Solutions by NN
EngSci 1T7

MAT194 Final Exam December 2009 – Detailed Solutions

Disclaimer: These are student produced solutions and are not official nor do they represent the marking scheme used.

Question 1

A) (i) $9x^2$ (ii) -3sin(3x) (iii) 1/2x (iv) $-2x(e^{-x^2})$ (v) $3^{x^2}x\log 9$

B) (i) $3x^4/4$ (ii) sin(3x)/3 (iii) not required (iv) arctan(x/3) (v) 3^x/ln(3)

Question 2

a) We need a δ such that for $0 < |x-3| < \delta$ implies $|x^2-9| < 7$. Note that we can factor out the quadratic as a difference of squares: $|x+3||x-3| < 7$ .If we can bound the "x-3" term, then we have control over "x+3". By inspection, the largest possible δ we can use is 1. If $0 < |x-3| < 1$, this implies that $|x+3| < 7$ and $|x+3||x-3| < 7$ as required. There is no smallest possible δ, there will always exist ε to challenge it.
b) This question uses the essence of the part (a). Essentially we are looking at the following: $0 < |x-3| < \delta \implies |x+3||x-3| < \epsilon$. We need to bound the "x-3" term to have control over the inequality. This bound can be chosen arbitrarily. Let us choose to bound "x-3" by 1, this implies that the "x+3" will be bounded by 4. The inequality then reduces to: $0 < |x-3| < \delta \implies 4|x-3| < \epsilon$. We can see, since the "x-3" exists in both inequalities that δ = ε/4. We are now ready to write the proof, which is as follows:

$$Let\ \epsilon > 0. Let\ \delta = \min\left\{1, \frac{\epsilon}{4}\right\}$$
$$If\ 0 < |x-3| < \delta \implies |x^2-9| < \epsilon$$
$$Then\ if\ 0 < |x-3| < \delta \implies 4|x-3| < 4\left(\frac{\epsilon}{4}\right) = 4\ (by\ previous\ work)$$

Question 3: *Sketch the curve $f(x) = 1 - \frac{1}{x} - \frac{6}{x^2}$, indicating all important features.*

Use Wolfram Alpha to check your answer but the solution is as follows:

$$f(x) = 1 - \frac{1}{x} - \frac{6}{x^2}$$

$$f'(x) = \frac{1}{x^2} + \frac{12}{x^3}$$

$$f''(x) = -\frac{2}{x^3} - \frac{36}{x^4}$$

Right off the bat, it is unclear if f(x) has zeros, but we can see clearly that the function gets heated as x approaches zero: (the limit holds for both the right and left sides)

$$\lim_{x\to 0} 1 - \frac{1}{x} - \frac{6}{x^2} = -\infty$$

We know there exists a vertical asymptote at zero. Let us now find the horizontal asymptote (if it exists):

$$\lim_{x\to\infty} 1 - \frac{1}{x} - \frac{6}{x^2} = 1$$

This limit holds for both negative and positive infinity. So, we are aware that a horizontal asymptote at 1 exists and a vertical asymptote exists at 0.

Let's find the zeros to f(x) by multiplying through by x^2 and setting to zero to get the following quadratic equation:

$$x^2 - x - 6 = 0$$

Which factors nicely into:

$$(x+2)(x-3) = 0$$

Giving us zeros of x = -2 and x = 3.

Setting f'(x) = 0 gives us a critical point of -12. By inspection, this value of x gives us an f''(x) which is negative, hence this is a maximum. Now setting f''(x) = 0 gives a single inflection point of x = -18.

We now have the following points:

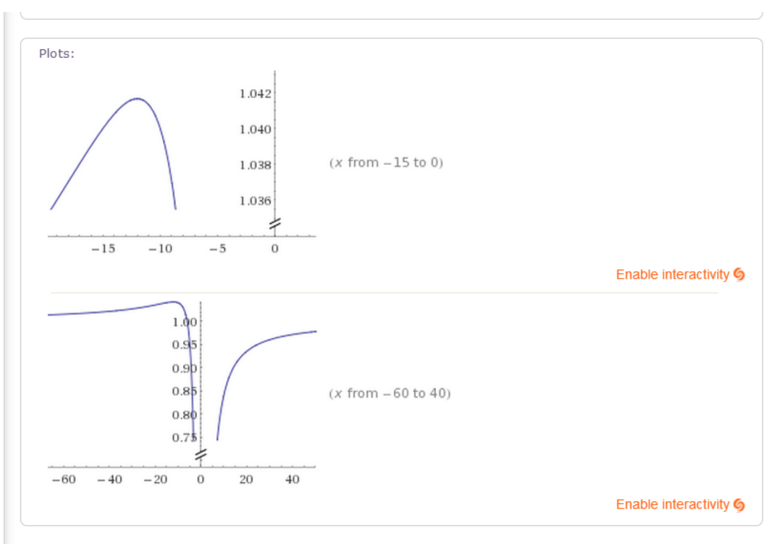
Zeros: x = -2, 3

Maximum: (-12, 1) (Note: substituting -12 into f(x) gives a value very close, but not equal to one)

