University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – December, 2019
DURATION: 2 and 1/2 hrs

SECOND YEAR – ENGINEERING SCIENCE
AER210F VECTOR CALCULUS and FLUID MECHANICS
Examiner: A. Ekmekci

Instructions:
(1) Closed book examination; except for a non-programmable calculator, no aids are permitted.
(2) Write your name and student number in the space provided below.
(3) Answer as many questions as you can. Parts of questions may be answered.
(4) Questions are NOT assigned equal marks.
(5) Use the overleaf side of pages for additional or preliminary work.
(6) Do not separate or remove any pages from this exam booklet.
(7) You may use $g = 10$ m/s^2, $\rho_{water} = 1000$ kg/m^3 where appropriate.

Family Name:______________________________

Given Name:______________________________

Student Number:______________________________

FOR MARKER USE ONLY		
Question	Mark	Earned
1	12	
2	10	
3	8	
4	10	
5	10	
6	12	
7	13	
8	9	
9	7	
10	9	
11	8	
12	12	
13	10	
TOTAL	130	

The following integrals and formulae may be useful:

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \quad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C;$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA; \quad \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS; \quad \oiint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

$$c = \sqrt{\gamma RT}; \quad \tau = \mu\frac{du}{dy}; \quad E_V = -dP/(d\forall/\forall); \quad M = V/c; \quad dA/A = -(1-M^2)dV/V; \quad Re = \frac{\rho VD}{\mu}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z}; \quad -\nabla p + \rho\vec{g} = \rho\vec{a}; \quad \frac{p}{\rho} + \frac{V^2}{2} + gz = constant \text{ (Bernoulli eqn).}$$

$$\frac{\partial}{\partial t}\oiiint_{\forall} \rho\, d\forall + \oiint \rho\vec{V}\cdot\overrightarrow{dA} = 0\ ; \ \frac{\partial\rho}{\partial t} + \nabla\cdot\left(\rho\vec{V}\right) = 0\ ; \ \frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = 0 \text{ (Different forms of continuity)}$$

$$\Sigma\overrightarrow{F_{CV}} = \frac{d}{dt}\iiint_{CV}\vec{V}(\rho d\forall) + \oiint_S \vec{V}(\rho\vec{V}\cdot\overrightarrow{dA}); \quad \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V} = \rho\frac{D\vec{V}}{Dt} \text{ (Navier-Stokes eqn.)}$$

1) a) [4 marks] Evaluate the following integral by reversing the order of integration. Provide a sketch of the region.

$$\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\, dy dx$$

b) [5 marks] Forming the appropriate double integral, find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x + y = 2$, $x = 0$, $y = 0$ and $z = 0$.

d) [3 marks] Change the order of integration to show that:

$$\int_0^1 \int_0^t f(u)\, du dt = \int_0^1 (1-u) f(u)\, du$$

2) a) [4 marks] Show that the following line integral is independent of path, then find a potential function and evaluate the integral using this potential function along any path C from (2,0) to $(1, \pi)$:

$$\int_C \sin y \, dx + (x\cos y - \sin y)dy$$

b) [3 marks] If $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in a domain D, prove that $\int_{C_1} \vec{F} \cdot d\vec{r}$ has the same value for any piecewise smooth curve C_1 in D between two fixed points (say points P_o and P_1).

2) c) [3 marks] Evaluate the line integral $\int_C (3x - y)ds$, where C is the line segment from point $(1, 2)$ to point (2, 3).

3) [8 marks] Using an appropriate coordinate transformation, evaluate the following integral

$$\iint_R y^2 dA$$

where R is the planar region bounded by the curves: $y = 1/x$, $y = 2/x$, $y^2 = 1/x$, $y^2 = 2/x$. Provide a sketch of the region.

4) [10 marks] Use a triple integral in cylindrical coordinates to find the volume of the solid region R inside the sphere $x^2+y^2+z^2 = 6$ and above the paraboloid $z = x^2+y^2$.

