ECE259: Electromagnetism

Term test 2, April 2nd, 2024

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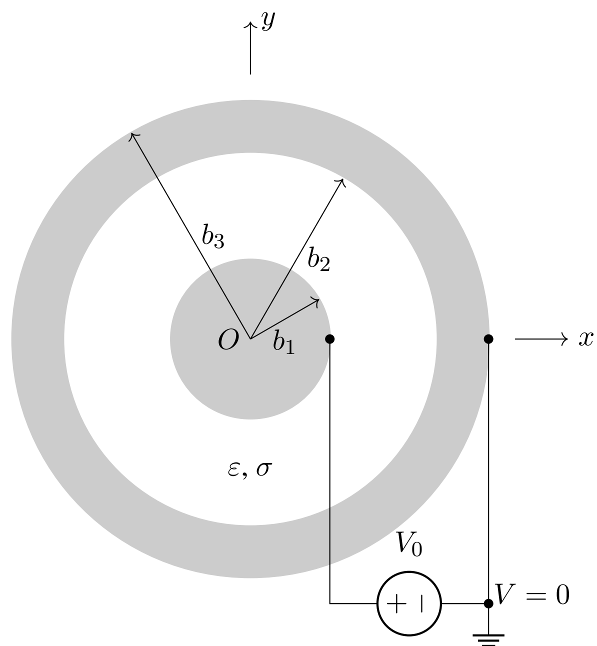
SOLUTION

Instructions

- Duration: 1 hour 30 minutes (9:10-10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**
- Please write with a **dark** pen or pencil. This test will be scanned.

Question 1 [20 points]



The structure shown in the figure above consists of

- an solid metallic *sphere* of radius b_1, shown in grey;
- a metallic *spherical* enclosure of inner radius b_2 and outer radius b_3, shown in grey;
- a filling material in between, shown in white. This material occupies the region $R \in [b_1, b_2]$.

The two metallic spheres (grey objects) can be considered perfect conductors. The filling material (white region) is mildly conductive, with conductivity σ and permittivity ε (both uniform). The structure is surrounded by air. A voltage source of value V_0 is applied as in the figure, with the "+" terminal connected to the inner sphere, and the "-" terminal connected to the enclosure.

a) Solve the Poisson equation to find an expression for potential $V(R)$ in the region $R \in [b_1, b_2]$. In this region, you can assume that potential is only function of the R coordinate, and that the density of free charge is zero in this region ($\rho_v = 0$) [12 points];

b) Derive an expression for the electric field $\mathbf{E}$ in the region $R \in [b_1, b_2]$ [2 points];

c) Derive an expression for the capacitance C between the inner sphere and the enclosure [6 points];

Please justify all your answers.

a) Poisson equation in filling material

$$\cancel{\varepsilon}\nabla^2 V = 0$$

Simplifying Poisson eq. correctly (ε uniform, $\rho_v = 0$) [2pts]

Since $V = V(R) \Rightarrow \frac{1}{\cancel{R^2}}\frac{\partial}{\partial R}\left[R^2 \frac{\partial V}{\partial R}\right] = 0$

$$R^2 \frac{\partial V}{\partial R} = c_1 \quad ; \quad \frac{\partial V}{\partial R} = \frac{c_1}{R^2} \quad ; \quad V(R) = -\frac{c_1}{R} + c_2$$

Solving for $V(R)$ [4 pts]

Boundary conditions | Right boundary cond's [2pt]

$$\boxed{\begin{cases} V(b_1) = V_0 \\ V(b_2) = 0 \end{cases}} \qquad \begin{cases} -\frac{c_1}{b_1} + c_2 = V_0 \\ -\frac{c_1}{b_2} + c_2 = 0 \end{cases} \qquad \begin{cases} -\frac{c_1}{b_1} + \frac{c_1}{b_2} = V_0 \\ \dots \end{cases}$$

$$\begin{cases} c_1\left(\frac{1}{b_2} - \frac{1}{b_1}\right) = V_0 \\ \dots \end{cases} \qquad c_1 = \frac{V_0}{\frac{1}{b_2} - \frac{1}{b_1}}$$

$$c_2 = \frac{c_1}{b_2} = \frac{V_0}{b_2\left(\frac{1}{b_2} - \frac{1}{b_1}\right)}$$

Finding c_1 and c_2 [2pt]

