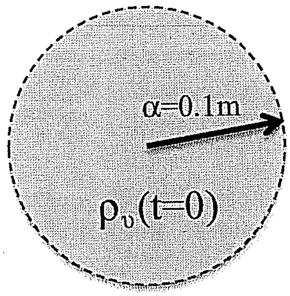
Question 1 [25 pts]

A. A lightning strike deposits, at t=0, total charge Q=1 mC <u>uniformly</u> within the volume of a lossy dielectric sphere with $\varepsilon=1.2\varepsilon_0$, σ=10 S/m and radius α=0.1m ($\varepsilon_0 \approx 8.854\times10^{-12}\,\mathrm{H/m}$).



1. At t=0, determine the electrostatic energy inside and outside the sphere, first determining the electric field inside and outside the sphere [4 pts], then the volume density of electric energy inside and outside the sphere [4 pts], and finally the total energy [2 pts].

Useful integrals: $\int_0^{\pi}\sin\theta\, d\theta = 2, \int R^p\, dR = \frac{R^{p+1}}{p+1}$ (p=integer). **[10 pts]**

$\rho_v = Q / \frac{4}{3}\pi\alpha^3$ (since distribution is uniform)

[4]

Inside sphere:

Gauss Law $\Rightarrow$ $E_R \cdot \cancel{4\pi} R^2 = \frac{1}{1.2\epsilon_0}\rho_v \cdot \frac{4}{3}\cancel{\pi} R^3 \Rightarrow E_R = \frac{\rho_v}{1.2\epsilon_0} R$

Outside

Gauss Law: $E_R \cdot 4\pi R^2 = \frac{1}{\epsilon_0} Q \Rightarrow E_R = \frac{Q}{4\pi\epsilon_0 R^2}$

Gauss Law on sphere, LHS $= E_R \cdot 4\pi R^2$ [2pts]

$Q_{inside} = \rho_v \cdot \frac{4}{3}\pi R^3$, $Q_{outside} = Q$

[1pt] [1pt]

$$w_e = \begin{cases} \frac{1}{2} \cdot 1.2\epsilon_0 \frac{\rho_v^2}{(1.2\epsilon_0)^2} \cdot R^2 = \frac{\rho_v^2 R^2}{2.4\epsilon_0} & \text{[2pts]} \\ \frac{1}{2}\epsilon_0 \frac{Q^2}{16\pi^2\epsilon_0^2 R^4} & \text{[2pts]} \end{cases}$$

[4]

$$W_{e,inside} = \frac{\rho_v^2}{2.4\epsilon_0} \int R^2 \cdot R^2 \sin\theta \, d\theta \, d\varphi \, dR$$

$$= \frac{\rho_v^2}{2.4\epsilon_0} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} \int_0^{\alpha} R^4 \, dR$$

$$= \frac{\rho_v^2}{2.4\epsilon_0} \cdot 4\pi \frac{\alpha^5}{5} = \frac{\pi \rho_v^2 \alpha^5}{3\epsilon_0} \qquad [1]$$

$$W_{e,outside} = \frac{Q^2}{32\pi^2\epsilon_0} \overbrace{\int_{\alpha}^{\infty} \frac{1}{R^2} \cdot dR}^{1/\alpha} \overbrace{\int_0^{\pi} \sin\theta \, d\theta}^{2} \overbrace{\int_0^{2\pi} d\varphi}^{2\pi}$$

$$= \frac{Q^2}{32\pi^{\cancel{2}}\alpha\epsilon_0} 4\cancel{\pi} = \frac{Q^2}{8\pi\alpha\epsilon_0} \qquad [1]$$

Total:

$$W_e = \frac{\rho_v^2 \pi \alpha^5}{3\epsilon_0} + \frac{Q^2}{8\pi\alpha\epsilon_0} = \frac{Q^2 \cancel{\pi\alpha^5}}{\frac{16}{\cancel{9}\,3}\pi^{\cancel{2}}\alpha^{\cancel{6}} \cdot 3\epsilon_0} + \frac{Q^2}{8\pi\alpha\epsilon_0}$$

$$= \frac{3 \cdot Q^2}{16\pi\alpha\epsilon_0} + \frac{Q^2}{8\pi\alpha\epsilon_0} = \frac{5Q^2}{16\pi\alpha\epsilon_0} = \frac{5 \times 10^{-3}\,C}{16\pi \times 0.1 \times 8.854 \times 10^{-12}}$$

$$= 1.1235 \times 10^8 \, J$$

2. Find the volume charge density as a function of time $\rho_\upsilon(t)$. **[5 pts]**

From $\nabla \cdot \bar{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow$

$$\nabla \cdot \left(\frac{\sigma}{\epsilon} \bar{D} \right) + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0$$

$$\boxed{\rho = \rho(t=0)\, e^{-\sigma t/\epsilon}}$$ [2 pts]

with $\sigma = 10$ S/m [1 pt]

