UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

Term Test II

MAT185H1S — Linear Algebra

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14 March 2019

Student Name:	Copy	Fair
	Last Name	First Names

Student No: ______ e-Address: ______

Signature: ______

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution. The total number of marks available is **50**.
3. Write solutions *only* in the boxed space provided for each question. *Do not* write solutions on the reverse side of pages. These will *not* be scanned and therefore will *not* be marked.
4. Two blank pages are provided at the end for rough work. Work on these back pages will *not* be marked unless clearly indicated; in such cases, clearly indicate on the question page(s) that the solution(s) is continued on a back page(s).
5. *Do not* write over the QR code on the top right-hand corner of each page.
6. *No* aid is permitted.
7. The duration of this test is 90 minutes.
8. There are 7 pages and 5 questions in this test paper.

A Note on Notation:

1. ${}^m\mathbb{R}^n = M_{m\times n}(\mathbb{R})$, the former notation is used in the Notes and the latter in Nicholson.

A. Definitions and Statements

Fill in the blanks. Note that definitions and statements of theorems are either correct or not; 2 marks will be given for correct answers and 0 for incorrect ones.

1(a). The *column space* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$ is defined as

As in Notes.

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1(b). The *rank* of a matrix $\mathbf{A} \in {}^m\mathbb{R}^n$ is defined as

As in Notes.

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1(c). The *coordinates* of a vector $\boldsymbol{v} \in \mathcal{V}$ relative to a basis $\{\boldsymbol{e}_1 \cdots \boldsymbol{e}_n\}$ are defined as

As in Notes.

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1(d). State the *Cauchy-Binet product theorem* for determinants.

As in Notes.

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1(e). The (i, j)-*cofactor* of a matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

As in Notes.

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B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. A correct answer earns 2 marks but 2 marks will be deducted for an incorrect answer; the minimum total mark for this section is 0.*

2(a). If **AB** is invertible, then so is **BA**. | F |

2(b). The rows of an upper triangular matrix are linearly independent. | F |

2(c). If $\dim \operatorname{null} \mathbf{A} = k$, where $\mathbf{A} \in {}^n\mathbb{R}^m$, then $\dim \operatorname{null} \mathbf{A}^T = m - n + k$. | F |

2(d). Let $\alpha = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be the basis of a 4-dimensional subspace $\mathcal{W}$ of a 7-dimensional vector space $\mathcal{V}$ and let $\beta = \{\boldsymbol{v}_3, \boldsymbol{v}_2, \boldsymbol{v}_4, \boldsymbol{v}_1\}$. The change-of-basis (transformation or transition) matrix for the coordinates of vectors in $\mathcal{W}$ from α to β is

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

| T |

2(e). For any matrix $\mathbf{A} \in {}^n\mathbb{R}^n$, $\det(\lambda \mathbf{A}) = \lambda \det \mathbf{A}$. | F |

C. Problems

3. Consider the subspace of $\mathbb{P}_3$ given by

$$\mathcal{T} = \operatorname{span}\{1 + x + x^2 + 2x^3, 1 + 3x - 2x^2 + x^3, 3 + 5x + x^3, 4x - 6x^2 + 2x^3\}$$

Find a basis for $\mathcal{T}$ consisting of vectors from the original spanning set.

Consider the coordinates (in ${}^n\mathbb{R}$) of the vectors:

$$p_1(x) = 1 + x + x^2 + 2x^3 \leftrightarrow \mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad p_2(x) = 1 + 3x - 2x^2 + x^3 \leftrightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

$$p_3(x) = 3 + 5x + x^3 \leftrightarrow \mathbf{p}_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \quad p_4(x) = 4x - 6x^2 + 2x^3 \leftrightarrow \mathbf{p}_4 = \begin{bmatrix} 0 \\ 4 \\ -6 \\ 2 \end{bmatrix}$$

Let us build the matrix $\mathbf{A}$ whose columns are the coordinates and row-reduce:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 3 & 5 & 4 \\ 1 & -2 & 0 & -6 \\ 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & -3 & -3 & -6 \\ 0 & -1 & -5 & 2 \end{bmatrix} \qquad \begin{aligned} \mathbf{r}_2' &= \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3' &= \mathbf{r}_3 - \mathbf{r}_1 \\ \mathbf{r}_4' &= \mathbf{r}_4 - 2\mathbf{r}_1 \end{aligned}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 5 & -2 \end{bmatrix} \qquad \begin{aligned} \mathbf{r}_2'' &= \tfrac{1}{2}\mathbf{r}_2' \\ \mathbf{r}_3'' &= -\tfrac{1}{3}\mathbf{r}_3' \\ \mathbf{r}_4'' &= -\mathbf{r}_4' \end{aligned}$$

$$\to \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -4 \end{bmatrix} \qquad \begin{aligned} \mathbf{r}_3''' &= \mathbf{r}_3'' - \mathbf{r}_2'' \\ \mathbf{r}_4''' &= \mathbf{r}_4''' - \mathbf{r}_2''' \end{aligned}$$