5) (10 marks) Verify Stoke's theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n}\, dS$ for the vector field

$$\vec{F}(x, y, z) = z\vec{\imath} + 2xz\,\vec{\jmath} + xy\,\vec{k}$$

over the part of the plane 2x + 4y + z = 8 in the first octant. Provide a sketch of the surface and the boundary curve.

6) (a) [2 marks] Show that the velocity vector field $\vec{V} = x\vec{\imath} + y\vec{\jmath} + (x^2 - 2z)\vec{k}$ satisfies the differential form of the continuity equation for an incompressible flow.

(b) [10 marks] By integrating over the closed surface formed by $z = 9 - x^2 - y^2$ and the plane $z = 0$, show that the velocity field also satisfies the integral form of continuity for incompressible flows:

$$\oiint_S \vec{V} \cdot \vec{n} \, dS = 0$$

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7) a) [9 marks] Indicate whether the statement is True (T) or False (F):

___ Sound travels faster in hot summer days compared to cold winter days.

___ As water flows upwards in a pipe of uniform diameter, its velocity decreases because of the influence of gravity.

___ A Pitot tube works on the principle of converting kinetic energy to potential energy.

___ A stagnation point occurs where the pressure becomes a minimum.

___ When a supersonic flow enters a converging duct, it decelerates at the exit of the duct.

___ Euler's equation arises when inertial effects are ignored, but viscous forces are included.

___ A floating body will be stable when its centre of gravity is above its centre of buoyancy.

___ In a flow where the velocity field is $\vec{V} = u\vec{i} + v\vec{j}$, if the only non-zero velocity gradient is $\frac{\partial u}{\partial x}$ then the fluid elements would undergo a volume change as the elements move from one location to another.

___ Consider a fisherman in a boat, which is anchored in a river some distance directly upstream of the point where a disturbance is creating free-surface waves in the river. The Froude number of the flow is more than 1. The waves created by the disturbance should eventually rock this boat.

b) [4 marks] You are the pilot of a hot air balloon which has a diameter of 25 m, and a total mass of 250 kg (including yourself). Unfortunately, the burner fails, the air in the balloon cools, the balloon loses its buoyancy and soon reaches a steady downward terminal velocity. (The spherical shape of the balloon is maintained during this fall). While you are prepared to endure a 3 m/s (11 km/hr) crash, if the terminal speed is any faster than 3 m/s (11 km/hr) you will use an emergency parachute. Having watched all the films on drag, you know that the drag coefficient ($C_D = (F_D/A)/(\frac{1}{2}\rho V^2)$, where A = projected area, F_D = drag force, $\rho = \rho_{air}$) of spheres at very high Reynolds number is about 0.2.

- Determine the terminal falling speed of the parachute? [2 marks]
- Indicate if the assumption of very-high-speed Reynolds number is reasonable. [1 mark]
- Based on your finding, will you use your parachute? [1 mark]

($\rho_{air} = 1$ kg/m^3, $\mu_{air} = 1.79 \times 10^{-5}$ Ns/m^2, g = 10 m/s^2, $Re = \frac{\rho V D}{\mu}$)

8) [9 marks] An Eulerian velocity vector field is described by $\vec{V} = 3xz\vec{j} + y\vec{k}$, where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors in the x-, y- and z- directions, respectively.

a) [1 mark] Is the flow one-, two- or three-dimensional?

b) [1 mark] What is the acceleration vector field?

c) [4 marks] If the gravity and viscous forces can be neglected, determine the pressure gradient in the x, y, and z directions.