Inflection point: (-16, 1) (Note: substituting -16 into f(x) gives a value very close to, but not equal to one)

Asymptotes: x = 0 (goes to negative infinity), y = 1.

We have all our important points, we can sketch the graph now and this should look like:



<u>Question 4:</u> *At which points on the curve $y = 1 + 40x^3 - 3x^5$ does the tangent line have the largest slope?*

The tangent line is given by the first derivative y'. But the slope of the tangent line is in fact given by y''. So we find the second derivative of y.

$$y = 1 + 40x^3 - 3x^5$$

$$y'' = 240x - 60x^3$$

We need to maximize this function, setting y'' to zero gives:

$$240x = 60x^3$$

$$x^2 = 4$$

$$x = \pm 2$$

Substituting our x values into y(x) gives us the points:

$$(-2, -253) \; and \; (2, 255)$$

<u>Question 5:</u>

Part A: *Calculate* $\lim_{\mathrm{x}\to 0} (1 + ax)^{\frac{1}{x}}$

Initially, this looks like a daunting limit. However we can use the laws of logarithms to help us. Let the limit equal some number L (if it exists):

$$\lim_{\mathrm{x}\to 0} (1 + ax)^{\frac{1}{x}} = L$$

Then by taking the natural log of both sides:

$$\lim_{\mathrm{x}\to 0} \left(\frac{1}{x}\right)(1 + ax) = \ln L$$

The numerator is going to 1, whilst the denominator is going towards zero. We have an odd situation of limits here of 1/0 equalling the log of some number. It is best to use L'Hopital's rule here which will greatly simplify matters.

$$\lim_{\mathrm{x}\to 0} (a) = \ln L$$

$$a = \ln L \implies L = e^a$$

Or you could notice that the limit is very similar to the definition of Euler's number:

$$e = \lim_{x\to 0} (1 + x)^{\frac{1}{x}}$$

And extend the notion.

Part B: *Calculate:*

$$\frac{d}{dx}\int_{x^3+3x}^{\sin(x)} \frac{\sin^{-1} t}{t^2}\,dt$$

This simply requires the application of the fundamental theorem of calculus. Let us use the properties of integrals to break up the given one – as FTC works only if the upper limit is variable.

$$\frac{d}{dx}\int_0^{\sin x} \frac{\sin^{-1} t}{t^2}dt - \frac{d}{dx}\int_0^{x^3+3x} \frac{\sin^{-1} t}{t^2}dt$$

Now to apply the FTC but not forgetting to use the chain rule; the upper limits are functions, and so we need to take their derivatives as well and multiply.

$$= \frac{\sin^{-1}\sin(x)\cos(x)}{\sin^2 x} - \frac{\sin^{-1}(x^3+3x)(3x^2+3)}{(x^3+3x)^2}$$

$$= x\cot x\csc x - \frac{\sin^{-1}(x^3+3x)(3x^2+3)}{(x^3+3x)^2}$$

Further simplification is up to you.

Question 6: *Let $\mathfrak{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the following quantities:*

Part A: *The area of* $\mathfrak{R}$.

Let us find the intersection(s) of the two curves first:

$$x^3 = 2x - x^2$$

$$x(x^2) = x(2-x)$$

Before we divide through, we must realize that x = 0 is a solution.

$$x^2 + x - 2 = 0$$

$$(x-1)(x+2) = 0$$

So, the three solutions are x = 1, x = 0 and x = -2 of which we only need x =1 and x = 0 as this lies within our domain.

$$A = \int_0^1 2x - x^2 - x^3\,dx \;\; = x^2 - \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1$$

$$= \frac{5}{12}$$

Part B: *The volume by rotating $\mathfrak{R}$ about the x-axis.*

It is best to use the washer method here, as it will yield the result much more quickly without complications.

