**University of Toronto**

**Faculty of Applied Science and Engineering**

**Final Examination, April, 2013**

**STA286S: Probability and Statistics**

Examiners: B. Donmez and K. Knight

Name:___________________________________________

Student Number:____________________________________

Circle your **lecture** and **tutorial** sections:

| Lecture (circle one) | Tutorial (circle one) | Time | Location | TA |
|---|---|---|---|---|
| LEC 01 (Prof. Knight) | TUT01 | Mon 1-2 pm | BA2159 | Zheng Li |
| | TUT02 | Mon 1-2 pm | BA3008 | Zhen Qin |
| | TUT03 | Th 5-6 pm | BA2155 | Maryam Merrikhpour |
| | TUT04 | Wed 1-2 pm | WB144 | Zhen Qin |
| LEC 02 (Prof. Donmez) | TUT05 | Mon 2-3 pm | BA3116 | Zheng Li |
| | TUT06 | Th 5-6 pm | WB144 | Wayne Giang |
| | TUT07 | Tues 1-2 pm | BA3008 | Maryam Merrikhpour |
| | TUT08 | Tues 1-2 pm | BA3012 | Wayne Giang |

**Instructions:**

- **Time allowed:** 2 ½ hours.
- **Aids:** a non-programmable calculator and a double-sided A4 size aid sheet.
- There are 8 questions. Carefully proportion your time among them.
- If you do not understand a question, or are having some other difficulty, do not hesitate to ask your instructor or TA for clarification.
- There are 17 pages including this page. The last three pages contain the probability tables. Please ensure that you are not missing any pages.
- Points for each question are indicated in parentheses. Total points: 100.

| **Question** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **Total** |
|---|---|---|---|---|---|---|---|---|---|
| **Max** | **15** | **10** | **16** | **16** | **10** | **15** | **13** | **5** | **100** |
| **Score** | | | | | | | | | |

**GOOD LUCK** ☺

**Question 1**

Suppose that X and Y are independent Poisson random variables with means μ and $\lambda$. Define $Z = X + Y$.

(a) (5 pts) Show that Z has a Poisson distribution with mean μ + $\lambda$. (Hint: $P(Z = z) = \sum_{x=0}^{z} P(X = x, Y = z - x)$.)

**We learned two ways of solving this problem. One is using pmf's, the other is through mgf's. Both solutions are acceptable.**

**Through pmf:**

$$P(Z=z) = \sum_{x=0}^{z} \frac{e^{-\mu}\mu^x}{x!}\frac{e^{-\lambda}\lambda^{z-x}}{(z-x)!} = e^{-(\mu+\lambda)}\sum_{x=0}^{z}\frac{\mu^x}{x!}\frac{\lambda^{z-x}}{(z-x)!} = \frac{e^{-(\mu+\lambda)}}{z!}\sum_{x=0}^{z}\frac{z!}{x!\,(z-x)!}\mu^x\lambda^{z-x}$$

**Using binomial expansion**

$$P(Z=z) = \frac{e^{-(\mu+\lambda)}}{z!}(\mu+\lambda)^z$$

**Through mgf:**

$$M_X(t) = e^{\mu(e^t-1)}$$

$$M_Y(t) = e^{\lambda(e^t-1)}$$

$$M_z(t) = e^{\mu(e^t-1)}e^{\lambda(e^t-1)} = e^{(\mu+\lambda)(e^t-1)}$$

**This is the mgf of a Poisson random variable with mean $\mu + \lambda$.**

(b) (5 pts) Find the conditional probability mass function of $X$ given $Z = z$; that is, find $P(X = x|Z = z)$ for $x = 0, \dots, z$.

$$\boldsymbol{P(X = x|Z = z) = \frac{P(X = x, Y + X = z)}{P(Y + X = z)} = \frac{P(X = x, Y = z - x)}{P(Y + X = z)} = \frac{P(X = x)P(Y = z - x)}{P(Y + X = z)}}$$

$$\boldsymbol{= \frac{\frac{e^{-\mu}\mu^x}{x!}\frac{e^{-\lambda}\lambda^{z-x}}{(z-x)!}}{\frac{e^{-(\mu+\lambda)}(\mu+\lambda)^z}{z!}} = \frac{z!}{x!\,(z-x)!}\frac{\mu^x\lambda^{z-x}}{(\mu+\lambda)^z}}$$

