ESC195 - Calculus II
Final Exam April 2023

Instructor: J. W. Davis

First name (please write as legibly as possible within the boxes)

Last name

Student number

Closed book, no aid sheets, no calculators
There are 12 questions; each question is worth 10 marks.

1. Evaluate the integrals:

a) $\displaystyle\int \frac{lnx}{x^2}\,dx$ b) $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx$ c) $\displaystyle\int \frac{x}{(x+4)(2x-1)}\,dx$

2. Sketch the curves and regions indicated and find an integral(s) representing the area of the region. Do not evaluate the integrals.

   (a) Inside one loop of the curve: $r = 4\cos 3\theta$.

   (b) Inside both $r^2 = \sin 2\theta$ and $r^2 = \cos 2\theta$.

   (c) Inside the cardioid $r = 1 + \cos\theta$, but outside the circle $r = 3\cos\theta$

3. Evaluate each limit in two ways: (i) by l'Hospital's rule, (ii) by expanding the functions as Taylor series.

a) $\lim_{x \to 0} \frac{e^x - 1}{x}$ b) $\lim_{x \to 0} \frac{2\cos 2x - 2 + 4x^2}{2x^4}$ c) $\lim_{x \to 0} \frac{e^{-2x} - 4e^{-x/2} + 3}{2x^2}$

4. (a) Determine whether the series is convergent or divergent:

$$\text{i)}\ \sum_{k=1}^{\infty} ke^{-k^2} \qquad\qquad \text{ii)}\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

(b) Suppose $a_n > 0$ and $\dfrac{a_{n+1}}{a_n} \geq \dfrac{n}{n+1}$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ diverges.

5. Given the series: $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

   (a) Find the radius and interval of convergence.

   (b) Show that the series is not convergent (conditionally or otherwise) for $x < -2$.

6. Determine from first principles (that is by taking derivatives) the Taylor series for the function $f(x) = \dfrac{1}{x^2}$ about $x = 1$. Express the series in $\sum$ notation, and find the radius and interval of convergence.

7. (a) Find an equation for the tangent plane to the surface $xy^2 + 2z^2 = 12$ at the point $(1, 2, 2)$.

(b) The curve $\vec{r} = \frac{1}{2}t^2\,\hat{\imath} + \frac{4}{t}\,\hat{\jmath} + (\frac{1}{2}t - t^2)\,\hat{k}$ intersects the hyperbolic paraboloid $x^2 - 4y^2 - 4z = 0$ at the point $(2, 2, -3)$. Find the angle of intersection.

8. Use the formal definition for the derivative of a multivariable function (the $o(h)$ formulation) to find the gradient of: $f(x, y, z) = xy + yz$. Show that all remainder terms are $o(h)$.

9. Find the maximum and minimum values of $f(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$ on the the upper half plane, $y \geq 0$. Provide a sketch of the region, and identify and show the locations of all critical points.

10. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Use Lagrange Multipliers to find the the points on this ellipse that are nearest and furthest from the origin.

11. Given $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{1 - y\cos\theta}\, d\theta = 2\pi\, \frac{1 - \sqrt{1 - y^2}}{y\sqrt{1 - y^2}}$, where $0 < y < 1$, find $\displaystyle\int_0^{2\pi} \ln\left(1 - y\cos\theta\right) d\theta$

12. Show that $\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \frac{1}{6^6} + \cdots$.