AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 1

2 October 2017 8:45 am - 9:45 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis

Given Name: Solutions

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	7	
2	9	
3	6	
4	5	
5	10	
6	10	
7	10	
TOTAL	57	/ 53

Note: The following integrals may be useful.

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

1) Evaluate the double integrals by inspection (ie., without any calculations). Provide sketches and outline your reasoning.

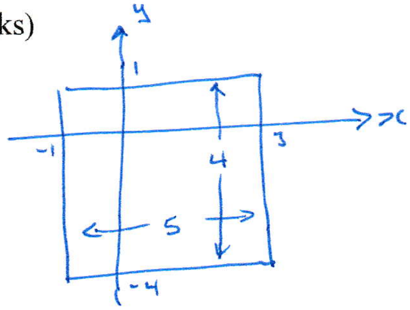
a) $\int_R dR$ Where R is the rectangle $-1 \le x \le 3$, $-4 \le y \le 1$

b) $\int_R (x+3)\,dR$ Where R is the half disk $0 \le y \le \sqrt{4-x^2}$

c) $\int_{x^2+y^2 \le a^2} \sqrt{a^2 - x^2 - y^2}\, dR$

(7 marks)

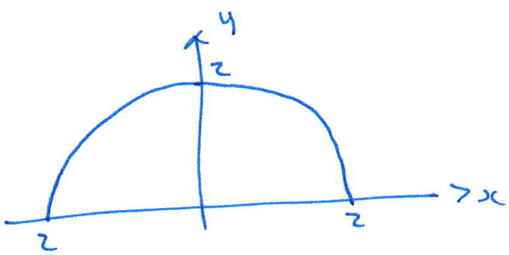
a)



$\int_R dR$ gives the area of the region R

$\therefore$ Area $= 4 \times 5 = 20$

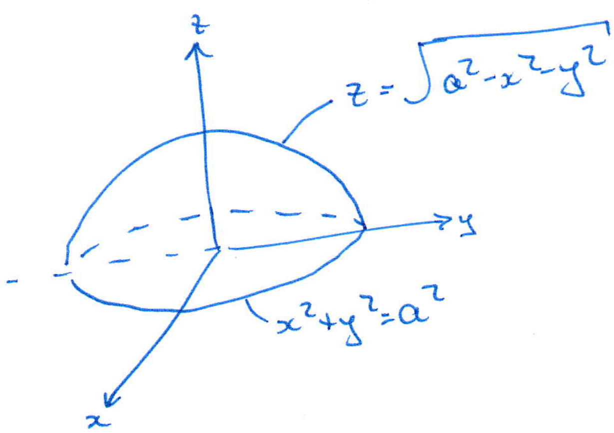
b)



$\Rightarrow$ by symmetry, $\int_R x\,dx$ will cancel; values to left of y-axis are negative.

$\therefore \int_R (x+3)\,dR = \int_R 3\,dR = 3\left(\frac{1}{2}\pi 2^2\right) = 6\pi$

c)



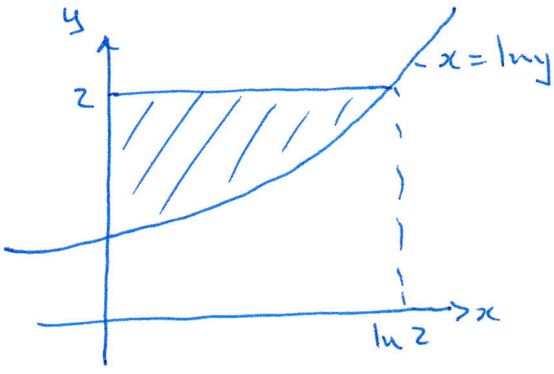
$\rightarrow$ The integral calculates the volume of the hemisphere of radius a

$\therefore \int_{x^2+y^2 \le a^2} \sqrt{a^2-x^2-y^2}\, dR = \frac{1}{2}\left(\frac{4}{3}\pi a^3\right) = \frac{2}{3}\pi a^3$

2) a) Sketch the region R that gives rise to the repeated integral, change the order of integration and then evaluate.

