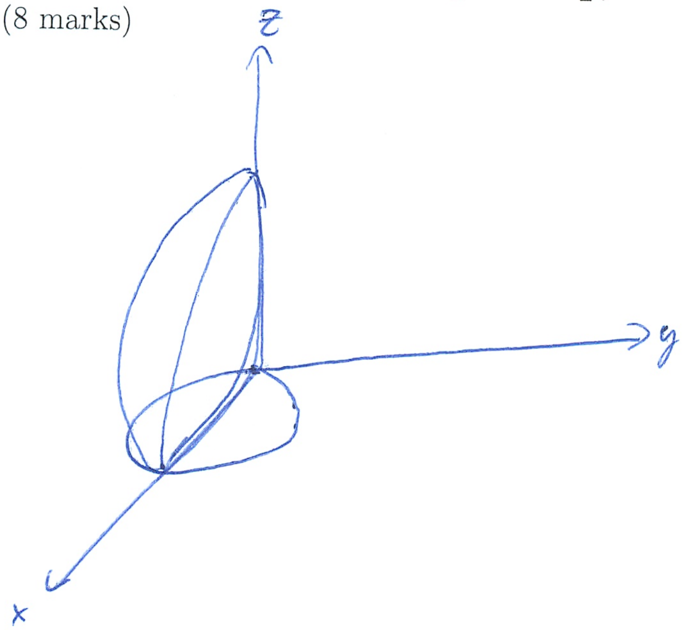
$$= 2\pi\left[-\frac{1}{12}9^{3/2} - 1 + \frac{1}{12}13^{3/2}\right]$$

$$= 2\pi\left[\frac{13^{3/2}}{12} - \frac{3^3}{12} - 1\right]$$

$$= \pi\left[\frac{13^{3/2}}{6} - \frac{9}{2} - 2\right]$$

$$= \pi\left[\frac{13^{3/2}}{6} - \frac{13}{2}\right]$$

4. Find the volume of the portion of the sphere $x^2 + y^2 + z^2 = 1$ which lies above the region $R = \left\{(r, \theta) \,\middle|\, r = \cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$. Provide a sketch of the volume. (8 marks)



$r = \cos\theta$ is a cylinder. If students drew a petal shape we will also accept that since it is difficult to make the distinction in polar coordinates and the distinction is not important to the question.

$$V = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta}\int_0^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^{\cos\theta} r\sqrt{1-r^2}\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} (1-r^2)^{3/2}\left(\frac{2}{3}\right)\left(\frac{-1}{3}\right)\Bigg|_{r=0}^{\cos\theta} d\theta$$

$\sin^3\theta = (1-\cos^2\theta)\sin\theta = \sin\theta - \cos^2\theta\sin\theta$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{-1}{3}\right)\left[(1-\cos^2\theta)^{3/2} - 1\right]d\theta = \frac{-1}{3}\int_{-\pi/2}^{\pi/2} \underbrace{|\sin^3\theta|}_{\text{Note abs. values}} - 1\,d\theta = \underbrace{\frac{-2}{3}\int_0^{\pi/2}}_{\text{Integrate in the half of the region where } \sin\theta \geq 0 \text{ and multiply by } 2.} \sin\theta - \cos^2\theta\sin\theta - 1\,d\theta$$

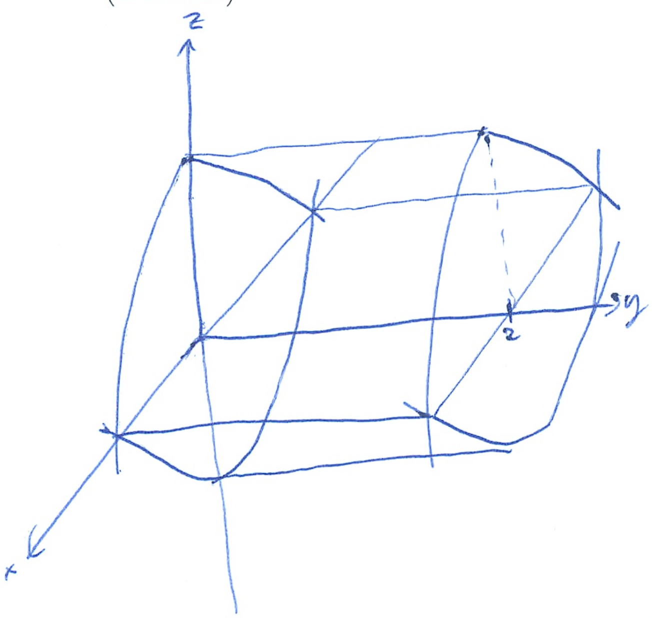
$$= \frac{-2}{3}\left[-\cos\theta + \frac{1}{3}\cos^3\theta - \theta\,\Bigg|_{\theta=0}^{\pi/2}\right]$$

