The Edward S. Rogers Sr. Department of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 1 - Thursday February 11, 2016

Instructors: Costas Sarris (LEC01) , Piero Triverio (LEC02)

Last name: ....

First name: ....

Student number: ....

Tutorial section number: ....

TUT Section	Day	Time	Room
1	Thursday	16:00-17:00	BA2155
2	Thursday	16:00-17:00	HA401
3	Thursday	16:00-17:00	BA2139
4	Thursday	16:00-17:00	BA3116
5	Thursday	16:00-17:00	BA2195
6	Tuesday	14:00-15:00	BA2195
7	Tuesday	14:00-15:00	BA2175
8	Tuesday	14:00-15:00	BA2165

Instructions

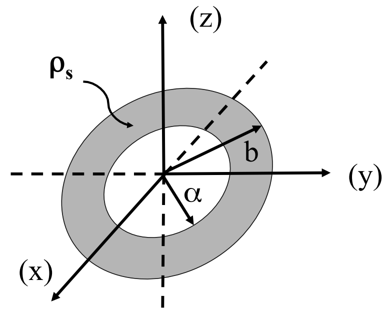
- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full credit!**

Marks:

Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1

1. Consider a charged ring of inner radius α and outer radius b, on the $z = 0$ plane, with surface charge density $\rho_s = r_{s,0}\cos^2\phi$.



The electric field of this ring and its potential can be expressed in terms of the superposition integrals:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\iint_{ring}\frac{dQ}{|\mathbf{R}-\mathbf{R}'|^3}\left(\mathbf{R}-\mathbf{R}'\right),$$

$$V = \frac{1}{4\pi\epsilon_0}\iint_{ring}\frac{dQ}{|\mathbf{R}-\mathbf{R}'|}$$

a) Specify these integrals for an arbitrary observation point (x, y, z). You do not need to evaluate these integrals, just determine dQ, $\mathbf{R}$, $\mathbf{R}'$, $\mathbf{R}-\mathbf{R}'$, $|\mathbf{R}-\mathbf{R}'|$ and the limits of integration. Substitute them to provide the final expressions for the integrals. **(6 pts)**

$$dQ = \rho_s dS' = \left(r_{s,0}\cos^2\phi'\right) r' d\phi' dr'$$

(1 pt)

$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

(0.5 pt)

$$\mathbf{R}' = r'\mathbf{a}_{r'} = r'\left(\mathbf{a}_x\cos\phi' + \mathbf{a}_y\sin\phi'\right)$$

(1 pt)

hence,

$$\mathbf{R}-\mathbf{R}' = (x - r'\cos\phi')\mathbf{a}_x + \left(y - r'\sin\phi'\right)\mathbf{a}_y + z\mathbf{a}_z$$

(0.5 pt)

and:

$$|\mathbf{R}-\mathbf{R}'| = \sqrt{(x - r'\cos\phi')^2 + (y - r'\sin\phi')^2 + z^2}$$

(1 pt)

So, just substituting into the integral and recognizing that the integration is carried out with respect to ϕ' from 0 to 2π and r' from α to b, we have:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_\alpha^b\int_0^{2\pi}\frac{\left(r_{s,0}\cos^2\phi'\right)r'd\phi'dr'\left((x-r'\cos\phi')\mathbf{a}_x+(y-r'\sin\phi')\,\mathbf{a}_y+z\mathbf{a}_z\right)}{\left((x-r'\cos\phi')^2+(y-r'\sin\phi')^2+z^2\right)^{3/2}}$$

(1 pt) Likewise,

$$V = \frac{1}{4\pi\epsilon_0}\int_\alpha^b\int_0^{2\pi}\frac{\left(r_{s,0}\cos^2\phi'\right)r'd\phi'dr'}{\sqrt{(x-r'\cos\phi')^2+(y-r'\sin\phi')^2+z^2}}$$

(1 pt)

b) Now, consider an observation point on the $z-$axis, $(0,0,z)$. Determine the electric field and the electric potential at this point. To determine these, you may need the following integrals:

$$\int\frac{rdr}{(r^2+z^2)^p} = \frac{1}{2}\frac{\left(r^2+z^2\right)^{-p+1}}{-p+1}$$

$$\int_0^{2\pi}\cos^2\phi d\phi = \pi$$

Hint: The electric field has only a $z-$component. Justify why and use it to expedite your calculations. **(8 pts)**

