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PHY 180 Midterm Solutions

1a) Two objects of mass $m = 2\,kg$ collide. Their initial velocities are

$$\vec{u}_1 = (5\,m/s)\,\hat{i} \quad \text{and} \quad \vec{u}_2 = (5\,m/s)\,\hat{j}$$

After the collision the velocity of object 1 is

$$\vec{v}_1 = (5\,m/s)\,\hat{i} - (5\,m/s)\,\hat{j}$$

By conservation of (linear) momentum

$$\begin{aligned}
\Sigma \vec{p}_i &= \Sigma \vec{p}_f \\
m\vec{u}_1 + m\vec{u}_2 &= m\vec{v}_1 + m\vec{v}_2 \\
\vec{v}_2 &= \vec{u}_1 + \vec{u}_2 - \vec{v}_1 \\
\vec{v}_2 &= (5\,m/s)\,\hat{i} + (5\,m/s)\,\hat{j} - [(5\,m/s)\,\hat{i} - (5\,m/s)\,\hat{j}] \\
\vec{v}_2 &= (10\,m/s)\,\hat{j}
\end{aligned}$$

1b) An elastic collision is one for which the net kinetic energy is unchanged. In this system the initial and final kinetic energies are K_i and K_f so

$$\begin{aligned}
\Delta K &= K_f - K_i \\
&= \tfrac{1}{2}m|\vec{v}_1|^2 + \tfrac{1}{2}m|\vec{v}_2|^2 - [\tfrac{1}{2}m|\vec{u}_1|^2 + \tfrac{1}{2}m|\vec{u}_2|^2] \\
&= \tfrac{1}{2}m[\vec{v}_1\cdot\vec{v}_1 + \vec{v}_2\cdot\vec{v}_2 - \vec{u}_1\cdot\vec{u}_1 - \vec{u}_2\cdot\vec{u}_2] \\
&= \tfrac{1}{2}(2\,kg)[\{(5\,m/s)^2 + (5\,m/s)^2\} + (10\,m/s)^2 - (5\,m/s)^2 - (5\,m/s)^2] \\
&= 100\,kg\,m^2\,s^{-2} \\
&= 100\,J
\end{aligned}$$

$\therefore$ the collision is <u>NOT</u> elastic, as 100 J of energy is gained by the system.

2a) Given the potential $V(x) = \frac{1}{2}Ax^2 - \frac{1}{4}Bx^4$ where x and $V(x)$ have units of length and energy respectively

$$[x] = m \qquad [V(x)] = J = kg\,m^2\,s^{-2}$$

so

$$[V(x)] = [\tfrac{1}{2}Ax^2 - \tfrac{1}{4}Bx^4]$$

$$\begin{aligned}
kg\,m^2\,s^{-2} &= [A][x]^2, & kg\,m^2\,s^{-2} &= [B][x]^4 \\
kg\,m^2\,s^{-2} &= [A]\,m^2 & kg\,m^2\,s^{-2} &= [B]\,m^4 \\
[A] &= kg\,s^{-2} & [B] &= kg\,m^{-2}\,s^{-2}
\end{aligned}$$

So A has units kg/s^2 and B has units $kg/(m^2\,s^2)$

This is the mathematical approach to dimensional analysis. This system is simple enough that you could just find T & L by inspection.

2b) We want to construct quantities T and L with dimensions of time $[T] = s$ and length $[L] = m$. Let's try something of the form

$$T = m^a A^b B^c$$
$$[T] = [m^a A^b B^c]$$
$$s = kg^a (kg\ s^{-2})^b (kg\ m^{-2} s^{-2})^c$$
$$s^{-1} \cdot s = kg^a\ kg^b\ s^{-2b}\ kg^c\ m^{-2c} s^{-2c} \cdot s^{-1}$$
$$1 = kg^{a+b+c}\ s^{-2b-2c-1}\ m^{-2c}$$

Since we have units on one side and none on the other, each exponent must sum to zero, giving us three equations and three unknowns

