AER210 VECTOR CALCULUS and FLUID MECHANICS

Quiz 4

Duration: 70 minutes

26 November 2018

Closed Book, no aid sheets

Non-programmable calculators allowed

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Family Name: Ekmekci

Given Name: Alis

Student #: ______________________

TA Name/Tutorial #: ______________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	8	
2	10	
3	12	
4	9	
5	12	
TOTAL	51	/50

Hints:

$$\sum \vec{F}_{CV} = \frac{d}{dt}\iiint_{CV} \vec{V}(\rho d\forall) + \oiint_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A})$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z}$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

$$\nabla.\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

(a) [**3 marks**] Bernoulli equation is given below. Indicate the meaning of each term on the left hand side of this equation:

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

$\frac{p}{\rho}$ = energy due to pressure per unit mass

$\frac{V^2}{2}$ = kinetic energy per unit mass

gz = potential energy per unit mass

Note:

$$\left[\frac{p}{\rho}\right] = \left[\frac{V^2}{2}\right] = [gz] = \frac{m^2}{s^2} = kg\,\frac{m^2}{s^2}\,\frac{1}{kg}$$

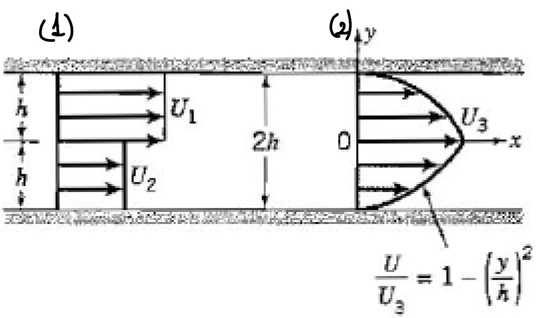
$$= kg\,\frac{m}{s^2}\,\frac{m}{kg}$$ ← N

$$= \frac{N\,m}{kg}$$ ← J

$$= \frac{J}{kg}$$

= energy per unit mass

(b) [**5 marks**] In the rectangular duct shown below, two parallel streams of a constant density gas enter on the left with constant velocities U_1 and U_2. After mixing, the gas exits on the right with a parabolic profile $U = U_3(1-(y/h)^2)$ where U_3 is the maximum velocity value. Find U_3 in terms of U_1 and U_2.



$$\boxed{\dot{m}_1 = U_1 h w + U_2 h w}$$

(here w is the dimension in z direction)

$$\dot{m}_2 = \int_{y=-h}^{y=h} U\,w\,dy = \int_{-h}^{h} \left(U_3 - \left(\frac{y}{h}\right)^2 U_3\right) w\,dy$$

$$= U_3 w y \Big|_{-h}^{h} - U_3 w \frac{y^3}{3h^2}\Big|_{-h}^{h}$$

$$= U_3 w \underbrace{(h+h)}_{2h} - U_3 \frac{w}{3h^2} \underbrace{(h^3+h^3)}_{2h^3}$$

$$= 2U_3 w h - \frac{2}{3} U_3 w h$$

$$\boxed{\dot{m}_2 = \frac{4}{3} U_3 w h}$$

From continuity:

$$\dot{m}_1 = \dot{m}_2 \Rightarrow U_1 \cancel{h}\cancel{w} + U_2 \cancel{h}\cancel{w} = \frac{4}{3} U_3 \cancel{h}\cancel{w} \Rightarrow \boxed{U_3 = \frac{3}{4}(U_1 + U_2)}$$

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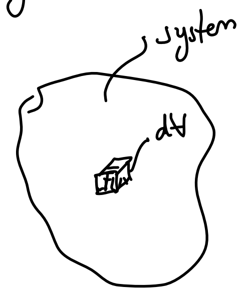
2) (a) [**5 marks**] Using the Reynolds Transport theorem, derive the conservation of mass equation for a control volume (in other words, the integral form of the continuity equation). <u>*Hint*</u>*: Start with the conservation of mass equation for a fluid system. Then use the Reynolds Transport Theorem to convert the conservation of mass equation for a control volume.*
The Reynolds Transport Theorem for a fluid parameter $\boldsymbol{B = mb}$ *can be written as:*

