The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 1 - Thursday February 8, 2018
Instructors: Profs. Micah Stickel and Piero Triverio

Last name: . . . .

First name: . . . .

Student number: . . . .

Tutorial section number: . . . .

Section	**Day**	**Time**	**Room**	**TA name**
TUT0101	Monday	14:00-15:00	BA 3012	Shashwat
TUT0102	Monday	14:00-15:00	RS 310	Gengyu (Paul)
TUT0103	Monday	14:00-15:00	BA 2159	Sameer
TUT0104	Monday	14:00-15:00	BA 3116	Fadime
TUT0105	Wednesday	13:00-14:00	BA 3012	Shashwat
TUT0106	Wednesday	13:00-14:00	WB 144	Gengyu (Paul)
TUT0107	Wednesday	13:00-14:00	BA 2159	Sameer
TUT0108	Wednesday	13:00-14:00	BA 3116	Fadime

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

Marks:

Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1

A line charge lies in free space along the z-axis from $z = -h$ to $z = 0$, and is charged with a linear charge density given by $\rho_l(z) = \rho_{l0} z$, where ρ_{l0} is a positive constant.

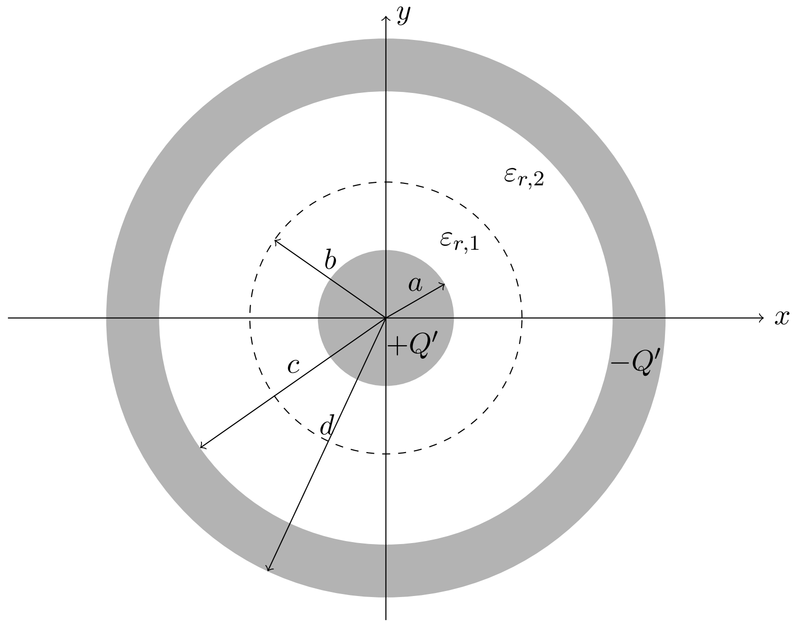
1. Find the electric scalar potential function at an arbitrary point in the xy-plane, i.e., find $V(r, \phi, 0)$.
   [9 points]

2. Is the electric scalar potential at $P_1(r = h, \phi = 0°, z = 0)$ positive or negative? Briefly explain the physical meaning of the value of $V(r = h, \phi = 0°, z = 0)$. [3 points]

3. For the line charge described above, find the r-component of the electric field intensity at an arbitrary point in the xy-plane. [4 points]

4. An electron is introduced to the system at $P_2(r = h, \phi = 0°, z = -h/2)$. Briefly describe *the direction* of the electric force this charge would experience at P_2. Briefly justify your answer. You do NOT need to determine the exact expression for the force at this point, and your answer can be in qualitative terms. [4 points]

Question 2



1. Consider the structure shown in the figure above, which is infinitely long along the z axis. The structure consists of:

   - an inner solid cylinder of radius a. This cylinder is made of a perfect electric conductor, and is positively charged. The charge per unit length is $+Q'$;
   - a first layer of a perfect dielectric with relative permittivity $\varepsilon_{r,1}$;
   - a first layer of a perfect dielectric with relative permittivity $\varepsilon_{r,2}$;
   - an outer hollow cylinder, with inner radius c and outer radius d. This cylinder is also made of a perfect electric conductor, and is negatively charged. The charge per unit length is $-Q'$;

