Written Questions

1. A uniform wire of mass 100 g and length 0.50 m is used to suspend a 10 kg mass from a ceiling.
   (a) [6 marks] Calculate the speed at which waves can propagate along this wire.

- We know that the wave speed is given by $c = \sqrt{\frac{T}{\mu}}$ [1 mark].
- The tension comes from the 10 kg mass suspended in gravity (9.8m/s^2) giving 98 N (100N is almost ok) . [1 mark].
- The mass density of this wire is μ= 100g/0.5 m = 0.2 kg/m. [1 mark].
- So, the wave speed is $c = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{98}{0.2}} = 70m/s$. [3 marks].

   (b) [4 marks] Determine the lowest-frequency standing wave that can exist on this wire.

- The lowest frequency wave on this wire, will be the one with the **longest** wavelength.
- In fact, if we have nodes at both ends of the wire and no nodes in between, we'll have $\frac{\lambda}{2}$setup over the 0.5 m length of wire. Giving λ= 1m. [1 mark].
- But we know $\nu = \frac{c}{\lambda}$ so $\nu = {}^{70}/_{1}\, Hz = 70\ Hz$ [3 marks].
- Could also give this in radians ($\omega = 2\pi\nu \approx 440\ rad/s$).

Now a second uniform wire, also of length 0.50 m, but having a mass of 400 g, is spliced (joined seamlessly) to the first wire, producing a single wire of length 1.00 m. The resulting wire is again used to suspend the 10 kg mass from a ceiling. A wave pulse of amplitude 0.50 cm is introduced at the top of the new wire and propagates downward.

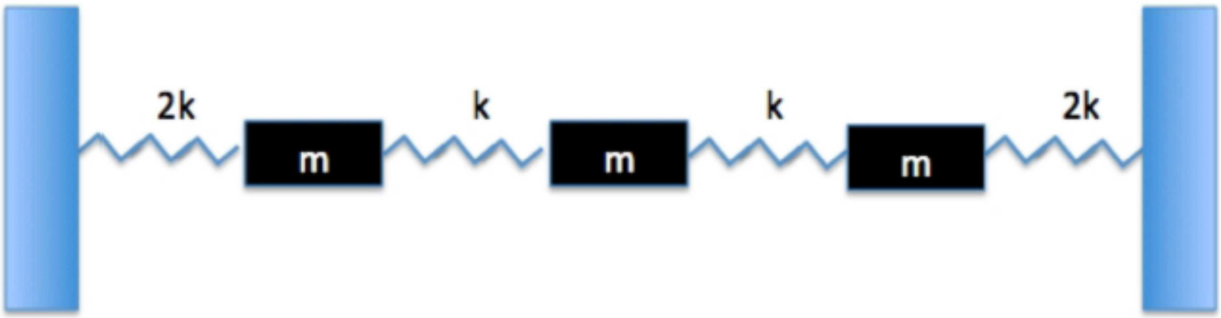
   (c) [6 marks] What is the amplitude of the wave pulse which is reflected back from the joint between the two wires?

- The impedance for the first part of the combined wire is $Z_1 = \sqrt{T\mu_1} = \sqrt{98N \cdot 0.2kg/m} = 4.43$. [1 mark].
- The impedance for the second part of the wire is $Z_2 = \sqrt{T\mu_2} = \sqrt{98N \cdot 0.8kg/m} = 8.85$[1 mark].
- From the formula sheet the reflected **amplitude** is $r = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{4.43-8.85}{4.43+8.85} = \frac{-4.42}{13.28} = 0.33$. [4 marks].
- **Note the reflected waves has the** <u>opposite</u> **amplitude from the incident wave.**

   (d)[4 marks] The amplitude of the wave pulse that continues downwards beyond the joint.

- Using the impedances computed above and the formula from the equation sheet we get $t = \frac{2Z_1}{Z_1+Z_2} = \frac{2\cdot 4.43}{4.43+8.85} = 0.66$ [4 marks].
- The transmitted wave has the **same** sign as the incident wave

2. Three equal masses m are connected to fixed walls at either end by four springs of varying stiffness (k or 2k) as shown below. Ignore gravity.



