MAT194 Final Exam 2013 Solutions

Note: These are student solutions.

1. Too easy, you can calculus.
2. Standard delta-epsilon proof
3. See Stewart Calculus 7th Ed. Chapter 3 Review Questions # 41 (page 227).

4. Let $\boldsymbol{f(x) = \frac{1}{\sqrt[4]{1+|x|x}}}$.

a) What is the domain of *f*? Where is *f* continuous? Where is *f* differentiable?

The modulus sign is giving us trouble with the function. The function is in fact piecewise defined:

$$f(x) = \begin{cases} \dfrac{1}{\sqrt[4]{1+x^2}} & x \geq 0 \\ \dfrac{1}{\sqrt[4]{1-x^2}} & -1 < x < 0 \end{cases}$$

For x greater than or equal to zero, f(x) doesn't care what x is given as the argument because it will always be positive. For x less than zero f(x) is more picky, especially as x approaches -1, then f(x) shoots off to infinity giving a discontinuity. F(x) also ceases to exist in the reals after -1.

So, the domain of *f* is $(-1, \infty)$.

F is continuous everywhere apart from 2 hotspots, -1 and 0. We have taken care of -1, now to take care of 0. The limits of f(x) from 0^+ and 0^- are both 1, hence f(x) is continuous at 0 by definition.

F(x) is continuous for x>-1.

F(x) is also differentiable for x>-1.

b) Sketch the curve of *f* indicating all important features.

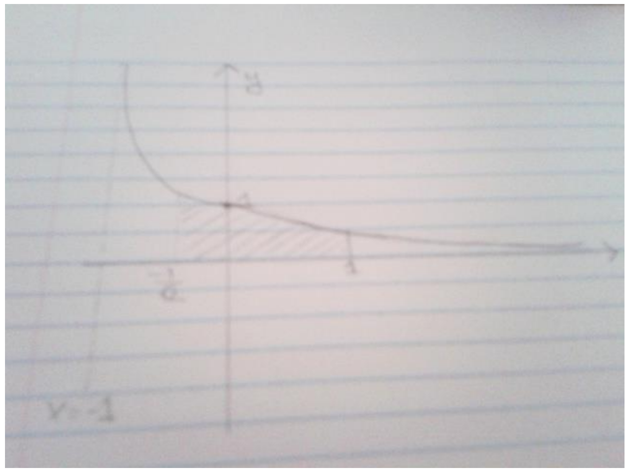
We need to find the derivatives of f(x):

$$f'(x) = \begin{cases} \dfrac{-x}{\sqrt{1+x^2}} & x \geq 0 \\ \dfrac{x}{\sqrt{1-x^2}} & -1 < x < 0 \end{cases}$$

The y-intercepts are both 1. We see that f(x) is strictly decreasing since the derivatives are always negative. The derivative is 0 at x = 0, so we have a horizontal inflection point (since f(x) is always decreasing, there is no maximum or minimum, hence this must be an infection point). The limit of as f(x) goes to infinity is 0, also the same limit for f'(x) is zero, implying that f(x) is "levelling off" as x goes to infinity. There is also a vertical asymptote at x = -1. Taking the second derivative:

$$f''(x) = \begin{cases} -\dfrac{1}{\sqrt{1+x^2}} + \dfrac{x^2}{(1+x^2)^{\frac{3}{2}}} \\ -\dfrac{1}{\sqrt{1+x^2}} - \dfrac{x^2}{(1+x^2)^{\frac{3}{2}}} \end{cases}$$

Setting f’’(x) to zero gives x = 0 as a solution.



c) Calculate the volume of revolution by rotating the region bounded by *y = f(x), y =0, x =* $\frac{-1}{\sqrt{3}}$ and *x = 1*, around the *x* axis. Simplify your answer as much as possible.

