MAT195S CALCULUS II

Midterm Test #2

25 March 2014 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: P. Athavale and J. W. Davis

Family Name: JW Davis

Given Name: Solutions.

Student #:

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Question	Marks	Earned
1	12	
2	10	
3	10	
4	10	
5	12	
6	6	
7	10	
TOTAL	70	/ 65

Tutorial Section:

TA Name:

1) Test the series for convergence or divergence:

a) $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ b) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ c) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

(12 marks)

a) $\frac{k!}{k^k} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot (k-1) \cdot k}{k \cdot k \cdot k \cdot k \cdot \ldots \cdot k \cdot k} < \frac{1}{k} \cdot \frac{2}{k} \cdot 1 \cdot 1 \cdot 1 \cdot \ldots \cdot 1 < \frac{2}{k^2}$

now $\sum \frac{2}{k^2}$ converges (p-series, $p > 1$)

$\therefore \sum \frac{k!}{k^k}$ converges by comparison test

b) $\Rightarrow$ show $\frac{\ln n}{\sqrt{n}}$ is decreasing : let $f(x) = \frac{\ln x}{\sqrt{x}}$

$\Rightarrow f'(x) = \ln x\left(-\frac{1}{2} x^{-3/2}\right) + x^{-3/2} = \frac{1 - \frac{1}{2}\ln x}{x^{3/2}}$

$\Rightarrow f'(x) < 0$ for $\ln x > 2$ or $x > e^2$

$\therefore f(x)$ is decreasing $\therefore \frac{\ln n}{\sqrt{n}}$ is decreasing

$\Rightarrow$ show $\frac{\ln n}{\sqrt{n}} \longrightarrow 0$: $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} \stackrel{*}{=} \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$ is convergent by Alternating series test

c) root test : $(a_n)^{1/n} = \frac{n!}{n^4} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)!}{n \cdot n \cdot n \cdot n}$

$= 1 \cdot \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) \cdot (n-4)!$

$\longrightarrow (n-4)! \longrightarrow \infty$

$\lim_{n\to\infty} (a_n)^{1/n} \longrightarrow \infty \quad \therefore$ divergent

2) Suppose that the series Σa_n is conditionally convergent.

a) Prove that $\Sigma n^2 a_n$ is divergent. (Hint: use a proof by contradiction.)

b) Knowing that Σa_n is conditionally convergent is not sufficient to determine whether $\Sigma n a_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\Sigma n a_n$ converges, and an example where $\Sigma n a_n$ diverges.

(10 marks)

a) Assume that $\sum n^2 a_n$ is convergent.

$\therefore n^2 |a_n| \to 0$ as $n \to \infty$ (test for divergence)

$\therefore$ for $n > N$, $n^2 |a_n| < \epsilon$ for some N.

or $|a_n| < \frac{\epsilon}{n^2}$

But $\sum \frac{\epsilon}{n^2}$ is convergent (p-series, $p > 1$) which means that $\sum |a_n|$ must also be convergent, which is a contradiction of the initial statement that $\sum a_n$ is conditionally convergent. $\Rightarrow \therefore \sum n^2 a_n$ must be divergent.

b) i) consider $a_n = (-1)^n \cdot \frac{1}{n}$ which is conditionally convergent

$\Rightarrow \sum \frac{1}{n}$ diverges by integral test: $\int_1^\infty \frac{dx}{x} = [\ln x]_1^\infty \to \infty$

$\Rightarrow \sum \frac{(-1)^n}{n}$ converges by alt series test: $\frac{1}{n+1} < \frac{1}{n}$; $\frac{1}{n} \to 0$

$\Rightarrow \sum n a_n = \sum (-1)^n$ diverges: $|n a_n| \not\to 0$

ii) consider $b_n = (-1)^n \frac{1}{n \ln n}$ which is also conditionally convergent

$\Rightarrow \sum \frac{1}{n \ln n}$ diverges by integral test: $\int_2^\infty \frac{dx}{x \ln x} = [\ln \ln x]_2^\infty \to \infty$

$\Rightarrow \sum \frac{(-1)^n}{n \ln n}$ converges by alt series test: $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$; $\frac{1}{n \ln n} \to 0$

$\Rightarrow \sum n b_n = \sum \frac{(-1)^n}{\ln n}$; $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ & $\frac{1}{\ln n} \to 0$

$\therefore$ convergent by alt series test.

