MAT195S CALCULUS II

Midterm Test #1

12 February 2019 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

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Family Name: J Davis

Given Name: Solutions

Student #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	11	
2	9	
3	12	
4	8	
5	8	
6	12	
7	9	
8	7	
TOTAL	76	/70

Tutorial Section:

TA Name:

1) Evaluate the following integrals.

a) $\int \tan x \sec^4 x \, dx$

b) $\int \frac{10}{(x-1)(x^2+9)} dx$

c) $\int (\arcsin x)^2 \, dx$

(11 marks)

a) $\int \tan x \sec^4 x \, dx = \int \tan x (1+\tan^2 x) \sec^2 x \, dx$

$= \int \tan x \sec^2 x \, dx + \int \tan^3 x \sec^2 x \, dx$

$= \frac{1}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C$

b) $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \Rightarrow 10 = A(x^2+9) + (Bx+C)(x-1)$

$= Ax^2 + 9A + Bx^2 + Cx - Bx - C$

let $x=1 \Rightarrow 10 = A + 9A \Rightarrow A = 1$

$x^2: \; 1 + B = 0 \Rightarrow B = -1$

$1: \; 10 = 9 - C \Rightarrow C = -1$

$\int \frac{10}{(x-1)(x^2+9)} dx = \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+9} dx = \ln|x-1| - \frac{1}{2} \int \frac{2x\,dx}{x^2+9} - \int \frac{dx}{x^2+9}$

$= \ln|x-1| - \frac{1}{2} \ln(x^2+9) - \frac{1}{3} \arctan \frac{x}{3} + C$

c) $\int (\arcsin x)^2 dx$ let $u = \arcsin^2 x$ $\quad dv = dx$

$du = 2 \arcsin x \cdot \frac{dx}{\sqrt{1-x^2}}$ $\quad v = x$

$= x \arcsin^2 x - \int 2 \arcsin x \cdot \frac{x\,dx}{\sqrt{1-x^2}}$ let $u = \arcsin x$ $\quad dv = \frac{x\,dx}{\sqrt{1-x^2}}$

$du = \frac{dx}{\sqrt{1-x^2}}$ $\quad v = -\sqrt{1-x^2}$

$= x \arcsin^2 x + 2\sqrt{1-x^2} \arcsin x - 2 \int \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} dx$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2} \arcsin x - 2x + C$

2) a) Determine whether the integral $\int_0^1 \frac{3\sec^2 x}{x^3}dx$ converges or diverges. If it converges, determine its value.
(4 marks)

$\lim_{x\to 0} \frac{3\sec^2 x}{x^3}$ DNE $\therefore$ improper integral.

Now $\cos x < 1$ for $0 < x < 1$ $\therefore$ $\sec x > 1$ $\therefore$ $\sec^2 x > 1$

$\therefore \frac{3\sec^2 x}{x^3} > \frac{1}{x^3} \quad 0 < x < 1$

$\Rightarrow \int_0^1 \frac{dx}{x^3} = \lim_{t\to 0^+} \left[\frac{-1}{3x^2}\right]_t^1 \longrightarrow \infty$

$\therefore \int_0^1 \frac{3\sec^2 x}{x^3}dx$ is divergent by comparison

b) The Gamma function is defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$. Show that $\Gamma(4) = 3!$.

(5 marks)

$\Gamma(4) = \int_0^\infty t^3 e^{-t}\,dt$ let $u = t^3$, $du = 3t^2dt$; $dv = e^{-t}dt$, $v = -e^{-t}$

$= \left[-t^3e^{-t}\right]_0^\infty + 3\int_0^\infty t^2e^{-t}dt$

$= \underbrace{\lim_{t\to\infty}\left(-\frac{t^3}{e^t}\right)} + 0 + 3\int_0^\infty t^2e^{-t}dt = 3\int_0^\infty t^2e^{-t}dt.$

$\stackrel{*}{=} \lim_{t\to\infty} \frac{-3t^2}{e^t} \stackrel{*}{=} \lim_{t\to\infty}\frac{-6t}{e^t} \stackrel{*}{=} \lim_{t\to\infty}\frac{-6}{e^t} \to 0$

$\Rightarrow 3\int_0^\infty t^2e^{-t}dt = 3\left[-t^2e^{-t}\right]_0^\infty + 3\cdot 2\int_0^\infty te^{-t}dt = 6\int_0^\infty te^{-t}dt$