① $0 = a + b + c$

② $0 = -2b - 2c - 1$

③ $0 = -2c$

③ gives $c = 0$

③ and ② give ④: $0 = -2b - 0 - 1$

$b = -\frac{1}{2}$

① and ④ give

$0 = a + (-\frac{1}{2}) + 0$

$a = \frac{1}{2}$

Therefore

$$T = m^{1/2} A^{-1/2} B^0$$
$$T = \sqrt{\frac{m}{A}}$$

Similarly for L:

$$L = m^d A^e B^f$$
$$[L] = [m]^d [A]^e [B]^f$$
$$m = kg^d (kg\ s^{-2})^e (kg\ m^{-2} s^{-2})^f$$
$$1 = kg^{d+e+f}\ s^{-2e-2f}\ m^{-1-2f}$$

so

① $0 = d + e + f$

② $0 = e + f$

③ $0 = 1 + 2f$

① & ② give $d = 0$

③ gives $f = -\frac{1}{2}$

② gives $e = \frac{1}{2}$

Therefore

$$L = m^0 A^{1/2} B^{-1/2}$$
$$L = \sqrt{\frac{A}{B}}$$

2c) In one dimension, force is the spatial derivative of potential (times -1) so for this system

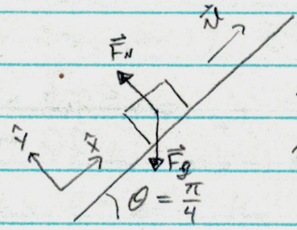
$$F(x) = -\frac{d}{dx}V(x) = -\frac{d}{dx}\left(\frac{1}{2}Ax^2 - \frac{1}{4}Bx^4\right) = -Ax + Bx^3$$

the force goes to zero at points given by

$$0 = -Ax + Bx^3$$
$$0 = x(Bx^2 - A)$$
$$0 = Bx\left[x^2 - \left(\sqrt{A/B}\right)^2\right]$$
$$0 = Bx\left(x - \sqrt{\tfrac{A}{B}}\right)\left(x + \sqrt{\tfrac{A}{B}}\right)$$

this equation has solutions $x = 0$, $x = \pm\sqrt{A/B}$

3a) A block of mass m is launched up an incline as shown.



$\vec{u}_0 = \hat{x}(1\,m/s)$

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Assuming no friction, the vector sum of forces on any massive object is equal to mass times acceleration so

$$m\vec{a} = \Sigma\vec{F}$$
$$m\vec{a} = \vec{F}_g + \vec{F}_N$$
$$m\vec{a} = mg(-\hat{y}\cos\theta - \hat{x}\sin\theta) + \hat{y}mg\cos\theta$$
$$\vec{a} = -\hat{x}g\sin\theta$$
$$a_x = -g\sin\theta$$
$$\frac{du_x}{dt} = -g\sin\left(\frac{\pi}{4}\right)$$
$$\int_{u_0}^{u_x(t)} du_x' = -\frac{g}{\sqrt{2}}\int_0^t dt'$$
$$u_x(t) - u_0 = -\frac{g}{\sqrt{2}}t$$
$$\frac{dx}{dt} = u_0 - \frac{g}{\sqrt{2}}t$$
$$\int_0^{x(t)} dx' = \int_0^t \left(u_0 - \frac{g}{\sqrt{2}}t'\right)dt'$$
$$x(t) = \left(u_0 t' - \frac{g}{2\sqrt{2}}(t')^2\right)\Big|_0^t$$

$$x(t) = u_0 t - \frac{g}{2\sqrt{2}} t^2$$

When the block returns to its initial position $x(t_f) = 0$ so

$$0 = u_0 t_f - \frac{g}{2\sqrt{2}} t_f^2$$

$$0 = t_f \left(u_0 - \frac{g}{2\sqrt{2}} t_f \right)$$

$$0 = -\frac{g}{2\sqrt{2}} t_f \left(t_f - \frac{2\sqrt{2}\, u_0}{g} \right)$$

This has a solution for $t_f = 0$ (the starting point) or $t = 2\sqrt{2}u_0/g$ (the point we're actually interested in) so

$$t_f = \frac{2\sqrt{2}\,(1\,m/s)}{10\,m/s^2} = \frac{\sqrt{2}}{5}\,s \simeq 0.2828\,s$$

