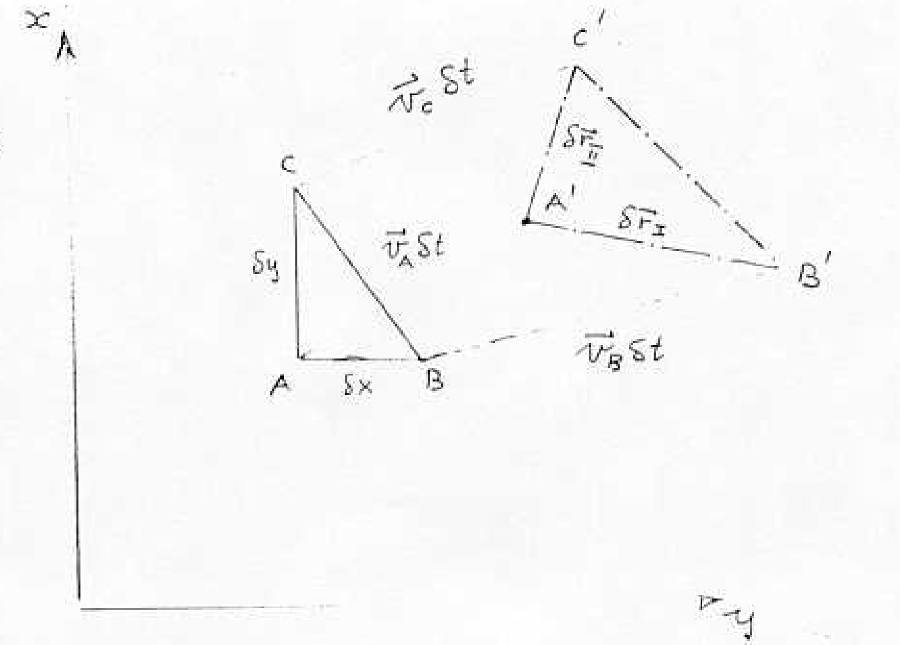
6.1



(i) In the limit as $\delta t \to 0$ for both steady and unsteady flow, we can write

$$\vec{r}_{A'} - \vec{r}_A = \vec{v}_A(r, t)\, \delta t$$

with corresponding expressions for the points B and C. Hence in time δt the triangle ABC moves and deforms as shown. Thus the area of ABC is $A = \frac{1}{2}\delta x \delta y$, and – for δx and δy small enough – the lines A'B' and A'C' can still be approximated as straight, so that the area A^* of the region A'B'C' can be expressed as

$$A^* = \frac{1}{2}\left| \delta \vec{r}_I \times \delta \vec{r}_{II} \right|$$

(ii) Now calculate $\delta \vec{r}_I$ and δr_{II}. We have

$$\vec{v}_A \delta t + \delta \vec{r}_I = \delta x\, \hat{i}_x + \vec{v}_B \delta t \implies \delta \vec{r}_I = \delta x\, \hat{i}_x + (\vec{v}_B - \vec{v}_A)\,\delta t$$

But, again using a differential approximation, in order to introduce velocity gradients, we write

$$\vec{v}_B - \vec{v}_A = (u_B - u_A)\,\hat{i}_x + (v_B - v_A)\,\hat{i}_y$$

$$= \left[u(x+\delta x, y, t) - u(x,y,t)\right]\hat{i}_x + \left[v(x+\delta x, y, t) - v(x,y,t)\right]$$

$$= \frac{\partial u}{\partial x}\delta x\, \hat{i}_x + \frac{\partial v}{\partial x}\delta x\, \hat{i}_y$$

where the derivatives can be evaluated anywhere between A and B, but we use the point A.

