UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test II

First Year — Program 5

MAT185H1S — Linear Algebra

Examiners: G M T D'Eleuterio

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Student Name:	Fair Copy
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Student Number:		Tutorial Section:	LEC

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 6 pages and 4 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
D		
4	20	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). State the *Fundamental Theorem of Linear Algebra*.

Chapter 6, Theorem II.

/2

1(b). The set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$ is defined as *linearly independent* if

$\sum_i \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$ *implies that all* $\lambda_i = 0$. *(Alternate acceptable definition: A set of vectors is linearly independent if the span of any subset is smaller than the span of the entire set.)*

/2

1(c). The *rank* of $\mathbf{A} \in {}^n\mathbb{R}^n$ is defined as

the common dimension of its column or row space.

/2

1(d). The properties defining a *determinant function* $\Delta : {}^n\mathbb{R}^n \to \mathbb{R}$ are

I. $\Delta[\mathbf{E}(1; i, j)\mathbf{A}] = \Delta(\mathbf{A})$
II. $\Delta[\mathbf{E}(\lambda; i)\mathbf{A}] = \lambda\Delta(\mathbf{A})$

/2

1(e). State the *Maclaurin-Cramer rule*.

The solution to $\mathbf{Ax} = \mathbf{b}$*, where* $\mathbf{A} \in {}^n\mathbb{R}^n$ *is invertible and* $\mathbf{b} \in {}^n\mathbb{R}$*, is given by* $x_i = \det \mathbf{A}_i / \det \mathbf{A}$*, where* x_i *is the* i*th entry in* $\mathbf{x}$ *and* $\mathbf{A}_i$ *is* $\mathbf{A}$ *with the* i*th column replaced by* $\mathbf{b}$*.*

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "***T***" (for true) and "***F***" (for false) in the box beside the question. The value of each question is 1 mark.*

2(a). Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2 \cdots \boldsymbol{e}_n\}$ be a basis for a vector space $\mathcal{V}$ and let $\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_r \in \mathcal{V}$ have coordinates $\mathbf{v}_1, \mathbf{v}_2 \cdots \mathbf{v}_r \in {}^n\mathbb{R}$ with respect to the basis E for any integers n and r. Then the dimension of span$\{\boldsymbol{e}_1, \boldsymbol{e}_2 \cdots \boldsymbol{e}_n\}$ is equal to the dimension of span$\{\mathbf{v}_1, \mathbf{v}_2 \cdots \mathbf{v}_r\}$. F

2(b). Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\} \subset \mathcal{V}$ be linearly independent. Then for a vector $\boldsymbol{v} \in \mathcal{V}$, $\{\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$ is linearly independent if $\boldsymbol{v} \notin$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$. T

2(c). Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\} \subset \mathcal{V}$ be linearly independent. Then for a vector $\boldsymbol{v} \in \mathcal{V}$, $\boldsymbol{v} \in$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$ if $\{\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2 \cdots \boldsymbol{v}_n\}$ is linearly dependent. T

2(d). If the rows of $\mathbf{A} \in {}^m\mathbb{R}^n$ are linearly independent then $\mathbf{Ax} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$. F

2(e). Let $\mathbf{U} \in {}^m\mathbb{R}^m, \mathbf{V} \in {}^n\mathbb{R}^n$ be invertible. Then $\{\mathbf{A}_1, \mathbf{A}_2 \cdots \mathbf{A}_r\} \subset {}^m\mathbb{R}^n$ is linearly independent if and only if $\{\mathbf{UA}_1\mathbf{V}, \mathbf{UA}_2\mathbf{V} \cdots \mathbf{UA}_r\mathbf{V}\}$ is linearly independent. T

2(f). Let $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\mathcal{U} = \{\mathbf{X} \in {}^n\mathbb{R}^n \mid \mathbf{AX} = \mathbf{O}\}$. Then $\dim \mathcal{U} = n^2 - n\,\text{rank}\,\mathbf{A}$. T

2(g). For $\mathbf{A}, \mathbf{B} \in {}^n\mathbb{R}^n$, $\det \mathbf{AB} = \det \mathbf{BA}$. T

2(h). The absolute value of the determinant of a 3×3 matrix can be geometrically interpreted as the volume of a parallelepiped [*corrected*] where the rows are interpreted as vectors in $\mathbb{R}^3$ representing the sides of the parallelepiped [*corrected*]. T

2(i). Let $\mathbf{A} \in {}^n\mathbb{R}^n$. Then $\det \text{adj}\,(\mu\mathbf{A}) = \mu^n \det \mathbf{A}$ for any $\mu \in \mathbb{R}$. F

2(j). Let $\mathbf{A}, \mathbf{B} \in {}^n\mathbb{R}^n$ be invertible. Then $\text{adj}\,\mathbf{AB} = \text{adj}\,\mathbf{B}\,\text{adj}\,\mathbf{A}$. T

C. Just the Answer

Provide just the answers. The value of each question is 2 marks.

