CHE 260: THERMODYNAMICS AND HEAT TRANSFER

QUIZ FOR HEAT TRANSFER

23rd NOVEMBER 2015

Q.1. [17 points] HOT SPOT IN A HEATED FLUID

Consider a sphere of radius *a* whose surface is maintained at a uniform temperature T_0. The sphere is placed in a fluid medium that is at a constant temperature T_0 far away from the sphere. The thermal conductivity of the fluid is k_f. The fluid is stagnant everywhere. If convection effects and radiation can be ignored, conduction is the only mechanism for heat transfer from the sphere surface into the fluid. Now, present in the fluid is a heat source that releases heat according to the relationship, $\dot{S} = \dot{S}_0 \frac{a^4}{r^4}$, where $\dot{S}_0$ has units of W/m^3. As can be seen, the heat source term depends on the radial position. It is largest at the sphere surface, and decays away rapidly far away from the sphere according to a $1/r^4$ relationship.

Answer the following questions:

(a) **[10 points]** Beginning from the energy conservation equation in the spherical co-ordinate system [see last page], determine the *steady-state* temperature distribution *in the fluid*. Specify the governing equations and boundary conditions clearly. Note that the domain for the governing equation will be $a \le r < \infty$, so boundary conditions have to be applied at $r = a$ and for $r \to \infty$.

Solution:

$$\rho C_p \frac{\partial T}{\partial t} = k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2}\right] + \dot{S}$$

At steady state, for spherically symmetric temperature distribution, T = T(r).

$$k\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)+\dot{S}_0\frac{a^4}{r^4}=0.$$

The boundary conditions are

$$T\big|_{r=a}=T\big|_{r\to\infty}=T_0.$$

Rearranging the governing equation, we get

$$\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=-\frac{\dot{S}_0}{k}\frac{a^4}{r^2}.$$

Integrating the governing equation once, we have

$$r^2\frac{dT}{dr}=\frac{\dot{S}_0}{k}\frac{a^4}{r}+c_1.$$

Dividing by r^2 and integrating, we get

$$T=-\frac{\dot{S}_0}{2k}\frac{a^4}{r^2}-\frac{c_1}{r}+c_2.$$

Applying $T\big|_{r\to\infty}=T_0$, we get $c_2=T_0$.

$$T=-\frac{\dot{S}_0}{2k}\frac{a^4}{r^2}-\frac{c_1}{r}+T_0.$$

Applying $T\big|_{r=a}=T_0$, we get

$$T_0=-\frac{\dot{S}_0}{2k}a^2-\frac{c_1}{a}+T_0.$$

Thus, $c_1=-\frac{\dot{S}_0}{2k}a^3$.

The temperature distribution is, thus,

$$T=-\frac{\dot{S}_0}{2k}\frac{a^4}{r^2}+\frac{\dot{S}_0}{2k}\frac{a^3}{r}+T_0,$$

or

$$T=\frac{\dot{S}_0a^2}{2k}\left(-\frac{a^2}{r^2}+\frac{a}{r}\right)+T_0,$$

Arguments leading up to governing equation: 3 points

Boundary conditions: 2 points

Integration and reaching general form of temperature distribution: 2 points

Application of boundary conditions to get constants: 2 points

Final temperature distribution: 1 point

(b) **[3 points]** Determine the radial heat flux, $\dot{q}_r$, at the surface of the sphere, $r = a$. Is heat being lost to the ambient or being gained from the ambient?

Solution:

The radial heat flux at the surface of the sphere is

$$\dot{q}_r = -k\frac{\partial T}{\partial r} = -k\frac{\dot{S}_0 a^2}{2k}\left(+\frac{2a^2}{r^3} - \frac{a}{r^2}\right) = -\frac{\dot{S}_0 a}{2}\left(+\frac{2a^3}{r^3} - \frac{a^2}{r^2}\right),$$

which, on the sphere surface, is

$$\dot{q}_r\big|_{r=a} = -\frac{1}{2}\dot{S}_0 a.$$

Since the radial outward heat flux $\dot{q}_r\big|_{r=a}$ is negative, energy is gained by the sphere from the ambient.

Definition and expression for the surface flux: 2 points

Heat gained argument: 1 point

(c) **[4 points]** Determine if there can be a hot spot in the fluid, i.e., if there is a radial position where the temperature profile shows a maximum. Find the radial location of the hot spot. What is the temperature at the hot spot?