$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \dot{B}_{out} - \dot{B}_{in} \quad \text{or} \quad \frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} + \oint\!\!\oint_{CS} b\rho\vec{V}.\overrightarrow{dA}$$

$$\frac{d}{dt}(m_{sys}) = 0 \Leftarrow \text{(conservation of mass for a fluid system)}$$



$$B = m\,b \qquad \underset{m}{B}, \quad \underset{1}{b}$$

$$\underbrace{\frac{d\cancel{m}_{sys}}{dt}}_{0} = \frac{dm_{cv}}{dt} + \oint\!\!\oint_{CS} \rho\,\vec{V}.d\vec{A}$$

$$0 = \frac{d}{dt}\left(\iiint_{\forall} \rho\, d\forall\right) + \oint\!\!\oint_{A} \rho\vec{V}.d\vec{A}$$

(or CV) (or CS)

$$\boxed{\frac{d}{dt}\iiint_{\forall} \rho\, d\forall + \iint_{A} \rho\,\vec{V}.d\vec{A} = 0}$$

Continuity eqn in integral form

(b) [**5 marks**] The x, y and z components of velocity for a certain incompressible steady flow field are respectively:

u = 3x+y

v = xy + z

w = ?

Find the z component (w) required to satisfy the continuity equation.

Continuity eqn. for incompressible flow $\Rightarrow \vec{\nabla}.\vec{V} = 0$

$$\underbrace{\frac{\partial u}{\partial x}}_{3} + \underbrace{\frac{\partial v}{\partial y}}_{x} + \frac{\partial w}{\partial z} = 0 \Rightarrow \frac{\partial w}{\partial z} = -x - 3 \xrightarrow[\text{integrate}]{} \boxed{w = -xz - 3z + f(x,y)}$$

$f(x,y)$ can have any form & conservation of mass will still be satisfied.

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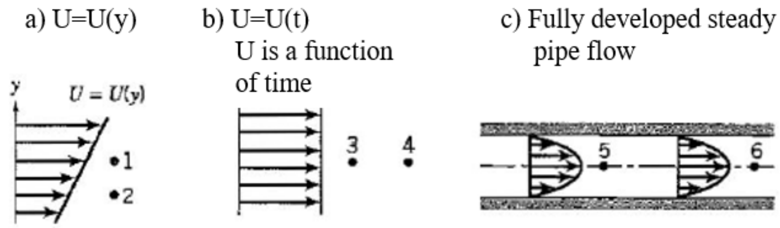
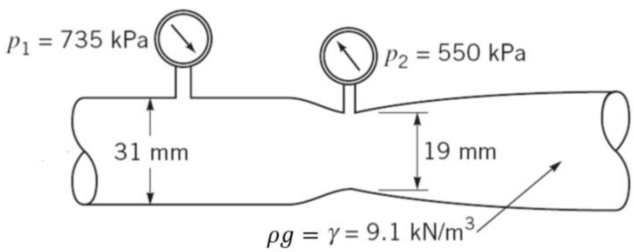
3) (a) [**6 marks**] Indicate TRUE (T) or FALSE (F) to the statements blow:



F Bernoulli can be used between points 1 and 2 in (a) above.
F Bernoulli can be used between points 3 and 4 in (b) above.
T Bernoulli can be used between points 5 and 6 in (c) above.
F The divergence of the velocity vector is the rate of outflow of volume per unit mass.
T A tiny neutrally buoyant electronic pressure probe is released into the inlet pipe of a water pump and transmits 2000 pressure readings per second as it passes through the pump. This is a Lagrangian measurement.
T In a steady incompressible flow, substantial derivative of density is zero.

(b) [**6 marks**] Gauge pressures measured for a pipe are given below for the circular cross-sections 1 and 2. If d_1 = 31 mm and d_2 = 19 mm for sections 1 and 2 respectively, determine the velocity at section 2. Specific weight of the fluid is given as $\gamma = \rho g$ = 9.1kN/m^3.



$$\boxed{\frac{P_1}{\rho}+\frac{V_1^2}{2}+\cancel{gz_1} = \frac{P_2}{\rho}+\frac{V_2^2}{2}+\cancel{gz_2}} \quad (1)$$

$z_1 = z_2$

Continuity $\Rightarrow V_1 A_1 = V_2 A_2$

or

$$V_1 = V_2\frac{A_2}{A_1} \Rightarrow \boxed{V_1 = \left(\frac{D_2}{D_1}\right)^2 V_2} \quad (2)$$