   Use Gauss' law to find the electric field $\mathbf{E}_1(r)$ in the first dielectric layer ($r \in [a, b]$) and the electric field $\mathbf{E}_2(r)$ in the second dielectric layer ($r \in [b, c]$). [10 points]

2. Find the voltage ΔV between the inner conductor and the outer conductor, i.e., find $\Delta V = V_{inner} - V_{outer}$. Express ΔV in terms of Q'. [4 points]

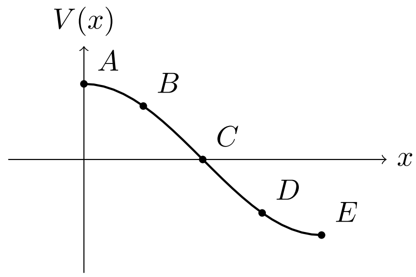
3. Now, assume that dimensions are: $a = 1\,\text{mm}$, $b = 2\,\text{mm}$, $c = 4\,\text{mm}$, $d = 6\,\text{mm}$. Dielectrics have the following characteristics:

   - first dielectric layer: $\varepsilon_{r,1} = 2$ and dielectric strength $E_{br,1} = 40\,\text{MV/m}$;
   - second dielectric layer: $\varepsilon_{r,2} = 6$ and dielectric strength $E_{br,2} = 20\,\text{MV/m}$.

   Remember that $\varepsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}$. Find the maximum charge per unit length Q' that can be placed on the conductors without causing any damage to the dielectrics. Express Q' in μC/m. [4 points]

4. Find the maximum voltage ΔV_{max} that the structure can withstand without damaging the dielectrics. Express ΔV_{max} in kV. [2 points]

Question 3.1



The electrostatic potential V in a region depends only on the x coordinate, and is given in the graph above. The electric field intensity $|\mathbf{E}|$ is:

(a) maximal at point A, minimal at point E;

(b) minimal at point A, maximal at point E;

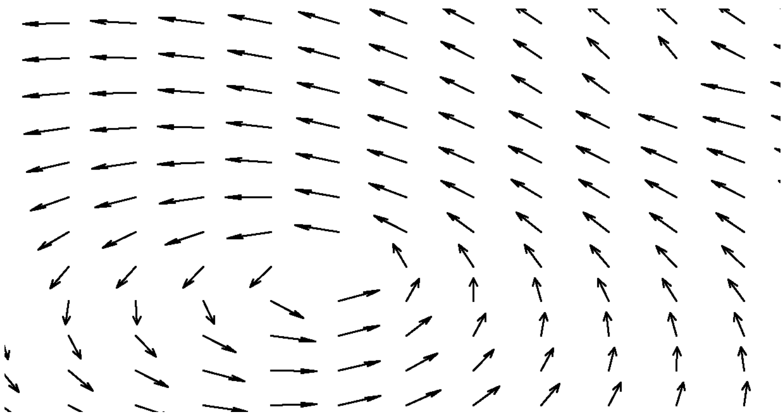
(c) minimal at points A and E, maximal at point C;

(d) maximal at points A and E, minimal at point C;

(e) none of the above.

Briefly justify your answer. [4 points]

Question 3.2



Consider the vector field **F** depicted in the figure above. Can **F** be the electric field produced by a static distribution of charge in vacuum (ie, there is anything else apart from the charges)?

(a) yes;

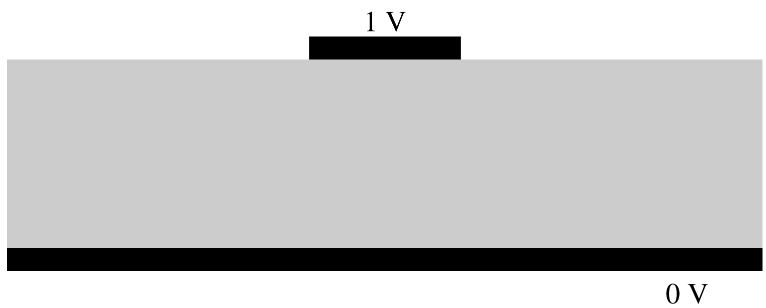
(b) no;

(c) more information is needed to answer this question.

Briefly justify your answer. [2 points]

Question 3.3

A very common element in printed circuit boards is the microstrip line, which consists of a dielectric substrate with a thin rectangular conductor on top and a wide conductive plane at the bottom which serves as ground plane. Both conductors are made of a highly conductive material. The cross section is shown in the figure below.



