MAT195S CALCULUS II

Midterm Test #2

24 March 2016 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: Y. Cher and J. W. Davis

Family Name: J W Davis

Given Name: Solutions

Student #: ______

FOR MARKER USE ONLY		
Question	Marks	Earned
1	6	
2	9	
3	9	
4	12	
5	11	
6	6	
7	8	
8	11	
TOTAL	72	/ 65

Tutorial Section: ______

TA Name: ______

1) Test the series for convergence or divergence:

a) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ b) $\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

(6 marks)

a) root test: $(a_n)^{1/n} = \frac{n!}{n^4} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!}{n \cdot n \cdot n \cdot n}$

$\lim_{n \to \infty} \frac{n!}{n^4} = \lim_{n \to \infty} (1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n}) \cdot (n-4)! \longrightarrow \infty$

$\therefore \sum \frac{(n!)^n}{n^{4n}}$ diverges

b) limit comparison test with $1/n$

$\lim_{n \to \infty} \frac{2^{1/n} - 1}{1/n} = \lim_{x \to 0} \frac{2^x - 1}{x} \overset{*}{=} \lim_{x \to 0} \frac{2^x \ln 2}{1} = \ln 2$

since $\sum \frac{1}{n}$ diverges, so does $\sum (2^{1/n} - 1)$

2) a) Show by example that $\sum a_n b_n$ may diverge even if $\sum a_n$, $\sum b_n$ both converge.

(4 marks)

Consider $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$

$\Rightarrow |a_n| \to 0$ & $|a_{n+1}| < |a_n|$

$\therefore \sum a_n$ converges by the Alt. Series test

But $\sum a_n b_n = \sum \frac{1}{n}$ which diverges (harmonic series)

b) Prove that if both $\sum a_n$ and $\sum b_n$ are both convergent with positive terms, then $\sum a_n b_n$ is convergent.

(5 marks)

Given that $\sum a_n$ converges, $a_n \to 0$

$\therefore$ for $n > N$, $a_n < 1$

$\therefore$ For $n > N$ $a_n \cdot b_n < b_n$

$\Rightarrow$ all positive terms : $0 < a_n b_n < b_n$

$0 < \sum a_n b_n < \sum b_n$

Since $\sum b_n$ converges, $\sum a_n b_n$ converges by pinching theorem.

3) a) Find the sum of the series: $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Hint: start with the geometric series $\sum x^n$

(4 marks)

Geometric series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{x}\sum_{n=1}^{\infty} n x^n$$

$$\Rightarrow \sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad |x| < 1$$

let $x = \frac{1}{2}$ $\therefore$ $\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2$

b) For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$ converge absolutely? Conditionally? Give the radius and interval of convergence.

(5 marks)

ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n4^n}{(x+2)^n}\right| = \frac{1}{4}\frac{n}{n+1}|x+2| \longrightarrow \frac{|x+2|}{4}$

$\Rightarrow$ convergence for $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ $\Rightarrow R = 4$

$\Rightarrow x \in (-6, 2)$ Absolutely convergent

test $x = -6$: $\sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ conditionally convergent alternating harmonic series

test $x = 2$: $\sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent harmonic series

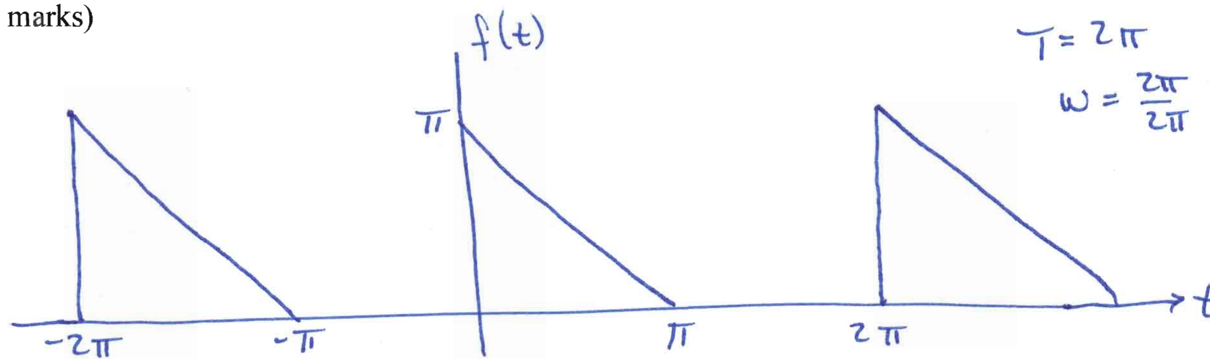
$\therefore \sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$ converges for $x \in [-6, 2)$

4) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(t) = \begin{cases} 0 & \text{if} \quad -\pi \le t < 0 \\ \pi - t & \text{if} \quad 0 \le t < \pi \end{cases}$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

(12 marks)



$T = 2\pi$

$\omega = \frac{2\pi}{2\pi} = 1$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi}(\pi - t)\cos nt\,dt$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - t)\,dt = \frac{1}{\pi}\left[\pi t - \frac{t^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \int_0^{\pi}\cos nt\,dt - \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt$$

let $u = t$ $\quad dv = \cos nt$

$du = dt$ $\quad v = \frac{1}{n}\sin nt$

$$= 0 - \frac{1}{\pi}\underbrace{\left[\frac{t}{n}\sin nt\right]_0^{\pi}}_{0} + \frac{1}{\pi}\int_0^{\pi}\frac{1}{n}\sin nt\,dt = \frac{-1}{\pi n^2}\left[\cos nt\right]_0^{\pi} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{\pi n^2} & n \text{ odd} \end{cases}$$

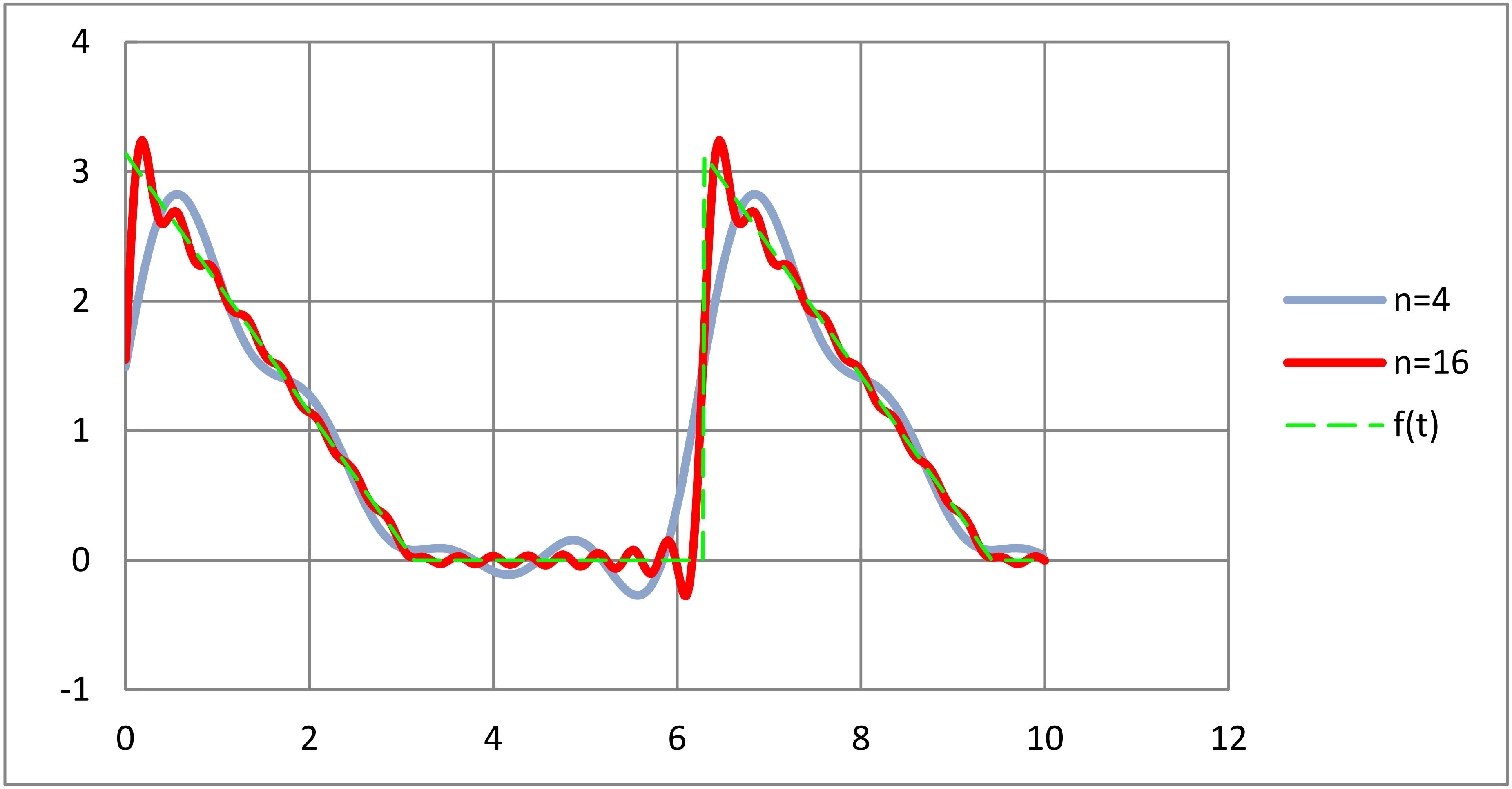
$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \int_0^{\pi}\sin nt\,dt - \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt$$

