UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Division of Engineering Science
The Edward S. Rogers Sr. Department of
Electrical and Computer Engineering

UofT

ECE 259H1S, ELECTRICITY AND MAGNETISM

Final Exam

Thursday, April 19, 2012, 9:30 am-12 noon

Examiners: Piero Triverio and Costas Sarris

Calculator Type: 2
All non-programmable electronic calculators are allowed.

Exam Paper Type: A
Closed book, no aid sheets.

Marks for each question are shown. All questions are independent. Only answers that are fully justified will be given full credit.

ANSWERS SHOULD BE WRITTEN IN THE SPACE BELOW EACH QUESTION.

NAME ____

STUDENT # ____

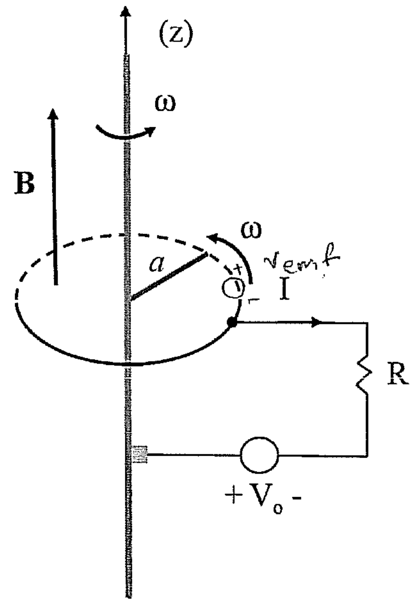
MARKS

Q.1	Q.2	Q.3	Q.4	TOTAL
/30	/20	/15	/15	/80

GOOD LUCK!

Question 1 [30 pts]

A. An electric motor consists of a metallic rod of length a, which is rotating around the z-axis with angular frequency ω within a constant magnetic field $\mathbf{B}=B_0\mathbf{a}_z$. The one end of the rod is connected to a rotating axis, and the other end is sliding along a circular metallic rail ($r=a$), as shown in the figure below. The rail is also connected to an external voltage source V_0 and a resistor R.

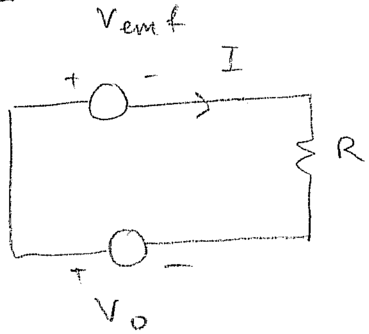


1. Find the current I shown in the figure, using Faraday's law: $V_{emf} = -\frac{d\Phi(t)}{dt}$. **[6 pts]**

$$\underbrace{\Phi(t) = B \cdot S(t)}_{1\text{ pt}} = B \cdot \frac{\overbrace{a^2}^{1\text{pt}}\varphi}{2} \Rightarrow d\Phi/dt = +\frac{Ba^2\omega}{2} \Rightarrow$$

$$\left.V_{emf} = -\frac{Ba^2\omega}{2}\right\} 2\text{pts}$$

Circuit:



$$V_{emf} + IR - V_0 = 0 \Rightarrow$$

$$\left.I = \frac{V_0 - V_{emf}}{R}\right\} 2\text{pts} \Rightarrow$$

$$I = \frac{V_0}{R}\left[1 + \frac{Ba^2\omega}{2V_0}\right].$$

2. Repeat the calculation of the current I, now employing the formula for the motional electromotive force: $V_{\text{emf, motional}} = \oint_C \boldsymbol{\upsilon}\times\mathbf{B}\cdot d\boldsymbol{\ell}$. **[6 pts]**