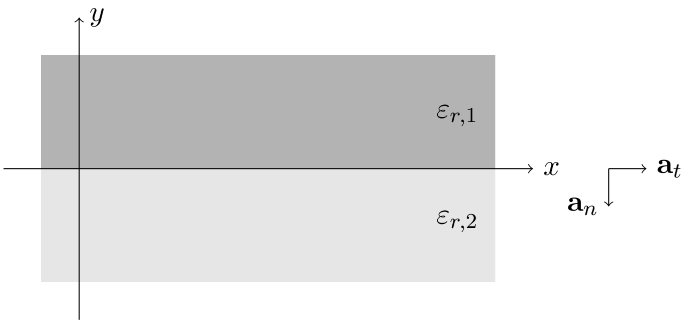
Assuming that the top conductor is at a potential of 1 V with respect to the ground plane, sketch in the figure:

- the electric field lines, indicating their direction;
- the equipotential lines.

Briefly justify your answer. [4 points]

Question 3.4

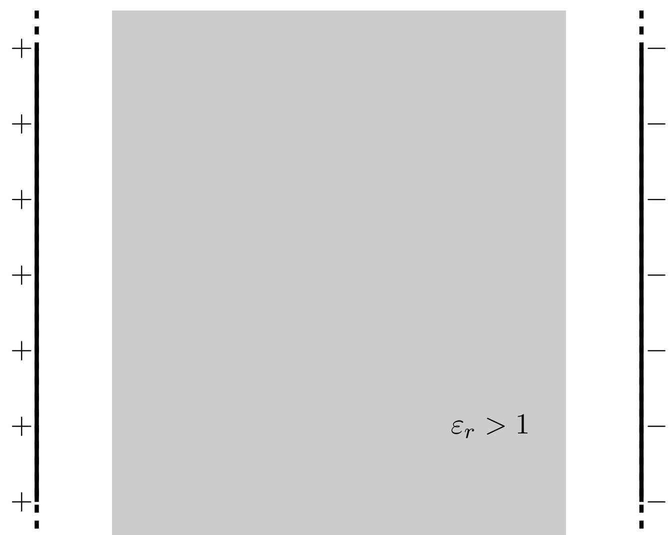
The plane $y = 0$ is the interface between two perfect dielectrics with $\varepsilon_{r,1} = 3$ and $\varepsilon_{r,2} = 6$, as shown in the figure below. Given that the field right above the interface ($y = 0^+$) is $\mathbf{E}_1 = 2\mathbf{a}_x + 4\mathbf{a}_y$ V/m, complete the table below, and briefly explain your results. [6 points]



<table>
<tr><th></th><th>Tangential component
(measured along $+\mathbf{a}_t$)</th><th>Normal component
(measured along $+\mathbf{a}_n$)</th><th>Unit</th></tr>
<tr><td>Electric field $\mathbf{E}_1$ in medium 1</td><td></td><td></td><td rowspan="2">V/m</td></tr>
<tr><td>Electric field $\mathbf{E}_2$ in medium 2</td><td></td><td></td></tr>
<tr><td>Electric flux density $\mathbf{D}_1$ in medium 1</td><td></td><td></td><td rowspan="2"></td></tr>
<tr><td>Electric flux density $\mathbf{D}_2$ in medium 2</td><td></td><td></td></tr>
<tr><td>Net bound surface charge density at the interface</td><td colspan="2"></td><td></td></tr>
</table>