Thus

$$\delta \vec{r}_I = \delta x \, \hat{\imath}_x + \left(\frac{\partial u}{\partial x} \delta x \, \hat{\imath}_x + \frac{\partial v}{\partial x} \delta x \, \hat{\imath}_y \right) \delta t$$

$$= \delta x \left\{ \left(1 + \frac{\partial u}{\partial x} \delta t\right) \hat{\imath}_x + \frac{\partial v}{\partial x} \delta t \, \hat{\imath}_y \right.$$

By same argument, interchanging x and y, and u and v we have

$$\delta \vec{r}_{II} = \delta y \left\{ \frac{\partial u}{\partial y} \delta t \, \hat{\imath}_x + \left(1 + \frac{\partial v}{\partial y} \delta t\right) \hat{\imath}_y \right.$$

(iii) We have then

$$A^* = \frac{1}{2} \delta x \delta y \left[\left\{ \left(1 + \frac{\partial u}{\partial x} \delta t\right) \hat{\imath}_x + \frac{\partial v}{\partial x} \delta t \, \hat{\imath}_y \right\} \times \left\{ \frac{\partial u}{\partial y} \delta t \, \hat{\imath}_x + \left(1 + \frac{\partial v}{\partial y} \delta t\right) \hat{\imath}_y \right\} \right]$$

But if $\vec{a} = (a_x, a_y, 0)$, $\vec{b} = (b_x, b_y, 0)$ $|\vec{a} \times \vec{b}| = | a_x b_y - a_y b_x |$

so

$$A^* = \frac{1}{2} \delta x \delta y \left| \left[\left(1 + \frac{\partial u}{\partial x} \delta t\right) \left(1 + \frac{\partial v}{\partial y} \delta t\right) - \frac{\partial v}{\partial x} \delta t \, \frac{\partial u}{\partial y} \delta t \right] \right|$$

$$= A \left| \left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta t + \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \right) \delta t^2 \right] \right|$$

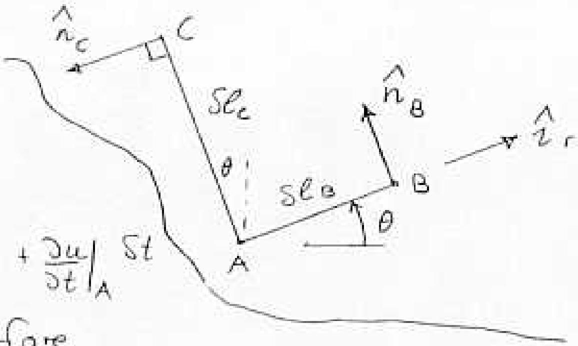
(iv) Again, as $\delta t \to 0$

$$A^* \to A \left[1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \delta t \right]$$

Whence if $A^* = A$, for small, but arbitrary, δt we must have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

6.2



In general, with $\vec{v}(\vec{r},t) = u\,\hat{i}_x + v\,\hat{i}_y$, we have, with $u = u(x,y,t)$

$$u - u_A = \delta u = \left.\frac{\partial u}{\partial x}\right|_A \delta x + \left.\frac{\partial u}{\partial y}\right|_A \delta y + \left.\frac{\partial u}{\partial t}\right|_A \delta t$$

At the given instant t, therefore

$$\delta u_B = \left.\frac{\partial u}{\partial x}\right|_A \delta x_B + \left.\frac{\partial u}{\partial y}\right|_A \delta y_B \equiv \frac{\partial u}{\partial x}\,\delta\ell_B\cos\theta + \frac{\partial u}{\partial y}\,\delta\ell_B\sin\theta$$

where, in the subsequent analysis, all derivatives are evaluated at A at time "t". Thus, as $\delta\ell_B \to 0$

$$(\vec{v}_B - \vec{v}_A) \to \delta\ell_B\left\{\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)\hat{i}_x + \left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\hat{i}_y\right\}$$

But with $\hat{i}_r = \cos\theta\,\hat{i}_x + \sin\theta\,\hat{i}_y \Rightarrow \hat{n}_B = \cos\left(\theta + \frac{\pi}{2}\right)\hat{i}_x + \sin\left(\theta + \frac{\pi}{2}\right)\hat{i}_y$

$$\Rightarrow \hat{n}_B = -\sin\theta\,\hat{i}_x + \cos\theta\,\hat{i}_y$$

$$\Rightarrow \omega_{AB} = \frac{(\vec{v}_B - \vec{v}_A)\cdot\hat{n}_B}{\delta\ell_B} = -\frac{\partial u}{\partial x}\cos\theta\sin\theta - \frac{\partial u}{\partial y}\sin^2\theta + \frac{\partial v}{\partial x}\cos^2\theta + \frac{\partial v}{\partial y}\sin\theta\cos\theta$$

