ECE259H1: Electromagnetism

Homework Review Quiz 2 – Friday March 10, 2023

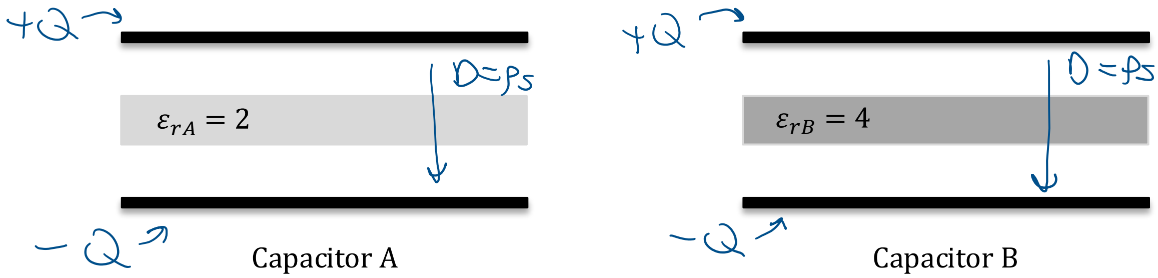
$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The quiz is worth 20 marks and has two questions. Question 1 is worth 6 marks, and Question 2 is work 14 marks.
- Show all of your work.
- The final page has some reference material that you might find helpful.
- Take a deep breath and relax 😊.

Solution #1

Question #1 (6 marks)

Consider the two capacitors shown below. In both cases, the dielectric piece takes up one-third of the spacing between the plates and centered between the plates. The plate sizes are the same for both, and you can ignore the effects of fringing fields. Both capacitors have $\pm Q$ on their plates.



How do the voltages and stored energies compare between these two capacitors? Briefly justify your answers.

Since the capacitor's have the same charge, and the air/dielectric interfaces are perpendicular to the field $\bar{D}$ is constant throughout.

$$\therefore D_1 = D_2 = \rho_s = \frac{Q}{S}$$

$\therefore$ E then is given as: $E_A = \begin{cases} \frac{Q}{\varepsilon_0 S} & \text{air regions} \\ \frac{Q}{2\varepsilon_0 S} & \text{dielectric region} \end{cases}$ & $E_B = \begin{cases} \frac{Q}{\varepsilon_0 S} & \text{air regions} \\ \frac{Q}{4\varepsilon_0 S} & \text{dielectric region} \end{cases}$

Since $V = E_{air}\left(\frac{2d}{3}\right) + E_{diel}\left(\frac{d}{3}\right)$

$$V_A = \frac{2dQ}{3\varepsilon_0 S} + \frac{dQ}{6\varepsilon_0 S} = \frac{5dQ}{6\varepsilon_0 S} > V_B = \frac{2dQ}{3\varepsilon_0 S} + \frac{dQ}{12\varepsilon_0 S} = \frac{9dQ}{12\varepsilon_0 S} = \frac{3dQ}{4\varepsilon_0 S}$$

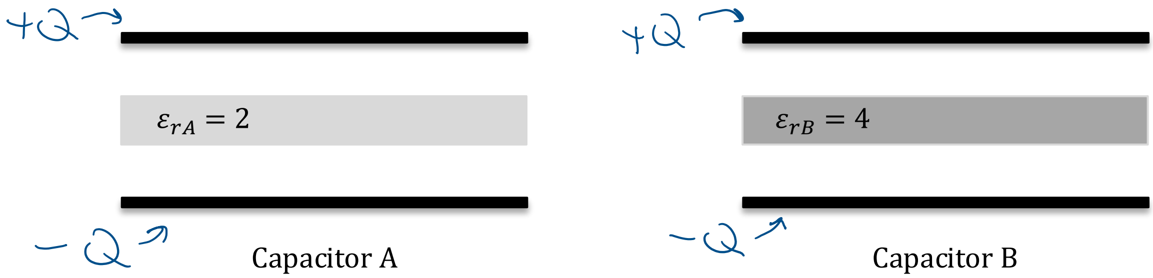
And since $E_{dielA} > E_{dielB}$ the volume integrals of the fields would result in

$$W_{eA} = \frac{1}{2}\iiint_{Vol} \bar{D}\cdot\bar{E}_A \, dv > W_{eB} = \frac{1}{2}\iiint_{Vol} \bar{D}\cdot\bar{E}_B \, dv$$

Solution #2

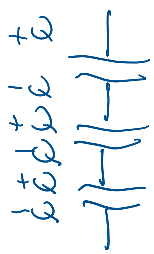
Question #1 (6 marks)

Consider the two capacitors shown below. In both cases, the dielectric piece takes up one-third of the spacing between the plates and centered between the plates. The plate sizes are the same for both, and you can ignore the effects of fringing fields. Both capacitors have $\pm Q$ on their plates.



How do the voltages and stored energies compare between these two capacitors? Briefly justify your answers.

