NAME:______________________________

STUDENT NUMBER:________________________________

TUTORIAL GROUP: DAY OF THE WEEK:______________________TIME:_________

Time: 60 minutes

This is a closed-book exam worth a total of 60 points. Please answer all questions.

ONLY THE FOLLOWING ITEMS ARE ALLOWED ON YOUR DESK DURING THE EXAM:

1. **THIS EXAM BOOK –** It contains this cover page, three question pages and a formula sheet (which may be torn off). Make sure you start by putting your **NAME, ID NUMBER,** and **TUTORIAL GROUP** on the front (cover) page of the exam.
   The entire exam book (minus the formula sheet) **will be handed in** at the end of the exam and marked.
   a. **FORMULA SHEET**, annotated on the printed page. The formula sheet has to be printed from the file provided on Quercus.
2. **A CALCULATOR**, which can be anything that does NOT connect to wifi and does NOT communicate with other devices. ACCEPTABLE calculators include programmable and graphing calculators, scientific calculators, etc. UNACCEPTABLE calculators include: cell phones, tablets, laptops, etc.
3. **A PEN OR PENCIL.**
4. **YOUR STUDENT ID CARD**, used to check your identity.

If you are missing anything from the above items, raise your hand and ask an exam supervisor to supply what is missing. If you are missing an item that should have been brought by you (e.g., calculator, pen/pencil) there is a limited supply of extras and are on a first come, first served basis.

The last page of the midterm (page #6) is empty and can be used for scrap notes, but only pages 2, 3, & 4 will be scanned for marking.

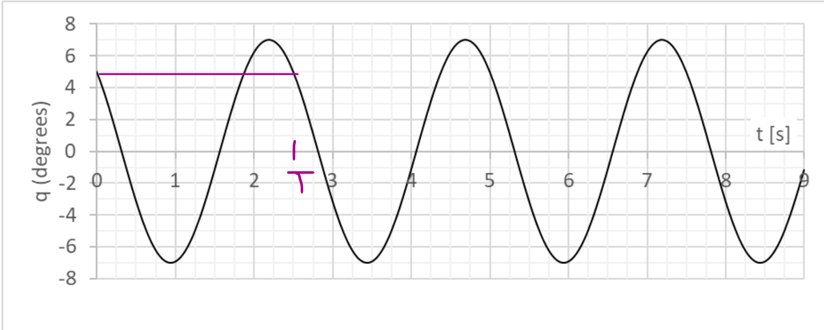
COMPLETE SOLUTION INCLUDES:

a) A sketch with a coordinate system, where appropriate. Where a diagram is provided, additional markings can be added to it.
b) Clear and logical workings, including quoting the equations that were used, proper substitutions of numbers (when necessary) with all steps clearly indicated.
c) The final answer with units and three significant figures.

FOR OFFICE USE ONLY				
QUESTION	I	II	III	TOTAL
MARK				
MAXIMUM	20	20	20	60

Question I

1. A simple pendulum consists of a string of length L and a point-like mass $m = 0.055$ kg attached to it. The graph shows the angular position of the pendulum, $\theta(t) = \theta_A \cos(\omega t + \theta_i)$, as a function of time.



   a. What is the natural frequency of the pendulum? [2 points]

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2.6s} = 2.41 \text{ rad/s}$$

   b. What is the initial phase constant of the pendulum? [6 points]

$$\theta(0) = 5deg = 7\,\text{deg}\cos(\theta_i) \rightarrow \cos\theta_i = \frac{5}{7}$$

$$\theta_i = \pm 0.775rad$$

Velocity of the pendulum is negative, therefore $v(0) = -\theta_A \omega \sin\theta_i < 0$

$$\sin\theta_i > 0 \therefore \theta_i = 0.775 \text{ rad}$$

   c. What is the total energy of this system? Indicate any assumptions/approximations you used in your calculations. [6 points]

Correct assumptions: at max height, K=0, at min height U=0, K=max, small angle approximation therefore $U \propto x^2$ (or θ^2),assuming no damping (total energy stays constant) – any correct set of approximations relevant to the calculations

Multiple ways: e.g At max height $E_k = 0, U_g = mgy_o = mgL(1 - cos\theta_A)$, $T = 2\pi\sqrt{\frac{L}{g}}$

Small angle approximation, $1 - cos\theta_A = 1 - \left(1 - \frac{\theta_A^2}{2}\right)$

If remember, can use the potential energy for the pendulum with small θ_A (approx.)