$$= \frac{-2}{3}\left[\left(0 + 0 - \frac{\pi}{2}\right) - \left(-1 + \frac{1}{3} - 0\right)\right] = \frac{-2}{3}\left[-\frac{\pi}{2} + \frac{2}{3}\right]$$

$$= \frac{\pi}{3} - \frac{4}{9}$$

5. Evaluate $\iiint_E x - y \, dV$, where E is the region enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$, and $y = 2$. Provide a sketch of the region.

(8 marks)



$$\iiint_E x-y \, dV = \int_0^2 \int_{-1}^1 \int_{x^2-1}^{1-x^2} x-y \, dz\,dx\,dy$$

$$= \int_0^2 \int_{-1}^1 (1-x^2)(x-y) - (x^2-1)(x-y) \, dx\,dy$$

$$= \int_0^2 \int_{-1}^1 2(1-x^2)(x-y) \, dx\,dy = 2\int_0^2\int_{-1}^1 x - x^3 - y + x^2 y \, dx\,dy$$

$$= 2\int_0^2 \frac{1}{2}x^2 - \frac{1}{4}x^4 - xy + \frac{1}{3}x^3 y \Big|_{x=-1}^{1} dy$$

$$= 2\int_0^2 \left(\frac{1}{2} - \frac{1}{4} - y + \frac{1}{3}y\right) - \left(\frac{1}{2} - \frac{1}{4} + y - \frac{1}{3}y\right) = 2\int_0^2 -2y + \frac{2}{3}y \, dy$$

$$= 2\int_0^2 -\frac{4}{3}y \, dy = -\frac{8}{3}\int_0^2 y \, dy = -\frac{4}{3}y^2 \Big|_{y=0}^{2} = \frac{-16}{3}$$

6. Consider the region in the first octant bounded by the cones $z^2 = x^2 + y^2$ and $z^2 = 3x^2 + 3y^2$ and the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$.

(a) Calculate the mass of a fluid occupying this region if the density $D(x, y, z)$ of the fluid is given by $D(x, y, z) = xyz$. Provide a sketch of the region.

(9 marks)

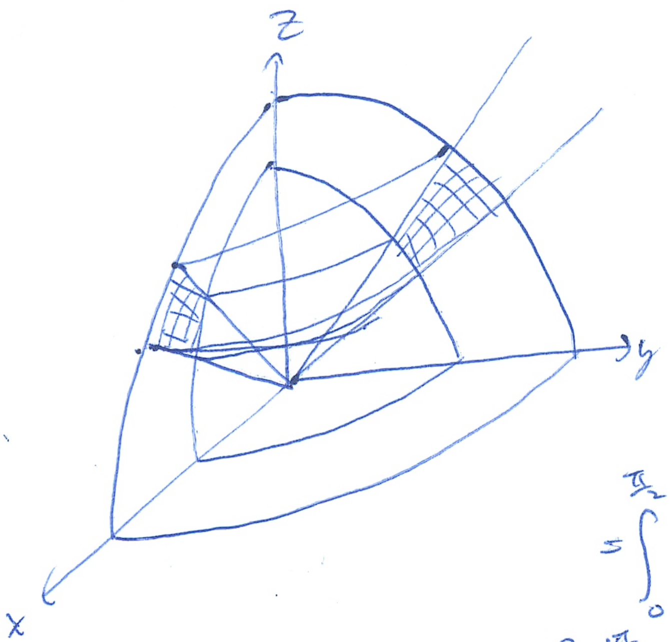
Spherical coordinates.