(c) (5 pts) Name the distribution that you derived in part b (it is one that you have learned in this course) and state its parameter value.

$$\boldsymbol{P(X = x|Z = z) = \frac{z!}{x!\,(z-x)!}\frac{\mu^x\lambda^{z-x}}{(\mu+\lambda)^z} = \binom{z}{x}\frac{\mu^x\lambda^{z-x}}{(\mu+\lambda)^{z-x}(\mu+\lambda)^x} = \binom{z}{x}\frac{\mu^x}{(\mu+\lambda)^x}\frac{\lambda^{z-x}}{(\mu+\lambda)^{z-x}}}$$

$$\boldsymbol{= \binom{z}{x}\left(\frac{\mu}{\mu+\lambda}\right)^x\left(1 - \frac{\mu}{\mu+\lambda}\right)^{z-x}}$$

**Binomial pmf with $\boldsymbol{p = \frac{\mu}{\mu+\lambda}}$**

**Question 2**

Suppose that $X_1, \ldots, X_{200}$ are independent continuous random variables with common density function

$$f(x) = \begin{cases} 3x^2 & for\ 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

(a) (5 pts) Compute the mean and variance of $X_i$

$$\boldsymbol{E(X) = \int_0^1 3x^3 dx = 3/4}$$

$$\boldsymbol{Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2 = \int_0^1 3x^4 dx - 9/16 = 3/80}$$

(b) (5 pts) Suppose that $Y_i = 1$ if $X_i > 0.5$ and $Y_i = 0$ otherwise. Define $S = Y_1 + \cdots + Y_{200}$. Find an approximate value for $P(S \ge 145)$.

$$\boldsymbol{P(X_i > 0.5) = \int_{0.5}^1 3x^2 dx = 1 - 0.5^3 = 0.875}$$

**S is binomial with p = 0.875. Using normal approximation to binomial with continuity correction,**

$$\boldsymbol{P(S \ge 145) \cong P\left(Z \ge \frac{145 + 0.5 - 200 \times 0.875}{\sqrt{200 \times 0.875 \times 0.125}}\right) = P\left(Z \ge \frac{145 - 200 \times 0.875}{\sqrt{200 \times 0.875 \times 0.125}}\right)}$$

$$\boldsymbol{= P(Z \ge -6.307) \cong 1}$$

**Question 3**

Suppose that $\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n}$ are independent discrete random variables with common probability function

$$f_X(x;\theta) = \theta(1-\theta)^x \quad for\ x = 0, 1, 2, \dots$$

(a) (6 pts) Find the maximum likelihood estimator of $\theta$ based on $\mathrm{X}_1, \dots, \mathrm{X}_\mathrm{n}$.

**The likelihood function is given by**

$$\boldsymbol{L(\theta|x_1, \dots, x_n) = \prod_1^n \theta(1-\theta)^{x_i} = \theta^n \prod_1^n (1-\theta)^{x_i}}$$

**The log-likelihood function is given by**

$$\boldsymbol{l(\theta) = ln\theta^n + x_1 ln(1-\theta) + x_2 ln(1-\theta) + \cdots = nln\theta + \sum_1^n x_i\, ln(1-\theta)}$$

