3.2

(i) In Newton's Law of Gravitation, $F = G\frac{m_1 m_2}{r^2} \Rightarrow$

$$\mathcal{D}[G] = \mathcal{D}\left[\frac{\text{force} \times (\text{length})^2}{(\text{mass})^2}\right] \quad M^{-1} L^3 T^{-2}$$

(ii) In $T_c = C R^x M^y G^z$, $\mathcal{D}[T_c] = T = L^x M^y (M^{-1} L^3 T^{-2})^z$

$\Rightarrow L: x + 3z = 0; \quad M: y - z = 0; \quad T: -2z = 1$

$\Rightarrow \quad z = -1/2; \; y = -1/2; \; x = 3/2 \Rightarrow T_c = \frac{C R^{3/2}}{M^{1/2} G^{1/2}}$

(iii) For Sun, with $C \simeq 1$,

$$T_c \simeq \frac{(6.96 \times 10^8)^{3/2}}{(2 \times 10^{30} \times 6.67 \times 10^{-11})^{1/2}} = 1589 \text{ secs} \approx \frac{1}{2} \text{ hr}$$

3.4

(i) $$F_D = \sum_{j=1}^{\infty} C_j \rho^{x_j} D^{y_j} U^{z_j} \mu^{\omega_j}$$

$$MLT^{-2} = (ML^{-3})^{x_j} L^{y_j} (LT^{-1})^{z_j} (ML^{-1}T^{-1})^{\omega_j}$$

$$\left.\begin{array}{llll} L: & -3x_j + y_j + z_j - \omega_j = 1 & \Rightarrow & -3x_j + y_j + z_j = 1 + \omega_j \\ M: & x_j + \omega_j = 1 & & x_j = 1 - \omega_j \\ T: & -z_j - \omega_j = -2 & & z_j = 2 - \omega_j \end{array}\right\} y_j = 2 - \omega_j$$

Thus $$F_D = \sum_{j=1}^{\infty} C_j \rho^{1-\omega_j} D^{2-\omega_j} U^{2-\omega_j} \mu^{\omega_j}$$

$$\Rightarrow \frac{F_D}{\rho D^2 U^2} = \sum_{j=1}^{\infty} C_j \left(\frac{\mu}{\rho D U}\right)^{\omega_j}$$ which implies $\frac{F_D}{\rho D^2 U^2} = f\left(\frac{\mu}{\rho D U}\right)$

(ii) Pi-Theorem $(F_D, \rho, D, U, \mu) \rightarrow$ 5 variables containing 2 dimensions $\rightarrow$ 2 dimensionless groups. Check

$$\mathcal{D}\left[\frac{F_D}{\rho D^2 U^2}\right] = \frac{MLT^{-2}}{(ML^{-3})L^2(LT^{-1})^2} = M^0L^0T^0 \quad \mathcal{D}\left[\frac{\mu}{\rho L U}\right] = M^0L^0T^0$$ also

(iii) If $F_D = f(\mu, D, U)$ only, using

$$F_D = \sum_{j=1}^{\infty} C_j \mu^{x_j} D^{y_j} U^{x_j}$$

gives

$$F_D = \sum_{j=1}^{\infty} C_j \mu D U \Rightarrow \frac{F_D}{\mu D U} = \sum_{j=1}^{N} C_j \equiv K.$$

(iv) If $$F_D = K \mu D U \Rightarrow \frac{F_D}{\rho D^2 U^2} = \frac{K(\mu D U)}{\rho D^2 U^2} = \frac{K\mu}{\rho D U} = \frac{K}{Re}$$

where $Re \equiv \frac{\rho D U}{\mu}$ is the Reynolds number

3.6

(i) With ρ, g, λ as reference quantities containing between them M, L and T, we have, by T10

$$\frac{c}{\rho^{a_1} g^{b_1} \lambda^{c_1}} = f\left(\frac{\gamma}{\rho^{a_2} g^{b_2} \lambda^{c_2}}\right)$$

	ρ	g	λ	c	γ
M	1	0	0	0	1
L	-3	1	1	1	0
T	0	-2	0	-1	-2
	1	0	0	0	1
	0	1	0	½	1
	0	0	1	½	2

$$= \frac{c}{\sqrt{g\lambda}} = f\left(\frac{\gamma}{\rho g \lambda^2}\right)$$

$$\Pi_c = f(\Pi_\gamma)$$

(ii) For sea water $\Pi_\gamma = \dfrac{7.3 \times 10^{-2}}{1025 \times 9.81\, \lambda^2} = 7.26 \times 10^{-6} \lambda^{-2}$

At $\lambda = 1\,m$, $\Pi_\gamma = 7.26 \times 10^{-6}$, suggesting surface tension effects are negligible, so that

$$\Pi_c = \frac{c}{\sqrt{g\lambda}} = \text{constant} \Rightarrow c \propto \sqrt{g\lambda}$$

At $\lambda = 5\,mm$, $\Pi_\gamma = 0.2904$, suggesting surface tension cannot be ignored

(iii) If surface tension effects can be ignored, $c \propto \sqrt{g\lambda}$ implies

$$\frac{c_1}{c_2} = \sqrt{\frac{g_1 \lambda_1}{g_2 \lambda_2}} \; ; \text{ if } g_1 = g_2 = g \quad \frac{c_{100}}{c_1} = \sqrt{\frac{100}{1}} = 10$$

$\Rightarrow \quad c_{100} = 12.5\ m/sec = 45\ km/hr$. [Experiments confirm this!