$E_{k_{maximum}} = \frac{1}{2} I(\omega)^2, I = mL^2$

2. A disturbance on a string is described with an equation $y(x,t) = Y_o \sin(kx)\cos(\omega t + \phi_i)$. Determine whether this disturbance is a wave. [6 points]

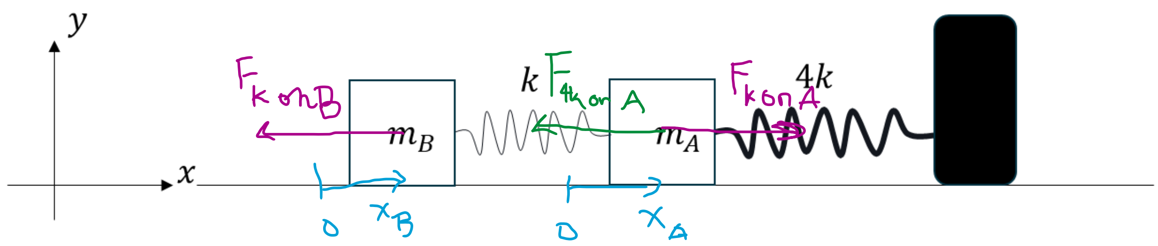
   Use wave equation:

$$\frac{\partial y}{\partial k} = Y_0 k \cos(kx)\cos(\omega t + \phi_i)$$
$$\frac{\partial^2 y}{\partial k^2} = -Y_0 k^2 \sin(kx)\cos(\omega t + \phi_i)$$
$$\frac{\partial y}{\partial t} = -Y_0 \omega \sin(kx)\sin(\omega t + \phi_i)$$
$$\frac{\partial^2 y}{\partial t^2} = -Y_0 \omega^2 \sin(kx)\cos(\omega t + \phi_i)$$

Compare second derivatives, with $\frac{\omega}{k} = v$

Question II

Consider a setup shown in the picture below: two identical masses m, m_B on the left and m_A on the right, connected to each other with a spring k. The mass on the right (m_A) is connected to a rigid support with a spring $4k$.



a. Assuming both masses are displaced from equilibrium of your choice by distances x_A and x_B, draw (in the picture) and clearly label the forces exerted on the masses by the spring. You can assume there is no dampening in the system and that the gravitational force on each mass is balanced by the normal force. [3 points]

b. Write the equation of motion of each mass. [4 points]

$$B:\ -k(x_B - x_A) = m_B \frac{d^2 x_B}{dt^2} = k x_B + k_{x_A}$$

$$A:\ -4k(x_A) + k(x_B - x_A) = m_A \frac{d^2 x_A}{dt^2} = +k x_B - 5k x_A$$

c. Determine the coefficient matrix for this system and the normal frequencies of the oscillation. [10 points]

$$x_A = A\cos(\omega t)\,, x_B = B\cos(\omega t)$$

A: $-m_A A\omega^2 \cos(\omega t) = -5kA\cos(\omega t) + kB\cos(\omega t)$

B: $-m_B B\omega^2 \cos(\omega t) = +kA\cos(\omega t) - kB\cos(\omega t)$

True for any time, and masses are identical ∴

A: $-mA\omega^2 = -5kA + kB$

B: $-mB\omega^2 = +kA - kB$

$$\begin{bmatrix} -5\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = -\omega^2 \begin{bmatrix} A \\ B \end{bmatrix} \rightarrow \begin{bmatrix} -5\frac{k}{m}+\omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m}+\omega^2 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = 0$$

$$det\begin{bmatrix} -5\frac{k}{m}+\omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m}+\omega^2 \end{bmatrix} = 0 \rightarrow \left(-5\frac{k}{m}+\omega^2\right)\left(-\frac{k}{m}+\omega^2\right) - \frac{k^2}{m^2} = 4\frac{k^2}{m^2} - 6\frac{k}{m}\omega^2 + \omega^4 = 0$$

$\omega^2 = \frac{6\frac{k}{m} \pm \sqrt{36\frac{k^2}{m^2} - 16\frac{k^2}{m^2}}}{2} = 3\frac{k}{m} \pm \sqrt{5}\frac{k}{m}$ (check, both positive)

$$\omega = \sqrt{3\frac{k}{m} + \sqrt{5}\frac{k}{m}} \text{ or } \sqrt{3\frac{k}{m} - \sqrt{5}\frac{k}{m}}$$

d. A driver applying force $F(t) = F_0\cos(\omega t)$ is attached to the left side of the set up. What are the amplitudes of the normal modes if the driving frequency is equal to the larger of the normal frequencies? [3 points]

$$q_1 = x_A + x_B, q_2 = x_A - x_B$$

$$q_1 = C_1\cos(\omega_1 t)\,, q_2 = C_2\cos(\omega_2 t)$$

q_1 is associated with the larger frequency as there is no dampening its amplitude will go infinity, while amplitude of q_2 will go to zero.

