ESC194 - Midterm Test #2
November 26, 2020
9:10 - 10:40 am, EST

The following materials are considered to be acceptable aids during the writing of this test:

- The Stewart textbook and the student solution manuals
- The Stangeby/Barbeau ESC194 Supplement
- Any course notes or problem solutions prepared by the student
- Any handouts or other materials posed on the ESC194 course website
- Any non-programmable calculator; graphing calculators are not allowed

All questions are worth 10 marks

1. Evaluate the integrals:

$$\text{a) } \int_0^2 (2-x)^7\,dx \qquad \text{b) } \int_0^2 (2-x)\sqrt{x}\,dx \qquad \text{c) } \int_{3/4}^3 \frac{dx}{x\sqrt{16x^2-9}}$$

$$\text{d) } \int_0^1 \frac{e^x+1}{e^x}\,dx \qquad \text{e) } \int_{-1}^1 \frac{\sin x}{1+x^2}\,dx$$

a) $\int_0^2 (2-x)^7\,dx$ let $u = 2-x$, $du = -dx$; $u(x=0) = 2$, $u(x=2) = 0$

$= \int_2^0 -u^7\,du = \int_0^2 u^7\,du = \left[\frac{u^8}{8}\right]_0^2 = \frac{256}{8} = 32$

b) $\int_0^2 (2-x)\sqrt{x}\,dx = \int_0^2 (2x^{1/2} - x^{3/2})\,dx = \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2$

$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}$

c) $\int_{3/4}^3 \frac{dx}{x\sqrt{16x^2-9}}$ let $u = 4x$, $du = 4\,dx$; $u(\frac{3}{4}) = 3$, $u(3) = 12$

$= \int_3^{12} \frac{du/4}{\frac{u}{4}\sqrt{u^2-3^2}} = \frac{1}{3}\left[\sec^{-1}\left(\frac{|u|}{3}\right)\right]_3^{12} = \frac{1}{3}\sec^{-1}4$

d) $\int_0^1 \frac{e^x+1}{e^x}\,dx = \int_0^1 (1+e^{-x})\,dx = \left[x - e^{-x}\right]_0^1 = (1-\frac{1}{e}) - (0-1)$

$= 2 - \frac{1}{e}$

e) $\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx = 0$ since the integrand is an odd function

2. Consider the function: $f(x) = e^{3x} - e^x$

a) Determine the domain of f and find any intercepts.

b) Find the intervals in which f increases or decreases.

c) Find the extreme values.

d) Determine the concavity of the graph, and find the inflection points.

e) Sketch the graph specifying the asymptotes, if any.

$f(x) = e^{3x} - e^x$ $\quad$ $f'(x) = 3e^{3x} - e^x$ $\quad$ $f''(x) = 9e^{3x} - e^x$

a) Domain: $x \in \mathbb{R}$

Intercepts: $x = 0 \Rightarrow y = 0$

$y = 0 \Rightarrow e^{3x} = e^x \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

b) $f'(x) = 0 \Rightarrow e^{2x} = \frac{1}{3} \Rightarrow 2x = \ln\frac{1}{3} \Rightarrow x = -\frac{\ln 3}{2}$

$f'(x) = e^x(3e^{2x} - 1)$ < 0 for $e^{2x} < \frac{1}{3}$ or $x \in (-\infty, -\frac{\ln 3}{2})$ Decr

> 0 for $e^{2x} > \frac{1}{3}$ $\quad x \in (-\frac{\ln 3}{2}, \infty)$ Incr

c) $f(-\frac{\ln 3}{2}) = e^{3(-\frac{\ln 3}{2})} - e^{(-\frac{\ln 3}{2})} = (\frac{1}{3})^{3/2} - (\frac{1}{3})^{1/2} = (\frac{1}{3})^{1/2}(\frac{1}{3} - 1) = \frac{-2}{3\sqrt{3}}$ = local and abs min

d) $f''(x) = 0 \Rightarrow e^{2x} = \frac{1}{9} \Rightarrow 2x = \ln\frac{1}{9} \Rightarrow x = \frac{-\ln 9}{2}$