Letting $x = 0 = y$, the formulas for the field and the potential become:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_\alpha^b\int_0^{2\pi}\frac{\left(r_{s,0}\cos^2\phi'\right)r'd\phi'dr'\left(-r'\cos\phi'\mathbf{a}_x - r'\sin\phi'\mathbf{a}_y+z\mathbf{a}_z\right)}{\left((r')^2+z^2\right)^{3/2}}$$

(1 pt)

$$V = \frac{1}{4\pi\epsilon_0}\int_\alpha^b\int_0^{2\pi}\frac{\left(r_{s,0}\cos^2\phi'\right)r'd\phi'dr'}{\sqrt{(r')^2+z^2}}$$

(1 pt)

For **E**, we can conclude that only its $z-$component is non-zero, either geometrically (recall similar arguments in class for the field of a charged ring or an infinite charged plane), or mathematically, since: $\int_0^{2\pi}\cos^3\phi'd\phi' = 0$ and $\int_0^{2\pi}\cos^2\phi'\sin\phi'd\phi' = 0$. **(2 pts)**

Hence:

$$\mathbf{E} = \mathbf{a}_z\frac{z}{4\pi\epsilon_0}\int_0^{2\pi}\left(r_{s,0}\cos^2\phi'\right)d\phi'\int_\alpha^b\frac{r'}{\left((r')^2+z^2\right)^{3/2}}$$

With:

$$\int_0^{2\pi} \left(r_{s,0} \cos^2 \phi'\right) d\phi' = r_{s,0}\pi$$

and (from the formula above with $p = 3/2$)

$$\int_\alpha^b \frac{r'}{((r')^2 + z^2)^{3/2}} = \frac{1}{2}\frac{\left((r')^2 + z^2\right)^{-0.5}}{-0.5} = -\frac{1}{\sqrt{(r')^2 + z^2}}|_{r'=\alpha}^{b} = \frac{1}{\sqrt{\alpha^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}}$$

we have:

$$\mathbf{E} = \mathbf{a}_z \frac{z r_{s,0}}{4\epsilon_0}\left(\frac{1}{\sqrt{\alpha^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}}\right)$$

(2 pts)

For V, the following integral is used (from the formula above with $p = 1/2$):

$$\int_\alpha^b \frac{r'}{\sqrt{(r')^2 + z^2}} dr' = \sqrt{b^2 + z^2} - \sqrt{\alpha^2 + z^2}$$

Hence:

$$V = \frac{r_{s,0}}{4\epsilon_0}\left(\sqrt{b^2 + z^2} - \sqrt{\alpha^2 + z^2}\right)$$

(2 pts)

Note that $E_z = -\dfrac{dV}{dz}\mathbf{a}_z$, consistently with $\mathbf{E} = -\nabla V$.

c) Can you find the electric field of the ring via Gauss' law ? Briefly explain. **(2 pts)**

No, the problem has no cylindrical, spherical or rectangular symmetry that would allow us to apply Gauss law. Just ask yourself: what surface would we choose, if we were to apply Gauss law ? With no firm knowledge of the field lines, we cannot choose a surface.

d) Find the electric field and the potential for a large distance $R >> b$ away from the origin. Sketch the field lines and the equi-potential surfaces. Briefly explain. **(4 pts)**

The key here is that the ring carries a positive amount of charge:

$$Q = \int_\alpha^b \int_0^{2\pi} r_{s,0} \cos^2 \phi r' dr' d\phi' = \pi \frac{b^2 - \alpha^2}{2} r_{s,0}$$

Hence, at $R >> b$ it will look like a positive point charge **(2 pts)**, with field lines in the $+\mathbf{a}_R$ direction **(1 pt)** and equi-potential surfaces being spheres centered at the origin **(1 pt)**.

<u>Addendum:</u> One to check the validity of our results for the electric field and the potential is by using the total charge Q of the ring and our intuition that from a far distance this ring should behave as a point charge, generating along the $+z$-axis:

$$\mathbf{E}(z) = \frac{Q}{4\pi\epsilon_0 z^2}\mathbf{a}_z$$

and

$$V(z) = \frac{Q}{4\pi\epsilon_0 z}$$

Indeed,

$$\mathbf{E} = \mathbf{a}_z \frac{z r_{s,0}}{4\epsilon_0}\left(\frac{1}{\sqrt{\alpha^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right) \to \frac{z r_{s,0}}{4\epsilon_0}\frac{1}{z}\left(1-\frac{\alpha^2}{2z^2} - 1 + \frac{b^2}{2z^2}\right) = \frac{z r_{s,0}}{4\epsilon_0 z}\frac{\pi\left(b^2-\alpha^2\right)}{2z^2}$$