Similarly, for AC, $\delta x_C = \delta\ell_C\cos(\theta + \pi/2) = -\delta\ell_C\sin\theta$, $\delta y_C = \delta\ell_C\cos\theta$

$$\Rightarrow \vec{v}_C - \vec{v}_A \to \delta\ell_C\left\{\left(-\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta\right)\hat{i}_x + \left(-\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta\right)\hat{i}_y\right\}$$

With $\hat{n}_C = \cos(\theta + \pi)\,\hat{i}_x + \sin(\theta + \pi)\,\hat{i}_y = -\cos\theta\,\hat{i}_x - \sin\theta\,\hat{i}_y$

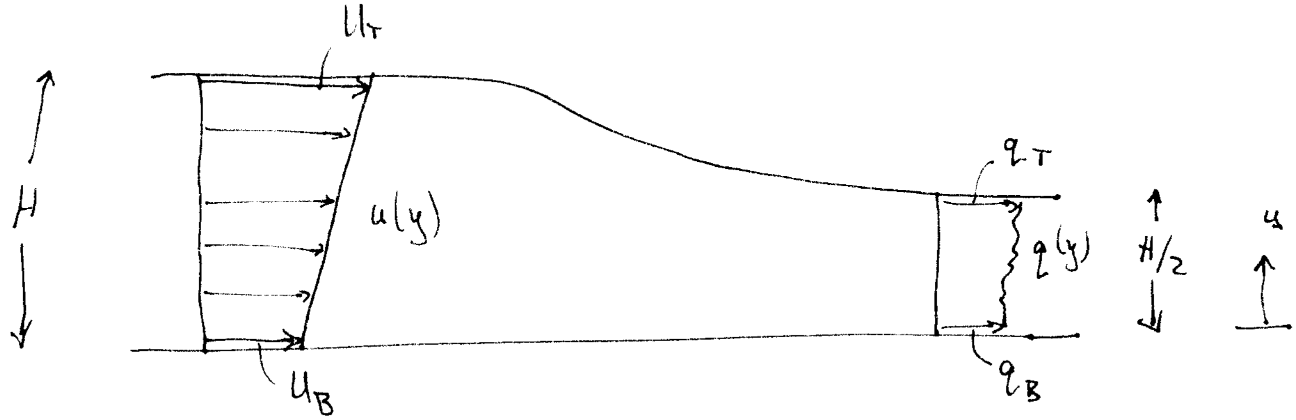
$$\omega_{AC} = \frac{(\vec{v}_C - \vec{v}_A)\cdot\hat{n}_C}{\delta\ell_C} = +\frac{\partial u}{\partial x}\cos\theta\sin\theta - \frac{\partial u}{\partial y}\cos^2\theta + \frac{\partial v}{\partial x}\sin^2\theta - \frac{\partial v}{\partial y}\sin\theta\cos\theta$$

Thus $\omega_{MEAN} = \dfrac{\omega_{AB} + \omega_{AC}}{2} = \dfrac{1}{2}\left\{\dfrac{\partial v}{\partial x}\left(\cos^2\theta + \sin^2\theta\right) - \dfrac{\partial u}{\partial y}\left(\sin^2\theta + \cos^2\theta\right)\right\}$

$$\Rightarrow \omega_{MEAN} = \frac{1}{2}\left\{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right\} = \frac{1}{2}(\nabla\times\vec{v})\cdot\hat{i}_z$$

Note: Because ω_{MEAN} is independent of θ, it is called an <u>invariant</u> of the motion

6.3 2-D Incompressible Steady Flow



Given linear velocity distribution: $u(y) = U_B + \frac{(U_T - U_B)\,y}{H}$

Continuity: $Q_w = \int_0^H u(y)\,dy = \text{const.}$

$$= \int_0^H \left(U_B + \frac{U_T - U_B}{H} y \right) dy$$

$$= U_B H + \frac{U_T - U_B}{H} \cdot \frac{H^2}{2} = \frac{H}{2}\left(U_B + U_T \right)$$

$$= \int_0^{H/2} q(y)\,dy$$

Helmholtz's Theorem: $\frac{D}{Dt}(\xi) = \frac{D}{Dt}(\nabla \times \vec{v}) = \frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

