UNIVERSITY OF TORONTO
ENGINEERING SCIENCE

2018 FINAL EXAM

MAT185

Duration - 2.5 hours

No Aids Allowed

Name: ______________________ Student Number: ______________

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name, student number and tutorial section on this page. Write clearly.
- You may ask us questions, but we cannot answer math questions or questions like "have I shown enough work?"
- There will be partial credit awarded for some questions, so show your work.
- If you need extra room to write a solution, use the back of the pages of the exam. Make sure you write "*CONTINUED ON BACK*" so that the grader knows where to look.
- Please try to write neatly and to express your ideas clearly. We cannot give points for solutions that we cannot read or understand.
- You may leave early. Put your completed exam on the front table and enjoy your day.

Grades:

Question 1: __________ (out of 16)

Question 2: __________ (out of 12)

Question 3: __________ (out of 10)

Question 4: __________ (out of 18)

Question 5: __________ (out of 16)

Question 6: __________ (out of 12)

Question 7: __________ (out of 16)

Total: __________ (out of 100)

Question 1

Determine whether the following claims are true or false and mark the appropriate box – you do not need to explain your answer. (2 points each)

(a) If U and W are two subspaces of a given vector space V, then the intersection $U \cap W$ is also a vector space.

☐ True ☐ False

(b) If $\mathbf{A} \in {}^n\mathbb{R}^n$ is diagonalizable with $\lambda_i = 0$ for $i = 1, \ldots, n$, then $\mathbf{A}$ is the zero matrix.

☐ True ☐ False

(c) The set of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0,1) = 2$, together with usual operations on functions, defines a vector space.

☐ True ☐ False

(d) All the eigenvectors and eigenvalues of a real matrix are real.

☐ True ☐ False

(e) Let $\mathbf{A} \in {}^n\mathbb{R}^n$ such that $\mathbf{A}^2$ is the zero matrix. Then $\mathbf{A}$ can have non-zero eigenvalues.

☐ True ☐ False

(f) For any matrix $\mathbf{A} \in {}^n\mathbb{R}^n$ and $\mu \in \mathbb{R}$, we have $\det_n(\mu\mathbf{A}) = \mu \det_n(\mathbf{A})$.

☐ True ☐ False

(g) The mapping $T(f(x)) = e^x f(x)$ is a linear transformation.

☐ True ☐ False

(h) It is possible to find a 2×2 real matrix with eigenvalues equal to 2 and $i \in \mathbb{C}$, where $i^2 = -1$.

☐ True ☐ False

Question 2

(a) Which *triangular* matrices $\mathbf{A} \in {}^n\mathbb{R}^n$ are invertible? Justify your answer. (4 points)

Recall that the set of *integers* is the set

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},$$

and that ${}^n\mathbb{Z}^n$ denotes the set of $n \times n$ matrices all of whose entries are integers. For questions (b-c) below, you may use the fact (without proving it) that if all entries of a matrix are integers, then the determinant of that matrix is also an integer.

(b) Which matrices from ${}^n\mathbb{Z}^n$ have an inverse in ${}^n\mathbb{Z}^n$? Justify your answer. (4 points)

(c) Let $\mathbf{A} \in {}^n\mathbb{Z}^n$ such that $\det_n(\mathbf{A}) = 1$. Show that for any $\mathbf{b} \in {}^n\mathbb{Z}$, the solution to $\mathbf{Ax} = \mathbf{b}$ also belongs to ${}^n\mathbb{Z}$. (4 points)

Question 3

Let $\mathbf{A} \in {}^n\mathbb{R}^n$. Consider the set $U = \left\{ \mathbf{x} \in {}^n\mathbb{R} \text{ such that } \mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{x} \right\}$.

(a) Show that U is a vector space. (4 points)

(b) Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 5 & 6 & 7 & 3 \\ 4 & 6 & 8 & 2 & 7 \\ 2 & 7 & 3 & 4 & 2 \\ 1 & 3 & 7 & 2 & 5 \end{pmatrix}.$$

Find a basis for U. (6 points)

Question 4

Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1001 & 2 & 3 & 4 & 5 \\ 1 & 1002 & 3 & 4 & 5 \\ 1 & 2 & 1003 & 4 & 5 \\ 1 & 2 & 3 & 1004 & 5 \\ 1 & 2 & 3 & 4 & 1005 \end{pmatrix}.$$

(a) Let $\mathbf{B} = \mathbf{A} - 1000\,\mathbf{I}$, where $\mathbf{I}$ is the 5×5 identity matrix. Show that $\lambda_1 = 0$ is an eigenvalue of $\mathbf{B}$. (4 points)

(b) Determine the geometric multiplicity of λ_1 and find a basis of the corresponding eigenspace of $\mathbf{B}$. (4 points)

(c) Show that $\lambda_2 = 15$ is also an eigenvalue for $\mathbf{B}$, and give an eigenvector associated to λ_2. (3 points)

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(d) Show that the algebraic multiplicity of λ_2 is equal to 1 (Hint: use the geometric multiplicity of λ_1 obtained in part **(b)**). Is $\mathbf{B}$ diagonalizable? (4 points)

(e) Deduce the eigenvalues and eigenvectors of $\mathbf{A}$. (3 points)

Question 5

Consider the space $\mathbb{P}_2$ of polynomials of degree at most 2.

(a) Show that the set v given by $\{v_1 = 1, v_2 = 1 + x, v_3 = 1 + x^2\}$ forms a basis of $\mathbb{P}_2$. (6 points)

(b) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be a linear transformation whose matrix (in the basis of the previous question) is

$$[T]_v^v = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

What is the polynomial $T(x^2)$? (6 points)

(c) Compute $[T]_w^w$ where $w = \{w_1 = 1, w_2 = x, w_3 = x^2\}$. (4 points)

Question 6

(a) Let $\mathbf{A}$ be a 3×3 matrix with eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (with eigenvalue $\lambda_1 = 2$) and $v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ (with eigenvalue $\lambda_2 = -1$). Compute $\mathbf{A} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. (6 points)

(b) Let $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, and let $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute $\mathbf{B}^{1000} v$. (6 points)

Question 7

Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices, such that $\mathbf{AB} = \mathbf{BA}$.

(a) Show that if x is an eigenvector of $\mathbf{A}$, then $\mathbf{B}x$ is also an eigenvector of $\mathbf{A}$, with the same eigenvalue. (6 points)

(b) Suppose ${}^n\mathbb{R}$ has a basis $\{v_1, \ldots, v_n\}$ of eigenvectors of $\mathbf{A}$, with **distinct** eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that each v_i is also an eigenvector for $\mathbf{B}$. (6 points)

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Question 7, (continued)

(c) Show that given the assumptions of the previous question, **B** is also diagonalizable. (4 points)

You're done with MAT 185! To protect the exam, please restrict your fallen tears of joy to this box. $\longrightarrow$ $\boxed{}$