MAT292 - Fall 2019

Term Test 2 - November 14, 2019

Time allotted: 105 minutes | Aids permitted: None

Total marks: 66

Full Name: ______________________________
Last | First

Student Number: ______________________________

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- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- In the first section, only answers and brief justifications are required.
- In the second section, justify your answers fully.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 10–12 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 10–12.
- No calculators, cellphones, or any other electronic gadgets are allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.

Question	Q1-Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	**Total**
Marks	10	5	5	10	8	10	8	10	**66**

HAVE FUN!

SECTION I   No explanation is necessary.   (10 marks)

1. **(2 marks)** Use Euler's method with step size $h = 0.5$ for the IVP $y' = t + y$, $y(0) = 1$.

Final answer, *no justification required*: $y(1) \approx$ ___2.5___

2. **(2 marks)** Select all properties that apply to the solution of $y'' - y' = 0$, $y(0) = 1$, $y'(0) = 1$.

○ bounded   ⊗ always positive   ○ periodic   ⊗ increasing

3. **(2 marks)** Consider the IVP $y' = f(t, y)$, $y(0) = y_0$. You estimate $y(1)$ two ways:

- Using Euler's Method with two steps ($h = 0.5$). You get $y(1) \approx y_{\text{Euler}}$.
- Using the Runge-Kutta Method with two steps ($h = 0.5$). You get $y(1) \approx y_{\text{RK}}$.

Make a choice:

○ $|y(1) - y_{\text{RK}}| \leqslant |y(1) - y_{\text{Euler}}|$ ○ $|y(1) - y_{\text{RK}}| \geqslant |y(1) - y_{\text{Euler}}|$ ⊗ This can not be decided.

Justify **briefly**: In typical situations, the Runge-Kutta method is better than Euler's Method. However depending on the ODE, sometimes we "get lucky" and Euler's Method actually gives a better approximation. To see that, one could carefully craft a "curious" direction field for which this happens (not required for full points).

4. **(2 marks)** Consider this statement: For a homogeneous system of three linear first-order ODEs, if we found three distinct solutions $\vec{x}_1$, $\vec{x}_2$, $\vec{x}_3$, then any other solution can be written as $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$ for some coefficients c_1, c_2, c_3.

Make a choice: ○ The statement is TRUE.   ⊗ The statement is FALSE.

Justify **briefly**: This statement would be true if also $W[\vec{x}_1, \vec{x}_2, \vec{x}_3] \neq 0$. Otherwise, the distinct solutions could, for example, be multiples of each other and not form a general solution.

5. **(2 marks)** Consider this statement: **There is NO** second order homogeneous linear ODE with constant coefficients such that $y(t) = 3e^{2t} + 5\cos(2t)$ is a solution.

Make a choice: ⊗ The statement is TRUE.   ○ The statement is FALSE.

Justify **briefly**: There are only three cases for the characteristic equation of a second order homogeneous ODE with constant, **real** coefficients: Two real roots, one real root, two complex roots. e^{2t} can only appear in the case of one or two real roots. $\cos(2t)$ can only appear in the case of complex roots. These cases are mutually exclusive. Alternatively, substitute into $y'' + ay' + by = 0$ to obtain inconsistent equations: $12 + 6a + 3b = 0, -20 + 5b = 0, -10a = 0$.

SECTION II **Justify** your answers. **(56 marks)**

6. **(5 marks)** Show the following: If $Y_1(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_1(t)$, and $Y_2(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_2(t)$, then $Y(t) = Y_1(t) + Y_2(t)$ is a particular solution of $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$

Solution: Since $Y_1(t)$ and $Y_2(t)$ are solutions, we have

$$Y_1'' + p(t)Y_1' + q(t)Y_1 = g_1(t),$$
$$Y_2'' + p(t)Y_2' + q(t)Y_2 = g_2(t).$$

Adding the equations gives

$$Y_1'' + Y_2'' + p(t)Y_1' + p(t)Y_2' + q(t)Y_1 + q(t)Y_2 = g_1(t) + g_2(t),$$

which can be written as

$$(Y_1 + Y_2)'' + p(t)(Y_1 + Y_2)' + q(t)(Y_1 + Y_2) = g_1(t) + g_2(t),$$

since differentiation and multiplication by a scalar are linear. This shows that $Y(t) = Y_1(t) + Y_2(t)$ is a solution to $y'' + p(t)y' + q(t)y = g_1(t) + g_2(t)$.

7. **(5 marks)** Consider an initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ such that $y'' > 0$ at all times.