Solution:

At the location of the hot spot,

$$\frac{dT}{dr} = -k\frac{\dot{S}_0 a^2}{2k}\left(+\frac{2a^2}{r^3} - \frac{a}{r^2}\right) = 0$$

This occurs at $r = 2a$.

The temperature at the hot spot is

$$T = \frac{\dot{S}_0 a^2}{2k}\left(-\frac{a^2}{4a^2} + \frac{a}{2a}\right) + T_0 = \frac{\dot{S}_0 a^2}{8k} + T_0.$$

Determination of position of hot spot: 3 points

Temperature at hot spot: 1 point

2. [15 points] HEAT LOSSES FROM A STEAM PIPE

A 10 cm OD by 6 cm ID *cylindrical* steam line delivers superheated steam at 1000^{o}C. The line is steel wrapped with 10 cm thick cylindrical shell of asbestos, and 1 cm of cylindrical plaster shell over the asbestos. The thermal conductivities of steel, asbestos and plaster are 14 W/m-K, 0.156 W/m-K and 0.107 W/m-K, respectively. The convective heat transfer coefficients for the steam side and the air side are 2500 W/m^2 and 7 W/m^2. Using the thermal resistance approach, determine the rate of heat loss per unit length of the cylindrical steam line at steady state. Account for radiative losses from the surface as well, assuming a plaster emissivity of 0.85, and that the 'surrounding' surface receiving the radiation to be at a temperature of 30^{o}C. The ambient air temperature is also 30^{o}C. The Stefan Boltzmann constant is 5.67×10^{-8} W/m^{2}K^4.

Note: The Newton-Raphson iterative formula for finding the root x^* of a function $f(x)$, such that $f(x^*) = 0$, is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}.$$

Solution:

Assume a pipe length of $L = 1$ m.

The resistance formulation is

The various resistances are as follows:

$$R_{\mathrm{conv}_\mathrm{i}} = \frac{1}{h_i \pi D_i L} = \frac{1}{2500\times\pi\times 6\times 10^{-2}\times 1} = 2.1221\times 10^{-3}\ {}^{\mathrm{o}}\mathrm{C/W}.$$

$$R_{\mathrm{steel}} = \frac{1}{2\pi k_{\mathrm{steel}} L}\ln\left(\frac{r_o}{r_i}\right) = \frac{1}{2\pi\times 14\times 1}\ln\left(\frac{5\times 10^{-2}}{3\times 10^{-2}}\right) = 5.8072\times 10^{-3}\ {}^{\mathrm{o}}\mathrm{C/W}.$$

$$R_{\mathrm{asb}} = \frac{1}{2\pi k_{\mathrm{asb}} L}\ln\left(\frac{r_o}{r_i}\right) = \frac{1}{2\pi\times 0.156\times 1}\ln\left(\frac{15\times 10^{-2}}{5\times 10^{-2}}\right) = 1.1208\ {}^{\mathrm{o}}\mathrm{C/W}.$$

$$R_{\mathrm{plas}} = \frac{1}{2\pi k_{\mathrm{plas}} L}\ln\left(\frac{r_o}{r_i}\right) = \frac{1}{2\pi\times 0.107\times 1}\ln\left(\frac{16\times 10^{-2}}{15\times 10^{-2}}\right) = 6.5844\times 10^{-2}\ {}^{\mathrm{o}}\mathrm{C/W}.$$

The total resistance prior to the heat transfer off the surface of the plaster is

$$\begin{aligned} R_{\mathrm{eff}\,1} &= R_{\mathrm{conv}_\mathrm{i}} + R_{\mathrm{steel}} + R_{\mathrm{asb}} + R_{\mathrm{plas}} \\ &= 2.1221\times 10^{-3} + 5.8072\times 10^{-3} + 1.1208 + 6.5844\times 10^{-2} = 1.1946\ {}^{\mathrm{o}}\mathrm{C/W}. \end{aligned}$$

An energy balance at steady state yields

$$\dot{Q} = \frac{(T_i - T_S)}{R_{\mathrm{eff}\,1}} = h_o A_o (T_S - T_\infty) + \sigma\varepsilon A_o\left[(T_S + 273.16)^4 - (T_{surr} + 273.16)^4\right]$$

where all temperatures are in degrees Celcius.