$\epsilon = 1.2\, \epsilon_0$ [1 pt]

$$\rho(t=0) = \frac{Q}{\frac{4}{3}\pi a^3}$$ [1 pt]

3. Does the electrostatic energy stored in the sphere change with time? If it does, what happens to this energy? Does the electrostatic energy stored outside the sphere change with time? **[5 pts]**

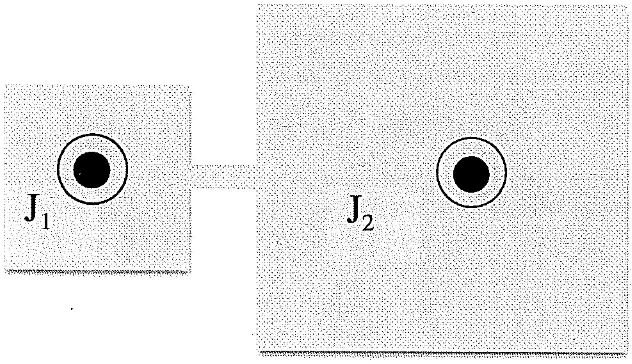
As $\rho_\upsilon \to 0$, $W_e^{inside} \to 0$ [2 pts]

Converted to heat [2 pts]

Energy stored outside does NOT change with time (sphere behaves like a charge Q at the origin for $R > a$) [1 pt].

B. The following questions are independent from each other.

1. The following figure shows the cross section of a long conductor with uniform conductivity σ that carries a steady current, as shown in the figure. The volume current densities J_1 and J_2 in the two parts of the conductor are related at steady state as: $J_1<J_2$, $J_1>J_2$ or $J_1=J_2$? Explain. **[2 pts]**



$J_1 = J_2$ due to $E_1 = E_2$ (tangential E - continuous)

J tangential hence continuous
[1pt] [1pt]

2. The voltage between the two conductors of a capacitor with a dielectric of constant permittivity ε is V_0. If the electric flux density at every point of the dielectric is doubled, what is the new voltage between the conductors of the capacitor? **[1 pt]**

$$V = \int_{+}^{-} \bar{E} \cdot d\bar{l} = \underbrace{\int_{+}^{-} \frac{\bar{D}}{\epsilon} \cdot d\bar{l}}_{1\ pt.} = 2\, V_{initial}$$

Consider an electrostatic system. If we change the reference point adopted for the potential, how will the following quantities change?

a. The electric potential **[1 pt]**
b. The electric potential difference (voltage) **[1 pt]**

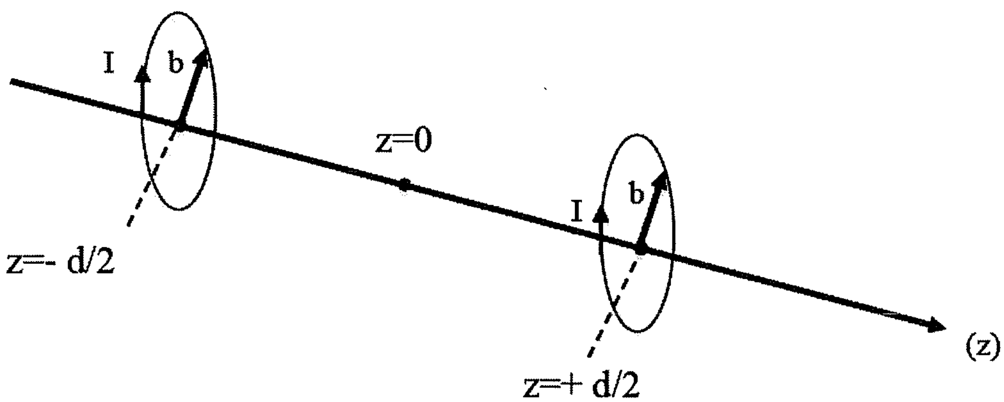
Potential: changes because it is referred to different point [1]