3.10

With $\Delta p = f(\rho, D, R, Q, \mu)$, there $N = 6$ variables involving three dimensions $\Rightarrow$ There are 3 independent Pi's. Choose, ρ, D, and Q as reference dimensions to obtain

$$\Pi_{\Delta p} = \frac{\Delta p}{\rho^{x_1} D^{y_1} Q^{z_1}} \; ; \quad \Pi_R = \frac{R}{\rho^{x_2} D^{y_2} Q^{z_2}} \; ; \quad \Pi_\mu = \frac{\mu}{\rho^{x_3} D^{y_3} Q^{z_3}}$$

with the x_i, y_i and z_i are chosen to make the Π's dimensionless. By inspection $\Pi_R = R/D$, but we proceed formally, arranging the linear equations as a dimensional matrix:

	$\rho(x_i)$	$D(y_i)$	$Q(z_i)$	Δp	R	μ
M	1	0	0	1	0	1
L	-3	1	3	-1	1	-1
T	0	0	-1	-2	0	-1

$\Rightarrow$

ρ	D	Q	Δp	R	μ
1	0	0	1	0	1
0	1	0	-4	1	-1
0	0	1	2	0	1

$$\Pi_{\Delta p} \Rightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \Rightarrow \Pi_{\Delta p} = \frac{\Delta p}{\rho D^{-4} Q^2} \; ;$$

$$\Pi_R \Rightarrow \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ (surprise!) } \Rightarrow \Pi_R = \frac{R}{D}$$

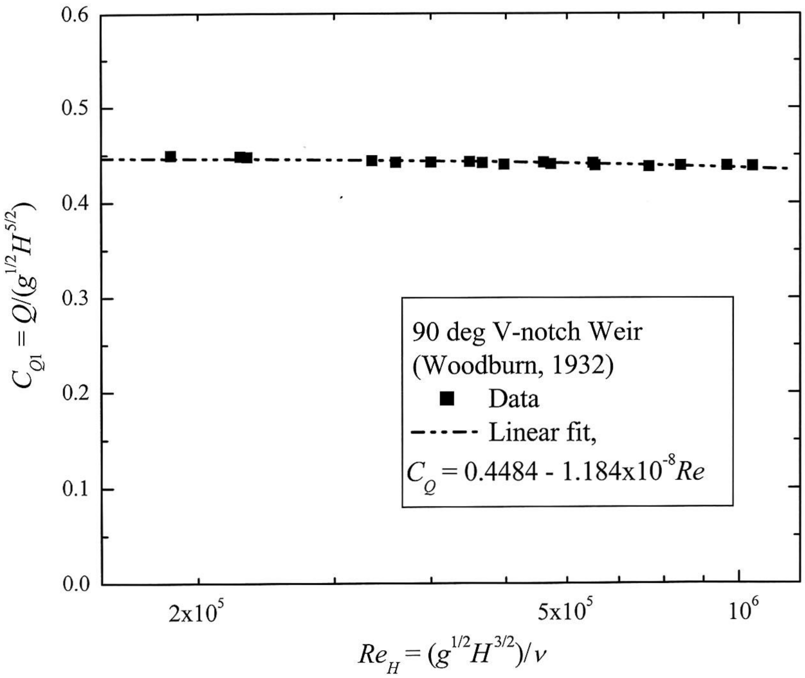
$$\Pi_\mu = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \Pi_\mu = \frac{\mu}{\rho D^{-1} Q}$$

$$\Rightarrow \frac{\Delta p D^4}{\rho Q^2} = f\left(\frac{R}{D}, \frac{\mu D}{\rho Q}\right)$$

CHAPTER 3: PROBLEM ANSWERS AND HINTS.