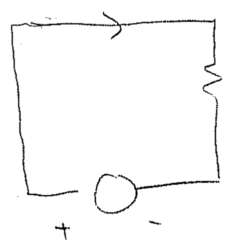
$$V_{emf} = \oint_C (\bar{\upsilon}\times\bar{B})\cdot d\bar{\ell} = \int_{r=a}^{0 \,\leftarrow\, \text{integral over rod} \quad 2\text{pts}} (\overbrace{\omega r\, \bar{a}_\varphi}\times \underbrace{\bar{a}_\varphi \times \bar{a}_z}_{\bar{a}_r} B_0)\cdot \underbrace{d\bar{\ell}}_{\bar{a}_r\, dr} \Big\} 1\text{pt}$$

$$= \omega B_0 \int_a^0 r\, dr = -\omega B_0 a^2/2 \Big\} 1\text{pt}$$

Again, $\underbrace{I = \dfrac{V_0 - V_{emf}}{R}}_{2\text{pts}} = \dfrac{-V_0}{R}\left[1 + \dfrac{Ba^2\omega}{2V_0}\right]$.

3. Draw and explain the direction of the induced current flowing in the loop (i.e. the current that would flow if V_0=0) in terms of Lenz's law. **[3 pts]**

Direction is:



Since I is positive, hence consistent with the direction shown in the figure:

$$I_{ind} = \frac{Ba^2 w}{2R}$$

$\underbrace{\text{Note that this direction creates a field opposing the external } \bar{B}\text{, consistent}}_{2\text{pts}}$ w/ Lenz law $\underbrace{(d\Phi/dt > 0 \text{ always})}_{1}$.

B. The following questions are independent from each other.

1. In a lossy dielectric medium with dielectric permittivity ε and conductivity σ, the (conduction) electric current density is $J_0 \cos(\omega t)\, \mathbf{a}_x$. Write the expression of the displacement current in the same medium. **[3 pts]**

← 1pt

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\epsilon \bar{J}}{6}\right) = \frac{\epsilon}{6}(-\omega) J_0 \sin \omega t\, \bar{a}_x$$

↑ 1pt

$$= -\frac{\omega \epsilon}{6} J_0 \sin \omega t\, \bar{a}_x$$ 1pt for setting up formula.

2. Can the displacement current exist in vacuum? Explain. **[3 pts]**

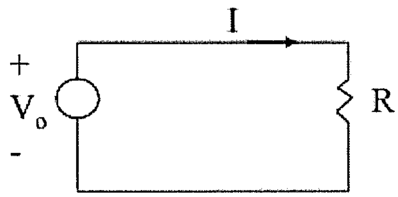
Yes, $\overbrace{\bar{J}_D = \frac{\partial \bar{D}}{\partial t}}^{1\,pt}$, as long as $\underbrace{\bar{D} = \bar{D}(t)}_{2\,pts}$, there will be $\bar{J}_D$.

3. In magnetostatics, we showed that the normal components of the magnetic flux are continuous at the interface between two media, i.e. $\mathbf{a}_n \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$. Is that generally true for time-varying magnetic fields? Explain. **[3 pts]**

Yes, because the condition is based on $\oint \bar{B} \cdot d\bar{s} = 0$ [2pts)

which holds in general.
[1pt]

4. Consider the following electric circuit. Is it always true that V_0=IR (Kirchhoff's voltage law)? **[3 pts]**



No, if there is V_{emf}, due to $d\Phi/dt$ through the circuit.

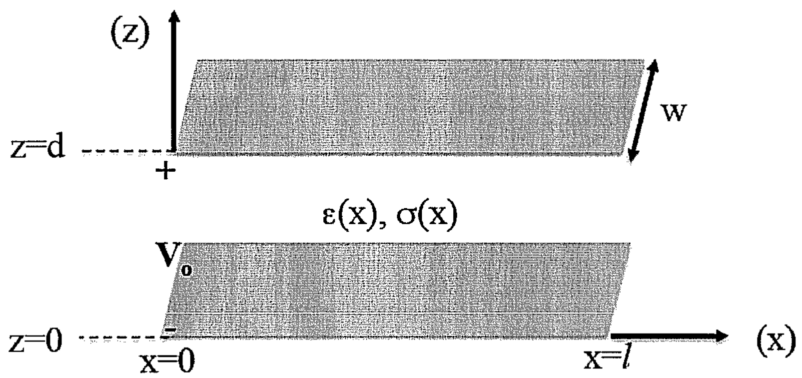
5. Digital Subscriber Line (DSL) technology is based on the deployment of bundles of multiple twisted copper pairs that connect the Central Office of a telecommunication service provider to customer premises. A significant limitation of DSL is "crosstalk", when one customer can listen to someone else's conversation. Use Faraday's law to explain how one's telephone circuit can receive someone else's signal. **[3 pts]**

A user's loop current creates a magnetic field that can induce an emf onto another user's loop.