Voltage stays the same $(V = W/q)$ [1]

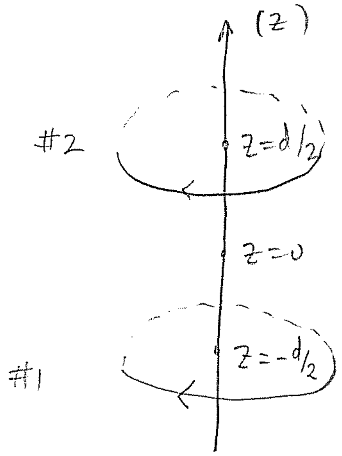
--- End of Question 1

Question 2 [25 pts]

A. A Helmholtz coil, widely used in open MRI systems, is a pair of two identical coils of radius b each, at distance d from each other in free space ($\mu_0 = 4\pi \times 10^{-7}\,\mathrm{H/m}$), as shown in the figure below. Each coil carries current I in the same direction.



1. Using the Biot-Savart law, determine the magnetic flux density **B** at any point on the z-axis. *Optional*: To test the correctness of your result, you can confirm that d|**B**| /dz=0 at z=0 (i.e. magnetic flux is uniform around the position of the patient). **[15 pts]**



$$d\bar{B} = \frac{\mu_0}{4\pi} \frac{I\, d\bar{l}' \times (\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|^3}$$

Coil 1, 2: $I\, d\bar{l}' = -I\, b\, d\varphi'\, \bar{a}_{\varphi'}$ [2pts]

$\bar{R} = z\bar{a}_z$

$\bar{R}_1' = b\bar{a}_{r'} - \frac{d}{2}\bar{a}_z$

$\bar{R}_2' = b\bar{a}_{r'} + \frac{d}{2}\bar{a}_z$

$\underbrace{|\bar{R} - \bar{R}_{1,2}'| = \sqrt{b^2 + \left(z \pm \frac{d}{2}\right)^2}}_{[4pts]}$, $\quad \bar{R} - \bar{R}_{1,2}' = \left(z \pm \frac{d}{2}\right)\bar{a}_z - b\bar{a}_{r'}$

$$-I\, b\, d\varphi'\, \bar{a}_{\varphi'} \times \left\{\left(z \pm \frac{d}{2}\right)\bar{a}_z - b\bar{a}_{r'}\right\} = -I\, b\, d\varphi' \left\{\left(z \pm \frac{d}{2}\right)\bar{a}_{r'} + b\bar{a}_z\right\} \text{ [4pts]}$$

$$\bar{B} = \frac{\mu_0}{4\pi}\left\{ \underbrace{\int_0^{2\pi}}_{[2pts]} \frac{-Ib\,d\varphi'\{(z+\frac{d}{2})\bar{a}_{r'} + b\bar{a}_z\}}{[b^2+(z+\frac{d}{2})^2]^{3/2}} + \int_0^{2\pi} \frac{-Ib\,d\varphi'\{(z-\frac{d}{2})\bar{a}_{r'} + b\bar{a}_z\}}{[b^2+(z-\frac{d}{2})^2]^{3/2}} \right\}$$

$\underline{a_{r'}}$ integrated to 0

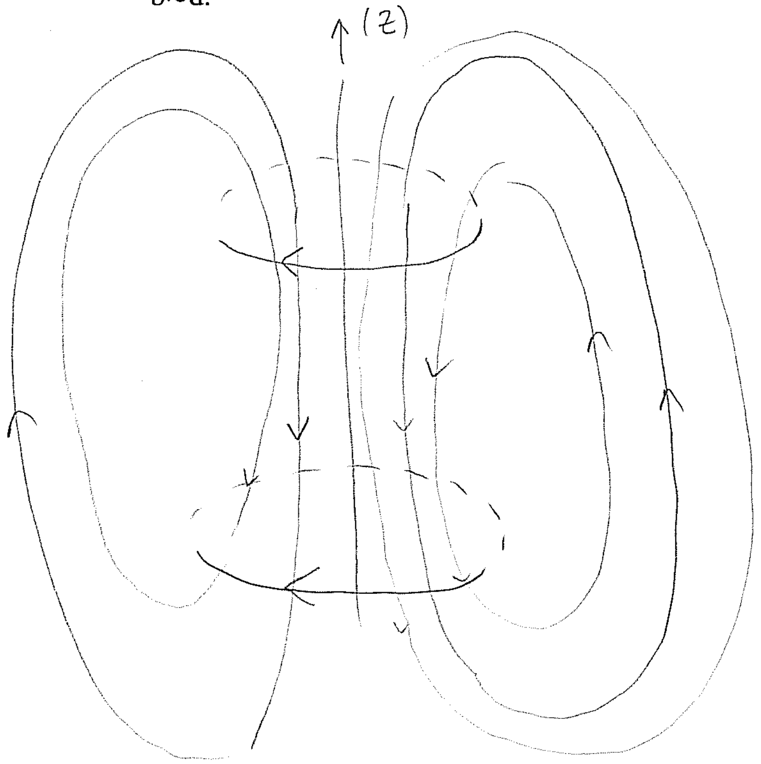
$$\frac{\mu_0}{4\pi}(-Ib^2\bar{a}_z)\left[2\pi\frac{1}{[b^2+(z+\frac{d}{2})^2]^{3/2}} + 2\pi\frac{1}{[b^2+(z-\frac{d}{2})^2]^{3/2}}\right]$$