It is best to split up the volumes as the functions are – before and after *x = 0*. Using method of disks we get:

$$V_1 = \pi \int_{-\frac{1}{\sqrt{3}}}^{0} \frac{1}{\sqrt{1-x^2}} dx$$

$$V_2 = \pi \int_{0}^{1} \frac{dx}{\sqrt{1+x^2}}$$

V_1 is the arc-sine function, and directly evaluating V_1 gives:

$$V_1 = \pi \arcsin \frac{1}{\sqrt{3}}$$

V_2 is trickier to evaluate. We use a trigonometric substitution of $x = \sec u$.

$$V_2 = \pi \int \frac{\sec u \tan u}{\sqrt{1+\sec^2 u}} du$$

$$V_2 = \pi \int \sec u \, du$$

This is an integral which one must know:

$$V_2 = \pi \ln|\sec u + \tan u|$$

Using basic trigonometry, we get *x* back into the expression:

$$V_2 = \pi \ln\left|x + \sqrt{x^2 - 1}\right|$$

Now, evaluating between our limits:

$$V_2 = \pi \ln 2$$

So the total volume is:

$$V_T = \pi(\arcsin\frac{1}{\sqrt{3}} + \ln 2)$$

5. Calculate the following limits:

a) $\lim_{x\to 0}(x + \sin x) = 0$

Pretty simple, x and sin(x) are both continuous, so we can sub in 0, yielding 0.

b) $\lim_{x\to 1^+} \frac{x^2-9}{x^2+2x-3} = -\infty$

Numerator and denominator both are continuous so we can sub in the limit. The denominator approaches 0^+ as x approaches 1^+. Numerator approaches -10 as x approaches 1^+. Thus the limit overall goes towards negative infinity.

c) $\lim_{x\to 0} \frac{e^x-1-x}{x^2} = \frac{1}{2}$

Use L'Hopital's Rule twice; differentiating both numerator and denominator. This will eliminate the x^2 in the denominator, reducing it to 2. The numerator will reduce to e^x. Since this function is continuous, we can sub in 0, and this yields ½.

d) $\lim_{x\to\infty}(x^3 + \sqrt{x^6 + x^3 + 1}) = \infty$

A trickier limit to evaluate. It is best to start off by multiplying by the conjugate:

$$x^3 + \sqrt{x^6 + x^3 + 1} \times \frac{x^3 - \sqrt{x^6 + x^3 + 1}}{x^3 - \sqrt{x^6 + x^3 + 1}}$$

$$= \frac{x^6 - x^6 + x^3 + 1}{x^3 - \sqrt{x^6 + x^3 + 1}} = \frac{x^3 + 1}{x^3 - \sqrt{x^6 + x^3 + 1}}$$

Now dividing by the highest power:

$$\frac{\left(1 + \frac{1}{x^3}\right)}{1 - \sqrt{1 + \frac{1}{x^3} + \frac{1}{x^6}}}$$

Now by taking the limit, we see that all the x terms vanish and we are left with a 1/0 limit which is, in fact, infinity.

e) $\lim_{x\to\infty}(\sqrt[3]{x^3 + x^2} - \sqrt[3]{x^3 - x^2}) = 0$

We can't utilize the conjugate technique as in part (d) because we have cube roots. Let us factor out a x^3 inside the root, this is because we have a cube root:

$$\sqrt[3]{x^3\left(1+\frac{1}{x}\right)}-\sqrt[3]{x^3\left(1+\frac{1}{x}\right)}$$

$$\Rightarrow x\left(\sqrt[3]{1+\frac{1}{x}}-\sqrt[3]{1-\frac{1}{x}}\right)$$

Now, the 1/x terms will vanish as x approaches infinity making the term inside the brackets approach 0. The x outside the brackets wants to run off to infinity, but the brackets will pull it back to zero, eventually winning. Hence, the limit is 0.

6. Consider the differential equation:

$$y''+y'+y=xe^{-x}$$

a) Find the most general solution *y*.

We start with finding the characteristic equation which is:

$$r^2+r+1=0$$

Using the quadratic formula we get the roots:

$$r=\ -\frac{1}{2}\pm\frac{\sqrt{1-4}}{2}$$

$$\Rightarrow r=\ -\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$$

So the homogenous (or complementary) solution is:

$$y_C=e^{-\frac{x}{2}}(c_1\sin\left(\frac{\sqrt{3}x}{2}\right)+c_2\cos\left(\frac{\sqrt{3}x}{2}\right))$$

Now to find the particular solution y_P:

Using method of undetermined coefficients, we try a solution of the form: $y_P=\ (Ax+B)e^{-x}$

$$y_p'=-(Ax+B)e^{-x}+Ae^{-x}$$

$$y_p''=(Ax+B)e^{-x}-Ae^{-x}-Ae^{-x}=(Ax+B)e^{-x}-2Ae^{-x}$$

Substituting back into the original differential equation:

$$(Ax+B)e^{-x}-(Ax+B)e^{-x}+Ae^{-x}+(Ax+B)e^{-x}-2Ae^{-x}=xe^{-x}$$

Dividing through by e^{-x} (which is never 0, so we are allowed to do this):

$$Ax+B-Ax-B+A+Ax+B-2A=x$$

$$Ax+B-A=x$$

$$\Rightarrow A=1,B=\ -1$$

$$\Rightarrow y_P = (x-1)e^{-x}$$

Thus the general solution, *y*, is:

$$y = e^{-\frac{x}{2}}(c_1 \sin\left(\frac{\sqrt{3}x}{2}\right) + c_2 \cos\left(\frac{\sqrt{3}x}{2}\right)) + (x-1)e^{-x}$$

b) Determine $\lim_{x\to\infty} y$ for any solution y.

Using limit laws, we observe that the limits:

$$\lim_{x\to\infty} e^{-\frac{x}{2}} = 0$$

..and

$$\lim_{x\to\infty} e^{-x} = 0$$

Despite the fact that the limit does not exist for the sin and cos operators inside the brackets, multiplying by zero automatically makes the entire limit zero. This applies to both terms. Hence:

$$\lim_{x\to\infty} y = 0$$

7. Compute:

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} \frac{n}{j^2 + n^2}$$

…any way you can.

As it stands, this looks like a very unusual sum. We know that this should be a Riemann sum, given the nature of the question, which will eventually boil down to a definite integral. The general Riemann definition of the integral is (which should be familiar):

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} f(x_i^*)\Delta x = \lim_{n\to\infty} \sum_{j=1}^{\infty} f(a + j\Delta x)\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{j=1}^{\infty} f\left(a + j\left(\frac{\mathrm{b-a}}{\mathrm{n}}\right)\right)(\mathrm{b-a}) \quad (1)$$

So we need to find the values of *a*, *b* and deduce the function *f(x)*. We want to pull out a “1/n” term outside to get it into the form of equation (1). Some initial manipulation yields:

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{j=1}^{\infty} \frac{n}{\frac{j^2}{n} + n}$$

… the *n* in the numerator cancels with the *1/n* outside:

$$\lim_{n\to\infty}\sum_{j=1}^{\infty}\frac{1}{\frac{j^2}{n}+n}$$

Perhaps we can do some more manipulation to get it back into the form of (1) by factoring out an *n* downstairs:

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{j=1}^{\infty}\frac{1}{\frac{j^2}{n^2}+1}$$

The form of equation (1) is now apparent, and we can deduce that (at least at this point) that (b-a) is equal to 1. This is deduced by realizing that the sum is not multiplied by any factor i.e. there is only the *1* upstairs, and the *1/n* is pulled out. Looking back at (1) we see that $f(x_i^*)$ is $f(a + j(b-a)/n)$. But inside the sum we have just $(j/n)^2$, thus implying that a = 0, and hence b = 1. By inspection we also see that the function $f(x)$ is:

$$f(x)=\frac{1}{x^2+1}$$

Hence the sum is, in fact, the integral:

$$\int_0^1\frac{1}{x^2+1}dx\ =\arctan x\,|_0^1=\frac{\pi}{4}-0=\frac{\pi}{4}$$

A tricky question, with some foresight required.

8. Let $f(x)=x^3+3x+1$ and $g(x)=\arctan x$. How many real roots does *f(g(x))* have? Justify your answer with a proof.

Firstly we want to obtain the actual composite function *f(g(x))*, call it *h(x)*:

$$h(x)=(\arctan x)^3+3(\arctan x)+1$$

The domain of *h(x)* is $(-\infty,\infty)$, as the inverse tan function will take in any value of *x*. The range *h(x)* is determined by the range of the inverse tan function; it will spit out a number between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Now, using these range limits, we can obtain the maximum and minimum possible values of *h(x)*:

$$\min h(x)=\left(-\frac{\pi}{2}\right)^3+3\left(-\frac{\pi}{2}\right)+1=\ -\frac{\pi^3}{8}-\frac{3\pi}{2}+1<0$$

$$\max h(x)=\frac{\pi^3}{8}+\frac{3\pi}{2}+1>0$$

By looking at the maximum and minimum values, by the Intermediate Value Theorem (a proved theorem), there must be a point where *h(x)* = 0. Hence, *h(x)* must have *at least* one real root.

