University of Toronto – Faculty of Applied Science & Engineering

MAT292 – Final Examination – December 14, 2022

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Exam Type: C
1 page handwritten "cheatsheet" allowed

Time alotted: 150 minutes

Calculator Type: none

Full Name ______________________________

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DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This exam contains 21 pages (including this title page). Once the exam starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages 17–18 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

! ! ! ! THERE IS A TABLE ON THE LAST PAGE ! ! ! !

GOOD LUCK! YOU GOT THIS!

SECTION I Provide the final answer. No justification necessary.

1. **(2 marks)** Consider the initial value problem:

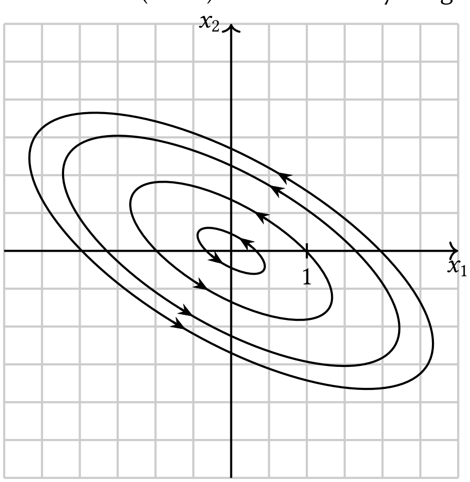
$$\begin{cases} y'(t) = e^{-y^2} y(y-1)(y-10) \\ y(0) = \beta \end{cases}$$

$(a, b) =$ $\boxed{}$

Determine an interval $I = (a, b)$ such that if $\beta \in (a, b)$ then $\lim_{t\to\infty} y(t) = 1$.

Solution: The equilibrium solutions are $y = 0, 1, 10$. By looking at the 1D phase portrait, we can determine that $y = 1$ is a stable equilibrium. Thus, we can conldue that we need $\beta \in (0, 10)$ for $\lim_{t\to\infty} y(t) = 1$.

2. **(2 marks)** The phase portrait of the system $\frac{d\vec{x}}{dt} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \vec{x}(t)$ is given below. Is $a < 0$ or $a > 0$ or is there not enough information (N.E.I.) to conclude anything?



$\boxed{}$

Solution: If we substitute in $\vec{x}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get that $\frac{d\vec{x}}{dt} = \begin{pmatrix} a \\ c \end{pmatrix}$ at the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the phase portrait. Looking at the phase portrait, we can see that at this point, the solution is decreasing in the x_1 direction. Thus, we can conclude that $a < 0$.

3. **(2 marks)** You are asked to model the population of squirrels on UofT campus. We have witnessed that once there are over 5,000 squirrels, their population decreases due to territorial disputes caused by severe space limitations at UofT. Furthermore, it is estimated that if the population falls below 10, the harsh winter conditions will not allow the squirrels to survive anymore. Propose a first order autonomous ODE that models the population that captures these three facts. You may include a parameter in your answer.

Solution: We use the logitistic equation with critical threshold. From lecture, we know that the ODE

$$y' = -r\left(1 - \frac{y}{10}\right)\left(1 - \frac{y}{5000}\right)y$$

captures the critical threshold of 10 and the saturation level of 5000. The $r > 0$ is the unspecified growth rate of the population of the squirrels.

SECTION II

For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

4. (2 marks) The initial value problem

$$\begin{cases} y' = y^{2/3} \\ y(0) = 0 \end{cases}$$

has a unique solution.

Solution: This is False **however**, we can not just say that this is because Existence and Uniqueness does not hold. To get full marks, we need to find two solutions that solve this IVP that are not equal. To do this, begin by noticing that $y(t) = 0$ is a solution to the IVP. To find another solution, notice that the equation is separable so we get that:

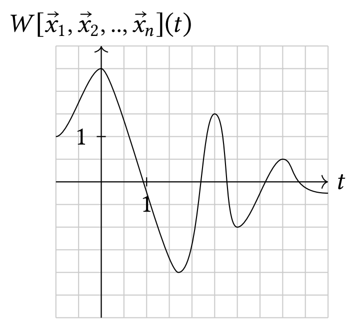
$$3y^{1/3} = t + C \Rightarrow y(t) = \left(\frac{1}{3}t + C\right)^3$$

Plugging in the initial condition, we get that $C = 0$ so

$$y(t) = \frac{1}{27}t^3$$

which is clearly not the same solution as $y(0) = 0$ in any neighborhood around $t = 0$.

