Family Name, Given Name (Please print)	Student Number	Tutorial Leader's Name

PHY293 – Oscillations – Midterm (Solutions)

Thursday October 13, 2011

Duration - 50 minutes

PLEASE read carefully the following instructions.

Aids allowed: A non-programmable calculator without text storage.

Before starting, please **print** your name, tutorial group, and student number **at the top of this page and at the top of the answer sheet**.

There are three questions on this midterm test. Each question is worth one-third of the total grade.

Partial credit will be given for partially correct answers, so show any intermediate calculations that you do and write down, **in a clear fashion**, any relevant assumptions you are making along the way.

POSSIBLY USEFUL EQUATIONS:

	Amplitude	Velocity	Power
Peak Frequency	$\omega = \omega' = \omega_0\sqrt{1 - 1/(2Q^2)}$	$\omega = \omega_0$	$\omega = \omega_0$
Peak Value	$a_{\max} = \frac{a_0 Q}{\sqrt{1-\frac{1}{4Q^2}}}$	$v_{\max} = a_0\omega_0 Q$	$P_{\max} = \frac{1}{2}ma_0^2\omega_0^3 Q$
General	$a(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}}$	$v(\omega) = \frac{a_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}}$	$< P(\omega) >= P_{\max}\frac{\gamma^2}{(\omega_0^2-\omega^2)^2/\omega^2+\gamma^2}$
	$\tan\delta = \frac{\omega\gamma}{(\omega_0^2-\omega^2)}$		$< P >= P_{\max}\frac{\gamma^2/4}{(\omega_0-\omega)^2+\gamma^2/4}$ $\quad Q \gg 1$

Do no separate the two stapled sheets of the question paper. Hand in the question sheets with your exam booklet at the end of the test.

Good luck!

1. Explain succinctly (ie. in three sentences or less) the meaning *and* significance of each of the following, in the context of harmonic oscillations we've discussed in this class. Your answer should make clear not only what the term, or concept, *is*, but also put it in the context of this course and give an example of how it affects any every-day situation.

   The whole question was worth [10] points. [2.5] points for each of the definitions/descriptions requested. Most people got 1.5 points quite easily by mentioning two (or more) definitional properties of the concept in the question). to get the final point you needed some kind of every-day example which was easy for the first two, a little harder for the 3rd one and very hard for the last one.

   (a) Natural frequency;

   The natural frequency of a system is the frequency as which it will oscillate if there is no resistive damping or driving force. When damping and driving are considered the system will move at maximum velocity (though not at maximum amplitude) at this frequency. The natural frequency of the system is a property of the system itself and not of any external forces, to change it general requires changing one of the intrinsic properties of the oscillator (eg. length of pendulum string, mass of oscillator, density of water in U-tube, etc.). This is done in grandfather clocks, where the pendulum is calibrated to swing once per second or some fixed number of times per minute.

   (b) Critically damped oscillator;

   When $\gamma \geq 2\omega_o$ then an oscillator actually ceases oscillating and the motion just becomes a damped exponential that returns the oscillator to its equilibrium position. When the equality: $\gamma = 2\omega_0$; holds the oscillator returns to its equilibrium position fastest, while over-damped oscillators actually take longer to stabilise back at their equilibrium position. Thus if one is designing a shock-absorber or some other object to have minimal oscillations, the damping is chosen to produce critically damped oscillator (ie. one that has $\gamma = 2\omega_0$).

