MAT292 - Calculus III - Fall 2014

Term Test 2 - November 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

Full Name: ______________________________
Last First

Student ID: ______________________________

Email: ______________________________ @mail.utoronto.ca

<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 20 pages (including this title page). Make sure you have all of them.
- You can use pages 19–20 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 19–20.

GOOD LUCK!

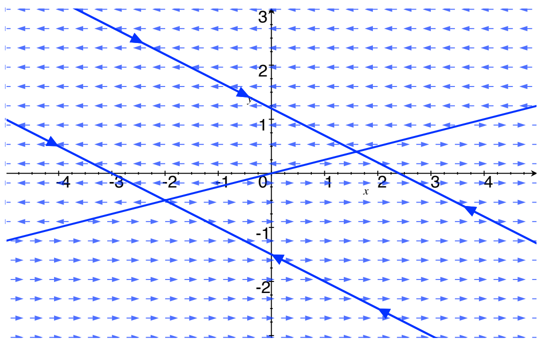
PART I No explanation is necessary.

For questions 1–4, consider the following systems of differential equations: **(4 marks)**

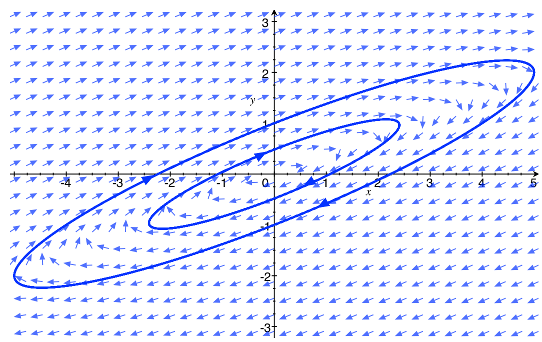
Letter	System Matrix		
a	$\mathbf{A} = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix}$	$\lambda_1 = -3, \vec{\xi_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = 0, \vec{\xi_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
b	$\mathbf{A} = \begin{pmatrix} -1 & 4 \\ \frac{1}{2} & -2 \end{pmatrix}$	$\lambda_1 = 0, \vec{\xi_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = -3, \vec{\xi_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
c	$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix}$	$\lambda_1 = -i, \vec{\xi_1} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = i, \vec{\xi_2} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
d	$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$	$\lambda_1 = i, \vec{\xi_1} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = -i, \vec{\xi_2} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
e	$\mathbf{A} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$	$\lambda_1 = -\frac{7}{4}, \vec{\xi_1} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{1}{8}, \vec{\xi_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
f	$\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{13}{8} \end{pmatrix}$	$\lambda_1 = -\frac{1}{8}, \vec{\xi_1} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{7}{4}, \vec{\xi_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
g	$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\lambda_1 = -1, \vec{\xi_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\lambda_2 = -1, \vec{\xi_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\lambda_1 = 2, \vec{\xi_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\lambda_2 = 2, \vec{\xi_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Continued...

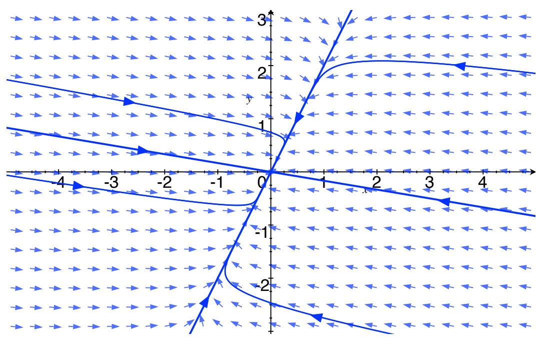
Next to each phase plane diagram, write the letter of the corresponding system of differential equations.



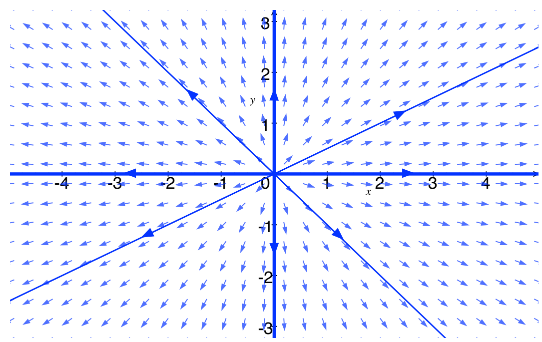
1. This is system ___________



2. This is system ___________



3. This is system ___________



4. This is system ___________

5. Write a differential equation whose complementary solution is **(2 marks)**

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 t^2 e^{-2t} + c_4$$

____

6. Consider the ODE $y^{(6)} + 2y^{(4)} + y^{(2)} = \cos(t) + t^2$. When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* have the form (**select all that apply**): **(2 marks)**