Substituting the known values, we get

$$\begin{aligned} \dot{Q} &= \frac{(1000 - T_S)}{1.1946} \\ &= 7\left(2\pi\times 16\times 10^{-2}\times 1\right)(T_S - 30) + 5.67\times 10^{-8}\left(2\pi\times 16\times 10^{-2}\times 1\right)\left[(T_S + 273.16)^4 - (30 + 273.16)^4\right] \end{aligned}$$

Rearranging the equation and evaluating all the prefactors, we get

$$8.4066(T_S - 30) + 6.8093\times 10^{-8}\left[(T_S + 273.16)^4 - 303.16^4\right] - (1000 - T_S) \quad = 0$$

or

$$9.4066 T_S + 6.8093\times 10^{-8}(T_S + 273.16)^4 - 1827.36 \quad = 0\,.$$

The derivative of the function on the left is $9.4066 + 6.8093\times 10^{-8}\times 4(T_S + 273.16)^3$.

Applying the Newton Raphson method, we get Ts =80.7$^{\mathrm{o}}$C. The rate of heat transfer is

$$\dot{Q} = \frac{(1000 - T_S)}{1.1946} = \frac{(1000 - 80.7)}{1.1946} = 770 \text{ W}.$$

1.5 points each for Rconv$_i$, Rsteel, Rasb, Rplas (6 points in all)
1 point for effective resistance prior to surface.
3 points for the energy balance at steady state
Nonlinear equation in T_s : 2 points.
Solution of nonlinear equation in T_s : 2 points.
Rate of heat transfer : 1 points.

3. [18 points] COMPOSITE MATERIAL

Use the resistance network approach to solve this problem (see table on page 6).

In class, you have always consider isotropic materials with a constant thermal conductivity, irrespective of the direction in which temperature gradient is applied. However, this is not always true, particularly for composite materials. Consider a slab of a composite material made of alternating planar layers of two different materials A and B in the x-direction of thicknesses W_A and W_B, and thermal conductivities k_A and k_B, respectively (see figure to the left). The total width of the material in the x direction is W [with $W = n\ (W_A+W_B)$, where n is the number of pairs of the A-B layers], and the height in the y-direction is H. The depth into the plane of the paper is D.

Answer the following questions:

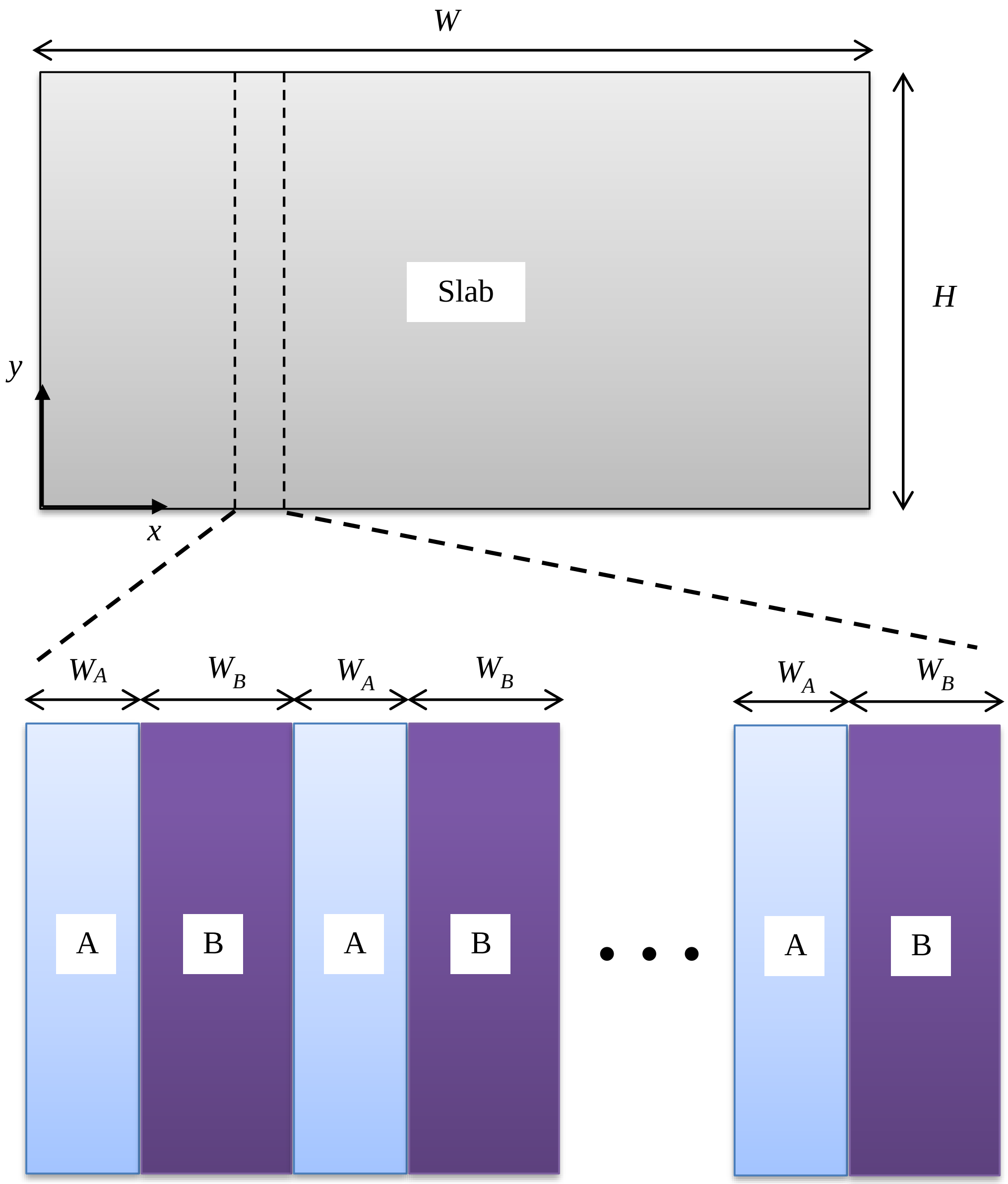
(a) [6 points] If a temperature difference is imposed across the material in the x direction, with the face at $x = 0$ maintained at a temperature T_1, and the face at $x = W$ maintained at a temperature T_2, determine the total rate of heat transfer by conduction in the x-direction through the slab. Assume that there is no temperature variation in the y direction.