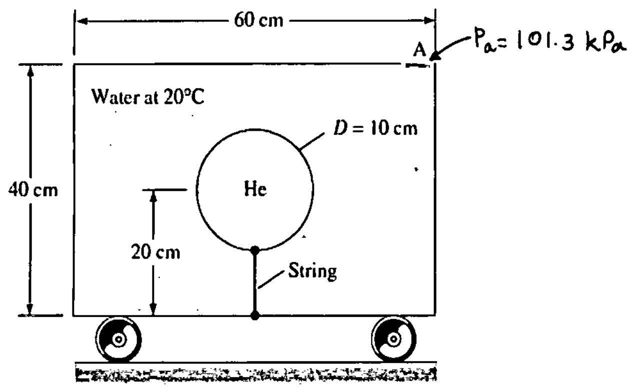
d) [1 mark] Can you define the stream function? If yes, determine the stream function. If not, explain the reason why?

e) [2 marks] Find the vorticity and indicate if this flow is rotational or irrotational.

9) [7 marks] The tank in the figure below is filled fully with water and has a vent hole at point A, which is open to atmospheric pressure P_a. The tank is 1 m wide into the paper. Inside the tank, a 10-cm balloon, filled with helium at 130 kPa, is tethered centrally by a string. If this tank accelerates to the right at 5 m/s^2 in rigid-body motion, neglecting the weight of the balloon's material:

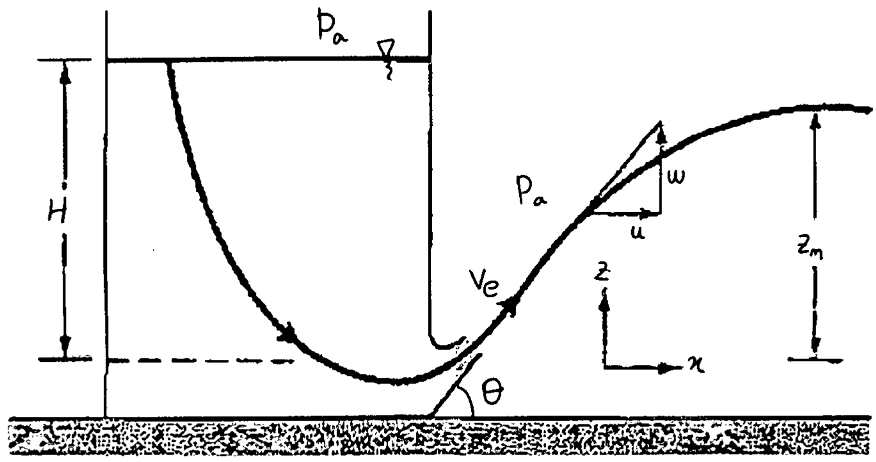
a) At what angle will the balloon lean? [5 marks]
b) Will it lean to the right or to the left? [2 marks]

The gravitational acceleration is $g = 10$ m/s^2.

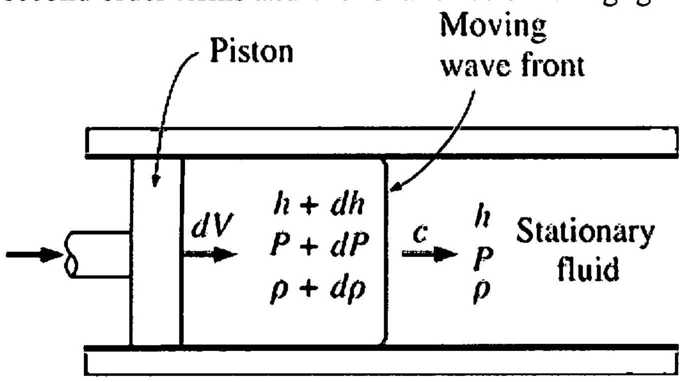


10) [9 marks] Consider a tank draining through a small orifice, where the orifice outlet points up at an angle as shown in the figure below. As the jet issues into the atmosphere, the vertical component of the velocity, w, decreases under the action of gravity. At the top of the jet trajectory $w = 0$. On the other hand, the horizontal component of the fluid velocity, u, remains constant throughout the trajectory when air friction is neglected, because the only force acting is due to gravity in the vertical direction. Neglecting the frictional effects, find z_m, the maximum height to which the jet rises in terms of H and θ.

