The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 2 - Thursday March 15, 2018

Instructors: Profs. Micah Stickel and Piero Triverio

Last name: SOLUTION

First name: ....

Student number: ....

Tutorial section number: ....

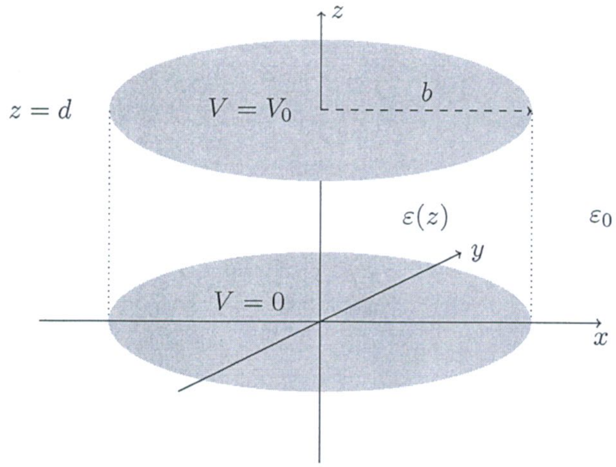
Section	Day	Time	Room	TA name
TUT0101	Monday	14:00-15:00	BA 3012	Shashwat
TUT0102	Monday	14:00-15:00	RS 310	Gengyu (Paul)
TUT0103	Monday	14:00-15:00	BA 2159	Sameer
TUT0104	Monday	14:00-15:00	BA 3116	Fadime
TUT0105	Wednesday	13:00-14:00	BA 3012	Shashwat
TUT0106	Wednesday	13:00-14:00	WB 144	Gengyu (Paul)
TUT0107	Wednesday	13:00-14:00	BA 2159	Sameer
TUT0108	Wednesday	13:00-14:00	BA 3116	Fadime

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

Marks:	Q1: /20	Q2: /12	Q3: /8	Q4: /20	TOTAL: /60

Question 1



Consider the structure shown in the figure above. We have two conducting circular disks of radius b, one in the plane $z = d$ and the other in the plane $z = 0$. The volume between the disks is filled by a dielectric which is ideal ($\sigma = 0$) and inhomogeneous, with permittivity

$$\varepsilon(z) = \varepsilon_0 \left(1 + \frac{z}{d}\right)$$

The upper disk is held at potential $V = V_0$ while the lower disk is held at potential $V = 0$, and it is known that the volume density of free charge inside the dielectric material is zero.

(a) Assume that the potential inside the structure is only function of z, i.e. $V = V(z)$. Solve Poisson's equation to prove that $V(z) = \frac{V_0}{\ln(2)} \ln\left(\frac{d+z}{d}\right)$. [8 points]

Poisson's equation $\nabla \cdot (\varepsilon \nabla V) = \boxed{-\cancel{\rho_v} = 0}$ 1

Since $V = V(z) \Rightarrow \nabla V = \bar{a}_z \frac{\partial V}{\partial z}$

$$\nabla \cdot \left(\varepsilon_0 \left(1 + \frac{z}{d}\right) \bar{a}_z \frac{\partial V}{\partial z} \right) = 0$$

$$\frac{\partial V}{\partial z} \left[\cancel{\varepsilon_0} \left(1 + \frac{z}{d}\right) \frac{\partial V}{\partial z} \right] = 0$$

integrate once 4

$$\left(1 + \frac{z}{d}\right) \frac{\partial V}{\partial z} = C_1$$

$$\frac{\partial V}{\partial z} = \frac{C_1 d}{z + d}$$

Integrate again $\rightarrow$ $V(z) = C_1 d \ln(z+d) + C_2$

Boundary cond's

$$\begin{cases} V(d) = V_0 \\ V(0) = 0 \end{cases} \quad \begin{cases} C_1 d \ln(2d) + C_2 = V_0 \\ C_1 d \ln(d) + C_2 = 0 \end{cases} \quad \begin{cases} C_1 d \ln(2) = V_0 \ ; \quad C_1 = \dfrac{V_0}{d \ln(2)} \\ C_2 = -\dfrac{V_0}{\cancel{d} \ln(2)} \cdot \cancel{d} \ln(d) = -V_0 \dfrac{\ln d}{\ln 2} \end{cases}$$

