## ESC103F Engineering Mathematics and Computation: Tutorial #4

**Question 1:**

i) Find a 3x3 matrix with 3 independent columns and all nine entries equal to 1 or 2.

ii) In part (i), what is the maximum possible number of 1's?

**Solution:**

i) One example is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$.

ii) To maintain 3 independent columns, seven 1's is the maximum possible number of 1's. With eight 1's, two of the columns will be equal.

**Question 2:**

Suppose matrix $A$ is 5x2, with column vectors $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$. We are now going to add one more column to produce matrix $B$, now 5x3. Do $A$ and $B$ have the same column space if:

i) the new column is the zero vector?

ii) the new column is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

iii) the new column is $\overrightarrow{a_2} - \overrightarrow{a_1}$?

**Solution:**

i) In this case, $A$ and $B$ have the same column space defined by combinations of $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$.

ii) If $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is already in the column space of $A$, then $A$ and $B$ have the same column space.

If $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not in the column space of $A$, then $A$ and $B$ do not have the same column space.

iii) Since the new column is $\overrightarrow{a_2} - \overrightarrow{a_1}$ is a combination of $\overrightarrow{a_1}$ and $\overrightarrow{a_2}$, then $A$ and $B$ have the same column space.

**Question 3:**

Assume the vectors $\vec{v}$ and $\vec{w}$ are nonzero and non-parallel. Which of the following sets of vectors are linearly independent:

i) $\{\vec{0}, \vec{v}, \vec{w}\}$
ii) $\{\vec{v}, \vec{w}, 3\vec{v} - 4\vec{w}\}$
iii) $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$

**Solution:**

i) To answer this question, we need to know if there are constants $c_1, c_2, c_3$ that satisfy:

$$c_1\vec{0} + c_2\vec{v} + c_3\vec{w} = \vec{0}$$

other than $c_1 = c_2 = c_3 = 0$. If we choose $c_1 \neq 0$ and $c_2 = c_3 = 0$, this equation is satisfied. Therefore, the vectors are not independent.

ii) To answer this question, we need to know if there are constants $c_1, c_2, c_3$ that satisfy:

$$c_1\vec{v} + c_2\vec{w} + c_3(3\vec{v} - 4\vec{w}) = \vec{0}$$

other than $c_1 = c_2 = c_3 = 0$. If we choose $c_1 = -3, c_2 = 4, c_3 = 1$, this equation is satisfied. Therefore, the vectors are not independent.

iii) To answer this question, we need to know if there are constants $c_1, c_2, c_3$ that satisfy:

$$c_1\vec{v} + c_2\vec{w} + c_3(\vec{v} \times \vec{w}) = \vec{0}$$

other than $c_1 = c_2 = c_3 = 0$. Let's begin by trying to solve for $c_1, c_2, c_3$ using what we know about cross product:

$(\vec{v} \times \vec{w}) \cdot \left( c_1\vec{v} + c_2\vec{w} + c_3(\vec{v} \times \vec{w})\right) = (\vec{v} \times \vec{w}) \cdot \vec{0} = 0$

Since $\vec{v}$ and $\vec{w}$ are orthogonal to $\vec{v} \times \vec{w}$:

$(\vec{v} \times \vec{w}) \cdot \left( c_1\vec{v} + c_2\vec{w} + c_3(\vec{v} \times \vec{w})\right) = 0 + 0 + c_3\|\vec{v} \times \vec{w}\|^2 = 0$

Since $\vec{v}$ and $\vec{w}$ are nonzero, non-parallel vectors:

$\|\vec{v} \times \vec{w}\|^2 \neq 0$

$\therefore c_3 = 0$

$\therefore\ c_1\vec{v} + c_2\vec{w} = \vec{0}$

However, since $\vec{v}$ and $\vec{w}$ are not parallel:

$c_1 = c_2 = 0$

Therefore, the vectors are independent.

**Question 4:**

If two 5x2 matrices $A$ and $B$ each have independent columns, so does the matrix $A + B$. Is this statement true or false?

**Solution:**

This statement is false. For example, if $B = -A$, then $A + B$ has two zero column vectors that are dependent.

**Question 5:**

i) Solve this system of equations $S\vec{y} = \vec{c}$ for the unknowns in $\vec{y}$ in terms of the constants in $\vec{c}$:

$$S\vec{y} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

ii) Write the solution in part (i) for $\vec{y}$ as a matrix $T$ times the vector $\vec{c}$.

iii) The matrix $T$ is called the inverse of matrix $S$. Are the columns of $S$ independent or dependent?

**Solution:**

i) $y_1 = c_1$

$$y_1 + y_2 = c_2$$
$$\therefore y_2 = c_2 - c_1$$
$$y_1 + y_2 + y_3 = c_3$$
$$\therefore y_3 = c_3 - c_1 - (c_2 - c_1) = c_3 - c_2$$

ii) $$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = T\vec{c}$$

iii) The columns of matrix $S$ are independent.