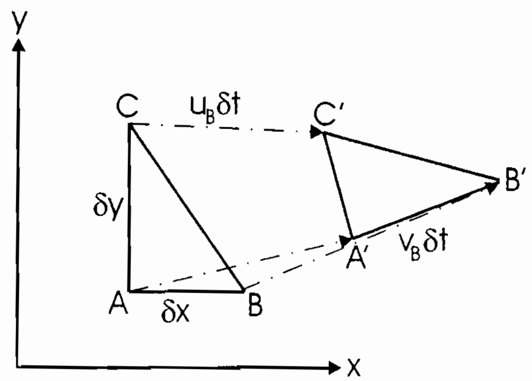
Exercises for Chapter 6

(6.1) A *two-dimensional flow* has a velocity field of the form

$$\mathbf{v}(\mathbf{r},t) = u(x,y,t)\,\mathbf{i_x} + v(x,y,t)\,\mathbf{i_y}$$

If such a flow is also in compressible, the volume of the triangular particle depicted in the diagram should not change as it subsequently moves and deforms. By considering the motion over an interval of time small enough to allow a the differential approximation to be used, show that this requires

$$\nabla\cdot\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



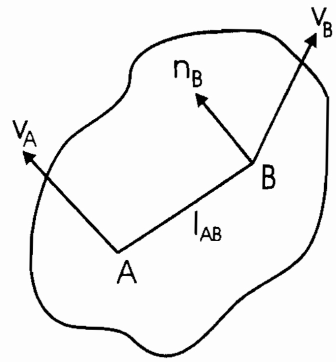
(6.2) For motion of a body in a plane such as occurs in two-dimenional flow the angular velocity ω_{AB} of one point B in the body relative to another point A can be defined as the component of the velocity of B relative to A normal to the straight line between A and B, divded by the length of this straight line. Thus, as depicted in the diagram to the right,

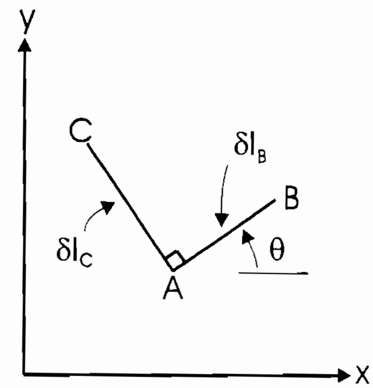
$$\omega_{AB} = \frac{1}{l_{AB}}(\mathbf{v_B} - \mathbf{v_A})\cdot\mathbf{n_B}$$

For a rigid body ω_{AB} is the same regardless of the location of A and B but, when deformation occurs, such as in a body fluid, ω_{AB} varies with the locations of A and B choice of points. Show that, for two straight lines of fluid particles, AB and AC respectively, which are in the diagram, if th elines are short enough to aloow the use of differential approximations to calculate relative velocities

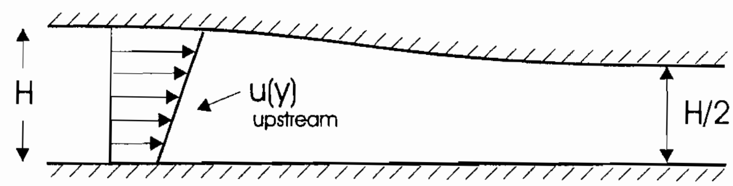
$$\omega_{MEAN} = \frac{1}{2}(\omega_{AB} + \omega_{BC}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Note that $\mathbf{n_B}$ and $\mathbf{n_C}$ are chosen to make counterclockwise rotation postive.





(6.3) A channel of width H is connected to a contraction which reduces the width to $H/2$ as depicted in the diagram. The incompressible fluid approaching the contraction has the velocity distribution $\mathbf{v}(\mathbf{r},t) = u(y)\mathbf{i_x}$ where U increases linearly from U_B at $y = 0$ to U_T at $y = H$. By assuming that the flow is frictionless and steady that, downstream of the contraction the streamlines are parallel to $\mathbf{i_x}$, find the pressure decrease $p_U - p_D$ through the contraction.