3

4

$$V(z) = V_0 \frac{\ln(z+d)}{\ln(2)} - V_0 \frac{\ln d}{\ln 2} = \frac{V_0}{\ln 2} \ln\left(\frac{z+d}{d}\right)$$

(b) Find the electric field **E** in the dielectric. [4 points]

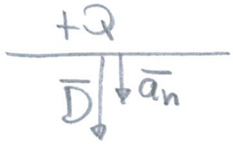
$$\bar{E} = -\nabla V = -\bar{a}_z \frac{\partial V}{\partial z} =$$

$$= -\bar{a}_z \frac{V_0}{\ln 2} \frac{\cancel{d}}{z+d} \cdot \frac{1}{\cancel{d}} = -\frac{V_0}{(\ln 2)(z+d)} \bar{a}_z$$

(c) Find the capacitance C between the plates. [8 points]

$$C = \frac{Q}{V_0}$$

Q: charge on top plate



$$\rho_s = \bar{a}_n \cdot \bar{D} = (-\bar{a}_z) \cdot \varepsilon_0 \left(1 + \frac{d}{d}\right) \left(-\frac{V_0}{(d+d) \ln 2}\right) \bar{a}_z =$$

$$= \varepsilon_0 \cdot \cancel{2} \frac{V_0}{\cancel{2} d \ln 2}$$

4

$$Q = \frac{\varepsilon_0 V_0}{d \ln 2} \cdot \pi b^2$$

2

$$\boxed{C = \frac{Q}{V_0} = \frac{\varepsilon_0 \pi b^2}{d \ln(2)}}$$

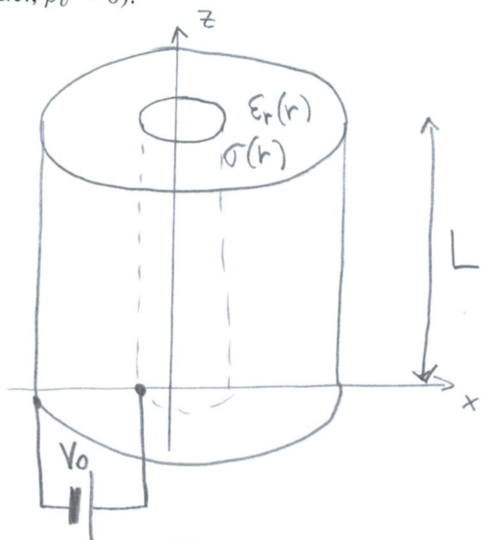
Final answer

2

Question 2

A cylindrical resistor lies along the z-axis and consists of two concentric cylinders of radii a and b ($a < b$), and has a total length L. The perfectly-conducting cylinders are separated by a dielectric material with a non-uniform relative permittivity, $\epsilon_r = \epsilon_{r0} r$, and a non-uniform conductivity given by $\sigma = \sigma_0 r$, where both ϵ_{r0} and σ_0 are positive constants. The resistor is connected to a battery with voltage V_0, and the inner conductor is at a higher electric potential than the outer conductor. You can ignore the effects of fringing fields in this question and you can assume that the free charge density inside the dielectric is zero (i.e., $\rho_v = 0$).