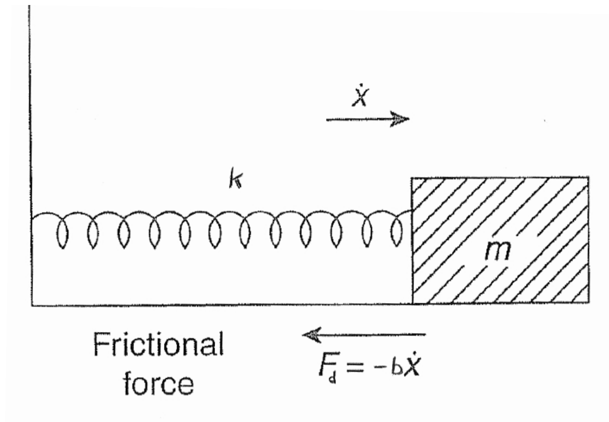
   (c) Power transfer in an LRC circuit;

   An LRC circuit obeys the same differential equation as a damped, forced, harmonic oscillator. All of the properties we learned about power transfer in a forced/damped oscillating system apply to such a circuit. The waveform generator that drives the circuit is the only source of power going into the circuit and the resistor (the electrical analog of the damping) removes power from the circuit. The Q of an LRC circuit is given by $\omega_0/\gamma = \frac{1}{R}\sqrt{L/C}$ (or more simply is proportional to $1/R$). The higher the Q, or the lower the R the sharper the power resonance and the more efficiently the circuit will transfer power from the inpu to the output – at the very specific, resonant, frequency. High Q resonating LRC circuits can be tuned to amplify the voltage of signals near the resonant frequency. This is the basis for analog radio signal reception where a very small electrical signal (the radio signal itself) is amplified in the receiver to produce audible sound.

   (d) The superposition of normal modes in a system of coupled oscillators.

   The solution of a coupled oscillator problem, involving two or more discrete objects oscillating in a collective way gives one natural frequency for each degree of freedom (number of objects in the system). These eigenvalues of the characteristic equation each have a corresponding eigenmode that describes the overall state of the objects at that frequency. If the system is excited in a particular eigenmode (say the two coupled pendula swinging together) it will never leave that mode, or exhibit any of the characteristics of any of the other modes as the eigenmodes are orthogonal to one another. To describe an arbitrarily complicated motion one constructs the linear superposition of all eigenmodes and determines the amplitude for each mode. This works because the equations of motion are, themselves, linear. To compute the amount of energy in a system of coupled oscillators one need only compute the energy in each mode as this remains constant for all times. For example in a hydro-electric generator the motion of the turbine is coupled (through magnets) to the electric generator and operates at 60 Hz generating AC power for the grid. Other modes (at other frequencies) are present but damped as much as possible to avoid wasting energy at frequencies that don't result in the generation of electricity that can be used by the grid.

2. A mass of 0.03 kg rests on a horizontal table and is attached to one end of a spring that has a spring constant of $k = 12$ N/m. The other end of the spring is subjected to a harmonic driving force $F = F_0 \cos(\omega t)$, where $F_0 = 0.15$ N. The mass also feels a damping force $F_\mathrm{d} = -b\dot{x}$, where $b = 0.06$ kg/s, from friction through contact with the table.