$f''(x) = e^x(9e^{2x} - 1)$ < 0 for $e^{2x} < \frac{1}{9}$ or $x \in (-\infty, -\frac{\ln 9}{2})$ CD

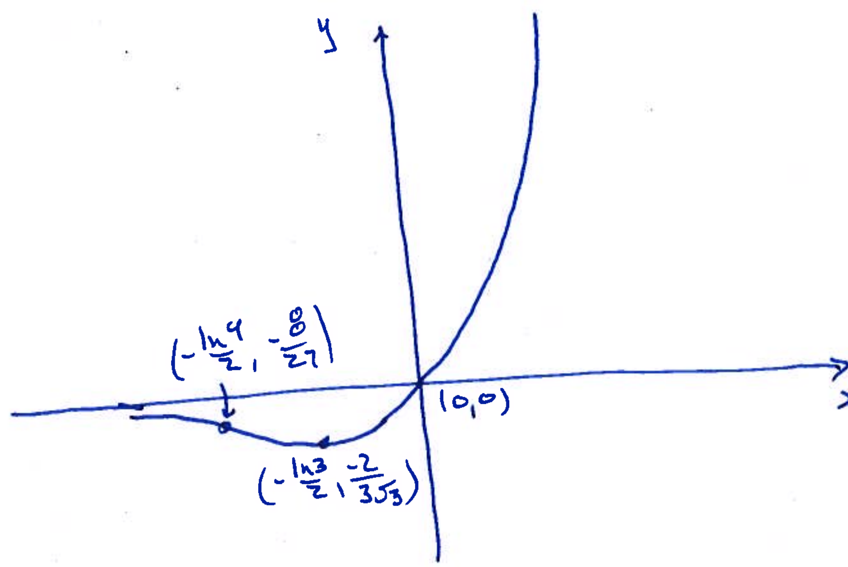
> 0 for $e^{2x} > \frac{1}{9}$ or $x \in (-\frac{\ln 9}{2}, \infty)$ CU

point of inflection at $x = -\frac{\ln 9}{2}$, $y = \frac{1}{3}(\frac{1}{9} - 1) = -\frac{8}{27}$

e) $\lim_{x \to -\infty} e^{3x} - e^x = 0$

$\therefore y = 0$ is a HA

No vertical asymptotes



3. a) Consider the region bounded by the curves $y = x^2$ and $y = x^{1/3}$. Find, but do NOT solve integrals which represent the volume formed when:

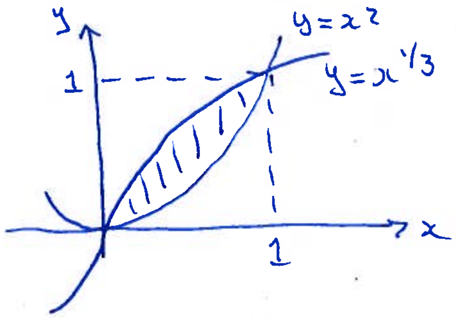
   i) the region is rotated about the x-axis, formulated by the washer method.

   ii) the region is rotated about the y-axis, formulated by the washer method

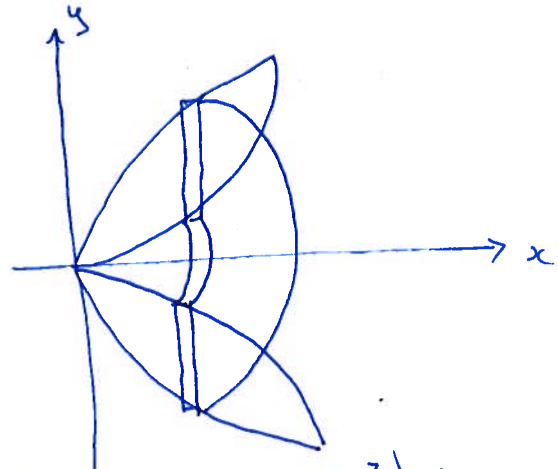
   b) Consider the region bounded by the curves $y = \sqrt{x}$ and $y = x^3$. Use the shell method to find, but do NOT solve, the integral which represents the volume formed when the region is rotated about the x-axis.