(a) Determine the expression for resistance of this structure. [6 points]



Find V using Poisson's equation

$$\nabla \cdot (\varepsilon(r) \nabla V) = -\cancel{\rho_v} = 0$$

$$\nabla \cdot (\cancel{\varepsilon_{r0}} r \nabla V(r)) = 0$$

$$\cancel{\frac{1}{r}} \frac{\partial}{\partial r}\left[r\, r \frac{\partial V}{\partial r} \right] = 0 \rightarrow r^2 \frac{\partial V}{\partial r} = C_1$$

1

$$\frac{\partial V}{\partial r} = \frac{C_1}{r^2} \; ; \quad V(r) = -\frac{C_1}{r} + C_2$$

Impose $\begin{cases} V(r=a) = V_0 \\ V(r=b) = 0 \end{cases}$

$$\begin{cases} -\frac{C_1}{a} + C_2 = V_0 \\ -\frac{C_1}{b} + C_2 = 0 \end{cases} \quad C_1\left(\frac{-1}{a} + \frac{1}{b}\right) = V_0 \; ; \quad C_1 = -\frac{V_0 ab}{b-a}$$

$$C_2 = \frac{C_1}{b} = -\frac{V_0 a}{b-a}$$

$$V(r) = +\frac{V_0 ab}{b-a} \cdot \frac{1}{r} - \frac{V_0 a}{b-a} = \frac{V_0 a}{b-a}\left(\frac{b}{r} - 1\right)$$

B.C.'s & V

1

$$\bar{E} = -\frac{\partial V}{\partial r} \bar{a}_r = -\frac{V_0 a}{b-a}\left(-\frac{b}{r^2}\right)\bar{a}_r = \frac{V_0 ab}{(b-a) r^2} \bar{a}_r$$ 1

$$\bar{J} = \sigma \bar{E} = \sigma_0 \cancel{r} \frac{V_0 ab}{(b-a) r^{\cancel{2}}} \bar{a}_r = \frac{\sigma_0 V_0 ab}{(b-a) r} \bar{a}_r$$ 1

$$I = \int_S \bar{J} \cdot d\bar{S} = \frac{\sigma_0 V_0 ab \; 2\pi \cancel{r} L}{(b-a) \cancel{r}} = \frac{2\pi \sigma_0 ab V_0 L}{(b-a)}$$ 1

$$\boxed{R = \frac{b-a}{2\pi \sigma_0 ab L}}$$ 1

Question 2

A cylindrical resistor lies along the z-axis and consists of two concentric cylinders of radii a and b ($a < b$), and has a total length L. The perfectly-conducting cylinders are separated by a dielectric material with a non-uniform relative permittivity, $\epsilon_r = \epsilon_{r0} r$, and a non-uniform conductivity given by $\sigma = \sigma_0 r$, where both ϵ_{r0} and σ_0 are positive constants. The resistor is connected to a battery with voltage V_0, and the inner conductor is at a higher electric potential than the outer conductor. You can ignore the effects of fringing fields in this question and you can assume that the free charge density inside the dielectric is zero (i.e., $\rho_v = 0$).

(a) Determine the expression for resistance of this structure. [6 points]

ALTERNATIVE SOLUTIONS

5

Sol'n #1: $R = \int dR = \int_a^b \frac{dr}{\sigma(r) S} = \int_a^b \frac{dr}{(\sigma_0 r)(2\pi r L)} = \frac{1}{2\pi\sigma_0 L}\int_a^b \frac{dr}{r^2}$

$\therefore R = \frac{1}{2\pi\sigma_0 L}\left[-\frac{1}{r}\Big|_a^b\right] = \frac{1}{2\pi\sigma_0 L}\left(\frac{1}{a}-\frac{1}{b}\right) = \underline{\underline{\frac{1}{2\pi\sigma_0 L}\left(\frac{b-a}{ab}\right)}}$

1

Sol'n #2: $R = \frac{V}{I} = \frac{\left|-\int \bar{E}\cdot d\bar{l}\right|}{\iint_S \bar{J}\cdot d\bar{S}} = \frac{\int \bar{E}\cdot d\bar{l}}{\iint_S \sigma\bar{E}\cdot d\bar{S}} = \frac{V_0}{\iint_S \sigma \bar{E}\cdot d\bar{S}}$

