UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Term Test

October 28, 2021

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Closed book.

All questions are of equal value.

Permitted calculators (any suffix is acceptable):

- Sharp EL-W516
- Casio FX-991

This test contains 22 pages including this page and the cover page, printed two-sided. Do <u>not</u> tear any pages from this test. Present complete solutions in the space provided.

Given information:

$$cos\,\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\, \vec{u} = \frac{\vec{u}\cdot\vec{d}}{||\,\vec{d}\,||^2}\vec{d}$$

$$det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q1: State whether each statement is true or false. Give a proof using vectors if true or find a counterexample if false. Assume all vectors in this question are nonzero vectors.

a) Let $\vec{u}, \vec{v}, \vec{w} \in R^3$. If $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is orthogonal to $\vec{v}$ and $\vec{w}$, then $\vec{v}$ is parallel to $\vec{w}$.

b) Let $\vec{u}, \vec{v}, \vec{w} \in R^3$. If $\vec{u}$ is orthogonal to $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is orthogonal to $\vec{v} + 2\vec{w}$.

c) Let $\vec{u}, \vec{v} \in R^3$. If $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors, then $\|\vec{u} - \vec{v}\| = \sqrt{2}$.

Solutions:

a) False by counterexample:

Let $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Note: Any two vectors that are orthogonal to $\vec{u}$ but are not parallel to each other will work.

Both vectors are orthogonal to $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e.

$\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$. However, $\vec{v}$ is not parallel to $\vec{w}$ because $\vec{v}$ cannot be expressed as a scalar multiple of $\vec{w}$.

b) True by proof:

Given $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$,

$\therefore \vec{u} \cdot (\vec{v} + 2\vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot (2\vec{w}) = \vec{u} \cdot \vec{v} + 2(\vec{u} \cdot \vec{w}) = 0 + 2(0) = 0$

Note: $\vec{u} \cdot (2\vec{w}) = 2(\vec{u} \cdot \vec{w})$ must be shown for full marks.

c) True by proof:

Given $\vec{u} \cdot \vec{v} = 0$,

$\|\vec{u} - \vec{v}\|^2 = (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2$

Since $\vec{u}$ and $\vec{v}$ are unit vectors,

$\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 = 1 + 1 = 2$

$\therefore \|\vec{u} - \vec{v}\| = \sqrt{2}$

Note: Because we are in R^3, answers will be accepted that state that this is a right-angle triangle with the two sides making the right angle both having unit length. Therefore, by the Pythagorean theorem, the hypotenuse is $\sqrt{2}$. However, to get full marks, the answers must show the two sides of the right angle as vectors $\vec{u}$ and $\vec{v}$ with both their tails at the right-angle, and the hypotenuse shown as vector $\vec{u} - \vec{v}$ or $\vec{v} - \vec{u}$.

Q2: Consider the linear combination of three vectors:

$$c\vec{u} + d\vec{v} + e\vec{w}$$

$$\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

c, d, e are scalars

a) Find a linear combination of these three vectors that equals $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, i.e., find values for $c, d,$ and e.

b) Express the set of equations used to solve part (a) in the standard form $A\vec{x} = \vec{b}$ where A is a $3x3$ matrix, $\vec{x} = \begin{bmatrix} c \\ d \\ e \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

c) Give a geometric interpretation of the row picture associated with the system of linear equations in part (b)?

d) Building on your answer to part (c), give a geometric interpretation of the solution to this system of linear equations found in part (a).

Solutions:

a) Find values for $c, d,$ and e such that $c\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Corresponding linear equations that need to be solved:

$$2c - d = 1 \rightarrow d = 2c - 1 \quad (1)$$

$$-c + 2d - e = 0 \rightarrow 2d = c + e \quad (2)$$

$$-d + 2e = 0 \rightarrow d = 2e \quad (3)$$

Substituting (3) → (2),

$$4e = c + e \rightarrow c = 3e \quad (4)$$

Substituting (3) and (4) into (1),

$$2e = 2(3e) - 1 \rightarrow 4e = 1 \rightarrow e = \frac{1}{4}$$

$\therefore c = \frac{3}{4}, d = \frac{2}{4}$ and $e = \frac{1}{4}$.

Note: Any solution method used here is acceptable.

b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

c) The row picture is 3 planes in R^3 with the following scalar equations,

$$2c - d + 0e = 1$$

$$-c + 2d - e = 0$$

$$0c - d + 2e = 0$$

d) The solution in part (a) corresponds to a single point of intersection of the 3 planes, namely $\left(\frac{3}{4}, \frac{2}{4}, \frac{1}{4}\right)$.

