University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – December, 2017

SECOND YEAR – ENGINEERING SCIENCE
Program 5
AER210F VECTOR CALCULUS and FLUID MECHANICS
Examiners: P. McCarthy and J. W. Davis

Instructions:
(1) Closed book examination; except for a non-programmable calculator, no aids are permitted.
(2) Write your name and student number in the space provided below.
(3) Answer as many questions as you can. Parts of questions may be answered.
(4) Questions are NOT assigned equal marks.
(5) Use the overleaf side of pages for additional or preliminary work.
(6) Do not separate or remove any pages from this exam booklet.
(7) You may use $g = 9.81$ m/s^2, $\rho_{water} = 1000$ kg/m^3 where appropriate.

Family Name:______________________________

Given Name:______________________________

Student Number:______________________________

FOR MARKER USE ONLY		
Question	Mark	Earned
1	12	
2	11	
3	10	
4	10	
5	18	
6	9	
7	11	
8	17	
9	6	
10	11	
11	8	
12	10	
TOTAL	133	

The following integrals and formulae may be useful:

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \quad \oiint_S \bar{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\bar{F}\, dV; \quad \oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot d\bar{S}$$

$$-\nabla p + \rho\vec{g} = \rho\vec{a}; \qquad For\ a\ circle -\ I_{xx} = \frac{\pi R^4}{4}; \quad Centroid\ occurs\ at\ centre\ of\ circle$$

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}; \quad c_v = \frac{R}{\gamma-1}; \quad c_p = \frac{\gamma R}{\gamma-1}; \quad d\tilde{u} = c_v dT; \ dh = c_p dT$$

$$\frac{d}{dt}\iiint_{CV} \rho dV + \oiint \rho(\vec{U}\cdot\hat{n})dA = 0; \qquad \sum \vec{F} = \frac{d}{dt}\iiint_{CV} \rho\vec{U}\, dV + \oiint \rho\vec{U}(\vec{U}\cdot\hat{n})dA$$

1) a) [4 marks] Evaluate the integral $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3+1}\; dy\, dx$ by reversing the order of integration. Provide a sketch of the region.

b) [4 marks] Use polar coordinates to find the volume below the cone $z = \sqrt{x^2+y^2}$ and above the ring $1 \le x^2 + y^2 \le 4$. Provide a sketch of the region.

c) [4 marks] Evaluate the integral $\int_{-a}^{a} \int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} (x^2z + y^2z + z^3) dzdxdy$, by changing to spherical coordinates.

2) a) [5 marks] Solve the integral equation: $y(x) = 2 + \int_1^x (y(t))^2 dt$

b) [6 marks] Let $y = f(x)$ be a curve in the x - y plane with f continuous, and $f(x) > 0$, for $a \leq x \leq b$. Let S be the surface generated when the graph of f on $[a, b]$ is revolved about the x - axis. Show by means of a sketch that $\vec{r}(u, v) = u\hat{\imath} + f(u)cosv\hat{\jmath} + f(u)sinv\hat{k}$, for $a \leq u \leq b$, $0 \leq v \leq 2\pi$, gives a parametric description of the surface. Use this description to find an integral representing the surface area.

3) [10 marks] Given a function $z = f(x, y)$ and a transformation $x = x(u, v)$, $y = y(u, v)$ such that $z = g(u, v)$, show that:

$$\iint f(x,y)dydx = \iint g(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$

4) [10 marks] Evaluate (without using the Divergence theorem) the surface integral $\int_S \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = y\hat{\jmath} - z\hat{k}$, where S consists of the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$, and the disk $x^2 + z^2 \leq 1, y = 1$.

5) [18 marks] Evaluate the line integral $\oint_C xzdx - 3xdy + yzdz$ where C is the curve of intersection between the sphere $x^2 + y^2 + z^2 = 2$ and the paraboloid $z = (x^2 + y^2)$, directed clockwise as viewed from the origin, in three ways:

a) directly as a line integral

b) using Stokes' Theorem on the part of $z = (x^2 + y^2)$ bounded by C

c) using Stokes' theorem on the surface $z = 1$ bounded by C.

question 5 continued

6) [9 Marks] Fill in the blanks or circle the correct answer:

a) [1 Mark] For incompressible flow, the differential form of the continuity equation reduces to:

b) [1 Mark] The ratio $\frac{U}{\sqrt{gL}}$ is known as ______________________________,
where U is velocity, g is gravitational acceleration and L is the characteristic length.

c) [1 Mark] What terms are neglected in the Navier-Stokes equations to give the Euler equations?