Upstream: $v = 0 \quad \therefore \quad \xi = -\frac{\partial u}{\partial y} = -\frac{U_T - U_B}{H}$

$$\Longrightarrow \xi = -\frac{U_T - U_B}{H} \quad \text{everywhere}$$

Downstream: $v = 0$ $\quad \therefore$ continuity : $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 0$

$$\Rightarrow u = u(y) \text{ only}$$
$$= q(y)$$

$$\therefore \xi = -\frac{\partial q}{\partial y} = -\frac{U_T - U_B}{H}$$

$$\Rightarrow q = \frac{U_T - U_B}{H} y + C_1 = \frac{U_T - U_B}{H} y + \underset{\substack{\uparrow \\ q(y=0)}}{q_B}$$

Continuity: $Q_w = \displaystyle\int_0^{H/2} \left(q_B + \left(\frac{U_T - U_B}{H} \right) y \right) dy = q_B \frac{H}{2} + \frac{U_T - U_B}{2H} \frac{H^2}{4}$

$$\Rightarrow q_B \frac{H}{2} + (U_T - U_B) \frac{H}{8} = \frac{H}{2} (U_B + U_T)$$

$$\therefore q_B = U_B + U_T + \frac{U_B}{4} - \frac{U_T}{4} = \frac{5}{4} U_B + \frac{3}{4} U_T$$

$$\therefore q(y) = \frac{5}{4} U_B + \frac{3}{4} U_T + \frac{U_T - U_B}{H} y$$

Bernoulli:
(along bottom streamline)

$$P_{up} + \frac{1}{2} \rho U_B^2 = P_{Down} + \frac{1}{2} \rho q_B^2$$

$$\therefore P_{up} - P_{Down} = \frac{1}{2} \rho \left(q_B^2 - U_B^2 \right)$$
$$= \frac{1}{2} \rho \left(\left(\frac{5}{4} U_B + \frac{3}{4} U_T \right)^2 - U_B^2 \right)$$
$$= \frac{\rho}{32} \left(9 U_B^2 + 30 U_B U_T + 9 U_T^2 \right)$$

6.4

a) With $\vec{v}(\vec{r},t) = u(x,y,t)\hat{i}_x + v(x,y,t)\hat{i}_y$, we have

$$u = 2xy(1+\alpha t), \quad v = (ax^2 + by^2)(1 + et + ft^2)$$

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow 2y(1+\alpha t) + 2by(1+et+ft^2) = 0$

Choose $b = -1,\ e = \alpha,\ f = 0 \Rightarrow v = (ax^2 - y^2)(1+\alpha t)$

(b) For irrotational flow, $\zeta = \partial v/\partial x - \partial u/\partial y = 0$

$$\frac{\partial u}{\partial y} = 2x(1+\alpha t) \Rightarrow \frac{\partial v}{\partial x} = 2x(1+\alpha t) \Rightarrow v = x^2(1+\alpha t) + f(y) \Rightarrow a = 1$$

$$\boxed{u = 2xy(1+\alpha t)\ ;\ v = (x^2 - y^2)(1+\alpha t)}$$

(c) The flow satisfying $\nabla\cdot\vec{v} = 0$ but not $\zeta = 0$ has $u = 2xy(1+\alpha t)$ and $v = (ax^2 - y^2)(1+\alpha t)$. For a velocity potential $\phi(x,y,t)$ to exist we must have $u = \partial\phi/\partial x$ and $u = \partial\phi/\partial y$.

Integrating $u = \frac{\partial \phi}{\partial x}$ leads to $\phi_1 = x^2 y(1+\alpha t) + f_1(y,t)$

Integrating $u = \frac{\partial\phi}{\partial y}$ leads to $\phi_2 = (ax^2y - y^3/3)(1+\alpha t) + f_2(x,t)$

Thus $\phi_1 = \phi_2$ only if $a = 1 \Rightarrow \phi = (x^2y - y^3/3)(1+\alpha t)$

(d) If the flow is to be made steady, we set $\alpha = 0 \Rightarrow$

$u = 2xy,\ v = x^2 - y^2$. Then

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow \frac{\partial p}{\partial x} = -\rho\left[2xy\cdot 2y + (x^2-y^2)2x\right] = -\rho\left[2x^3 + 2xy^2\right]$$