Note that in the picture, H is the distance from the free surface of the water in the tank to the orifice exit, P_a is the atmospheric pressure, V_e is the exit velocity at the orifice, and z_m is the distance between the highest point the jet reaches and the orifice exit.



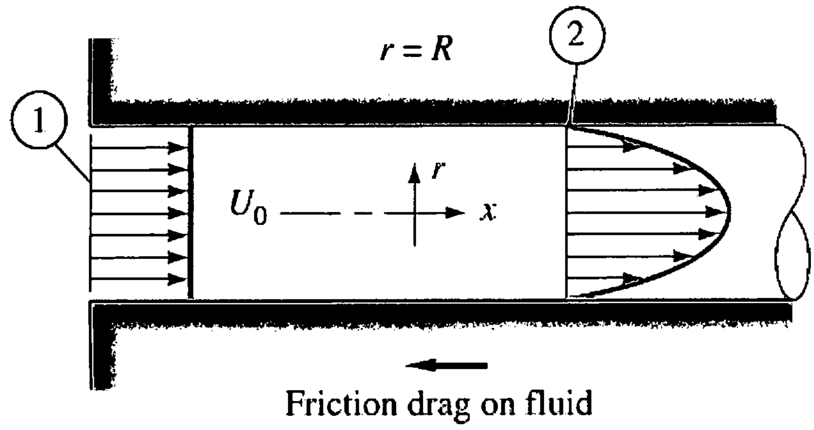
11) [8 marks] Consider a duct that is filled with a fluid at rest. A piston, fitted in the duct as shown in the figure, is initially at rest. This piston suddenly starts moving at a constant incremental velocity, dV, to the right, creating a weak pressure pulse moving at the speed of sound, c, through the stationary fluid. Applying the continuity and momentum equations to a control volume enclosing the wave front and moving with the speed of sound, derive an expression for the speed of sound. Assume that the second order terms and frictional effects are negligible during your derivation.



12) (12 marks) Consider incompressible flow in the entrance of a circular tube, as in the figure below. The inlet flow is uniform, $u_1 = U_0$. The flow at section 2 is fully developed pipe flow. Find the wall drag force F as a function of p_1, p_2, ρ, U_0, R if the flow velocity at section 2 is

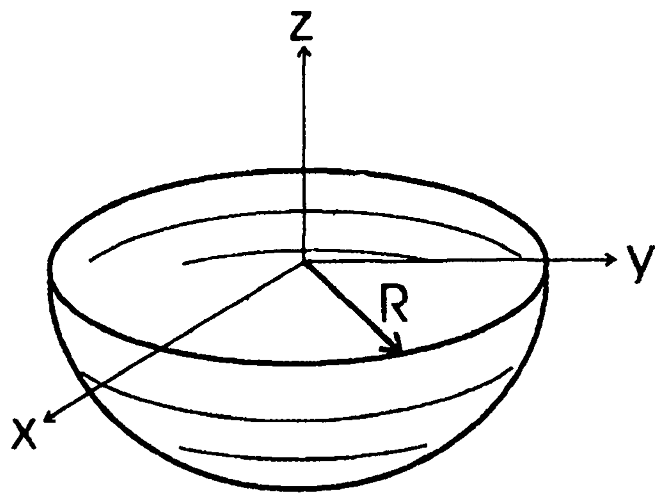
$$u_2 = u_{max}\left(1 - \frac{r^2}{R^2}\right)$$

Here, R shows the radius of the circular pipe, ρ is the fluid density, and p_1 and p_2 are the uniformly distributed pressures at sections 1 and 2, respectively.



13) [10 marks] An open container in the form of a hemisphere of radius R, as depicted in the diagram below, is fully filled with liquid of density ρ. By direct integration of the pressure forces acting on the surface of the container, find the resultant pressure force, $\overrightarrow{F_P}$, acting on the container.

(Note that during this calculation, the use of any method other than the direct integration of the pressure forces will not receive any credit.)



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