**Further, taking derivative of *l*(θ) with respect to θ and setting it to 0 we get:**

$$\boldsymbol{l'(\theta) = \frac{n}{\theta} - \frac{\sum_1^n x_i}{1-\theta} = 0}$$

$$\boldsymbol{\widehat{\theta} = \frac{1}{1+\bar{x}}}$$

**Checking second derivative:**

$$\boldsymbol{l''(\theta) = -\frac{n}{\theta^2} - \frac{\sum_1^n x_i}{(1-\theta)^2}}$$

$$\boldsymbol{l''(\widehat{\theta})} < 0$$

**Therefore,** $\boldsymbol{\widehat{\theta} = \frac{1}{1+\bar{x}}}$ **is the MLE of θ.**

(b) (4 pts) Suppose that we cannot observe $X_1, \dots, X_n$ but rather only whether a given $X_i$ is 0 or not. Define random variables $Y_1, \dots, Y_n$ so that $Y_i = 0$ if $X_i = 0$ and $Y_i = 1$ if $X_i > 0$. Show that the probability function of $Y_i$ is

$$p_y(y;\theta) = (1-\theta)^y \theta^{1-y} \quad for\ y = 0, 1$$

$$\boldsymbol{P(Y_i = 0) = P(X_i = 0) = \theta}$$

$$\boldsymbol{P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \theta}$$

**Thus,** $\boldsymbol{p_y(y;\theta) = (1-\theta)^y \theta^{1-y} \quad for\ y = 0, 1}$

(c) (6 pts) Find the maximum likelihood estimator of $\theta$ based on $Y_1, \dots, Y_n$.

**The likelihood function is given by**

$$\boldsymbol{L(\theta|\mathbf{y}, \dots, y_n) = \prod_1^n (1-\theta)^{y_i} \theta^{1-y_i} = (1-\theta)^{\sum_1^n y_i}\ \theta^{\sum_1^n 1-y_i}}$$

**The log-likelihood function is given by**

$$\boldsymbol{l(\theta) = \sum_1^n y_i\, ln(1-\theta) + \sum_1^n (1-y_i)\, ln\theta}$$

**Further, taking derivative of *l*(θ) with respect to θ and setting it to 0 we get:**

$$\boldsymbol{l'(\theta) = -\frac{\sum_1^n y_i}{1-\theta} + \frac{\sum_1^n (1-y_i)}{\theta} = 0}$$

$$\boldsymbol{\hat{\theta} = 1 - \bar{y}}$$

**Checking second derivative:**

$$\boldsymbol{l''(\theta) = -\frac{\sum_1^n y_i}{(1-\theta)^2} - \frac{n - \sum_1^n y_i}{\theta^2}}$$

$$\boldsymbol{l''(\hat{\theta})} < 0$$

**Therefore,** $\boldsymbol{\hat{\theta} = 1 - \bar{y}}$ **is the MLE of θ.**

**Question 4**

A machine produces metal rods used in an automobile suspension system. A random sample of 21 rods is selected and the diameter of each (in millimetres) is measured; the data are given below:

| | | | | | | |
|---|---|---|---|---|---|---|
| 8.236 | 8.228 | 8.195 | 8.209 | 8.202 | 8.278 | 8.228 |
| 8.264 | 8.241 | 8.251 | 8.193 | 8.251 | 8.256 | 8.234 |
| 8.239 | 8.183 | 8.231 | 8.248 | 8.201 | 8.243 | 8.261 |

The sample mean is $\bar{x} = 8.232$ mm and the sample variance is $s^2 = 6.76 \times 10^{-4}$.

(a) (4 pts) Assuming that the measurements are normally distributed, find a 95% confidence interval for the mean diameter.

$$\boldsymbol{\bar{x} \pm t_{\frac{\alpha}{2},n-1} \frac{s}{\sqrt{n}}}$$

$$\mathbf{8.232 \pm t_{0.025\,,20} \frac{0.026}{\sqrt{21}}}$$

$$\mathbf{8.232 \pm 2.086 \frac{0.026}{\sqrt{21}}} = [\mathbf{8.220, 8.244}]$$

(b) (6 pts) How might you assess the assumption of normality? State two different alternatives. For each alternative, indicate what would suggest normality and what would suggest non-normality.

**Normal probability plot or quantile-quantile plot: look for a straight line**

**Conduct a formal hypothesis test: e.g., Chi-sq goodness of fit or Kruskal Wallis. If you reject then there is evidence to suggest non-normality.**

(c) (6 pts) Find an approximate 90% confidence interval for the proportion of rods whose diameter is less than 8.2 mm.