When using Euler's Method to numerically approximate the solution to this IVP, explain why, at every step, you would underestimate $y(t)$.

Solution: Consider the first step. Euler's Method estimates

$$y(t_1) \approx y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

This means the step follows the line tangent to the graph of y. By assumption, y is concave up. Therefore the tangent line lies below the graph and the same is true for our estimate.

Now, for the second step, note that Euler's method follows the line tangent to a solution $\overline{y}$ with $\overline{y}(t_1) = y_1$. This solution is also concave up, for sufficiently small steps. By the same argument as above, the step from Euler's Method will stay below the graph of this solution, and therefore also below the graph of the actual solution of the IVP. The argument for the second step extends to all subsequent steps. For large step sizes, it is possible for $\overline{y}$ to not be concave up and therefore the numerical estimates might not be underestimates. See page 10 for an example.

8. Solve the initial value problem

$$y'' + y' - 2y = -10\sin(t), \quad y(0) = 1, \;\; y'(0) = 0,$$

using the following three steps.

(a) **(3 marks)** Use the characteristic polynomial to solve the complementary equation.

Solution: The characteristic polynomial $\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2) = 0$ so that the complementary equation has the general solution $y_{\mathrm{c}} = c_1 e^t + c_2 e^{-2t}$.

(b) **(4 marks)** Use the method of undetermined coefficients to find a particular solution.

Solution: Since $\sin(t)$ is not a solution to the complementary equation, we try the form for the particular solution $y_{\mathrm{p}} = A\sin(t) + B\cos(t)$. Substitution gives

$$\begin{aligned} y_{\mathrm{p}}'' + y_{\mathrm{p}}' - 2y_{\mathrm{p}} &= -A\sin(t) - B\cos(t) + A\cos(t) - B\sin(t) - 2A\sin(t) - 2B\cos(t) \\ &= (-3A - B)\sin(t) + (-3B + A)\cos(t) \\ &= -10\sin(t) + 0\cos(t) \end{aligned}$$

Matching coefficients gives $-3A - B = -10, -3B + A = 0$, which has the unique solution $A = 3, B = 1$ giving the general solution

$$y = c_1 e^t + c_2 e^{-2t} + 3\sin(t) + \cos(t).$$

(c) **(3 marks)** Now solve the initial value problem.

Solution: The initial conditions give $c_1 + c_2 + 1 = 1, c_1 - 2c_2 + 3 = 0$ with solution $c_1 = -1, c_2 = 1$. Therefore the solution is

$$y = -e^t + e^{-2t} + 3\sin(t) + \cos(t).$$

9. Consider the ODE problem

$$x_1'(t) = x_2(t), \quad x_2'(t) = -\frac{\pi^2}{4} x_1(t), \quad x_1(0) = 0, \quad x_2(1) = 0.$$

Note in the conditions that x_1 is evaluated at $t = 0$ and x_2 is evaluated at $t = 1$.

(a) **(4 marks)** Verify that it has solutions $x_1(t) = A \sin\left(\frac{\pi}{2} t\right), x_2(t) = A \frac{\pi}{2} \cos\left(\frac{\pi}{2} t\right)$ for any $A \in \mathbb{R}$.

Solution: Computing the derivatives, we see that

$$x_1'(t) = A \cos\left(\frac{\pi}{2} t\right) \frac{\pi}{2} = x_2(t),$$

and

$$x_2'(t) = -A \frac{\pi}{2} \sin\left(\frac{\pi}{2} t\right) \frac{\pi}{2} = -\frac{\pi^2}{4} x_1(t).$$

Checking the conditions,

$$x_1(0) = A \sin(0) = 0, \quad x_2(1) = A \frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) = 0.$$

(b) **(4 marks)** Why does this result not violate the existence-uniqueness theorem? Explain.

Solution: The existence-uniqueness theorem applies to initial value problems. In an initial value problem, the unknown functions (and an appropriate number of their derivatives) must be given at a single time. The above problem specifies the unknowns at different times $t = 0$ and $t = 1$. The existence-uniqueness theorem does not apply to the above problem.

10. A MAT292 student implemented the improved Euler method for an initial value problem $x'(t) = f(t,x),\ x(0) = 0,\ t \in [0,1]$. The student's MATLAB code below produced the results in the table for various values of the stepsize h and number of steps N

N	h	estimated $x(1)$	estimated error
10	0.10000	-0.07394	0.00666
20	0.05000	-0.07791	0.00269
40	0.02500	-0.07942	0.00118
80	0.01250	-0.08005	0.00055
160	0.00625	-0.08034	0.00026
320	0.00313	-0.08047	0.00013

```
t = 0; x = 0; h = 1/N;
for n=1:N
   k1 = f(t,x);
   k2 = f(t,x+h*k1);
   x = x + h*(k1+k2)/2;
   t = t + h;
end
```

(a) **(4 marks)** Considering the table on the left, for what order k do you estimate that the error can be bounded in terms of $C \cdot h^k$, for some constant C? Explain.