Hence:

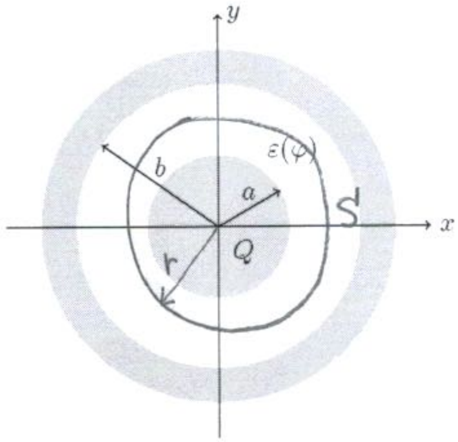
$$\mathbf{E} \equiv \frac{Q}{4\pi\epsilon_0 z^2}\mathbf{a}_z$$

where we used:

$$\frac{1}{\sqrt{z^2+b^2}} = \frac{1}{z}\frac{1}{\sqrt{1+(b/z)^2}} \to \frac{1}{z}\left(1-\frac{b^2}{2z^2}\right)$$

for $z >> b$. Similar for V.

Question 2



1. Consider the coaxial cable shown in the figure above. The cable consists of an inhomogeneous dielectric surrounded by two perfect conductors of radii a and b. The permittivity of the dielectric is $\varepsilon(\varphi) = (3 + \sin\varphi)\varepsilon_0$, where $\varphi \in [0, 2\pi]$. The charge per unit length on the inner conductor is Q. Assuming $\mathbf{E} = E_r(r)\mathbf{a}_r$, use Gauss' law to find the electric field in the dielectric [6 points].

Generalized Gauss' law $\int_S \bar{D} \cdot d\bar{S} = Q_{enc}$

As S, we use a cylindrical surface of radius r and length L. [1pt]

Flux from top and bottom faces of S is zero because there $\bar{D} \perp d\bar{S}$ [1pt]

[2pt]

[1pt] for correct charge enclosed

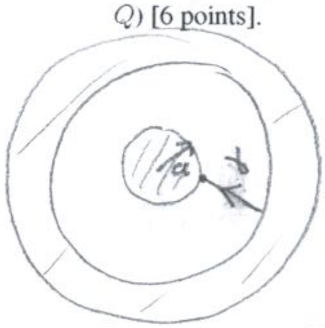
$$\int_{\varphi=0}^{2\pi}\int_{z=0}^{L} \varepsilon(\varphi) E_r(r)\, \bar{a}_r \cdot \bar{a}_r \, r \, d\varphi \, dz = Q \cdot L$$

$$r E_r(r) \cancel{L} \underbrace{\int_{\varphi=0}^{2\pi} \varepsilon(\varphi)\, d\varphi}_{3\varepsilon_0 \cdot 2\pi} = Q \cdot \cancel{L}$$

Final answer: [1pt]

$$\mathbf{E}(r) = \frac{Q}{6\pi \varepsilon_0 r}\,\bar{a}_r$$

2. Find $V(r=a) - V(r=b)$, the voltage between the outer conductor and the inner conductor (in terms of Q) [6 points].



$$V(a) - V(b) = -\int_{r=b}^{r=a} \bar{E}\cdot d\bar{l} =$$

Right use of V definition [2pt]

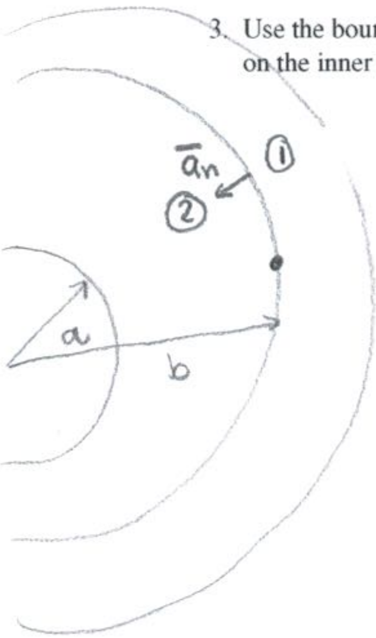
$$= \int_{r=a}^{b} \frac{Q}{6\pi\varepsilon_0 r}\,\underbrace{\bar{a}_r \cdot \bar{a}_r}_{1}\,dr = \frac{Q}{6\pi\varepsilon_0}\ln\frac{b}{a}$$