5. (2 marks) There exists a linear system $\frac{d\vec{x}}{dt} = A\vec{x}$ such that the graph of the Wronskian of its solutions looks like:



Solution: This is False. We have seen that if the $\vec{x}_1, ..., \vec{x}_n$ are solutions to a homogeneous linear system, then the Wronskian is either identially 0 or never zero. It can not be zero at some points or non-zero at others!

6. (2 marks) You can apply the Laplace Transform on any ODE to get an expression for the solution in the s-domain
Solution: This is False. We can only take the Laplace transform on ODEs that involve terms that are of exponential order and piecewise continuous. In other words, we can only take the Laplace Transforms of ODEs where the Laplace Transform is well-defined. An example of ODE where we can not do this is:

$$y''(t) - y'(t) + y(t) = e^{t^2}$$

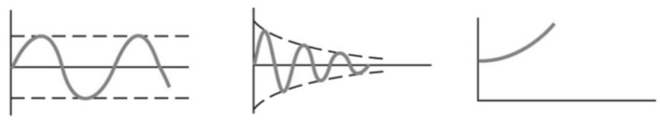
7. **(2 marks)** Consider the ODEs:

$$a_1 y'' + b_1 y' + c_1 y = g(t)$$
$$a_2 y'' + b_2 y' + c_2 y = g(t)$$

Assuming the ODEs have equivalent impulse responses, then for any forcing function they must have equivalent forced responses.

Solution: The forced response is the convolution of the impulse response with the forcing function. If the impulse responses are equal then for any forcing function the forced responses will be equal.

8. **(3 marks)** Recall that Re(s) and Im(s) stand for the real and imaginary part of a complex number s. The Laplace transform of each of the following functions has singularities (points in which the function diverges). Denoting by s such a singularity, explain what do we know about Re(s) and Im(s).



Solution: The first function is a sine. Its Laplace transform is $\frac{\omega}{s^2+\omega^2}$. Therefore, there's a singularity satisfying

$$s^2 = -\omega^2 \iff s = \pm i\omega$$

i.e., $\text{Re}(s) = 0, \text{Im}(s) \neq 0$.

The second function is an exponentially decaying sine wave. Its Laplace transform is $\frac{\omega}{(s+a)^2+\omega^2}$ for a positive and real a. There's a singularity satisfying

$$(s+a)^2 + \omega^2 = 0 \iff (s+a)^2 = -\omega^2 \iff s = \pm i\omega - a$$

i.e., $\text{Re}(s) < 0, \text{Im}(s) \neq 0$.

The third function is exponentially growing. Its Laplace transform is $\frac{\omega}{s-a}$ with a positive real a. There's a singularity at $s = a$, i.e., $\text{Re}(s) > 0, \text{Im}(s) = 0$.

SECTION III **Justify** all your answers, unless it **specifically** says that you do not need to justify.

9. Consider the following linear system:

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(a) **(2 marks)** Find the general solution to the system for the case where $b_1 = b_2 = 0$.

Solution: We find the eigenvalues and eigenvectors of the homogeneous system. If we do this, we see that the system has two distinct real eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$ with corresponding eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ respectively. Thus, we can conclude that the general solution to the system is:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(b) **(2 marks)** Let $\vec{x}_1$ and $\vec{x}_2$ be two linearly independent solutions you found in part a). Compute their Wronskian $W[\vec{x}_1, \vec{x}_2](t)$.