[1pt]

$$=\bar{a}_z\frac{\mu_0}{2}(-Ib^2)\left[\frac{1}{(b^2+(z+\frac{d}{2})^2)^{3/2}} + \frac{1}{(b^2+(z-\frac{d}{2})^2)^{3/2}}\right]$$

[2pts]

2. Sketch the magnetic flux lines produced by this system of coils, assuming that b≈d. **[3 pts]**



Lines pointing DOWNWARDS [1]

Closed [2]

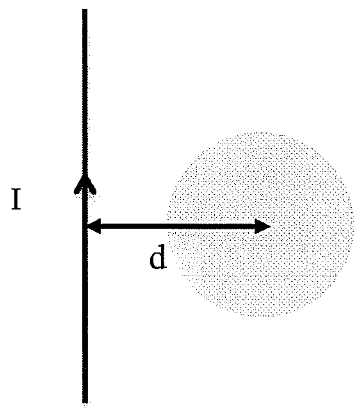
3. What is the force between the coils (i.e. repulsive or attractive)? **[2 pts]**

Parallel currents $\Rightarrow$ attractive

[1pt] [1pt]

B. The following questions are independent from each other.

1. A sphere of radius α is placed in free space near a wire carrying a constant current I, as shown in the figure below. The distance of the wire from the center of the sphere is d. Does the total magnetic flux through the sphere depend on I, d, α? **[1 pt]**



No. $\oint \bar{B} \cdot d\bar{s} = 0$

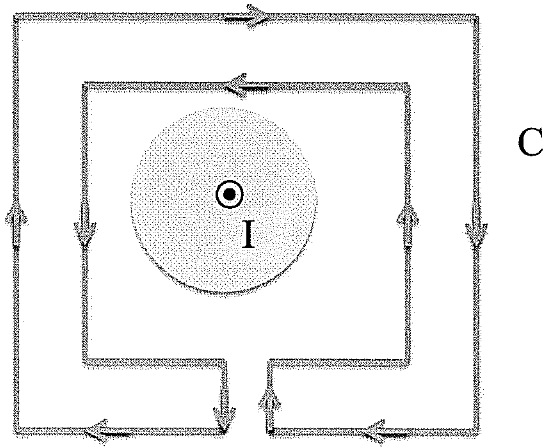
regardless.

2. In a region of space, the magnetic flux density is uniform and constant equal to $\mathbf{B}_0$. What is the volume current density in that region? **[2 pts]**

$$\nabla \times \bar{B} = \bar{J} \Rightarrow \bar{J} = 0 \text{ since } \nabla \times \bar{B}_0 = 0$$

[1pt] [1pt]

3. A cylindrical copper conductor carries a constant current I, as shown in the figure below. What is $\oint_C \mathbf{H} \cdot \mathbf{d}\ell$ along the closed path C as defined below? **[2 pts]**

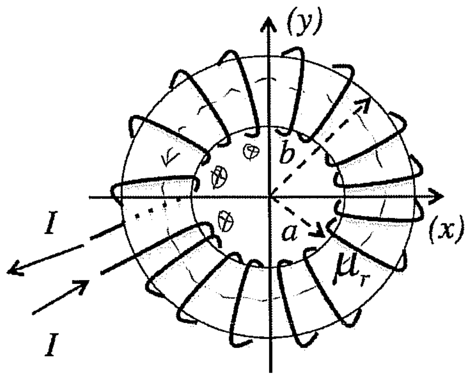


$$\underbrace{\text{No enclosed}}_{1} \text{ current} \Rightarrow \underbrace{\oint \bar{H} \cdot d\bar{l} = 0}_{1}$$