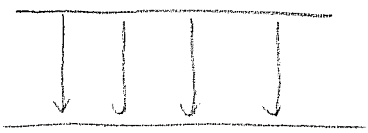
■ End of question 1

Question 2 [20 pts]

A. The following inhomogeneous parallel-plate lossy capacitor is filled with a medium of permittivity $\varepsilon(x)=\varepsilon_0\left(1+\frac{x}{\ell}\right)$ and conductivity $\sigma(x)=\sigma_0\left(1+\frac{x}{\ell}\right)$. A voltage source keeps the voltage difference between the two plates constant and equal to V_0. In answering the following questions, you can disregard the "edge effects".



1. Invoking electric field boundary conditions, show that the electric field is uniform throughout the capacitor and provide its expression. *Hint: Recall the case of "parallel connection" of capacitors.* **[2 pts]**



$\bar{E} = \frac{V_0}{d}\bar{a}_z$ is consistent with boundary [1pt] conditions as ε changes in x, and $\bar{E}$-lines run parallel to interfaces [1pt]

between changing ε.

2. Find the ohmic power dissipated and the resistance of this lossy capacitor. **[~~4~~ 5 pts]**

$$\frac{dp}{dv} = \sigma|\bar{E}|^2 = \sigma V_0^2/d^2 \quad 1\text{pt} \Rightarrow$$

$$P = \int \underbrace{\frac{\sigma V_0^2}{d^2}\cdot dx\,dy\,dz}_{1\text{pt}} = \frac{V_0^2}{d^2}(w\cdot d)\int_0^{\ell}\sigma(x)\,dx$$

$$= \frac{\sigma_0 V_0^2}{d}\cdot w\cdot \underbrace{\int_0^{\ell}\left(1+\frac{x}{\ell}\right)dx}_{\ell+\frac{\ell^2}{2\ell}=\frac{3\ell}{2}} = \frac{3\ell}{2}\,\frac{\sigma_0 w\, V_0^2}{d}$$

1 pt calculus, result

$$I^2 R = \frac{V_0^2}{R} = \frac{3\ell}{2}\,\frac{\sigma_0 W}{d}\,V_0^2 \Rightarrow \boxed{R^{-1} = \frac{3\ell\,\sigma_0 W}{2d}}$$ 2pts

OR by calculating current $I = \underbrace{\int \bar{J}\cdot d\bar{s}}_{1pt} = \underbrace{\int \sigma_0\left(1+\frac{x}{\ell}\right)\frac{V_0}{d}}_{1pt\ J=\sigma E}\,dx\,dy$

$$= \underbrace{\frac{V_0\cdot w\,\sigma_0}{d}\,\frac{3\ell}{2} \Rightarrow \frac{V_0}{I} = R = \frac{2d}{3\sigma_0 \ell w}}_{1pt\ calculus,\ result}, \quad P = \frac{V_0^2}{R}$$ [2pts]