Spheres: $\rho = 1, \rho = 2$

Cones:

$\rho^2\cos^2\phi = 3\rho^2\sin^2\phi \Rightarrow \frac{1}{\sqrt{3}} = \tan\phi \Rightarrow \phi = \frac{\pi}{6}$

$\rho^2\cos^2\phi = \rho^2\sin^2\phi \Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{3}$



$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_1^{\sqrt{2}} \rho^2\sin\phi(\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\,d\rho\, d\phi\, d\theta.$$

$$= \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_1^{\sqrt{2}} \rho^5\sin^3\phi\cos\phi\cos\theta\sin\theta\, d\rho\, d\phi\, d\theta.$$

$$= \frac{1}{6}\rho^6\frac{1}{4}\sin^4\phi\frac{1}{2}\sin^2\theta\Big|_{\rho=1}^{\sqrt{2}}\Big|_{\phi=\frac{\pi}{6}}^{\frac{\pi}{3}}\Big|_{\theta=0}^{\frac{\pi}{2}} = \frac{1}{6}(2^3-1)\frac{1}{4}\left(\frac{1}{2^2}-\frac{1}{2^4}\right)\frac{1}{2}(1-0)$$

$$= \frac{1}{3\cdot 2^4}(7)\left(\frac{4-1}{16}\right) = \frac{7\cdot 3}{3\cdot 2^8} = \frac{7}{256}$$

	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$= \frac{\sin}{\cos}$

(b) What coordinate system would you use if the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$ in this problem were replaced with the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 2$?

(1 mark) Cylindrical

7. By directly calculating partial derivatives, find the third-degree polynomial approximation to the function $f(x, y) = \dfrac{x}{y}$ near the point $(-1, 2)$.

(8 marks) @ $(-1,2)$

$f_x = \frac{1}{y}$	$\frac{1}{2}$
$f_{xx} = 0$	0
$f_y = \frac{-x}{y^2}$	$\frac{1}{4}$
$f_{yy} = \frac{2x}{y^3}$	$\frac{-2}{8} = -\frac{1}{4}$
$f_{xy} = \frac{-1}{y^2}$	$\frac{-1}{4}$
$f_{xxx} = 0$	0
$f_{xxy} = 0$	0
$f_{xyy} = \frac{-2}{y^3}$	$\frac{-2}{8} = -\frac{1}{4}$
$f_{yyy} = \frac{-6x}{y^4}$	$\frac{6}{16} = \frac{3}{8}$

$$
\begin{aligned}
f(x,y) \approx & -\tfrac{1}{2} \\
& + \left(\tfrac{1}{2}(x+1) + \tfrac{1}{4}(y-2)\right) \\
& + \tfrac{1}{2}\left(0 - \tfrac{1}{2}(x+1)(y-2) - \tfrac{1}{4}(y-2)^2\right) \\
& + \tfrac{1}{6}\left(0 + 0 + \tfrac{-3}{4}(x+1)(y-2)^2 + \tfrac{3}{8}(y-2)^3\right)
\end{aligned}
$$

1

$\overset{x}{1}$ $\overset{y}{1}$

$\overset{xx}{1}$ $\overset{xy}{2}$ $\overset{yy}{1}$

$\overset{xxx}{1}$ $\overset{xxy}{3}$ $\overset{xyy}{3}$ $\overset{yyy}{1}$

8. Show that the area of the part of the plane $z = ax + by + c$ that projects onto a region with area A in the xy-plane is $\sqrt{a^2 + b^2 + 1}A$.

(2 marks)

$$SA = \iint_R \sqrt{1 + z_x^2 + z_y^2}\, dA = \iint_R \sqrt{1 + a^2 + b^2}\, dA = \sqrt{1 + a^2 + b^2} \iint_R dA$$

$$= \sqrt{1 + a^2 + b^2}\, A.$$