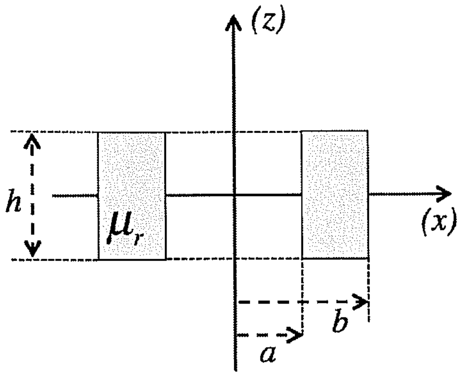
--- End of Question 2

Question 3 [25 pts]

A. Consider the toroidal coil shown below. The core, shown in the figure in grey, is made of a magnetic material with relative permeability $\mu_r = 3$ and has a rectangular cross section. The core has inner radius a, outer radius b, height h. The coil has N closely wound turns, where a dc current I flows.



Top view



Side view (cross section)

Find the magnetic energy stored in the coil and the self-inductance of the coil, in the following steps: using Ampere's law, find the magnetic flux density **B** in the core [5 pts]; find the volume density of magnetic energy and the total magnetic energy stored inside the core [10 pts]; find the inductance [5 pts]. **[20 pts]**

Ampere law:

$$\oint_C \bar{B} \cdot d\bar{l} = \mu_0 \mu_r I_{enclosed}$$

LHS: $\bar{B} = B_\varphi(r)$ [1 pt]

$$\Rightarrow \underbrace{\oint_C}_{\text{circle of radius } r} \bar{B} \cdot d\bar{l} = B_\varphi \cdot 2\pi r \quad [1pt]$$

[1pt]

$$\Rightarrow B_\varphi = -\frac{\mu_0 \mu_r \cdot N \cdot I}{2\pi r}$$

RHS $= (-)\mu_0 \mu_r NI$ [1pt]

[1pt]

$$w_m = \underbrace{\frac{1}{2}\frac{B^2}{\mu_0\mu_r}}_{[2pts]} = \frac{1}{2}\frac{\cancel{\mu_0^2}\mu_r^{\cancel{2}} N^2 \cdot I^2}{\cancel{\mu_0}\cancel{\mu_r}\cdot 4\pi^2 r^2} = \underbrace{\frac{1}{2}\mu_0\mu_r \frac{N^2 I^2}{4\pi^2 r^2}}_{[2pts]}$$

$$W_m = \frac{1}{2}\mu_0\mu_r \frac{N^2 I^2}{4\pi^2}\int \frac{1}{r^2}\cdot \overbrace{\underbrace{(r\,d\varphi\,dr\,dz)}_{dv \text{ in cylindrical system}}}^{3}$$

$$= \frac{1}{2}\mu_0\mu_r\frac{N^2I^2}{4\pi^2}\cdot \underbrace{\int_a^b \frac{dr}{r}}_{(1pt)} \underbrace{\int_0^{2\pi} d\varphi}_{(1pt)} \underbrace{\int_0^h dz}_{(1pt)}$$

$$= \frac{1}{\cancel{2}}\mu_0\mu_r\frac{N^2I^2}{4\pi^{\cancel{2}}}\ln\frac{b}{a}\cdot\cancel{2\pi}\cdot h$$

$$= \frac{1}{2}\cdot I^2\left\{\frac{\mu_0\mu_r N^2}{2\pi}\left(\ln\frac{b}{a}\right)\cdot h\right\}$$

$$\underbrace{W_m = \frac{1}{2}LI^2}_{[2pts]} \Rightarrow \underbrace{\boxed{L = \frac{\mu_0\mu_r}{2\pi}N^2\cdot h\cdot \ln\frac{b}{a}}}_{[3pts]}.$$

B. The following questions are independent from each other.

1. Consider a lossy medium with parameters ε_0, μ_0, σ, and volume density of conduction current $\mathbf{J}(t)=J_0 \cos(\omega t)\, \mathbf{a}_x$, where J_0 is a constant. What is the volume density of the displacement current? **[3 pts]**