$$\int_1^2 \int_0^{\ln y} e^{-x}\, dxdy$$

(4 marks)



$$\int_1^2 \int_0^{\ln y} e^{-x}\, dx\, dy = \int_0^{\ln 2} dx \int_{e^x}^{2} e^{-x}\, dy$$

$$= \int_0^{\ln 2} e^{-x} dx \left[y \right]_{e^x}^{2} = \int_0^{\ln 2} e^{-x}(2 - e^{x})\, dx$$

$$= \int_0^{\ln 2} (2e^{-x} - 1)\, dx = \left[-2e^{-x} - x \right]_0^{\ln 2}$$

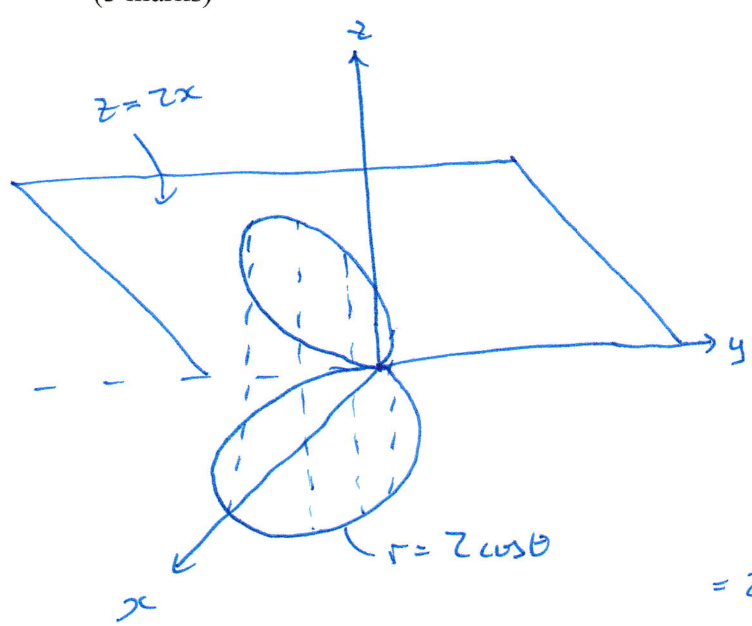
$$= -1 - \ln 2 + 2 + 0$$

$$= 1 - \ln 2$$

b) Use a double integral in polar coordinates to find the volume of the solid bounded above by the plane $z = 2x$ and below by the disk $(x-1)^2 + y^2 \le 1$. Provide a sketch of the volume.

Hint: $\int \cos^4 x\, dx = \frac{3}{8}x + \frac{3}{16}\sin x + \frac{1}{4}\cos^3 x \sin x + C$

(5 marks)



$$(x-1)^2 + y^2 = 1$$

$$\Rightarrow x^2 - 2x + 1 + y^2 = 1$$

$$x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta$$

$$r = 2\cos\theta$$

$$V = \int_0^{\pi} d\theta \int_0^{2\cos\theta} (2x - 0)\, r\, dr$$

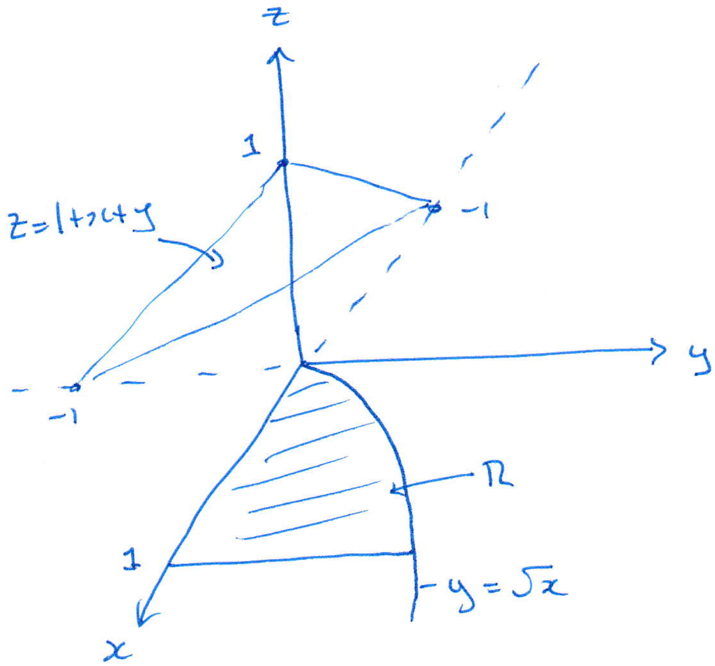
$$= \int_0^{\pi} d\theta \int_0^{2\cos\theta} 2r^2 \cos\theta\, dr$$

$$= 2\int_0^{\pi} \cos\theta\, d\theta \left[\frac{r^3}{3} \right]_0^{2\cos\theta} = 2\int_0^{\pi} \cos\theta \cdot \frac{8\cos^3\theta}{3}\, d\theta$$

$$V = \frac{16}{3}\int_0^{\pi} \cos^4\theta\, d\theta = \frac{16}{3}\left[\frac{3}{8}\theta + \frac{3}{16}\sin\theta + \frac{1}{4}\cos^3\theta \sin\theta \right]_0^{\pi} = \frac{16}{3} \cdot \frac{3\pi}{8} = 2\pi$$

3) Evaluate $\int_V 6xy\,dV$, where V lies under the plane $z = 1 + x + y$ and above the region in the x-y plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$. Provide a sketch of the volume.

