University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test
9:10 – 10:55, 22 November 2021
105 minutes
No calculators or aids
There are 9 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

JW Davis
Solutions

1) Evaluate the integrals:

a) $\int_1^4 (6x-1)\,dx$ $= \left[3x^2 - x\right]_1^4 = (48-4)-(3-1) = 42$

b) $\int_4^1 \sqrt{5x}\,dx$ $= \sqrt{5}\left[\frac{2}{3}x^{3/2}\right]_4^1 = \frac{2\sqrt{5}}{3}(1-8) = -\frac{14\sqrt{5}}{3}$

c) $\int_{-1}^5 |2x-3|\,dx$ $= \int_{-1}^{3/2}(3-2x)\,dx + \int_{3/2}^{5}(2x-3)\,dx$

$= \left[3x - x^2\right]_{-1}^{3/2} + \left[x^2 - 3x\right]_{3/2}^{5}$

$= \left(\frac{9}{2}-\frac{9}{4}\right)-(-3-1) + (25-15)-\left(\frac{9}{4}-\frac{9}{2}\right) = \frac{9}{2}+14 = \frac{37}{2}$

d) $\int_{-1}^1 (t^2-1)^3\,t\,dt$ $= \int_{t=-1}^{t=1}\frac{1}{2}u^3\,du = \left[\frac{1}{2}\frac{u^4}{4}\right]_{t=-1}^{t=1} = \frac{1}{8}\left[(t^2-1)^4\right]_{-1}^{1} = 0$

let $u = t^2-1$

$du = 2t\,dt$

$\Rightarrow$ or note that $t(t^2-1)^3$ is odd f'n

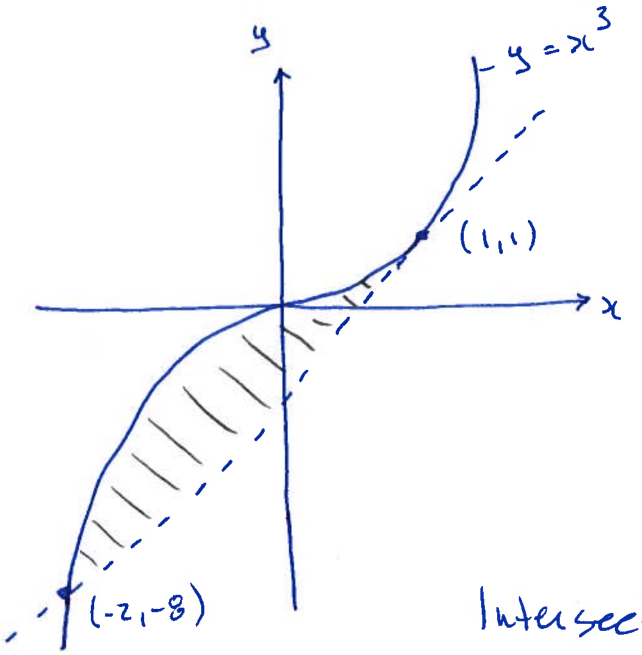
e) $\int \frac{6}{\sqrt{4-5t}}\,dt$ $= \int \frac{6}{\sqrt{u}}\cdot\frac{du}{(-5)} = -\frac{6}{5}\cdot 2u^{1/2} + C$

let $u = 4-5t$

$du = -5t$

$= -\frac{12}{5}\sqrt{4-5t} + C$

2) Find the area of the finite plane region bounded by the curve $y = x^3$ and the tangent line to the curve at the point $(1,1)$.