* Can consider these as the series combination of three capacitors



$C_{1A} = \frac{\varepsilon_0 S}{d/3}$

$C_{2A} = 2\varepsilon_0 S / d/3$

$C_{3A} = \frac{\varepsilon_0 S}{d/3}$

$$C_A = \left(\frac{1}{C_{1A}} + \frac{1}{C_{2A}} + \frac{1}{C_{3A}}\right)^{-1}$$

$$= \left(\frac{d}{3\varepsilon_0 S} + \frac{d}{6\varepsilon_0 S} + \frac{d}{3\varepsilon_0 S}\right)^{-1}$$

$$= \left(\frac{2d + d + 2d}{6\varepsilon_0 S}\right)^{-1} = \frac{6\varepsilon_0 S}{5d}$$

Similarly: $C_B = \left(\frac{d}{3\varepsilon_0 S} + \frac{d}{12\varepsilon_0 S} + \frac{d}{3\varepsilon_0 S}\right)^{-1} = \frac{12\varepsilon_0 S}{9d} = \frac{4\varepsilon_0 S}{3d}$

Since $C_A = \frac{Q}{V_A} \rightarrow V_A = \frac{5Qd}{6\varepsilon_0 S}$ and $C_B = \frac{Q}{V_B} \rightarrow V_B = \frac{3Qd}{4\varepsilon_0 S}$

$\therefore V_A > V_B$

$\therefore W_{eA} = \frac{1}{2} Q V_A > W_{eB} = \frac{1}{2} Q V_B$ Since the Q's are constant between both capacitors

Question #2 (14 marks)

Assume the xy-plane ($z = 0$), separates two lossless dielectric regions with $\varepsilon_{r1} = 2$ ($z > 0$) and $\varepsilon_{r2} = 4$ ($z < 0$). The electric field intensity in Region 1 is given by $\mathbf{E}_1(x, y, z) = (2x + y)\,\hat{\mathbf{a}}_x + (x - 2y + z)\,\hat{\mathbf{a}}_y + y\,\hat{\mathbf{a}}_z$. A free surface charge density $\rho_S = 6\varepsilon_0(x - y)$ exists at $z = 0$.

Find $\mathbf{E}_2$, $\mathbf{D}_2$, and the bound charge density (i.e., ρ_{sb2}), just below the interface in Region 2.

The field in Region 2 just below the interface is given by

$$\bar{E}_2 = \bar{E}_{t2} + \bar{E}_{n2}, \quad \bar{D}_2 = 4\varepsilon_0 \bar{E}_2$$

$\underline{E_{t1} = E_{t2}}$: $\bar{E}_{t1} = (2x+y)\hat{a}_x + (x-2y)\hat{a}_y$ @ $z=0$

$$= \bar{E}_{t2}$$

$\underline{D_{n1} - D_{n2} = \rho_S}$: $D_{n1} = 2\varepsilon_0 E_{n1} = +2\varepsilon_0 y$

$$\therefore D_{n2} = D_{n1} - \rho_S = +2\varepsilon_0 y - 6\varepsilon_0(x-y) = -6\varepsilon_0 x + 8\varepsilon_0 y$$

$$\therefore E_{n2} = \frac{D_{n2}}{4\varepsilon_0} = -\frac{3}{2}x + 2y$$

$$\therefore \bar{E}_2(z=0) = \bar{E}_{t1} + \bar{E}_{n2} = \underline{\underline{(2x+y)\hat{a}_x + (x-2y)\hat{a}_y + (2y - \tfrac{3}{2}x)\hat{a}_z}} \left[\frac{V}{m}\right]$$

$$\therefore \bar{D}_2(z=0^-) = 4\varepsilon_0 \bar{E}_2 = \underline{\underline{4\varepsilon_0(2x+y)\hat{a}_x + 4\varepsilon_0(x-2y)\hat{a}_y + 4\varepsilon_0(2y - \tfrac{3}{2}x)\hat{a}_z}} \left[\frac{C}{m^2}\right]$$

Question #2 (continued)

For ρ_{sb} just below the interface:

$$\rho_{sb} = \bar{P}_2 \cdot \hat{a}_n \quad \text{where } \hat{a}_n = \hat{a}_z \text{ in this case}$$

$$= \varepsilon_0(\varepsilon_{r_2} - 1)\bar{E}_2 \cdot \hat{a}_z$$

$$= \underline{\underline{3\varepsilon_0\left(2y - \tfrac{3}{2}x\right) \quad [C/m^2]}}$$

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Reference Formulae

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

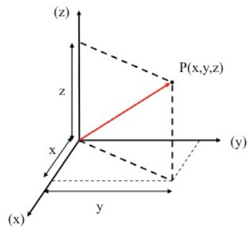
Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

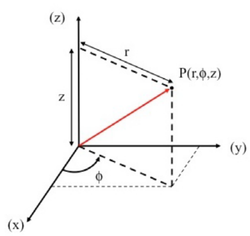
1.3 Spherical coordinates

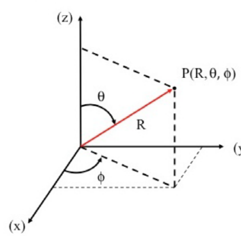
Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ







3. Dot products of unit vectors

·	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

5. Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$\begin{aligned}
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}$$

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	-$A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

Table 1 Electrostatics

$$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{|\mathbf{R}_2 - \mathbf{R}_1|^3}(\mathbf{R}_2 - \mathbf{R}_1) \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3}dQ'$$

$$\nabla\times\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{D} = \rho_v$$

$$\oint_C \mathbf{E}\cdot\mathbf{dl} = 0 \qquad \oint_S \mathbf{D}\cdot\mathbf{dS} = Q$$

$$\mathbf{E} = -\nabla V \qquad V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{|\mathbf{R} - \mathbf{R}'|}$$

$$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E}\cdot\mathbf{dl}$$

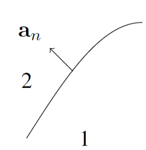
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E} \qquad \mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$$

$$\rho_{p,v} = -\nabla\cdot\mathbf{P}$$

$$\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{a}_n\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

$$E_{1,t} = E_{2,t}$$



$$Q = CV \qquad W_e = \frac{1}{2}QV$$

$$W_e = \frac{1}{2}\int_{vol}\rho_v V dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}dv$$

$$\nabla\cdot(\varepsilon\nabla V) = -\rho_v \qquad \nabla\cdot(\varepsilon\nabla V) = 0$$