Problem 1: The velocity components for a certain incompressible, steady flow field are

$$u = x^2 + y^2 + z^2$$

$$v = xy + yz + z$$

$$w = ?$$

Determine the form of z component of velocity required to satisfy the continuity equation.

Solution: Any physically possible velocity distribution must for an incompressible fluid satisfy conservation of mass as expressed by the continuity equation as:

$$\vec{\nabla} \cdot \vec{V} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For the given velocity distribution:

$$\frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial v}{\partial y} = x + z$$

So, the required expression for $\frac{\partial w}{\partial z}$ is

$$\frac{\partial w}{\partial z} = -2x - x - z = -3x - z$$

Integration with respect to z yields:

$$w = -3xz - \frac{z^2}{2} + f(x, y)$$

The third velocity component cannot be explicitely determined since the function $f(x,y)$ can have any form and conservation of mass will still be satisfied. The specific form of this function will be governed by the flow field described by these velocity components - that is, some additional information is needed to completely determine w.

Problem 2: Consider the steady, two-dimensional flow field given as:

$$\vec{V} = \left(\frac{V_0}{\ell}\right)(x\hat{i} - y\hat{j})$$

Determine the acceleration field for this flow.

Solution: In general, acceleration is substantial derivative of velocity vector:

$$\vec{a} = \frac{D\vec{V}}{Dt} = \underbrace{\cancel{\frac{\partial \vec{V}}{\partial t}}}_{\text{(steady)}} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \underbrace{\cancel{w}\cancel{\frac{\partial \vec{V}}{\partial z}}}_{\text{(2-D flow)}}$$

$$\vec{V} = \left(\frac{V_0}{\ell}\right)(x\hat{i} - y\hat{j}) \Rightarrow \quad u = \frac{V_0}{\ell}x \quad \text{and} \quad v = -\left(\frac{V_0}{\ell}\right)y$$

For steady flow $\Rightarrow \dfrac{\partial(.)}{\partial t} = 0$

For two-dimensional flow $\Rightarrow w = 0$ and $\dfrac{\partial(.)}{\partial z} = 0$

$$\vec{a} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j}$$

$$= \left[\left(\frac{V_0}{\ell}\right)(x)\left[\frac{V_0}{\ell}\right) + \left(-\frac{V_0}{\ell}\right)(y)(0)\right]\hat{i} + \left[\left(\frac{V_0}{\ell}\right)(x)(0) + \left(-\frac{V_0}{\ell}\right)(y)\left(-\frac{V_0}{\ell}\right)\right]\hat{j}$$

$$\vec{a} = \frac{V_0^2}{\ell^2}x\hat{i} + \frac{V_0^2}{\ell^2}y\hat{j}$$

$$a_x = \frac{V_0^2}{\ell^2}x \quad ; \quad a_y = \frac{V_0^2}{\ell^2}y$$

The fluid experiences acceleration in both x and y direction. Since the flow is steady, there is no local acceleration, that is, the fluid velocity at any given point is constant in time. However, there is convective acceleration due to the change in velocity from one point on the particle's pathline to another.

Problem 3: Consider a steady velocity field given by $\vec{V} = (u, v, w) = a(x^2y + y^2)\hat{i} + bxy^2\hat{j} + cx\hat{k}$ where a, b, and c are constants. Under what conditions is this flow field incompressible?

Solution:

If the flow is incompressible $\Rightarrow \vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \underbrace{\frac{\partial u}{\partial x}}_{2axy} + \underbrace{\frac{\partial v}{\partial y}}_{2bxy} + \underbrace{\frac{\partial w}{\partial z}}_{0} = 0$

$$\Downarrow$$

$$2axy + 2bxy = 0$$

Thus, to guarantee incompressibility, constraints a and b must be equal in magnitude but opposite in sign:

$\therefore$ Condition for incompressibility $\Rightarrow \boxed{a = -b}$

The velocity field is given by the formula:

$$\vec{V} = 4tx\hat{\imath} - 2t^2y\hat{\jmath} + 4xz\hat{k}$$

- Is this flow fields steady or unsteady?
- Is the flow two- or three- dimensional?
- At the point (x,y,z) = (-1, +1, 0), compute the acceleration vector.

- Flow is unsteady as $\vec{V}$ is a function of t.
- Flow is three dimensional as $\vec{V}$ is a function of x, y & z.

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\boxed{\frac{\partial \vec{V}}{\partial t} = (4x)\hat{\imath} - (4ty)\hat{\jmath}}$$

$$\frac{\partial \vec{V}}{\partial x} = 4t\hat{\imath} + 4z\hat{k}; \quad u\frac{\partial \vec{V}}{\partial x} = (4tx)(4t\hat{\imath} + 4z\hat{k}) = \boxed{(16t^2x)\hat{\imath} + (16txz)\hat{k} = u\frac{\partial \vec{V}}{\partial x}}$$

$$\frac{\partial \vec{V}}{\partial y} = -2t^2\hat{\jmath}; \quad v\frac{\partial \vec{V}}{\partial y} = (-2t^2y)(-2t^2)\hat{\jmath} = \boxed{4t^4y\hat{\jmath} = v\frac{\partial \vec{V}}{\partial y}}$$

$$\frac{\partial \vec{V}}{\partial z} = 4x\hat{k}; \quad w\frac{\partial \vec{V}}{\partial z} = (4xz)(4x)\hat{k} = \boxed{16x^2z\hat{k} = w\frac{\partial \vec{V}}{\partial z}}$$

$$\vec{a} = \frac{D\vec{V}}{Dt} = (4x + 16t^2x)\hat{\imath} + (-4ty + 4t^4y)\hat{\jmath} + (16txz + 16x^2z)\hat{k}$$

at $(x,y,z) = (-1,+1,0) \Rightarrow \boxed{\vec{a} = (-4 - 16t^2)\hat{\imath} + (4t^4 - 4t)\hat{\jmath} + 0\hat{k}}$