$$\frac{dy}{dx} = 3x^2 \Rightarrow y'(x=1) = 3$$

tangent line: $y = 3x + b$

$$\Rightarrow 1 = 3(1) + b \Rightarrow b = -2$$

$$\therefore\ y = 3x - 2$$

Intersection: $3x - 2 = x^3$

$$x^3 - 3x + 2 = 0$$

$$(x-1)(x^2 + x - 2) = 0$$

$$(x-1)(x-1)(x+2) = 0$$

$\therefore$ intersection at $(-2, -8)$

$$\text{Area} = \int_{-2}^{1} \left(x^3 - (3x-2) \right) dx$$

$$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \left(\frac{1}{4} - \frac{3}{2} + 2 \right) - \left(\frac{16}{4} - \frac{12}{2} - 4 \right)$$

$$= \frac{3}{4} + 6 = \frac{27}{4}$$

3) a) Without evaluating the integrals, explain why the area under the curve $y = \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}$ on [4, 9] equals the area under the curve $y = x^2$ on [1, 2].

b) Evaluate the integral: $\int_0^1 x\sqrt{1-\sqrt{x}}\,dx$

a) $\int_4^9 \frac{(\sqrt{x}-1)^2}{2\sqrt{x}}\,dx$ let $u = \sqrt{x}-1$ $u(x=4) = 1$

$x = (u+1)^2$ $u(x=9) = 2$

$dx = 2(u+1)\,du$

$$= \int_1^2 \frac{u^2}{2(u+1)} \cdot 2(u+1)\,du = \int_1^2 u^2\,du = \int_1^2 x^2\,dx$$

b) $\int_0^1 x\sqrt{1-\sqrt{x}}\,dx$ let $a = \sqrt{x} \Rightarrow x = a^2 \Rightarrow dx = 2a\,da$

$$= \int_0^1 a^2\sqrt{1-a} \cdot 2a\,da = \int_0^1 2a^3\sqrt{1-a}\,da$$

let $u = \sqrt{1-a} \Rightarrow u^2 = 1-a \Rightarrow a = 1-u^2$

$da = -2u\,du$

$$= 2\int_1^0 (1-u^2)^3 u(-2u\,du) = 4\int_0^1 u^2(1-u^2)^3\,du$$

$$= 4\int_0^1 u^2(1-3u^2+3u^4-u^6)\,du = 4\int_0^1 (u^2-3u^4+3u^6-u^8)\,du$$

$$= 4\left[\frac{u^3}{3} - \frac{3u^5}{5} + \frac{3u^7}{7} - \frac{u^9}{9}\right]_0^1 = 4\left(\frac{1}{3} - \frac{3}{5} + \frac{3}{7} - \frac{1}{9}\right)$$

$$= 4\left(\frac{2}{9} - \frac{6}{35}\right) = \frac{4 \cdot 16}{9 \cdot 35} = \frac{64}{315}$$

4) Let f be a function such that f' is continuous on $[a, b]$. Find $\int_a^b f(t)f'(t)dt$

Let $F(t) = f^2(t) \Rightarrow F'(t) = 2f(t)\, f'(t)\, dt$

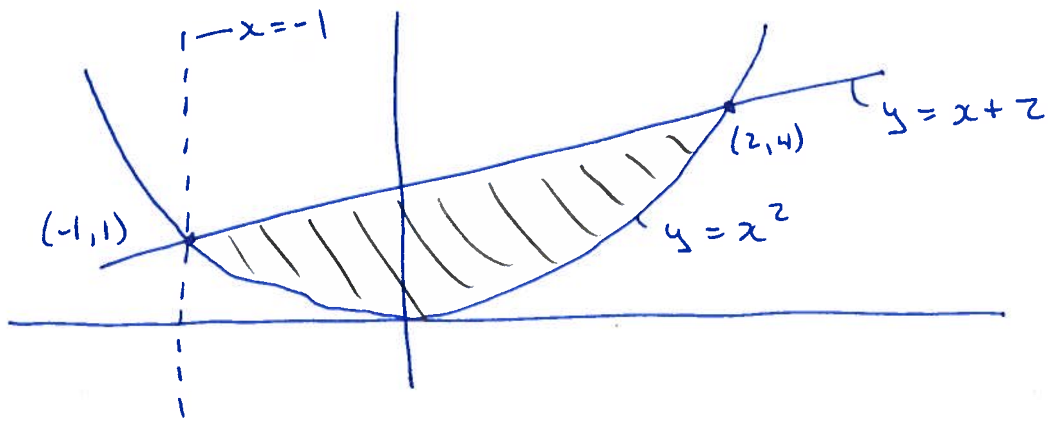
$\therefore \int_a^b f(t)\, f'(t)\, dt = \int_a^b \frac{1}{2} F'(t)\, dt = \left[\frac{1}{2} F(t)\right]_a^b$