Potential $$V(R) = -\frac{V_0}{\left(\frac{1}{b_2} - \frac{1}{b_1}\right)R} + \frac{V_0}{b_2\left(\frac{1}{b_2} - \frac{1}{b_1}\right)}$$

$$V(R) = \frac{V_0}{\frac{b_1 - b_2}{b_1 b_2}}\left(\frac{1}{b_2} - \frac{1}{R}\right) = \frac{V_0}{b_1 - b_2}\left(b_1 - \frac{b_1 b_2}{R}\right)$$

$$= \frac{V_0 b_1}{b_2 - b_1}\left(\frac{b_2}{R} - 1\right) \checkmark$$

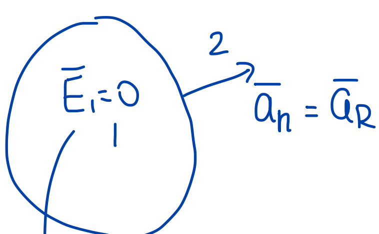
Finding V [2pts]

b)

$$\bar{E} = \underbrace{-\nabla V = -\bar{a}_R \frac{\partial V}{\partial R}}_{[1pt]} = \underbrace{-\frac{V_0 b_1}{b_2 - b_1}\left(-\frac{b_2}{R^2}\right)\bar{a}_R = V_0 \frac{b_1 b_2}{(b_2 - b_1)R^2}\bar{a}_R}_{[1pt]}$$

c) Capacitance $C = \dfrac{Q}{V_0}$

Find ρ_s on inner sphere using boundary conditions



since perfect cond

$$\rho_s = \bar{a}_R \cdot \left(\varepsilon \bar{E}(R = b_1^+) - 0\right) =$$

$$= \varepsilon \underbrace{\bar{a}_R \cdot \bar{a}_R}_{=1} V_0 \frac{\cancel{b_1} b_2}{(b_2 - b_1) b_1^{\cancel{2}}}$$

Find ρ_s [3pt]

$$\rho_s = \varepsilon V_0 \frac{b_1 b_2}{(b_2-b_1)\, b_1^2}$$

$$Q = \rho_s \cdot 4\pi b_1^2 = 4\pi \varepsilon_0 V_0 \frac{b_1^{\cancel{2}} b_2}{(b_2-b_1)\cancel{b_1}}$$

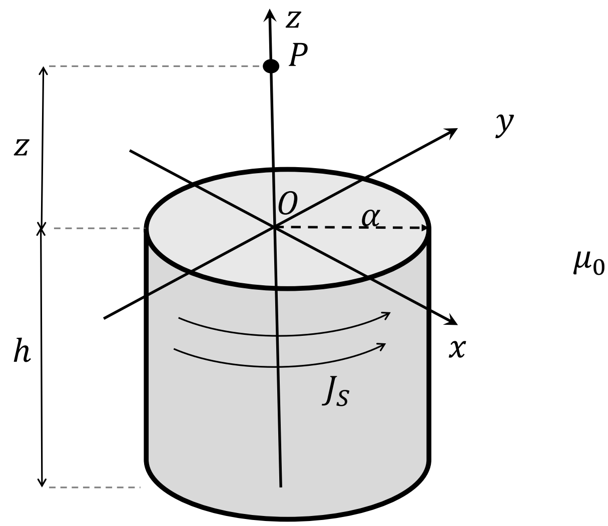
Find Q [2pt]

Capacitance

$$C = \frac{Q}{V_0} = 4\pi \varepsilon_0 \frac{b_1 b_2}{b_2-b_1}$$

Find C [1pt]

Question 2 [20 points]



We have a thin metallic **cylindrical surface** of radius α and height h. The cylindrical surface extends from $z = -h$ to $z = 0$, and its axis is on the z axis, as shown in the figure. The upper and lower ends are open (there are no “caps”). The cylinder is surrounded by air. On the surface, a current flows, with **surface** current density