3(a). Let $\mathcal{F}$ be the set of infinite sequences $(a_1, a_2, a_3 \cdots)$, where $a_i \in \mathbb{R}$ that satisfy

$$a_{i+3} = a_i + a_{i+1} + a_{i+2}$$

This describes a finite-dimensional vector space. Determine a basis for $\mathcal{F}$.

$$\boxed{\begin{matrix}(1, 0, 0 \cdots) \\ (0, 1, 0 \cdots) \\ (0, 0, 1 \cdots)\end{matrix}}$$

3(b). Let

$$\mathbf{A} = \mathbf{E}(\pi; 17, 3)\mathbf{E}(73, 3)\mathbf{E}(-2; 3)\mathbf{E}(17, 97)\mathbf{E}(13, 97)\mathbf{E}(4; 97)\mathbf{E}(3; 13, 17) \in {}^{100}\mathbb{R}^{100}$$

Determine the determinant of $\mathbf{A}$.

$$\boxed{\det \mathbf{A} = 8}$$

3(c). Let

$$\mathbf{A} = \begin{bmatrix} \alpha & 1 & 1 \\ -6 & \alpha & 0 \\ 5 & 0 & 1 \end{bmatrix}$$

Determine the values of α for which $\mathbf{A}$ is not invertible.

$$\boxed{\alpha = 2, 3}$$

3(d). Let

$$\begin{aligned} t_1(x) &= 2 - 4\sin x + 4\cos x \\ t_2(x) &= 1 + \sin x + 5\cos x + 3\tan x \\ t_3(x) &= 1 - \sin x + 3\cos x + \tan x \\ t_4(x) &= 1 + \sin x + \cos x + \tan x \end{aligned}$$

Determine a basis for $\mathcal{T} = \operatorname{span}\{t_1, t_2, t_3, t_4\}$ from among t_1, t_2, t_3, t_4. (Consider using coordinates.)

t_1, t_2, t_4

or

t_1, t_3, t_4

or

t_2, t_3, t_4

3(e). Let $\mathbf{p}, \mathbf{q}, \mathbf{s}, \mathbf{t}, \mathbf{z} \in \mathbb{R}^3$ and suppose

$$\alpha_1 = \det \begin{bmatrix} \mathbf{p} \\ \mathbf{s} \\ \mathbf{z} \end{bmatrix}, \quad \alpha_2 = \det \begin{bmatrix} \mathbf{p} \\ \mathbf{t} \\ \mathbf{z} \end{bmatrix}, \quad \alpha_3 = \det \begin{bmatrix} \mathbf{q} \\ \mathbf{s} \\ \mathbf{z} \end{bmatrix}, \quad \alpha_4 = \det \begin{bmatrix} \mathbf{q} \\ \mathbf{t} \\ \mathbf{z} \end{bmatrix}$$

Determine

$$\det \begin{bmatrix} \lambda_1\mathbf{p} + \lambda_2\mathbf{q} \\ \mu_1\mathbf{s} + \mu_2\mathbf{t} \\ \mathbf{z} \end{bmatrix}$$

in terms of the αs, λs and μs.

$$\alpha_1\lambda_1\mu_1 + \alpha_2\lambda_1\mu_2 + \alpha_3\lambda_2\mu_1 + \alpha_4\lambda_2\mu_2$$

D. Proving Ground

In each of the following questions, two statements A and B are given. Determine the relation between the two and indicate your answer in the box beside the question. There are four options:

If there is no relation	. . . indicate by. . .	"✗"
If A implies B	. . . indicate by. . .	"$\Rightarrow$"
If A is implied by B	. . . indicate by. . .	"$\Leftarrow$"
If A implies and is implied by B	. . . indicate by. . .	"$\Leftrightarrow$"

The value of each question is 4 marks. For 4 marks the complete answer is required while a partially correct answer will earn 2 marks.

4(a). Let $\mathbf{A} \in {}^m\mathbb{R}^n$.

A. $\mathbf{A}\mathbf{A}^T$ in invertible

B. $\mathbf{A}\mathbf{B} = \mathbf{1}$ for some $\mathbf{B} \in {}^n\mathbb{R}^m$

$\Leftrightarrow$

4(b). Let $\mathbf{A}, \mathbf{B} \in {}^m\mathbb{R}^n$.

A. $\operatorname{null}\mathbf{A} = \operatorname{null}\mathbf{B}$

B. $\operatorname{col}\tilde{\mathbf{A}} = \operatorname{col}\tilde{\mathbf{B}}$

$\Rightarrow$

4(c). Let $\mathbf{A} \in {}^n\mathbb{R}^n$ and let $B = \{\mathbf{b}_1, \mathbf{b}_2 \cdots \mathbf{b}_n\}$ be a basis for ${}^n\mathbb{R}$.

A. $\operatorname{rank}\mathbf{A} = \operatorname{rank}[\,\mathbf{A} \,|\, \mathbf{b}_i\,]$ for all i

B. $\mathbf{A}$ is invertible

$\Leftrightarrow$

4(d). Let $\mathbf{A} = [a_{ij}] \in {}^n\mathbb{R}^n$.

A. $a_{ij} \geq 0$ for all i, j

B. $\det\mathbf{A} \geq 0$

✗

4(e). Let $\mathbf{A} \in {}^n\mathbb{R}^n, n \geq 2$.

A. $\operatorname{rank}\mathbf{A} \leq n - 2$

B. $\operatorname{adj}\mathbf{A} = \mathbf{O}$

$\Leftrightarrow$