$= \frac{1}{2}\left(F(b) - F(a)\right)$

$= \frac{1}{2}\left(f^2(b) - f^2(a)\right)$

5) Consider the region defined by the curves $y = x^2$ and $y = x + 2$. Provide a sketch of the region and find (but do NOT solve) integrals representing the following:

a) The volume of rotation about the x-axis using the washer method.
b) The volume of rotation about the x-axis using the shell method.
c) The volume of rotation about the line $x = -1$ using the washer method.
d) The volume of rotation about the line $x = -1$ using the shell method.



a) $V = \int_{-1}^{2} \pi \left((x+2)^2 - (x^2)^2 \right) dx$

b) $V = \int_{0}^{1} 2\pi y \left(\sqrt{y} - (-\sqrt{y}) \right) dy + \int_{1}^{4} 2\pi y \left(\sqrt{y} - (y-2) \right) dy$

c) $V = \int_{0}^{1} \pi \left((\sqrt{y}+1)^2 - (-\sqrt{y}+1)^2 \right) dy + \int_{1}^{4} \pi \left((\sqrt{y}+1)^2 - (y-2+1)^2 \right) dy$

d) $V = \int_{-1}^{2} 2\pi (x+1) \left((x+2) - (x^2) \right) dx$

6) For each of the functions: a) $f(x) = \frac{\ln x}{x}$ b) $f(x) = \frac{\ln(x^{1/3})}{x}$ c) $f(x) = \frac{x}{\ln x}$

i) Determine the domain of f.
ii) Find the intervals in which f increases or decreases.
iii) Find the extreme values.
iv) Determine the concavity of the graph and find the inflection points.
v) Sketch the graph specifying the asymptotes, if any.

Hint: $\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to 0^+} \frac{x}{\ln x} = 0$

a) $f(x) = \frac{\ln x}{x}$

i) Domain: $x > 0$

ii) $f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2} \Rightarrow f' = 0$ for $x = e$; $f(e) = \frac{1}{e}$

$f'(x) < 0$ for $x > e$ ∴ f decr

$f'(x) > 0$ for $0 < x < e$ ∴ f incr

iii) ∴ $f(e) = 1/e$ is a local max $\Rightarrow$ range: $-\infty < f \le \frac{1}{e}$

iv) $f''(x) = \frac{-1}{x^3} + (1-\ln x)(-2)x^{-3} = \frac{2\ln x - 3}{x^3}$

$\Rightarrow x^3 > 0$ in domain of $f(x)$

$f'' > 0$ for $2\ln x - 3 > 0 \Rightarrow x > e^{3/2}$ concave up

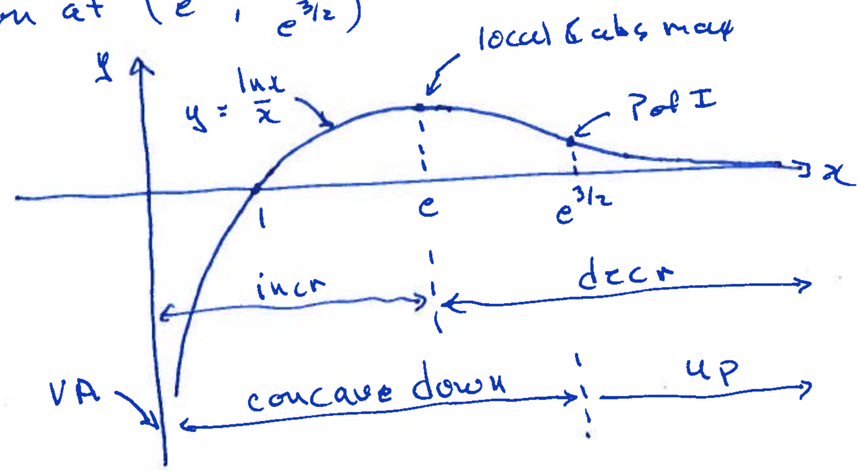
$f'' < 0$ for $2\ln x - 3 < 0 \Rightarrow 0 < x < e^{3/2}$ concave down

