UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING

FINAL EXAMINATION, April 17, 2019

Duration: 150 minutes

PHY294S - QUANTUM PHYSICS AND THERMAL PHYSICS

Exam Type A: Closed Book

Calculator Type 3: Non-programmable calculators without text storage

Examiners: Professor A. Paramekanti and Professor B. Netterfield

- Print in BLOCK LETTERS your name, student number and tutorial group on top of all examination booklets. If you use more than one booklet, enter the BOOK NO and TOTAL NUMBER OF BOOKS USED at the upper right corner of the booklet.
- Place your student ID on the desk.
- Aids allowed: only calculators, from a list of approved calculators as issued by the Faculty Registrar are allowed. No other aid (notes, textbook, dictionary) is allowed.
- Turn off any communication device (phone, pager, PDA, iPod, etc.) you may have and place it far from where you are sitting.
- There are two parts of the exam, one for Quantum Physics, and the other one for Thermal Physics. Each part is worth 50 marks in total.
- This examination paper consists of 19 questions. Answer all questions. Show all important steps.
- Part marks will be given for partially correct answers.
- Do not separate the stapled sheets of the question paper. Hand in the questions with your exam booklet at the end of the exam.
- The total number of marks is 100.

PHY294 Quantum Mechanics: Formula Sheet

Units, Constants, Identities

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x \\
\text{electron Volt } 1\text{eV} &= 1.6 \times 10^{-19}\text{J} \\
\text{Planck constant } h &= 6.63 \times 10^{-34} kg\ m^2/s
\end{aligned}$$

Time-independent Schrodinger equation in 1D

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

Particle in a 1D box

$$\begin{aligned}
E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2} \qquad (n = 1, 2\ldots) \\
\psi_n(x) &= \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}
\end{aligned}$$

Harmonic oscillator

$$\begin{aligned}
E_n &= (n + \frac{1}{2})\hbar\omega \qquad (n = 0, 1, 2\ldots) \\
\psi_0(x) &= (\frac{m\omega}{\pi\hbar})^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \\
\psi_1(x) &= (\frac{m\omega}{\pi\hbar})^{1/4}\sqrt{\frac{m\omega}{2\hbar}}\ 2x \exp\left[-\frac{m\omega x^2}{2\hbar}\right]
\end{aligned}$$

Fourier transforms

$$\begin{aligned}
\tilde{\psi}(k) &= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}}\ e^{-ikx}\ \psi(x) \\
\psi(x) &= \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2\pi}}\ e^{+ikx}\ \tilde{\psi}(k)
\end{aligned}$$

Integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2} &= \sqrt{\frac{\pi}{\alpha}} \\
\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\ e^{-\alpha x^2}\ e^{-ikx} &= \sqrt{\frac{1}{2\alpha}}\ e^{-\frac{k^2}{4\alpha}} \\
\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\ x e^{-\alpha x^2}\ e^{-ikx} &= -\sqrt{\frac{1}{2\alpha}}\ \frac{ik}{2\alpha}\ e^{-\frac{k^2}{4\alpha}} \\
\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\ x^2 e^{-\alpha x^2}\ e^{-ikx} &= \frac{1}{4\sqrt{2}\ \alpha^{5/2}}(2\alpha - k^2)\ e^{-\frac{k^2}{4\alpha}}
\end{aligned}$$

PHY294 Thermal Physics: Formula Sheet

Thermodynamics

$dU = TdS - PdV + \mu dN$ $\quad S = k\ln(\Omega)$ $\quad T = \left(\frac{\partial S}{\partial U}\right)^{-1}_{N,V}$ $\quad F = U - TS$

$C_v = \left(\frac{\partial U}{\partial T}\right)_V$ $\quad P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$ $\quad S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$ $\quad \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V} = -T\left(\frac{\partial S}{\partial N}\right)_{U,V}$

Multiplicity

Ideal gas: $\Omega = f(N)V^N U^{3N/2}$, where $f(N)$ is some function of only N.
Einstein Solid: $\Omega = (\frac{eq}{N})^N$.

Blackbodies

$\epsilon = 2.82kT$ when $u(\epsilon)$ is at maximum.
$u(\epsilon) = d(U/V)/d\epsilon = \frac{8\pi}{(hc)^3}\frac{\epsilon^3}{e^{\epsilon/kT}-1}$
Power $/A = \sigma T^4$

Partition Functions

$Z = \sum_s e^{-E(s)/kT}$
$P(s) = \frac{e^{-E(s)/kT}}{Z}$
$\frac{P(s_1)}{P(s_2)} = \frac{e^{-E(s_1)/kT}}{e^{-E(s_1)/kT}}$

Distribution Functions

Fermi-Dirac Distribution: $\frac{1}{e^{(\epsilon-\mu)/kT}+1}$
Bose-Einstein Distribution: $\frac{1}{e^{(\epsilon-\mu)/kT}-1}$
Planck Distribution: $\frac{1}{e^{\epsilon/kT}-1}$