3. Find the total electric energy stored and the capacitance. **[~~4~~ 5 pts]**

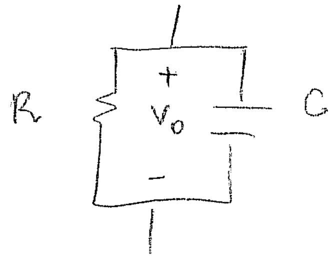
$$W_e = \frac{1}{2}\overbrace{\int \underbrace{\varepsilon(x)}_{1pt} E^2\,dv}^{1pt\ integral} = \frac{1}{2}\,\frac{V_0^2}{d^2}\cdot \ell\cdot d\cdot w\,\epsilon_0 \underbrace{\int\left(1+\frac{x}{\ell}\right)dx}_{3\ell/2}$$ 1pt calculus, result

$$\Rightarrow \overbrace{W_e = \cancel{\frac{1}{2}}CV_0^2}^{1pt} = \cancel{\frac{1}{2}}\,\frac{V_0^2}{d}\cdot w\cdot\epsilon_0\,\frac{3\ell}{2}$$

$$\boxed{C = \epsilon_0\,\frac{3\ell_0 w}{2d}}$$ 1pt result

OR from charge.

4. Draw an equivalent circuit diagram for this system and explain its derivation. **[2 pts]**

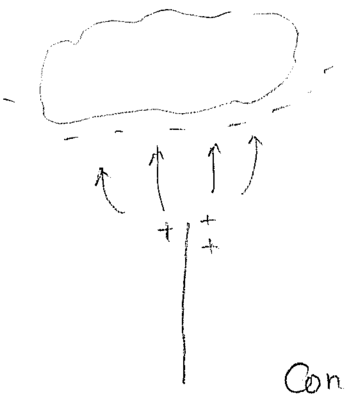


elements in parallel, since both [1pts]
connected to same voltage

1pts > R-C circuit.

B. The following questions are independent from each other.

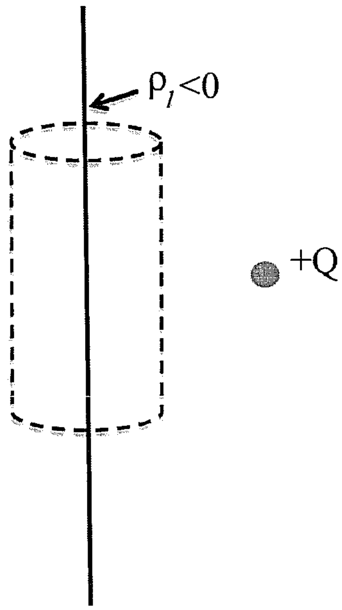
1. Briefly explain the operation of a lighting rod, employing all relevant course concepts. **[4 pts]**



Concept #1: Dielectric breakdown of air allows charged clouds to discharge through the rod. [2pts]

Concept #2: $|\vec{E}| \sim \rho_s \Rightarrow$ high ρ_s at tip allows the rod to create high fields that surpass the breakdown barrier of the air. [2pts]

2. Consider a finite negative line charge distribution. Is it possible to use a positive charge +Q, shown in the figure, to make the total flux $\oint_S \mathbf{D}\cdot d\mathbf{s}$ through the closed cylindrical surface (shown with dashed lines) positive, or zero? **[~~4~~ 2 pts]**



No. Gauss Law says

$\oint_S \bar{D}\cdot d\bar{s} = Q_{enclosed}$. [2pts]

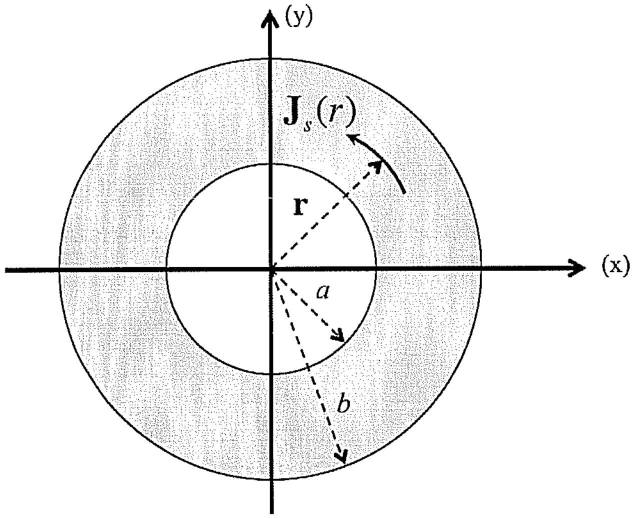
■ End of question 2

Question 3 [15 pts]