   Provide sketches of all volumes

a)

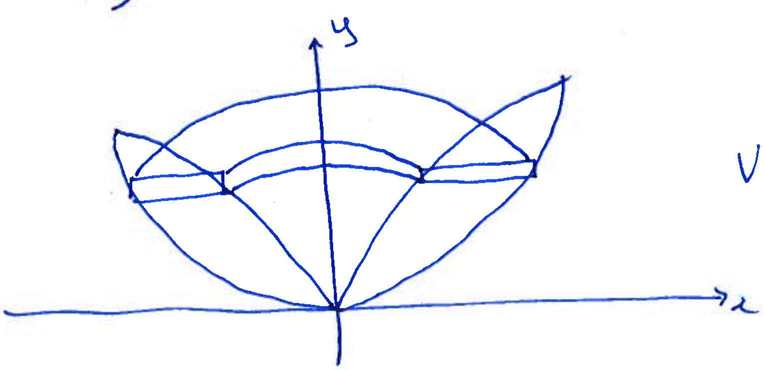


i)



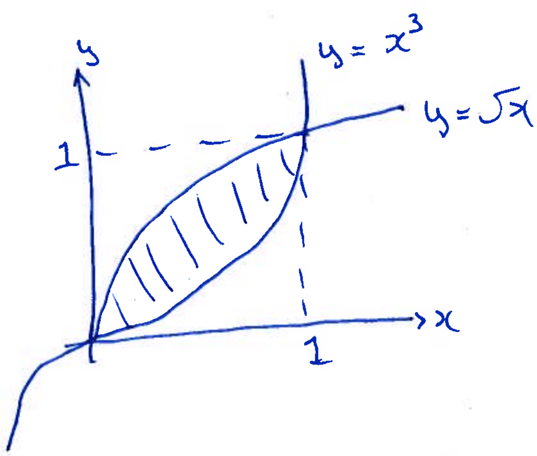
$$V = \int_0^1 \pi \left(\left(x^{1/3} \right)^2 - \left(x^2 \right)^2 \right) dx$$

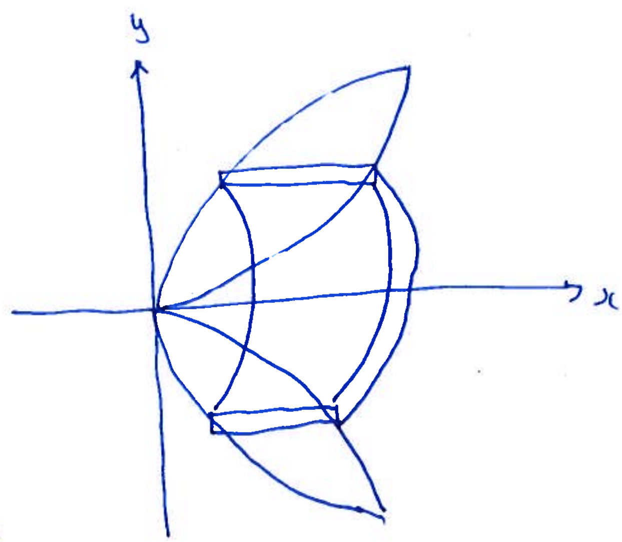
ii)



$$V = \int_0^1 \pi \left(\left(y^{1/2} \right)^2 - \left(y^3 \right)^2 \right) dy$$

b)





$$V = \int_0^1 2\pi y \left(y^{1/3} - y^2 \right) dy$$

4. a) Determine whether or not the function $f(x) = x^3 - 1$ is one-to-one and, if so, find its inverse. Give the domain of f^{-1} if it exists..

