UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

Term Test I

MAT185H1S — Linear Algebra

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27 February 2020

Student Name:	Copy	Fair
	Last Name	First Names

Student No: ______ e-Address: ______

Signature: ______

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution. The total number of marks available is **50**.
3. Write solutions *only* in the boxed space provided for each question. *Do not* write solutions on the reverse side of pages.
4. Two blank pages are provided at the end for rough work. Work on these back pages will *not* be marked unless clearly indicated; in such cases, clearly indicate on the question page(s) that the solution(s) is continued on a back page(s).
5. *Do not* write over the QR code on the top right-hand corner of each page.
6. *No* aid is permitted. You are to attempt the test independently.
7. You must scan it your solutions and submit them to Crowdmark no later that **Friday, 9:00 am EDT, 27 March 2020**. ***No solution will be accepted after this time***.
8. There are 6 pages and 5 questions in this test paper.

A. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the small box beside the question. If true, give a proof; if false give a counterexample. One (1) mark will be given for the correct true/false determination and four (4) marks for the explanation. No mark will be deducted for a wrong answer.*

1(a). Let $\mathbf{A}$ be an $m \times n$ matrix. If $\mathbf{Ax} = \mathbf{b}$ has a solution for every $m \times 1$ column $\mathbf{b}$ then $\mathbf{A}^T\mathbf{x} = \mathbf{c}$ has a solution for every $n \times 1$ column $\mathbf{c}$.

F

(a)

For $\mathbf{A} \in {}^m\mathbb{R}^n$*, if* $\mathbf{Ax} = \mathbf{b}$ *has a solution for every* $\mathbf{b} \in {}^m\mathbb{R}$*, then* $\operatorname{col}\mathbf{A}$ *must span* ${}^m\mathbb{R}$*, i.e.,* $\dim\operatorname{col}\mathbf{A} = \operatorname{rank}\mathbf{A} = m < n$*. That means* $\dim\operatorname{col}\mathbf{A}^T = m < n$ *and therefore* $\operatorname{col}\mathbf{A}^T$ *cannot span* ${}^n\mathbb{R}$*, which would be required if* $\mathbf{A}^T\mathbf{x} = \mathbf{c}$ *has a solution for every* $\mathbf{c} \in {}^n\mathbb{R}$*.*

Note: This solution explains why a counterexample should be possible. A good answer to this question would include a specific counterexample.

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1(b). If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ an $n \times m$ matrix, where $m > n$, then both $\det\mathbf{AB} = 0$ and $\det\mathbf{AB} \neq 0$ are possible.

F

(b)

If $m > n$ *then* $\det\mathbf{AB}$ *can never be nonzero. Note that* $\operatorname{rank}\mathbf{AB}$ *must be less than either* $\operatorname{rank}\mathbf{A}$ *or* $\operatorname{rank}\mathbf{B}$*, neither of which will be greater than* n*. Thus* $\operatorname{rank}\mathbf{AB} < m$ *and as such* $\det\mathbf{AB} = 0$*.*

Alternatively, as $m > n$ *and* $\operatorname{rank}\mathbf{B} \leq n < m$*,* $\dim\operatorname{null}\mathbf{B} > 0$*. This means that* $\dim\operatorname{null}\mathbf{AB} > 0$ *and thus* $m - \operatorname{rank}\mathbf{AB} > 0$*, i.e.,* $\operatorname{rank}\mathbf{AB} < m$ *and again we must have* $\det\mathbf{AB} = 0$*.*

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B. True or False

Continued...

2(a). A nonzero square matrix $\mathbf{A}$ is *nilpotent* if $\mathbf{A}^k = \mathbf{O}$ for some positive k. Then the determinant of any nilpotent matrix is zero.

$\boxed{T}$

(a)

We observe that $\det \mathbf{A}^k = \det \mathbf{O} = 0$. Then, by the Cauchy-Binet product rule,

$$\det \mathbf{A}^k = \underbrace{(\det \mathbf{A}) \times \cdots \times (\det \mathbf{A})}_{k \text{ times}} = (\det \mathbf{A})^k = 0$$

which implies $\det \mathbf{A} = 0$.