Solution: The empirical order of convergence appears to be one, i.e. $k = 1$, since halving h approximately halves the error.

(b) **(2 marks)** What order k did you expect?

Solution: The improved Euler method is expected to be second order, i.e. $k = 2$.

(c) **(4 marks)** Find the bug in the code that lead to the discrepancy between parts (a) and (b).

Solution: The bug in the code is that the time for which f is evaluated for `k2` should be `t+h`, not `t`. It is interesting that the method converged despite the bug. The results for the corrected code are below.

N	h	estimated $x(1)$	estimated error
10	0.10000	-0.08519	0.00459
20	0.05000	-0.08184	0.00124
40	0.02500	-0.08092	0.00032
80	0.01250	-0.08068	0.00008
160	0.00625	-0.08062	0.00002
320	0.00313	-0.08061	0.00000

11. Let $A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$.

(a) **(5 marks)** Use the definition of the matrix exponential to compute e^{At}.

Solution: Since $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A^p, p \geqslant 2$,

$$e^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + tA = \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix}.$$

(b) **(3 marks)** Substitute $-t$ for t to explicitly verify for this matrix A that $e^{At}e^{-At} = I$.

Solution:

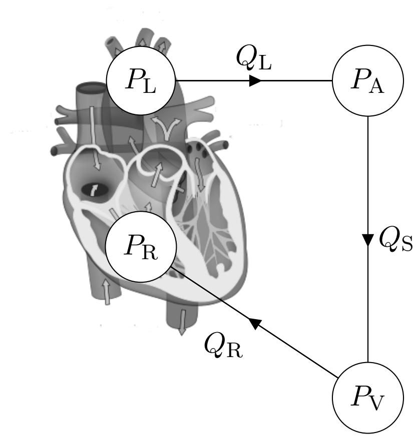
$$e^{-At} = \begin{pmatrix} 1-2t & t \\ -4t & 1+2t \end{pmatrix}$$

Explicitly computing the product gives

$$\begin{aligned} e^{At}e^{-At} &= \begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix} \begin{pmatrix} 1-2t & t \\ -4t & 1+2t \end{pmatrix} \\ &= \begin{pmatrix} (1+2t)(1-2t) - t(-4t) & (1+2t)t - t(1+2t) \\ 4t(1-2t) + (1-2t)(-4t) & 4t^2 + (1-2t)(1+2t) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

12. The human systemic circulatory system can be modelled with a system of differential equations based on the following assumptions. Time t is given in minutes.

There are *four compartments*: **L**eft heart, **A**rteries, **V**eins, and **R**ight heart. They are connected by *three vessel networks*. The flow through each network, given in L/min, is described by Q_L, Q_S and Q_R.



(a) **(1 mark)** The flow out of the heart is given as

$$Q_\mathrm{L}(t) = 5 + 5\sin\left(\frac{2\pi t}{T}\right).$$

What do you think is a realistic value for the parameter T?

Solution: T is the period of the heart beat. A typical heart rate is between 70 and 100 beats/min, so $T \approx 1/100$ min to $T \approx 1/70$ min.

(b) **(2 marks)** The pressure in the compartments A and V, measured in mmHg, is $P_\mathrm{A}(t)$ and $P_\mathrm{V}(t)$. The volume of blood, measured in litres, is $V_\mathrm{A}(t)$ and $V_\mathrm{V}(t)$. We assume that in each case, volume is proportional to pressure. Express this in two formulas introducing constants.

Solution: $P_\mathrm{A}(t) = c_\mathrm{A}V_\mathrm{A}(t)$ and $P_\mathrm{V}(t) = c_\mathrm{V}V_\mathrm{V}(t)$ for constants c_A and c_V.

(c) **(2 marks)** Blood flows between connected compartments at a rate proportional to their difference in pressures, from the high pressure to the low pressure compartment. Use this fact to express Q_S in terms of P_A and P_V, also introducing a constant.

Solution: $Q_\mathrm{S} = g_\mathrm{S} \cdot (P_\mathrm{A} - P_\mathrm{V})$ for constant $g_\mathrm{S} > 0$.