let $u = t$ $\quad dv = \sin nt$

$du = dt$ $\quad v = \frac{-1}{n}\cos nt$

$$= \left[\frac{-1}{n}\cos nt\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{t}{n}\cos nt\right]_0^{\pi} + \frac{1}{\pi}\underbrace{\int_0^{\pi}\frac{1}{n}\cos nt\,dt}_{0} = \begin{cases} 0 + \frac{1}{n} & n \text{ even} \\ \frac{2}{n} - \frac{1}{n} & n \text{ odd} \end{cases}$$

$$\therefore f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{2}{\pi(2n-1)^2}\cos(2n-1)t + \sum_{n=1}^{\infty}\frac{1}{n}\sin nt$$



4
3
2
1
0
-1
0
2
4
6
8
10
12
n=4
n=16
f(t)

5) a) Show that if P and Q are partitions of $[a,b]$, and f is the function to be integrated, then $L_f(P) \le U_f(Q)$

(4 marks)

We form a new partition $P \cup Q$ which contains all the points of subdivision of both P and Q.

$\therefore L_f(P) \le L_f(P \cup Q)$

$U_f(Q) \ge U_f(P \cup Q)$

$\therefore L_f(P) \le L_f(P \cup Q) \le U_f(P \cup Q) \le U_f(Q)$

b) Using the definition of integrability, show that for any a, $b > 0$, $\int_1^a \frac{dx}{x} = \int_b^{ab} \frac{dx}{x}$.

Provide a sketch, and show numerical calculations, for the specific case of $a = 2$, $b = 3$ and $N = 3$, where N is the number of elements in the partition.

Hint: Given a partition P of $[1, a]$, we can create a partition of $[b, ab]$ by multiplying all points in each element of P by b and vice versa.

(7 marks)

- Given a partition P of $[1, a]$, we create a partition of $[b, ab]$ by multiplying each element in $[1, a]$ by b:

$\therefore \Delta x_{[b,ab]_i} = b\, \Delta x_{[1,a]_i} = b\, \Delta x_i$

- Since $f(x) = \frac{1}{x}$ is decreasing for $x > 0$, $f(x_i^{left}) = f_i^{max}$, $f(x_i^{right}) = f_i^{min}$

$\therefore L^N_{[1,a]} = \sum_{n=1}^{N} \frac{\Delta x_i}{x_i^{right}}$ $\quad U^N_{[1,a]} = \sum_{n=1}^{N} \frac{\Delta x_i}{x_i^{left}}$

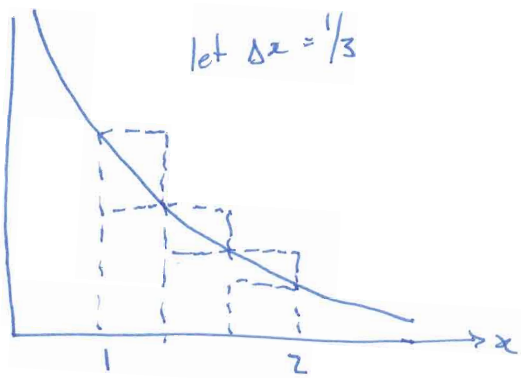
$\Rightarrow L^N_{[b,ab]} = \sum_{n=1}^{N} \frac{b\,\Delta x_i}{b\, x_i^{right}} = \sum_{n=1}^{N} \frac{\Delta x_i}{\Delta x_i^{right}} = L^N_{[1,a]}$

