Term Test 1

SOLUTIONS

1. *False.*

Consider $(0, 1, 1) \in W$ so $x - |y| + z = 0$. Now, by scalar multiplication,

$$(-1)(0, 1, 1) = (0, -1, -1)$$

However,

$$(0) - |-1| + (-1) = -2 \neq 0$$

Hence closure under scalar multiplication fails and accordingly W is not a subspace of $\mathbb{R}^3$. (Closure under vector addition is not satisfied either.)

2. *False.*

For $W_1 = W_2$, we must be able to find k such that the spanning set of W_1 can be written as a linear combination of the spanning set of W_2 and vice versa. Let's start by trying to write $(0, k, 1)$ as a linear combination of $(1, 1, 2)$ and $(1, 0, 1)$. This requires

$$(0, k, 1) = \lambda_1(1, 1, 2) + \lambda_2(1, 0, 1)$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. We must have then

$$\begin{aligned} 0 &= \lambda_1 + \lambda_2 \\ k &= \lambda_1 \\ 1 &= 2\lambda_1 + \lambda_2 \end{aligned}$$

From the first and third equations, we find that $\lambda_1 = 1$ and $\lambda_2 = -1$, which means that $k = 1$. But we still need to check the other linear combinations.

By inspection, $(1, -2, -1) = -2(1, 1, 2) + 3(1, 0, 1)$, from which we conclude that $W_2 \subseteq W_1$.

Also, $(1, 1, 2) = 3(0, 1, 1) + (1, -2, -1)$ and $(1, 0, 1) = 2(0, 1, 1) + (1, -2, -1)$, which shows that $W_1 \subseteq W_2$. Hence $k = 1$ makes $W_1 = W_2$.

3. *True.*

By contraposition, assume that r_1, r_2, r_3 are not all distinct meaning that at least two must be equal. Then $\{f_1, f_2, f_3\}$ is not linearly independent because at least two of the functions will be the same. Therefore, it cannot be a basis.

4(a). *Proof.* If $\boldsymbol{x} \notin \operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ then $\operatorname{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subsetneq \operatorname{span}\{\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$. By Theorem I, Chapter 6, we can claim that $\{\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent.

Let's now test for the linear independence of $\{\boldsymbol{v}_1 + \boldsymbol{x}, \boldsymbol{v}_2 + \boldsymbol{x}, \boldsymbol{v}_3 + \boldsymbol{x}\}$:

$$\lambda_1(\boldsymbol{v}_1 + \boldsymbol{x}) + \lambda_2(\boldsymbol{v}_2 + \boldsymbol{x}) + \lambda_3(\boldsymbol{v}_3 + \boldsymbol{x}) = \boldsymbol{0}$$

or

$$(\lambda_1 + \lambda_2 + \lambda_3)\boldsymbol{x} + \lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \lambda_3 \boldsymbol{v}_3 = \boldsymbol{0}$$

As $\{\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is linearly independent, all the coefficients of the vectors here must vanish. That is, only $\lambda_1 = \lambda_2 = \lambda_3 = 0$ satisfies the foregoing condition. Thus $\{\boldsymbol{v}_1 + \boldsymbol{x}, \boldsymbol{v}_2 + \boldsymbol{x}, \boldsymbol{v}_3 + \boldsymbol{x}\}$ is linearly independent.

4(b). *Proof*. From part (*a*), we know that $\{\boldsymbol{x}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \subset \mathcal{V}$ is linearly independent. This means that V cannot be spanned by fewer than 4 vectors; otherwise the Fundamental Theorem of Linear Algebra would be violated. Therefore $V \neq \mathrm{span}\{\boldsymbol{v}_1 + \boldsymbol{x}, \boldsymbol{v}_2 + \boldsymbol{x}, \boldsymbol{v}_3 + \boldsymbol{x}\}$.