So, that is part of the problem solved. We need to now determine how many roots exist. We shall use Rolle’s Theorem; if f(x) is continuous on [a,b] and differentiable on (a,b) then f’(x) is equal to zero at some point for there to be two or greater roots. Now there is a slight nuance here: f(x) is continuous everywhere, but we need a closed interval – infinity doesn't count! So we can let the interval be [-N,N] where N is some very large real number. Let us take the derivative of h(x):

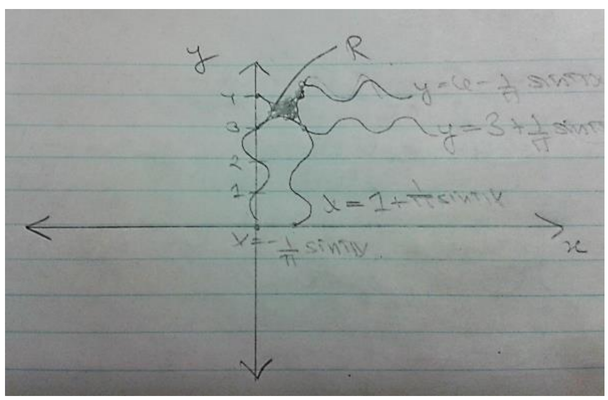
$$h'(x) = 3(\arctan x)^2 \left(\frac{1}{1+x^2}\right) + 3 > 0$$

The derivative is always greater than zero thanks to all the squared terms. Hence, there can only be 1 real root. ■

9. $\mathfrak{R}$ is the region bounded by the 4 curves $y = 3 + \frac{1}{\pi}\sin \pi x$ $\;y = 4 - \frac{1}{\pi}\sin \pi x$ $\;x = -\frac{1}{\pi}\sin \pi x$, $x = 1 + \frac{1}{\pi}\sin \pi x$.

a) Sketch $\mathfrak{R}$. You don't need to indicate important features.

Although you have to sketch 4 different functions, note that all are essentially the same function except they are all shifted or oriented different, or reversed. Note that the sin terms go to zero at every integer value of *x* or *y*, and reach a maximum at every ½ multiple of *x* or *y*. To sketch x as a function of y, simply rotate your page 90 degrees and treat it as y as a function of x. You should get the following region (perhaps not well sketched):

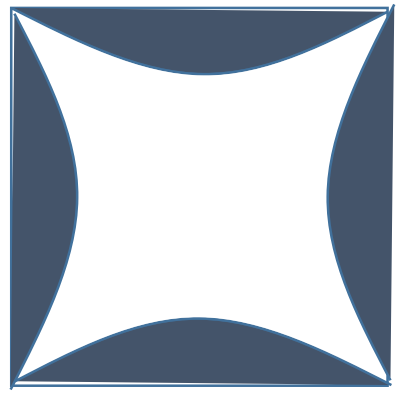


Note that R is symmetric (if drawn well, you can notice this by inspection).

b) Calculate the area of $\mathfrak{R}$.

First we need to determine the intersections. This is actually better determined by inspection as opposed to analysis. The intersections are in fact (0, 3), (0, 4), (1, 3) and (1, 4). This all relates to the fact that the functions are all the same, albeit oriented differently.

Before trying to integrate 4 functions across the x and y axis, look at the intersections; they form a square of length 1 around R, with coincident vertices. The figure below illustrates this, the unshaded region is R.



Note that each of the 4 shaded chords are of the <u>same area</u>. This area is simply that of half the period of the function (or 1 unit along either axis). This area is the area between the function $\frac{1}{\pi}\sin \pi x$ and the x axis (as can be seen in the figure). So the area of the region will be the difference of the area of the square and the area of the 4 chords: (Note: take the absolute value of the area obtained by the integral, a negative area will be meaningless in this approach to the problem)

$$A = 1 - 4\int_0^1 \frac{1}{\pi}\sin \pi x \, dx$$

$$A = 1 - \left|\frac{4}{\pi}\left(\frac{1}{\pi}\right)[\cos \pi - \cos 0]\right|$$

$$A = 1 - \frac{4}{\pi^2}(2)$$

$$A = \frac{\pi^2 - 8}{\pi^2}\ units^2$$

c) Calculate the volume obtained by rotating $\mathfrak{R}$ about the line $y = x$, using a calculus method.