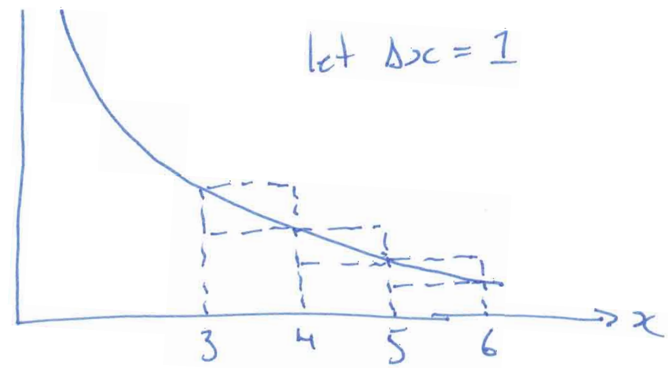
$U^N_{[b,ab]} = \sum_{n=1}^{N} \frac{b\,\Delta x_i}{b\, x_i^{left}} = \sum_{n=1}^{N} \frac{\Delta x}{\Delta x_i^{left}} = U^N_{[1,a]}$

5 b) continued

$$\therefore \int_1^a \frac{dx}{x} = \lim_{N\to\infty} L^N_{[1,a]} = \lim_{N\to\infty} U^N_{[1,a]}$$

$$= \lim_{N\to\infty} U^N_{[b,ab]} = \lim_{N\to\infty} L^N_{[b,ab]} = \int_b^{ab} \frac{dx}{x}$$





$$L^3_{[1,2]} = \frac{1/3}{4/3} + \frac{1/3}{5/3} + \frac{1/3}{6/3} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$U^3_{[1,2]} = \frac{1/3}{1} + \frac{1/3}{4/3} + \frac{1/3}{5/3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

$$L^3_{[3,6]} = \frac{1}{4} + \frac{1}{5} + \frac{1}{6}$$

$$U^3_{[3,6]} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

6) Find the curvature of $f(x) = \ln x$ and find the point at which it is a maximum. What is the maximum value?

(6 marks)

$$K(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}$$

$$f(x) = \ln x \quad ; \quad f'(x) = \frac{1}{x} \quad ; \quad f''(x) = \frac{-1}{x^2} \quad ; \quad x > 0$$

$$\therefore K(x) = \frac{\left|-\frac{1}{x^2}\right|}{\left(1 + \left(\frac{1}{x}\right)^2\right)^{3/2}} = \frac{\frac{1}{x^2}}{\left(1+\frac{1}{x^2}\right)^{3/2}} = \frac{x}{(x^2)^{3/2}\left(1+\frac{1}{x^2}\right)^{3/2}} = \frac{x}{(x^2+1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{1}{(x^2+1)^{3/2}} + x\left(-\frac{3}{2}\right)(x^2+1)^{-5/2}(2x) = \frac{x^2+1-3x^2}{(x^2+1)^{5/2}}$$

$$= \frac{1-2x^2}{(x^2+1)^{5/2}}$$

$$\frac{dK}{dx} = 0 \implies 1 - 2x^2 = 0 \implies x = 1/\sqrt{2}$$

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{1/\sqrt{2}}{\left(\frac{1}{2}+1\right)^{3/2}}$$

$$\left.\begin{array}{l} K' > 0 \text{ for } 0 < x < \frac{1}{\sqrt{2}} \\ K' < 0 \text{ for } x > \frac{1}{\sqrt{2}} \end{array}\right\} \therefore K\left(\frac{1}{\sqrt{2}}\right) \text{ is a maximum}$$

7) Given the definition of curvature: $\kappa \equiv \left\| \frac{d\vec{T}}{ds} \right\|$, show that for a 3-D curve given by the vector function $\vec{r}(t)$, that curvature can also be given by the formula: $\kappa = \frac{\left\| \vec{r}\,'(t) \times \vec{r}\,''(t) \right\|}{\left\| \vec{r}\,'(t) \right\|^3}$

(8 marks)