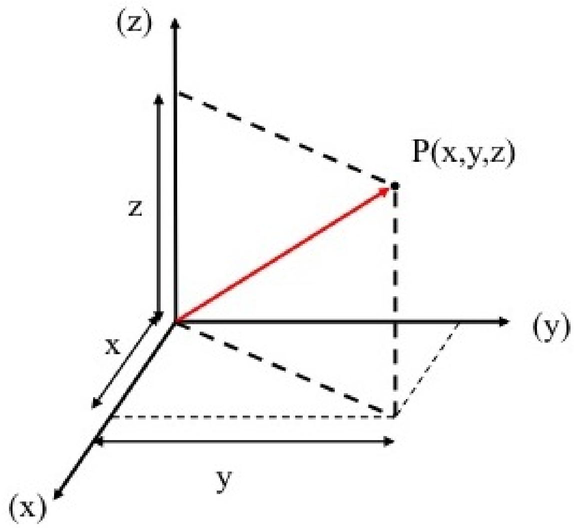
Question 3.5

We have two infinitely-wide charged planes, one positively charged and the other negatively charged, as shown in the figure. The gap between the planes is partially filled with a dielectric with relative permittivity $\varepsilon_r > 1$. Draw the densities of polarization charge ρ_p and $\rho_{p,s}$ that exist in the dielectric, using a "+" sign to indicate a positive charge density, a "-" sign to indicate a negative charge density, and a "0" to indicate a vanishing charge density. Justify your answer. [4 points]



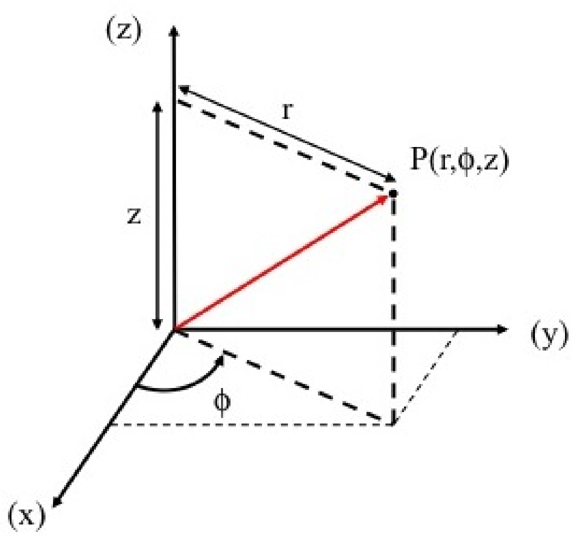
1 Coordinate Systems

1.1 Cartesian coordinates



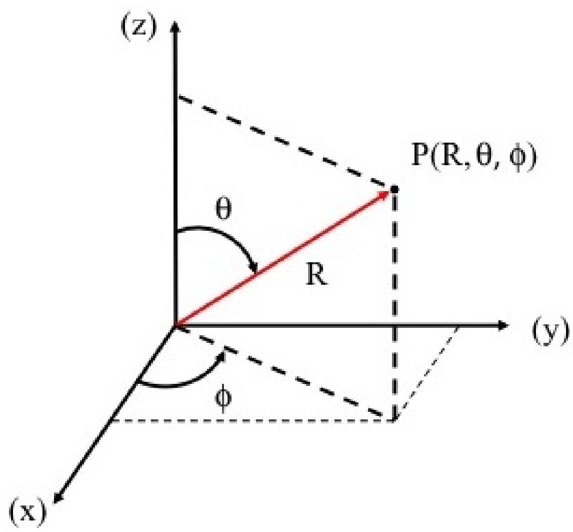
Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates



Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates



Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2 Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3 Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4 Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r \cos\phi - A_\phi \sin\phi$	$A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$
A_y	A_y	$A_r \sin\phi + A_\phi \cos\phi$	$A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi - A_\phi \cos\phi$
A_z	A_z	A_z	$A_R \cos\theta - A_\theta \sin\theta$
A_r	$A_x \cos\phi + A_y \sin\phi$	A_r	$A_R \sin\theta + A_\theta \cos\theta$
A_ϕ	-$A_x \sin\phi + A_y \cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R \cos\theta - A_\theta \sin\theta$
A_R	$A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$	$A_r \sin\theta + A_z \cos\theta$	A_R
A_θ	$A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$	$A_r \cos\theta - A_z \sin\theta$	A_θ
A_ϕ	$-A_x \sin\phi + A_y \cos\phi$	A_ϕ	A_ϕ

5 Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

5.4 Curl

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

6 Electromagnetic formulas

Table 1 Electrostatics

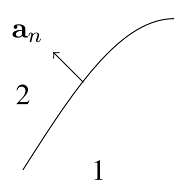
$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$	
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$	$\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$
$\rho_{p,v} = -\nabla \cdot \mathbf{P}$ $\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$ $\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$ $E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2}QV$
$W_e = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D} \cdot \mathbf{E} dv$	
$\nabla \cdot (\varepsilon\nabla V) = -\rho_v$	$\nabla \cdot (\varepsilon\nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$