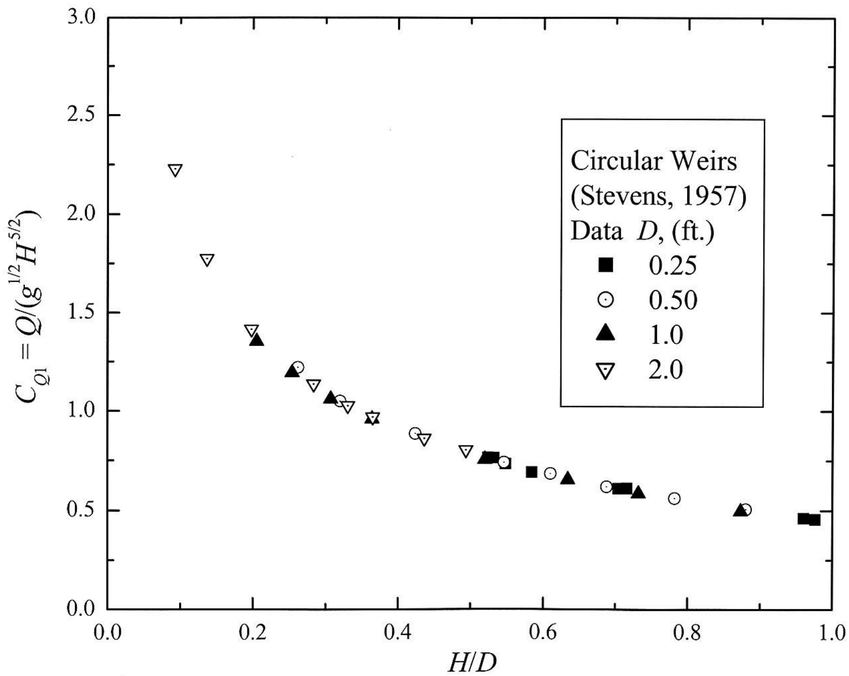
Problem 3.12: Dimensional analysis gives

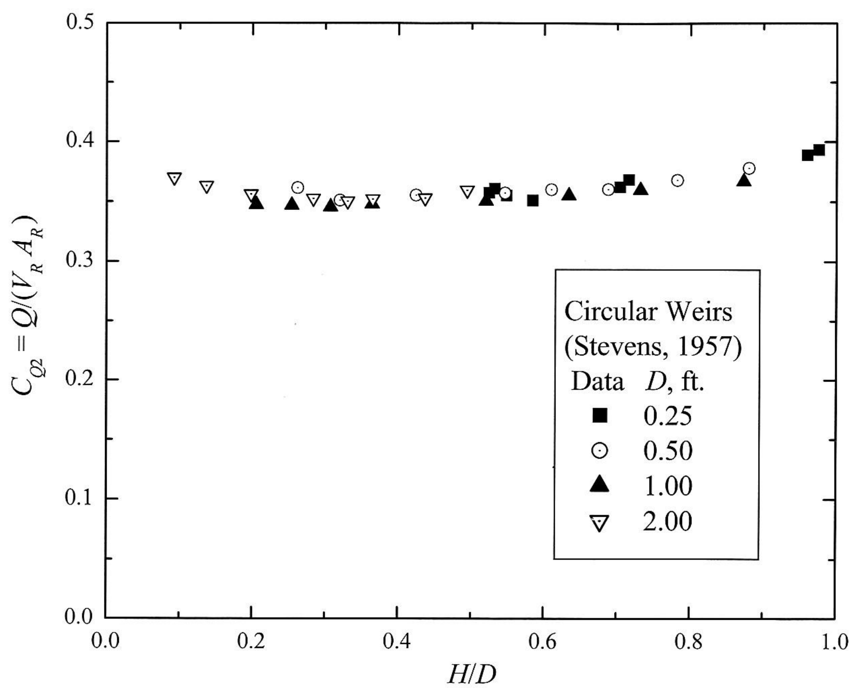
$$\Pi_Q \equiv C_{Q1} = \frac{Q}{g^{1/2}H^{5/2}} = f\left(\frac{\mu}{\rho g^{1/2}H^{3/2}}, \theta, \frac{V_0}{g^{1/2}H^{1/2}}\right).$$



With $Re_H \gg 1$, the flow should be predominantly frictionless. The graph confirms that, to a first approximation, $Q \propto H^{5/2}$ as predicted by dimensional analysis for frictionless flow with negligible V_0, but the least squares fit of the data shows that C_{Q1} decreases by about 2% (from about 0.4460 to 0.4366), over the available data range. Viscous phenomena such as boundary layer thicknesses should decrease as Re_H increases, causing Q and thus C_{Q1} to increase, so the source of the variation is probably another factor, such as V_0, notch edge sharpness or surface tension. Thus, for accurate measurements, in the absence of data on the effects of such factors, calibration of individual notches is required.

Problem 3.13: The plot of C_{Q1} against H/D, given below for the available D, shows the ability of dimensional analysis to account for geometric similarity. As an alternate, define $C_{Q2} \equiv Q/(V_R A_R)$, with V_R being the Torricellian speed $(2gH)^{1/2}$, and with A_R being the maximum available flow area based on H and the particular weir geometry. For a V-notch, $A_R = H^2 \tan(\theta/2)$, and $CQ_2 = C_{Q1}$ for $\theta = 90°$. For a circle A_R is the area of the lower segment defined by the segment semi-angle α = arcos$(1 - 2H/D)$; that is, $A_R = D^2/4[\alpha - \frac{1}{2}\sin 2\alpha]$ for $0 \le \alpha \le \pi$. This plot is also given below.





For the circular weir, whereas C_{Q1} varies strongly with *H*/*D*, C_{Q2} is nearly constant, with an average value of 0.359, which is about 20% below that for the V-notch weir. The stronger variation with *H*/*D* is probably caused by that variable's effect on the vena-contracta, an effect not present in the V-notch weir. Even then the variation is only about ± 7% about the mean value.