ESC195 - Midterm Test #1
February 8, 2022
9:10 - 10:50 am
Instructor: J. W. Davis

Closed book, no aid sheets, no calculators
There are 8 questions plus a bonus question.
All questions are worth 10 marks.

J W Davis

Solutions

1. Use l'hospital's rule to evaluate the following limits:

a) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin \theta}$ b) $\lim_{x \to 0} \frac{1 - \cos x^n}{x^{2n}}$ c) $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x}$

a) $\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin \theta} \left(\to \frac{0}{0}\right) \overset{*}{=} \lim_{\theta \to 0} \frac{3\cos 3\theta}{\cos \theta} = 3$

b) $\lim_{x \to 0} \frac{1-\cos x^n}{x^{2n}} \left(\to \frac{0}{0}\right) \overset{*}{=} \lim_{x \to 0} \frac{\sin x^n \cdot n x^{n-1}}{2n\, x^{2n-1}}$

$= \lim_{x \to 0} \frac{\sin x^n}{2x^n} \left(\to \frac{0}{0}\right) \overset{*}{=} \lim_{x \to 0} \frac{\cos x^n \cdot n x^{n-1}}{2n\, x^{n-1}}$

$= \lim_{x \to 0} \frac{\cos x^n}{2} = \frac{1}{2}$

c) $\lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x} \to \infty^0 \Rightarrow$ Consider $e^{\ln (\tan x)^{\cos x}}$

$\lim_{x \to \frac{\pi}{2}^-} \cos x \ln(\tan x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\ln(\tan x)}{\sec x} \to \frac{\infty}{\infty}$

$\overset{*}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\frac{\sec^2 x}{\tan x}}{\sec x \tan x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\sec x}{\tan^2 x} = \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{\sin^2 x} = 0$

$\therefore \lim_{x \to \frac{\pi}{2}^-} (\tan x)^{\cos x} = e^0 = 1$

2. Evaluate the integrals:

a) $\displaystyle\int \sin x \ln(\cos x)\, dx$ b) $\displaystyle\int \frac{1}{(9-25x^2)^{\frac{3}{2}}}\, dx$ c) $\displaystyle\int \frac{3x^2 - x + 8}{x^3+4x}\, dx$

a) $\int \sin x \ln(\cos x)\, dx$

let $u = \ln(\cos x)$ $dv = \sin x\, dx$

$du = \frac{-\sin x}{\cos x}\, dx$ $v = -\cos x$

$= -\cos x \ln(\cos x) - \int \sin x\, dx = -\cos x \ln(\cos x) + \cos x + C$

b) $\int \frac{dx}{(9-25x^2)^{3/2}}$

let $x = \frac{3}{5}\sin x$

$dx = \frac{3}{5}\cos\theta\, d\theta$

$\sqrt{9-25x^2} = 3\cos\theta$

$= \int \frac{\frac{3}{5}\cos\theta\, d\theta}{27\cos^3\theta} = \frac{1}{45}\int \sec^2\theta = \frac{1}{45}\tan\theta + C = \frac{1}{45}\,\frac{\frac{5}{3}x}{\frac{1}{3}\sqrt{9-25x^2}} + C$

$= \frac{1}{9}\,\frac{x}{\sqrt{9-25x^2}} + C$

c) $\frac{3x^2 - x + 8}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4} \Rightarrow 3x^2 - x + 8 = Ax^2 + 4A + Bx^2 + Cx$

$x^2 :\ 3 = A + B \Rightarrow B = 1$

$x :\ -1 = C$

$1 :\ 8 = 4A \Rightarrow A = 2$

$= \frac{2}{x} + \frac{x-1}{x^2+4}$

$\int \frac{3x^2-x+8}{x^3+4x}\, dx = \int \frac{2}{x}\, dx + \frac{1}{2}\int \frac{2x\, dx}{x^2+4} - \int \frac{dx}{x^2+4}$

