2012 EXAM SOLUTIONS

1.
$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 8 & -3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ R2+3R1 \\ \\ \\ \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 8 & -3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ R3-4R1 \\ \\ \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 1 & 0 \\ 8 & -3 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ R4-8R1 \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 & -8 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ R4+3R2 \end{array}\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 0 & 1 \end{array}\right]$$

$\Uparrow$

INVERSE

$E_1\ (R2+3R1)$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$E_2\ (R3-4R1)$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$E_3\ (R4-8R1)$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -8 & 0 & 0 & 1 \end{bmatrix}$$

$E_4\ (R4+3R2)$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

INVERSE $= E_4 E_3 E_2 E_1$

2. a)

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 200 \\ -25 \\ 175 \\ -150 \\ 200 \end{bmatrix}}_{B}$$

b)

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 200 \\ 1 & -1 & 0 & 0 & 0 & 0 & -25 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & -1 & 0 & -150 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \end{array}\right]$$

$$\Big\Downarrow \text{ RNF VIA GAUSSIAN ELIMINATION}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 0 & 1 & 0 & -1 & 175 \\ 0 & 0 & 1 & -1 & 0 & 1 & 50 \\ 0 & 0 & 0 & 0 & 1 & 1 & 200 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 150 \\ 175 \\ 50 \\ 0 \\ 200 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

x_4, x_6 FREE

c) $x_4 = 50 \qquad x_6 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 150 \\ 175 \\ 50 \\ 0 \\ 200 \\ 0 \end{bmatrix} + 50 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 100 \\ 125 \\ 100 \\ 50 \\ 200 \\ 0 \end{bmatrix}$$

3. a) FALSE

LET $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{u} \cdot (\vec{v} + \vec{w}) = 0$ $\quad$ $\vec{u}$ IS ORTHOGONAL TO $\vec{v} + \vec{w}$

$\vec{u} \cdot \vec{v} = 1 \neq 0$ $\quad$ $\therefore$ $\vec{u}$ IS NOT ORTHOGONAL TO $\vec{v}$

$\vec{u} \cdot \vec{w} = -1 \neq 0$ $\quad$ $\therefore$ $\vec{u}$ IS NOT ORTHOGONAL TO $\vec{w}$

3. b) TRUE

GIVEN $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v}$

$\therefore \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u}\cdot\vec{v} = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\therefore \vec{u}\cdot\vec{v} = 0$

$\therefore$ $\vec{u}$ AND $\vec{v}$ ARE ORTHOGONAL.

c) FALSE

LET $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $\vec{u}$ AND $\vec{v}$ ARE PARALLEL

$\|\vec{u}+\vec{v}\| = 0$

$\|\vec{u}\| + \|\vec{v}\| = 1 + 1 = 2$

$\therefore \|\vec{u}+\vec{v}\| \neq \|\vec{u}\| + \|\vec{v}\|$

4. a)

$z = 1/x$	y
1	7
1/3	3
1/6	1

$$AX = B$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1/3 \\ 1 & 1/6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 3 & 3/2 \\ 3/2 & 41/36 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 11 \\ 49/6 \end{bmatrix}$$

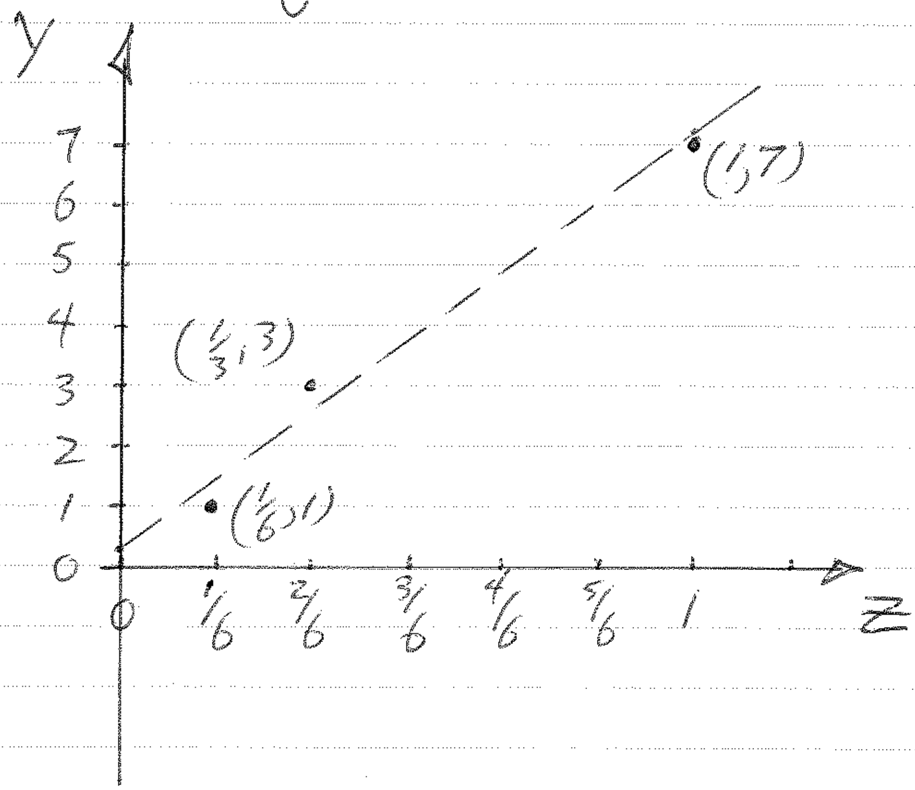
$$\left[\begin{array}{cc|c} 3 & 3/2 & 11 \\ 3/2 & 41/36 & 49/6 \end{array}\right]$$