(a) [12 marks] Find the eigenfrequencies for all normal modes of oscillation of this system.

The equations of motion come from Newton's laws $m\ddot{x}_i - F(x_i) = 0$.

- The forces on the mass on the left are:
  - x_1 increasing $\Rightarrow$ "2k" spring extends $\Rightarrow$ Force to left $\Rightarrow -2k\, x_1$
  - Other side: "k" spring compressed by a length $x_1 - x_2 \Rightarrow$ Force to right $\Rightarrow -k\,(x_1 - x_2)$
- $m\ddot{x}_1 - k(-3x_1 + x_2) = 0 \Rightarrow \ddot{x}_1 + \omega^2(3x_1 - x_2) = 0$ [2 marks]

Similarly

- The forces on the middle mass are:
  - $x_1 - x_2$ shrinking $\Rightarrow$ left "k" spring compresses $\Rightarrow$ Force to right $\Rightarrow k\,(x_1 - x_2)$
  - $x_2 - x_3$ shrinking $\Rightarrow$ right "k" spring compresses $\Rightarrow$ Force to left $\Rightarrow -k\,(x_2 - x_3)$
- $m\ddot{x}_2 - k\,(x_1 - 2\,k\,x_2 + k\,x_3) = 0 \Rightarrow \ddot{x}_2 - \omega^2(x_1 - 2x_2 + x_3) = 0$ [2 marks}

And finally

- The forces on the mass on the right are:
  - $x_2 - x_3$ increasing $\Rightarrow$ "k" spring extends $\Rightarrow$ Force to left $\Rightarrow -k\,(x_2 - x_3)$
  - x_3 increasing $\Rightarrow$ "2k" spring shrinks $\Rightarrow$ Force to left $\Rightarrow -2k\,x_3$
- $m\ddot{x}_3 - (k\,x_2 - 3k\,x_3) = 0 \Rightarrow \ddot{x}_1 + \omega^2(-x_2 + 3x_3) = 0$ [1 marks]

where $\omega^2 = k/m$. To get credit for this part of the question you must explain how the physical forces lead to these equations of motion. Notice the symmetry of the problem at this point. No Explanation where equations came from [-2 marks]. Wrong sign [-1 or -2 marks]

- From here we need to put the equations of motion into matrix form to find the determinant that leads to the eigenfrequencies.
- We have $[M]\ddot{\vec{x}} + [k]\vec{x} = 0$, or assuming harmonic solutions we can substitute directly for $\ddot{x}$ with $-\omega^2 x$ (as suggested at the end of each of the lines of the equations of motion above) to get:

$$\begin{bmatrix} 3k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 3k - m\omega^2 \end{bmatrix} \vec{x} = 0$$

- Which leads to the determinant that we want to vanish being:

$$\begin{vmatrix} 3k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 3k - m\omega^2 \end{vmatrix} = 0$$

- This leads to the characteristic equation: $(3k - m\omega^2)(4k^2 - 5m\omega^2 + m\omega^4) = 0$.
- Which gives three eigenfrequencies: $\omega_1^2 = {}^{3k}/_{m}$, $\omega_2^2 = {}^{k}/_{m}$ and $\omega_3^2 = {}^{4k}/_{m}$.
  - The latter two are the solutions to the quadratic in ω^2 [7 marks]

No attempt at solving characteristic equation [-3], no steps in solution [-2]

(b) Assume that all masses are stationary at t = 0: $\vec{v}$(t = 0) = 0; find a set of initial displacements $\vec{x}$ (t = 0), that would excite only the eigenmode with frequency $2\sqrt{\frac{k}{m}}$.

[8 marks]

- To answer this question we need to find the eigenmode associated with the eignefrequency $2\sqrt{\frac{k}{m}}$.
- $\begin{pmatrix} -k & -k & 0 \\ -k & -2k & -k \\ 0 & -k & -k \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0$ [4 marks]
- Substituting this frequency into the matrix equation of motion, we find the eigenvector of the form (1, -1, 1) . [2 marks]
- So, $\vec{x}(t = 0)$ = (1, -1, 1) cm, or any multiple will excite only the eigenmode with frequency $2\sqrt{\frac{k}{m}}$.[Should really include some unit in your answer.] [2 marks]

Problem 3

Part a) [6 points]: This is from Example 4.6 in the text.

