SOLUTIONS

Q1:

a)

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

b) IF $\begin{bmatrix} a \\ b \end{bmatrix}$ IS PARALLEL TO $\begin{bmatrix} c \\ d \end{bmatrix}$, THEN

$$\begin{bmatrix} a \\ b \end{bmatrix} = k \begin{bmatrix} c \\ d \end{bmatrix} \qquad \text{SCALAR } k$$

$$\therefore \quad a = kc \text{ AND } b = kd$$

$$\therefore \quad \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} kc \\ c \end{bmatrix} = c \begin{bmatrix} k \\ 1 \end{bmatrix}$$

AND

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} kd \\ d \end{bmatrix} = d \begin{bmatrix} k \\ 1 \end{bmatrix}$$

SINCE BOTH $\begin{bmatrix} a \\ c \end{bmatrix}$ AND $\begin{bmatrix} b \\ d \end{bmatrix}$ ARE SCALAR MULTIPLES OF $\begin{bmatrix} k \\ 1 \end{bmatrix}$, THEY ARE PARALLEL.

C) YES, THIS IS POSSIBLE. A NEGATIVE DOT PRODUCT INDICATES THAT THE ANGLE BETWEEN TWO VECTORS IS BETWEEN $\frac{\pi}{2}$ AND π. SINCE THE 3 ANGLES MUST ADD UP TO 2π, WE CAN HAVE 3 ANGLES THAT EACH ARE BETWEEN $\frac{\pi}{2}$ AND π AND STILL SUM TO 2π. FOR EXAMPLE, EACH ANGLE COULD BE $\frac{2\pi}{3}$.

Q2: a) START BY FINDING A NORMAL VECTOR TO $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ AND $\begin{bmatrix}1\\0\\1\end{bmatrix}$ USING CROSS PRODUCT.

$$\begin{bmatrix}1\\2\\-1\end{bmatrix} \times \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\-2\\-2\end{bmatrix}$$

RECOGNIZING THAT $\begin{bmatrix}2\\1\\1\end{bmatrix}$ IS ORTHOGONAL TO THIS NORMAL VECTOR (DOT PRODUCT IS ZERO), THIS MEANS THAT $\begin{bmatrix}2\\1\\1\end{bmatrix}$ IS IN THE SAME PLANE AS $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ AND $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

$\therefore$ $\begin{bmatrix}2\\1\\1\end{bmatrix} = c\begin{bmatrix}1\\2\\-1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\quad$ c, d SCALARS

$\therefore$ $1 = 2c$

$c = 0.5$

$\therefore$ $2 = c + d$

$d = 1.5$

Check: $1 = -c + d = -0.5 + 1.5 = 1$ ✓

$\therefore$ $c = 0.5$

$d = 1.5$

b) RECOGNIZING THAT $\begin{bmatrix}3\\1\\1\end{bmatrix}$ IS NOT ORTHOGONAL TO THE NORMAL VECTOR $\begin{bmatrix}2\\-2\\-2\end{bmatrix}$, THIS MEANS THAT $\begin{bmatrix}3\\1\\1\end{bmatrix}$ IS NOT IN THE SAME PLANE AS $\begin{bmatrix}1\\2\\-1\end{bmatrix}$ AND $\begin{bmatrix}1\\0\\1\end{bmatrix}$.

THEREFORE WE WANT TO FIND THE PROJECTION OF $\begin{bmatrix}3\\1\\1\end{bmatrix}$ IN THIS PLANE BY FIRST PROJECTING IT ON THE NORMAL VECTOR.

$$\text{proj}_{\begin{bmatrix}2\\-2\\-2\end{bmatrix}}\begin{bmatrix}3\\1\\1\end{bmatrix} = \frac{2}{(\sqrt{12})^2}\begin{bmatrix}2\\-2\\-2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}2\\-2\\-2\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix}$$

AND THEN WE CAN FIND THE PROJECTION ON THE PLANE BY VECTOR SUBTRACTION:

$$\begin{bmatrix}3\\1\\1\end{bmatrix} - \begin{bmatrix}\frac{1}{3}\\-\frac{1}{3}\\-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{8}{3}\\\frac{4}{3}\\\frac{4}{3}\end{bmatrix}$$