$= 2\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

3. Determine whether the integral is convergent or divergent. Evaluate the integrals that are convergent.

a) $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx$ b) $\displaystyle\int_0^5 \frac{x}{x-2}\,dx$ c) $\displaystyle\int_e^\infty \frac{dx}{\sqrt{x+1}\ln x}$

a) $\displaystyle\int_{-\infty}^{-1} x^{-1/3}\,dx = \lim_{t\to-\infty}\int_t^{-1} x^{-1/3}\,dx = \lim_{t\to-\infty}\left[\tfrac{3}{2}x^{2/3}\right]_t^{-1} \to -\infty$ $\therefore$ divergent

b) $\displaystyle\int_0^5 \frac{x}{x-2}\,dx = \int_0^2 \frac{x}{x-2}\,dx + \int_2^5 \frac{x}{x-2}\,dx$

$\displaystyle\Rightarrow \int_0^2 \frac{x\,dx}{x-2} = \lim_{t\to2^-}\int_0^t\left(1+\frac{2}{x-2}\right)dx = \lim_{t\to2^-}\left[x+2\ln|x-2|\right]_0^t$

$\displaystyle= \lim_{t\to2^-}\left(t+2\ln|t-2|-2\ln2\right) \to -\infty$

$\displaystyle\therefore \int_0^2 \frac{x}{x-2}\,dx$ diverges $\displaystyle\therefore \int_0^5 \frac{x}{x-2}\,dx$ diverges

c) $\ln x < \sqrt{x}$ (see note) $\displaystyle\therefore \frac{1}{\ln x} > \frac{1}{\sqrt{x}} \Rightarrow \frac{1}{\sqrt{x+1}\ln x} > \frac{1}{\sqrt{x+1}\sqrt{x}} > \frac{1}{\sqrt{x+1}\sqrt{x+1}} = \frac{1}{x+1}$

$\displaystyle\therefore \int_e^\infty \frac{dx}{\sqrt{x+1}\ln x} > \int_e^\infty \frac{dx}{x+1} = \lim_{t\to\infty}\left[\ln(x+1)\right]_e^t$

$\displaystyle= \lim_{t\to\infty}\left(\ln(t+1)-\ln(e+1)\right) \to \infty$

$\displaystyle\therefore \int_e^\infty \frac{dx}{\sqrt{x+1}\ln x}$ diverges by comparison test

Note: There are several ways to show this:

1) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} \overset{*}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2}x^{-1/2}} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}} \to 0$

$\therefore \sqrt{x} > \ln x$ for large x

2) let $f(x) = \sqrt{x} - \ln x$

$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{x - 2\sqrt{x}}{2x^{3/2}}$

$f'(x) > 0$ for $x > 2\sqrt{x}$ or $x > 4$

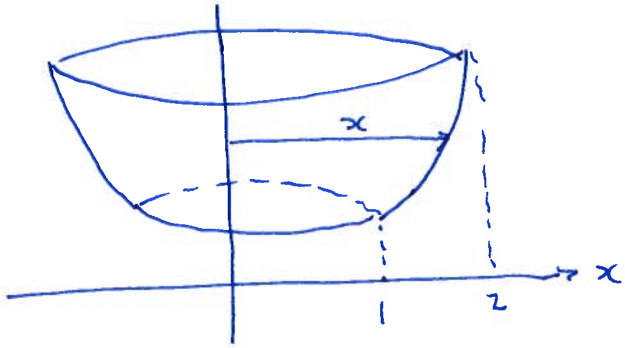
$\Rightarrow f(x)$ is increasing for $x > 4$

$\Rightarrow f(e^2) = e - 2\ln e = e - 2 > 0$

$\therefore \sqrt{x} > \ln x$ for $x > e^2$

In fact, $\sqrt{x} > \ln x$ for $x > 0$, but for our purposes here, it is only necessary to show $\sqrt{x} > \ln x$ as $x \to \infty$.

4. Find the area of the surface of revolution generated by revolving the curve $y = \frac{1}{5}x^5 + \frac{1}{12x^3}$, $1 \le x \le 2$ about the y-axis. Provide a sketch.



$$A = \int 2\pi x \, ds$$

$$= \int_1^2 2\pi x \sqrt{1 + (f'(x))^2} \, dx$$

$$y' = x^4 - \frac{1}{4x^4} = \frac{4x^8 - 1}{4x^4} \Rightarrow (y')^2 = \frac{16x^{16} - 8x^8 + 1}{16x^8}$$