**1 if the observation is less than 8.2; 0 otherwise.**

| | | | | | | |
|---|---|---|---|---|---|---|
| 8.236(0) | 8.228(0) | 8.195**(1)** | 8.209(0) | 8.202(0) | 8.278(0) | 8.228(0) |
| 8.264(0) | 8.241(0) | 8.251(0) | 8.193**(1)** | 8.251(0) | 8.256(0) | 8.234(0) |
| 8.239(0) | 8.183**(1)** | 8.231(0) | 8.248(0) | 8.201(0) | 8.243(0) | 8.261(0) |

$$\hat{p} \pm t_{\frac{\alpha}{2},n-1}\sqrt{\frac{\hat{p}\hat{q}}{n}}$$

$$\frac{1}{7} \pm t_{0.05\,,20}\sqrt{\frac{\frac{1}{7}\times\frac{6}{7}}{21}}$$

$$\frac{1}{7} \pm 1.725\sqrt{\frac{\frac{1}{7}\times\frac{6}{7}}{21}} = [0.011, 0.275]$$

**Question 5**

(10 pts) Suppose that $X_1, \ldots, X_n$ are independent Normal random variables with mean $\mu$ and variance 1. Suppose we want to test

$$H_0: \mu = 10 \text{ versus } H_1: \mu \neq 10$$

at the 5% level. We reject $H_0$ if either $\bar{X} \geq 10 + 1.96/\sqrt{n}$ or $\bar{X} \leq 10 - 1.96/\sqrt{n}$. Suppose the true value of $\mu$ is 10.1. How large must n be for this test to have power at least 0.90?

**Under the alternative** $\boldsymbol{\bar{X} \sim N\left(10.1, variance = \frac{1}{n}\right)}$

$$\boldsymbol{Power = P(reject|H_1\ true) = P\left(\bar{X} \geq 10 + \frac{1.96}{\sqrt{n}}\middle| H_1\ true\right) + P\left(\bar{X} \leq 10 - \frac{1.96}{\sqrt{n}}\middle| H_1\ true\right)}$$

$$\boldsymbol{= P\left(Z \geq \frac{10 + \frac{1.96}{\sqrt{n}} - 10.1}{\frac{1}{\sqrt{n}}}\right) + P\left(Z \leq \frac{10 - \frac{1.96}{\sqrt{n}} - 10.1}{\frac{1}{\sqrt{n}}}\right)}$$

$$\boldsymbol{= P\left(Z \geq -0.1\sqrt{n} + 1.96\right) + P\left(Z \leq -0.1\sqrt{n} - 1.96\right) \geq 0.9}$$

**This can be solved iteratively; you will notice that as n increases the term $\boldsymbol{P\left(Z \leq -0.1\sqrt{n} - 1.96\right)}$ approaches 0 fairly fast. Solving $\boldsymbol{P\left(Z \geq -0.1\sqrt{n} + 1.96\right) \geq 0.9}$ should give an approximate answer.**

$\boldsymbol{-0.1\sqrt{n} + 1.96 \geq -1.29;\quad n \geq 1056.25}$

**(note that -1.28 is not acceptable as it would give a probability less than 0.9; interpolation is acceptable)**

**n=1057**

**With exact standard normal probabilities, the answer is n = 1051.**

**Question 6**

To investigate the relationship between weight and blood pressure, a sample of 20 males (between ages 25 and 50) is taken and their weights and systolic blood pressures recorded. A scatterplot of the data reveals that the following model is reasonable:

$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

where $x_i$ and $y_i$ are the weight (in pounds) and blood pressure, respectively, of the i$^{th}$ subject. For these data, we have

$$\overline{x} = 185.5 \qquad \overline{y} = 130.6$$

$$\sum_{i=1}^{20}(x_i - \overline{x})^2 = 4142.9 \qquad \sum_{i=1}^{20}(y_i - \overline{y})^2 = 913.4$$

$$\sum_{i=1}^{20}(x_i - \overline{x})(y_i - \overline{y}) = 1291.6$$

(a) (5 pts) Evaluate the least squares estimates of $\beta_0$ and $\beta_1$.