Solution: We need to use our solution from part a) to get the fundamental matrix. Out of simplicity, we pick $c_1 = 1$ and $c_2 = 1$. Thus, we get the fundamental matrix:

$$X(t) = \begin{pmatrix} 1 & 3e^{5t} \\ 2 & e^{5t} \end{pmatrix}$$

To get the Wronskian, we take the determinant of $X(t)$ to get that $W[\vec{x}_1, \vec{x}_2](t) = e^{5t} - 6e^{5t} = -5e^{5t}$

(c) **(3 marks)** Find the general solution to the system for the case where $b_1 = 6$ and $b_2 = 2$ and sketch the phase portrait. Find an expression for the equilibrium solutions.

Solution: Since the matrix is non-invertible, we can not directly use the formula that $\vec{x}_{eq} = -A^{-1}\vec{b}$ where $A = \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix}$. However, we can see (by inspection) that $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$ is still in the column space of A. In fact, we have that

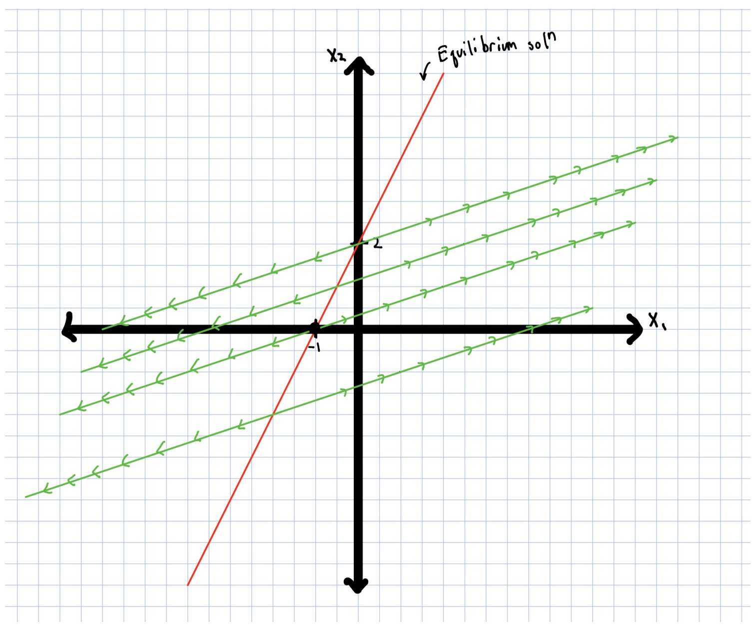
$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Thus, we get that $\vec{x}_{eq} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Using part a), we can then conclude that the general solution in this case is:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From this equation, we can see that the equilibrium solutions will be of the form:

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



(d) **(3 marks)** Find the general solution to the system for the case where $b_1 = 1$ and $b_2 = 3$. Does the system admit an equilibrium solution? If yes, find an expression for the equilibrium solutions.

Solution: Unlike part c), $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ does not lie in the column space of A. Thus, we need to use Variation of Parameters to find a particular solution:

$$\vec{x}_p(t) = X(t) \int X(t)^{-1} \vec{b}\, dt$$

Using part a) and part b), we have that

$$X(t)^{-1} = -\frac{e^{-5t}}{5} \begin{pmatrix} e^{5t} & -3e^{5t} \\ -2 & 1 \end{pmatrix}$$

Then

$$X(t)^{-1}\vec{b} = -\frac{e^{-5t}}{5} \begin{pmatrix} -8e^{5t} \\ 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 8 \\ -e^{-5t} \end{pmatrix}$$

If we integrate this term-by-term with respect to s, we get that

$$\int X(s)^{-1}\vec{b}\, ds = \frac{1}{5}\begin{pmatrix} 8t \\ \frac{1}{5}e^{-5t} \end{pmatrix}$$

Lastly, if multiply this vector against $X(t)$, we get that

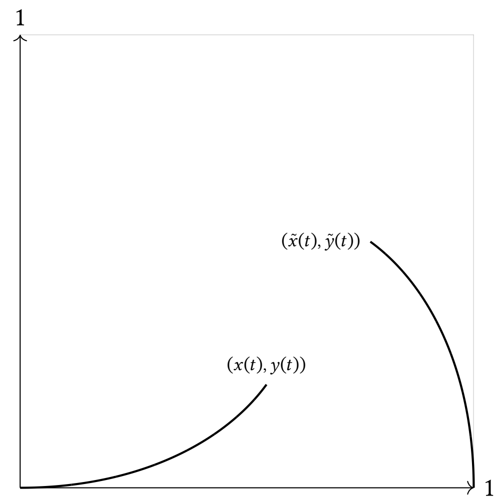
$$\vec{x}_p(t) = \frac{1}{5}\begin{pmatrix} 8t + \frac{3}{5} \\ 16t + \frac{1}{5} \end{pmatrix}$$

and that the general solution is:

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 8t + \frac{3}{5} \\ 16t + \frac{1}{5} \end{pmatrix}$$

Looking at this expression, we can see that we don't have any equilibrium solutions.

10. Four kittens are located in the four corners of a square room with a side length of one meter. The bottom left corner is the point $(0, 0)$. Each kitten runs towards the kitten to its right and all kittens are running at the same speed.



(a) **(3 marks)** Denote by $(x(t), y(t))$ the coordinates of the kitten starting in the bottom left at time t. What are the coordinates of the kitten starting in the bottom right at time t, i.e., $(\tilde{x}(t), \tilde{y}(t))$? **Hint:** symmetry in image above. **DO NOT ATTEMPT TO FIND AN EQUATION FOR THESE PATHS.**

Solution: $(1 - y, x)$

(b) **(3 marks)** **Explain in words** why the following ODE models the trajectory of the bottom left kitten.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - y$$
$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - x - y$$

Solution: The difference between the bottom left kitten and the bottom right kitten is given by

$$\Delta x = (1 - y) - x, \qquad \Delta y = x - y.$$

Therefore, the system is the one specified above.

(c) **(2 marks)** Find the Laplace transform of each of the ODEs above.
Solution:

$$sY = sX - sY$$
$$sX = \frac{1}{s} - X - Y$$

(d) **(2 marks)** Solve for $X(s)$ and $Y(s)$ (solutions should be in the s domain).
Solution: The first equation implies

$$X = (1+s)Y.$$

Together with the second equation, we get

$$Y = \frac{1}{s} - (1+s)X = \frac{1}{s} - (1+s)^2 Y$$

or equally

$$Y = \frac{1}{s(1+(1+s)^2)}.$$

Substituting the above back into the equation of X, we get

$$X = \frac{1+s}{s(1+(1+s)^2)}.$$

(e) **(2 marks)** Show that the solution of the system of ODEs is

$$y(t) = \frac{1}{2}\left[1 - e^{-t}\cos t - e^{-t}\sin t)\right]$$
$$x(t) = \frac{1}{2}\left[1 - e^{-t}\cos t + e^{-t}\sin t)\right]$$

Solution: Partial fractions and linearity of inverse Laplace transform.

(f) **(1 mark)** What can be said about the position of all four kittens after a very long time?
Solution: If we look at the solutions from part e) and take the limit as $t \to \infty$ we can see that the position of the cat is tending towards the point $\left(\frac{1}{2}, \frac{1}{2}\right)$. By symmetry, we can therefore conclude that all four kittens will be tending towards this midpoint.

11. Consider the following second order initial value problem:

$$\begin{cases} ty'' - ty' + y = g(t) \\ y(0) = y_0 \\ y'(0) = y_1 \end{cases}$$

(a) **(2 points)** Apply the Laplace Transform to this IVP and find an ODE that $\mathcal{L}\{y\}(s) = Y(s)$ must satisfy. You may express this ODE in terms of $\mathcal{L}\{g\}(s) = G(s)$

Solution: Using the look-up table, we know that $\mathcal{L}\{tf(t)\}(s) = -F'(s)$. Thus, we have that

$$\mathcal{L}\{ty''\} = \frac{d}{ds}(s^2Y(s) - sy(0) - y'(0)) = -2sY(s) - s^2Y'(s) + y(0)$$

and

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}(sY(s) - y(0)) = -Y(s) - sY'(s)$$

If we combine this together, we get that the ODE that $Y(s)$ satisfies is:

$$-2sY(s) - s^2Y'(s) + y_0 + Y(s) + sY'(s) + Y(s) = G(s)$$

We can simplify this and put this into standard form to get:

$$s(1-s)Y'(s) + 2(1-s)Y(s) = G(s) - y_0$$
$$\Rightarrow Y'(s) + \frac{2}{s}Y(s) = \frac{G(s) - y_0}{s(1-s)}$$

(b) For the remainder of this problem, assume that $y_0 = 3, y_1 = 2$ and $g(t) = 3$.

i. **(2 points)** Solve part a) to get an explicit expression for $Y(s)$.