(6 marks)



$$\int_V 6xy\,dV$$

$$= \int_0^1 dx \int_0^{\sqrt{x}} dy \int_0^{1+x+y} dz \cdot 6xy$$

$$= \int_0^1 dx \int_0^{\sqrt{x}} dy \cdot 6xy\,(1+x+y)$$

$$= \int_0^1 dx \int_0^{\sqrt{x}} dy \left(6xy + 6x^2y + 6xy^2\right) dy$$

$$= \int_0^1 dx \left[\frac{6xy^2}{2} + \frac{6x^2y^2}{2} + \frac{6xy^3}{3}\right]_0^{\sqrt{x}}$$

$$= \int_0^1 \left(3x^2 + 3x^3 + 2x^{5/2}\right) dx$$

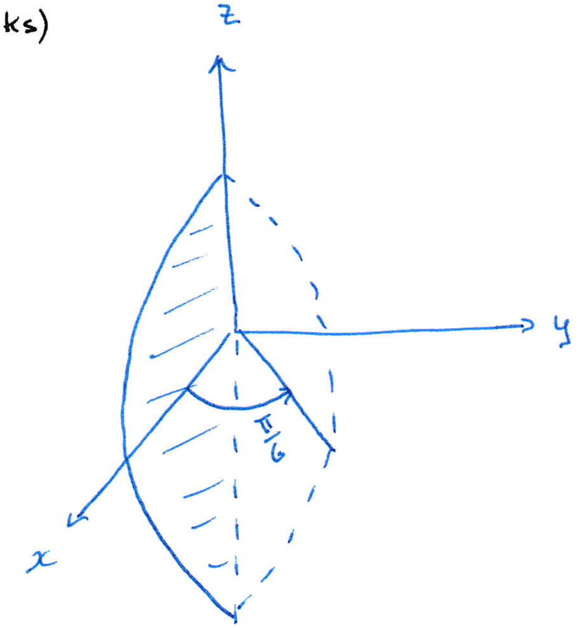
$$= \left[x^3 + \frac{3x^4}{4} + \frac{4}{7}x^{7/2}\right]_0^1$$

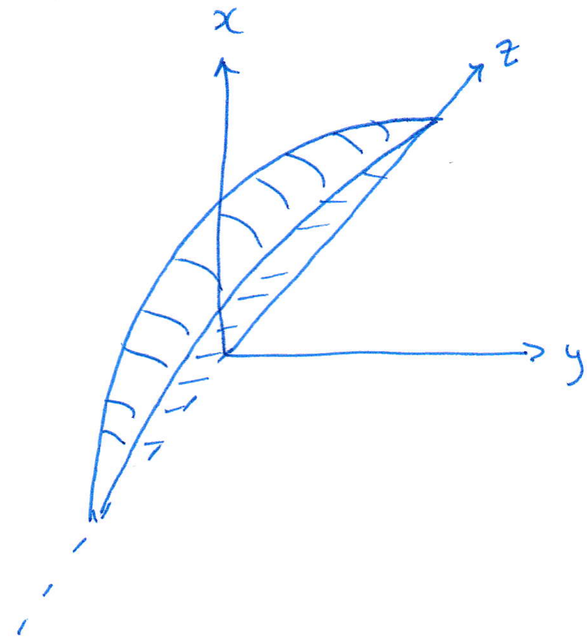
$$= 1 + \frac{3}{4} + \frac{4}{7}$$

$$= \frac{65}{28}$$

4) Use a triple integral in spherical coordinates to find the volume of the smaller wedge cut from a sphere of radius a by two planes that intersect along a diameter at an angle of $\pi/6$. Provide a sketch of the volume.

(5 marks)





$$V = \int_0^{\pi/6} d\theta \int_0^{\pi} d\phi \int_0^{a} \rho^2 \sin\phi \, d\rho$$

$$= \int_0^{\pi/6} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^{a} \rho^2 \, d\rho$$

$$= \left(\frac{\pi}{6}\right)(1+1)\left(\frac{a^3}{3}\right)$$

$$= \frac{1}{9}\pi a^3$$

5) Given $\int_0^{2\pi} \frac{\cos\theta}{1-y\cos\theta} d\theta = 2\pi \frac{1-\sqrt{1-y^2}}{y\sqrt{1-y^2}}$, where $0 < y < 1$, find $\int_0^{2\pi} \ln(1-y\cos\theta)\, d\theta$.