(6.4) Given the unsteady two-dimensional incompressible inviscid flow field

$$\mathbf{v}(\mathbf{r},t) = 2xy(1+\alpha t)\mathbf{i_x} + (ax^2 + by^2)(1 + et + ft^2)\mathbf{i_y}$$

choose the constants a, b, e and f so that (i), the equation of continuity is satisfied and (ii), the flow field is *irrotational*, that is, has the veloctiy $\zeta = 0$. Show that a scalar velocity potential $\phi(x,y)$ exists only if the flow is irrotational. For the particular case of steady flow, integrate the momentum equation in the case of zero gravitational forces to obtain the pressure distribution $p(x,y)$ subject to the condition that $p(0,0)$ is held constant at P_0. Verify that Bernoulli's equation applies to the entire flow field.

(6.5) In two-dimensional incompressible inviscid steady flow, an idealized vortex is modelled in polar (r,θ) coordinates as $\mathbf{v}(\mathbf{r}) = q(r)\mathbf{i}_\theta$. Geometrically, the streamlines are concentric circles with centres at the origin. With radial pressure gradients provding the necessary centrifugal acceleration, find $q(r)$ for the case that $E = p/\rho + \frac{1}{2}q^2$ is the same for all streamlines. Express the veloctiy field in Cartesian coordinates and then show that the vorticity $\zeta = \partial v/\partial x - \partial u/\partial y = 0$ for this flow, except at the origin. Take the motion to be in the horizontal plane, so that gravity can be omitted. Hint: Start by looking at Example 2.7 of the text.

(6.6) Describe the two-dimensional flow field represented by the complex variable

$$z = x + iy = w + e^w \quad \text{where} \quad w = \phi + i\psi$$

here ϕ and ψ are, respectively, the velocity potential and the stream function. Obtain expressions for the velocity components in terms of ϕ and ψ and locate any stagnation points. Plot the streamlines $\psi = 0,\ \pi/2$, and π. Hint: recall that points at which the speed $q = \sqrt{u^2 + v^2}$ becomes infinite occur at locations at which streamlines have discontinuities in slope.

(6.7) Describe the irrotational flow field modelled by the complex variable relationship

$$w + e^{(w+z)} = 0$$

In particular, find expressions for the shape of the streamlines n the form

$$x = f_x(\psi, \phi) = 0 \qquad y = f_y(\psi, \phi) = 0$$

Find the shape of the streamline $\psi = 0$, and plot this graph for $-3 \leq x \leq +3$. Also plot the streamlines $\psi = -1/4$ and $-1/2$. Locate the position of any stagnation points, and hence or otherwise describe the physical significance of the flow.

(6.8) For a small region around the origin of the (x, y) plane, a two-dimensional steady flow field can be expressed as a Taylor's series; that is,

$$u = a + bx + cy + \cdots \quad ; \quad v = d + ex + fy + \cdots$$

where b, c, e, and f are the appropriate derivatives. For this flow field, demonstrate

(6.9) Show that the complex potential $w = Az^{4/5}$ with A real, corresponds to flow over an obtuse angled corner as shown in the diagram below. What is the angle α and the direction of the flow on the body?

(6.10) If the complex potential $w = f(z)$, show that, with $q(x, y)$ the local speed at a given point, and θ, the direction of flow, measured relative to the positive x-axis, then

$$\frac{dw}{dz} = qe^{i(\pi-\theta)}$$

(6.11) Show that the vector field represented by

$$u = xyzt \quad , \quad v = -xyzt^2 \quad , \quad w = \frac{z^2}{2}(xt^2 - yt^2)$$

represents a possible incompressible flow field. Can it be represented by a potential? Find the component of the pressure gradient parallel to the $\hat{e}_x$ axis.