AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 2

20 October 2014 8:45 am - 9:45 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis.

Given Name: Solutions

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	12	
3	8	
4	8	
5	14	
TOTAL	52	/ 48

Note: The following integrals may be useful.

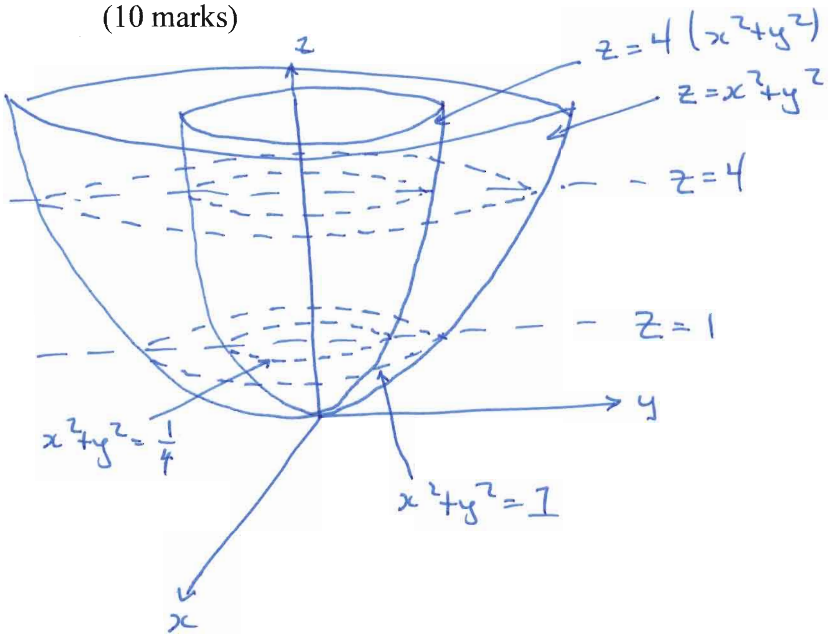
$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \qquad \oiint_S \bar{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \bar{F} \, dV; \qquad \oint_C \bar{F} \cdot d\bar{r} = \iint_S \nabla \times \bar{F} \cdot d\bar{S}$$

1) Use the coordinate transformation: $x = \frac{u}{v}\cos\theta,\ y = \frac{u}{v}\sin\theta,\ z = u^2$, to evaluate the triple integral $I = \int_V \frac{dV}{x^2 + y^2}$, where V is the volume that lies between the paraboloids $z = x^2 + y^2$, $z = 4(x^2 + y^2)$ and between the planes $z = 1$, $z = 4$. Provide a sketch of the volume.

Hint: While the limits for θ and u are easily found, the bounds for v are not so obvious. Consider the traces of the paraboloids in the $z = 1$ or $z = 4$ planes to help determine the limits on v.

(10 marks)



$0 \le \theta \le 2\pi$

$1 \le u \le 2$

on $z = 1$ $\therefore u = 1$

$\Rightarrow 1 = x^2 + y^2 \quad \therefore v = 1$

$\Rightarrow 1 = 4(x^2+y^2) \quad \therefore v = 2$

$\therefore 1 \le v \le 2$

$$\frac{\partial(x,y,z)}{\partial(u,v,\theta)} = \begin{vmatrix} \frac{1}{v}\cos\theta & -\frac{u}{v^2}\cos\theta & -\frac{u}{v}\sin\theta \\ \frac{1}{v}\sin\theta & -\frac{u}{v^2}\sin\theta & \frac{u}{v}\cos\theta \\ 2u & 0 & 0 \end{vmatrix} = 2u\left(-\frac{u^2}{v^3}\cos^2\theta - \frac{u^2}{v^3}\sin^2\theta\right)$$

$$= -\frac{2u^3}{v^3}$$

$$\therefore I = \int_0^{2\pi} d\theta \int_1^2 du \int_1^2 dv \left|-\frac{2u^3}{v^3}\right| \frac{1}{u^2/v^2} = 4\pi \int_1^2 u\,du \int_1^2 \frac{dv}{v}$$

$$= 4\pi \left[\frac{u^2}{2}\right]_1^2 \left[\ln v\right]_1^2 = 4\pi\left(2 - \frac{1}{2}\right)(\ln 2 - 0)$$

$$= 6\pi \ln 2$$

2) Evaluate the line integrals:

a) $\int_C x^2\,dx + y^2\,dy + z^2\,dz$, where C consists of the line segment from $(1,2,-1)$ to $(3,2,0)$.

b) $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = \sin x\,\hat{i} + \cos y\,\hat{j} + xz\,\hat{k}$ and $C\colon \bar{r}(t) = t^3\,\hat{i} - t^2\,\hat{j} + t\,\hat{k}$, $0 \le t \le 1$.

c) $\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = e^y\,\hat{i} + xe^y\,\hat{j} + (z+1)e^z\,\hat{k}$ and $C\colon \bar{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$, $0 \le t \le 1$.