3) Determine from first principles the Taylor series for the function $f(x) = \dfrac{1}{x^2}$ about $x = 1$, and determine the interval of convergence, including the status of the end points.

(10 marks)

$f(x) = \frac{1}{x^2}$ $\qquad f(1) = 1$

$f'(x) = -2x^{-3}$ $\qquad f'(1) = -2$

$f''(x) = 2 \cdot 3 x^{-4}$ $\qquad f''(1) = 3!$

$f'''(x) = -2 \cdot 3 \cdot 4 x^{-5}$ $\qquad f'''(1) = -4!$

$$\therefore \frac{1}{x^2} = f(1) + \frac{1}{1!} f'(1)(x-1) + \frac{1}{2!} f''(1)(x-1)^2 + \frac{1}{3!} f'''(1)(x-1)^3 + \ldots$$

$$= 1 - \frac{2!}{1!}(x-1) + \frac{3!}{2!}(x-1)^2 - \frac{4!}{3!}(x-1)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n \frac{(n+1)!}{n!} = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

Ratio test: $\left| \dfrac{a_{n+1}}{a_n} \right| = \left| \dfrac{(n+2)(x-1)^{n+1}}{(n+1)(x-1)^n} \right| = \dfrac{n+2}{n+1}|x-1| \longrightarrow |x-1|$

$\therefore$ require $|x-1| < 1 \Rightarrow R = 1$

at $x=2$: $\sum_{n=0}^{\infty} (-1)^n (n+1)(1)^n \longrightarrow \infty$; $a_n \not\to 0$

$x=0$: $\sum_{n=0}^{\infty} (-1)^n (n+1)(-1)^n \longrightarrow \infty$; $a_n \not\to 0$

$\therefore$ convergent for $x \in (0, 2)$

4) a) Give an ϵ-δ definition of uniform continuity of a function f defined for real numbers (essentially a statement of the "small-span theorem").

b) Using the ϵ-δ definition, prove that $f(x) = \cos x$ is uniformly continuous on $\mathbb{R}$. HINT: Use the mean value theorem.

(10 marks)

a) We say that a function f is continuous on $\mathbb{R}$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x_2 - x_1| < \delta$ then $|f(x_2) - f(x_1)| < \epsilon$.

b) Given $|x_2 - x_1| < \delta \longrightarrow$ show $|\cos x_2 - \cos x_1| < \epsilon$

Mean value theorem: $f'(z) = \dfrac{f(b) - f(a)}{b - a}$; $z \in (b, a)$

or $-\sin z = \dfrac{\cos x_2 - \cos x_1}{x_2 - x_1}$; $z \in (x_1, x_2)$

$\Rightarrow |\cos x_2 - \cos x_1| = |\sin z \,(x_2 - x_1)|$

Since $|\sin z| \leq 1 \Rightarrow |\cos x_2 - \cos x_1| \leq |x_2 - x_1|$

$\therefore$ let $\delta = \epsilon$ (or $\epsilon/2$, etc)

$\therefore$ For $|x_2 - x_1| < \delta = \epsilon$ then $|\cos x_2 - \cos x_1| < \epsilon$