$$\boldsymbol{b_1 = \frac{\sum_{i=1}^{20}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{20}(x_i - \overline{x})^2} = \frac{1291.6}{4142.9} = 0.3118}$$

$$\boldsymbol{b_0 = \overline{y} - b_1\overline{x} = 130.6 - 0.3118 \times 185.5 = 72.761}$$

(b) (5 pts) Find a 95% confidence interval for $\beta_1$. Do the data support the claim that systolic blood pressure does not depend on an individual's weight? Briefly explain why?

$$\boldsymbol{s = \frac{S_{yy} - b_1 S_{xy}}{n-2} = \frac{913.4 - 0.3118 \times 1291.6}{18} = 28.37}$$

$$\boldsymbol{b_1 \pm t_{\frac{\alpha}{2},n-2}\frac{s}{\sqrt{S_{xx}}}}$$

$$\boldsymbol{0.3118 \pm 2.101\frac{28.37}{\sqrt{4142.9}} = [-0.614, 1.238]}$$

**No the data do not support the claim as the confidence interval includes 0.**

(c) (5 pts) Give a 95% prediction interval for the mean systolic blood pressure of a man weighing 182 pounds. How would you interpret such an interval?

$$\hat{\boldsymbol{y}}_0 \pm \boldsymbol{t}_{\frac{\alpha}{2},n-2}\boldsymbol{s}\sqrt{\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{n}}+\frac{(\boldsymbol{x}_0-\overline{\boldsymbol{x}})^2}{\boldsymbol{S}_{xx}}}$$

$$(\mathbf{72.761}+\mathbf{0.3118}\times\mathbf{182})\pm\mathbf{2.101}\times\mathbf{28.37}\sqrt{\mathbf{1}+\frac{\mathbf{1}}{\mathbf{20}}+\frac{(\mathbf{182}-\mathbf{185.5})^2}{\mathbf{4142.9}}}=[\mathbf{68.345},\mathbf{190.672}]$$

**We are 95% confident that the blood pressure of a man who weighs 182 pounds would fall between 68.345 and 190.672.**

**In more detail, if we were to repeat this experiment infinitely many times, or in other words, if we were to repeatedly sample 20 men, fit a simple linear regression, and calculate the prediction interval for a man weighing 182 pounds, 95% of the prediction intervals that we calculate will include the true blood pressure value for the man.**

**<u>Unacceptable answer:</u> The probability that the true blood pressure of this man falls within** $[\mathbf{68.345},\mathbf{190.672}]$ **is 0.95.**

**Question 7**

In order to compare the running times of films produced by two motion-picture companies, two samples were collected. It is known that the running times of films produced by these two companies follow a normal distribution. The results are summarized in the following table.

| Company | Sample Size | Sample Mean | Sample Standard Deviation |
|---|---|---|---|
| A | 5 | 97 | 8.9 |
| B | 7 | 110 | 30.2 |

(a) (2 pts) The researcher who collected these data believes that there is a difference in the variability of running times of films produced by these two companies. What are the appropriate null and alternative hypotheses to test his claim?

**Let *X* be the running time of Company A movies and *Y* the running time of Company B movies**

$H_0 : \sigma_x^2 = \sigma_y^2$ **vs** $H_a : \sigma_x^2 \neq \sigma_y^2$

(b) (3 pts) What is the test statistic and critical (rejection) region of the test in part (a) (assume α = 0.1)? What conclusion can you draw based on this critical region?

**The test statistic is:** $F_{stat} = \frac{s_x^2}{s_y^2} = \frac{(30.2)^2}{(8.9)^2} = 11.514$**, it has a** $F_{(8,4)}$ **distribution.**

**Using α = 0.1, we reject $H_0$ if:**

$$F_{stat} > F_{(6,4);0.05} = 6.16 \textbf{ or } F_{stat} < F_{(6,4);0.95} = \frac{1}{F_{(4,6);0.05}} = \frac{1}{4.53} = 0.22$$

**Conclusion: since** $F_{stat} > F_{(6,4);0.05} = 6.16$**, i.e., it is in the rejection region**

**we reject $H_0$ and conclude that there is a difference in the variances.**

(c) (4 pts) Find a 95% confidence interval for the difference between the true mean lengths of films produced by these two companies. Assume equal variances.

