Last name: First name: ID number:

ECE 286
Midterm exam

March 10, 2022
6:30 – 8:00 pm

Circle your lecture section:

LEC0101 (Tuesday 11-12)

LEC0102 (Monday 12-1)

Guidelines:

- Write your answer in the space provided for each question. Show your work and use the back side of sheets as needed.
- The exam is Type D. You may use a non-programmable calculator and a one-sided, 8.5 by 11 handwritten note sheet (prepared by you).
- Only exams written in pen will be considered for regrades.

Problem	Score
1	/10
2	/10
3	/10
Total	/30

1. Consider a coin with $P(H) = 0.4$ and $P(T) = 0.6$, where H denotes 'Heads' and T denotes 'tails'. Justify your answers.

   (a) You flip the coin four times. Let X be the number of Heads.

      i. (2 points) Compute $P(X = 3)$.

      Solution: A sequence of coin flips is described by the binomial distribution. We have

$$\begin{aligned} b(3; 0.4, 4) &= \binom{4}{3} 0.4^3 0.6^1 \\ &= 0.1536. \end{aligned}$$

      ii. (2 points) Compute $P(X \geq 2)$.

      Solution: We have

$$\begin{aligned} P(X \geq 2) &= 1 - P(X \leq 1) \\ &= 0.6^4 + \binom{4}{1} 0.4^1 0.6^3 \\ &= 0.5248. \end{aligned}$$

   (b) (2 points) If you flip the coin six times, what is the probability of the sequence $HHTTTH$.

   Solution: It is $P(HHTTTH) = 0.4^3 0.6^3 = 0.0138$.

   (c) Suppose you win two dollars for each heads and lose a dollar for each tails. Let Y be the total amount of money you win or lose after twelve coin flips.

      i. (2 points) Compute $E[Y]$.

      Solution: We have

$$\begin{aligned} E[Y] &= 12(2P(H) - 1P(T)) \\ &= 12(2 \times 0.4 - 1 \times 0.6) \\ &= 2.4. \end{aligned}$$

      ii. (2 points) Compute the variance of Y.

      Solution: Because each coin flip is independent, we can add the variances of twelve flips. We have

$$\begin{aligned} \mathrm{var}[Y] &= 12\left((2 - 0.2)^2 P(H) + (-1 - 0.2)^2 P(T)\right) \\ &= 25.92. \end{aligned}$$

2. The sample space, S, contains the events A, B, and C. We know that $A \cup B \cup C = S$; $P(A) = 0.3$, $P(B) = 0.5$, and $P(C) = 0.6$; and $P(A \cap B) = P(A \cap C) = 0$. Justify your answers.

   (a) (2 points) What is $P(B \cap C)$?

   Solution: We known that

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C).$$

   Therefore, $P(B \cap C) = 0.3 + 0.5 + 0.6 - 1 = 0.4$.

   (b) (2 points) What is $P(A \mid C)$?

   Solution: Using the definition of conditional probability, $P(A \mid C) = 0$.

   (c) (2 points) What is $P(B \mid C)$?

   Solution: Using the definition of conditional probability, $P(B \mid C) = P(B \cap C)/P(C) = 0.4/0.6 = 2/3$.

   (d) (4 points) Consider the event $D \subset S$ with $P(A \cap D) = 0.1$, $P(B \cap D) = 0.2$ and $P(C \cap D) = 0.3$. Find the range of all possible values of $P(D)$.

   Solution: By the law of total probability,

   $$P(D) = P(A \cap D) + P(B \cap D) + P(C \cap D) - P(B \cap C \cap D) = 0.6 - P(B \cap C \cap D).$$

   We don't know $P(B \cap C \cap D)$. *It is upper bounded by each pairwise intersection. So taking the smallest,* $P(B \cap C \cap D) \leq P(B \cap D) = 0.2$. *Therefore, a lower bound is*

   $$P(A \cap D) + P(B \cap D) + P(C \cap D) - P(B \cap C \cap D) = 0.1 + 0.2 + 0.3 - 0.2 = 0.4.$$

   We can construct this D by putting as much of D as we can in $B \cap C$. We find the upper bound by putting as little of D as we can in $B \cap C$. Observe that the largest possible value of $P(B \cap D \cap \overline{C}) = 0.1$ *and similarly* $P(C \cap D \cap \overline{B}) = 0.2$. *This corresponds to* $P(B \cap C \cap D) = 0.1$, *for which* $P(D) = 0.5$. *Therefore,*

   $$0.4 \leq P(D) \leq 0.5.$$

3. X is a continuous random variable with PDF

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/\alpha & \text{if } 0 \le x < \alpha \\ e^{3(\alpha - x)} & \text{if } \alpha \le x \end{cases},$$

where $\alpha > 0$ is a constant. Justify your answers.

(a) (4 points) Find α.

Solution: The PDF must integrate to one. Integrating, we have

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \int_0^{\alpha} x/\alpha dx + \int_{\alpha}^{\infty} e^{3(\alpha-x)}dx \\ &= \alpha/2 - \frac{e^{3\alpha}}{3} e^{-3x}|_{\alpha}^{\infty} \\ &= \alpha/2 - \frac{e^{3\alpha}}{3}(0 - e^{-3\alpha}) \\ &= \alpha/2 + 1/3. \end{aligned}$$

Setting this equal to one, we have $\alpha = 4/3$.

(b) (2 points) Find $P(X \ge \alpha/2)$.

Solution: Observe that $P(X < \alpha/2) = \alpha/8$. *We have* $P(X \ge \alpha/2) = 1 - P(X < \alpha/2) = 1 - \alpha/8$. *Plugging in the numbers from the previous part, we have* $P(X \ge \alpha/2) = 5/6$.

(c) Let $Y = 2X$ be another random variable.

i. (2 points) Find the PDF of Y, $g(y)$.

Solution: We have

$$\begin{aligned} g(y) &= f(u^{-1}(y))\frac{du^{-1}(y)}{dy} \\ &= f(y/2)/2 \\ &= \begin{cases} 0 & \textit{if } y < 0 \\ y/(4\alpha) & \textit{if } 0 \le y < 2\alpha \\ e^{3(\alpha - y/2)}/2 & \textit{if } 2\alpha \le y \end{cases}. \end{aligned}$$

ii. (2 points) Find the correlation coefficient of X and Y, ρ_{XY}.

Solution: We know $\sigma_Y^2 = 4\sigma_X^2$. *We have*

$$\begin{aligned}
\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \\
&= \frac{E[(X-\mu_X)(Y-\mu_Y)]}{2\sigma_X \sigma_X} \\
&= \frac{2E[(X-\mu_X)(X-\mu_X)]}{2\sigma_X^2} \\
&= \frac{2E[(X-\mu_X)^2]}{2\sigma_X^2} \\
&= \frac{2\sigma_X^2}{2\sigma_X^2} \\
&= 1.
\end{aligned}$$