Solution: We have that $G(s) = \mathcal{L}\{3\} = \frac{3}{s}$. Then we have that the ODE becomes

$$Y'(s) + \frac{2}{s}Y(s) = \frac{3}{s^2}$$

This is a first-order linear equation so we can use method of integrating factor to solve this ODE. The integrating factor in this case is:

$$\mu(s) = \exp(2\ln(s)) = s^2$$

Then

$$\left(s^2Y(s)\right)' = 3$$

If we integrate both sides with respect to s, we get that:

$$s^2Y(s) = 3s + C \Rightarrow Y(s) = \frac{3}{s} + \frac{C}{s^2}$$

where $C \in \mathbb{R}$

ii. **(2 points)** Find an explicit expression for the solution in the t-domain. Your final solution should have no undefined constants.

Solution: Taking the inverse Laplace transform of the solution we got in part b) i), we get that

$$y(t) = 3 + Ct$$

To determine what C is, we use the initial conditions. We see that $y(0) = 3$ is already satisfied. Notice that $y'(t) = C$ and we need $y'(0) = 2$ so we need $C = 2$. Thus, we get that the solution is:

$$y(t) = 3 + 2t$$

12. **(a)** **(2 marks)** Consider the ODE:

$$x' = Bx$$

where

$$B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}.$$

Let t be some non-negative scalar. Compute the even and odd powers of tB:
Solution:

$$(tB)^{2k+1} = \begin{bmatrix} 0 & (-1)^k (tb)^{2k+1} \\ -(-1)^k (tb)^{2k+1} & 0 \end{bmatrix}$$
$$(tB)^{2k} = \begin{bmatrix} (-1)^k (tb)^{2k} & 0 \\ 0 & (-1)^k (tb)^{2k} \end{bmatrix}$$

(b) **(2 marks)** Recall the Taylor series $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ and $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$. Using your answer to the previous item, find e^{Bt}.
Solution:

$$\begin{aligned} e^{Bt} &= \sum_{n=0}^{\infty} \frac{1}{n!} (tB)^n \\ &= \sum_{k=0}^{\infty} \frac{1}{((2k+1)!} (tK)^{2k+1} + \sum_{k=0}^{\infty} \frac{1}{((2k)!} (tK)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{1}{((2k+1)!} \begin{bmatrix} 0 & (-1)^k (tb)^{2k+1} \\ -(-1)^k (tb)^{2k+1} & 0 \end{bmatrix} + \sum_{k=0}^{\infty} \frac{1}{(2k)!} \begin{bmatrix} (-1)^k (tb)^{2k} & 0 \\ 0 & (-1)^k (tb)^{2k} \end{bmatrix} \end{aligned}$$
$$= \begin{bmatrix} \cos tb & \sin tb \\ -\sin tb & \cos tb \end{bmatrix}$$

(c) **(3 marks)** Consider the alternative ODE:

$$x' = Ax$$

where

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Find e^{At} using the properties of matrix exponentials. Mention explicitly at each step what property of matrix exponentials you are invoking.
Solution: Notice that

$$e^{At} = e^{Bt + atI} = e^{Bt} e^{atI} = e^{Bt} e^{at} I.$$

The previous to last equality is correct since the matrices B and the identity commute. The last inequality is correct since all eigenvalues of atI are equal to at and the eigenvectors are the identity matrix. Therefore, $e^{atI} = I \operatorname{diag}(e^{at}, e^{at}, \ldots, e^{at}) I = e^{at} I$. We conclude

$$e^{At} = e^{at} \begin{bmatrix} \cos tb & \sin tb \\ -\sin tb & \cos tb \end{bmatrix}$$