This question in fact requires a very useful theorem known as Theorem of Pappus (taught in MAT195, and not mentioned in 194, but explained in the textbook):

$$V = 2\pi\bar{R}A$$

… where $\bar{R}$ is the distance from the rotation axis (y=x) to the <u>centroid</u> of the region, A is the area of this region.
The centroid of the region is exactly that, the centre of mass of the region. Thanks to the symmetry of R, this centroid can be written down directly, it is the centre of R, and hence the centre of the square: (1/2, 7/2). A is obtained from part (b). Finding the distance to the centroid from y = x involves a perpendicular distance, thus we need to form a perpendicular line passing through the centroid that intersects y=x at some point.

The gradient of this will be -1 (by properties of perpendicular lines) and so the equation of the line is (in point slope form, first):

$$y - \frac{7}{2} = -1\left(x - \frac{1}{2}\right)$$
$$\Rightarrow y = -x + 4$$

Now, equating this to y = x, we get the point of intersection (2,2).
We need to find the distance between (1/2, 7/2) and (2,2), which will yield R bar. Using the distance formula we get the distance to be:

$$\bar{R} = \frac{3\sqrt{2}}{2}$$

Multiplying all our terms together we get a total volume of:

$$V = \frac{3\sqrt{2}(\pi^2 - 8)}{\pi}\ units^3$$

An unusual question requiring out-of-course knowledge!

10. Let $f(x) = \int_2^x \frac{dt}{\ln t}$, $x \geq 2$

a) Show that there is a constant *b* such that $\int_b^{\ln x} \frac{e^t}{t} dt = f(x)$ and find the value of *b*.

On initial inspection, we see that the integrals are somewhat similar to each other in terms of e's and logarithms. The upper limit of the integral is $\ln x$, and so we want to transform $f(x)$ such that its upper limit is also $\ln x$.

Let us make the substitution $t = \ln u$ in the new integral – noting the changes in the limits of integration:

$$\int_{e^b}^{x} \frac{u}{\ln u}\left(\frac{1}{u}\right) du$$

$$\Rightarrow \int_{e^b}^{x} \frac{1}{\ln u} du = \int_2^x \frac{1}{\ln t} dt$$

$$\Rightarrow e^b = 2 \Leftrightarrow b = \ln 2$$

b) Let $g(x) = e^4 f(e^{2x-4}) - e^2 f(e^{2x-2}), x > 3$. Show that $g'(x) = e^{2x}(x^2 - 3x + 2)^{-1}$

Let us first expand this expression, substituting the arguments into $f(x)$.

$$g(x) = e^4 \int_2^{e^{2x-4}} \frac{dt}{\ln t} - e^2 \int_2^{e^{2x-2}} \frac{dt}{\ln t}$$

Using the fundamental theorem of calculus we obtain $g'(x)$ [don't forget to apply the chain rule as well, since the upper limit is a function!]

$$g'(x) = \frac{e^4(2)(e^{2x-4})}{2x-4} - \frac{e^2(2)(e^{2x-2})}{2x-2}$$

Dividing by 2, and cross multiplying to obtain a common denominator:

$$g'(x) = \frac{e^4(e^{2x-4})(x-2) - e^2(e^{2x-2})(x-1)}{x^2-3x+2}$$

$$g'(x) = \frac{xe^{2x} - 2e^{2x} - xe^{2x} + e^{2x}}{x^2-3x+2}$$

$$g'(x) = e^{2x}(x^2-3x+2)^{-1}$$

c) Express $\int_c^x \frac{e^{2t}}{t-1} dt$ in terms of $f(x)$, where $c = 1 + \frac{1}{2} \ln 2$.

Let $t = 1 + \frac{1}{2}u$:

By a careful treatment of the limits of integration, we get:

$$c = 1 + \frac{1}{2}\ln 2$$

$$c_u = \ln 2$$

$$x = x$$

$$1 + \frac{1}{2}u = x$$

$$u = 2x - 2$$

$$\int_{\ln 2}^{2x-2} \frac{e^{2\left(1+\frac{1}{2}u\right)}}{1+\frac{1}{2}u - 1} du$$

$$\Rightarrow \int_{\ln 2}^{2x-2} \frac{e^2 e^u}{u} du$$

$$\Rightarrow e^2 \int_{\ln 2}^{2x-2} \frac{e^u}{u} du$$

By inspection and from part (a), this is:

$$e^2 f(e^{2x-2})$$

…a tough question!