(10 marks)

let $F(y) = \int_0^{2\pi} \ln(1-y\cos\theta)\,d\theta$

$\therefore F'(y) = \frac{d}{dy}\int_0^{\pi} \ln(1-y\cos\theta)\,d\theta = \int_0^{2\pi} \frac{\partial}{\partial y}\left(\ln(1-y\cos\theta)\right) d\theta$

$= \int_0^{2\pi} \left[\frac{1}{1-y\cos\theta}\cdot(-\cos\theta)\right] d\theta = -2\pi \frac{1-\sqrt{1-y^2}}{y\sqrt{1-y^2}}$ (Given)

$\therefore F(y) = -2\pi \int \frac{1-\sqrt{1-y^2}}{y\sqrt{1-y^2}}\,dy$

let $x = \sqrt{1-y^2}$

$\Rightarrow y = \sqrt{1-x^2}$ $\quad y > 0$

$dy = \frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x)\,dx$

$= -2\pi \int \frac{1-x}{x\sqrt{1-x^2}}\cdot\frac{-x}{\sqrt{1-x^2}}\,dx = -2\pi \int \frac{x-1}{1-x^2}\,dx$

$= 2\pi \int \frac{1-x}{(1-x)(1+x)}\,dx = 2\pi \int \frac{dx}{1+x} = 2\pi \ln(1+x) + C$

$= 2\pi \ln\left(1+\sqrt{1-y^2}\right) + C$

Now $F(y) = \int_0^{2\pi} \ln(1-y\cos\theta)\,d\theta$ is continuous in y at $y=0$

$\Rightarrow F(y=0) = \int_0^{2\pi} \ln(1)\,d\theta = 0$

$\therefore 2\pi \ln\left(1+\sqrt{1-y^2}\right) + C = 0$ for $y = 0$

or $2\pi \ln(2) = -C \Rightarrow C = -2\pi \ln 2$

$\therefore F(y) = 2\pi \ln\left(\frac{1+\sqrt{1-y^2}}{2}\right)$

6) By directly calculating the partial derivatives, find the second degree polynomial approximation to the function $f(x,y)=\dfrac{1}{\sqrt{5x+y^2}}$ near the point $(1,2)$.

(10 marks)

$$f(x,y) = (5x+y^2)^{-1/2} \qquad f(1,2) = \frac{1}{3}$$

$$f_x = -\frac{1}{2}(5x+y^2)^{-3/2}\cdot 5 \qquad f_x(1,2) = -\frac{5}{2}\left(\frac{1}{3}\right)^3 = \frac{-5}{54}$$

$$f_y = -\frac{1}{2}(5x+y^2)^{-3/2}\cdot 2y \qquad f_y(1,2) = -\frac{4}{2}\left(\frac{1}{3}\right)^3 = \frac{-2}{27}$$

$$f_{xx} = -\frac{5}{2}\cdot\frac{-3}{2}(5x+y^2)^{-5/2}\cdot 5 \qquad f_{xx}(1,2) = \frac{75}{4}\left(\frac{1}{3}\right)^5 = \frac{25}{324}$$

$$f_{xy} = -\frac{5}{2}\cdot\frac{-3}{2}(5x+y^2)^{-5/2}\cdot 2y \qquad f_{xy}(1,2) = 15\left(\frac{1}{3}\right)^5 = \frac{5}{81}$$

$$f_{yy} = -(5x+y^2)^{-3/2} - y\left(-\frac{3}{2}\right)(5x+y^2)^{-5/2}\cdot 2y$$

$$= -(5x+y^2)^{-3/2} + 3y^2(5x+y^2)^{-5/2} \qquad f_{yy}(1,2) = \frac{-1}{27}+\frac{4}{81} = \frac{1}{81}$$