(12 marks)

a) parameterize line: $\bar{r}(t) = (1+2t,\ 2,\ t-1)$ $\quad 0 \le t \le 1$

$$I = \int_0^1 (1+2t)^2 \cdot 2dt + 4\cdot 0dt + (t-1)^2 \cdot dt$$

$$= \int_0^1 \left[2(1+4t+4t^2) + (t^2-2t+1)\right]dt = \int_0^1 (9t^2+6t+3)\,dt$$

$$= \left[3t^3 + 3t^2 + 3t\right]_0^1 = 9$$

b) $$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (\sin t^3,\ \cos(-t^2),\ t^3\cdot t)\cdot(3t^2dt,\ -2tdt,\ dt)$$

$$= \int_0^1 (3t^2\sin t^3 - 2t\cos(-t^2) + t^4)\,dt$$

$$= \left[-\cos t^3 + \sin(-t^2) + \frac{t^5}{5}\right]_0^1 = \left[1 - \cos 1 + \sin(-1) + \frac{1}{5}\right)$$

$$= \frac{6}{5} - \cos 1 - \sin 1$$

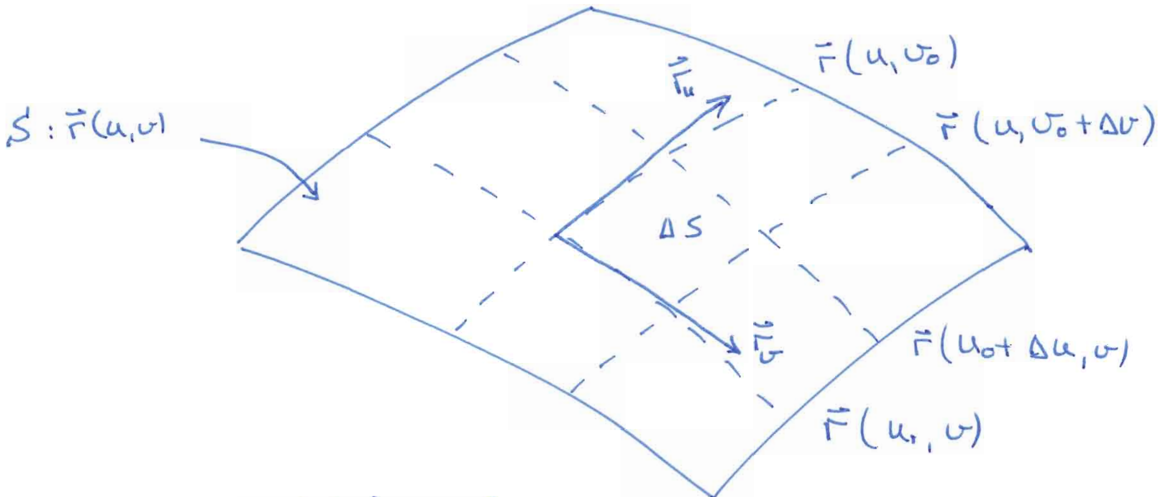
c) at $t=0$: $\bar{r}(0) = (0,0,0)$ $\qquad$ at $t=1$ $\bar{r}(1) = (1,1,1)$

let $f = xe^y + ze^z$ $\Rightarrow \nabla f = (e^y,\ xe^y,\ e^z + ze^z) = \bar{F}$

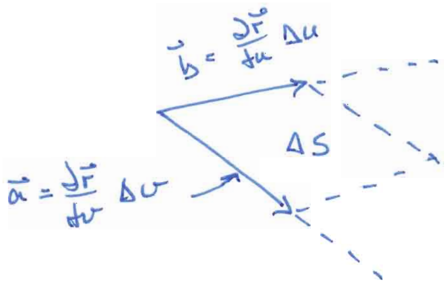
$$\therefore \int_C \bar{F}\,d\bar{r} = f(1,1,1) - f(0,0,0) = e + e - 0 - 0 = 2e$$

3) Given a surface defined parametrically by $\bar{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$, show that the surface area can be found from: $\int_S dS = \iint \|\bar{r}_u \times \bar{r}_v\| du\,dv$.