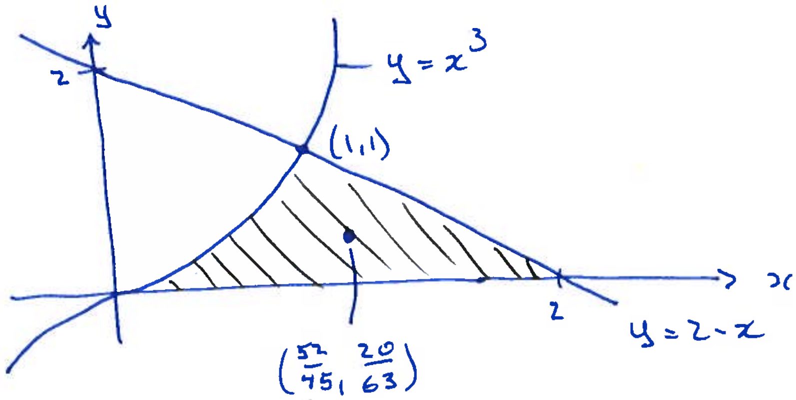
$$\Rightarrow 1 + (y')^2 = \frac{16x^8 + 16x^{16} - 8x^8 + 1}{16x^8} = \frac{16x^{16} + 8x^8 + 1}{16x^8} = \left(\frac{4x^8 + 1}{4x^4}\right)^2$$

$$\Rightarrow A = \int_1^2 2\pi x \left(\frac{4x^8+1}{4x^4}\right) dx = 2\pi \int_1^2 \left(x^5 + \frac{1}{4x^3}\right) dx$$

$$= 2\pi \left[\frac{x^6}{6} - \frac{1}{8x^3}\right]_1^2 = 2\pi\left(\frac{64}{6} - \frac{1}{32} - \frac{1}{6} + \frac{1}{8}\right) = \pi\left(\frac{1024}{48} - \frac{3}{48} - \frac{16}{48} + \frac{12}{48}\right)$$

$$= \pi \cdot \frac{1017}{48} = \pi \, \frac{339}{16}$$

5. Find the centroid of the region bounded by the curves: $y = x^3, \quad x + y = 2, \quad y = 0$
Provide a sketch of the region indicating the location of the centroid.



$$A = \int_0^1 x^3\,dx + \int_1^2 (2-x)\,dx = \left[\frac{x^4}{4}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{4} + \left(4-2-2+\frac{1}{2}\right) = \frac{3}{4}$$

$$\bar{x}A = \int_0^1 x \cdot x^3\,dx + \int_1^2 x(2-x)\,dx = \left[\frac{x^5}{5}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{5} + \left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right) = \frac{1}{5} + \frac{2}{3} = \frac{13}{15}$$

$$\Rightarrow \bar{x} = \frac{13}{15} \cdot \frac{4}{3} = \frac{52}{45}$$

$$\bar{y}A = \int_0^1 \frac{1}{2}(x^3)^2\,dx + \int_1^2 \frac{1}{2}(2-x)^2\,dx = \left[\frac{1}{2}\frac{x^7}{7}\right]_0^1 + \left[\frac{-1}{2}\frac{(2-x)^3}{3}\right]_1^2$$

$$= \frac{1}{14} + \frac{1}{6} = \frac{10}{42}$$

$$\Rightarrow \bar{y} = \frac{10}{42} \cdot \frac{4}{3} = \frac{20}{63}$$

6. a) Find the Cartesian equation for the polar curves:

i) $r = 4\sec\theta$ ii) $r^2 \sin 2\theta = 1$

b) Find a polar equation for the curves represented by the given Cartesian equations:

i) $y = -2x^2$ ii) $x^2 - y^2 = 4$

a) i) $r = 4\sec\theta \Rightarrow r\cos\theta = 4 \Rightarrow x = 4$

ii) $r^2 \sin 2\theta = 1 \Rightarrow r^2 \cdot 2\cos\theta\sin\theta = 1 \Rightarrow 2xy = 1 \Rightarrow y = \frac{1}{2x}$

b) i) $y = -2x^2 \Rightarrow r\sin\theta = -2r^2\cos^2\theta$

$\Rightarrow r = 0$ or $r = \frac{-\sin\theta}{2\cos^2\theta} = -\frac{1}{2}\sec\theta\tan\theta$

ii) $x^2 - y^2 = 4 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 4$

$\Rightarrow r^2\cos 2\theta = 4$

7. Sketch a graph of the parametric curve: $x = 3t^2 - t^3$, $y = t^2 - 2t$
Show all vertical and horizontal tangents, the tangents at $(2, 2)$, and identify the asymptotic behaviour.

