ESC195 - Final Exam
April, 2021
Part A: Q1 - Q9: time available - 150 minutes
Part B: Q10 - Q12: time available - 40 minutes (part of the 150 min total)

The following materials are considered to be acceptable aids during the writing of this test:

- The Stewart textbook and the student solution manuals
- Any course notes or problem solutions prepared by the student
- Any handouts or other materials posed on the ESC195 course website
- Any non-programmable, non-graphing calculator

All questions are worth 10 marks

Because of the availability of the text and other resources, it is important to show any work leading to your solutions. Final answers on their own will not be awarded any marks, even if they are correct.

1. Evaluate the integrals:

$$\text{a)} \int \sqrt{x} \ln x \, dx \qquad \text{b)} \int \tan^5 x \sec^3 x \, dx \qquad \text{c)} \int \frac{x^2+2}{2x^3+x} \, dx$$

2. a) Find the length of the parametric curve: $x = 3t - t^3$, $y = 3t^2$, $0 \leq t \leq 1$

b) Find the area of the surface generated by revolving the curve $r = 2 + 2\cos\theta$ about the polar axis.

3. Find the area inside the polar curve $r = 2\cos\theta$ but outside the curve $r = 2\sin 2\theta$. Provide a sketch of the area.

4. a) Determine whether the sequence converges or diverges; if it converges, find the limit:

$$\text{i) } a_n = 2 + (0.86)^n \qquad \text{ii) } a_n = \left(\frac{n}{n+5}\right)^n$$

b) Evaluate the double sum:

$$S = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots = \sum_{m=2}^{\infty} \sum_{k=2}^{\infty} \frac{1}{m^k}$$

5. a) A team of workers is constructing a tunnel under a river. Not surprisingly, the rate of construction decreases as the tunnel gets longer because rocks and earth must be removed a greater distance. Suppose that each week the team digs 0.9 of the distance it dug the previous week. In the first week, the team constucted 50 m of tunnel.

   i) How far does the team dig in 10 weeks? 20 weeks? N weeks?

   ii) What is the longest tunnel the team of workers can build at this rate?

b) For the series $\sum_{n=0}^{\infty}(n+1)^2\left(\frac{x}{x+2}\right)^n$, determine the radius and interval of convergence. When does the series converge absolutely? Conditionally?

6. a) Find the Maclaurin series for the error function defined by: $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\, dt$

b) Another way of finding the Maclaurin series for $\tan x$:

$$\text{Let} \qquad \tan x = \frac{\sin x}{\cos x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Then multiply both sides by $\cos x$ and replace both $\cos x$ and $\sin x$ by their Maclaurin series. The series for $\tan x$ can then be found by equating coefficients. Use this methiod to evaluate the coeficients of the Maclaurin series for $\tan x$ up to the x^5 term.

7. Determine the unit tangent verctor, $\vec{T}$, the principal unit normal, $\vec{N}$, the curvature, κ, and the tangential and normal components of accceleration at a general point on the curve $\vec{r}(t) = e^t\,\hat{\imath} + \sqrt{2}t\,\hat{\jmath} + e^{-t}\,\hat{k}$. Show that $\vec{a}_T + \vec{a}_N = \vec{r}\,''$.

8. Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = xz + y^2$ on the sphere $x^2 + y^2 + z^2 = 1$.

9. a) Determine whether the following vector function is a gradient $\nabla f(x, y)$. If so, find such a function f.

$$(e^x + 2xy)\hat{\imath} + (x^2 + \sin y)\hat{\jmath}$$

b) Solve the integral equation: $f(x) = 7 - 2x - 3\int_1^x e^{3(x-t)} f(t)dt$

10. Evaluate the following improper integrals, or show that they diverge:

a) $$\int_1^3 \frac{dx}{\sqrt{(x-1)(3-x)}}$$

b) $$\int_1^\infty \frac{dx}{e^{x+1}+e^{3-x}}$$

11. Use the formal definition for the derivative of a multivariable function (the $o(h)$ formulation) to find the gradient of: $f(x, y) = 2x^2y - \frac{1}{z}$. Show that all remainder terms are $o(h)$.

12. Find all of the critical points, and determine the absolute maximum and minimum values of:
$f(x, y) = (x+1)(x+y-2)(x-y-2)$ for $-2 \le x \le 3$, $-4 \le y \le 4$.