∴ pt. of inflection at $\left(e^{3/2}, \frac{3/2}{e^{3/2}}\right)$

v) $f(1) = 0$

$\lim_{x\to 0^+} f = -\infty$

$\lim_{x\to \infty} f = 0$



b) $\frac{\ln x^{1/3}}{x} = \frac{1}{3}\frac{\ln x}{x}$ ∴ same as part (a) but scaled by $1/3$.

c) $f(x) = \dfrac{x}{\ln x}$

i) Domain : $x > 0$, $x \neq 1$

ii) $f'(x) = \dfrac{1}{\ln x} + \dfrac{x(-1)\frac{1}{x}}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \Rightarrow f' = 0$ for $x = e$; $f(e) = e$

$f' > 0$ for $x > e$ $\therefore$ f incr

$f' < 0$ for $x < e$ $\therefore$ f decr

iii) $\therefore f(e) = e$ is a local min.

iv) $f''(x) = \dfrac{\frac{1}{x}}{(\ln x)^2} + (\ln x - 1)(-2)(\ln x)^{-3} \cdot \dfrac{1}{x} = \dfrac{\ln x - 2\ln x + 2}{x(\ln x)^3} = \dfrac{2 - \ln x}{x(\ln x)^3}$

$= \dfrac{\ln x(2 - \ln x)}{x(\ln x)^4}$ ← always $+^{ve}$ for $x > 0$, $x \neq 1$

$f'' = 0 \Rightarrow 2 - \ln x = 0 \Rightarrow x = e^2$, $f(e^2) = \dfrac{e^2}{2}$

