ESC103F Engineering Mathematics and Computation: Tutorial #6

Question 1: Consider the system of linear equations given below:

$$2x + 3y = 1$$
$$10x + 9y = 11$$

i) Using elimination, determine the equivalent upper triangular system.
ii) What are the two pivots associated with this upper triangular system?
iii) Use back substitution to find the unknowns (and check your solution).

Solution:

i) $2x + 3y = 1$
$0x - 6y = 6$ (R2-5R1)

$$\begin{bmatrix} 2 & 3 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

ii) 2 and -6

iii) $-6y = 6$
$\therefore y = -1$
$2x + 3(-1) = 1$
$\therefore x = 2$

Question 2: Consider the system of linear equations given below:

$$ax + by = f$$
$$cx + dy = g$$

We will assume that the first pivot a is nonzero.

i) Elimination produces what formula for the second pivot?
ii) What condition must hold for the second pivot to be nonzero?
iii) Assuming the condition in part (ii) holds, what is the value for y?

Solution:

i) $ax + by = f$
$0x + \left(d - \frac{bc}{a}\right) y = g - \frac{cf}{a}$ (R2$-\frac{c}{a}$R1)
Therefore, the second pivot is given by $d - \frac{bc}{a}$.

ii) For the second pivot to be nonzero,

$$d - \frac{bc}{a} \neq 0$$
$$\frac{ad - bc}{a} \neq 0$$
$$\therefore ad - bc \neq 0$$

iii) $y = \frac{g - \frac{cf}{a}}{d - \frac{bc}{a}} = \frac{ag - cf}{ad - bc}$

Question 3:

A system of linear equations cannot have just two solutions. Let's examine why this is the case.

i) Consider a system of linear equations with 3 unknowns. Assume that two solutions are known, (x, y, z) and (X, Y, Z). What is another solution?

ii) If you know that 25 planes in R^3 meet at two points, where else do they meet?

Solution:

i) A system of linear equations in 3 unknowns (x_1, x_2, x_3) will be made up of equations like:

$$ax_1 + bx_2 + cx_3 = d$$

Solutions to the system must satisfy all such equations, i.e.

$ax + by + cz = d$ and $aX + bY + cZ = d$

One example of another solution would be:

$$(\frac{2}{3}x + \frac{1}{3}X, \frac{2}{3}y + \frac{1}{3}Y, \frac{2}{3}z + \frac{1}{3}Z)$$

because:

$$a\left(\frac{2}{3}x + \frac{1}{3}X\right) + b\left(\frac{2}{3}y + \frac{1}{3}Y\right) + c\left(\frac{2}{3}z + s\frac{1}{3}Z\right)$$

$$= \frac{2}{3}(ax + by + cz) + \frac{1}{3}(aX + bY + cZ)$$
$$= \frac{2}{3}d + \frac{1}{3}d$$
$$= d$$

ii) They meet at all points along the line in R^3 that goes through those two points.

Question 4:

Consider the system of linear equations below:

$$\begin{aligned} 2x + 5y + z &= 0 \\ 4x + dy + z &= 2 \\ y - z &= 3 \end{aligned}$$

The objective is to determine the equivalent upper triangular system.

i) What value for d forces a row exchange?
ii) What is the upper triangular system for the value of d in part (i)?
iii) What value of d produces a zero pivot in row 3?
iv) Is there a solution to this system for the value of d in part (iii)?

Solution:

i)
$$\begin{bmatrix} 2 & 5 & 1 \\ 4 & d & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \text{(R2-2R1)}$$

If $d = 10$,

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

This forces a row exchange between row 2 and row 3.

ii) Exchanging rows 2 and 3 produces the following upper triangular system,

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right]$$

iii) If $d = 11$, this produces a zero pivot in row 3,

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \text{(R3-R2)}$$

iv) The last row represents the equation:

$$0x + 0y + 0z = 1$$

No values of (x, y, z) satisfy this equation. Therefore, there is no solution to this system when $d = 11$.

Question 5:

Consider matrix A given below:

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}$$

For which three values of a will elimination fail to produce 3 nonzero pivots?

Solution:

$a = 0$ will produce,

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

which cannot produce 3 nonzero pivots.

$$A = \begin{bmatrix} a & 2 & 3 \\ 0 & a-2 & 1 \\ 0 & a-2 & a-3 \end{bmatrix} \quad \text{(R2-R1) followed by (R3-R1)}$$

$a = 2$ will produce,

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

which cannot produce 3 nonzero pivots.

$a = 4$ will produce,

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(R3-R2)}$$

which cannot produce 3 nonzero pivots.

Question 6:

i) Determine three elimination matrices E_1, E_2, E_3 that put matrix A into upper triangular form U,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

where $E_3 E_2 E_1 A = U$.

ii) Solve for one elimination matrix $E = E_3 E_2 E_1$.

iii) Include $\vec{b}$ to produce the augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right]$$

With our goal being to solve this system, we want to use elimination to convert $A\vec{x} = \vec{b}$ into $U\vec{x} = \vec{c}$. Determine $\vec{c}$ from E and $\vec{b}$.

iv) Working with $U\vec{x} = \vec{c}$, solve for $\vec{x}$ using back substitution.

Solution:

i) $$\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{(R2-4R1)}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \quad \text{(R3+2R1)}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{(R3-2R2)}$$

$$\therefore U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Apply R2-4R1 to the identity matrix gives,

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply R3+2R1 to the identity matrix gives,

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

Apply R3-2R2 to the identity matrix gives,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

ii) $E = E_3 E_2 E_1$

$$E_3 E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} = E \quad \text{(check your answer E}A = U\text{)}$$

iii) $A\vec{x} = \vec{b}$

$$EA\vec{x} = E\vec{b}$$

$$U\vec{x} = E\vec{b}$$

$$\therefore \vec{c} = E\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$$

iv) $U\vec{x} = \vec{c}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$$

$$-2x_3 = 10 \rightarrow \boxed{x_3 = -5}$$

$$2x_2 + x_3 = -4 \rightarrow \boxed{x_2 = \frac{1}{2}}$$

$$x_1 + x_2 = 1 \rightarrow \boxed{x_1 = \frac{1}{2}}$$