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3. . . . *cont'd*

$$\mathbf{A} \rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \mathbf{r}_3'''' = \mathbf{r}_4''' \\ \mathbf{r}_4'''' = \mathbf{r}_3'''' \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{r}_3''''' = \tfrac{1}{4}\mathbf{r}_3''''$$

The columns with the leading "1"s are columns $1, 2, 3$. Therefore $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent in ${}^n\mathbb{R}$ and $\mathbf{p}_4$ is a linear dependent on these; accordingly, p_1, p_2, p_3 are linearly independent in $\mathcal{T}$ and p_4 is linearly dependent on these.

We conclude that $\{p_1, p_2, p_3\}$ form a basis for $\mathcal{T}$.

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4. Let

$$\mathbf{A} = \begin{bmatrix} 1 & a+b & c \\ 1 & a+c & b \\ 1 & b+c & a \end{bmatrix}$$

where a, b, c are not all equal.

(a) Determine rank $\mathbf{A}$.

(b) Find a basis for null $\mathbf{A}$.

(c) Find a basis for col $\mathbf{A}$.

Much will be revealed by row-reducing $\mathbf{A}$:

$$\mathbf{A} \to \begin{bmatrix} 1 & a+b & c \\ 0 & c-b & b-c \\ 0 & c-a & a-c \end{bmatrix} \qquad \begin{array}{l} \mathbf{r}_2' = \mathbf{r}_2 - \mathbf{r}_1 \\ \mathbf{r}_3' = \mathbf{r}_3 - \mathbf{r}_1 \end{array}$$

At least one of the last two rows cannot be zero because not all a, b, c are the same. Without loss in generality, assume it's $\mathbf{r}_2'$. Then we continue as follows:

$$\mathbf{A} \to \begin{bmatrix} 1 & a+b & c \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{l} \mathbf{r}_2'' = (c-b)^{-1}\mathbf{r}_2' \\ \mathbf{r}_3'' = \mathbf{r}_3' - (c-a)(c-b)^{-1}\mathbf{r}_2' \end{array}$$

(a) We immediately see that the rank of $\mathbf{A}$ is 2 as given by the number of nonzero rows in its row-reduced form.

(b) We also observe that

$$\text{null}\,\mathbf{A} = \text{span}\{\mathbf{x}_0\}, \qquad \mathbf{x}_0 = \begin{bmatrix} a+b+c \\ -1 \\ -1 \end{bmatrix}$$

from which we conclude that $\{\mathbf{x}_0\}$ is a basis for null $\mathbf{A}$. (We can check that one vector constitutes a basis because $\dim \text{null}\,\mathbf{A} = n - \text{rank}\,\mathbf{A} = 3 - 2 = 1$.)

(c) Finally, we observe that as rank $\mathbf{A} = 2$, $\dim \text{col}\,\mathbf{A} = 2$. Two linearly independent columns of $\mathbf{A}$ would thus suffice as a basis for col $\mathbf{A}$. We can see by inspection that the first and third columns are indeed linearly independent as not all a, b, c are identical. These then constitute a basis for col $\mathbf{A}$.

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5. Show that the equation of the line through distinct points (x_1, y_1) and (x_2, y_2) can be written as

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{bmatrix} = 0$$

The general equation of a line is

$$ax + by + c = 0$$

where not all a, b, c are zero. We require this line to pass through (x_1, y_1) and (x_2, y_2), i.e.,

$$\begin{aligned} ax_1 + by_1 + c &= 0 \\ ax_2 + by_2 + c &= 0 \end{aligned}$$

The preceding three equations must be satisfied simultaneously while admitting a nontrivial solution for a, b, c; that is,

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{0}$$

For a nontrivial solution,

$$\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{bmatrix} = 0$$

This condition therefore must describe the line.

Alternate Approach. It is sufficient to note that the determinant does indeed produce the equation of a line because, by expanding across the last row,

$$x \det \begin{bmatrix} y_1 & 1 \\ y_2 & 1 \end{bmatrix} - y \det \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} + \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = 0$$

It's not necessary to carry the computation further. We can immediately see that $(x, y) = (x_1, y_1)$ or (x_2, y_2) satisfies the equation because in either case two rows will be made identical making the 3×3 determinant vanish.

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