From Gauss's Law: $\bar{D} = \frac{Q}{2\pi r L}\hat{a}_r$, $\bar{E} = \frac{\bar{D}}{\varepsilon_r\varepsilon_0} = \frac{Q}{2\pi\varepsilon_{r0}\varepsilon_0 r^2 L}\hat{a}_r$

$\Delta V = V_0 = -\int_b^a \left(\frac{Q}{2\pi\varepsilon_{r0}\varepsilon_0 r^2 L}\right) dr = \frac{-Q}{2\pi\varepsilon_{r0}\varepsilon_0 L}\left[-\frac{1}{r}\Big|_b^a\right] = \frac{Q}{2\pi\varepsilon_{r0}\varepsilon_0 L}\left(\frac{b-a}{ab}\right)$

$\therefore Q = \frac{V_0 2\pi\varepsilon_{r0}\varepsilon_0 L(ab)}{(b-a)}$ $\quad \bar{E} = \frac{V_0 ab}{(b-a)r^2}\hat{a}_r$

E 3

$\therefore R = \underbrace{\frac{V_0}{\iint_S \sigma\bar{E}\cdot d\bar{S}} = \frac{\cancel{V_0}}{\int_0^L\int_0^{2\pi}(\sigma_0 \cancel{r})\frac{\cancel{V_0}(ab)}{(b-a)\cancel{r^2}}\cancel{r}\,d\phi\, dz} = \frac{(b-a)}{2\pi\sigma_0 L(ab)}}_{I\ 2} = \underline{\underline{\frac{1}{2\pi\sigma_0 L}\left(\frac{b-a}{ab}\right)}}$

1

(b) If $a = 1.4$ mm, $b = 2.1$ mm, $V_0 = 15$ V, $\epsilon_{r0} = 4.5$, and $\sigma_0 = 8 \times 10^{-8}$ S/m determine the maximum length possible for this resistor, such that the power dissipated in the resistor is less than 123 nW. [4 points]

$$P = V_o I = \frac{V_o^2}{R} = \frac{V_o^2 (2\pi \sigma_o L)(ab)}{(b-a)} \leq 123 \times 10^{-9} \text{ W}$$ 2

$$\therefore L \leq \frac{(123 \times 10^{-9})(2.1 \times 10^{-3} - 1.4 \times 10^{-3})}{(15)^2 (2\pi)(8 \times 10^{-8})(2.1 \times 10^{-3})(1.4 \times 10^{-3})}$$

$$\leq \underline{\underline{259 \text{ mm}}}$$ 2

(c) For the length of the resistor found in part (b), determine the energy dissipated over a 10 minute period. Briefly describe the physical mechanism behind this energy use. [2 points]

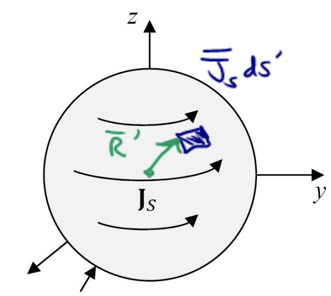
$$W_e = \int P\,dt = P(\Delta t) = (123 \times 10^{-9})(10)(60)$$

$$= \underline{\underline{73.8 \ \mu J}}$$ 1

* This energy is lost to heat caused as a result of the movement of excited charge carriers (i.e. electrons) within the dielectric atomic lattice. 1

Question 3

One way to model the movement of the valence electrons around an atom's nucleus is with a current density flowing on the surface of a sphere with radius $R = a$. As illustrated, this uniform surface current density can be expressed as $\mathbf{J}_S = J_0\mathbf{a}_\varphi$ for $R = a$, where J_0 is a constant with units of [A/m]. Determine the magnetic flux density, $\mathbf{B}$, at the origin of this sphere, which results from the current distribution $\mathbf{J}_S$. You can assume this atom exists in a non-magnetic material (i.e., $\mu_r = 1$). [8 points]



Sphere of radius $R = a$, centered about the origin.