Q3: Consider the following transformation, $\vec{u} = T(\vec{v})$ where $\vec{v}, \vec{u} \in R^2$ and

$$\vec{u} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = T(\vec{v}) = T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

Hint: For part (b) of this question, making a sketch showing the vectors in R^2 may help.

a) Show that under this transformation $\vec{v}$ and $\vec{u}$ are orthogonal.
b) State geometrically what this transformation does to every nonzero vector $\vec{v}$.
c) Show that $T(\vec{v})$ is a linear transformation.
d) Derive the matrix M_T associated with this linear transformation.
e) Calculate M_T^4. Explain geometrically, based on your answer to part (b), why $M_T^4 = I$.

Solutions:

a) Examine $\vec{u} \cdot \vec{v} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -v_2 v_1 + v_1 v_2 = 0$
   Therefore $\vec{v}$ and $\vec{u}$ are orthogonal.
b) This transformation rotates $\vec{v}$ counterclockwise by 90° to produce $\vec{u}$.
c) Check Property 1:

   $$cT(\vec{v}) = c\begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} -cv_2 \\ cv_1 \end{bmatrix}$$

   $$T(c\vec{v}) = T\left(\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}\right) = \begin{bmatrix} -cv_2 \\ cv_1 \end{bmatrix} = cT(\vec{v})$$

   $\therefore$ Property 1 is satisfied.

   Check Property 2:

   $$T(\vec{v}) + T(\vec{w}) = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix} + \begin{bmatrix} -w_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} -v_2 - w_2 \\ v_1 + w_1 \end{bmatrix}$$

   $$T(\vec{v} + \vec{w}) = T\left(\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}\right) = \begin{bmatrix} -(v_2 + w_2) \\ v_1 + w_1 \end{bmatrix} = T(\vec{v}) + T(\vec{w})$$

   $\therefore$ Property 2 is satisfied.

   Therefore $T(\vec{v})$ is a linear transformation.
d) Examine $T(\vec{\imath}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $T(\vec{\jmath}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$
   $\therefore M_T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
e) $M_T^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$$M_T^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$M_T^4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

M_T^4 summarizes the linear transformation that rotates $\vec{v}$ counterclockwise by $4 \times 90^{\circ} = 360^{\circ}$ which corresponds to $\vec{v}$ being returned to its original position, i.e., $\vec{v} = I\vec{v}$ and therefore $M_T^4 = I$.

Q4:

a) Find the eigenvalues of the matrix:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

b) Find the eigenvalues of the matrix:

$$A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

c) The eigenvalues of the matrices A and $A + I$ are related. Derive the general relationship between their eigenvalues and confirm that this relationship holds for the matrices found in parts (a) and (b).

Solutions:

a) To find the eigenvalues of matrix A, we need to solve,

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$$

$$\therefore \lambda = \frac{4 \pm \sqrt{36}}{2} = 5, -1$$

b) To find the eigenvalues of matrix $A + I$, we need to solve,

$$\det(A + I - \lambda^* I) = 0$$

$$A + I - \lambda^* I = \begin{bmatrix} 2 - \lambda^* & 4 \\ 2 & 4 - \lambda^* \end{bmatrix}$$

$$\det(A + I - \lambda^* I) = (2 - \lambda^*)(4 - \lambda^*) - 8 = {\lambda^*}^2 - 6\lambda^* + 0 = 0$$

$$\therefore \lambda^* = 6,0$$

c) Basic equation for matrix A is given by (where λ denotes the eigenvalues of A),

$$A\vec{w} = \lambda\vec{w} \quad \vec{w} \neq \vec{0} \quad (1)$$

Basic equation for matrix $A + I$ is given by (where λ^* denotes the eigenvalues of $A + I$),

$$(A + I)\vec{z} = \lambda^*\vec{z} \quad \vec{z} \neq \vec{0}$$

$$A\vec{z} + I\vec{z} = A\vec{z} + \vec{z} = \lambda^*\vec{z}$$

$$\therefore A\vec{z} = (\lambda^* - 1)\vec{z} \quad \vec{z} \neq \vec{0} \quad (2)$$

By comparison of (1) and (2), it may be concluded that,

$$\lambda = \lambda^* - 1$$

Note: Solutions that derive this result for the general 2x2 case $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are acceptable.