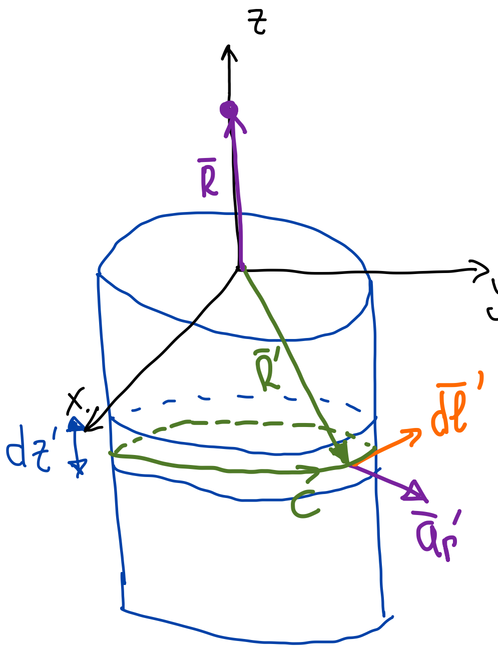
$$\mathbf{J}_S = J_0 \mathbf{a}_\phi \qquad \text{for } r = \alpha, z \in [-h, 0] \, .$$

Calculate the magnetic flux density vector $\mathbf{B}$ at a point P with coordinates $(0, 0, z > 0)$ on the **positive** z axis. *Hint:* divide the surface into thin rings of height dz' [20 points].

You may find the following integral useful

$$\int \frac{du}{(u^2 + w^2)^{\frac{3}{2}}} = \frac{u}{w^2 \sqrt{w^2 + u^2}} + C$$

Please justify all your answers.



Strips of width dz'

Current in each strip:

$dI = J_s\, dz = J_o\, dz'$ [2pts]

Since thin, treat each strip as a wire

Use Biot-Savart law. $\bar{dB}$ from each strip [2 pts]

$$d\bar{B} = \frac{\mu_o\, dI}{4\pi} \int_c \frac{\bar{d\ell}' \times (\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|^3}$$

$\bar{d\ell}' = \alpha\, d\varphi'\, \bar{a}_\varphi'$ [1 pt]

$\bar{R} = z\, \bar{a}_z$ [1 pt]

$\bar{R}' = z'\bar{a}_z + \alpha\, \bar{a}_r'$ [1 pt]

$\bar{R} - \bar{R}' = -\alpha\, \bar{a}_r'(\varphi') + (z - z')\, \bar{a}_z$ [1 pt]

$|\bar{R} - \bar{R}'| = \sqrt{\alpha^2 + (z - z')^2}$ [1 pt]

$$\bar{dl}' \times (\bar{R} - \bar{R}') = -\alpha\, d\varphi'\, \bar{a}_\varphi' \times \left[\alpha\, \bar{a}_r'(\varphi') + (z - z')\bar{a}_z\right] =$$

$$= \alpha^2 d\varphi' \,(+\bar{a}_z) - \alpha\, d\varphi' (z - z')\, \underbrace{\bar{a}_r'(\varphi')}_{\text{will integrate to zero}}$$

[3 pts]

[2pts]

$$d\bar{B} = \frac{\mu_0 J_0 dz'}{4\pi} \int_{\varphi'=0}^{2\pi} \frac{\alpha^2 d\varphi'\, \bar{a}_z}{\left[\alpha^2 + (z - z')^2\right]^{3/2}} =$$

Calculations for $d\bar{B}$

[3 pts]

$$= \frac{\mu_0 J_0 dz'}{\cancel{4\pi}_{2}} \frac{\alpha^2\, \bar{a}_z}{\left[\alpha^2 + (z - z')^2\right]^{3/2}} \cancel{2\pi} = \frac{\mu_0 J_0 \alpha^2 dz'}{2\left[\alpha^2 + (z - z')^2\right]^{3/2}} \bar{a}_z$$

Superposition along z' direction

$$\bar{B} = \int_{z'=-h}^{0} d\bar{B} = \frac{\mu_0 J_0 \alpha^2}{2} \bar{a}_z \int_{z'=-h}^{0} \frac{dz'}{\left[\underbrace{\alpha^2}_{w^2} + \underbrace{\underbrace{(z - z')^2}_{=(z'-z)^2}}_{u^2}\right]^{3/2}}$$