$\therefore$ $\cos x$ is uniformly continuous

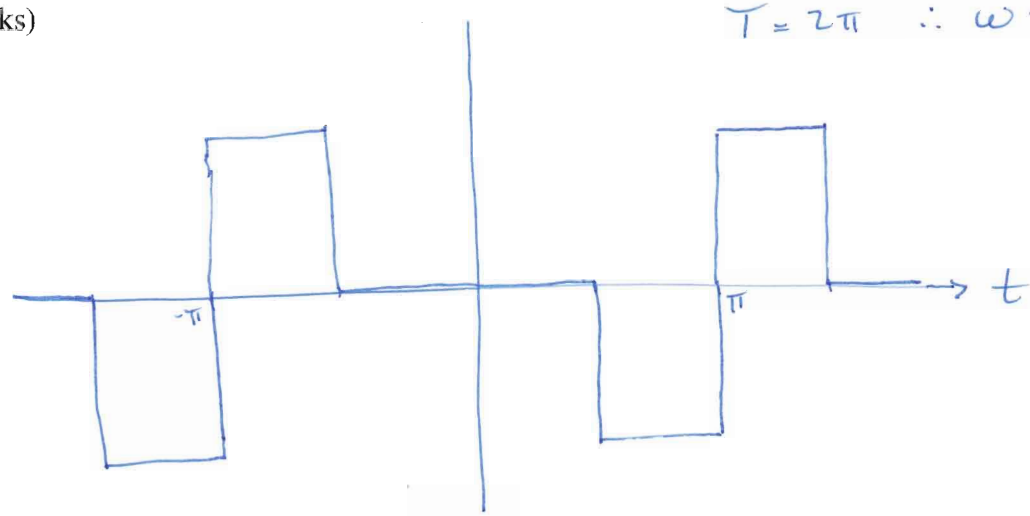
5) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(t) = \begin{cases} 1, & -\pi \le t \le -\pi/2 \\ 0, & -\pi/2 < t \le \pi/2 \\ -1, & \pi/2 < t \le \pi \end{cases}$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

(12 marks)

$T = 2\pi \quad \therefore \omega = 1$



$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} 1\,dt + \int_{-\pi/2}^{\pi/2} 0\,dt + \int_{\pi/2}^{\pi} (-1)\,dt\right] = \frac{1}{\pi}\left(\frac{\pi}{2} + 0 - \frac{\pi}{2}\right) = 0$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} \cos nt\,dt - \int_{\pi/2}^{\pi} \cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2} - \sin n\pi + \sin\frac{n\pi}{2}\right) = 0$$

$\Rightarrow f(t)$ is an odd function, so we expect $a_n = 0$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}\sin nt\,dt - \int_{\pi/2}^{\pi}\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

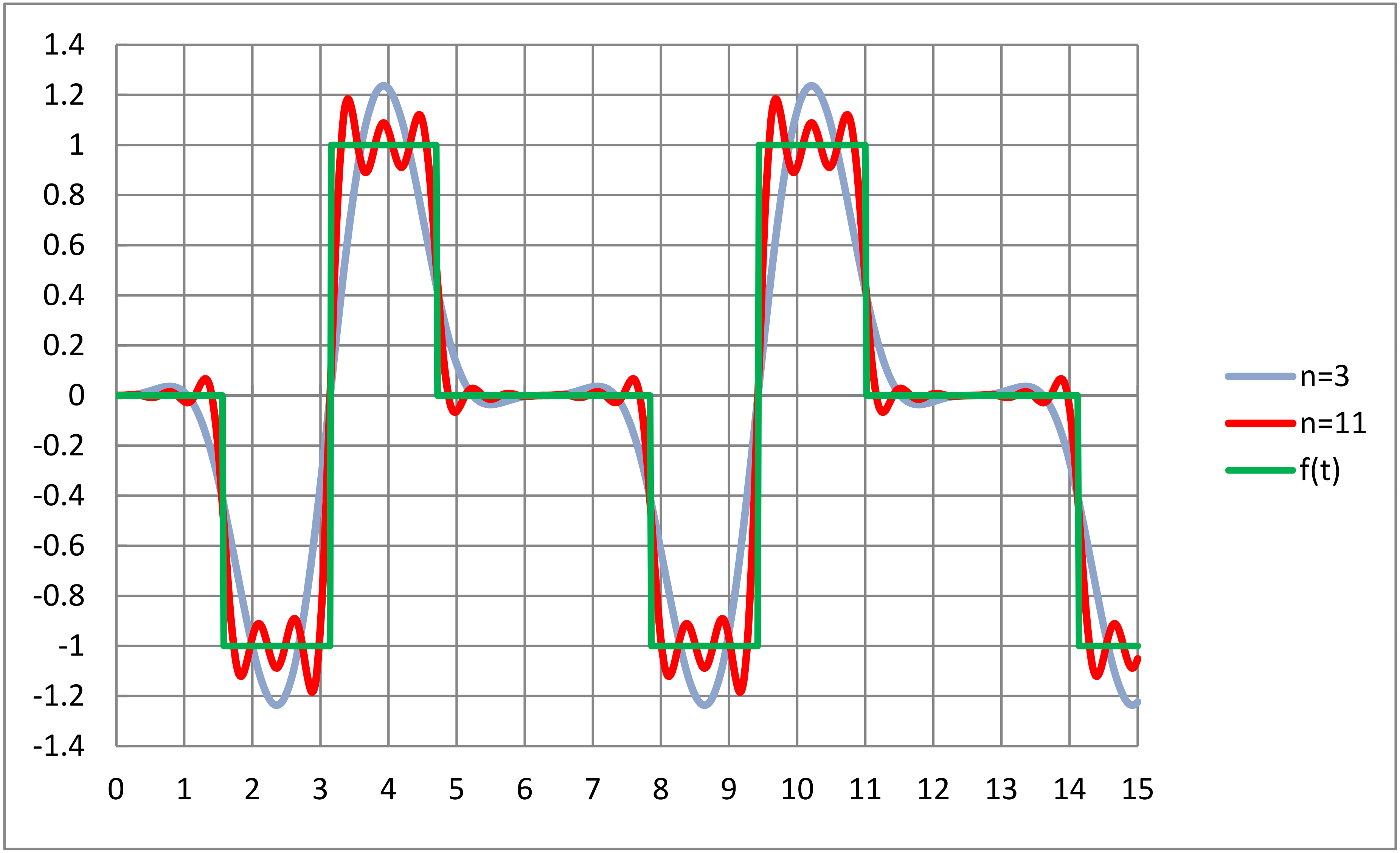
$$= \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2} + \cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$= \frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$\Rightarrow f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\sin nt$$