(8 marks)



We assume ΔS is small enough to be considered planar



$$\vec{a} = \vec{r}(u, v_0 + \Delta v) - \vec{r}(u, v_0)$$
$$\simeq \frac{\partial \vec{r}}{\partial v} \Delta v = \vec{r}_v \Delta v$$

$$\vec{b} = r(u_0 + \Delta u, v) - \vec{r}(u_0, v)$$
$$\simeq \frac{\partial \vec{r}}{\partial u} \Delta u = \vec{r}_u \Delta u$$

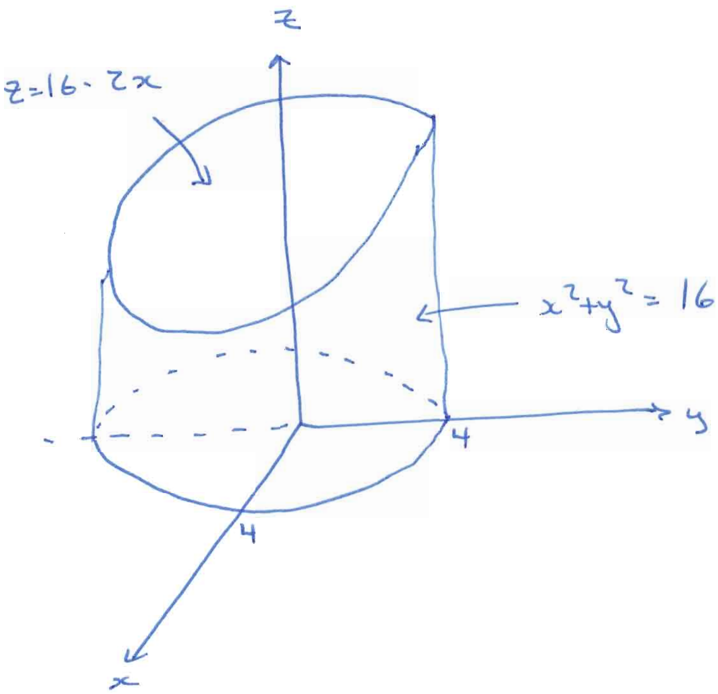
Area of parallelogram : $\Delta S \simeq \| \vec{a} \times \vec{b} \| = \| \vec{r}_v \times \vec{r}_u \| \Delta u \Delta v$

As $\Delta u, \Delta v \to 0$, $\Delta S \to dS = \| \vec{r}_u \times \vec{r}_v \| du dv$

$\therefore$ total surface area $= \int_S dS = \int_S \| \vec{r}_u \times \vec{r}_v \| du dv$

4) Use a parametric representation of the surface to find the surface area of the cylinder $x^2 + y^2 = 16$ between the planes $z = 0$ and $z = 16 - 2x$. Provide a sketch of the area.

(8 marks)



let $x = 4\cos\theta$

$y = 4\sin\theta$

$z = z$

$0 \le \theta \le 2\pi$

$0 \le z \le 16 - 2x$

$= 16 - 8\cos\theta$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -4\sin\theta & 4\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 4\cos\theta\,\hat{i} + 4\sin\theta\,\hat{j} + 0\,\hat{k}$$

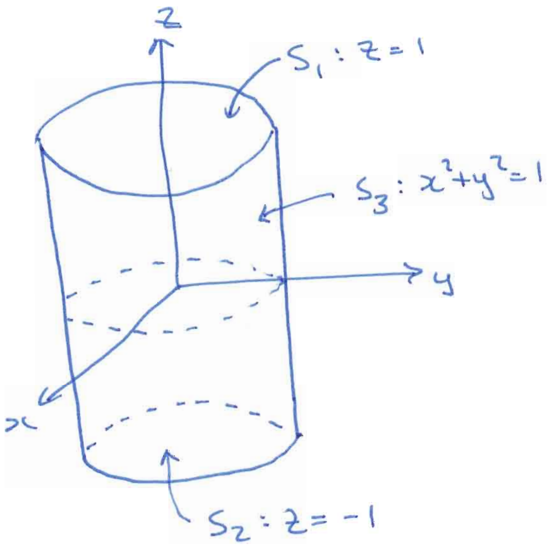
$$\therefore \|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{16\cos^2\theta + 16\sin^2\theta} = 4$$

