UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Term Test

October 29, 2020

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Q1: (all 3 parts are related)

a) Consider the linear combinations of $\vec{v} = \begin{bmatrix} +3 \\ -2 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -2 \\ +3 \\ -1 \end{bmatrix}$. Express the linear combination $c\vec{v} + d\vec{w}$ as a single vector. What is the sum of the three components of all linear combinations regardless of the values for c and d?

Solution:

Let c and d be scalars:

$$c\vec{v} + d\vec{w} = \begin{bmatrix} 3c \\ -2c \\ -c \end{bmatrix} + \begin{bmatrix} -2d \\ 3d \\ -d \end{bmatrix} = \begin{bmatrix} 3c - 2d \\ -2c + 3d \\ -c - d \end{bmatrix}$$

Sum of the 3 components:

$$3c - 2d - 2c + 3d - c - d = 0$$

b) Solve for the scalars c and d such that $c\vec{v} + d\vec{w} = \begin{bmatrix} +2 \\ -6 \\ +4 \end{bmatrix}$.

Solution:

We know there is a solution because the sum of the three components is $2 - 6 + 4 = 0$.

$$3c - 2d = 2 \quad (1)$$

$$-2c + 3d = -6 \quad (2)$$

$$-c - d = 4 \quad (3)$$

From (3):

$$c = -4 - d$$

(3)→(1):

$$3(-4 - d) - 2d = 2$$

$$-5d = 14$$

$$d = \frac{-14}{5}$$

$$\therefore c = -4 + \frac{14}{5} = \frac{-6}{5}$$

Check that (2) is satisfied:

$$-2\left(\frac{-6}{5}\right) + 3\left(\frac{-14}{5}\right) = \frac{12 - 42}{5} = \frac{-30}{5} = -6$$

Since (2) is satisfied:

$$c = \frac{-6}{5}$$

$$d = \frac{-14}{5}$$

c) Why is it not possible to find c and d such that $c\vec{v} + d\vec{w} = \begin{bmatrix} +2 \\ +6 \\ +4 \end{bmatrix}$? Answer this question by relating it to your answer in part a).

Solution: Check the sum of the 3 components:

$$2 + 6 + 4 = 12 \neq 0$$

Therefore, since the sum of the 3 components do not add up to 0, from part a), there is no linear combination of $\vec{v}$ and $\vec{w}$ such that $c\vec{v} + d\vec{w} = \begin{bmatrix} +2 \\ +6 \\ +4 \end{bmatrix}$.

Q2: (both parts are separate)

a) Consider any non-zero vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Let the angles between $\vec{v}$ and the unit vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$ be α, β, γ, respectively. Show that $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$.

Solution:

$$cos\alpha = \frac{\vec{v} \cdot \vec{\imath}}{\|\vec{v}\|\|\vec{\imath}\|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

Similarly:

$$cos\beta = \frac{\vec{v} \cdot \vec{\jmath}}{\|\vec{v}\|\|\vec{\jmath}\|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$cos\gamma = \frac{\vec{v} \cdot \vec{k}}{\|\vec{v}\|\|\vec{k}\|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$cos^2\alpha + cos^2\beta + cos^2\gamma = \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} = 1$$

b) Consider a vector of all-ones in 7 dimensions, i.e. $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

Find two vectors:

a. a unit vector pointing in the opposite direction to $\vec{v}$.

b. a unit vector that is orthogonal to $\vec{v}$.

Solution:

First find the unit vector pointing in the same direction as $\vec{v}$:

$$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

A unit vector pointing in the opposite direction to $\vec{v}$ is given by:

$$-\frac{1}{\sqrt{7}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

There are several unit vectors that are orthogonal to $\vec{v}$, so here is one of them. First find a vector that is orthogonal to $\vec{v}$:

$$\begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ 0 \end{bmatrix}$$

Therefore, a unit vector that is orthogonal to $\vec{v}$ is given by:

$$\frac{1}{\sqrt{6}}\begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ +1 \\ -1 \\ 0 \end{bmatrix}$$

Q3: (the first 2 parts are related, and the 3rd part is separate)

Consider the transformation:

$$T: R^2 \rightarrow R^1$$

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = v_1 - v_2$$

a) Is this transformation a linear transformation? (You must answer this question by determining if the transformation T satisfies both Property 1 and Property 2 for linear transformations.)

Solution:

Property 1: Does $T(c\vec{v}) = cT(\vec{v})$?

$$T(c\vec{v}) = T\left(\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}\right) = cv_1 - cv_2$$

$$cT(\vec{v}) = c(v_1 - v_2) = cv_1 - cv_2$$

∴ Property 1 is satisfied.

Property 2: Does $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\overrightarrow{w})$?

$$T(\vec{v} + \vec{w}) = T\left(\begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}\right) = v_1 + w_1 - (v_2 + w_2)$$

$$T(\vec{v}) + T(\overrightarrow{w}) = v_1 - v_2 + w_1 - w_2$$

∴ Property 2 is satisfied.

∴ T is a linear transformation.

b) If T is a linear transformation, determine the matrix that summarizes this transformation.

Solution:

Let $\vec{v} = \vec{\imath}$:

$$T(\vec{\imath}) = 1 - 0 = 1$$

Let $\vec{v} = \vec{\jmath}$:

$$T(\vec{\jmath}) = 0 - 1 = -1$$

$$\therefore M = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

c) A linear transformation must leave the zero vector unchanged, i.e. $T(\vec{0}) = \vec{0}$. Prove this from $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$ by choosing $\vec{w} =$ ___ and then finish the proof.