n	$\cos n$	$-$	$\cos\frac{n\pi}{2}$		
1	-1	$-$	0	$=$	-1
2	1	$-$	-1	$=$	2
3	-1	$-$	0	$=$	-1
4	1	$-$	1	$=$	0
5	-1	$-$	0	$=$	-1
6	1	$-$	-1	$=$	2
7	-1	$-$	0	$=$	-1
8	1	$-$	1	$=$	0

$$\therefore \cos n\pi - \cos\frac{n\pi}{2} = \begin{cases} -1, & n \text{ odd} \\ 2, & n \text{ even}, \; 2, 6, 10, \ldots \\ 0, & n \text{ even} \; 4, 8, 12, \ldots \end{cases}$$



1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
n=3
n=11
f(t)

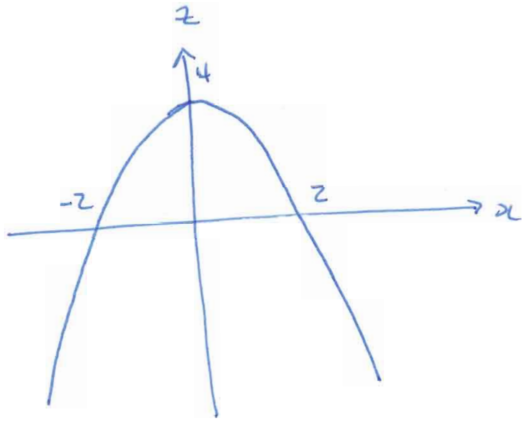
6) a) Sketch the parabola $z = 4 - x^2$ in the xz plane.

b) Sketch the quadric surface $z = 4 - x^2 - y^2$ in xyz coordinates.