[3pt]

$$V(r=a) - V(r=b) = \frac{Q}{6\pi\varepsilon_0}\ln\frac{b}{a}$$

Final [1pt]

3. Use the boundary condition between two generic media to find $\rho_{s,b}(\varphi)$, the surface density of *free* charge on the inner boundary of the outer conductor [4 points].



$$\bar{a}_n \cdot (\bar{D}_2 - \bar{D}_1) = \rho_s$$

$$\bar{a}_n = -\bar{a}_r \quad \text{[1pt]} \leftarrow \text{for right normal / sign of } \rho_s$$

$$\bar{D}_1 = 0 \leftarrow \text{[1pt]}$$

$$\bar{D}_2 = \varepsilon \cdot \bar{E}(b) = (3+\sin\varphi)\cancel{\varepsilon_0} \frac{Q}{6\pi \cancel{\varepsilon_0} b} \bar{a}_r \quad \leftarrow \text{[1pt]}$$

$$\boxed{\rho_{s,b}(\varphi) = \bar{a}_n \cdot \bar{D}_2 = -\frac{Q}{6\pi b}(3+\sin\varphi)} \quad \text{[1pt]}$$

4. Find the polarization vector **P** in the dielectric [4 points].

$$\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$$

$$\underbrace{\bar{P} = \bar{D} - \varepsilon_0 \bar{E}}_{\text{[2pt]}} = (\varepsilon - \varepsilon_0)\bar{E} = (2+\sin\varphi)\varepsilon_0 \bar{E}$$

Right approach

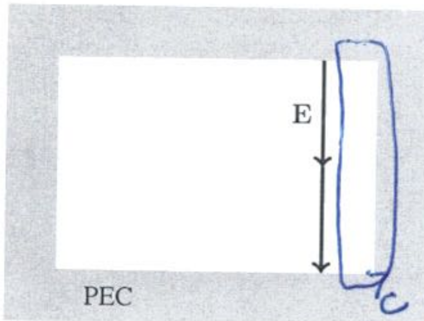
Calculations [1pt]

$$\boxed{\mathbf{P} = (2+\sin\varphi)\frac{Q}{6\pi r}\bar{a}_r}$$

Final answer [1pt]

Question 3

Q3.1) A block of perfect conductor (PEC) has a rectangular cavity, as shown in the figure below.



In the cavity there are no charges. Show that an electrostatic field **E** with the direction depicted in the figure cannot exist [4 points].

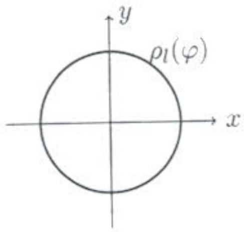
Such $\bar{E}$ violates $\oint_C \bar{E} \cdot \bar{d\ell} = 0$ [1pt]

Take path c shown in figure path : [1pt]

In ~~metal~~ PEC, $\bar{E} = 0$ ⟶ [1pt]

$$\oint_c \bar{E} \cdot \bar{d\ell} = \underbrace{\int_{PEC} \bar{E} \cdot \bar{d\ell}}_{=0} + \underbrace{\int_{cavity} \bar{E} \cdot \bar{d\ell}}_{[1pt]} \neq 0$$

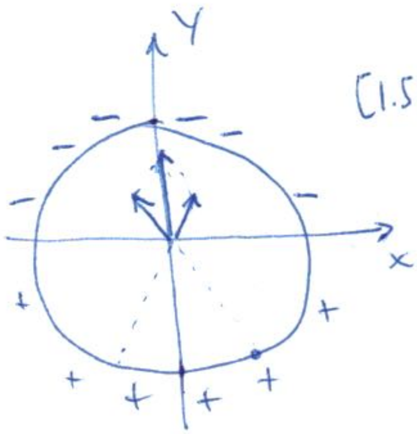
Q3.2) Consider the ring shown in the figure below.



The ring is charged with line charge density $\rho_l(\varphi) = -2\sin(\varphi)\,\mathrm{nC/m}$. At the origin, the charge distribution causes an electric field that points:

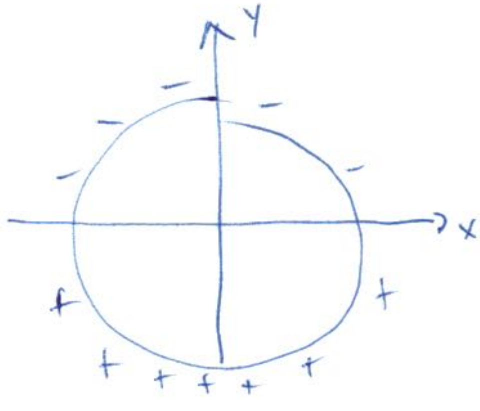
1. in the direction of $+\mathbf{a}_x$;
2. in the direction of $-\mathbf{a}_x$;
3. in the direction of $+\mathbf{a}_y$; ← Right answer [1pt]
4. in the direction of $-\mathbf{a}_y$;
5. the field is actually zero.