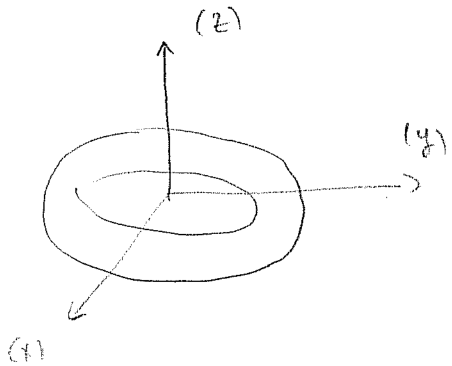
A. A surface current density

$$\mathbf{J}_s(\mathrm{r}) = \mathrm{J}_0 \frac{\mathrm{a}}{\mathrm{r}} \mathbf{a}_\varphi$$

exists on the circular surface shown in the figure. The surface lies on the plane $z = 0$, and has its center at the origin. Compute the magnetic flux density vector B(z) at a point on the positive z-axis ($z > 0$). **[9 pts]**



$$d\bar B = \frac{\mu_0}{4\pi} \frac{I\, d\bar l' \times (\bar R - \bar R')}{|\bar R - \bar R'|^3} \qquad I = J_s\, dr$$



$$I\, d\bar l' = \underbrace{\bar J_s\, dr}_{1}\, \underbrace{\bar a_\varphi}_{1}\, \underbrace{(r\, d\varphi)}_{1} \Big\} \text{ 3pts}$$

$d\bar l \parallel \bar a_\varphi'$

$|d\bar l| = r' d\varphi'$

$$= \frac{J_0 a}{\cancel{r'}}\, \bar a_{\varphi'} \left[\cancel{r'}\, d\varphi'\, dr'\right]$$

$$= J_0 a\, d\varphi'\, dr' \left[-\bar a_x \sin\varphi' + \bar a_y \cos\varphi'\right]$$

[2pts] $\left[\bar R = z\bar a_z \quad \bar R' = r'\bar a_r' \Rightarrow \bar R - \bar R' = z\bar a_z - r'\bar a_r', \quad |\bar R - \bar R'| = \sqrt{z^2 + (r')^2}\right]$

$$\Rightarrow \bar B = \frac{\mu_0}{4\pi} \cdot J_0 a \int \frac{\bar a_{\varphi'}\, dr'\, d\varphi' \times (z\bar a_z - r'\bar a_{r'})}{\left[z^2 + (r')^2\right]^{3/2}} =$$

← setting up integral [2pts]

$$= \frac{\mu_0}{4\pi} J_0 a \int \frac{dr'\, d\varphi'}{\left[z^2 + (r')^2\right]^{3/2}} \left(z\bar a_{r'} + r'\bar a_z\right)$$

Since $\int_0^{2\pi} \bar{a}_{r'}\, d\varphi' = 0$, only the z-component is $\neq 0 \Rightarrow$

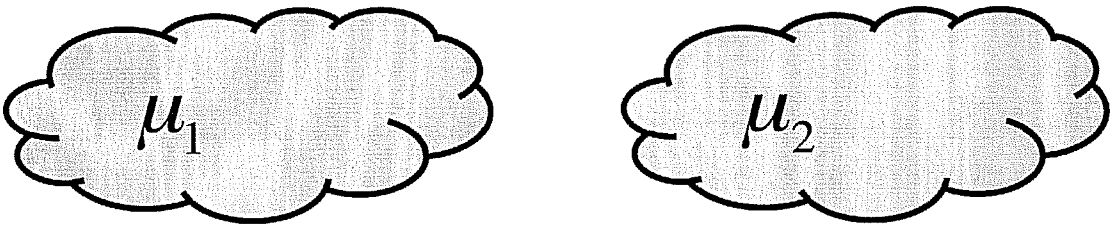
$$\bar{B} = \bar{a}_z \frac{\mu_0}{4\pi} J_0 a \int_{r'=a}^{b} \int_{\varphi'=0}^{2\pi} \frac{dr'\, d\varphi'}{\left[z^2 + (r')^2\right]^{3/2}}\, r'$$

$$= \bar{a}_z \frac{\mu_0 \cdot J_0 a}{2} \int_a^b \frac{r'\, dr'}{\left(z^2 + (r')^2\right)^{3/2}}$$

