First name (please write as legibly as possible within the boxes)

Last name

Student ID number

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

December 18, 2018

Instructor – Professor W.R. Cluett

Closed book.

All questions are of equal value.

Permitted calculators:

- Sharp EL-520X
- Sharp EL-520W
- Casio FX-991
- Casio FX-991EX
- Casio FX-991ES Plus
- Casio FX-991MS

This test contains 20 pages including this page and the cover page, printed two-sided. Do not tear any page from this test.

Present complete solutions in the space provided. Page 20 is for rough work. If you want anything on page 20 to be marked, you must indicate in the relevant previous question that the solution continues on page 20.

Given information:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\,\vec{u} = \frac{\vec{u}\cdot\vec{d}}{||\vec{d}\,||^2}\vec{d}$$

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The normal system of equations corresponding to $A\vec{x} = \vec{b}$ is given by:

$$A^T A\vec{x} = A^T\vec{b}$$

Euler's method for solving a first order differential equation $y'(t) = f(t, y(t))$ is given by:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

Q1:

a) Construct a matrix A where the solution to $A\vec{x} = \vec{0}$ consists of only the linear combinations of $[2\ 2\ 1\ 0]^T$ and $[3\ 1\ 0\ 1]^T$.

b) If A is a 4x4 invertible matrix, determine all solutions to the homogeneous system given by $B\vec{x} = \vec{0}$ where B is the 4x8 matrix given by $B = [A \;\; A]$.

Q2:

a) Say that you have collected the following ten (x, y) data points:

$\{(1,0), (2,0), (3,0), (4,0), (5,0), (6,0), (7,0), (8,0), (9,0), (10,40)\}$

Find the best horizontal line $y = c$, where c is a constant, for each of the following three different measures for the error:

(i) $min\{e_1^2 + \cdots + e_{10}^2\}$ (least sum of squared errors),

(ii) $min\{max_{i=1,\ldots,10}|e_i|\}$ (least maximum absolute error),

(iii) $min\{|e_1| + \cdots + |e_{10}|\}$ (least sum of absolute errors).

Hint: part (i) should be done using a least squares approach; part (ii) may be done by inspection; part (iii) may be done by examining how the sum of absolute errors changes for different c values.

b) (Taken from Q2 b) of the term test)

Find what linear combination of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is closest to $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ using a least squares approach.

Q3:

a) Find A^{-1} by Gaussian elimination of $[A \mid I]$ where

$$A = \begin{bmatrix} +1 & -1 & +1 & -1 \\ 0 & +1 & -1 & +1 \\ 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & +1 \end{bmatrix}$$

To receive any marks for this part you must show that $AA^{-1} = I$.

b) Solve, using A^{-1} from part a), for $\vec{x}$ in $A\vec{x} = [1\ 1\ 1\ 1]^T$.

c) Given that the determinant of an upper diagonal matrix is equal to the product of its diagonal elements, find the eigenvalues of matrix A and the associated eigenvectors.

Q4:

a) Compute L and U for the symmetric matrix A given by

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

To receive any marks for this part you must show that $A = LU$.

b) State the four conditions on a, b, c, d that are required in order to get $A = LU$.

Q5:

a) Find the conditions on b_1, b_2, b_3 such that vector $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the column space of matrix A, where

i. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$

ii. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

b) What conditions on b_1, b_2, b_3, b_4 make the system given below solvable? Find x_1 and x_2 as a function of b_1, b_2, b_3, b_4 under these conditions.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Q6:

a) A fundamental source of error in Euler's method for numerically solving initial value problems is that the derivative at the beginning of the interval is assumed to apply across the entire interval. The improved Euler's method is an attempt to improve the estimate of the slope over the interval by calculating the average of the derivative at the initial point and the derivative at the end point. Another simple modification of Euler's method, called the modified Euler's method (MEM), is described here.

Given $y'(t) = f(t, y(t))$, this technique uses Euler's method to predict a value of y at the midpoint of the interval $(\Delta t/2)$

$$y_{n+1/2} = y_n + (\frac{\Delta t}{2})f(t_n, y_n)$$

Then, this predicted value is used to calculate a slope at the midpoint of the interval

$$y'_{n+1/2} = f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$

Finally, this slope is used to estimate y_{n+1}

$$y_{n+1} = y_n + \Delta t f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$$

Consider the following first order differential equation

$$y'(t) = f(t, y) = -2t^3 + 12t^2 - 20t + 8.5$$

With initial conditions $y(t = 0) = 1$.

Calculate using the MEM the estimate value for $y(t = 0.5)$ using a step size of $\Delta t = 0.5$ and compare this to the analytical solution $y(t) = -0.5t^4 + 4t^3 - 10t^2 + 8.5t + 1$ at $t = 0.5$.

b) Engineering applications can involve thousands of simultaneous differential equations. In each case, the procedure for solving such a system of equations simply involves applying the techniques developed for solving single equations simultaneously to multiple equations.

Consider the system of two first order differential equations

$$\frac{dy_1}{dt} = -0.5y_1$$

$$\frac{dy_2}{dt} = 4 - 0.3y_2 - 0.1y_1$$

With initial conditions $y_1(t = 0) = 4$ and $y_2(t = 0) = 6$.

Using Euler's method (EM), write the update equations for $y_{1,n+1}$ and $y_{2,n+1}$.

This page is for rough work or for extra space to finish a previous question. This page will not be marked unless you have indicated in a previous question to look at this page.