Combining (1) & (2)

$$\frac{P_1}{\rho} + \frac{\left(\frac{D_2}{D_1}\right)^4 V_2^2}{2} = \frac{P_2}{\rho} + \frac{V_2^2}{2}$$

$$\gamma = \rho g \Rightarrow \left(\rho = \frac{\gamma}{g}\right)$$

$$\frac{V_2^2}{2}\left(1 - \left(\frac{D_2}{D_1}\right)^4\right) = \frac{P_1 - P_2}{\rho}$$

$$V_2 = \sqrt{\frac{2\left(\frac{P_1-P_2}{\rho}\right)}{1-\left(\frac{D_2}{D_1}\right)^4}} = \sqrt{\frac{2g\left(\frac{P_1-P_2}{\gamma}\right)}{1-\left(\frac{D_2}{D_1}\right)^4}} = \sqrt{\frac{2\left(9.81\frac{m}{s^2}\right)\frac{(735-550)\,kPa}{9.1\,kN/m^3}}{1-\left(\frac{19mm}{31mm}\right)^4}} = 21.5\,m/s$$

$$\boxed{V_2 = 21.5\,m/s}$$

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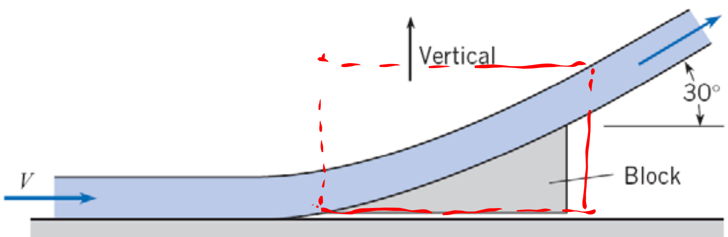
4) [**9 marks**] Water strikes a block as shown and is deflected 30°. The mass flow rate of the water is 1 kg/s, and the inlet velocity is V = 10 m/s. The mass of the block is 1 kg. The block will move if the friction between the block and the surface exceeds 1.48N.

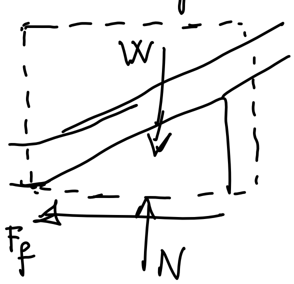
a) Determine normal and horizontal forces acting on the block (8 marks);

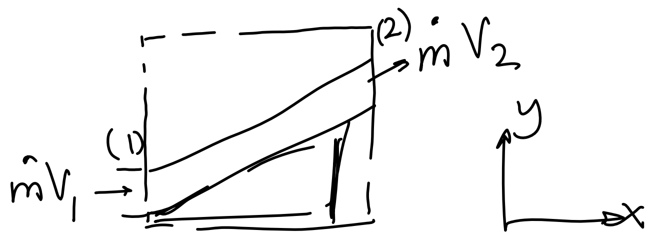
b) Determine whether the block will move (1 mark).

Neglect the weight of the water. Also, as the jet passes over the block neglect elevation changes. (Gravitational acceleration g = 10 m/s^2 and the density of water is ρ = 1000 kg/m^3).



Force Diagram





Bernoulli :

$$\cancel{\frac{P_1}{\rho}} + \frac{V_1^2}{2} + \cancel{g z_1} = \cancel{\frac{P_2}{\rho}} + \frac{V_2^2}{2} + \cancel{g z_2} \Rightarrow V_1 = V_2 = V = 10\ m/s$$

Momentum:

$$\Sigma \vec{F} = \dot{m}\vec{V}_2 - \dot{m}\vec{V}_1$$

$$(-W + N)\vec{j} - F_f\vec{i} = \dot{m}\left(V\cos 30\,\vec{i} + V\sin 30\,\vec{j}\right) - \dot{m}V\vec{i}$$

x-direction:

$$-F_f = \dot{m}V\cos 30 - \dot{m}V$$

$$= \dot{m}V(1 - \cos 30)$$

$$= 1 \times 10 \times (1 - \cos 30) \Rightarrow \boxed{F_f = 1.34N}$$

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EXTRA PAGE

y-direction:

$$-W + N = V \sin 30^\circ$$

$$N = W + V \sin 30$$

$$= mg + V \sin 30$$

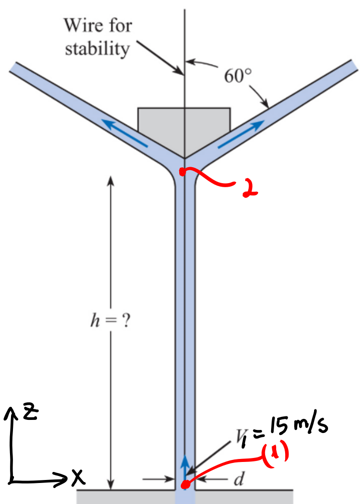
$$= 1 \times 10 + 1 \times 10 \times \sin 30$$