(a) Coulomb's law gives us the electrostatic force, and the acceleration of a particle in circular motion is v^2/r.

$$F = ma \rightarrow \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m\frac{v^2}{r} \Rightarrow v^2 = \frac{e^2}{4\pi\varepsilon_0 mr}$$

Thus,

$$E_{\substack{\text{classical}\\\text{particle}}} = \frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 mr}\right) - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \underbrace{\frac{e^2}{8\pi\varepsilon_0 r}}_{\text{Kinetic}} - \underbrace{\frac{e^2}{4\pi\varepsilon_0 r}}_{\text{Potential}} = -\frac{e^2}{8\pi\varepsilon_0 r}$$

We see that the negative potential energy is of greater magnitude than the positive kinetic, and the total strictly decreases (becomes more negative) as r decreases. There is no minimum energy.

I would give **two points** for the first line of equations above, **two points** for the second line of equations, **one point** for recognizing that since the energy of a bound state is negative, and there is (classically) no limit how small r can become, the energy has no minimum, and **one point** for saying that if this were true quantum mechanically there would be no stable atoms.

Part b) [3 points] Here they should make reference to the hypothesis that the electron in a hydrogen atom represents a standing wave (**one point**), make a requirement on the allowed wavelengths (e.g. $2\pi r = n\lambda$), based on this (**one point**) and state that the relevant wavelength is the deBroglie wavelength of the electron (**one point**). Note that they may also talk about the requirement on the angular momentum here. That's fine, but you should mark that as part of part c.

Part c) [8 points] Here they need to apply what is described in Part b). The following is from page 123 of the text, where this is done

Now we add Bohr's main postulate: The electron's angular momentum L may take on only the values

$$L = n\hbar \qquad \text{where } n = 1, 2, 3, \ldots$$

Because $L = mvr$ in a circular orbit, this condition may also be written

$$mvr = n\hbar \qquad n = 1, 2, 3, \ldots \tag{4-18}$$

Between equations (4-16) and (4-18), we may eliminate v and obtain a condition restricting r only to certain values.

$$r = \frac{(4\pi\varepsilon_0)\hbar^2}{me^2} n^2 \qquad n = 1, 2, 3, \ldots \tag{4-19}$$

or

$$r = a_0 n^2 \qquad \text{where } a_0 \equiv \frac{(4\pi\varepsilon_0)\hbar^2}{me^2} = 0.0529 \text{ nm}$$

According to Bohr's theory, the electron orbits at certain radii that are multiples of the **Bohr radius** a_0. Energy, in turn, is also quantized. Inserting equation (4-19) into (4-17),

$$E = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2} \frac{1}{n^2} = -13.6 \text{ eV} \frac{1}{n^2} \qquad n = 1, 2, 3, \ldots \tag{4-20}$$

The allowed values of the electron's energy depend on the integer n, known as the **principal quantum number**. As noted, these agree with the experimental evidence.

I suggest **2 points** for Eqn. 4-18, **2 points** for 4-19, **3 points** for 4-20 and **1 point** for the energy of the first excited state, which is obtained from Eqn. 4-20 with n = 2.

Part d) [3 points] Here the difference is only in the charge of the nucleus which, which enters the above equation squared. I guess they either get this or don't. So it's **3 points** for a correct answer, **2 points** if they somehow just make a small error, **1 point** if they do something wrong that you think is worth a point and **zero** otherwise.

Problem 4

Part a) [10 points]: This is problem 5.30 from the text for which the solution is given below:

Wave function outside must be zero. Inside: $\psi(x) = A\sin kx + B\cos kx$. Must be 0 both at $x = +\frac{1}{2}a$ and $-\frac{1}{2}a$. $A\sin\left(k(-\frac{1}{2}a)\right) + B\cos\left(k(-\frac{1}{2}a)\right) = 0$ and $A\sin\left(k(+\frac{1}{2}a)\right) + B\cos\left(k(+\frac{1}{2}a)\right) = 0$. Or, $B\cos(\frac{1}{2}a) \pm A\sin(\frac{1}{2}a) = 0$. Both $A\sin(\frac{1}{2}a)$ and $B\cos(\frac{1}{2}a)$ have to be zero! We cannot have both A and B zero at once, or we would have no wave! And sine and cosine are never zero at same place, so we cannot have both A and B nonzero. The only possibilities are: (1) cosine is zero when A is zero, and (2) sine is zero when B is zero.