3b) Work done by a nonconservative force such as friction is equal to the change in kinetic energy (times -1 because $W = \int \vec{F}_f \cdot d\vec{x}$ and $\vec{F}_f$ acts in the opposite direction to infinitesimal displacement $d\vec{x}$)

$$\begin{aligned} W &= -\Delta K \\ &= -\left(\tfrac{1}{2} m u^2 - \tfrac{1}{2} m v^2\right) \\ &= \tfrac{1}{2} m (v^2 - u^2) \\ &= \tfrac{1}{2}(1\,kg)\left[(0.3\,m/s)^2 - (1\,m/s)^2\right] \\ &= \tfrac{1}{2}(0.09 - 1)\ J \\ &= -0.455\ J \end{aligned}$$

Because the initial and final points occur at the same height, we ignore potential energy

4a) A projectile is launched with initial velocity $\vec{u} = (10\,m/s)\hat{i} + (10\,m/s)\hat{j}$ where gravitational force is $\vec{F}_g = -\hat{j} mg$. At the highest point the projectile explodes and one piece acquires an additional momentum $\Delta\vec{p}_A = (50\,kg\,m/s)\hat{i}$.

At the highest point (just before the explosion) the projectile has $\vec{V} = (10\,m/s)\hat{i}$ so

$$\Sigma \vec{F} = m\vec{a}$$
$$m\vec{a} = -\hat{j}mg$$
$$\frac{d\vec{v}}{dt} = -\hat{j}g$$
$$\int_{\vec{u}}^{\vec{v}} d\vec{v}' = -\hat{j}g\int_0^t dt'$$
$$\vec{v} - \vec{u} = -\hat{j}gt$$
$$(10\ m/s)\hat{i} - (10\ m/s)\hat{i} - (10\ m/s)\hat{j} = -\hat{j}gt$$
$$t = (10\ m/s)\ g^{-1}$$
$$t = \frac{10\ m/s}{10\ m/s^2}$$
$$t = 1\ s$$

Because there are no forces acting in x, the x-component of velocity is constant, and thus displacement in x is

$$\Delta x = v_x \Delta t$$
$$= (10\ m/s)(1\ s)$$
$$= (10\ m)$$

4b) Conservation of momentum tells us that during the explosion the other half of the projectile must acquire an extra momentum $\Delta\vec{p}_B$ such that the total change is zero

$$\Sigma \vec{p} = 0$$
$$0 = \Delta\vec{p}_A + \Delta\vec{p}_B$$
$$\Delta\vec{p}_B = -\Delta\vec{p}_A$$
$$= -(50\ kg\ m\ s^{-1})\hat{i}$$

4c) $m_A = m_B$ so $M = m_A + m_B$ ∴ $m_A = m_B = 5\ kg$. Therefore, after the explosion, the fragments have

$$\vec{u}_A = \vec{v} + \frac{\Delta\vec{p}_A}{m_A}$$
$$= (10\ m/s)\hat{i} + \frac{(-50\ kg\ m/s)\hat{i}}{5\ kg}$$
$$= 0$$

$$\vec{u}_B = \vec{v} + \frac{\Delta\vec{p}_B}{m_B}$$
$$= (10\ m/s)\hat{i} + \frac{(50\ kg\ m/s)\hat{i}}{5\ kg}$$
$$= (20\ m/s)\hat{i}$$

Therefore fragment A is stopped dead and drops straight down to land at $x = 10\ m$.

Since fragment B acquires only velocity in the x-direction and continues to accelerate downwards at $10\ m/s^2$, it takes just as long

to fall to the ground from the explosion as it took to reach its highest point, $\Delta t = 1s$.
Velocity in x is still constant so

$$\Delta x_B = V_B \Delta t$$
$$\Delta x_B = (20\ m/s)(1\ s)$$
$$\Delta x_B = 20\ m$$

So displacement of B when it hits the ground is

$$x_B = \Delta x + \Delta x_B$$
$$= 10\ m + 20\ m$$
$$= 30\ m$$