Solution:

Given that T is a linear transformation:

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$$

Let $\vec{w} = -\vec{v}$:

$$T(\vec{v} - \vec{v}) = T(\vec{0}) = T(\vec{v}) + T(-\vec{v})$$

From Property 2:

$$T(-\vec{v}) = -T(\vec{v})$$

$$\therefore T(\vec{0}) = T(\vec{v}) - T(\vec{v}) = \vec{0}$$

This completes the proof.

Q4: (both parts are related)

a) Find the elements c and d of the second row of matrix A to give eigenvalues 4 and 7.

$$A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$$

Solution:

$\lambda = 4$:

$$\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 4 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$w_2 = 4w_1$$

$$cw_1 + dw_2 = 4w_2$$

$$\therefore cw_1 + d(4w_1) = 4(4w_1)$$

$$(c + 4d - 16)w_1 = 0$$

Since $\vec{w} \neq \vec{0}$:

$$c + 4d - 16 = 0 \quad (1)$$

$\lambda = 7$:

$$\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 7 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$w_2 = 7w_1$$

$$cw_1 + dw_2 = 7w_2$$

$$\therefore cw_1 + d(7w_1) = 7(7w_1)$$

$$(c + 7d - 49)w_1 = 0$$

Since $\vec{w} \neq \vec{0}$:

$$c + 7d - 49 = 0 \quad (2)$$

Solving (1) and (2) for c and d:

$$16 - 4d + 7d - 49 = 0$$

$$3d = 33$$

$$d = 11$$

$$c = 16 - 4(11) = -28$$

$$\therefore A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$$

b) For matrix A found in part a), find the eigenvectors associated with the eigenvalue equal to 4.

Solution:

$$\begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 4 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$w_2 = 4w_1$$

$$-28w_1 + 11w_2 = 4w_2$$

$$\therefore \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ 4w_1 \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Q5: (the 1st part is separate, and the 2nd and 3rd parts are related)

a) Consider the linear combination:

$$3\vec{v} + 6\vec{w} + 9\vec{z} = \vec{b}$$

where:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};\ \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};\ \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Express $\vec{b}$ as a matrix-vector product $A\vec{x}$ where:

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and the columns of matrix A are parallel to $\vec{v}, \vec{w}, \vec{z}$.

Solution:

Given:

$$3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 9\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{b}$$

This can be re-expressed as:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \vec{b}$$

This can be also be written as:

$$\begin{bmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \vec{b}$$

Therefore:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

b) Find a linear combination $x_1\vec{v} + x_2\vec{w} + x_3\vec{z} = \vec{0}$ with $x_1 = 1$, where:

$$\vec{v} = \begin{bmatrix}1\\2\\3\end{bmatrix};\ \vec{w} = \begin{bmatrix}4\\5\\6\end{bmatrix};\ \vec{z} = \begin{bmatrix}7\\8\\9\end{bmatrix}$$

(Note: The vectors $\vec{v}, \vec{w}, \vec{z}$ are linearly independent if and only if the only linear combination among the 3 vectors:

$$x_1\vec{v} + x_2\vec{w} + x_3\vec{z} = \vec{0}$$

is the trivial one, i.e. scalars $x_1 = x_2 = x_3 = 0$. Therefore, your answer to this part of the question shows that these 3 vectors are dependent.)

Solution:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} + x_2\begin{bmatrix}4\\5\\6\end{bmatrix} + x_3\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$4x_2 + 7x_3 = -1 \quad (1)$$

$$5x_2 + 8x_3 = -2 \quad (2)$$

$$6x_2 + 9x_3 = -3 \quad (3)$$

$(1) \rightarrow (2)$:

$$\frac{5(-1 - 7x_3)}{4} + 8x_3 = -2$$

$$-5 - 35x_3 + 32x_3 = -8$$

$$-3x_3 = -3$$

$$x_3 = 1$$

$\rightarrow (1)$:

$$x_2 = \frac{-1 - 7(1)}{4} = -2$$

Check (3):

$$6(-2) + 9(1) = -3$$

Therefore:

$$\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

c) Because the 3 vectors $\vec{v}, \vec{w}, \vec{z}$ in part (b) are linearly dependent and are not parallel to each other, they lie in a plane that goes through the origin. Find the scalar equation for this plane.

Solution:

Use cross product of any of the two vectors to find a normal to the plane:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ +6 \\ -3 \end{bmatrix}$$

Scalar equation of a plane with this normal:

$$-3x + 6y - 3z = d$$

Given that the plane goes through the origin:

$$-3(0) + 6(0) - 3(0) = d = 0$$

Therefore:

$$-3x + 6y - 3z = 0$$

Or:

$$x - 2y + z = 0$$

Q6: (both parts are related)

a) Consider the system of three linear equations written in the form $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} +1 & 0 & 0 \\ -1 & 1 & 0 \\ +1 & -1 & +1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The inverse problem is to find $\vec{x}$, given A and $\vec{b}$. Solve the inverse problem by expressing the solution to this system for x_1, x_2, x_3 in terms of b_1, b_2, b_3.

Solution:

$$x_1 = b_1 \quad (1)$$

$$-x_1 + x_2 = b_2 \quad (2)$$

$$x_1 - x_2 + x_3 = b_3 \quad (3)$$

$(1) \rightarrow (2)$:

$$x_2 = b_2 + b_1 \quad (4)$$

$(1) and\ (4) \rightarrow (3)$:

$$x_3 = b_3 - b_1 + b_2 + b_1 = b_3 + b_2$$

Therefore:

$$x_1 = b_1$$

$$x_2 = b_2 + b_1$$

$$x_3 = b_3 + b_2$$

b) Express your answer to part (a) in the following form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = B \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where B is a $3x3$ matrix.

Solution:

From part (a):

$$x_1 = b_1$$

$$x_2 = b_2 + b_1$$

$$x_3 = b_3 + b_2$$

This may be re-expressed as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Therefore:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$