Name: ______________________________ Student No.: ____________________

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, April 25, 2019
DURATION: 2.5 hours

Second Year – Engineering Science

ECE286H1 S – PROBABILITY AND APPLICATIONS

Calculabor Type: 2

Exam Type: C

Examiner – Deepa Kundur

- Answer **all** questions in the space provided in this question booklet. You may use the backs of pages if needed, but please specify the question you are answering clearly. There are **8 pages in total** to this exam.
- Allowable aids: (1) a 8.5”x11” double-sided *handwritten or typed* aid sheet and (2) Type 2 (nonprogrammable) calculator. Please note that important tables are given at the end of this exam.
- If a particular question seems unclear, please explicitly state any reasonable assumptions and proceed with the problem.
- Please properly label all points of interest on sketches and graphs that you are requested to draw, so that there is no ambiguity.
- **For full marks, show all steps and present results clearly**.

Question	1	2	3	4	5	Total
Value of Ques.	10	10	10	10	10	**50**
Earned						

1. **(10 marks)**

(a) **(5 marks)** Impurities in a batch of final product of a manufacturing process often reflect a serious problem. From considerable manufacturing data gathered, it is known that the proportion Y of impurities in a batch has a density function given by:

$$f(y) = \begin{cases} c(1-y)^9 & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}.$$

i. Find the constant c in the above density function.

ii. A batch is considered not sellable and then not acceptable if the percentage of impurities exceeds 60%. With the current quality of the process, what is the percentage of batches that are not acceptable?

(b) **(5 marks)** The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport is a random variable $Y = 3X - 2$, where X has the density function:

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & x > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Find the mean and variance of the random variable Y.

Solution:

(a) See solution manual for Question 3.73.

(b) See solution manual for Question 4.43.

2. **(10 marks)**

   (a) **(5 marks)** You are a participant in a game show. Denote A, B and C to be the events that a grand prize is behind doors A, B and C respectively. Suppose you picked door A. The game show host opened door B to and showed no prize was behind it. Now the host offers you the option of either staying at the door that you picked (A) or switching to the remaining door (C). Use probability to explain whether you should switch or not. Assume that the host knows where the prize is, and as such will never at first open the door that you pick or the door with the prize behind it.

   (b) **(5 marks)** Suppose you need to toss a fair coin to determine the serve order in a badminton game, but you only have two *biased* coins. The first coin has a probability of heads $p_1 = 1/3$ and the other coin has probability of heads $p_2 = 1/4$. Further suppose you **cannot** distinguish the two coins. How could you generate the equivalent of a fair coin flip? Please assume that the two coins are independent.

Solution:

(a) Initially, it is clear that $P(A) = P(B) = P(C) = \frac{1}{3}$.

Let H_*: The event that the host opens door *.

You initially pick door A, and the host opens door B. Given that the host has opened door B for you, and there is no prize behind it, what is the probability of the prize being behind A? We want $P(A|H_B)$. For this, we use Bayes Rule.

$$
\begin{aligned}
P(A|H_B) &= \frac{P(A \cap H_B)}{P(H_B)} \\
&= \frac{P(H_B|A)P(A)}{P(H_B|A)P(A) + P(H_B|B)P(B) + P(H_B|C)P(C)}
\end{aligned}
$$

Let's proceed with determining what we need:

- If the prize is behind door A, then the host can choose to open either door B or door C. Therefore, the probability of the host opening door B given that the prize is behind A is
$$P(H_B|A) = \frac{1}{2}$$
- If the prize is behind door B, then the host would have not opened door B.
$$P(H_B|B) = 0$$
- If the prize is behind door C, then the host has no choice but to open door B.
$$P(H_B|C) = 1$$

.