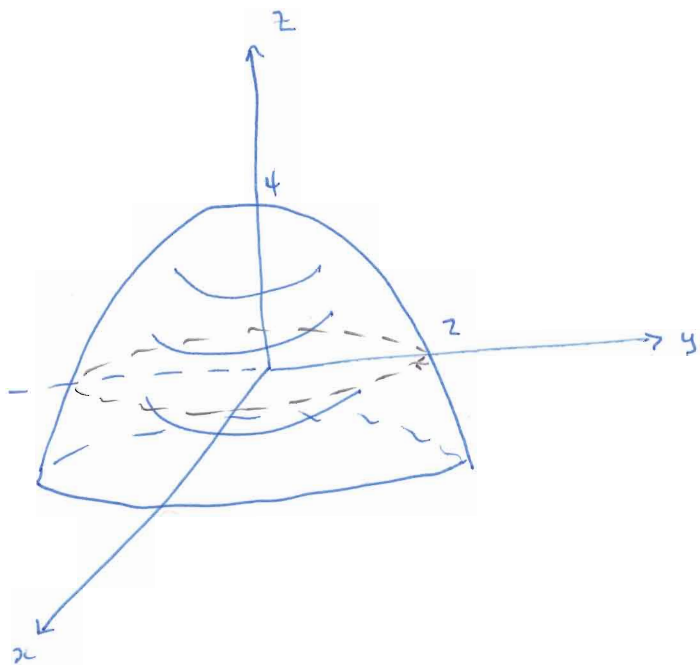
c) How are the sketches in parts (a) and (b) related?

(6 marks)

a)



b)



c) The 3-D paraboloid in (b) is a rotation of the parabola in (a) about the z-axis.

7) a) Let a curve be parameterized by $\bar{r}(t)$. Let $\bar{T}(t)$ denote the unit tangent vector. Show that $\bar{T}'(t)$ is perpendicular to $\bar{T}(t)$.

b) The curvature κ is defined as: $\kappa = \dfrac{\|\bar{T}'(t)\|}{\|\bar{r}'(t)\|}$

Beginning with $\bar{r}'(t) = \dfrac{ds}{dt}\bar{T}$, show that $\|\bar{r}' \times \bar{r}''\| = \left(\dfrac{ds}{dt}\right)^2 \|\bar{T}'\|$, and hence,

$$\kappa = \frac{\|\bar{r}'(t) \times \bar{r}''(t)\|}{\|\bar{r}'(t)\|^3}$$

(10 marks)

a) $\bar{T} = \dfrac{\bar{r}'(t)}{\|\bar{r}'(t)\|}$ is a unit vector, hence $\bar{T}\cdot\bar{T} = 1$

Differentiating: $\bar{T}\cdot\bar{T}' + \bar{T}'\cdot\bar{T} = 0 \Rightarrow \bar{T}\cdot\bar{T}' = 0$

$\therefore \bar{T} \perp \bar{T}'$

b) $\bar{r}'(t) = \dfrac{ds}{dt}\bar{T} \Rightarrow \bar{r}''(t) = \left(\dfrac{ds}{dt}\right)'\bar{T} + \dfrac{ds}{dt}\bar{T}'$

$= \dfrac{d^2s}{dt^2}\bar{T} + \dfrac{ds}{dt}\bar{T}'$

now $\bar{r}' \times \bar{r}'' = \dfrac{ds}{dt}\bar{T} \times \left(\dfrac{d^2s}{dt^2}\bar{T} + \left(\dfrac{ds}{dt}\right)\bar{T}'\right) = \left(\dfrac{ds}{dt}\right)^2 \bar{T}\times\bar{T}'$

$\therefore \|\bar{r}' \times \bar{r}''| = \left(\dfrac{ds}{dt}\right)^2 \|\bar{T}\times\bar{T}'\|$

But from part (a), $\bar{T} \perp \bar{T}'$ $\therefore \|\bar{T}\times\bar{T}'\| = \|\bar{T}\|\,\|\bar{T}'\| \sin\theta = \|\bar{T}'\|$

$\theta = 90°$

$\therefore \|\bar{r}' \times \bar{r}''\| = \left(\dfrac{ds}{dt}\right)^2 \|\bar{T}'\| \Rightarrow \|\bar{T}'\| = \dfrac{\|\bar{r}'\times\bar{r}''\|}{(ds/dt)^2} = \dfrac{\|\bar{r}'\times\bar{r}''\|}{\|\bar{r}'\|^2}$

$\therefore \dfrac{\|\bar{T}'\|}{\|\bar{r}'\|} = \dfrac{\|\bar{r}'\times\bar{r}''\|}{\|\bar{r}'\|^3}$