[2pts for deducing $\bar{a}_z$ component]

B. The following questions are independent from each other.

1. Consider two bodies with the same shape and different permeability μ_1 and μ_2, with $\mu_2 > \mu_1$



The free current density vector **J** is exactly the same in the two bodies. In which one the magnetic flux density **B** is larger? Explain. **[3 pts]**

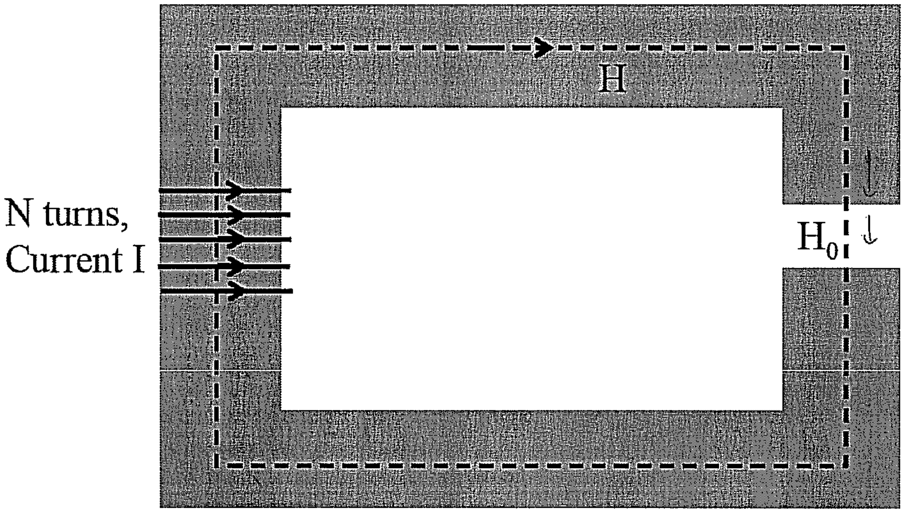
$\nabla \times \bar{H} = \bar{J} \Rightarrow \bar{H}$ $\underbrace{\text{same}}_{1}$ $\Rightarrow$ $\left.\begin{array}{l} \bar{B}_1 = \mu_1 \bar{H}_1 \\ \bar{B}_2 = \mu_2 \bar{H}_2 \end{array}\right\} 1$

with $\mu_2 > \mu_1$, $|\bar{B}_2|$ stronger than $|\bar{B}_1|$

1

2. For the magnetic circuit shown below, find the ratio H/H_0 of the magnetic field intensity H in the core of permeability μ to the magnetic field intensity H_0 in the air gap. If the total length of the core is Lc and the length of the gap is Lg, express H and H_0 in terms of N, I, Lc, Lg.

[3 pts]



$$NI = H \cdot L_c + H_o L_g \quad (1)$$

$$\mu H = \mu_o H_o \Rightarrow \boxed{\frac{H}{H_o} = \frac{\mu_o}{\mu}} \quad (2) \quad \text{[1 pt]} \rightarrow \text{boundary conditions}$$