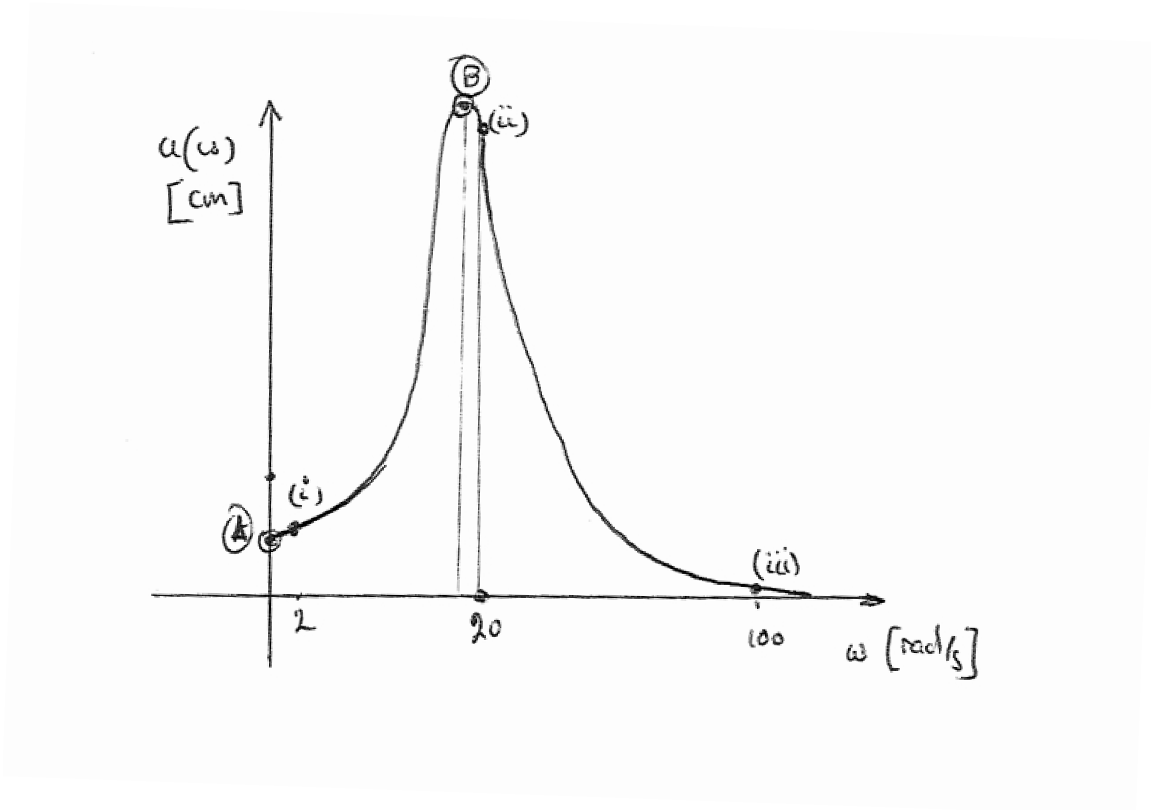
(a) Determine the amplitude of oscillation and phase angle between the driving force and the displacement of the mass for steady-state oscillations at frequencies of i) 2 rad/s, ii) 20 rad/s and iii) 100 rad/s.

- From the values given in the problem we can compute $a_0 = F_0/k$ which is 1.25 cm. To use the formulae on the front of the exam sheet we also need $\omega_0^2 = k/m$ which gives $\omega_0 = 20$ rad/s and $\gamma = b/m$ which gives $\gamma = 2$ rad/s. ([1.5]).
- We can then just plug these values in to the 3rd equation in the first row on the front of the paper to get the amplitudes: 1.26 cm, 12.50 cm and 0.052 cm. ([1.5])
- Similarly we can use the 4th equation in the first row to compute the phases as: 0.6°, 90° and 178 ° (also 0.01, $\pi/2$, 3.12 radians). ([2.5])

(b) What is the Q of this oscillator? The Q is just given by $\omega_0/\gamma = 10$. ([0.5])

(c) Sketch the amplitude vs. frequency curve for this system. I'm not looking for a super accurate drawing, but you should clearly indicate (by labelling them with (ω, $a(\omega)$) points on your sketch) where the three points from part (a) lie along with at least one other point on the curve.

Of course the three frequencies chosen for your to compute were not completely random. They correspond to the stiffness limit (low frequency), near the resonance (where the phase is almost exactly 90 degrees) and the inertia limit (high frequency)[1]. You could have chosen any other point to add to your plot, but two of the most common/simple ones to compute are the amplitude at $\omega = 0$, this is just $a_0 = 1.25$ cm (at the point labelled "A" on the accompanying figure) or the amplitude at resonance (which is at 19.98 rad/s and corresponds to an amplitude of 12.51 cm) – the point labelled "B" on the accompanying figure (note that this point is just to the left of point ii) computed above).[1] While the former is not very different from "i)" and the latter is not very different from "ii)", stating one of their values was all you needed to get the first two points for this part. The remaining two points were for the shape of the curve (doesn't go to 0 at 0 frequency) axis labels, units etc. ([2])



3. Two simple pendulums, each of length 0.300 m and mass 0.950 kg, are coupled by attaching a light, horizontal spring of spring constant $k = 1.50$ N/m to the masses.