* $\bar{B} = \iint_S d\bar{B} = \iint_S \frac{\mu_0}{4\pi}\frac{\bar{J}_s ds' \times (\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3}$

* Find $\bar{J}_s ds'$, $\bar{R}$, $\bar{R}'$ & $\bar{R}-\bar{R}'$

$\bar{J}_s ds' = J_0 \hat{a}_\phi (a^2 \sin\theta' \, d\theta' d\phi')$

$\bar{R} = 0$ (origin)

$\bar{R}' = a\hat{a}_R$

$\bar{R}-\bar{R}' = -a\hat{a}_R$

2

* Simplify $d\bar{B}$

$\hat{a}_\phi \times (-\hat{a}_R) = -\hat{a}_\theta$

$$d\bar{B} = \frac{\mu_0}{4\pi}\frac{\bar{J}_s ds' \times (\bar{R}-\bar{R}')}{|\bar{R}-\bar{R}'|^3} = \frac{\mu_0}{4\pi}\frac{J_0(\cancel{a^2}\sin\theta' d\theta' d\phi')\,\hat{a}_\phi \times (-\cancel{a}\hat{a}_R)}{\cancel{a^3}}$$

$$= \frac{\mu_0 J_0 \sin\theta'}{4\pi}(-\hat{a}_\theta)\, d\theta' d\phi'$$

1

$\Rightarrow$ Since $\hat{a}_\theta$ changes position as we integrate over the sphere, we need to convert to Cartesian coordinates:

$$\therefore d\bar{B} = \frac{-\mu_0 J_0}{4\pi}\sin\theta'\left[\cos\theta'\cos\phi'\,\hat{a}_x + \cos\theta'\sin\phi'\,\hat{a}_y - \sin\theta'\,\hat{a}_z\right] d\theta' d\phi'$$

2

* Integrate:

$$\bar{B} = -\frac{\mu_0 J_0}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\underbrace{\sin\theta'\cos\theta'\cos\phi'\,\hat{a}_x}_{} + \underbrace{\sin\theta'\cos\theta'\sin\phi'\,\hat{a}_y}_{} - \sin^2\theta'\,\hat{a}_z\right] d\theta' d\phi'$$

These components will integrate to zero with the ϕ' integral.

Also makes sense from the symmetry of the problem and the Right-Hand-Rule.

ϕ int.

$$= +\frac{\mu_0 J_0 \hat{a}_z}{4\pi}(2\pi)\underbrace{\int_0^{\pi}\sin^2\theta\, d\theta'}$$

$$= \frac{\mu_0 J_0 (\cancel{2\pi})}{\cancel{4\pi}_{\,2}}\left(\frac{\pi}{2}\right)\hat{a}_z \Rightarrow \bar{B} = \frac{\mu_0 J_0 \pi}{4}\hat{a}_z \quad [T]$$

3

Question 4.1

Marking scheme for all points of Q4.1:
1 pt right answer
1 pt complete and correct justification

A parallel-plate capacitor is filled with a dielectric with a relative permittivity of ϵ_r and has a plate area of S and a plate-to-plate spacing of d. The capacitor is attached to a battery and charged up, such that the plates have charges $\pm Q$. This is Case A. The capacitor is left for a long time and then the dielectric piece is removed from the capacitor with the battery still attached. This is Case B. Determine whether each of the following statements are **True** or **False**. Justify each of your answers. You can ignore the effects of fringing fields for this question. [8 points]

(a) (True / False) The capacitance of Case B is larger than the capacitance of Case A (i.e., $C_B > C_A$).

$$C_B = \frac{\varepsilon_0 S}{d} < C_A = \frac{\varepsilon_r \varepsilon_0 S}{d}$$

(b) (True / False) The charge on the positively-charged plate is the same in both cases (i.e., $Q_A = Q_B$).