This holds for the matrices in parts (a) and (b),

$$5 = 6 - 1$$

$$-1 = 0 - 1$$

Q5:

a) Consider the linear transformation T_1 that transforms $\vec{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\vec{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find the matrix M_{T_1} associated with this linear transformation.

b) Consider the linear transformation T_2 that transforms $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the matrix M_{T_2} associated with this linear transformation.

c) What is the relationship between these two matrices M_{T_1} and M_{T_2}? Provide evidence to support your answer.

Solutions:

a) $M_{T_1} = [T_1(\vec{\imath}) \quad T_1(\vec{\jmath})] = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

Note: Answers that simply present the final matrix without some sort of explanation will not receive full marks.

b) Let $M_{T_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where a, b, c, d are unknowns.

Given $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$,

$$2a + 5b = 1 \quad (1)$$
$$2c + 5d = 0 \quad (2)$$
$$a + 3b = 0 \quad (3)$$
$$c + 3d = 1 \quad (4)$$

Combining (1) and (3),

$$2(-3b) + 5b = 1 \rightarrow b = -1 \rightarrow a = 3$$

Combining (2) and (4),

$$2(1 - 3d) + 5d = 0 \rightarrow d = 2 \rightarrow c = -5$$

$$\therefore M_{T_2} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

c) M_{T_1} and M_{T_2} are inverses of each other,

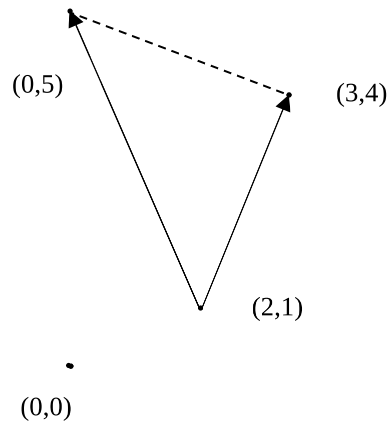
$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = I$$

Q6:
Hint #1: area of a parallelogram = base x height; area of a triangle = (base x height)/2
Hint #2: for both parts (a) and (b) of this question, making a sketch showing the points in R^2 may help.

a) The corners of a triangle are $(2,1)$, $(3,4)$ and $(0,5)$. Use a vector approach to find the area of this triangle.

b) Add a corner at $(-1,0)$ to make a lopsided four-sided region. Use a vector approach to find the area of this four-sided region.

Solutions:



a) Let's determine the two R^2 vectors shown above,

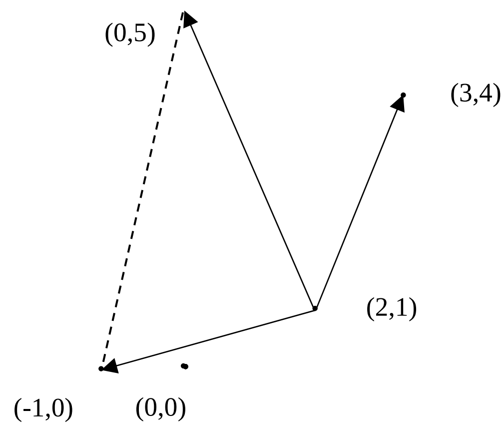
$\vec{v} = \begin{bmatrix} 0-2 \\ 5-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 3-2 \\ 4-1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

We can represent these same two vectors in R^3 as follows,

$\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Now, we can use cross product to calculate the area of the triangle,

$$Area = \frac{1}{2}\|\vec{v} \times \vec{w}\| = \frac{1}{2}\left\|\begin{bmatrix} 0 \\ 0 \\ (-2)(3) - (1)(4) \end{bmatrix}\right\| = \frac{1}{2}(10) = 5$$



b) The area of the four-sided region is equal to the area of the triangle in part (a) plus the area of the triangle shown in the figure above.

Let's determine the two R^2 vectors shown above,

$$\vec{v} = \begin{bmatrix} 0-2 \\ 5-1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -1-2 \\ 0-1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

We can represent these same two vectors in R^3 as follows,

$$\vec{v} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}$$

Now, we can use cross product to calculate the area of the triangle,

$$Area = \frac{1}{2}\|\vec{v} \times \vec{z}\| = \frac{1}{2}\left\|\begin{bmatrix} 0 \\ 0 \\ (-2)(-1)-(-3)(4) \end{bmatrix}\right\| = \frac{1}{2}(14) = 7$$

Therefore, the area of the four-sided region is $5 + 7 = 12$.

Note: This problem can be solved using projections but it is considerably more work.