$= 6\left[-te^{-t}\right]_0^\infty + 3\cdot 2\int_0^\infty e^{-t}dt$

$= 3\cdot 2\left[-e^{-t}\right]_0^\infty = 3\cdot 2\cdot 1 = 3!$

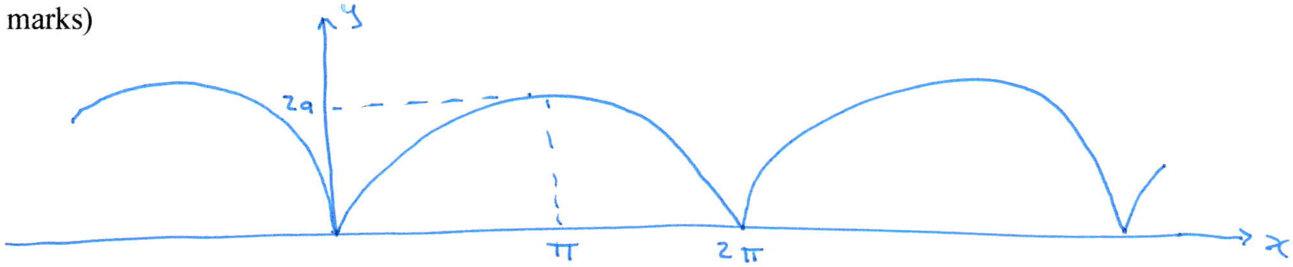
3) Given the parametric equations for a cycloid: $x(\theta) = a(\theta - \sin\theta)$; $y(\theta) = a(1-\cos\theta)$
   a) Provide a sketch of the curve
   b) Find the length of one arch of the cycloid.
   c) Find the area under one arch of the cycloid
   d) Find the surface area of revolution of one arch of the cycloid about the x-axis.

Hints: 1) $\sin^2\phi = \frac{1}{2}(1-\cos 2\phi)$

2) $\int \sin^3 x\,dx = -\frac{1}{3}(2+\sin^2 x)\cos x + C$

(12 marks)

a)



b) $x(\theta) = a(\theta - \sin\theta)$, $x'(\theta) = a - a\cos\theta$; $y(\theta) = a(1-\cos\theta)$, $y'(\theta) = a\sin\theta$

$$s = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\,d\theta = \int_0^{2\pi} a\sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}\,d\theta = a\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos\theta}\,d\theta$$

$\Rightarrow$ let $\theta = 2x$ $\quad\therefore\ \cos\theta = \cos 2x = 1 - 2\sin^2 x$

$d\theta = 2dx$ $\quad$ or $2\sin^2 x = 1 - \cos 2x = 1 - \cos\theta$

$$= a\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2 x}\,d\theta = 2a\int_0^{2\pi}\sin x\,d\theta = 2a\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= 2a\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}\cdot 2 = 4a(-(-1)+1) = \boxed{8a}$$

c) $$A = \int_0^{2\pi} y(\theta)\,x'(\theta)\,d\theta = \int_0^{2\pi} a(1-\cos\theta)\cdot a(1-\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)\,d\theta = a^2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \boxed{3\pi a^2}$$

d) $$A = \int 2\pi y\,ds = \int_0^{2\pi} 2\pi a(1-\cos\theta)\cdot\sqrt{2}a\sqrt{1-\cos\theta}\,d\theta = 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos\theta)^{3/2}\,d\theta$$

$\Rightarrow$ let $\theta = 2x$ $\Rightarrow$ $1 - \cos 2x = 2\sin^2 x$

$$= 2\sqrt{2}\pi a^2\int_0^{2\pi} 2^{3/2}\sin^3 x\,d\theta = 8\pi a^2\int_0^{2\pi}\sin^3\frac{\theta}{2}\,d\theta$$

$$= 8\pi a^2\left[2\left(-\frac{1}{3}\left(2+\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\right]_0^{2\pi} = 16\pi a^2\left(-\frac{1}{3}(2+0)(-1) + \frac{1}{3}(2+0)(1)\right)$$

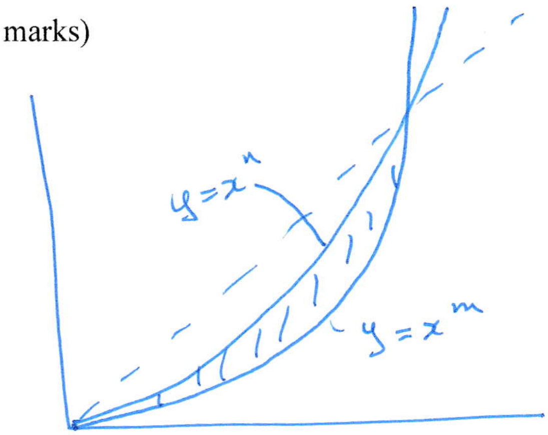
$$= 16\pi a^2\left(\frac{2}{3}+\frac{2}{3}\right) = \boxed{\frac{64}{3}\pi a^2}$$

4) Let $\Re$ be the region that lies between the curves: $y = x^m$, $y = x^n$, $0 \le x \le 1$, where m and n are integers with $0 \le n < m$.

a) Sketch the region $\Re$.
b) Find the coordinates of the centroid (centre of mass) of $\Re$.
c) Show that for $n = 3$ and $m = 4$, the centroid lies outside the region $\Re$.