$$\boxed{N = 15N}$$

$F_f = 1.34N < 1.48N \Rightarrow$ block will not slip

5) [**12 marks**] A cone that is held stable by a wire is free to move in the vertical direction and has a jet of water striking it from below. The cone weighs 30 N. The initial speed of the jet as it comes from the circular orifice is 15 m/s, and the initial jet diameter is d = 2 cm. Find the height to which the cone will rise and remain stationary. (Gravitational acceleration: g = 10 m/s^2, water density ρ = 1000kg/m^3)

Notes: - The wire is only for stability and should not enter into your calculations.
- Assume that the height of the wedge is negligibly small.
- Notice that the water jet spreads around the cone three dimensionally (i.e., not only in the paper plane).



Apply Bernoulli between (1) & (2):

$$\frac{V_1^2}{2} + g z_1 + \cancel{\frac{P_1}{\rho}} = \frac{V_2^2}{2} + g z_2 + \cancel{\frac{P_2}{\rho}}$$

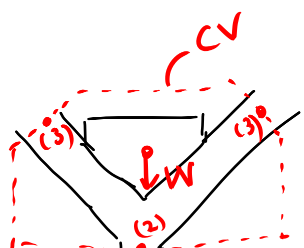
($P_1 = P_2 = 0$ atmospheric pressure)

$$V_2^2 = V_1^2 + 2g(\underbrace{z_1 - z_2}_{-h})$$

$$\boxed{V_2^2 = V_1^2 - 2gh} \quad \text{(Eqn. 1)}$$

Let's apply momentum eqn. in y-direction.

Select a stationary control volume surrounding the cone.



Force diagram

$$\sum \vec{F}_{CV} = \dot m_3 \vec{V}_3 - \dot m_2 \vec{V}_2$$

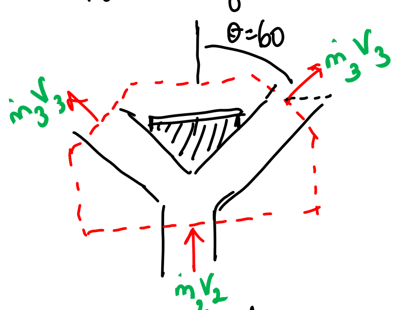
From continuity

$$\boxed{\dot m_2 = \dot m_3 = \dot m}$$

in y direction: △

$$-W = \dot m(V_3 \underbrace{\cos 60°} - V_2)$$

$\cos 60° = 0.5$



Momentum diagram

$$\dot m = \rho_2 V_2 A_2 = \rho_1 V_1 A_1$$

$$\boxed{-30 = \overbrace{(1000)}^{\rho}\overbrace{(15)}^{V_1}\overbrace{\pi\frac{(0.02)^2}{4}}^{A_1}\left(V_3(0.5) - V_2\right)} \quad \text{eqn. 2}$$

Apply Bernoulli between (2) & (3)

(with $z_2 \approx z_3$) $$\frac{V_2^2}{2} + \cancel{g z_2} + \cancel{\frac{P_2}{\rho}} = \frac{V_3^2}{2} + \cancel{g z_3} + \cancel{\frac{P_3}{\rho}} \rightarrow \boxed{V_2 = V_3} \quad \text{eqn. 3}$$

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Inserting eqn. 3 into eqn. 2, we get

$$-30 = (1000)(15)\,\pi\,\frac{(0.02)^2}{4}\left(V_2 \cdot (0.5) - V_2\right)$$

Solve for V_2:

$$V_2 = \frac{(30)\,(4)}{(1000)\,(15)\,\pi\,(0.02)^2\,(1-0.5)} = 12.73 \text{ m/s}$$

$$\boxed{V_2 = 12.73 \text{ m/s}}$$

From eqn. 1

$$V_2^2 = V_1^2 - 2gh \Rightarrow h = \frac{V_1^2 - V_2^2}{2g} = \frac{15^2 - (12.73)^2}{2 \times 10} = 3.15\text{m}.$$

$$\boxed{h = 3.15 \text{ m}}$$