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2(b). Suppose $\mathbf{A}$ is an $m \times n$ matrix and nullity $\mathbf{A} = k$ (i.e., $\dim \operatorname{null} \mathbf{A} = k$) then nullity $\mathbf{A}^T = n - m + k$ (i.e., $\dim \operatorname{null} \mathbf{A}^T = n - m + k$).

$\boxed{F}$

(b)

We have by the dimension formula, $\operatorname{nullity} \mathbf{A} = n - \operatorname{rank} \mathbf{A}$. So $\operatorname{rank} \mathbf{A} = n - \operatorname{nullity} \mathbf{A} = n - k$. Then

$$\dim \operatorname{null} \mathbf{A}^T = m - \operatorname{rank} \mathbf{A}^T = m - \operatorname{rank} \mathbf{A} = m - (n-k) = m - n + k \neq n - m + k$$

if $m \neq n$.

Note: This solution explains why a counterexample should be possible. A good answer to this question would include a specific counterexample.

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C. Problems

3. Let $\beta_1 = \{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$ and $\beta_2 = \{\boldsymbol{u}_1, \boldsymbol{u}_2 \cdots \boldsymbol{u}_n\}$ be two bases for some vector space $\mathcal{V}$. If the coordinates for every vector in $\mathcal{V}$ are identical with respect to both bases, does it follows that $\boldsymbol{v}_i = \boldsymbol{u}_i$, $i = 1, 2 \cdots n$? Explain.

According to the given premise, if $\mathbf{x}$ *represents the coordinates with respect to either basis,*

$$\mathbf{x} = \mathbf{P}\mathbf{x}$$

for every $\mathbf{x} \in {}^n\mathbb{R}$ *where* $\mathbf{P}$ *is the transition (transformation) matrix from* β_2 *to* β_1*, say. Let* $\mathbf{x} = \mathbf{e}_i$*, the* i*th member of the standard basis for* ${}^n\mathbb{R}$*. Then the* i*th column of* $\mathbf{P}$ *must be* $\mathbf{e}_i$*, that is, the* i*th column of the identity matrix. Do this for all* i *and we have* $\mathbf{P} = \mathbf{1}$*. The basis vectors transform as the transpose of* $\mathbf{P}$*, i.e.,*

$$\begin{aligned}
\boldsymbol{v}_1 &= p_{11}\boldsymbol{u}_1 + p_{21}\boldsymbol{u}_2 = \cdots + p_{n1}\boldsymbol{u}_n \\
\boldsymbol{v}_2 &= p_{12}\boldsymbol{u}_1 + p_{22}\boldsymbol{u}_2 = \cdots + p_{n2}\boldsymbol{u}_n \\
&\vdots \\
\boldsymbol{v}_n &= p_{1n}\boldsymbol{u}_1 + p_{2n}\boldsymbol{u}_2 = \cdots + p_{nn}\boldsymbol{u}_n
\end{aligned}$$

But all $p_{ij} = 0$ *except for the diagonal entries. Hence* $\boldsymbol{v}_i = \boldsymbol{u}_i$*,* $i = 1, 2 \cdots n$*.*

Another way to argue it is that any $\boldsymbol{v} \in \mathcal{V}$ *can be written in either basis using the same coordinates* x_i*, that is,*

$$\boldsymbol{v} = x_1\boldsymbol{v}_1 + \cdots + x_n\boldsymbol{v}_n = x_1\boldsymbol{u}_1 + \cdots + x_n\boldsymbol{u}_n$$

Let $x_i \neq 0$ *in turn while all the others are zero. Then*

$$x_i\boldsymbol{v}_i = x_i\boldsymbol{u}_i$$

Thus $\boldsymbol{v}_i = \boldsymbol{u}_i$*.*

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4. Show that an $n \times n$ matrix $\mathbf{A}$ is not invertible and $\operatorname{rank}\mathbf{A} + \operatorname{rank}(\operatorname{adj}\mathbf{A}) = n$ if and only the column space of $\operatorname{adj}\mathbf{A}$ is identical to the null space of $\mathbf{A}$, *i.e.*, $\operatorname{col}(\operatorname{adj}\mathbf{A}) = \operatorname{null}\mathbf{A}$.