$w = \alpha$

$u = z' - z$

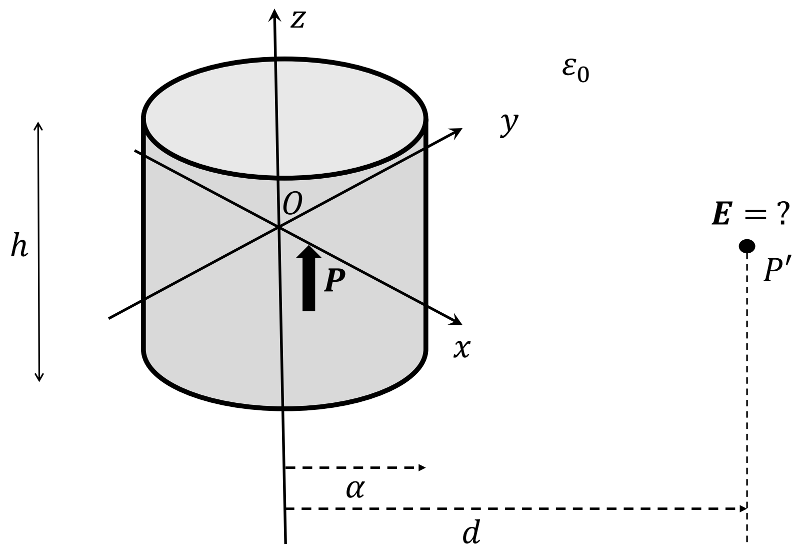
$$\bar{B} = \frac{\mu_0 J_0 \alpha^2}{2} \bar{a}_z \left[\frac{z' - z}{\alpha^2 \sqrt{(z'-z)^2 + \alpha^2}} \right]_{z'=-h}^{z'=0}$$

$$= \frac{\mu_0 J_0 \alpha^2}{2} \bar{a}_z \left[-\frac{z}{\alpha^2 \sqrt{z^2 + \alpha^2}} - \frac{-h - z}{\alpha^2 \sqrt{(h+z)^2 + \alpha^2}} \right] =$$

$$= \frac{\mu_0 J_0 \alpha^2}{2} \bar{a}_z \left[\frac{z + h}{\alpha^2 \sqrt{(h+z)^2 + \alpha^2}} - \frac{z}{\alpha^2 \sqrt{z^2 + \alpha^2}} \right]$$

Calculations for $\bar{B}$ [3pts]

Question 3.1 [8 points]



We have a cylindrical object of height h and radius α centered at the origin and aligned with the z axis. The object is permanently polarized, with polarization vector given by

$$\mathbf{P} = P_0 \mathbf{a}_z \quad \text{(in the volume of the cylinder)}\,.$$

The cylinder is surrounded by air. Let $\mathbf{E}$ be the electric field produced by the polarized cylinder at a point P' in the surrounding air.

a) Find all the densities of polarization (bound) charge present in the cylinder. [4 points]

b) Can $\mathbf{E}$ be approximated by the field produced by an electric dipole? The possible answers are: [4 points]

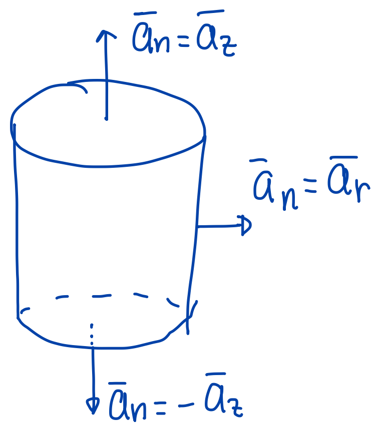
   i) **Yes, always**. Explain why, and find the equivalent electric dipole moment $\mathbf{p}$;

   ii) **Yes, but only under some conditions**. Explain under which conditions the approximation is reasonable, and find the equivalent electric dipole moment $\mathbf{p}$;

   iii) **No**. Explain why not.

a) Volume density of polarization charge

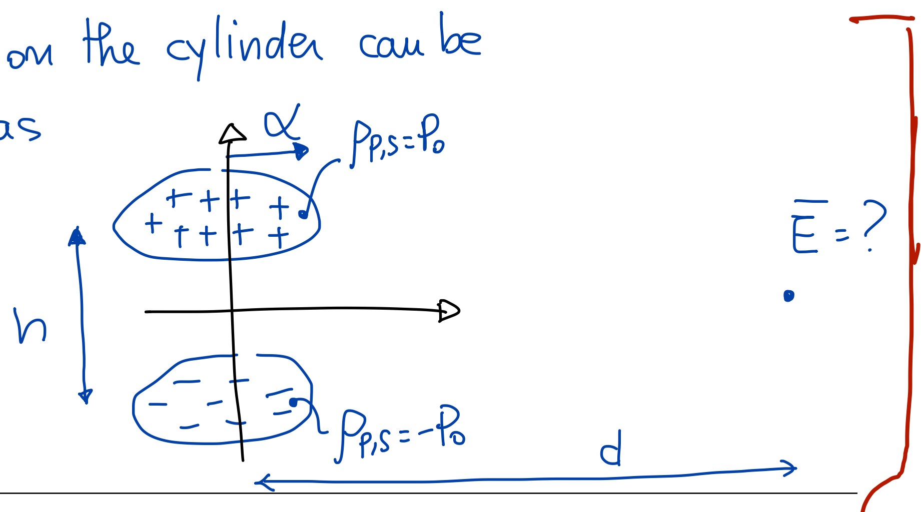
$$\rho_{p,v} = -\nabla \cdot \bar{P} = 0 \quad [1\,pt]$$