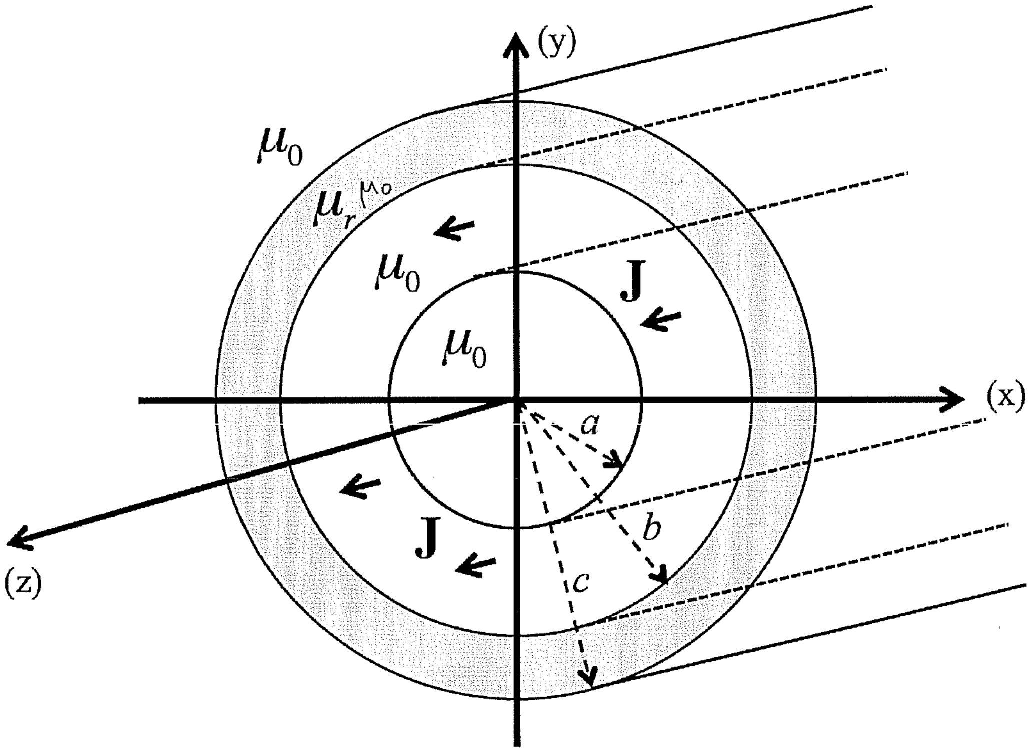
$$\Rightarrow \underbrace{H\left[L_c + \frac{\mu}{\mu_o} L_g\right] = N \cdot I}_{\text{[1pt] Amp's Law}} \Rightarrow H = \frac{NI}{L_c + \frac{\mu L_g}{\mu_o}}$$

$$H_o = \frac{\mu}{\mu_o} \frac{NI}{L_c + \mu \frac{L_g}{\mu_o}} = \frac{\mu NI}{\mu_o L_c + \mu L_g} \quad (3) \quad \text{[1 pt]}$$

■ End of question 3

Question 4 [15 pts]

A. Consider the hollow cylindrical conductor of inner radius a and outer radius b shown below. The conductor is infinitely long and is coated by a layer of magnetic material with permeability $\mu_r = 5$. A uniform current density $\mathbf{J} = J_0 \mathbf{a}_z$ flows inside the conductor $\alpha<r<b$.



1. Find the magnetic flux density vector **B** for $\alpha<r<c$. Hint: as the current distribution is cylindrically symmetric, the magnetic field will be in the form $\mathbf{H} = H_\varphi(r)\mathbf{a}_\varphi$. **[6 pts]**

$a < r < b$: $H_\varphi \cdot 2\cancel{\pi} r = J_0 \cdot \cancel{\pi}(r^2 - a^2) \Rightarrow H_\varphi = \frac{J_0}{2} \frac{r^2 - a^2}{r}$