Question III

Signal $V_1(t) = V_A(t)\cos\left(\omega t - \frac{3}{4}\pi\right)$ is observed across a resistor $R = 50.0\ \Omega$ in RLC circuit that also includes capacitor $C = 2.0\ \mu\text{F}$ and inductor $L = 0.20\ \text{H}$. At time $t = 0$ $V_1(0) = 2.0\ \text{V}$.

a. What is the natural frequency of this circuit? [1 point]

$$\omega_0 = 1\sqrt{LC} = 1580\ \text{rad/s}$$

b. What is the damping factor γ? [1 point]

$$\gamma = \frac{R}{L} = \frac{50}{.2} = 250\frac{1}{s}$$

c. What type of behaviour would this system exibit? Briefly justify your answer. [2 points]

As $\frac{\omega_0}{\gamma} = 6.32$ (or $\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = 1575 \approx \omega_0 \rightarrow light\ dampening$

d. Determine the expressions for <u>current in the circuit</u> and <u>potential difference across the inductor</u>. You do not have to substitute all values, but you need to be very clear with the symbols you are using. [8 points]

Many ways to do it but:

$$i_R(t) = \frac{V_1(t)}{R} = V_1(t) = \frac{V_A(t)}{R}\cos\left(\omega t - \frac{3}{4}\pi\right)$$
$$V_{L(t)} = L\frac{di}{dt} = \frac{L}{R}\left(\frac{dV_A}{dt}\right)\cos\left(\omega t - \frac{3}{4}\pi\right) - \frac{L}{R}\omega V_A(t)\sin\left(\omega t - \frac{3}{4}\pi\right)$$

Using $V_A(t) = V_A\exp\left(-\frac{\gamma}{2}t\right)$ also correct

e. When the signal measured across the resistor is added a the second, unknown signal $V_2(t) = V_B(t)\cos(\omega t + \phi_B)$ a sinusoidal wave $V(t) = 3.0\ \text{V}\exp\left(-\frac{\gamma}{2}t\right)\sin(\omega t)$ is recorded. Determine the amplitude (at time $t = 0$) and the phase constant ϕ_B of the second wave. [8 points]

$$V(t) = 3.0\ V\exp\left(-\frac{\gamma}{2}t\right)\sin(\omega t) = 3.0\ V\exp\left(-\frac{\gamma}{2}t\right)\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$\tilde{V}(0) = 3\exp\left(-\mathbb{j}\left(-\frac{\pi}{2}\right)\right) = 3\cos\left(-\frac{\pi}{2}\right) + \mathbb{j}\,3\sin\left(-\frac{\pi}{2}\right) = -3\mathbb{j}$$
$$\tilde{V}_1(0) = 2\cos\left(-\frac{3}{4}\pi\right) + \mathbb{j}2\sin\left(-\frac{3}{4}\pi\right)$$
$$\tilde{V}_1 + \tilde{V}_2 = \tilde{V}$$

$$Re(\tilde{V}_2) = 0 - 2\cos\left(-\frac{3}{4}\pi\right) = 1.414$$
$$Im(\tilde{V}_2) = -3 - 2\cos\left(-\frac{3}{4}\pi\right) = -4.414$$

Amplitude: $4.37\ V$, angle is $\text{atan}\left(-\frac{4.414}{1.414}\right) \rightarrow 4th\ quadrant$