Recall that the washer method expresses the volume as:

$$V = \pi \int_a^b [f(x)]^2 dx$$

Here we need to take the difference of volumes (NOT difference of areas); one volume from rotating y = 2x – x^2 and the second from rotating y = x^3. Hence:

$$V = \pi \int_0^1 [2x - x^2]^2 dx - \pi \int_0^1 [x^3]^2 \, dx$$

$$V = \pi \int_0^1 4x^2 - 4x^3 + x^4 - x^6 \, dx$$

$$V = \frac{41\pi}{105} \; units^3$$

Part C: *The volume by rotating $\mathfrak{R}$ about the y-axis.*

Here, the cylinder method is best to use. Recall that the cylinder method expresses the volume as:

$$V = 2\pi \int_a^b x \, H(x) dx$$

The height function, H(x), can be expressed as the difference in functions:

$$H(x) = 2x - x^2 - x^3$$

$$V = 2\pi \int_0^1 2x^2 - x^3 - x^4 \, dx$$

$$V = \frac{13\pi}{60} \; units^3$$

Question 7: *Evaluate:*

Part A: $\lim_{t\to 0}\frac{e^{3t}-1}{t}$

We get an indeterminate form of 0/0 here, so we have to use L'Hopital's Rule. Differentiating numerator and denominator independently and then taking the same limit we get:

$$\lim_{t\to 0} 3e^{3t} = 3$$

Part B: $\lim_{x\to a^+}\frac{[\cos x][\ln(x-a)]}{\ln(e^x-e^a)}$

We get another indeterminate form of negative infinity over negative infinity. Before using L'Hopital's Rule, let us try to simplify the function.

The laws of logarithms allow us to do the following:

$$\lim_{x\to a^+}\frac{\cos x \ln\left(\frac{x}{a}\right)}{(\ln e^x/e^a)}$$

Now we can use L'Hopital's rule to help resolve the 0/0 indeterminacy.

$$\frac{\lim_{x\to a^+}(-\sin x \ln\left(\frac{x}{a}\right) + \cos x \ \left(\frac{a}{x}\right))}{\frac{e^a}{e^x}}$$

$$= \cos a$$

Part C: $\lim_{t\to 0^+}\frac{\ln t}{t}$

We have an odd negative infinity over zero form. Let us make the substitution of w = 1/t. This changes the limit to:

$$\lim_{w\to\infty} \ln\left(\frac{1}{w}\right) w = -\infty$$

We know that ln(1/w) approaches negative infinity whilst w goes towards infinity. The result is negative infinity times positive infinity which is negative infinity.

Question 8: *Solve the differential equation $y'' - y = e^x$ using either (a) undetermined coefficients or (b) variation of parameters. y(0) = 0, y'(0) = 2. Bonus 5 marks for both methods.*

The most straightforward method with time constraints is undetermined coefficients, but we will look at both.

The auxiliary equation is:

$$r^2 - 1 = 0$$

$$r = \pm 1$$

The solution to the homogenous case is:

$$y_h = c_1 e^x + c_2 e^{-x}$$

In finding the particular solution, we notice that the driving force G(x) appears in the auxiliary equation and though initially we would want to try $y_p = Ae^x$, we need to multiply this by x. Hence:

$$y_p = Axe^x$$

$$y_p' = A(xe^x + e^x)$$

$$y_p'' = A(xe^x + 2e^x)$$

Substituting into the differential equation:

$$Axe^x + Axe^x - 2Ae^x = xe^x$$

$$A = -\frac{1}{2}$$

$$y(x) = c_1 e^x + c_2 e^{-x} - \frac{xe^{-x}}{2}$$

Now to use our initial conditions to obtain the constants, c_1 and c_2:

$$y(0) = 0 = c_1 + c_2$$

$$y'(0) = 2 = c_1 - c_2 - \frac{1}{2}(xe^x - e^x) = 2$$

$$c_1 - c_2 = \frac{3}{2}$$

$$2c_2 = -\frac{3}{2} \Rightarrow c_2 = -\frac{3}{4}$$

$$c_1 = \frac{3}{4}$$

$$y(x) = \frac{3}{4}e^x - \frac{3}{4}e^{-x} + \frac{xe^x}{2}$$

Alternatively you could use the variation of parameters:

First writing the homogenous solution in the standard form:

$$y_p = u_1 e^x + u_2 e^{-x}$$

And now taking the first and second derivatives:

$$y_p' = u_1' e^x + u_1 e^x + u_2' e^{-x} - u_2 e^{-x}$$

We set $u_1' e^x + u_2' e^{-x} = 0$, as u_1 and u_2 are, in essence, arbitrary functions.