$\vec{T}$ is the unit tangent vector: $\|\vec{T}\| = 1$

$\vec{T}'$ is $\perp$ to $\vec{T}$ $\therefore$ $\vec{T} \cdot \vec{T}' = 0$

& $\|\vec{T} \times \vec{T}'\| = \|\vec{T}\| \, \|\vec{T}'\| \, |\sin\theta| = \|\vec{T}'\|$

Now $\vec{T} = \frac{\vec{r}\,'}{\|\vec{r}\,'\|}$ $\therefore$ $\vec{T}' = \frac{\vec{r}\,''}{\|\vec{r}\,'\|} - \frac{\vec{r}\,'}{\|\vec{r}\,'\|^2} \left(\|\vec{r}\,'\| \right)'$

$\therefore \vec{T} \times \vec{T}' = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} \times \frac{\vec{r}\,''}{\|\vec{r}\,'\|}$ since $\vec{r}\,' \times \vec{r}\,' = 0$

$\therefore \|\vec{T} \times \vec{T}'\| = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^2} = \|\vec{T}'\|$

$\therefore \kappa = \left\| \frac{d\vec{T}}{ds} \right\| = \left\| \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right\| = \frac{\left\| \frac{d\vec{T}}{dt} \right\|}{\left| \frac{ds}{dt} \right|} = \frac{\|\vec{T}'\|}{\|\vec{r}\,'\|} = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3}$

8) a) Find all of the second partial derivatives for the function: $f(x,y)=\sqrt{1+xy^2}$

(5 marks)

$f(x,y) = (1+xy^2)^{1/2}$

$f_x = \frac{1}{2}(1+xy^2)^{-1/2} \cdot y^2$

$f_{xx} = -\frac{1}{4}(1+xy^2)^{-3/2} \cdot y^4$

$f_y = \frac{1}{2}(1+xy^2)^{-1/2} \cdot 2xy$

$f_{yy} = -\frac{1}{4}(1+xy^2)^{-3/2} \cdot 4x^2y^2 + \frac{1}{2}(1+xy^2)^{-1/2} \cdot 2x$

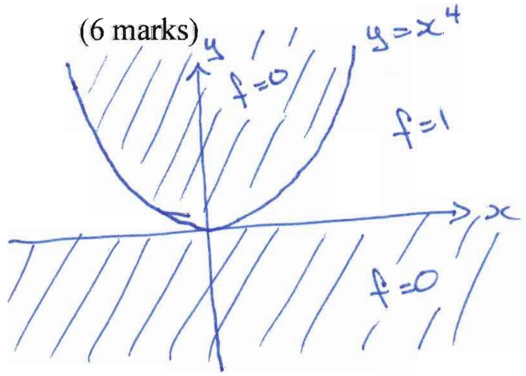
$f_{xy} = f_{yx} = -\frac{1}{4}(1+xy^2)^{-3/2} \cdot y^2 \cdot 2xy + \frac{1}{2}(1+xy^2)^{-1/2} \cdot 2y$

b) Let $f(x.y)=\begin{cases} 0 & \text{if} \quad y \le 0 \quad \text{or} \quad y \ge x^4 \\ 1 & \text{if} \quad 0 < y < x^4 \end{cases}$

i) Show that $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along any path through $(0,0)$ of the form $y = mx^a$ with $0 < a < 4$.

ii) Despite part (i), show that f is discontinuous at $(0,0)$.

(6 marks)



let $y = mx^a \qquad 0 < a < 4$

consider $x \to 0 \quad f(x, mx^a)$

$\Rightarrow$ for $0 < a < 4$, $\quad mx^a > x^4 \quad$ as $x \to 0$

$\Rightarrow \frac{mx^a}{x^4} = m x^{(a-4)} > m$ for $0 \le a < 4$, $x < 1$

$\therefore$ for any finite value of m, it will always be possible to find a value of x small enough to make $mx^a > x^4$

$\therefore f(x,y) = 0$ along the path near $x=0$

$\therefore \lim_{x \to 0} f(x, mx^a) = 0 \qquad 0 < a < 4$

Consider path : $y = x^6$

for $-1 < x < 1 \qquad x^6 < x^4$

except at $x = 0$

$\therefore f(x,y) = 1$

$\therefore \lim_{x \to 0} f(x, x^6) = 1$

$\therefore f(x,y)$ is discontinuous at $(0,0)$