NOW WE NEED TO SOLVE FOR C AND d

$$\begin{bmatrix}\frac{8}{3}\\\frac{4}{3}\\\frac{4}{3}\end{bmatrix} = c\begin{bmatrix}1\\2\\-1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\therefore \quad \frac{4}{3} = 2c$$

$$c = \frac{2}{3}$$

$$\therefore \quad \frac{8}{3} = c + d$$

$$d = \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$$

Check: $\frac{4}{3} = -c + d = -\frac{2}{3} + \frac{6}{3} = \frac{4}{3}$ ✓

$$\therefore \quad c = \frac{2}{3}$$

$$d = 2$$

Q3: $T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ EXCEPT $T\left(\begin{bmatrix} 0 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

a) $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$T\left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$T\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

b) $T\left(\begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix}\right) = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = cT\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$

$T\left(c\begin{bmatrix} 0 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 0 \\ cv_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = cT\left(\begin{bmatrix} 0 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So $T(cv) = cT(v)$

LET $v = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$ AND $w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

$T(v+w) = T\begin{pmatrix} w_1 \\ v_2+w_2 \end{pmatrix} = \begin{bmatrix} w_1 \\ v_2+w_2 \end{bmatrix}$

$T(v)+T(w) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$

So $T(v+w) \neq T(v) + T(w)$

Q4:

a)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$$

IF $a+b = c+d$, THEN

$$\begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} a+b \\ a+b \end{bmatrix} = (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS AN EIGENVECTOR WITH AN EIGENVALUE EQUAL TO $a+b$.

b) $$A - \lambda I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (a-\lambda)(d-\lambda) - bc$$
$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(c-b)^2+4bc}}{2}$$

$$= \frac{(a+d) \pm \sqrt{c^2+2bc+b^2}}{2}$$

$$= \frac{(a+d) \pm \sqrt{(c+b)^2}}{2}$$

$$= \frac{(a+d) \pm (c+b)}{2}$$

$$= \frac{a+d+c+b}{2}, \frac{a+d-c-b}{2}$$

$$= \frac{2(a+b)}{2}, \frac{2(a-c)}{2}$$

$$= (a+b), (a-c)$$

c) $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda = a-c$ $\left[\begin{array}{cc|c} a-a+c & b & 0 \\ c & d-a+c & 0 \end{array}\right]$

$$\left[\begin{array}{cc|c} c & b & 0 \\ c & b & 0 \end{array}\right]$$

$$cv_1 + bv_2 = 0$$

$$v_1 = -\frac{b}{c} v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{b}{c} v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -b/c \\ 1 \end{bmatrix}$$

EIGENVECTOR: $\vec{V} = \begin{bmatrix} -b/c \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -b \\ c \end{bmatrix}$

Q5:

a) $$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] R2-2R1$$

IF $d=10$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

THEN WE WOULD INTERCHANGE ROWS 2 AND 3.

b) $$\to \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 2 \end{array}\right] R2 \rightleftarrows R3$$

$$\to \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right] R3 \times (-1)$$

$$\to \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] R2+R3$$

$$\to \left[\begin{array}{ccc|c} 2 & 5 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] R1-R3$$

$$\to \left[\begin{array}{ccc|c} 2 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] R1-5R2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1.5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad (RNF)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1.5 \\ 1 \\ -2 \end{bmatrix}$$

c)

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & d & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad R2-2R1$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 0 & -1+\frac{1}{d-10} & 3-\frac{2}{d-10} \end{array}\right] \quad R3-\frac{1}{d-10}R2 \qquad d \neq 10$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & d-10 & -1 & 2 \\ 0 & 0 & \frac{11-d}{d-10} & \frac{3d-32}{d-10} \end{array}\right]$$

IF $d=11$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

SYSTEM IS INCONSISTENT.

Q6:

a)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix} \begin{matrix} \\ R2-R1 \\ R3-R1 \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R2 \times \frac{1}{4} \\ R3 \times \frac{1}{8} \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R3-R2 \end{matrix} \quad (RNF)$$

$\text{rank}(A) = 2$

b) $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

c) $$A = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$