$\Downarrow$ RNF VIA GAUSSIAN ELIMINATION

$$\left[\begin{array}{cc|c} 1 & 0 & 5/21 \\ 0 & 1 & 48/7 \end{array}\right]$$

$$\checkmark \; y = \frac{5}{21} + \frac{48}{7}\left(\frac{1}{x}\right)$$

4. b)



5. a)

$$Y' = F(t, Y)$$

$$Y = \begin{bmatrix} x \\ y \end{bmatrix} \quad Y' = \begin{bmatrix} x' \\ y' \end{bmatrix} \quad Y_0 = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$$

$$Y' = \begin{bmatrix} ax - bxy \\ -cy + dxy \end{bmatrix} = F(t, Y)$$

EULER'S METHOD:

$$t_{n+1} = t_n + \Delta t$$

$$Y_{n+1} = Y_n + \Delta t \underbrace{\begin{bmatrix} ax_n - bx_ny_n \\ -cy_n + dx_ny_n \end{bmatrix}}_{F(t_n, Y_n)}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \Delta t \begin{bmatrix} ax_n - bx_ny_n \\ -cy_n + dx_ny_n \end{bmatrix}$$

5. b) $Y_0 = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} + 0.1 \begin{bmatrix} 1.2x_n - 0.6x_n y_n \\ -0.8y_n + 0.3x_n y_n \end{bmatrix}$$

t_n	x_n	y_n
0	2	1
0.1	2.12	.98
0.2	2.250	.964
0.3	2.390	.952
0.4	2.540	.944
0.5	2.701	.940

6. a)

TOTAL HEAT ABSORBED

$$= \varepsilon A \int_0^t q \, dt$$

$$\approx \varepsilon A \Delta t \frac{1}{2}\left[q(0) + 2q(2) + 2q(4) + 2q(6) + 2q(8) + 2q(10) + 2q(12) + q(14)\right]$$

$$= (0.45)(150\times10^3)(2)\left(\frac{1}{2}\right)\left[0.10 + 2(5.32) + 2(7.80) + 2(8.00) + 2(8.03) + 2(6.27) + 2(3.54) + 0.20\right]$$

$$= 67.5\times10^3\left[78.22\right]$$

$$= 5280 \times 10^3 \text{ cal}$$

$$= 5280 \text{ kcal}$$

6. b)

FIT A 7^{TH} ORDER POLYNOMIAL THROUGH THE 8 POINTS

$$q = a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6 + ht^7$$

$$AX = B$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 & 2^7 \\ 1 & 4 & 4^2 & 4^3 & 4^4 & 4^5 & 4^6 & 4^7 \\ 1 & 6 & 6^2 & 6^3 & 6^4 & 6^5 & 6^6 & 6^7 \\ 1 & 8 & 8^2 & 8^3 & 8^4 & 8^5 & 8^6 & 8^7 \\ 1 & 10 & 10^2 & 10^3 & 10^4 & 10^5 & 10^6 & 10^7 \\ 1 & 12 & 12^2 & 12^3 & 12^4 & 12^5 & 12^6 & 12^7 \\ 1 & 14 & 14^2 & 14^3 & 14^4 & 14^5 & 14^6 & 14^7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 0.1 \\ 5.32 \\ 7.80 \\ 8.00 \\ 8.03 \\ 6.27 \\ 3.54 \\ 0.2 \end{bmatrix}$$