$$y_p'' = u_1' e^x + u_1 e^x - u_2' e^{-x} + u_2 e^{-x}$$

Now, substitute the expressions back into the original differential equation:

$$u_1' e^x + \cancel{u_1 e^x} - u_2' e^{-x} \cancel{+ u_2 e^{-x}} - \cancel{u_1 e^x} - \cancel{u_2 e^{-x}} = e^{-x}$$

We get:

$$u_1' e^x - u_2' e^{-x} = e^{-x}$$

Not forgetting we also have:

$$u_1' e^x + u_2' e^{-x} = 0$$

We now have two functional simultaneous equations:

$$u_1' = -u_2' e^{-2x}$$

$$-u_2' e^{-x} - u_2' e^{-x} = e^{-x}$$

$$-2u_2' e^{-x} = e^{-x}$$

$$u_2' = -\frac{1}{2} \Rightarrow u_2 = -\frac{x}{2}$$

$$u_1' e^x + \frac{1}{2} e^{-x} = e^{-x}$$

$$u_1' = \frac{1}{2} e^{-2x} \Rightarrow \frac{1}{4} e^{-2x}$$

Writing our solutions we have:

$$y(x) = c_1 e^x + c_2 e^{-x} - \frac{x e^{-x}}{2}$$

<u>Question 9:</u> *Show that for x>0,* $\frac{x}{1+x^2} < \tan^{-1} x < x$

Immediately, it is not clear how the three functions are related. Important to note is that if we want show that one function is always greater than the other, we should take the derivative of the function, and if one derivative is always greater than the other, this implies that the function is always greater than the other (within a given domain, in this case, x > 0).

We need to show 3 cases:

1) $\frac{x}{1+x^2} < \tan^{-1} x$, 2) $\tan^{-1} x < x$, 3) $\frac{x}{1+x^2} < x$

Now, let us take derivatives and check all three cases.

Case I: Taking the derivatives of $\frac{x}{1+x^2}$ and $\tan^{-1} x$, the inequality still holds.

$$\frac{1 + x^2 - 2x^2}{(1 + x^2)^2} < \frac{1}{1 + x^2}$$

$$\frac{1 - x^2}{(1 + x^2)^2} < \frac{1}{1 + x^2}$$

$$1 - x^2 < 1 + x^2$$

$$0 < 2x^2$$

The last inequality holds true for any value of x not equal to zero.

Case II: Checking $\tan^{-1} x < x$

We already have the derivative for $\tan^{-1}(x)$ and the derivative of x is just 1.

$$\frac{1}{1+x^2} < 1$$

$$1 < 1 + x^2$$

Once again, the last equality holds true for any value of x not equal to zero.

Case III: Checking $\frac{x}{1+x^2} < x$

Although this last case may be obvious, it is always a good idea to be thorough and rigorous.

$$\frac{1-x^2}{(1+x^2)^2} < 1$$

$$1 - x^2 < (1 + x^2)^2$$

We already showed that $1 < 1 + x^2$, so by inspection, we see that this inequality holds true for any value of x not equal to zero.

And since we have shown all three cases, we can conclude that the given equalities hold for any $x > 0$. Remember that if we have a negative value for x, the last inequality breaks down, so we restrict our domain to greater than zero even if the derivatives are always less than each other.

<u>Question 10:</u> *For what value of $c > 0$ does the equation $x^2 = c \ln(x)$ have exactly one solution?*

It is tempting to set $x^2 - c\ln(x) = 0$. But that does not help us in any way. It would be a good idea to sketch the graphs of x^2 and $c \ln(x)$ to help visualize the problem. Note that the two functions have opposite concavity, and so a single solution would be represented by one graph being tangent to the other, that is, both graphs have a common tangent – the derivatives are the same at some point, and the value of the function is the same at this same point.

So we take derivatives of both functions:

$$2x = \frac{c}{x}$$

Let the tangent be at some point, "a". The derivatives must be equal at this point:

$$2a = \frac{c}{a}$$

$$a = \sqrt{\frac{c}{2}}$$

Substituting into the original equations (writing "a" a bit differently) – note that the two functions are also equal to each other at this point "a".

$$\frac{c}{2} = c \ln\left(c^{\frac{1}{2}} 2^{-\frac{1}{2}}\right)$$

Using the logarithm laws to help us, and simplifying yields value of c.

$$\frac{c}{2} = c \left(\frac{1}{2}\ln c - \frac{1}{2}\ln 2\right)$$

$$\frac{1}{2} = \frac{1}{2}\ln c - \frac{1}{2}\ln 2$$

$$\ln c = \ln 2$$

$$c = 2$$