Misc

Equipartition theorem: The average energy per quadratic degree of freedom is $\frac{1}{2}kT$
$nR = Nk$
Photons: $\epsilon = h\nu$
Photons: $\lambda = c/\nu$
Photon in a box: $\epsilon = \frac{hc}{2L}\sqrt{j_x^2 + j_y^2 + j_z^2}$

Constants

Avogadro's number: $N_A = 6.022 \cdot 10^{23}$ per mole
Planck's constant: $h = 6.626 \cdot 10^{-34}$ J s
Boltzmann's constant: $k = 1.380 \cdot 10^{-23}$ J/K
Speed of light: $c = 3.00 \cdot 10^8$ m/s
Stefan-Boltzmann constant: $\sigma = 5.67 \cdot 10^{-8} W m^{-2} K^4$

Quantum Mechanics: Show all steps in your work

1. Spin - (10 points)
Consider a spin-1/2 particle in the ground state of the Hamiltonian (where $b_x > 0, b_z > 0$)

$$H = \begin{pmatrix} b_z & b_x \\ b_x & -b_z \end{pmatrix}$$

- What frequency photons can this particle absorb to make a transition into its excited spin state?
- If we make many measurements on this particle in its ground state, to check if it has spin-$\uparrow$ or spin-$\downarrow$, what fraction of them would show that it has spin-$\uparrow$, which corresponds to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

2. Identical particles - (10 points)
Consider 2 identical bosons of mass m confined to a box of size L. Let x_1, x_2 denote the position of the two bosons, with $0 < x_1 < L$ and $0 < x_2 < L$.

What are the normalized wavefunctions and energies for
- the ground state of this two-particle system?
- the first excited state of this two-particle system?

3. Momentum - (10 points)
Consider a particle in the second excited state of a harmonic oscillator, with the normalized wavefunction

$$\psi(x) = (\frac{m\omega}{4\pi\hbar})^{1/4}(\frac{2m\omega}{\hbar}x^2 - 1)\ \exp\left[-\frac{m\omega x^2}{2\hbar}\right]$$

- Obtain and schematically sketch the momentum distribution of this particle.
- What value can you never get when you measure the momentum of this particle?

4. Tunneling - (20 points)
Consider a particle of mass m which experiences a potential energy (with $V_0 > 0$):

$$V(x) = V_0\delta(x - L) + V_0\delta(x + L).$$

- Schematically sketch $V(x)$.
- Derive the transmission coefficient T for the particle incident from the far left with energy E.
- Schematically sketch T as a function of $k \equiv (1/\hbar)\sqrt{2mE}$.

(*Hint*: Recall that a delta function potential leads to a discontinuity in the derivative of the wavefunction).

Thermal Physics: Show all steps in your work

1. Equipartition (6 points)

Use the equipartition theorem to calculate the heat capacity of 1 mole (207 g) of lead. Assume that lead is well modeled as an Einstein solid with 6 degrees of freedom per atom.

2. Multiplicity (10 points)

Consider a box of gas with N molecules. What is the probability that, at some moment, one would find that the left 1% of the box is completely devoid of gas molecules. Calculate the probability for

a) $N = 100$
b) $N = 10^4$
c) $N = 10^{23}$

If the number is too big for your calculator, write down $\log_{10}(P)$ instead.

3. Entropy (12 points)

Consider two Einstein Solids with N_1 and N_2 oscillators each. They have q_1 and q_2 units of energy each. Recall that for Einstein solids, $\Omega = (\frac{eq}{N})^N$, where q is the number of units of energy in the oscillators. So $U = Dq$ where $D = (k_s/m)^{1/2}/2\pi$. In terms of N_1, N_2, q_1, q_2 and D,

a) Calculate the Entropy of the two solids.
b) Calculate the temperature of the two solids.
c) Using your expression for temperature, calculate q_1 at thermal equilibrium.

4. Partition Functions (12 points)

Consider protons and neutrons to be two different states of the same particle (a nucleon). The mass of a neutron is $1.6749 \cdot 10^{-27}$ kg and the mass of a proton is $2.3 \cdot 10^{-30}$ kg less. Assume that, in the early Universe, the conditions allowed neutrons to be readily converted into protons, and vice versa. In the early Universe, at $T = 10^{11}$ K, what fraction of the nucleons were protons, and what fraction were neutrons?

5. Blackbody radiation (10 points)

Consider a cylindrical person of height 1.6 m, diameter 0.3 m, mass 70 kg, and temperature 37C. Assume that the person radiates as a black body.

a) How much total power do they radiate?
b) At what wavelength is $u = d(U/V)/d\epsilon$ at its maximum?
c) How much power does this person emit per unit mass? Compare this with the sun. (The sun's mass is $2 \cdot 10^{30}$ kg, and emits $3.9 \cdot 10^{26}$ W.)