[⇒] Consider an arbitrary element $\mathbf{y} \in \operatorname{col}(\operatorname{adj}\mathbf{A})$. *It can be written as* $\mathbf{y} = (\operatorname{adj}\mathbf{A})\mathbf{x}$ *for some* $\mathbf{x} \in {}^n\mathbb{R}$. *Now*

$$\mathbf{A}\mathbf{y} = \mathbf{A}(\operatorname{adj}\mathbf{A})\mathbf{x} = \mathbf{0}$$

because $\mathbf{A}$ *is not invertible making* $\det\mathbf{A} = 0$ *and as a consequence* $\mathbf{A}\operatorname{adj}\mathbf{A} = \mathbf{O}$. *So* $\mathbf{y} \in \operatorname{null}\mathbf{A}$, *i.e.*, $\operatorname{col}(\operatorname{adj}\mathbf{A}) \subseteq \operatorname{null}\mathbf{A}$. *Furthermore,*

$$\dim\operatorname{null}\mathbf{A} = \underbrace{n - \operatorname{rank}\mathbf{A} = \operatorname{rank}(\operatorname{adj}\mathbf{A})}_{\textit{given in the premise}} = \dim\operatorname{col}(\operatorname{adj}\mathbf{A})$$

and therefore $\operatorname{col}(\operatorname{adj}\mathbf{A}) = \operatorname{null}\mathbf{A}$ *(Chapter 6, Theorem VI, Property 3).*

[⇐] If $\operatorname{col}(\operatorname{adj}\mathbf{A}) = \operatorname{null}\mathbf{A}$ *then for any* $\mathbf{y} = \operatorname{adj}(\mathbf{A})\mathbf{x} \in \operatorname{col}(\operatorname{adj}\mathbf{A})$,

$$\mathbf{A}(\operatorname{adj}\mathbf{A})\mathbf{x} = \mathbf{0}$$

In particular $\mathbf{A}(\operatorname{adj}\mathbf{A})\mathbf{e}_i = \mathbf{0}$ *where* $\mathbf{e}_i$ *is a member of the standard basis of* ${}^n\mathbb{R}$. *Accordingly,*

$$\mathbf{A}(\operatorname{adj}\mathbf{A})\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \cdots \mathbf{e}_n \end{bmatrix} = \mathbf{A}(\operatorname{adj}\mathbf{A})\mathbf{1} = \mathbf{A}(\operatorname{adj}\mathbf{A}) = \mathbf{O}$$

That is, determinant of $\mathbf{A}$ *is zero so* $\mathbf{A}$ *is not invertible. Moreover, as* $\operatorname{col}(\operatorname{adj}\mathbf{A}) = \operatorname{null}\mathbf{A}$, *their dimensions are equal. So*

$$\operatorname{rank}(\operatorname{adj}\mathbf{A}) = \dim\operatorname{col}(\operatorname{adj}\mathbf{A}) = \dim\operatorname{null}\mathbf{A} = n - \operatorname{rank}\mathbf{A}$$

which leads to $\operatorname{rank}\mathbf{A} + \operatorname{rank}(\operatorname{adj}\mathbf{A}) = n$, *and that completes the proof.*

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5. If a matrix $\mathbf{A}$ has a zero row show that the maximum rank of adj $\mathbf{A}$ is 1.

The adjoint of $\mathbf{A}$ *is the transpose of its cofactor matrix* $\mathbf{C} = [c_{ij}]$*. Now*

$$c_{ij} = (-1)^{i+j} \det \mathbf{M}_{ij}(\mathbf{A})$$

where $\mathbf{M}_{ij}(\mathbf{A})$ *is the* (i, j) *minor of* $\mathbf{A}$*. Let the* k*th row of* $\mathbf{A}$ *be the zero row. Then* $\mathbf{M}_{ij}(\mathbf{A})$ *will have a zero row for any* $i \neq k$*. Accordingly,* $\det \mathbf{M}_{ij}(\mathbf{A}) = 0$ *and* $c_{ij} = 0$ *for any* $i \neq k$*. That is, all the rows of* $\mathbf{C}$ *must be zero except possibly the* k*th row. Hence the maximum rank of* adj $\mathbf{A}$ *must be 1.*

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