(i) acceleration terms

(ii) body force terms

(iii) pressure force terms

(iv) viscous force terms

d) [2 Marks] What is the difference between a streakline and a pathline? [2 Marks]

e) [2 Marks] What are the dimensions of Kinematic Viscosity?

(i) L^2T^{-1}

(ii) $ML^{-1}T^{-1}$

(iii) MLT^{-2}

(iv) None of these

f) [1 Mark] In a converging duct, a compressible fluid will be accelerated more than an incompressible fluid?

(i) True

(ii) False

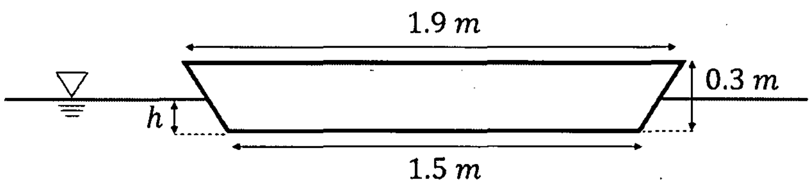
(iii) They will have equal acceleration

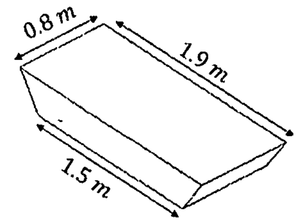
g) [1 Mark] The Bernoulli equation derived for flow along a streamline, $p + \frac{1}{2}\rho U^2 + \rho g z = constant$, is suitable for steady, inviscid, compressible flow in a duct?

(i) True

(ii) False

7) [11 Marks] A simple flat bottom boat, with dimensions shown in the image, is used to navigate in a shallow lake.



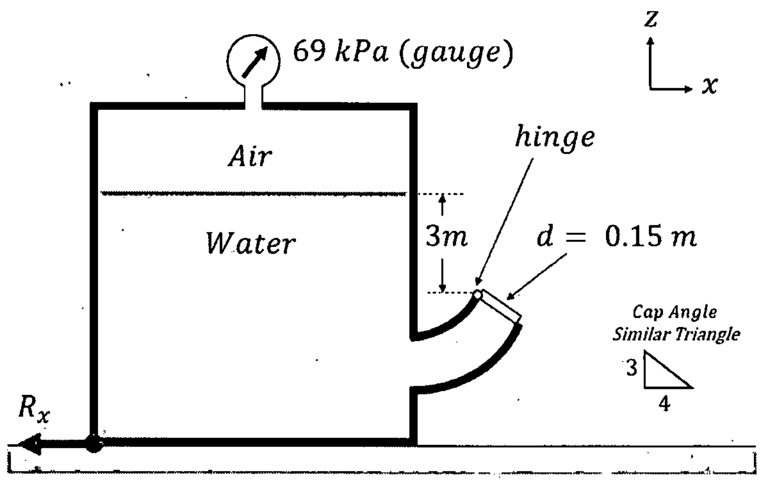


(a) [6 Marks] Find the depth of the bottom of the boat relative to the free surface, h, when the total mass of the boat (including all passengers inside it) is 250 kg.

b) [2 Marks] What is the minimum mass of the boat (including passengers) that would cause the boat to sink? i.e. when the water level reaches the top of the boat sides?

c) [3 Marks] The boat owner wants to use the boat on an artificial lake that has a constant depth, horizontal bottom. She wants to know if the water is deep enough for the boat, however, she does not have a ruler but she does have a stop watch and a stone. She throws the stone into the lake and records the time for the first ripple to reach the edge. It takes 5.4 seconds for the first ripple to reach the edge, travelling a total distance of 8.8 metres.
How deep is the lake?
How much distance would there be between the bottom of the boat and the lake floor, when the total mass of the boat is 250kg?