Solution:

If a temperature difference is imposed in the x direction, there are n combinations of slabs of A and B in series. The cross-sectional area for heat flow is H*D. The effective thermal resistance is

$$R_{\text{eff}_x} = n\left(\frac{W_A}{k_A(HD)} + \frac{W_B}{k_B(HD)}\right) = \frac{n}{(HD)}\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right) = \frac{1}{(HD)}\frac{W}{(W_A + W_B)}\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right).$$

Depth of slab into the plane of the paper is *D*



The rate of heat transfer in the x direction is

$$\dot{Q}_x = \frac{T_1 - T_2}{R_{\text{eff}_x}} = \frac{T_1 - T_2}{\frac{1}{(HD)}\frac{W}{(W_A + W_B)}\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right)} = \frac{(W_A + W_B)}{\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right)}(HD)\frac{(T_1 - T_2)}{W}.$$

Identifying that resistances are in series: 2 points

Identifying thickness and cross-sectional area for flow: 2 point

Correct resistance: 1 point

Correct rate of heat transfer: 1 point

(b) [2 points] Equate the rate of heat transfer obtained in part (a) to a slab of effective thermal conductivity k_x in the x-direction. Hence determine k_x.

Solution:

The total rate of heat transfer through a slab of effective thermal conductivity k_x is

$$\dot{Q}_x = k_x (HD)\frac{(T_1 - T_2)}{W}$$

Equating this to the expression for $\dot{Q}_x$ from part (a), we get

$$k_x (HD)\frac{(T_1 - T_2)}{W} = \frac{HD(W_A + W_B)(T_1 - T_2)}{W\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right)}.$$

The effective thermal conductivity in the x direction is, therefore,

$$k_x = \frac{(W_A + W_B)}{\left(\frac{W_A}{k_A} + \frac{W_B}{k_B}\right)}.$$

Equating rate of heat transfer: 1 point

Correct k_x: 1 point

(c) [6 points] If a temperature difference is imposed across the material in the y direction, with the face at y = 0 maintained at a temperature T_1, and the face at x = H maintained at a temperature T_2, determine the total rate of heat transfer by conduction in the y-direction through the slab.   Assume that there is no temperature variation in the x direction.

