UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test III

First Year — Program 5

MAT185H1S — Linear Algebra

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Last Name First Names

Student Number: Tutorial Section: TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 10 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). The *coordinates* of $\boldsymbol{v} \in \mathcal{V}$ in a basis $B = \{\boldsymbol{b}_1 \cdots \boldsymbol{b}_n\}$ are defined as

the coefficients $\alpha_i, i = 1 \cdots n$, such that $\boldsymbol{v} = \alpha_1 \boldsymbol{b}_1 + \cdots + \alpha_n \boldsymbol{b}_n$.

/2

1(b). The (i, j)-*minor* of $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

$\mathbf{M}_{ij} \in {}^{n-1}\mathbb{R}^{n-1}$ obtained from $\mathbf{A}$ by eliminating the ith row and jth column.

/2

1(c). The (i, j)-*cofactor* of $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

$c_{ij} = (-1)^{i+j} \det \mathbf{M}_{ij}$, where $\mathbf{M}_{ij}$ is the (i, j)-minor of $\mathbf{A}$.

/2

1(d). The *Laplace expansion* of the determinant of $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} c_{ij}$$

where a_{ij} is the (i, j) entry in $\mathbf{A}$ and c_{ij} is the (i, j)-cofactor of $\mathbf{A}$.

/2

1(e). State the *Maclaurin-Cramer rule*.

The solution to $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in {}^n\mathbb{R}^n$ is invertible, can be expressed as

$$x_i = \det \mathbf{A}_i / \det \mathbf{A}$$

where x_i is the ith variable in $\mathbf{x} \in {}^n\mathbb{R}$ and $\mathbf{A}_i \in {}^n\mathbb{R}^n$ is obtained from $\mathbf{A}$ by replacing the ith column with $\mathbf{b}$.

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). The matrix

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cannot be a transformation (transition) matrix.

F

2(b). Let $\boldsymbol{v}_1 \cdots \boldsymbol{v}_m \in \mathcal{V}$, a vector space of dimension n, with coordinates $\mathbf{v}_1 \cdots \mathbf{v}_m$ in some basis. Then $\{\boldsymbol{v}_1 \cdots \boldsymbol{v}_m\}$ is linearly dependent in $\mathcal{V}$ if and only if $\{\mathbf{v}_1 \cdots \mathbf{v}_m\}$ is linearly dependent in ${}^n\mathbb{R}$.

T

2(c). The determinant of

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 \end{bmatrix}$$

is 9.

F

2(d). If $\mathbf{A}^T = -\mathbf{A} \neq \mathbf{O}$ then $\det \mathbf{A} = -1$.

F

2(e). For any $\mathbf{A} \in {}^n\mathbb{R}^n$, $\operatorname{adj}(\lambda\mathbf{A}) = \lambda^{n-1}\operatorname{adj}\mathbf{A}$ for any $\lambda \in \mathbb{R}$.

T

C. Problems

3. The first three *Legendre polynomials* are $L = \{1, x, \frac{1}{2}(3x^2 - 1)\}$ and the first three *Gegenbauer polynomials* are $G = \{1, 2x, 4x^2 - 1\}$. Note that L and G are bases for $\mathbb{P}_2$.

(a) Determine the transformation (transition) matrix from G to L.

(b) If $p(x) = 2g_1(x) + g_2(x) - 3g_3(x)$, where g_1, g_2, g_3 are the basis elements of G as given above, determine the coordinates of p in L.

3(a). Determine the transformation (transition) matrix from G to L.

We write G in terms of L:

$$\begin{aligned}
1 &= 1\cdot 1 + 0\cdot x + 0\cdot \tfrac{1}{2}(3x^2-1)\\
2x &= 0\cdot 1 + 2\cdot x + 0\cdot \tfrac{1}{2}(3x^2-1)\\
4x^2-1 &= \tfrac{1}{3}\cdot 1 + 0\cdot x + \tfrac{8}{3}\cdot \tfrac{1}{2}(3x^2-1)
\end{aligned}$$

Thus, the transformation matrix from G to L is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 2 & 0 \\ 0 & 0 & \frac{8}{3} \end{bmatrix}$$

/6

3(b). If $p(x) = 2g_1(x) + g_2(x) - 3g_3(x)$, where g_1, g_2, g_3 are the basis elements of G as given above, determine the coordinates of p in L.