(d) **(1 mark)** Using the same assumptions, now also express Q_R in terms of P_V and P_R.

Solution: $Q_\mathrm{R} = g_\mathrm{R} \cdot (P_\mathrm{V} - P_\mathrm{R})$ for constant $g_\mathrm{R} > 0$.

For the remainder of the question, consider that the net flow of blood into a compartment changes its volume. For example, $\frac{d}{dt}V_{\mathrm{A}} = Q_{\mathrm{L}} - Q_{\mathrm{S}}$.

Also note that the pressure in compartment R can be assumed to be constant at $P_{\mathrm{R}} = 5\mathrm{mmHg}$.

(e) **(3 marks)** Use the assumptions and diagram above to write down a system of differential equations for $P_{\mathrm{V}}(t)$ and $P_{\mathrm{A}}(t)$.

Solution: Combining the above equations, we have

$$\frac{d}{dt}V_{\mathrm{A}} = \frac{d}{dt}(c_{\mathrm{A}}P_{\mathrm{A}}) = Q_{\mathrm{L}} - Q_{\mathrm{S}},$$
$$\frac{d}{dt}V_{\mathrm{V}} = \frac{d}{dt}(c_{\mathrm{V}}P_{\mathrm{V}}) = Q_{\mathrm{S}} - Q_{\mathrm{R}},$$

which can be re-written as

$$\frac{d}{dt}P_{\mathrm{A}} = \frac{Q_{\mathrm{L}}}{c_{\mathrm{A}}} + \frac{g_{\mathrm{S}}}{c_{\mathrm{A}}}(P_{\mathrm{V}} - P_{\mathrm{A}}),$$
$$\frac{d}{dt}P_{\mathrm{V}} = \frac{g_{\mathrm{S}}}{c_{\mathrm{V}}}(P_{\mathrm{A}} - P_{\mathrm{V}}) + \frac{g_{\mathrm{R}}}{c_{\mathrm{V}}}(P_{\mathrm{R}} - P_{\mathrm{V}}).$$

(f) **(1 mark)** If a person, due to injury, were to lose a significant amount of blood, how would you account for this within the model?

Solution: There are many ways to answer this question. If the blood loss was quick, you could instantaneously drop V_{V} or V_{A} (or both), which would cause instantaneous drops to P_{V} and P_{A}. If the blood loss takes place over a prolonged duration then a flow can be added exiting the arteries or the veins (or both). The flows could be modelled as being proportional to $P_{\mathrm{V}} - P_0$ and $P_{\mathrm{A}} - P_0$ for P_0, the atmospheric pressure. During blood loss, the heart beats faster, which is represented in the model by decreasing T.

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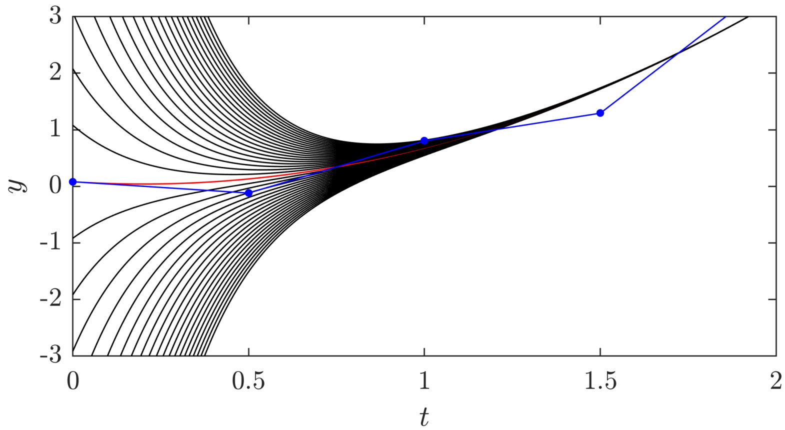
With respect to question 7, consider the differential equation $y' = f(t, y) = -5y + 5t^2$ with general solution $y = t^2 - \frac{2}{5}t + \frac{2}{25} + Ce^{-5t}$. For the initial condition $y(0) = \frac{2}{25}$, $C = 0$ and $y'' > 0$. The first step of Euler's method will always produce an underestimate: $y_1 = y_0 + \Delta t f(t_0, y_0) \leqslant y(t_0 + h)$. The exact solution $\bar{y}$ passing through the perturbed point $(t_0 + h, y_1)$ need not be concave up, which means that subsequent steps may not be underestimates. The plot below shows Euler's method in blue with $\Delta t = 0.5$, the true solution in red, and neighbouring solutions in black. The Euler steps alternate being above and below the true solution.



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