$\Rightarrow B_\varphi = \frac{\mu_0 J_0}{2r}(r^2 - a^2)$

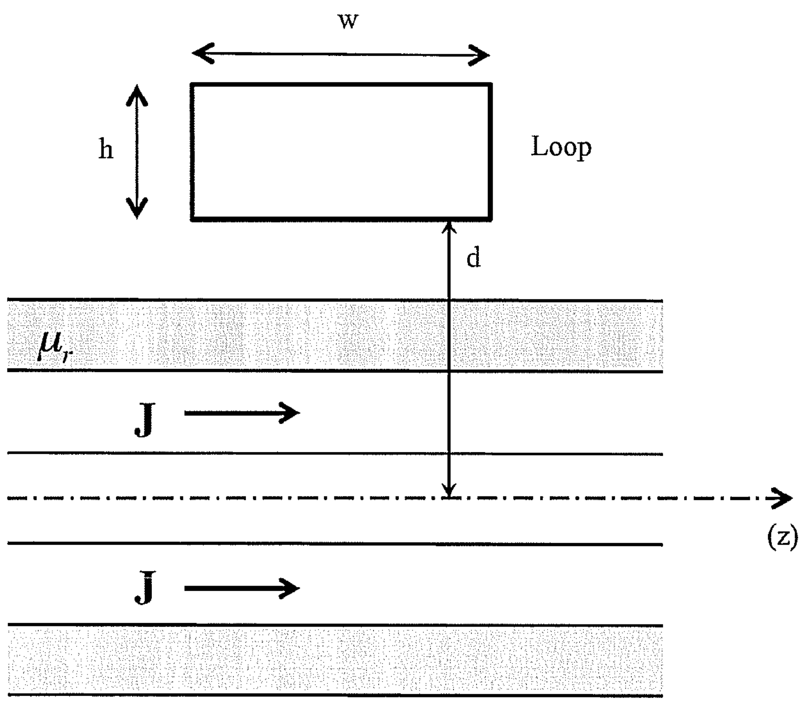
$r \geq b$: $H_\varphi \cdot 2\pi r = J_0 (b^2 - a^2) \Rightarrow H_\varphi = \frac{J_0 (b^2 - a^2)}{2\pi r}$

$$\Rightarrow B_\varphi = \begin{cases} \mu_r \mu_0 J_0 (b^2 - a^2)/2\pi r, & b < r < c \\ \frac{\mu_0 J_0 (b^2 - a^2)}{2\pi r} & r > c \end{cases}$$

Ampere law:
$a < r < b$: 2pts
~~$r \geq b$~~: 2pts
$b < r < c$

$B = \mu H$ [2pts]

2. A rectangular conducting loop is placed at a distance d from the conductor axis, as shown in the figure below. Find the mutual inductance between the conductor and the loop. **[6 pts]**



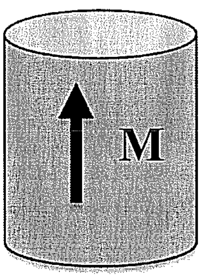
(2pts)

$$\bar{\Phi} = \underbrace{\int \bar{B}\cdot d\bar{s}}_{[1pt]} = \int \frac{\mu_0 J_0 (b^2-a^2)}{2\pi r}\, \bar{a}_\varphi \cdot \boxed{\bar{a}_\varphi\, dr\, dz} \quad , d\bar{s} = 2pts$$

$$= \frac{\mu_0 J_0 (b^2-a^2)}{2\pi} \underbrace{\int dz}_{w} \underbrace{\int dr}_{\ln \frac{d+h}{d}} = \frac{\mu_0 \cdot I}{2\pi} w \ln \frac{d+h}{d} \Rightarrow$$

$$\underbrace{L_{12}}_{1\,pt} = \frac{\Phi_{12}}{I} = \frac{\mu_0}{2\pi} w \ln \frac{d+h}{d}.$$

B. Consider a permanently magnetized cylindrical bar magnet, with uniform magnetization $\mathbf{M}=M_0\mathbf{a}_z$. Do any surface or volume magnetization currents exist in this magnet? Explain. **[3 pts]**



$\bar{J}_M = \nabla \times \bar{M} = 0 \Rightarrow$ no volume mag. current [1]

$\bar{J}_{M,s} = \bar{M} \times \bar{a}_n = \begin{cases} 0, \text{ top / bottom [1]} \\ M_0\,\bar{a}_\varphi \text{ on the side. [1]} \end{cases}$

■ <u>End of question 4</u>