   (a) Determine the frequencies of the two normal modes of this system of oscillators.

   To get full marks you didn't need to re-derive the characteristic equation, but you needed some explanation of the physics that leads to the eigenfrequencies.

   - Equations of motion:

$$m_A \ddot{x_A} + (m_A g/l + k)x_A - kx_B = 0$$
$$m_B \ddot{x_B} + (M_B g/l + k)x_B - kx_A = 0$$

     - Can write these in a vector form using a **state vector**: $\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$
     - Using this notation we can re-write the equations of motion as:

$$\begin{bmatrix} m_A & 0 \\ 0 & m_B \end{bmatrix} \frac{d^2}{dt^2} \begin{bmatrix} x_A \\ x_B \end{bmatrix} + \begin{bmatrix} m_A g/l + k & -k \\ -k & m_B g/l + k \end{bmatrix} \begin{bmatrix} x_A \\ x_B \end{bmatrix} = 0$$

     - Assume harmonic response $[M]\ddot{\vec{x}} = -\omega^2[M]\vec{x}$
     - Makes the equations of motion $-\omega^2[M] + [K])\vec{x} = 0$

- To solve this need the determinant: $\det|-\omega^2[M]+[K]|=0$

$$\det\left|-\omega^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}+\begin{bmatrix} mg/l+k & -k \\ -k & mg/l+k \end{bmatrix}\right|=0$$
$$\det\begin{vmatrix} -\omega^2 m+mg/l+k & -k \\ -k & -\omega^2 m+mg/l+k \end{vmatrix}=0$$
$$(-\omega^2 m+mg/l+k)^2-k^2=0[2]$$

- Thus we have two equations for ω^2 (one for the $+$ root and the other for the $-$ root) giving:

$$-\omega_1^2 m+mg/l=0 \quad \text{or} \quad \omega_1^2=g/l \quad \text{for } + \text{ root}[1]$$
$$-\omega_2^2 m+mg/l+2k=0 \quad \text{or} \quad \omega_2^2=g/l+2k/m \quad \text{for } - \text{ root}[1]$$

- Using g =9.8m/s^2 I get $\omega_1=5.71$ rad/s and $\omega_2=5.99$ rad/s)([2])
- (But we also accepted: $\omega_1=5.77$ rad/s and $\omega_2=6.04$ rad/s if you used g=10 m/s^2)

One of the pendulums is held a small distance away from its equilibrium position while the other pendulum is held at its equilibrium position. The two pendulums are then released simultaneously. As we saw in class the motion of the initially displaced pendulum eventually dies away and comes to rest (at least momentarily).

(b) Determine how long after the pendulums are released the initially displaced pendulum first comes to rest.

- The complete solution (with one mass fully displaced) and the other at it's equilibrium point is given by:
$$\vec{X}=A\Big[(1,1)\cos(\omega_1 t)+(1,-1)\cos(\omega_2 t)\Big] \qquad [1]$$
- Where $2A$ is determined by the 'small distance' the first mass is displaced. We are not given this in the problem, and don't need it for the solution.
- We are interested in when the first mass comes to rest so we can just look at the top component of the state vector:

$$\begin{aligned} x_1 &= A\Big(\cos(\omega_1 t)+\cos(\omega_2 t)\Big) \\ &= 2A\Big(\cos(\frac{[\omega_1+\omega_2]}{2}t)\cos(\frac{[\omega_2-\omega_1]}{2}t)\Big) \qquad [1] \end{aligned}$$

- Where we have used the trig identity discussed in class to turn the sum of two $\cos$ into a product of two $\cos$.
- But the first mass only comes to rest when the lower frequency $\cos$ comes back to 0. [1]
- This happens when $\frac{(\omega_1-\omega_2)}{2}t_0=\pi/2$ or for $t_0=\pi/(\omega_2-\omega_1)$.
- Plugging in constants I find: $t_0=11.2$ s. (also accepted $t_0=11.6$ s is you used g=10 m/s^2). [1]