$f'' < 0$ for $0 < x < 1$ $\therefore$ concave down

$f'' > 0$ for $1 < x < e^2$ $\therefore$ concave up

$f'' < 0$ for $x > e^2$ $\therefore$ concave down

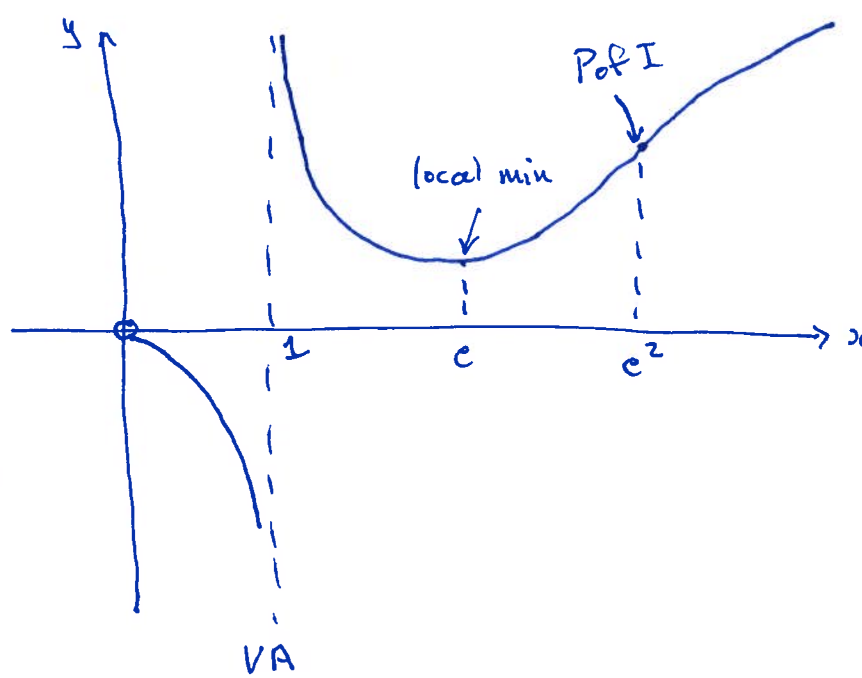
$f(e^2) = \dfrac{e^2}{2}$ is P of I

v) $\lim_{x \to 1^-} f = -\infty$

$\lim_{x \to 1^+} f = +\infty$

$\lim_{x \to \infty} f = \infty$

$\lim_{x \to 0^+} f = 0$



7) Show that the function $g(x) = \sqrt{2x+1}$ is one-to-one and find its inverse. Provide a simple sketch of $g(x)$ and $g^{-1}(x)$.

$g(x) = \sqrt{2x+1} \quad \Rightarrow \quad x \geq -\frac{1}{2}$ implied

If $g(x_1) = g(x_2) \Rightarrow \sqrt{2x_1+1} = \sqrt{2x_2+1}$

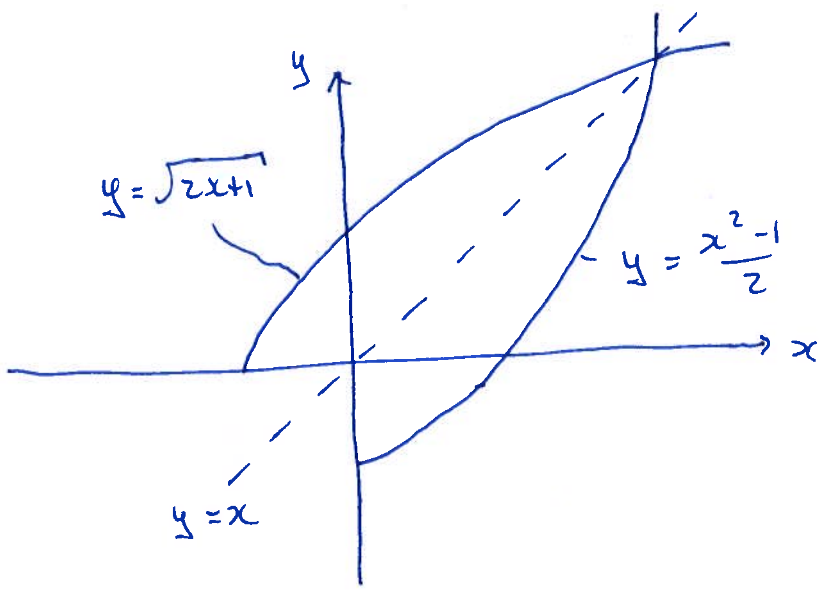
$\Rightarrow 2x_1+1 = 2x_2+1$

$\Rightarrow x_1 = x_2 \quad \therefore$ 1-1

Let $y = g^{-1}(x) \Rightarrow x = g(y) = \sqrt{2y+1}$

for $x \geq 0 \Rightarrow x^2 = 2y+1 \Rightarrow y = \frac{x^2-1}{2}$

$\therefore g^{-1}(x) = \frac{x^2-1}{2}, \quad x \geq 0$



8) Consider the function $f(x) = x^3,\ x \in [0,1]$

a) Approximate the area between $f(x)$ and the x-axis with a Reimann sum, S_n^U, with a regular partition, and with $x_i^* = x_i$ being the right-hand endpoint of each Δx_i. As $f(x)$ is an increasing function, this will be an overestimate of the area.

b) Find a similar underestimate of the area, S_n^L, where $x_i^* = x_{i-1}$ is the left-hand endpoint.

c) Given $\epsilon > 0$ determine the value of n required to satisfy: $|S_n^U - S_n^L| < \epsilon$.