$x = 3t^2 - t^3$
$x' = 6t - 3t^2$
$x' = 0 \Rightarrow t = 0,\ t = 2$
$\therefore$ vertical tangents $(0,0), (4,0)$

$y = t^2 - 2t$
$y' = 2t - 2$
$y' = 0 \Rightarrow t = 1$
$\therefore$ horizontal tangent $(2,-1)$

Intercepts: $y = 0 \Rightarrow t = 0 \Rightarrow (0,0)$
$t = 2 \Rightarrow (4,0)$

$x = 0 \Rightarrow t = 0 \Rightarrow (0,0)$
$t = 3 \Rightarrow (0,3)$

Slope at $(2,2)$: $y = 2 \Rightarrow t^2 - 2t - 2 = 0$

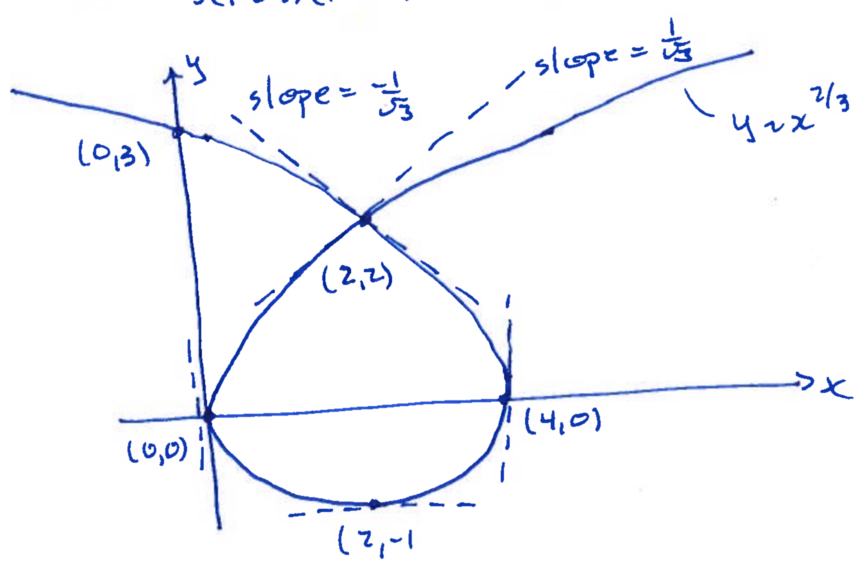
$$t = \frac{2 \pm \sqrt{4+8}}{2} = 1 \pm \sqrt{3}$$

$$\Rightarrow x(1\pm\sqrt{3}) = 3(1\pm\sqrt{3})^2 - (1\pm\sqrt{3})^3 = (1\pm\sqrt{3})^2(3 - 1 \mp \sqrt{3})$$
$$= (1 \pm 2\sqrt{3} + 3)(2 \mp \sqrt{3}) = 8 \pm 4\sqrt{3} \mp 4\sqrt{3} - 6 = 2$$

$$\frac{dy}{dx} = \frac{2t-2}{6t-3t^2} \quad \text{at } t = 1+\sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{2\sqrt{3}}{3(1+\sqrt{3})(1-\sqrt{3})} = \frac{-1}{\sqrt{3}}$$

$$t = 1-\sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{3}}{3(1-\sqrt{3})(1+\sqrt{3})} = \frac{1}{\sqrt{3}}$$

Asymptotic behaviour:
as $t \to \pm\infty$, $x \to -t^3$
or $t \to (-x)^{1/3}$
$\therefore y \to t^2 \to x^{2/3}$



8. Find the length of the curve $f(x) = \ln(x + \sqrt{x^2 - 1})$, $x \in [1, \sqrt{2}]$. Formulate your solution in two ways: a) by integrating with respect to x; b) by integrating with respect to y.

$$L = \int ds = \int \sqrt{1 + (f'(x))^2}\, dx = \int \sqrt{1 + (g'(y))^2}\, dy$$

a) $$f'(x) = \frac{1 + x(x^2-1)^{-1/2}}{x + \sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1} + x}{x + \sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}} \to \infty \text{ at } x = 1$$