(8 marks)

a)



$$A = \int_0^1 (x^n - x^m)\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$$

b)
$$\bar{x}A = \int_0^1 x(x^n - x^m)\,dx = \left[\frac{x^{n+2}}{n+2} - \frac{x^{m+2}}{m+2}\right]_0^1 = \frac{1}{n+2} - \frac{1}{m+2} = \frac{m-n}{(n+2)(m+2)}$$

$$\therefore \bar{x} = \frac{(n+1)(m+1)}{(n+2)(m+2)}$$

$$\bar{y}A = \int_0^1 \frac{1}{2}\left(x^{2n} - x^{2m}\right)dx = \frac{1}{2}\left[\frac{x^{2n+1}}{2n+1} - \frac{x^{2m+1}}{2m+1}\right]_0^1 = \frac{1}{2}\left(\frac{1}{2n+1} - \frac{1}{2m+1}\right) = \frac{m-n}{(2n+1)(2m+1)}$$

$$\therefore \bar{y} = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

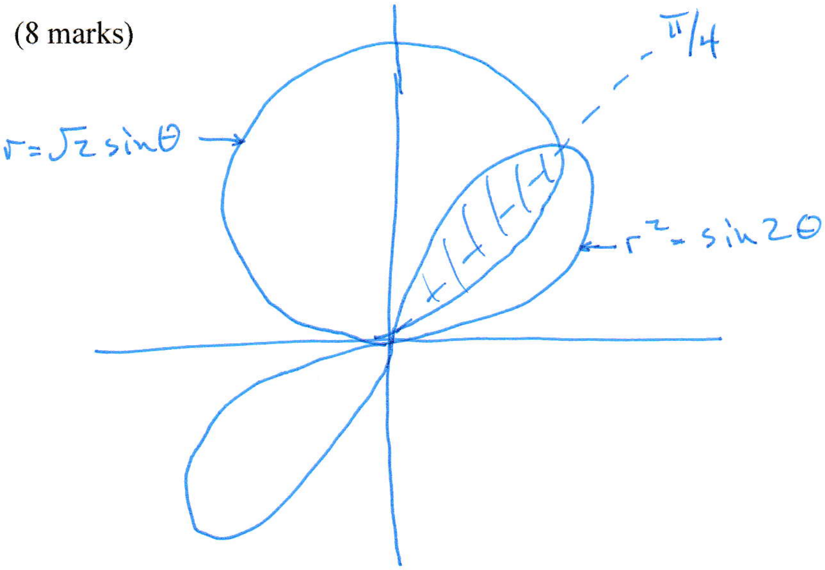
c) $n=3,\ m=4 \Rightarrow \bar{x} = \dfrac{4\cdot 5}{5\cdot 6} = \dfrac{2}{3}$ $\quad \bar{y} = \dfrac{4\cdot 5}{7\cdot 9} = \dfrac{20}{63}$

now $\left(\dfrac{2}{3}\right)^3 = \dfrac{8}{27} < \dfrac{20}{63}$ $\quad\therefore$ the centroid is above the upper curve, $y=x^n$, and thus is outside the region.

$y=x^3$

5) Find the area of the region that lies inside both the circle $r = \sqrt{2}\sin\theta$ and inside the lemniscate $r^2 = \sin 2\theta$. Provide a sketch of the region.

(8 marks)



points of intersection: $2\sin^2\theta = \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin\theta = \cos\theta \Rightarrow \theta = \frac{\pi}{4}$

$$A = \frac{1}{2}\int_0^{\pi/4} (\sqrt{2}\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin 2\theta\, d\theta$$

$$= \left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/4} - \frac{1}{4}\left[\cos 2\theta\right]_{\pi/4}^{\pi/2}$$

$$= \frac{\pi}{8} - \frac{1}{4} - 0 + 0 - \frac{1}{4}(-1-0)$$

$$= \frac{\pi}{8}$$

6) Determine whether the sequence converges or diverges. If it converges, find the limit:

(i) $a_n = \dfrac{(-1)^n n^2 \cos n}{n^4+1}$ (ii) $a_n = \dfrac{\ln(n^{10})}{n^{1/10}}$