Solution:

If a temperature difference is imposed in the *y* direction, there are *n* thermal resistors each of A and B in parallel. The cross-sectional area for heat flow is $W_A D$, and that for B is $W_B D$. The thickness for either resistor is H. The cross-sectional area for heat flow is H*D. The effective thermal resistance is, therefore,

$$\frac{1}{R_{\mathrm{eff}_y}} = n\left(\frac{1}{\dfrac{H}{k_A\left(W_A D\right)}} + \frac{1}{\dfrac{H}{k_B\left(W_B D\right)}}\right) = \frac{nD}{H}\left(k_A W_A + k_B W_B\right) = \frac{D}{H}\frac{W}{\left(W_A + W_B\right)}\left(k_A W_A + k_B W_B\right).$$

$$\dot{Q}_y = \frac{T_1 - T_2}{R_{\mathrm{eff}_y}} = \left(T_1 - T_2\right)\frac{D}{H}\frac{W}{\left(W_A + W_B\right)}\left(k_A W_A + k_B W_B\right) = \frac{\left(k_A W_A + k_B W_B\right)}{\left(W_A + W_B\right)}\left(WD\right)\frac{\left(T_1 - T_2\right)}{H}.$$

<u>Identifying that resistances are in parallels: 2 points</u>

<u>Identifying thickness and cross-sectional area for flow: 2 points</u>

<u>Correct resistance: 1 point</u>

<u>Correct rate of heat transfer: 1 point</u>

(d) [2 points] Equate the rate of heat transfer obtained in part (c) to a slab of effective thermal conductivity k_y in the *y*-direction. Hence determine k_y.

$$\dot{Q}_y = k_y\left(WD\right)\frac{\left(T_1 - T_2\right)}{H}.$$

Comparing the above equation with the result in part (c), we get

$$k_y = \frac{\left(k_A W_A + k_B W_B\right)}{\left(W_A + W_B\right)}.$$

<u>Equating rate of heat transfer: 1 point</u>

<u>Correct k_y: 1 point</u>

(e) [2 points] If W_A=W_B, which of k_x or k_y is greater?

Solution:

$$\frac{k_x}{k_y}=\frac{\left(W_A+W_B\right)^2}{\left(\frac{W_A}{k_A}+\frac{W_B}{k_B}\right)\left(k_AW_A+k_BW_B\right)}=\frac{k_Ak_B\left(W_A+W_B\right)^2}{\left(k_AW_A+k_BW_B\right)\left(k_AW_B+k_BW_A\right)}.$$

If $W_A=W_B=W_0$,

$$\frac{k_x}{k_y}=\frac{k_Ak_B\left(2W_0\right)^2}{W_0\left(k_A+k_B\right)W_0\left(k_A+k_B\right)}=\frac{4k_Ak_B}{\left(k_A+k_B\right)^2}=\frac{\left(2k_Ak_B\right)/\left(k_A+k_B\right)}{\left(k_A+k_B\right)/2}.$$

$\left(2k_Ak_B\right)/\left(k_A+k_B\right)$ is less than both k_A and k_B (unless $k_A = k_B$), while $\left(k_A+k_B\right)/2$ has a value in between k_A and k_B. The ratio k_x / k_y is always less than or equal to 1. Hence k_x is always less than k_y.

Derivation of k_x/k_y or k_x-k_y - 1 point

Using hint to deduce which is greater – 1 point

Bonus [3 points]: For isotropic materials, Fourier's first law of heat conduction is written as $\vec{q}=-k\vec{\nabla}T$, and therefore, the conductive heat flux is always parallel to the direction of the temperature gradient. How should Fourier's law of heat conduction be written for this composite slab? Will the direction of heat transfer always be parallel to the direction of the gradient?

Solution:

Fourier's law of heat conduction needs to be written as

$$\dot{\vec{q}}=-\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix}\cdot\vec{\nabla}T=-\mathbf{K}\cdot\vec{\nabla}T.$$

The individual components are

$$\dot{q}_x=-k_x\frac{\partial T}{\partial x},\quad \text{and}\quad \dot{q}_y=-k_y\frac{\partial T}{\partial y}.$$

When the thermal conductivities are the same, i.e., when $k_x = k_y = k$, or $\mathbf{K} = k\ \mathbf{I}$,

$$\vec{q} = -\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \cdot \vec{\nabla} T = -k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \vec{\nabla} T = -k\ \mathbf{I} \cdot\ \vec{\nabla} T = -k\ \vec{\nabla} T.$$

This recovers the form of Fourier's law of heat conduction for homogeneous materials.

The direction of heat transfer is not always parallel to the direction of the temperature gradient. For example, if $\vec{\nabla} T = \hat{i} + \hat{j}\ \ {}^{\text{o}}\text{C/m}$, then $\dot{q}_x = -k_x$, and $\dot{q}_y = -k_y$. Thus $\vec{q} = -k_x \hat{i} - k_y \hat{j}$, which is parallel to $\vec{\nabla} T = \hat{i} + \hat{j}$ only if $k_x = k_y$.

<u>3 points iff all arguments are correct.</u>