b) Verify that the function $f(x) = 2x + \cos x$ has an inverse and find $(f^{-1})'(\pi)$.

a) $f(x) = x^3 - 1 \quad \Rightarrow \quad f'(x) = 3x^2 > 0 \; , \; x \neq 0, \; \therefore$ increasing

at $x=0$: $\quad f(x=-\epsilon) = -\epsilon^3 - 1 < -1 < \epsilon^3 - 1 = f(x=+\epsilon)$

$\epsilon > 0$

$\therefore$ increasing $x \in \mathbb{R}$. $\quad \therefore$ one-to-one

let $y = f^{-1}(x) \Rightarrow x = y^3 - 1 \Rightarrow y = f^{-1}(x) = (x+1)^{1/3}$

$x \in \mathbb{R}$

b) $f(x) = 2x + \cos x \quad \Rightarrow \quad f'(x) = 2 - \sin x > 0 \quad x \in \mathbb{R}$

$\therefore$ increasing, $\therefore$ one-to-one

$f\left(\frac{\pi}{2}\right) = \pi \; , \; f'\left(\frac{\pi}{2}\right) = 1$

$\therefore (f^{-1})'(\pi) = \dfrac{1}{f'\left(\frac{\pi}{2}\right)} = 1$

5. a) If f is a continuous function such that $\int_a^x f(t)dt = x^2 \tan^2 x + \int_a^x \frac{f(t)}{(1+t^3)}dt$ for all x, find an explicit formula for $f(x)$.

b) If $x^3 \tan(\pi x) = \int_a^{x^3} f(t)dt$, where f is a continuous function, find $f(8)$ and find a value for the constant a that will satisfy the expression.

a) $\int_a^x f(t)\,dt = x^2 \tan^2 x + \int_a^x \frac{f(t)}{(1+t^3)}\,dt$

Differentiating:

$f(x) = 2x\tan^2 x + x^2 \cdot 2\tan x \cdot \sec^2 x + \frac{f(x)}{(1+x^3)}$

$f(x)\left(1 - \frac{1}{1+x^3}\right) = 2x\left(\tan^2 x + x\tan x \sec^2 x\right)$

$\Rightarrow f(x) = \frac{2(1+x^3)(\tan^2 x + x\tan x \sec^2 x)}{x^2}$

b) $x^3 \tan(\pi x) = \int_a^{x^3} f(t)\,dt$

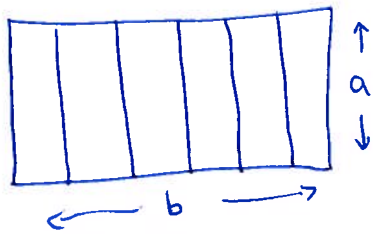
$3x^2 \tan(\pi x) + x^3 \sec^2(\pi x)\cdot\pi = f(x^3)\cdot 3x^2$ by Chain Rule

or $f(x^3) = \tan(\pi x) + \frac{\pi x}{3}\sec^2(\pi x)$

for $x = 2$: $f(8) = \tan(2\pi) + \frac{2\pi}{3}\sec^2(2\pi) = \frac{2\pi}{3}$

for $x = 0$: $0 = \int_a^0 f(t)\,dt \Rightarrow a = 0$ is a sol'n.

6. A veterinarian has 100 m of fencing, and wishes to construct six dog kennels by first building a fence around a rectangular region, and then subdividing the region into six smaller rectangles by placing five fences parallel to one of the sides. What dimensions of the regions will maximize the total area.