Now we have everything we need.

$$
\begin{aligned}
P(A|H_B) &= \frac{P(H_B|A)\frac{1}{3}}{P(H_B|A)\frac{1}{3} + P(H_B|B)\frac{1}{3} + P(H_B|C)\frac{1}{3}} \\
&= \frac{P(H_B|A)}{P(H_B|A) + P(H_B|B) + P(H_B|C)} \\
&= \frac{\frac{1}{2}}{\frac{1}{2} + 0 + 1} = \frac{1}{3}. \\
P(C|H_B) &= \frac{P(H_B|C)P(C)}{P(H_B|A)P(A) + P(H_B|B)P(B) + P(H_B|C)P(C)} \\
&= \frac{P(H_B|C)\frac{1}{3}}{P(H_B|A)\frac{1}{3} + P(H_B|B)\frac{1}{3} + P(H_B|C)\frac{1}{3}} \\
&= \frac{P(H_B|C)}{P(H_B|A) + P(H_B|B) + P(H_B|C)} \\
&= \frac{1}{\frac{1}{2} + 0 + 1} = \frac{2}{3}.
\end{aligned}
$$

Since $P(A|V) < P(C|V)$, you should switch the door.

(b) Here we start by denoting the sample space resulting from tossing these two coins in order, i.e. all the potential outcomes.

$$\Omega = \{HH, HT, TH, TT\},$$

where the event $\{HT\}$ denotes that the first coin flipped to heads, while the second coin to tails.

Here we will assume the two coins are independent, and compute the probability of all 4 outcomes

$$\mathbb{P}(HH) = \frac{1}{12}, \mathbb{P}(HT) = \frac{1}{4}, \mathbb{P}(TH) = \frac{1}{6}, \mathbb{P}(TT) = \frac{1}{2}.$$

At this point, we observe since the event $\{TT\}$ has probability exactly $1/2$, we can choose this event to be the equivalent event to a coin flipped to tails, with the complement as heads. Note with this choice, we do not need to distinguish the two coins at all.

3. **(10 marks)**

(a) **(5 marks)** Find the probability of observing 3 red cards in 5 draws from an ordinary deck of 52 playing cards:

i. without replacement of cards after each draw.

ii. with replacement of cards after each draw.

(b) **(5 marks)** A pair of dice is rolled 180 times. What is the probability that a total of 7 occurs:

i. at least 25 times?

ii. between 33 and 41 times inclusive?

Hint: Consider approximating the actual distribution.

Solution:

(a) i. The answer is given by a hypergeometric distribution:

$$\frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}} = 0.3251$$

ii. With replacement, we have a binomial distribution:

$$\binom{5}{3}(0.5)^2(0.5)^3 - 0.3125$$

(b) See solution manual for Question 6.34.

4. **(10 marks)**

(a) **(5 marks)** A random sample size 25 is taken from a normal population having a mean of 80 and a standard deviation of 5. A second random sample of size 36 is taken from a different normal population having a mean of 75 and a standard deviation of 3. Find the probability that the sample mean computed from the 25 measurements will exceed the sample mean computed from the 36 measurements by at least 3.4, but less than 5.9. Assume the difference of the means to be measured to the nearest tenth.

(b) **(5 marks)** A chemical engineer claims that the population mean yield of a certain batch process is 500 grams per millilitre of raw material. To check this claim he samples 25 batches each month. If the computed t-value falls between $-t_{0.05}$ and $t_{0.05}$, he is satisfied with his claim.

i. What conclusion should he draw from a sample that has mean $\overline{X} = 518$ grams per millilitre and a sample standard deviation $s = 40$ grams. Assume the distribution of yields to be approximately normal.

ii. What is a satisfiable claim for the population mean yield of the batch process that the engineer can make?

Solution:

(a) See solution manual for Question 8.28.

(b) See Example 8.11 of textbook.

5. **(10 marks)**

(a) **(6 marks)** A random sample of 100 automobile owners in Toronto shows that an automobile is driven on average $23,500$ kilometres per year with a standard deviation of 3900 kilometres. Assume the distribution of measurements to be approximately normal.

i. Construct a 99% confidence interval for the average number of kilometres an automobile is driven annually in Toronto.

ii. What can we assert with 99% confidence about the possible size of our error if we estimate the average number of kilometres driven by car owners in Toronto to be $23,500$ kilometres per year?

(b) **(4 marks)** An efficiency expert wishes to determine the average time that it takes to drill three holes in a certain concrete structure. How large a sample will she need to be 95% confident that her sample mean will be within 15 seconds of the true mean? Assume that it is known from previous studies that $\sigma = 40$ seconds.

Solution:

(a) See solution manual for Question 9.5. Note: there was omission of a page and the table value was not provided on the exam, so it's fine for the students to use a variable instead of the numerical value to complete the question. They were told this.

(b) See solution manual for Question 9.8.