$\therefore$ improper integral

$$L = \int_1^{\sqrt{2}} \sqrt{1 + \left(\frac{1}{\sqrt{x^2+1}}\right)^2}\, dx = \int_1^{\sqrt{2}} \sqrt{\frac{x^2-1+1}{x^2-1}}\, dx = \int_1^{\sqrt{2}} \frac{x\,dx}{\sqrt{x^2-1}} = \left[\sqrt{x^2-1}\right]_1^{\sqrt{2}} = 1$$

b) $$y = \ln(x + \sqrt{x^2-1}) \Rightarrow e^y = x + \sqrt{x^2-1} \Rightarrow e^y - x = \sqrt{x^2-1}$$

$$e^{2y} - 2xe^y + x^2 = x^2 - 1$$

$$e^{2y} + 1 = 2xe^y$$

$$\Rightarrow x = \frac{e^y + e^{-y}}{2} \qquad y \in [0, \ln(\sqrt{2}+1)]$$

$$x' = \frac{1}{2}(e^y - e^{-y})$$

$$(x')^2 = \frac{1}{4}(e^{2y} - 2 + e^{-2y})$$

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + \frac{1}{4}(e^{2y} - 2 + e^{-2y})}\, dy$$

$$= \int_0^{\ln(\sqrt{2}+1)} \frac{1}{2}\sqrt{4 + e^{2y} - 2 + e^{-2y}}\, dy$$

$$= \frac{1}{2}\int_0^{\ln(\sqrt{2}+1)} (e^y + e^{-y})\, dy = \frac{1}{2}\left[e^y - e^{-y}\right]_0^{\ln(\sqrt{2}+1)} = \frac{1}{2}\left(\sqrt{2}+1 - \frac{1}{\sqrt{2}+1} + 1 - 1\right)$$

$$= \frac{1}{2}\left(\frac{(\sqrt{2}+1)^2 - 1}{\sqrt{2}+1}\right) = \frac{1}{2}\left(\frac{2 + 2\sqrt{2} + 1 - 1}{\sqrt{2}+1}\right) = \frac{1+\sqrt{2}}{\sqrt{2}+1} = 1$$

9. Bonus Question

Evaluate the integral: $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$ and use the properties of integrals to show $\pi < \dfrac{22}{7}$

$$x^4(1-x)^4 = x^4(1-2x+x^2)^2 = x^4(1-2x+x^2-2x+4x^2-2x^3+x^2-2x^3+x^4)$$

$$= x^4(1-4x+6x^2-4x^3+x^4) = x^8-4x^7+6x^6-4x^5+x^4$$

$$\begin{array}{r|l}
 & x^6 - 4x^5 + 5x^4 - 4x^2 + 4 \\
\hline
x^2+1 & x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 \\
 & x^8 \qquad\quad + x^6 \\
\hline
 & -4x^7 + 5x^6 - 4x^5 + x^4 \\
 & -4x^7 \qquad\quad - 4x^5 \\
\hline
 & 5x^6 \qquad\quad + x^4 \\
 & 5x^6 \qquad\quad 5x^4 \\
\hline
 & -4x^4 \\
 & -4x^4 - 4x^2 \\
\hline
 & -4x^2 \\
 & +4x^2 + 4 \\
\hline
 & -4
\end{array}$$

$$\Rightarrow \frac{x^4(1-x)^4}{1+x^2} = x^6-4x^5+5x^4-4x^2+4 - \frac{4}{1+x^2}$$

$$\therefore \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$= \frac{1}{7} - \frac{4}{6} + 1 - \frac{4}{3} + 4 - 4\left(\frac{\pi}{4}\right) = \frac{3-14+21-28+84}{21} - \pi$$

$$= \frac{66}{21} - \pi = \frac{22}{7} - \pi$$

Since $\dfrac{x^4(1-x)^4}{1+x^2} > 0$ for $0 < x < 1$ $\Rightarrow \displaystyle\int_0^1 \frac{x^4(1-x)^4}{1+x^2} > 0$

$\therefore \dfrac{22}{7} - \pi > 0$ or $\pi < \dfrac{22}{7}$