The coordinates for $p(x)$ using the basis G are

$$\mathbf{v}_G = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

Therefore the coordinates in L are

$$\mathbf{v}_L = \mathbf{P}\mathbf{v}_G = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

/4

4. Define $\Delta_{\mathbf{B}}(\mathbf{A}) = \det_n \mathbf{B}\mathbf{A}\mathbf{B}^T$ for $\mathbf{A} \in {}^m\mathbb{R}^m$ and $\mathbf{B} \in {}^n\mathbb{R}^m$.

(a) Let $m < n$. Prove that $\Delta_{\mathbf{B}}(\mathbf{A}) = 0$ for all $\mathbf{A}, \mathbf{B}$.

(b) Let $m = n$. Prove that $\Delta_{\mathbf{B}}$ is a determinant function for all $\mathbf{B}$.

(c) Let $m > n$. Disprove that, for all $\mathbf{B}$, $\Delta_{\mathbf{B}}$ is a determinant function.

4(a). Let $m < n$. Prove that $\Delta_{\mathbf{B}}(\mathbf{A}) = 0$ for all $\mathbf{A}, \mathbf{B}$.

If $m < n$ *then*

$$\mathbf{B}^T\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. (Formally, this is given by the fact that $\dim \operatorname{null} \mathbf{B}^T = n - $ *rank* $\mathbf{B}$ *and rank* $\mathbf{B} \leq m$ *making* $\dim \operatorname{null} \mathbf{B}^T > 0$*.)*

Hence,

$$\mathbf{B}\mathbf{A}\mathbf{B}^T\mathbf{x} = \mathbf{0}$$

has a nontrivial solution which implies that $\mathbf{B}^T\mathbf{A}\mathbf{B}^T$ *is singular. (Formally, Theorem III, Chapter 7.)*

Therefore, $\det \mathbf{B}^T\mathbf{A}\mathbf{B}^T = 0$ *regardless of* $\mathbf{A}$ *or* $\mathbf{B}$ *and so* $\Delta_{\mathbf{B}} = 0$*.*

/4

4(b). Let $m = n$. Prove that $\Delta_{\mathbf{B}}$ is a determinant function for all $\mathbf{B}$.

If $m = n$, $\mathbf{A}$ and $\mathbf{B}$ are square. We can thus use the Cauchy-Binet product rule for determinants.

Now, we check that properties DI and DII hold:

DI. Note that

$$\Delta_{\mathbf{B}}[\mathbf{E}(1;i,j)\mathbf{A}] = \det \mathbf{B}\mathbf{E}(1;i,j)\mathbf{A}\mathbf{B}^T = \det \mathbf{B} \det \mathbf{E}(1;i,j)\mathbf{A} \det \mathbf{B}^T$$

but $\det \mathbf{E}(1;i,j)\mathbf{A} = \det \mathbf{A}$. *Hence,*

$$\Delta_{\mathbf{B}}[\mathbf{E}(1;i,j)\mathbf{A}] = \det \mathbf{B} \det \mathbf{A} \det \mathbf{B}^T = \det \mathbf{B}\mathbf{A}\mathbf{B}^T = \Delta_{\mathbf{B}}(\mathbf{A})$$

DII. Similarly,

$$\Delta_{\mathbf{B}}[\mathbf{E}(\lambda;i)\mathbf{A}] = \det \mathbf{B}\mathbf{E}(\lambda;i)\mathbf{A}\mathbf{B}^T = \det \mathbf{B} \det \mathbf{E}(\lambda;i)\mathbf{A} \det \mathbf{B}^T$$

but $\det \mathbf{E}(\lambda;i)\mathbf{A} = \lambda \det \mathbf{A}$. *Hence,*

$$\Delta_{\mathbf{B}}[\mathbf{E}(\lambda;i)\mathbf{A}] = \lambda \det \mathbf{B} \det \mathbf{A} \det \mathbf{B}^T = \lambda \det \mathbf{B} \det \mathbf{A} \det \mathbf{B}^T = \lambda \Delta_{\mathbf{B}}(\mathbf{A})$$

As DI and DII are satisfied, $\Delta_{\mathbf{B}}$ is a determinant function when $m = n$.