OSCILLATIONS

$\omega = 2\pi f = \frac{2\pi}{T}$	$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{mgd}{I}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$	
$x(t) = A\cos(\omega t + \phi_i)$	$x(t) = A_0 \exp\left(-\frac{\gamma t}{2}\right)\cos(\omega t + \phi_i))$	$x(t) = A(\omega)\cos(\omega t - \delta)$		
$x(t) = A\exp\left(-\frac{\gamma t}{2}\right) + Bt\exp\left(-\frac{\gamma t}{2}\right)$	$x(t) = A\exp\left(\left(-\frac{\gamma}{2} + \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right) + B\exp\left(\left(-\frac{\gamma}{2} - \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right)$			
$q_0(\omega) = \frac{\varepsilon_0}{\omega Z}$	$q(t) = q_0(\omega)\cos(\omega t - \delta)$	$Z = \sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$	$i = \frac{dq}{dt}$	
$V_R = i(t)R$	$V_C = \frac{q}{C}$	$V_L = L\frac{di}{dt}$		
$K = \frac{1}{2}mv^2$	$U = \frac{1}{2}kx^2$	$E(t) = E_0\exp(-\gamma t)$	$P = \frac{dE}{dt} = Fv$	
$Q = \frac{\omega_0}{\gamma}$	$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$	$\gamma = \frac{b}{m}$	$\gamma = \frac{R}{L}$	
$A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$	$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$			
$\bar{P}(\omega) = \frac{b[v_0(\omega)]^2}{2}$	$\bar{P}_{max} = \frac{F_0^2}{2m\gamma}$	$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma\left[\frac{4(\Delta\omega)^2}{\gamma^2} + 1\right]}$		

WAVES

$v = \lambda f$	$y(x,t) = f(x \pm vt)$	$y(x,t) = A\cos(kx \pm \omega t + \phi_i)$				
$k = \frac{2\pi}{\lambda}$	$y(x,t) = (A\sin(kx) + B\cos(kx))\cos(\omega t)$					
$v = \sqrt{\frac{F_T}{\mu}}$	$v = \sqrt{\frac{B}{\rho}}$	$v = \sqrt{\frac{Y}{\rho}}$	$v = \sqrt{\frac{\gamma RT}{M}}$	$\frac{\partial y^2}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$	$\omega = \frac{2\pi}{T}$	$f = \frac{1}{T}$
$\omega_n = \frac{n\pi v}{L}$	$\omega_n = \frac{n\pi v}{2L}$	$f_n = nf_1$	$E_n = \frac{1}{4}\mu\omega_n^2 A_n^2 L$			
$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$	$P_{ave} = \frac{1}{2}ZA^2\omega^2$	$I_{ave} = \frac{1}{2}Z_a A^2\omega^2$	$P = \mu v A^2\omega^2\sin^2(kx - \omega t + \phi_0)$			
$Z = \sqrt{\mu\tau}$	$Z_a = \sqrt{Y\rho}$	$Z_a = \sqrt{B\rho}$	$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$	$T = \frac{2Z_1}{Z_1 + Z_2}$		
$I(r) = I(r_0)\left[e^{-\alpha(r - r_0)}\right]\left(\frac{r_0}{r}\right)^{N-1}$	$v_g = \frac{d\omega}{dk}\Big\rvert_{k=k_0}$	$v_g = v - \lambda\frac{dv}{d\lambda}$	$R^2 + \frac{Z_2}{Z_1}T^2 = 1$			
$c = (\mu_0\varepsilon_0)^{-\frac{1}{2}}$	$v = \frac{c}{n}$		$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$			

MATHEMATICAL FORMULAE

$\cos\alpha + \cos\beta = 2\cos\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]$	$\cos\alpha - \cos\beta = -2\sin\left[\frac{\alpha+\beta}{2}\right]\sin\left[\frac{\alpha-\beta}{2}\right]$
$\sin\alpha + \sin\beta = 2\sin\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]$	$\sin\alpha - \sin\beta = 2\cos\left[\frac{\alpha+\beta}{2}\right]\sin\left[\frac{\alpha-\beta}{2}\right]$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$	$\tan^{-1}(x) = \{\theta, \theta + \pi\} + 2\pi n$ $\cos^{-1}(x) = \pm\theta + 2\pi n$
$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$	$\sin^{-1}(x) = \{\theta, \pi - \theta\} + 2\pi n$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$	$\tilde{A} = Ae^{j\theta} = A(\cos\theta + j\sin\theta)$
$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)$	$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

CONSTANTS

$c = 3.00 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T}\cdot\frac{\mathrm{m}}{\mathrm{A}}$	$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{N}\cdot\frac{\mathrm{m}}{\mathrm{C}^2}$	$g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}}$
$v_{sound\ at\ 20℃} = 343\,\frac{\mathrm{m}}{\mathrm{s}}$	$T_K = T_{℃} + 273.15℃$		