**Use the pooled two sample *t* CI.**

**The CI is** $(\bar{x}-\bar{y}) \pm t_{(n_1+n_2-2);\alpha/2} \cdot s_p \sqrt{\frac{1}{n_1}+\frac{1}{n_2}}$

**The confidence level 1- α = 0.95 and so** $\alpha/2 = 0.025$**.**

$t_{(n_1+n_2-2);\alpha/2} = t_{10;0.025} = 2.228$

**The pooled estimate of the common variance is:**

$$s_p^2 = \frac{(n_1-1)s_x^2 + (n_2-1)s_y^2}{n_1+n_2-2} = \frac{(4)(8.9)^2 + (6)(30.2)^2}{5+7-2} = 578.9$$

**Substituting into the formula we get:**

$$(110-97) \pm (2.228) \cdot \sqrt{578.9\left(\frac{1}{5}+\frac{1}{7}\right)} = (-18.39, 44.39)$$

(d) (2 pts) If the sample sizes were 50 and 70 (rather than 5 and 7), would you expect the interval you found in (c) to be narrower or wider? Explain.

**Narrower. Since the standard error of the difference will be smaller**

(e) (2 pts) Does the interval you found in part (c) provide evidence at $\alpha = 0.05$ of a difference between the true mean lengths of films produced by the two companies? Explain.

**Want to test:** $H_0: \mu_x = \mu_y \Rightarrow \mu_x - \mu_y = 0$ **vs** $H_a: \mu_x \neq \mu_y \Rightarrow \mu_x - \mu_y \neq 0$

**Since the 95% CI includes 0 we cannot reject $H_0$ at** $\alpha = 0.05$ **and we must have that the P-value > 0.05.**

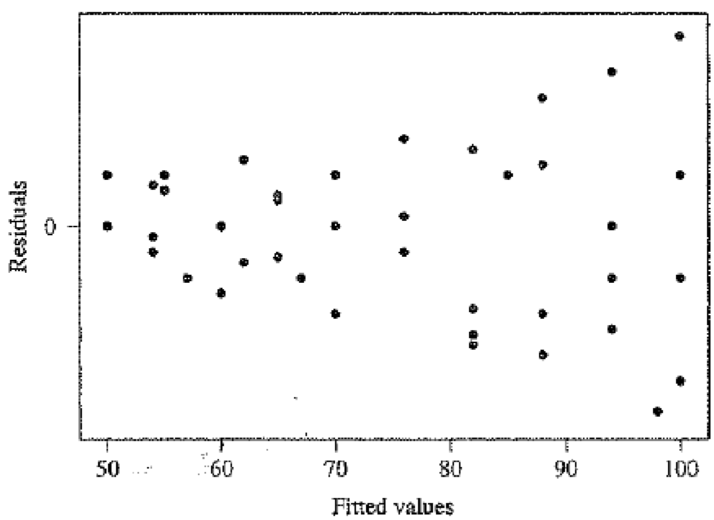
**We conclude that there is no evidence of a difference between the true means.**

## Question 8

(a) (4 pts) State the assumptions of simple linear regression.

- **There is a linear relationship between the mean response and the predictor.**
- **The deviations $\varepsilon_i$ are assumed to be independent.**
- **$\varepsilon_i$ are Normally distributed with mean 0 and constant standard deviation σ.**

(b) (1 pt) The following plot of residuals vs. fitted values was obtained from a simple linear regression model. What assumption is not met?