(a) $A\cos t$	**(d)** $D\sin t$	**(g)** G	**(j)** Jt^3	**(m)** Me^t	**(p)** Pe^{-t}
(b) $Bt\cos t$	**(e)** $Et\sin t$	**(h)** Ht	**(k)** Kt^4	**(n)** Nte^t	**(q)** Qte^{-t}
(c) $Ct^2\cos t$	**(f)** $Ft^2\sin t$	**(i)** It^2	**(l)** Lt^5	**(o)** Ot^2e^t	**(r)** Rt^2e^{-t}

For questions 7 and 8, consider the ODE: **(2 marks)**

$$ay'' + by' + cy = 0,$$

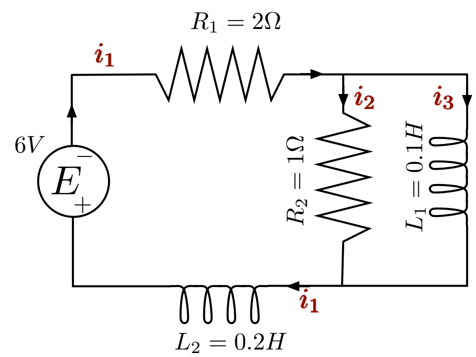
with $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$.

7. The solutions decay while oscillating if ______________________________.

8. The solutions grow while oscillating if ______________________________.

PART II **Justify** your answers.

9. Consider the following parallel circuit. **(10 marks)**



Using Kirchhoff's First Law, we deduce that $i_1 = i_2 + i_3$, so we consider only the currents i_1 and i_2.
Using Kirchhoff's Second Law, we can show that this parallel circuit is modelled by

$$\begin{cases} \dfrac{di_1}{dt} = -10i_1 - 5i_2 + 30 \\ \dfrac{di_2}{dt} = -10i_1 - 15i_2 + 30 \end{cases}$$

(a) The system of differential equations above in *non-homogeneous*, so we can "change variables" to transform the system into a *homogeneous* system of differential equations.

Consider a vector $\vec{x} = \vec{i} + \vec{b}$, with $\vec{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$.

Find $\vec{b}$ so that $\vec{x}$ is the solution of *homogeneous* system of differential equations.

(b) The new system is

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix} \vec{x}.$$

Find the general solution $\vec{x}$.

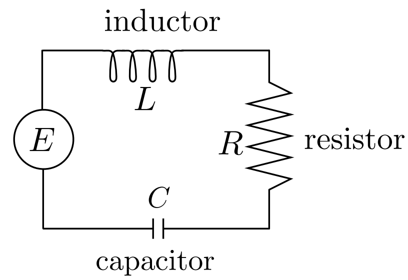
Continued...

(c) Given the initial conditions $\vec{i}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, what is the solution $\vec{i}$ of the original system?

(d) What is i_3?

Continued...

10. Consider the following electrical circuit. **(10 marks)**



The charge on the capacitor $q(t)$ is modelled by

$$Lq'' + Rq' + \frac{1}{C}q = E(t),$$

(a) Give a condition on the constants L, R, C that guarantees that the solution oscillates. Justify your answer.

(b) Let $L = 1$, $R = 0$, and $C = \frac{1}{4}$, and $E(t) = \sin(2t)$. Also assume that the capacitor starts with no charge and the circuit starts with no current. Find the solution of this initial-value problem. (**Hint.** Recall that current $i(t) = q'(t)$)

(c) How does the solution to **(b)** behave (grow / decay / oscillate) as t becomes larger and larger? Justify your answer.

(**Hint.** You don't need to have solved **(b)** to answer this question)

Continued...

11. Consider the ODE **(10 marks)**

$$y'' - (3 + 2t)y' + (6t - 2)y = 0. \qquad (\star)$$

(a) Show that $y_1(t) = e^{t^2}$ is a solution of this differential equation.

(b) Using reduction of order, consider a second solution of the form

$$y_2(t) = u(t)y_1(t).$$

Deduce a differential equation for $u(t)$.

(c) Find $u(t)$.

(**Hint.** You can leave u in the form of an integral)

Continued...

(d) Write the second solution $y_2(t)$ of $(\star)$ using a definite integral between 0 and t. Show that y_1 and y_2 form a fundamental set of solutions.

(e) What is the general solution of the differential equation $(\star)$?

Continued...

12. Consider the system of differential equations: **(10 marks)**

$$\vec{x}' = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \vec{x}.$$

(a) Find two linearly independent solutions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$.

(b) Consider the eigenvectors found in **(a)**. Construct a matrix $\mathbf{T}$ by putting each eigenvector as a column.

Find the matrix $\mathbf{T}^{-1}$.

$\left(\textbf{Hint.}\quad \text{For the forgetful ones, } \mathbf{A}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)$

(c) Consider a new variable $\vec{y}$ such that $\vec{x} = \mathbf{T}\,\vec{y}$. Which system of differential equations does it satisfy?

Continued...

(d) Find $\vec{y}$.

(e) What is the special fundamental matrix $\mathbf{\Phi}$ for the system of differential equations in **(c)**?

Continued...

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.

Continued...

USE THIS PAGE TO CONTINUE OTHER QUESTIONS.