Since the battery remains attached when the dielectric is removed:

$$Q_B = C_B V_0 < Q_A = C_A V_0$$

(c) (True / False) The magnitude of the electric flux density inside the capacitor for Case A is larger than that for Case B, (i.e., $|\mathbf{D}_A| > |\mathbf{D}_B|$)

From part (b) $Q_A = \rho_{sA} S > Q_B = \rho_{sB} S \rightarrow \therefore \rho_{sA} > \rho_{sB}$

Since from the boundary condition at the metal to dielectric/air interface we have $|\bar{D}_A| = \rho_{sA} > |\bar{D}_B| = \rho_{sB}$

(d) (True / False) The stored energy within the capacitor is larger in Case A than it is in Case B.

$$W_{eA} = \frac{1}{2} C_A V_0^2 > W_{eB} = \frac{1}{2} C_B V_0^2$$

Question 4.2

1 pt right answer
3 pt for justification

Consider a capacitor with a dielectric and constant charge Q on its plates (i.e., it has been connected to a battery for a long time and then the battery is disconnected before any changes are made). The dielectric is then taken out. After the dielectric is removed, does the stored energy: [4 points]

(a) Increase. Explain where the additional energy came from.

(b) Decrease. Explain where the "lost" energy went

(c) Remain the same. Explain why energy remains the same.

Briefly justify your answer.

$$W_e = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C} \qquad \left(V = \frac{Q}{C} \right)$$

$\Rightarrow$ Since Q is constant as the dielectric is removed (i.e. battery is disconnected) then we have to use $W_e = \frac{1}{2}\frac{Q^2}{C}$ not $W_e = \frac{1}{2} C V^2$

$$W_{e\,before} = \frac{1}{2} \frac{Q^2}{\frac{\varepsilon_r \varepsilon_0 S}{d}} = \frac{1}{2} \frac{Q^2 d}{\varepsilon_r \varepsilon_0 S}$$

$$W_{e\,after} = \frac{1}{2} \frac{Q^2}{\frac{\varepsilon_0 S}{d}} = \frac{1}{2} \frac{Q^2 d}{\varepsilon_0 S}$$

$\therefore W_{e\,after} > W_{e\,before}$

* The addition energy stored in the air capacitor is from the fact that some of the "field energy" is not being "stored" in the polarized atoms of the dielectric.

Question 4.3

1 pt right answer
3 pt for justification

A steady current of 5 A is carried from left to right by a 20 cm section of copper cylindrical wire A ($\sigma_c = 5.8 \times 10^7$ S/m), which has a circular cross-section with radius 1 cm. An aluminum wire B ($\sigma_a = 3.5 \times 10^7$ S/m) with total length 15 cm is connected to the right of wire A, and also has a circular cross-section with radius of 1 cm. The two wires share the same central axis. Which of the following statements is true? You may assume that the current is uniformly distributed in both wires and they are part of a larger circuit that is connected to a source which sustains the steady 5 A current. Recall, that the relative permittivity for both copper and aluminum can be assumed to be one, i.e., $\epsilon_r = 1$. Briefly justify your answer. [4 points]

(a) The current density, J, in wire A is different from that in wire B.

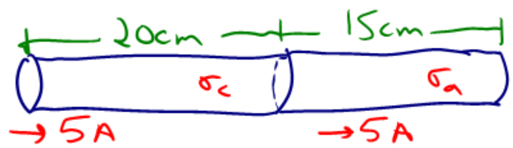
(b) The electric field intensity, $\mathbf{E}$, in wire A is the same as it is in wire B.

(c) The total current, I, in wire A is different than that in wire B.

(d) The total resistance of the length of wire A is greater than the total resistance of the length of wire B.

(e) A positive free charge density exists at the interface of the two wires.

(f) None of the above statements are true.