$l = 100\,m = 7a + 2b$

maximize total area: $A = a \times b$

$\Rightarrow A = a \cdot \left(\frac{100 - 7a}{2}\right) = 50a - \frac{7}{2}a^2$

$\frac{dA}{da} = 50 - 7a$

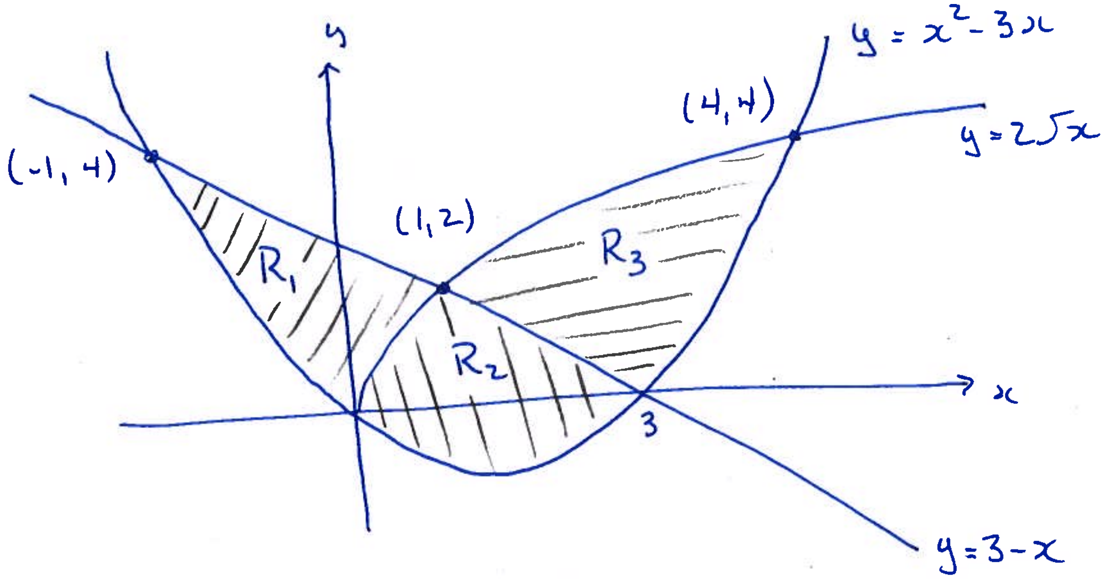
$\frac{dA}{da} = 0 \Rightarrow a = \frac{50}{7}$

$\therefore b = \frac{100 - 7(50/7)}{2} = 25$

$\frac{d^2A}{da^2} = -7 < 0 \therefore$ concave down

$\therefore A(\frac{50}{7}, 25)$ is a maximum

7. Three regions are enclosed by the curves: $y = 2\sqrt{x}$, $y = 3 - x$, and $y = x(x-3)$. Find integrals representing the areas of each of these regions. Do NOT evaluate the integrals.



Intersections: $3 - x = x(x-3) \Rightarrow x = 3,\ y = 0$
$x = -1,\ y = 4$

$3 - x = 2\sqrt{x} \Rightarrow 9 - 6x + x^2 = 4x$
$x^2 - 10x + 9 = 0 \Rightarrow (x-1)(x-9) = 0$
$\therefore\ x = 1,\ y = 2$
or $x = 9$ (require $x < 3$)

$x^2 - 3x = 2\sqrt{x} \Rightarrow x^3 - 6x^2 - 9x - 4 = 0$
$(x-1)^2(x-4) = 0 \Rightarrow x = 4, y = 4$
(require $x > 3$)

$$R_1 = \int_{-1}^{0} (3 - x - (x^2 - 3x))\,dx + \int_0^1 (3 - x - 2\sqrt{x})\,dx$$

