FIRST NAME: __________________ LAST NAME: __________________

ID#: __________________ TUTORIAL SECTION: __________________

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

Midterm Examination, 9:10am - 11:00am, March 5, 2020

ECE 286S — Probability and Statistics

Examination Type: D
Examiner: Raviraj Adve

Instructions

- You are allowed one *one-sided* 8.5×11 handwritten sheet of notes and a non-programmable calculator.
- Remember to enter your name on the first page. If there are loose pages, please put your name on every page that you turn in.
- Please make sure to enter the number of your **tutorial section**
- Use the space provided to enter your answers. You may use the back of pages for rough work.
- Show intermediate steps for partial credit. *Answers without justification will not be accepted.*
- This exam is nominally out of 60 marks.

MARKS

Page	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
Value	6	14	10	5	10	9	6	60
Mark								

1 Short answer questions.

(a) (**3 marks**) Let X be a random variable with PDF,

$$f(x) = \begin{cases} C\cos(x), & -\pi/2 \le x \le \pi/2; \\ 0, & \text{Otherwise.} \end{cases}$$

What is the value of C?

(b) (**3 marks**) Events $A_i, i = 1, \ldots, n$ form a partition on the sample space S. True or False: This implies that all these events are mutually independent.

6 marks

(c) (**4 marks**) Consider a random variable X such that $S_X = \{x_0, x_1, x_2, x_3\}$ with corresponding probabilities {1/2, 1/4, 1/8, 1/8} respectively, i.e., $f(x_0) = 1/2$, $f(x_1) = 1/4$ etc. Find $E(\log_2(f(X)))$.
Note the log is with respect to base 2. As usual, $f(x)$ denotes the PMF of X.

(d) (**8 marks** total, 2 marks each) Random variable X has mean μ_X and standard deviation σ_X while Y has mean μ_Y and standard deviation σ_Y. The two RVs are independent. State true or false. Justify your answer:

(i) If $Z = X + Y$, then $\mu_Z = \mu_X + \mu_Y$.

(ii) For the same Z, the variance, $\mathrm{var}(Z) = \sigma_X^2 + \sigma_Y^2$.

(iii) If $W = XY$, then $\mu_Z = \mu_X \mu_Y$.

(iv) For the same W, the variance, $\mathrm{var}(W) = \sigma_X^2 \sigma_Y^2$.

14 marks

2 In class I had promised to set this question. A company has a test for cancer with the following performance metrics: if the patient has cancer (event C), then the test is positive (event P) with probability 0.9. If the patient does not have cancer (event C', the complementary event to C), the event P happens with probability 0.01. The prior probability of C is $p = 10^{-4}$.

(a) **(4 marks)** This part we did in class. A patient is tested and the test is positive. What is the probability that the patient has cancer?

(b) **(6 marks)** The patient takes the **same** test again. The probability of getting the **same result** in both tests is 0.8. The patient gets a positive test in the first time s/he takes the test, but a negative result the second time. What is the probability the patient has cancer?

10 marks

(c) (**5 marks**) Forget part (b) of this question. Now the patient takes an **independent** test which has the following performance: if the patient has cancer (event C), then the test is positive (event P) with probability 0.7. If the patient does not have cancer (event C', the complementary event to C), the event P happens with probability 0.001. The patient gets a positive result in the first test and a negative result in the second test. What is the probability the patient has cancer?

5 marks

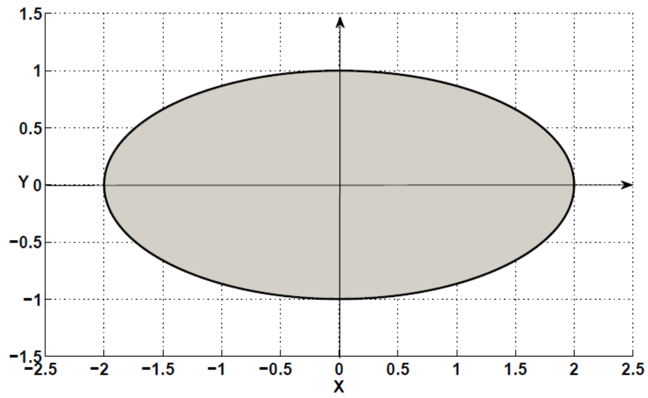
3 The other question I had promised in class: Orders arrive at a warehouse independently at a **average** rate of 10 orders per minute.

(a) (**3 marks**) We pretty much did this in class: in this part of the question, every worker in the warehouse processes exactly 100 orders an hour. How many workers do we need such that with probability 0.95 the worker process all orders received in every hour? Leave the answer in a format similar to how we did in class.

(b) (**7 marks**) Now, the number of orders a worker can process is a uniform random variable 97 and 103 (both not inclusive). So, if we denote this random variable as X, $S_X = \{98, 99, 100, 101, 102\}$. Repeat the question from part (a): How many workers do we need such that with probability 0.95 the worker process all orders received in every hour? *Hint: You remember that in the last class I said something about the total probability law?*

10 marks

4 The random variables X and Y are uniformly distributed within the ellipse as shown below. Note



that an ellipse is defined by the equation $x^2/a^2 + y^2/b^2 = 1$. The area of an ellipse is πab. Here $a = 2, b = 1$.

(a) (**5 marks**) What is the joint pdf, $f(x, y)$ of X and Y. Make sure that your answer covers the entire 2D plane.

(b) (**4 marks**) What is $E(X|Y = y)$, conditional expectation of X given $Y = y$? If you can find a way to answer this without integration, do so. Make sure you justify your answer.

9 marks

(c) (**6 marks**) Find $g(y|X = x)$, the conditional pdf of Y given $X = x$.

6 marks