(a),(c) For steady currents $J_{n1} = J_{n2}$ at the interface so $J_A = J_B$
↳ as well since the current of 5 A has to be continuous and the radii are the same $J_A = \frac{5}{\pi (1cm)^2} = J_B$

↳ ∴ Statements (a) & (c) are false

(b) Since $J_A = J_B = J_0 \rightarrow E_A = \frac{J_0}{\sigma_c} \neq \frac{J_0}{\sigma_a} = E_B$

↳ Statement (b) is false

(d) $$\frac{R_A}{R_B} = \frac{\frac{\cancel{20cm}\ 4}{(5.8\times\cancel{10^7})(\cancel{\pi\, 1cm^2})}}{\frac{\cancel{15cm}\ 3}{(3.5\times\cancel{10^7})(\cancel{\pi\, 1cm^2})}} = \frac{4(3.5)}{3(5.8)} = 0.8 \rightarrow R_A < R_B$$

(d) is false

(e) From $D_{n2} - D_{n1} = \rho_s \rightarrow \frac{\varepsilon_{rc}\varepsilon_0}{\sigma_c} J_n - \frac{\varepsilon_{ra}\varepsilon_0}{\sigma_a} J_n = \rho_s$

Since $\varepsilon_{rc} = \varepsilon_{ra} = 1 \rightarrow \rho_s = \left(\frac{\varepsilon_0}{\sigma_c} - \frac{\varepsilon_0}{\sigma_a}\right) J_n$ which must be positive and non-zero.

∴ Statement (e) is true.

Question 4.4

1 pt right answer
3 pt for justification

Consider an electric dipole that is located along the z-axis with a dipole moment given by $\mathbf{p} = Qd\mathbf{a}_z$ and is centered about the point $z = d$. This dipole is placed above an infinitely large perfectly conducting plane that is located in the xy-plane. Which statement below correctly describes the electric field at the origin? Briefly justify your answer. [4 points]

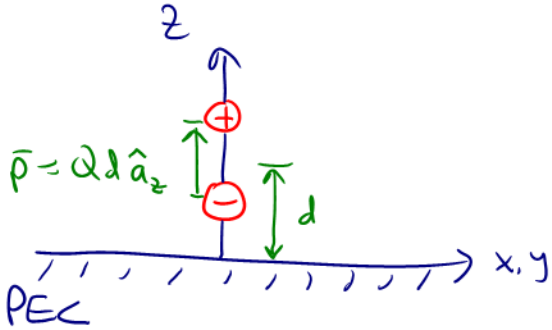
(a) The electric field is zero at the origin.

(b) The electric field points only in the positive z-direction at the origin.

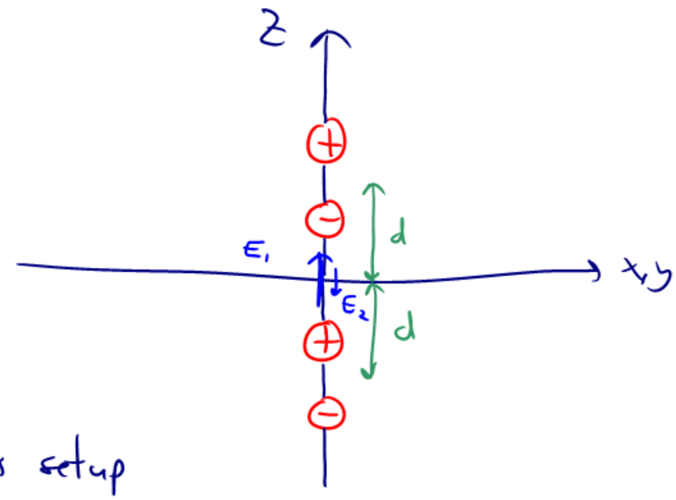
(c) The electric field points only in the negative z-direction at the origin.

(d) The electric field has x, y, and z components at the origin.

(e) More information is needed to know the nature of the electric field at the origin.



Through image theory =



* Through image theory we can reframe this setup as the combination of two new dipoles:

Dipole #1 → $\bar{p}_1 = -Qd\,\hat{a}_z$, $\bar{p}_2 = Q(3d)\,\hat{a}_z$

⇒ At the origin the field of dipole #1 will dominate in the +ve z-direction

⇒ As well, we know that $\bar{E}$ must be perpendicular to the xy-plane so $\bar{E}$ can only have a z-component.

∴ Statement (b) is correct.