/4

4(c). Let $m > n$. Disprove that, for all $\mathbf{B}$, $\Delta_{\mathbf{B}}$ is a determinant function.

To disprove the result, all we have to do is find a counterexample. Consider

$$\mathbf{A} = \mathbf{1} \in {}^2\mathbb{R}^2, \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

and note that

$$\Delta_{\mathbf{B}}[\mathbf{E}(\lambda;2)\mathbf{A}] = \det \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1$$

Yet $\lambda \Delta_{\mathbf{B}}(\mathbf{A}) = \lambda$. *Clearly,* $\Delta_{\mathbf{B}}[\mathbf{E}(\lambda;2)\mathbf{A}] \neq \lambda \Delta_{\mathbf{B}}(\mathbf{A})$ *if* $\lambda \neq 1$. *Hence,* $\Delta_{\mathbf{B}}$ *fails property DII and therefore cannot be a determinant function.*

/2

5. (a) If $\mathbf{A}$ is singular, show that $\operatorname{adj}\mathbf{A}$ is also singular.

(b) Show that $(\det \mathbf{A})\operatorname{adj}\operatorname{adj}\mathbf{A} = (\det \operatorname{adj}\mathbf{A})\mathbf{A}$.

5(a). If $\mathbf{A}$ is singular, show that $\operatorname{adj}\mathbf{A}$ is also singular.

If $\mathbf{A} = \mathbf{O}$, *clearly from the definition of the adjoint,* $\operatorname{adj}\mathbf{A} = \mathbf{O}$ *and hence is singular.*

If $\mathbf{A} \neq \mathbf{O}$ *but is nonetheless singular, consider that*

$$(\operatorname{adj}\mathbf{A})\mathbf{A} = \mathbf{O}$$

because $\det\mathbf{A} = 0$. *Furthermore,*

$$(\operatorname{adj}\mathbf{A})\mathbf{A}\mathbf{x} = \mathbf{0}$$

where for some $\mathbf{x} \in {}^n\mathbb{R}$, $\mathbf{A}\mathbf{x} \neq \mathbf{0}$. *(Note* $\operatorname{col}\mathbf{A}$ *must contain nonzero vectors.) Therefore, setting* $\mathbf{z} = \mathbf{A}\mathbf{x}$,

$$(\operatorname{adj}\mathbf{A})\mathbf{z} = \mathbf{0}$$

has a nontrivial solution which means that $\operatorname{adj}\mathbf{A}$ *cannot be invertible.*

/6

5(b). Show that $(\det \mathbf{A})\operatorname{adj}\operatorname{adj}\mathbf{A} = (\det \operatorname{adj}\mathbf{A})\mathbf{A}$.

From Theorem VIII, Chapter 9 (interpreting $\mathbf{A}$ *as* $\operatorname{adj}\mathbf{A}$*),*

$$(\operatorname{adj}\mathbf{A})\operatorname{adj}\operatorname{adj}\mathbf{A} = (\det \operatorname{adj}\mathbf{A})\mathbf{1}$$

Premultiply by $\mathbf{A}$*:*

$$\mathbf{A}(\operatorname{adj}\mathbf{A})\operatorname{adj}\operatorname{adj}\mathbf{A} = (\det \operatorname{adj}\mathbf{A})\mathbf{A}$$

But $\mathbf{A}\operatorname{adj}\mathbf{A} = (\det \mathbf{A})\mathbf{1}$*, which upon substitution gives the desired result.*

/4