(d) **(3 marks)** Suppose A has a repeated eigenvalue λ and assume $(A - \lambda I)^k = 0$ for $k \geq 2$. Show that

$$e^{At} = e^{\lambda t}(I + (A - \lambda I)t)$$

Solution: Notice that

$$\begin{aligned} & e^{(A-\lambda I)t} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (t(A - \lambda I)t)^n \\ &= I + (A - \lambda I)t + \frac{1}{2}(A - \lambda I)^2 t^2 + \ldots \\ &= I + (A - \lambda I)t \end{aligned}$$

The left-hand side can be equally written as

$$e^{-\lambda t}e^{At} = e^{(A-\lambda I)t} = I + (A - \lambda I)t.$$

Multiplying both sides by $e^{\lambda t}$ we obtain the result.

13. An engineering science student decided to construct from scratch a Positron Emission Tomography (PET) system as a summer project. The student is currently testing a key component of this system: a high energy photon detector. To test the system, the student "shoots" photons at the photon detector and measures its output. Each photon *shocks* the photon detector for an *instantaneous* period of time. The student first shot a single photon at the system and obtained an exponential decay output:

$$\begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The student then shot two photons at the system and noticed that the output is doubled. The student then became more adventurous and shot at times $t = 1, 2, 3, \dots, n$ a number of photons equal to $c_1, c_2, \dots, c_n$, respectively. The engineering science student, who previously learned MAT292, decided to model the input-output relation of the photon detector through the following ODE:

$$ay'' + by' + cy = f(t), \qquad y(0) = y_0, \qquad y'(0) = y_1.$$

The following questions concern the very last experiment in which the student shot many photons.

(a) **(2 marks)** What is the forcing function of the photon detector?
Solution:

$$f(t) = \sum_{i=1}^{n} c_i \delta(t - i)$$

(b) **(2 marks)** What is the Laplace transform of the forcing function of the photon detector?
Solution:

$$F(s) = \sum_{i=1}^{n} c_i e^{-si}$$

(c) **(3 marks)** What is the Laplace transform of the forcing function assuming $c_1 = c_2 = \cdots = c_n = 1$ and $n \to \infty$? Your answer should not contain an infinite sum.
Solution: The function becomes periodic with a period of 1. The Laplace transform of the periodic function is

$$\frac{1}{1 - e^{-s}} \mathcal{L}\delta = \frac{1}{1 - e^{-s}}$$

(d) **(2 marks)** What is the impulse response of the photon detector? The solution should not depend on a, b, c.
Solution:

$$h(t) = e^{-t} u_0(t)$$

where $u_0(t)$ is the Heaviside function.

(e) **(2 marks)** What is the transfer function of the photon detector? The solution should not depend on a, b, c.
Solution:

$$H(s) = \frac{1}{s + 1}$$

(f) **(2 marks)** Using the two previous items, find the constants a, b, c in the ODE.

Solution: On the one hand, we know from the previous item that

$$H(s) = \frac{1}{s+1}.$$

On the other hand, we know from the ODE that

$$H(s) = \frac{1}{as^2 + bs + c}.$$

Comparing the coefficients, we conclude $a = 0$, $b = 1$, $c = 1$.

(g) **(2 marks)** What is the free/transient response of the system? Simplify the answer as much as you can.

Solution:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1} b y_0\right\} = y_0 e^{-t} u_0(t)$$

(h) **(2 marks)** What is the forced response of the system? Simplify the answer as much as you can.