$$\therefore S = \int_0^{2\pi} d\theta \int_0^{16-8\cos\theta} 4\,dz = 4\int_0^{2\pi} (16 - 8\cos\theta)\,d\theta$$

$$= 4\left[16\theta - 8\sin\theta\right]_0^{2\pi} = 128\pi$$

5) Verify the divergence theorem for the vector field $\bar{F}(x,y,z) = xyz\hat{i} + x^2y\hat{j} + x^2z\hat{k}$ and S is the surface formed by the cylinder $x^2 + y^2 = 1$ and the planes $z = -1$ and $z = 1$.

(14 marks)



$$\int_V \nabla\cdot\vec{F}\,dV = \int_S \vec{F}\cdot\hat{n}\,dS$$

a) surface integral

$S_1 : \hat{n} = \hat{k},\ z = 1,\ x^2+y^2 \le 1$

$\Rightarrow \vec{F}\cdot\vec{n} = x^2 z = x^2$

$$\int_{S_1} \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi} d\theta \int_0^1 r\,dr \cdot r^2\cos^2\theta$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^1 = \boxed{\frac{\pi}{4}}$$

$S_2 : \hat{n} = -\hat{k},\ z = -1;\ \vec{F}\cdot\hat{n} = -x^2 z = x^2;\ x^2+y^2 \le 1$

$$\int_{S_2} \vec{F}\cdot\hat{n}\,dS = \int_0^{2\pi} d\theta \int_0^1 r\,dr\cdot r^2\cos^2\theta = \boxed{\frac{\pi}{4}}$$

S_3: let $\vec{r}(u,v) = (\cos u, \sin u, v)$ $\quad 0 \le u \le 2\pi;\ -1 \le v \le 1$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos u\,\hat{i} + \sin u\,\hat{j} + 0\,\hat{k}$$

$$\therefore \int_{S_3} \vec{F}\cdot\vec{N}\,du\,dv = \int_0^{2\pi} du \int_{-1}^1 dv\,(\cos u\sin u\,v,\ \cos^2 u\sin u,\ \cos^2 u\,v)\cdot(\cos u, \sin u, 0)$$

$$= \int_0^{2\pi} du \int_{-1}^1 dv\,(v\cos^2 u\sin u + \cos^2 u\sin^2 u) = \int_0^{2\pi} du\left[\cos^2 u\sin u\,\frac{z^2}{2} + \cos^2 u\sin^2 u\,z\right]_{-1}^1$$

$$= 2\int_0^{2\pi} \cos^2 u\sin^2 u\,du = 2\int_0^{2\pi}\frac{1}{4}\sin^2 2u = \frac{1}{2}\left[\frac{u}{2} - \frac{1}{4}\sin 4u\right]_0^{2\pi} = \boxed{\frac{\pi}{2}}$$

$$\Rightarrow \int_S \vec{F}\cdot\hat{n}\,dS = \frac{\pi}{4} + \frac{\pi}{4} + \frac{\pi}{2} = \pi$$

b) volume integral

$$\vec{F} = (xyz,\ x^2y,\ x^2z) \Rightarrow \nabla\cdot\vec{F} = yz + x^2 + x^2 = yz + 2x^2$$

Cylindrical coordinates: $x = r\cos\theta$ $\quad 0 \le r \le 1 \quad dV = r\,dr\,d\theta\,dz$

$y = r\sin\theta \quad 0 \le \theta \le 2\pi$

$z = z \quad -1 \le z \le 1$

$$\int_V \nabla\cdot\vec{F}\,dV = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{-1}^1 dz\,(rz\sin\theta + 2r^2\cos^2\theta)$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr \left[\frac{z^2}{2} r\sin\theta + z\cdot 2r^2\cos^2\theta\right]_{-1}^1$$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr\,(4r^2\cos^2\theta)$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} 4\left[\frac{r^4}{4}\right]_0^1$$

$$= \boxed{\pi}$$

$$= \int_S \vec{F}\cdot\hat{n}\,dS$$