TERM TEST SOLUTIONS

Q1: FIND A POINT ON EACH PLANE

P_1 ON $ax+by+cz=d_1$

$$\left(x_1, y_1, \frac{d_1-ax_1-by_1}{c}\right)$$

P_2 ON $ax+by+cz=d_2$

$$\left(x_1, y_1, \frac{d_2-ax_1-by_1}{c}\right)$$

FIND $\overrightarrow{P_1P_2}$

$$\overrightarrow{P_1P_2} = \begin{bmatrix} 0 \\ 0 \\ \frac{d_2-ax_1-by_1}{c} - \frac{d_1-ax_1-by_1}{c} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{d_2-d_1}{c} \end{bmatrix}$$

$\left|\operatorname{proj}_{\vec{n}} \overrightarrow{P_1P_2}\right|$ IS THE SHORTEST DISTANCE BETWEEN THE TWO PLANES WHERE $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$\left|\operatorname{proj}_{\vec{n}} \overrightarrow{P_1P_2}\right| = \frac{\left|\overrightarrow{P_1P_2} \cdot \vec{n}\right|}{\|\vec{n}\|}$$

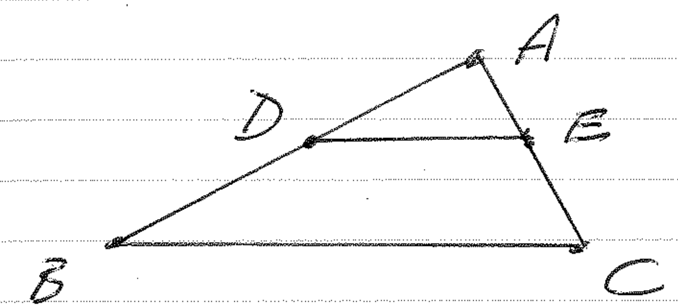
$$\vec{P_1P_2} \cdot \vec{n} = d_2 - d_1$$

$$\|\vec{n}\| = \sqrt{a^2+b^2+c^2}$$

$\therefore$ SHORTEST DISTANCE BETWEEN THE TWO

PLANES IS $\dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$

Q2:



GIVEN INFORMATION:

$$2\vec{DA} = \vec{BA} \qquad \therefore \vec{DA} = \frac{1}{2}\vec{BA}$$

AND

$$2\vec{EA} = \vec{CA} \qquad \vec{EA} = \frac{1}{2}\vec{CA}$$

BY VECTOR ADDITION:

$$\vec{ED} = \vec{EA} + \vec{AD} = \frac{1}{2}\vec{CA} - \frac{1}{2}\vec{BA} = \frac{1}{2}\left(\vec{CA} - \vec{BA}\right)$$

$$\vec{CB} = \vec{CA} + \vec{AB} = \vec{CA} - \vec{BA}$$

$$\therefore \quad \vec{ED} = \frac{1}{2}\vec{CB}$$

$\therefore$ LINE SEGMENT DE IS PARALLEL TO BC AND IS HALF AS LONG.

Q3:

AUGMENTED MATRIX:

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 3 & -1 & 13 & 2 \\ 4 & 1 & a^2 & a+3 \end{array}\right]$$

$$\begin{array}{l} \\ R2-3R1 \\ R3-4R1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 0 & -7 & 25 & -10 \\ 0 & -7 & a^2+16 & a+3-16 \end{array}\right]$$

$$\begin{array}{l} \\ R2\div(-7) \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 0 & 1 & -25/7 & 10/7 \\ 0 & -7 & a^2+16 & a-13 \end{array}\right]$$

$$\begin{array}{l} \\ \\ R3+7R2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 0 & 1 & -25/7 & 10/7 \\ 0 & 0 & a^2-9 & a-3 \end{array}\right]$$

WE CAN STOP HERE AND EXAMINE THE LAST ROW.

