Question 1 [6 marks]:

Consider a system of two Einstein Solids, A and B, each containing N oscillators, sharing a total of q units of energy. Assume the solids are weakly coupled, and that the total energy is fixed. For this question, express your answers in terms of binomial coefficients, $\binom{n}{r}$ where appropriate, so we don't have to bother with large number approximations. Do not assume $N>>1$, and do not assume $q>>N$.

a) How many different macrostates are available to the combined system, if we define a macrostate by the energy in each of the two solids? (1 pt)

Since q is shared between, both can have 0 to q energy so total macro states is $q+1$

b) How many different microstates are available to the combined system? (1 pt)

This is the total combined oscillators 2N choose q so $\binom{q+2N-1}{q}$

c) Assuming the system is in thermal equilibrium, what is the probability of finding all the energy in solid A? (2 pts)

This is $\binom{0+N-1}{0} \times \binom{q+N-1}{q}$ microstates for a probability of $\dfrac{\binom{q+N-1}{q}}{\binom{q+2N-1}{q}}$

$\binom{0+N-1}{0} = 1$

d) Assuming the system is in thermal equilibrium, what is the probability of finding one third of the energy in solid A? (2 pts)

$q_A = \frac{q}{3}$ $\quad q_B = \frac{2q}{3}$ so $\binom{\frac{q}{3}+N-1}{\frac{q}{3}} \times \binom{\frac{2q}{3}+N-1}{\frac{2q}{3}}$ microstates for a probability of

$$\frac{\binom{\frac{q}{3}+N-1}{\frac{q}{3}} \cdot \binom{\frac{2q}{3}+N-1}{\frac{2q}{3}}}{\binom{q+2N-1}{q}}$$

Question 2 [4 marks]:

a) Suppose you flip 800 coins. What is the probability of getting exactly 400 heads and 400 tails? Expand any binomial coefficients, $\binom{n}{r}$, and use Stirling's approximation to get this answer into a form that doesn't need a calculator that can work with numbers larger than 10^{99}. (You don't need to evaluate the expression with a calculator, however). (4 pts)

Total microstates: 2^{800} $\quad \Omega(800,400) = \binom{800}{400} = \dfrac{800!}{(400!)^2}$

$$= \frac{\sqrt{2\pi}\cdot\sqrt{800}\cdot\left(\frac{800}{e}\right)^{800}}{\sqrt{2\pi}^2\cdot\sqrt{400}^2\cdot\left(\frac{400}{e}\right)^{400\cdot 2}} = \frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{800}}{400}\cdot\frac{e^{800}}{e^{800}}\cdot\frac{800^{800}}{400^{800}} = \frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{2}}{20}\cdot 2^{800}$$

$$\sqrt{\frac{800}{400^2}}\ \sqrt{\frac{800}{400}\cdot\frac{1}{400}} = \sqrt{\frac{2}{400}} = \sqrt{\frac{1}{200}} \qquad \Rightarrow \Omega = \frac{2^{800}}{\sqrt{400\pi}}$$

$$P = \left(\frac{2^{800}}{\sqrt{400\pi}}\right) \div 2^{800} = \frac{1}{\sqrt{400\pi}}$$

$$2^{800} \qquad \frac{800!}{(400!)^2} = \frac{\sqrt{2\pi 800}\cdot\left(\frac{800}{e}\right)^{800}}{\sqrt{2\pi 400}^2\left(\frac{400}{e}\right)^{800}} = \sqrt{\frac{800}{2\pi 400^2}}\cdot\left(\frac{800}{400}\right)^{800} = \sqrt{\frac{1}{2\pi 200}}\cdot 2^{800}$$

$$\Rightarrow P = \sqrt{\frac{1}{2\pi 200}} = \frac{1}{\sqrt{400\pi}}\ \checkmark$$ yep, really was that easy

Question 3 [6 marks]:

Consider a totally contrived system where the multiplicity is given by $\Omega = \epsilon U^4$ at constant volume and constant number of particles.

a) What is the expression for entropy, S, for this system, as a function of ϵ and U? (2 pts)

$$S = k\ln(\Omega) = k\ln(\epsilon U^4)$$

b) What is the expression for temperature, T for this system as a function of ϵ and U? (2 pt)

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{\partial}{\partial U} k\ln(\epsilon U^4) = k \cdot \frac{4\epsilon U^3}{\cancel{\epsilon} U^4} = \frac{4k}{U} \Rightarrow T = \frac{U}{4k}$$

c) What is the expression for the heat capacity at constant volume for this system, as a function of ϵ and T? (2 pt)

$$U = 4kT \Rightarrow C_V = \frac{\partial U}{\partial T} = 4k$$

Question 4 [2 marks]:

According to kinetic theory and the equipartition theorem, what is the expression for the average speed of a molecule of an ideal gas? (2 pts)

average energy per molecule

$E = \frac{1}{2} m \bar{v}^2 \qquad U = D\frac{kT}{2} = Nf\frac{kT}{2}$ or for a single particle just $f\frac{kT}{2}$

$E = \frac{1}{2} m\bar{v}^2 = f\frac{kT}{2} \Rightarrow \bar{v}^2 = \frac{fkT}{m} \Rightarrow \bar{v} = \sqrt{\frac{fkT}{m}}$

For an ideal monatomic gas this equals $\sqrt{\frac{3kT}{m}}$ and for diatomic $\sqrt{\frac{5kT}{m}}$ at room temperature.

Question 5 [2 marks]:

According to the equipartition theorem, what is the expression for the heat capacity per molecule of a hot diatomic gas with 2 active rotational degrees of freedom and 2 active vibrational degrees of freedom? (2 pts)

$U = D\frac{kT}{2} = Nf\frac{kT}{2}$ so for one molecule $U = f\frac{kT}{2}$

f: 3 velocity DOFs + 2 rotational DOFs + 2 vibrational DOFs = 7 total DOFs

$\Rightarrow U = \frac{7}{2}kT \qquad C_V = \frac{\partial U}{\partial T} = \frac{7}{2}k$