Hint: $$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

a) Uniform partition: $\Delta x = \frac{1}{n} \Rightarrow x_i^* = x_i = \frac{i}{n}$

$$\therefore S_n^u = \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} \frac{1}{n}(x_i^*)^3 = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^3$$

$$= \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}\,\frac{n^2+2n+1}{n^2}$$

b) $x_i^* = x_{i-1} = \frac{i-1}{n}$

$$\therefore S_n^L = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i-1}{n}\right)^3 = \sum_{i=0}^{n-1}\left(\frac{i}{n}\right)^3\cdot\frac{1}{n} = \frac{1}{n^4}\sum_{i=1}^{n-1} i^3$$

$$= \frac{1}{n^4}\left[\left(\frac{n(n+1)}{2}\right)^2 - n^3\right] = \frac{1}{4}\,\frac{n^2+2n+1}{n^2} - \frac{1}{n}$$

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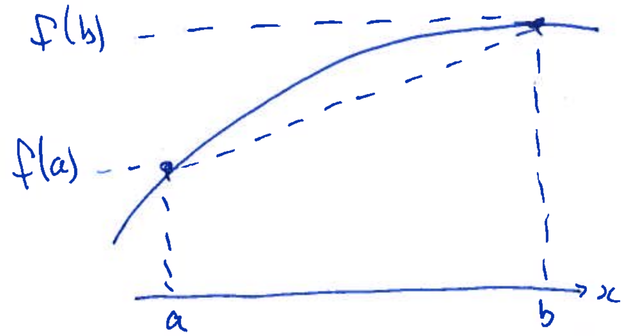
c) $$|S_n^u - S_n^L| = \left|\frac{n^2+2n+1}{4n^2} - \frac{n^2+2n+1}{4n^2} + \frac{1}{n}\right| = \frac{1}{n} < \epsilon$$

$\therefore$ choose $n > \frac{1}{\epsilon}$

9) a) Given $f(x) > 0$, suppose that f is continuous on $[a, b]$ with $f'' < 0$ on the interval. Use geometry to show: $(b-a)\frac{f(a)+f(b)}{2} \leq \int_a^b f(x)dx \leq (b-a)f\left(\frac{b+a}{2}\right)$

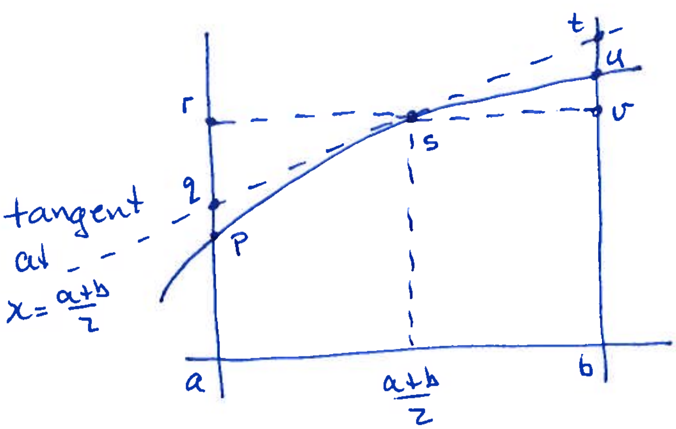
b) Divide the inequalities by $(b-a)$ and interpret the resulting inequalities in terms of the average value of f on $[a, b]$.

a)



since $f''<0$, the area under the line connecting $(a, f(a))$ and $(b, f(b))$ is less than the area under $f(x)$ (order properties of integrals).

or $(b-a)\frac{f(b)-f(a)}{2} < \int_a^b f(x)dx$



- since $f''<0$, tangent line qt lies above $f(x)$.
- area of triangle qrs = area of triangle stv.
- $\therefore$ area under line rv = area under line qt.
- area under line rv is greater than area under $f(x)$

or $(b-a)f\left(\frac{a+b}{2}\right) > \int_a^b f(x)dx$

b) $\underbrace{\frac{f(a)+f(b)}{2} < \frac{1}{b-a}\int_a^b f(x)dx}_{\text{For } f''<0 \text{, average value of } f(x) \text{ is > mid point of line connecting } f(a) \text{ \& } f(b)} < f\left(\frac{a+b}{2}\right)$

For $f''<0$, average value of $f(x)$ is > mid point of line connecting $f(a)$ & $f(b)$

For $f''<0$, average value of $f(x)$ is < the mid point value of the function