8) [17 Marks] A closed tank contains water, with air above the water at a pressure of 69 kPa gauge. One side of the tank has a spout that is covered by a $d = 0.15$m diameter circular cap, hinged along the top. The horizontal axis of the hinge is located 3m below the water surface.



a) [10 Marks] Determine the minimum moment that must be applied at the hinge to keep the cap closed. Neglect the weight of the cap and friction at the hinge. Atmospheric pressure acts on the outside of the cap and the cap angle is given by the similar triangle provided in the illustration.

b) [7 Marks] The same tank is resting on a smooth horizontal surface. The cap covering the spout is removed and a jet of water exits the tank from the spout. Find the restraining force, R_x, required to prevent the tank from moving in the x-direction. Assume the outlet velocity is one-dimensional across the outlet area and the jet angle is normal to the outlet area.

9) [6 Marks] Show that the isentropic steady flow energy equation, expressed in the form:

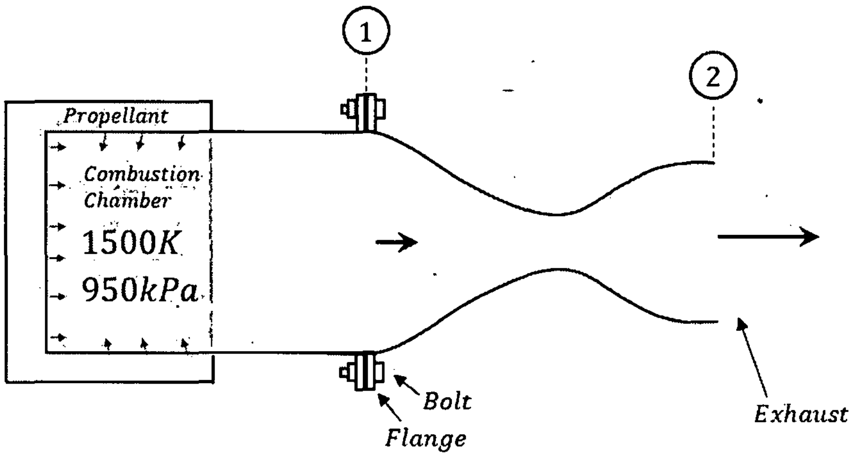
$$h + \frac{1}{2}U^2 + gz = constant$$

can be used to derive an expression relating flow Mach number with the ratio of stagnation temperature to static temperature:

$$\frac{T_0}{T} = \left[1 + \frac{(\gamma - 1)}{2} M^2\right]$$

State any assumptions that are made to arrive at the final equation.

10) [11 Marks] A solid propellant rocket is being tested at a facility in Toronto. Within the rocket combustion chamber, the flow can be considered stagnant with a pressure of 950 kPa (absolute) and temperature of 1500K. The rocket nozzle is axisymmetric and perfectly sized so that the exhaust pressure is equal to atmospheric pressure, $p_2 = 101\ 325$ Pa.
Assume one-dimensional isentropic properties across each flow area and constant values for both the ratio of specific heats $\gamma = 1.33$ and specific gas constant $R = 287$ J/kg.K.



a) [4 Marks] Find the Mach number, the static temperature and the exhaust flow velocity at the nozzle exit.

b) [7 Marks] The rocket nozzle is attached to the combustion chamber by 6 bolts, located equally around the flange (point 1 in the diagram). The diameter of the rocket nozzle at point 1 is $d_1 = 0.5$m and the pressure and temperature at this location are equal to the combustion chamber values. At the nozzle exhaust (point 2), the diameter of the nozzle is $d_2 = 0.2$m and the pressure, temperature and flow velocity are equal to those calculated in question *10) a)*. Calculate the force acting on each bolt.

Note: The velocity U_1 is small but not zero.

11) [8 Marks] Consider the flow field defined by the velocity vector $\vec{U} = xy^2\hat{\imath} - \frac{1}{3}y^3\hat{\jmath} + xy\hat{k}$

a) [1 Mark] Determine the number of flow dimensions.

b) [2 Marks] Determine if the flow is incompressible?

c) [5 Marks] Determine the acceleration of a fluid particle at point $x = 1, y = 2, z = 3$.

12) [10 Marks] The stream function for an incompressible, two-dimensional flow field is given by

$$\psi = Axy + Ay^2$$

a) [2 Marks] Determine the velocity vector for this flow.

b) [2 Marks] Determine the vorticity of the flow.

c) [6 Marks] For $A = 1$, sketch the streamlines for $\psi = 0$, $\psi = 6$ and $\psi = -2$.