Surface densities

- top base : $\rho_{p,s} = \bar{a}_n \cdot \bar{P} = \bar{a}_z \cdot P_0 \bar{a}_z = P_0$ [1 pt]
- lower base : $\rho_{p,s} = \bar{a}_n \cdot \bar{P} = -\bar{a}_z \cdot P_0 \bar{a}_z = -P_0$ [1 pt]
- side $\rho_{p,s} = \bar{a}_n \cdot \bar{P} = \bar{a}_r \cdot P_0 \bar{a}_z = 0$ [1 pt]

b) The charges on the cylinder can be depicted as



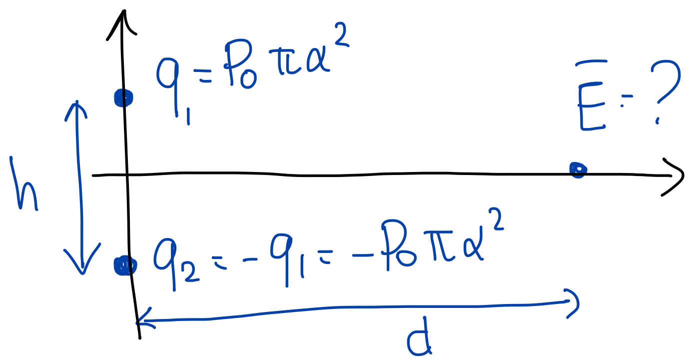
CONDITION OF VALIDITY OF APPROXIMATION: [1pt]

JUSTIFICATION: [1pt]

SUFFICIENT CONDITION TO GET 1pt

This is not a exactly a dipole.

However, if the distance of the observer (d) is much larger than the size of the cylinder, we can lump all charge in two points, as in a dipole



(Precise condition: $d \gg a$) Not required, but welcome if written!

Dipole moment: $\bar{p} = P_0 \pi a^2 h \bar{a}_z$] $\bar{p}$ value: [1pt]

So, answer is "Yes, but only under some conditions"

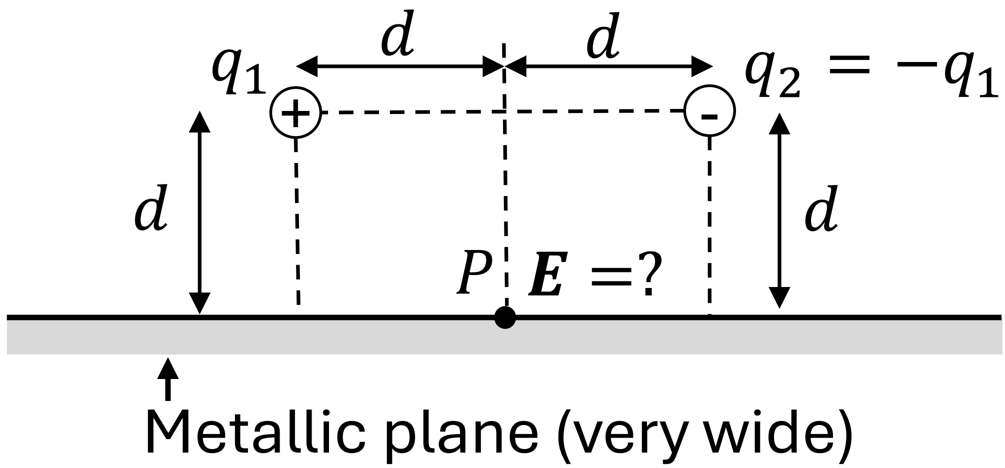
Right answer: [1 pt]

Note: if a student answers "i) Yes, always"

will receive points as:

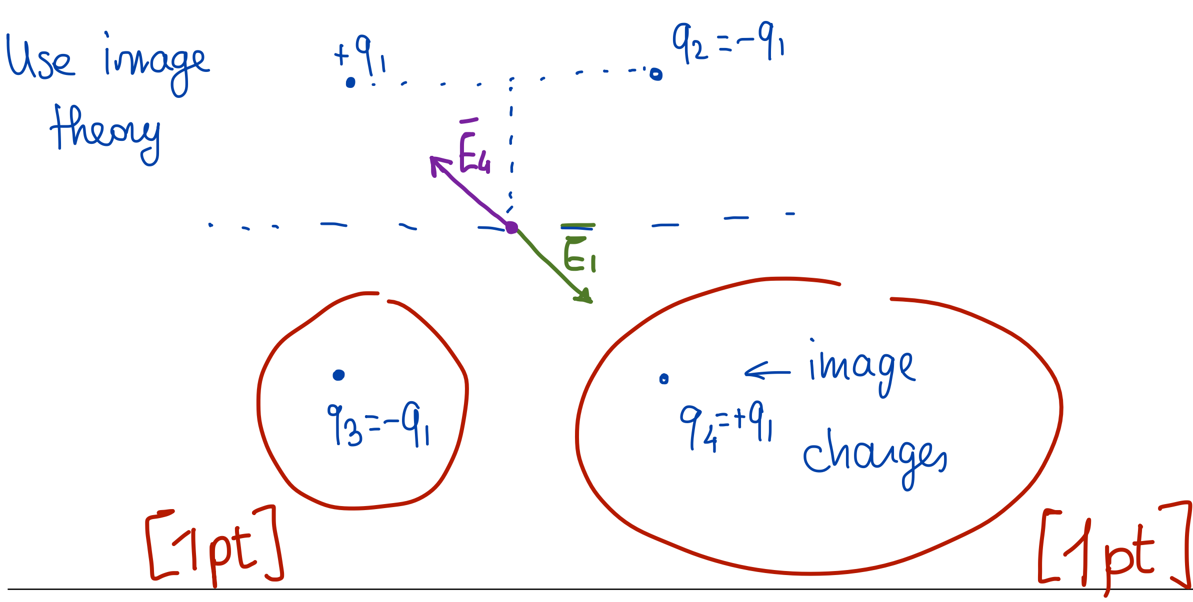
- justification: up to 1pt if reasonable
- condition : 0pt (typically missing)
- $\bar{p}$: up to 1pt if correct
- right answer : 0 pt

Question 3.2 [4 points]



We have two point charges of value $q_1 > 0$ and $q_2 = -q_1$, in air, located over an infinitely-wide metallic plane, as shown in the figure. The plane can be considered a perfect conductor. Find the electric field vector $\mathbf{E}$ at the point P shown in the figure.

Fully justify your answer.



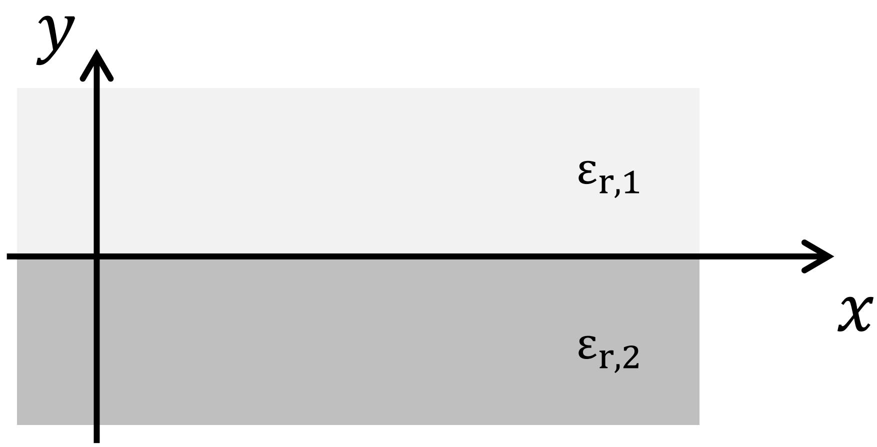
q_4 and q_1 are equal,
same distance,
aligned to observation
point → produce
equal and opposite
contributions

Same for q_3 and q_2

$\Rightarrow \bar{E} = 0$

[2pt]

Question 3.3 [8 points]