FOR INFINITE SOLUTIONS (AT LEAST ONE FREE VARIABLE):

$a^2-9=0$ AND $a-3=0$

$a^2=9$ AND $a=3$

$\therefore\ a=3$

FOR UNIQUE SOLUTION (ALL VARIABLES LEADING):

$$a^2 - 9 \neq 0$$
$$a^2 \neq 9$$

$$\therefore a \neq 3, -3$$

FOR NO SOLUTION (LAST ROW $[0 \ 0 \ 0 \mid *]$):

$$a^2 - 9 = 0 \text{ AND } a - 3 \neq 0$$

$$a^2 = 9 \text{ AND } a \neq 3$$

$$\therefore a = -3$$

Q4:

ASSUME $P\varphi = \varphi P$

$$(P+\varphi)^2 = (P+\varphi)(P+\varphi)$$
$$= P^2 + P\varphi + \varphi P + \varphi^2$$
$$= P^2 + P\varphi + P\varphi + \varphi^2$$
$$= P^2 + 2P\varphi + \varphi^2$$

AS DESIRED

ASSUME $(P+\varphi)^2 = P^2 + 2P\varphi + \varphi^2$

$$\cancel{P^2} + P\varphi + \varphi P + \cancel{\varphi^2} = \cancel{P^2} + 2P\varphi + \cancel{\varphi^2}$$
$$P\varphi + \varphi P = 2P\varphi$$
$$\varphi P = 2P\varphi - P\varphi$$
$$\varphi P = P\varphi$$

AS DESIRED

Q5:

a) $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix} \quad T(\vec{u}) = \begin{bmatrix} x+2y \\ xy \end{bmatrix}$

CHECK PROPERTY a)

$$T(k\vec{u}) = T\left(\begin{bmatrix} kx \\ ky \end{bmatrix}\right) = \begin{bmatrix} kx + 2ky \\ k^2xy \end{bmatrix}$$

$$\neq k\begin{bmatrix} x+2y \\ xy \end{bmatrix}$$

$\therefore$ T IS NOT A LINEAR TRANSFORMATION.

b) $D\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-2y \\ x \end{bmatrix}$

IF D IS A LINEAR TRANSFORMATION, WE CAN FIND MATRIX M_D SUCH THAT:

$$\begin{bmatrix} x+y \\ x-2y \\ x \end{bmatrix} = M_D\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore\ M_D = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$

c) $$E\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y+z \\ 2x+y+z \end{bmatrix}$$

IF E IS A LINEAR TRANSFORMATION, WE CAN FIND MATRIX M_E SUCH THAT:

$$\begin{bmatrix} x \\ x+y+z \\ 2x+y+z \end{bmatrix} = M_E \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore\ M_E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

c)

$$E\left(D\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = M_E M_D \begin{bmatrix} x \\ y \end{bmatrix}$$

$$3\times 3 \quad 3\times 2 \quad 2\times 1$$

$\therefore$ COMPOSITION IS DEFINED.

$$M_E M_D = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ 4 & 0 \end{bmatrix}$$

$$E\left(E\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right) = M_E M_E \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$3\times 3 \quad 3\times 3 \quad 3\times 1$$

$\therefore$ COMPOSITION IS DEFINED.

$$M_E M_E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 2 \\ 5 & 2 & 2 \end{bmatrix}$$

Q6:

A IS $n \times n$ WITH DISTINCT EIGENVALUES $\lambda_1, \lambda_2, \ldots, \lambda_n$

a) k IS A POSITIVE INTEGER

$\therefore$ A^k IS ALSO AN $n \times n$ MATRIX

$\therefore$ A^k HAS n EIGENVALUES.

b) EIGENVALUES OF A SATISFY:

$$A\vec{u} = \lambda \vec{u}$$

$$\therefore A^2\vec{u} = A\lambda\vec{u} = \lambda A\vec{u} = \lambda^2\vec{u}$$

$$A^3\vec{u} = A\lambda^2\vec{u} = \lambda^2 A\vec{u} = \lambda^3\vec{u}$$

$$\vdots$$

$$A^k\vec{u} = \lambda^k\vec{u}$$

$\therefore$ A^k HAS EIGENVALUES $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$.