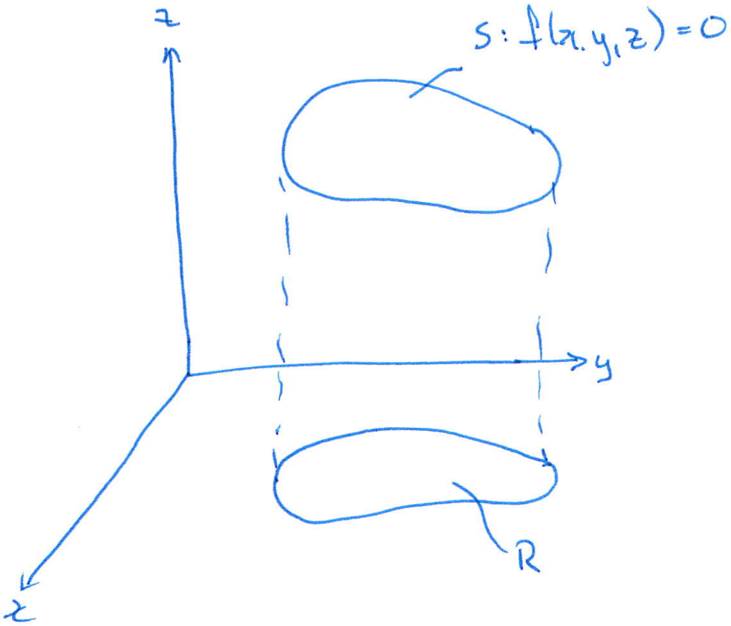
$$\therefore (5x+y^2)^{-1/2} \simeq \frac{1}{3} - \frac{5}{54}(x-1) - \frac{2}{27}(y-2)$$

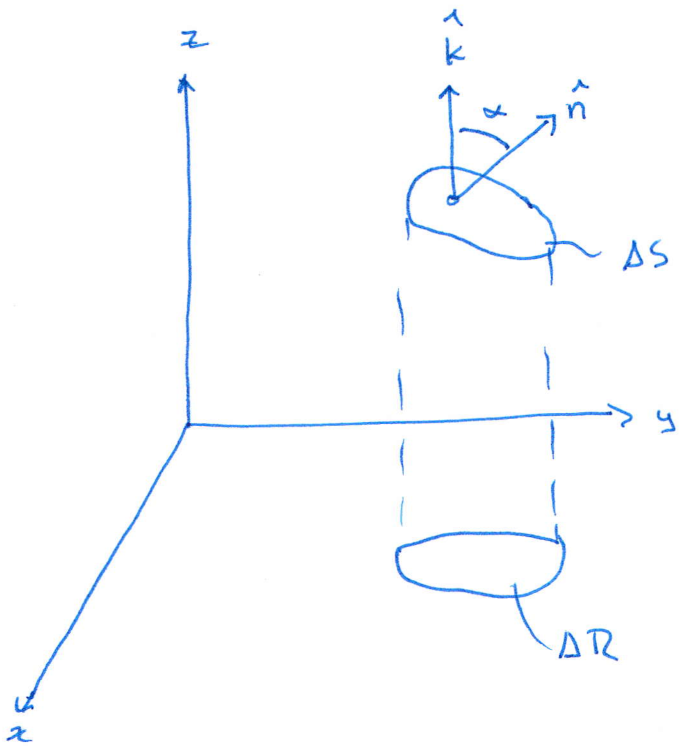
$$+\frac{1}{2!}\left(\frac{25}{324}(x-1)^2 + \frac{10}{81}(x-1)(y-2) + \frac{1}{81}(y-2)^2\right)$$

7) Beginning with a surface given by $f(x,y,z)=0$, outline the steps involved in deriving a formulation for the surface area of f above a region R in the x-y plane by means of projecting the surface onto the x-y plane. That is, show:

$$S=\int_R \frac{\sqrt{(\partial f/\partial x)^2+(\partial f/\partial y)^2+(\partial f/\partial z)^2}}{|\partial f/\partial z|}\,dR$$

(10 marks)





① Begin with a surface $f(x,y,z)=0$ which has a projection R in the x-y plane

② Divide the surface into small elements ΔS, such that all ΔS are small enough to be considered planar.

③ The projected area of ΔS is ΔR, and is related by $\Delta R = \Delta S\,|\cos\alpha|$, where α is the angle between $\hat{n}$ and $\hat{k}$, and $\hat{n}$ is the unit normal vector to ΔS.

④ $|\hat{k}\cdot\hat{n}| = \|\hat{k}\|\,\|\hat{n}\|\,|\cos\alpha| \Rightarrow |\cos\alpha| = |\hat{k}\cdot\hat{n}|$

⑤ ∇f is normal vector to surface $\rightarrow \hat{n} = \nabla f/\|\nabla f\|$

$\therefore |\cos\alpha| = |\nabla f\cdot\hat{k}|/\|\nabla f\| = |\partial f/\partial z|/\|\nabla f\|$

⑥ $\therefore S = \int dS = \int_R \frac{dR}{|\cos\alpha|} = \int_R \frac{\|\nabla f\|}{|\partial f/\partial z|}\,dR = \int_R \frac{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2}}{|\partial f/\partial z|}\,dR$