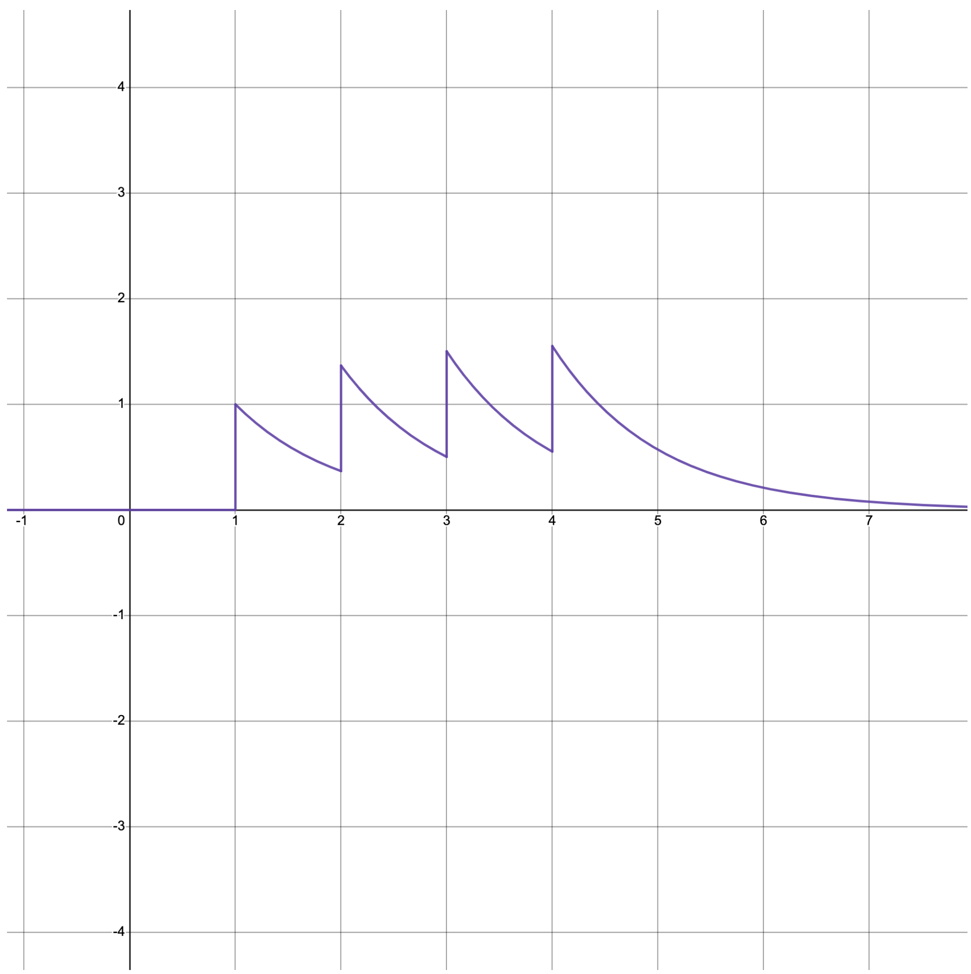
Solution:

$$h(t) * f(t) = (e^{-t}u_0(t)) * \sum_{i=1}^{n} c_i \delta(t-i) = \sum_{i=1}^{n} c_i (e^{-(t-i)} u_0(t-i))$$

(i) **(4 marks)** Draw a sketch of the forced response assuming $n = 4$ and $c_1 = c_2 = \cdots = c_4 = 1$. Use the approximation $1/e \approx \frac{1}{3}$.

Solution:

Note: The students will approximate $\frac{1}{e}$ with $\frac{1}{3}$ so it will look slightly different then the plot below which is the exact graph of the forcing function



14. Consider the 2D Laplace Equation with boundary conditions:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 \le x \le L, \quad 0 \le y \le H \\ u(x,0) = 0 = u(x,H) \\ u(0,y) = f(y), \quad u(L,y) = 0 \end{cases}$$

Assume the solution factorizes as follows: $u(x,y) = X(x)Y(y)$.

(a) (2 marks) Using the PDE, fill in the three blanks.
Solution:

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda > 0$$

(b) (2 marks) Find two ODEs, as well as one boundary condition for $X(x)$ and two boundary conditions for $Y(y)$. These three boundary conditions should be separable (i.e. only depend on X or only depend on Y).
Solution:

$$X''(x) - \lambda X(x) = 0 \qquad X(L) = 0$$

$$Y''(y) + \lambda Y(y) = 0 \qquad Y(0) = 0 \quad Y(H) = 0$$

(c) (2 marks) Show that the solutions for $Y(y)$ and λ must be of the form

$$Y_n(y) = c_1 \sin\left(\frac{n\pi}{H} y\right) \qquad \lambda_n = \frac{n^2\pi^2}{H^2} \qquad \text{where } n = 1, 2, \ldots$$