**Constant variance. It appears that the variance of the errors increases.**

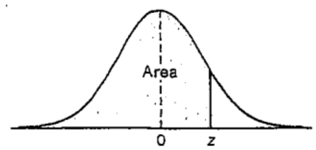



Table **A.3** Areas under the Normal Curve

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **−3.4** | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| **−3.3** | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| **−3.2** | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| **−3.1** | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| **−3.0** | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| **−2.9** | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| **−2.8** | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| **−2.7** | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| **−2.6** | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| **−2.5** | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| **−2.4** | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| **−2.3** | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| **−2.2** | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| **−2.1** | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| **−2.0** | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| **−1.9** | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| **−1.8** | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| **−1.7** | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| **−1.6** | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| **−1.5** | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| **−1.4** | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| **−1.3** | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| **−1.2** | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| **−1.1** | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| **−1.0** | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| **−0.9** | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| **−0.8** | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| **−0.7** | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| **−0.6** | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| **−0.5** | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| **−0.4** | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| **−0.3** | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| **−0.2** | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| **−0.1** | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| **−0.0** | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

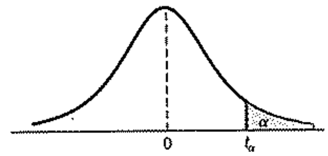



**Table A.4** Critical Values of the $t$-Distribution

| | $\alpha$ | | | | | | |
|---|---|---|---|---|---|---|---|
| $v$ | **0.40** | **0.30** | **0.20** | **0.15** | **0.10** | **0.05** | **0.025** |
| **1** | 0.325 | 0.727 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 |
| **2** | 0.289 | 0.617 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 |
| **3** | 0.277 | 0.584 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 |
| **4** | 0.271 | 0.569 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 |
| **5** | 0.267 | 0.559 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 |
| **6** | 0.265 | 0.553 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 |
| **7** | 0.263 | 0.549 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 |
| **8** | 0.262 | 0.546 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 |
| **9** | 0.261 | 0.543 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 |
| **10** | 0.260 | 0.542 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 |
| **11** | 0.260 | 0.540 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 |
| **12** | 0.259 | 0.539 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 |
| **13** | 0.259 | 0.538 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 |
| **14** | 0.258 | 0.537 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 |
| **15** | 0.258 | 0.536 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 |
| **16** | 0.258 | 0.535 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 |
| **17** | 0.257 | 0.534 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 |
| **18** | 0.257 | 0.534 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 |
| **19** | 0.257 | 0.533 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 |
| **20** | 0.257 | 0.533 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 |
| **21** | 0.257 | 0.532 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 |
| **22** | 0.256 | 0.532 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 |
| **23** | 0.256 | 0.532 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 |
| **24** | 0.256 | 0.531 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 |
| **25** | 0.256 | 0.531 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 |
| **26** | 0.256 | 0.531 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 |
| **27** | 0.256 | 0.531 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 |
| **28** | 0.256 | 0.530 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 |
| **29** | 0.256 | 0.530 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 |
| **30** | 0.256 | 0.530 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 |
| **40** | 0.255 | 0.529 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 |
| **60** | 0.254 | 0.527 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 |
| **120** | 0.254 | 0.526 | 0.845 | 1.041 | 1.289 | 1.658 | 1.980 |
| $\infty$ | 0.253 | 0.524 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 |

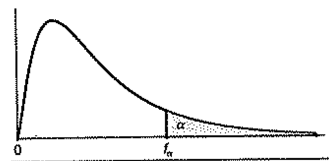



**Table A.6*** Critical Values of the $F$-Distribution

$f_{0.05}(v_1, v_2)$

| $v_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **1** | 161.45 | 199.50 | 215.71 | 224.58 | 230.16 | 233.99 | 236.77 | 238.88 | 240.54 |
| **2** | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| **3** | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| **4** | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| **5** | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| **6** | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| **7** | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| **8** | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| **9** | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| **10** | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| **11** | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| **12** | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| **13** | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| **14** | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| **15** | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| **16** | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| **17** | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| **18** | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| **19** | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| **20** | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| **21** | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| **22** | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| **23** | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| **24** | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| **25** | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| **26** | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| **27** | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| **28** | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| **29** | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| **30** | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| **40** | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| **60** | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| **120** | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 |
| **$\infty$** | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

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