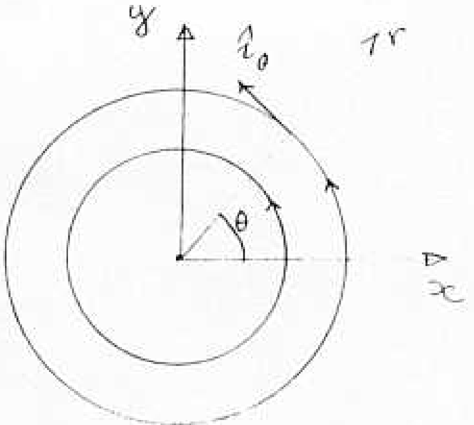
$$\Rightarrow p = -\rho\left[\frac{x^4}{2} + x^2y^2\right] + f_1(y)\ ;\ \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \Rightarrow \frac{\partial p}{\partial y} = -\rho\left[2x^2y + 2y^3\right]$$

$$\Rightarrow p = -\rho\left[x^2y^2 + \frac{y^4}{2}\right] + f_2(x)$$

Comparing $p = f_1(y) - \rho\left[\frac{x^4}{2} + x^2y^2\right]$ and $p = f_2(x) - \rho\left[x^2y^2 + \frac{y^4}{2}\right]$

we conclude that this can only be true if

6.5



(i) Here, in the absence of gravity the equation of motion is $-\nabla p/\rho = \vec{a}$ reduces to

$$-\frac{\nabla p}{\rho} = -\frac{q^2}{r}\hat{i}_r$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial p}{\partial r} = -\frac{q^2}{r} \; ; \; \frac{\partial p}{\partial \theta} = 0$$

$\Rightarrow \boxed{\frac{dp}{dr} = \frac{\rho q^2}{r}}$ But if $\frac{p}{\rho} + \frac{1}{2}q^2 = E =$ constant througho the entire field, $\boxed{\frac{dp}{dr} + \rho q\frac{dq}{dr} = 0}$ Equating these two expressions gives $\frac{dq}{dr} = -\frac{q}{r} \Rightarrow \frac{\delta q}{q} = -\frac{\delta r}{r}$

$\Rightarrow \ln q = -\ln r + C \Rightarrow q = \frac{D}{r}$ where C and D are constan

(ii) Thus $\vec{v}(\vec{r}) = \frac{D}{r}\hat{i}_\theta$ where $\hat{i}_\theta = -\sin\theta\,\hat{i}_x + \cos\theta\,\hat{i}_y$

$$\Rightarrow \vec{v}(\vec{r}) = -\frac{D\sin\theta}{r}\hat{i}_x + \frac{D\cos\theta}{r}\hat{i}_y = -\frac{Dy}{r^2}\hat{i}_x + \frac{Dx}{r^2}\hat{i}_y$$

$$\Rightarrow \boxed{\vec{v}(\vec{r}) = -\frac{Dy}{(x^2+y^2)}\hat{i}_x + \frac{Dx}{(x^2+y^2)}\hat{i}_y}$$

(iii) $$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{D}{(x^2+y^2)} - \frac{2Dx^2}{(x^2+y^2)^2} - \left(-\frac{D}{x^2+y^2} + \frac{2Dy^2}{(x^2+y^2)^2}\right)$$

$$= \frac{2D}{x^2+y^2} - \frac{2D}{(x^2+y^2)^2}[x^2+y^2] = 0$$

Thus individual particles do not rotate, even though the flow as a whole does.

$$p(x,y) = C - \rho\left[\frac{x^4}{2} + x^2y^2 + \frac{y^4}{2}\right]$$

where C is a constant. If $p \equiv P_o$ at the origin, $C = P_o$

(e) To show that Bernoulli's Theorem applies to the entire flow field, calculate

$$g(x,y) = \frac{p}{\rho} + \frac{1}{2}q^2 = \frac{p}{\rho} + \frac{1}{2}\left[u^2 + v^2\right]$$

$$= \frac{P_o}{\rho} - \left[\frac{x^4}{2} + x^2y^2 + \frac{y^4}{2}\right] + \frac{1}{2}\left[4x^2y^2 + (x^2-y^2)^2\right]$$

Thus the x and y dependence cancels, leaving $g(x,y) = \frac{P_o}{\rho}$ for the entire flow