Solution: We solve the ODE we found in part b) with boundary conditions. We see that the characteristic polynomial is:

$$Y^2 + \lambda = 0 \Rightarrow (Y + i\sqrt{\lambda})(Y - i\sqrt{\lambda}) = 0$$

Then we get that

$$Y(y) = c_1 \sin\left(\sqrt{\lambda} y\right) + c_2 \cos\left(\sqrt{\lambda} y\right)$$

Using the boundary condition that $Y(0) = 0$, we get that $c_2 = 0$. Using the boundary condition that $Y(H) = 0$, we get that:

$$c_1 \sin\left(\sqrt{\lambda} H\right) = 0$$

which implies that $\lambda = \frac{n^2\pi^2}{H^2}$ for $n = 1, 2, \ldots.$
Thus, we can conclude that for $n = 1, 2, \ldots$

$$Y_n(y) = c_1 \sin\left(\frac{n\pi}{H} y\right), \quad \lambda_n = \frac{n^2\pi^2}{H^2}$$

(d) (2 marks) Verify that for $n = 1, 2, ...$

$$X_n(x) = \sinh\left(\frac{n\pi(x-L)}{H}\right) = \frac{e^{n\pi(x-L)/H} - e^{-n\pi(x-L)/H}}{2}$$

is a solution to the following ODE:

$$\begin{cases} X'' - \frac{n^2\pi^2}{H^2}X = 0 \\ X(L) = 0 \end{cases}$$

Solution: We have that

$$X''(x) = \frac{n^2\pi^2}{H^2}\sinh\left(\frac{n\pi(x-L)}{H}\right)$$

Then $X''(x) - \frac{n^2\pi^2}{H^2}X(x) = 0$. We check the initial condition and we see that $X(L) = \sinh(0) = 0$

(e) (2 marks) Briefly justify why

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi}{H}y\right)$$

solves the following PDE:

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 \le x \le L, \quad 0 \le y \le H \\ u(x,0) = 0 = u(x,H) \\ u(L,y) = 0 \end{cases}$$

Solution: This follows by the superposition principle since the PDE is linear. Furthermore, the above boundary conditions are all homogeneous. Thus, we get that the linear combination of solutions will also be a solution.

(f) (3 marks) Using Fourier Series, determine a formula for $c_n \in \mathbb{R}$ such that

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi}{H}y\right)$$

and

$$u(0,y) = f(y)$$

Solution: We want $u(0,y) = f(y)$. If plug this into the formula of the solution, we have that we want

$$\sum_{n=1}^{\infty} c_n \sinh\left(-\frac{n\pi L}{H}\right)\sin\left(\frac{n\pi}{H}y\right) = f(y)$$

If we take the odd extension of f, we know that f can be represented by the Fourier series:

$$f(y) = \sum_{n=1}^{\infty} \tilde{c}_n \sin\left(\frac{n\pi}{H}y\right)$$

where

$$\tilde{c}_n = \frac{2}{H}\int_0^H f(y)\sin\left(\frac{n\pi}{H}y\right)\,dy$$

Comparing this with the equality we want, we get that

$$c_n = \frac{2}{H\sinh\left(-\frac{n\pi L}{H}\right)}\int_0^H f(y)\sin\left(\frac{n\pi}{H}y\right)\,dy$$

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TABLE 5.3.1 Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}, \quad s > 0$
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\dfrac{a}{s^2 + a^2}, \quad s > 0$
6.	$\cos at$	$\dfrac{s}{s^2 + a^2}, \quad s > 0$
7.	$\sinh at$	$\dfrac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\dfrac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
9.	$e^{at} \sin bt$	$\dfrac{b}{(s-a)^2 + b^2}, \quad s > a$
10.	$e^{at} \cos bt$	$\dfrac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t) f(t-c)$	$e^{-cs} F(s)$
14.	$e^{ct} f(t)$	$F(s-c)$
15.	$\displaystyle\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s) G(s)$
16.	$\delta(t-c)$	e^{-cs}
17.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
18.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$