(iii) $a_n = \sqrt{n+1} - \sqrt{n-1}$ (iv) $a_n = 2^n \cdot 3^{1-n}$

(12 marks)

i) $\dfrac{-n^2}{n^4+1} \le \dfrac{(-1)^n n^2 \cos n}{n^4+1} \le \dfrac{n^2}{n^4+1}$

$\lim_{n\to\infty} \dfrac{n^2}{n^4+1} = \lim_{n\to\infty} \dfrac{1}{n^2+1/n^2} \to 0$

$\therefore \dfrac{(-1)^n n^2 \cos n}{n^4+1} \to 0$ by pinching theorem

ii) $\dfrac{\ln n^{10}}{n^{1/10}} = \dfrac{10 \ln n}{n^{1/10}}$

$\lim_{x\to\infty} \dfrac{10 \ln x}{x^{1/10}} \overset{*}{=} \lim_{x\to\infty} \dfrac{10/x}{\frac{1}{10} x^{-9/10}} = \lim_{x\to\infty} 100\, x^{-1/10} \to 0$

$\therefore \dfrac{\ln n^{10}}{n^{1/10}} \to 0$

iii) $\sqrt{n+1} - \sqrt{n-1} \cdot \dfrac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n+1}} = \dfrac{(n+1)-(n-1)}{\sqrt{n+1}+\sqrt{n-1}} = \dfrac{2}{\sqrt{n+1}+\sqrt{n-1}} \to 0$

iv) $\dfrac{2^n}{3^{n-1}} = 3\left(\dfrac{2}{3}\right)^n \to 0$ $\left(x^n \text{ with } x < 1\right)$

7) a) Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty}(a_n b_n) = 0$.

(5 marks)

Given $\{b_n\}$ is a bounded sequence: $|b_n| \leq M \therefore |a_n||b_n| \leq |a_n| M$

Given $\epsilon > 0$, $\lim_{n\to\infty} a_n = 0$ means that $|a_n - 0| < \epsilon$ for $n > N$

similarly, $|a_n - 0| < \frac{\epsilon}{M}$ for $n > N'$

Then $|a_n b_n - 0| = |a_n b_n| = |a_n||b_n| \leq |a_n| M = |a_n - 0| M < \frac{\epsilon}{M} M = \epsilon$

for all $n > N'$

Since this is valid for all $\epsilon > 0$, $\lim_{n\to\infty}(a_n b_n) = 0$

b) Suppose that $\sum_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ is known to be a convergent series. Prove that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is a divergent series.

(4 marks)

If $\sum a_n$ is convergent, then $\lim_{n\to\infty} a_n \to 0$

If $\lim_{n\to\infty} a_n = 0$ then $\lim_{n\to\infty} \frac{1}{a_n} \neq 0$

$\therefore \sum \frac{1}{a_n}$ diverges by the test for divergence

8) a) A series $\sum_{k=1}^{\infty} a_k$ has partial sums, s_n, given by $s_n = \dfrac{7n-2}{n}$

i) Is $\sum_{k=1}^{\infty} a_k$ convergent? If it is, find the sum.

ii) Find $\lim_{k\to\infty} a_k$

iii) Find $\sum_{k=1}^{200} a_k$

(3 marks)

i) $\lim_{n\to\infty} S_n = 7 \quad \therefore \sum_{k=1}^{\infty} a_k = 7$ is convergent

ii) Since $\sum a_k$ is convergent, $\lim_{k\to\infty} a_k = 0$

iii) $\sum_{k=1}^{200} a_k = \dfrac{7(200)-2}{200} = \dfrac{1398}{200}$

b) Find the value of c if: $\sum_{n=2}^{\infty} \left(1-\dfrac{1}{c}\right)^n = \dfrac{1}{2}$

(4 marks)

$\Rightarrow$ require $\left|1-\frac{1}{c}\right| < 1 \Rightarrow \frac{1}{2} < c$

geometric series: $\sum_{n=0}^{\infty}\left(1-\frac{1}{c}\right)^n = \dfrac{1}{1-\left(1-\frac{1}{c}\right)} = \dfrac{c}{c-c+1} = c$

$\sum_{n=2}^{\infty}\left(1-\frac{1}{c}\right)^n = \sum_{n=0}^{\infty}\left(1-\frac{1}{c}\right)^n - 1 - \left(1-\frac{1}{c}\right) = c-1-1+\frac{1}{c} = \frac{1}{2}$

$\Rightarrow 2c^2 - 4c + 2 = c$

$2c^2 - 5c + 2 = 0$

$\therefore c = \dfrac{5 \pm \sqrt{25-16}}{4} = \dfrac{5\pm 3}{4} = \cancel{\tfrac{1}{2}}$ or 2

$\boxed{c = 2}$