Briefly justify your answer [4 points].



[1.5pt] Because of symmetry across y axis:

$\bar{E}$ can have only $\pm\bar{a}_y$ component

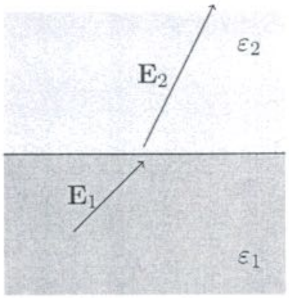


contribution of negative charges for $y>0$ $\rightarrow + \bar{a}_y$

contribution of positive charges for $y<0$ $\rightarrow + \bar{a}_y$ [1.5pt]

Hence ~~[illegible]~~ $\bar{E} \parallel +\bar{a}_y$

Q3.3) The figure below depicts the **E** field just below and just above the interface between two dielectrics, with permittivity ε_1 and ε_2.



There is no free charge at the interface. Which one of the following statements is true?

1. $\boxed{\varepsilon_1 > \varepsilon_2;}$ Right answer [1pt]
2. $\varepsilon_1 < \varepsilon_2$;
3. $\varepsilon_1 = \varepsilon_2$;
4. the given fields are not realistic, since they do not satisfy boundary conditions

Briefly justify your answer [4 points].

Boundary conditions:

$E_{1,t} = E_{2,t}$ → satisfied

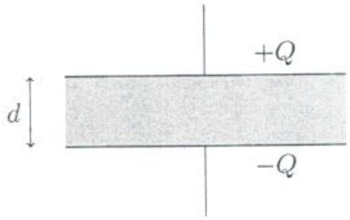
$D_{1,n} = D_{2,n}$

$\varepsilon_1 E_{1,n} = \varepsilon_2 E_{2,n}$

] use of boundary conditions: [1pt]

Since $E_{2,n} > E_{1,n} \Rightarrow \varepsilon_2 < \varepsilon_1$] Correct deduction $D \rightarrow E \rightarrow \varepsilon$ [2pt]

Q3.4) Consider the parallel plate capacitor shown in the figure below.



The capacitor has capacitance $C = 1\,\mathrm{nF}$. The dielectric is $d = 2\,\mathrm{mm}$ thick, and its dielectric strength is $E_{max} = 20\,\mathrm{kV/mm}$. What is the maximum amount of charge Q that the capacitor can store without breaking down? [4 points].

$$C = \frac{Q}{V}$$

Max voltage sustainable by dielectric:

$$\underbrace{V = E_{max} \cdot d}_{[1pt]} = \underbrace{40\,kV}_{[1pt]}$$

Max charge

$$\underbrace{Q = C \cdot V}_{[1pt]} = 10^{-9} \cdot 40 \cdot 10^{3} = 40 \cdot 10^{-6} = \underbrace{40\,\mu C}_{[1pt]}$$

Q3.5) The potential in an electrostatic system is

$$V(x, y, z) = e^{-x^2}$$

An electron is placed at the point $P(x = -1, y = 0, z = 0)$. The force acting on the electron is:

1. zero;
2. directed towards $+\mathbf{a}_x$; [Right answer: 1pt]
3. directed towards $-\mathbf{a}_x$;
4. directed towards $+\mathbf{a}_y$;
5. directed towards $-\mathbf{a}_y$;
6. directed towards $+\mathbf{a}_z$;
7. directed towards $-\mathbf{a}_z$;

Briefly justify your answer [4 points].

$$\bar{E} = -\nabla V = -\left(\frac{\partial}{\partial x} e^{-x^2}\right)\bar{a}_x = 2x\, e^{-x^2}\bar{a}_x$$

$$\bar{E}(x=-1) = -2e^{-1}\,\bar{a}_x$$

$\bar{E} \leftrightarrow V$ relation [1.5 pt]

$\bar{F} = q\bar{E} \longrightarrow$ direction: $+\bar{a}_x$

$(q<0)$

[1.5pt