The plane $y = 0$ is the interface between two *perfect* dielectrics with relative permittivities $\varepsilon_{r,1} = 3$ and $\varepsilon_{r,2} = 6$, as shown in the figure above. The electric field right above the interface $(y = 0^+)$ is $\mathbf{E}_1 = 2\mathbf{a}_x + 4\mathbf{a}_y$ V/m. Find an expression for:

a) the surface density of free charge ρ_S at the interface $y = 0$ [2 points];

b) the electric field $\mathbf{E}_2$ right below the interface $(y = 0^-)$ [2 points];

c) the electric flux density $\mathbf{D}_1$ right above the interface $(y = 0^+)$ [2 points];

d) the electric flux density $\mathbf{D}_2$ right below the interface $(y = 0^-)$ [2 points].

You can leave the results expressed in terms of ε_0. Justify all your answers.

a) $\rho_s = 0$ since both perfect dielectrics $\Rightarrow$ no free charge anywhere [2pt]

b) $\bar{E}_{2,t} = \bar{E}_{1,t} = 2\bar{a}_x \quad V/m$ Tangential comp. [0.5pt] of $\bar{E}_2$

$E_{1,n} = 4 \quad V/m$

$D_{1,n} = \varepsilon_{r,1}\varepsilon_0 E_{1,n} = 3\varepsilon_0 \cdot 4 = 12\,\varepsilon_0 \quad C/m^2$

$D_{2,n} = D_{1,n} = 12\,\varepsilon_0 \quad C/m^2$

$E_{2,n} = \dfrac{D_{2,n}}{6\varepsilon_0} = \dfrac{12\cancel{\varepsilon_0}}{6\cancel{\varepsilon_0}} = 2 \quad V/m$

Normal comp [1.5pt] of $\bar{E}_2$

$\bar{E}_2 = \bar{E}_{2,t} + E_{2,n}\,\bar{a}_y = 2\bar{a}_x + 2\bar{a}_y \quad V/m$ [2pt]

c) $\bar{D}_1 = \varepsilon_{r,1}\cdot\varepsilon_0\cdot\bar{E}_1 = 3\varepsilon_0(2\bar{a}_x + 4\bar{a}_y) = 6\varepsilon_0\,\bar{a}_x + 12\,\varepsilon_0\,\bar{a}_y \quad C/m^2$

d) $\bar{D}_2 = \varepsilon_{r,2}\cdot\varepsilon_0\,\bar{E}_2 = 6\,\varepsilon_0\,\bar{E}_2 = 12\,\varepsilon_0\,\bar{a}_x + 12\,\varepsilon_0\,\bar{a}_y \quad C/m^2$

[2pt]

For this question only: if units are missing, NO POINTS DEDUCTED.

Reason: unit of $\bar{D}$'s depend on unit used for ε_0

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2\sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Differential operators

4.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial R}\vec{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

4.2 Divergence

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

4.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

4.4 Curl

$$\begin{aligned}
\nabla\times\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\vec{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\vec{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\vec{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\vec{a}_\phi
\end{aligned}$$

5. Electromagnetic formulas

Table 1 Electrostatics

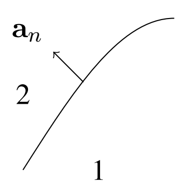
$\mathbf{F} = \dfrac{1}{4\pi\varepsilon} \dfrac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \displaystyle\int_v \dfrac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3} dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\displaystyle\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$	$\displaystyle\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$
$\mathbf{E} = -\nabla V$	$V = \dfrac{1}{4\pi\varepsilon} \displaystyle\int_v \dfrac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \dfrac{W}{Q} = -\displaystyle\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}$	
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$	$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$
$\rho_{p,v} = -\nabla \cdot \mathbf{P}$	
$\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$	
$\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$	
$E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \dfrac{1}{2} QV$
$W_e = \dfrac{1}{2} \displaystyle\int_v \rho_v V dv = \dfrac{1}{2} \int_{\mathbb{R}^3} \mathbf{D} \cdot \mathbf{E} dv$	
$\nabla \cdot (\varepsilon \nabla V) = -\rho_v$	$\nabla \cdot (\varepsilon \nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$	
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{B} = \nabla \times \mathbf{A}$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$	$\nabla \cdot \mathbf{A} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\mathbf{A} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$	$\mathbf{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}}{\lvert\mathbf{R} - \mathbf{R}'\rvert} dv'$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$	
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$	
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$	
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$	
$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$	
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$	
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$	

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$