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t} = \overbrace{\frac{\partial}{\partial t}\left(\epsilon \bar{J}/\sigma\right)}^{[2pts]}$$

$$= \underbrace{\frac{\epsilon J_0}{\sigma}\left(-\omega \sin \omega t\right)}_{[1pt]} \bar{a}_x$$

2. Does the displacement current exist in air or not? If it does, can you think of an example? **[2 pts]**

Yes, (1pt) air filled capacitor (1pt)

--- **End of Question 3**

Question 4 [25 pts]

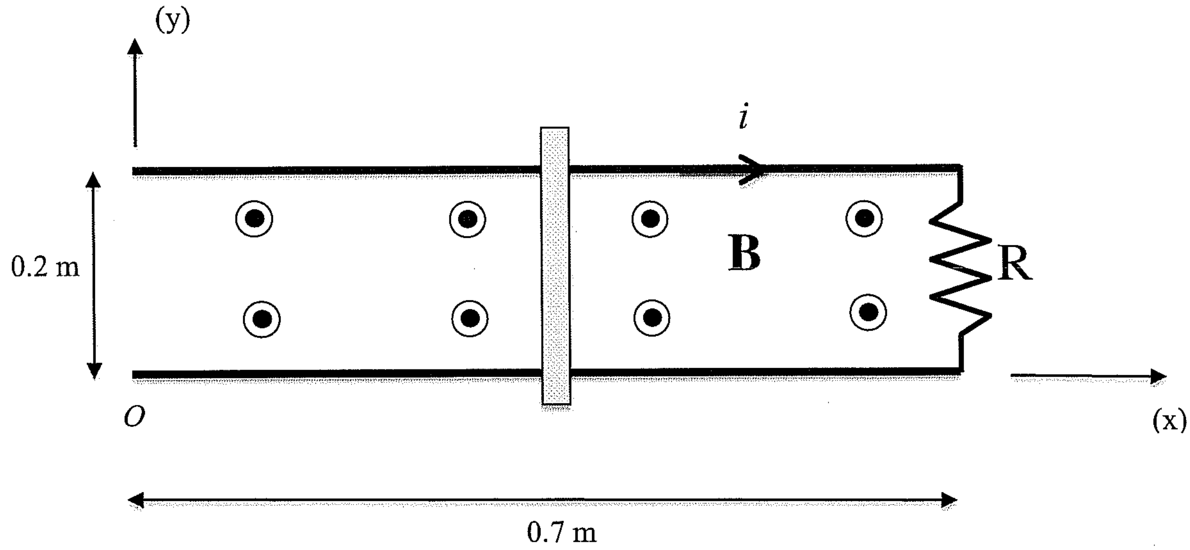
A. A conducting sliding bar oscillates over two parallel conducting rails in a sinusoidally varying magnetic field:

$$\mathbf{B} = \mathbf{a}_z 5\cos\omega t \;\; \text{mT}$$

as shown in the figure below. The position of the sliding bar is given by:

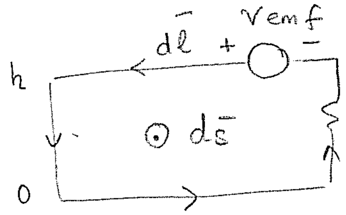
$$x = 0.35(1 - \cos\omega t) \;\; \text{m}$$

and the rails are terminated in a resistance R = 0.2 Ohm.



1. Determine the motional and the transformer electromotive forces (emf's) in this circuit, clearly showing their polarity. **[10 pts]**

Polarity:



$d\bar{l}$, $d\bar{s}$ consistent with RH-rule

This is the polarity for the emf's found below.

MOTIONAL: $\oint(\bar{v} \times \bar{B}) \cdot d\bar{l} = \int_h^0 \left(\frac{dx}{dt} \bar{a}_x \right) \times \left(\bar{a}_z\, 5\times10^{-3} \cos(\omega t) \right) \cdot \bar{a}_y \, dy$

(since rod is the only moving part)

$= \int_h^0 \left[0.35\omega \sin(\omega t) \times 5\times10^{-3} \cos(\omega t) \right] (-1)\, dy = +\, 1.75\times10^{-3} \times \omega \times \sin(\omega t) \cdot \cos(\omega t) = \frac{1.75}{2} \times 10^{-3} \times \omega \times \sin(2\omega t) \times 0.2 = 1.75\times10^{-4}\, \omega \times \sin(2\omega t)$