(1) $\cos(\frac{1}{2}a) = 0 \Rightarrow \frac{1}{2}ka = n\frac{\pi}{2}(n \text{ odd}) \Rightarrow k = \frac{n\pi}{a}$ but, $k = \frac{\sqrt{2mE}}{\hbar} \Rightarrow \frac{n^2\pi^2}{a^2} = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

(2) $\sin(\frac{1}{2}a) = 0 \Rightarrow \frac{1}{2}ka = n\frac{\pi}{2}(n \text{ even})$. This gives again $\mathrm{E} = \frac{n^2\pi^2\hbar^2}{2ma^2}$ and just fills in the even n.

Normalize: $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} A^2\sin^2\frac{n\pi x}{a}dx = A^2\frac{a}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{a}}$ and $\int_{-\frac{1}{2}a}^{+\frac{1}{2}a} B^2\cos^2\frac{n\pi x}{a}dx = B^2\frac{a}{2} = 1 \Rightarrow B = \sqrt{\frac{2}{a}}$

$\boldsymbol{\psi(\mathbf{x}) = \sqrt{\frac{2}{a}}\cos\frac{n\pi x}{a}(n\ \text{odd}),\ \psi(\mathbf{x}) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}(n\ \text{even}),\ E = \frac{n^2\pi^2\hbar^2}{2ma^2}}$. When plotted, these look like infinite well wave functions, because it *is* an infinite well; it's just moved sideways $\frac{1}{2}L$.

3 points for recognizing that both sine and cosine solutions are needed.
3 points for getting the correct wave-functions for all n.
1 point for the wave-function normalization.
3 points for getting the correct energies for all n.

Part b) [4 points]: This is essentially 5.26 from the text, slightly reworded.
The energy levels are given by:

$$E = \frac{\pi^2\hbar^2 n^2}{2mL^2} = \frac{\pi^2(1.055\times10^{-34}\,\mathrm{J\cdot s})^2 n^2}{2(1.67\times10^{-27}\,\mathrm{kg})(15\times10^{-15}\,\mathrm{m})^2} = 1.5\times10^{-13}\mathrm{J}\times n^2 \cong 1\mathrm{MeV}\times n^2.$$

The student needs to calculate the difference between the energies for n=2 and n=1. Here they either get it or they don't. If they use the equation above (from part a) with L = 10^{-15} m and the neutron mass, and take the difference between n=2 and n=1 they should get **4/4** unless they make some sort of calculation error, in which case they get **3/4.** If they make an incorrect attempt that you think is worth some marks, you can give them **1 or 2 points** if you wish, depending on what they did: for instance if they use some mass other than the neutron mass, or don't choose a transitions from n=2 to n=1 (I'm not sure what else could go wrong).

Part c) [6 points]: Here the main point is for them to realize that this is NOT a stationary state. If they write down the wave-function correctly (both the spatial and temporal parts) the should get **4 points.** The remaining **2 points** should be assigned depending on how nicely they simplify the solution. For full marks their result should be explicitly real.

Looking at how the question was phrased, it does not actually explicitly ask them to write out the form of the spatial wave-functions $\psi_1(x)$ and $\psi_2(x)$ or the forms of the energies E_1 and E_2 so if they simply write something like

$$\left| \frac{1}{\sqrt{2}} \psi_1(x) e^{-i\frac{E_1}{\hbar}t} + \frac{1}{\sqrt{2}} \psi_2(x) e^{-i\frac{E_2}{\hbar}t} \right|^2$$

and properly evaluate this, they should get the 4 points referred to above. Again, for full grades they should simplify to a form for the answer that is explicitly real.