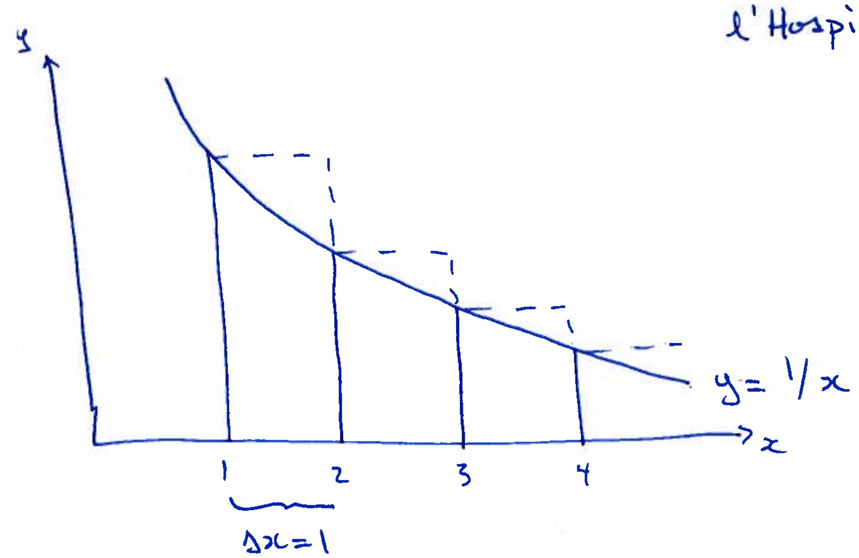
$$R_2 = \int_0^1 (2\sqrt{x} - (x^2 - 3x))\,dx + \int_1^3 (3 - x - (x^2 - 3x))\,dx$$

$$R_3 = \int_1^3 (2x - (3 - x))\,dx + \int_3^4 (2\sqrt{x} - (x^2 - 3x))\,dx$$

8. a) In Chapter 10, we will encounter the harmonic series: $S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Use a left-hand endpoint Riemann sum to approximate $\int_1^{n+1} \frac{dx}{x}$ (with unit spaciong between grid pooints) to show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \ln(n+1)$. What can you conclude about the sum: $\sum_{i=1}^{n} \frac{1}{i}$ as $n \to \infty$? Provide a sketch to support your reasoning.

b) Given n a positive integer, prove $e^x > x^n$ for all x sufficiently large. (Do not use l'Hospital's rule.)

a)



Area of rectangles $= 1 \times 1 + \frac{1}{2} \times 1 + \frac{1}{3} \times 1 + \dots + \frac{1}{n} \times 1$

Area under curve $= \int_1^{n+1} \frac{1}{x}\,dx$ < area of rectangles

$\therefore \int_1^{n+1} \frac{dx}{x} = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \sum_{i=1}^{n} \frac{1}{i}$

$\lim_{n\to\infty} \ln(n+1) = \infty \quad \therefore \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i} = \infty$

b) For $x > (n+1)!$: $e^x > 1 + x + \dots + \frac{x^{n+1}}{(n+1)!} > \frac{x^{n+1}}{(n+1)!} = x^n \left[\frac{x}{(n+1)!}\right] > x^n$

9. The equation $2y \ln y = x^2 - 4$ desribes a curve in the upper half plane that has two branches. Determine how close thse two branches come together.

$\Rightarrow$ Curves are symmetric wrt x: $x = \pm\sqrt{2y\ln y + 4}$

$\therefore$ curves will be closest together for a minimum value of x

$\Rightarrow 2x\,dx = 2\ln y\,dy + 2dy \Rightarrow \dfrac{dx}{dy} = \dfrac{\ln y + 1}{x}$

consider +ve branch, $x > 0$

$$\therefore \frac{dy}{dx} = \frac{\ln y + 1}{\sqrt{2y\ln y + 4}}$$

$\Rightarrow \dfrac{dx}{dy} = 0 \Rightarrow \ln y + 1 = 0 \Rightarrow y = \dfrac{1}{e}$

$\dfrac{dx}{dy} > 0,\ y > \dfrac{1}{e}$ $\therefore$ increasing

$\dfrac{dx}{dy} < 0,\ y < \dfrac{1}{e}$ $\therefore$ decreasing

$\Rightarrow$ there is a single minimum at $y = \dfrac{1}{e}$

$\therefore$ minimum distance separating curves

$$= 2x\left(\tfrac{1}{e}\right) = 2\sqrt{2\cdot\tfrac{1}{e}\ln\tfrac{1}{e} + 4} = 2\sqrt{4 - \tfrac{2}{e}}$$