TRANSFORMER

$$-\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} = -\int \left(-\bar{a}_z\, 5\omega \sin(\omega t) \times 10^{-3}\right) \cdot \bar{a}_z \, dx\, dy$$

$$= 5\omega \cdot \sin(\omega t) \times 10^{-3} \left(0.7 - x(t)\right) \cdot h$$

$5h = 5 \times 0.2 = 1$

$$= \omega \cdot \sin(\omega t) \times 10^{-3} \left[\overbrace{0.75 - 0.35}^{0.35} + 0.35 \cos(\omega t)\right]$$

$$= 0.35\omega \cdot \sin(\omega t) \cdot 10^{-3} + 0.35 \times 10^{-3} \times \omega \times \sin(\omega t) \cos(\omega t)$$

$$= 0.35\, \omega \sin(\omega t) \times 10^{-3} + 1.75 \times 10^{-4} \times \omega \times \sin(2\omega t).$$

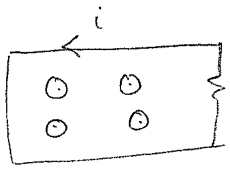
2. Determine the current $i(t)$. Explain its direction in terms of Lenz's law for $0<\omega t<\pi/2$. **[10 pts]**

$$\text{Total emf} = 0.35\,\omega \left(\sin(\omega t) + \sin(2\omega t)\right) \text{ mV}$$

$$+V_{emf} + iR = 0 \Rightarrow i = -\frac{V_{emf}}{R} = -1.75\,\omega \left(\sin(\omega t) + \sin(2\omega t)\right)$$

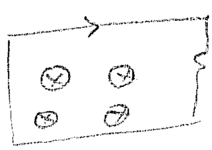
For $V_{emf} > 0 \Rightarrow d\Phi/dt < 0 \Rightarrow i < 0$

Hence, when $d\Phi/dt < 0$, the current flows as:



producing $\bar{B}$ that REINFORCES the external $\bar{B}$ $\Rightarrow$ resists the decay in flux.

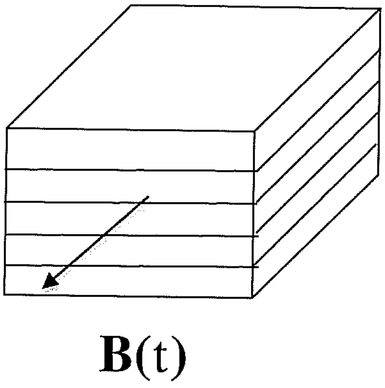
When $d\Phi/dt > 0$, the current flows as:

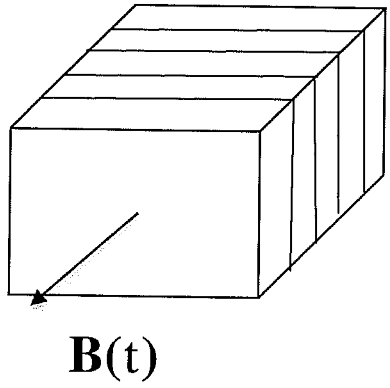


producing $\bar{B}$ that OPPOSES the external $\bar{B}$ $\Rightarrow$ resists the increase in flux

(all the above for $0 < \omega t < \pi/2$, $\bar{B} \parallel \bar{a}_z$).

B. Laminated cores, made by mutually insulated thin layers of magnetic material, are often used in AC transformers. Two possible choices of mutual orientations between the layers and the magnetic field are shown below.





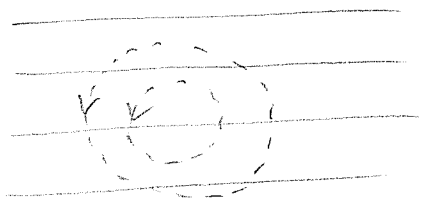
Explain:

1. Why the orientation of the layers influences the efficiency of the transformer. **[3 pts]**
2. Which orientation is preferable. **[2 pts]**

Eddy currents produced by $\bar{B}(t)$ due to the induced V_{emf} ⇒ induced $E(t)$ ⇒ $\bar{J}_{eddy} = \sigma \bar{E}(t)$.

By symmetry $\bar{E} \parallel \bar{a}_\varphi$

The first configuration "open-circuits" the path of the eddy currents, hence it is preferable.



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