TERM TEST SOLUTIONS

Q1

a) $\vec{d}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{d}_2 = \begin{bmatrix} 5 \\ 2.5 \\ 2.5 \end{bmatrix}$

SINCE $\vec{d}_2 = 2.5\,\vec{d}_1$, LINE 1 IS PARALLEL TO LINE 2.

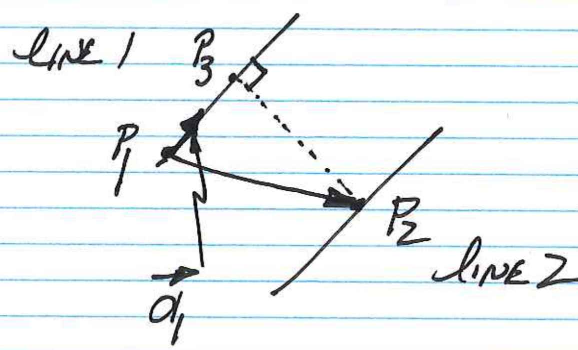
b) CHOOSE A POINT ON LINE 1 AND SEE IF IT LIES ON LINE 2.

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 5 \\ 2.5 \\ 2.5 \end{bmatrix}$$

NO VALUE OF s WILL SATISFY THIS EQUATION.

$\therefore$ LINE 1 $\neq$ LINE 2

c)



FIND $\|\overrightarrow{P_3P_2}\|$

TAKE $P_1(2,1,2)$ AND $P_2(0,-1,-1)$

THEN $\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix}$

$$\vec{P_1P_3} = \text{proj}_{\vec{d_1}} \vec{P_1P_2} = \frac{\vec{P_1P_2} \cdot \vec{d_1}}{\|\vec{d_1}\|^2} \vec{d_1}$$

$$\vec{P_1P_2} \cdot \vec{d_1} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = -4 - 2 - 3 = -9$$

$$\|\vec{d_1}\|^2 = 4 + 1 + 1 = 6$$

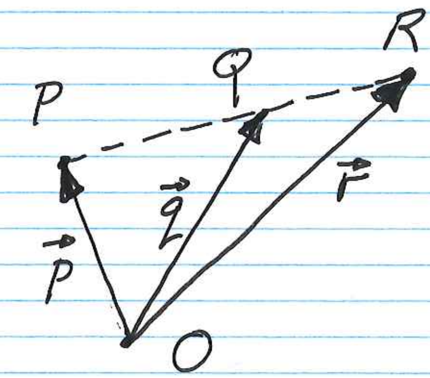
$$\therefore \quad \vec{P_1P_3} = \frac{-9}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1.5 \\ -1.5 \end{bmatrix}$$

$$\vec{P_3P_2} = \vec{P_1P_2} - \vec{P_1P_3} = \begin{bmatrix} -2 \\ -2 \\ -3 \end{bmatrix} - \begin{bmatrix} -3 \\ -1.5 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -.5 \\ -1.5 \end{bmatrix}$$

$$P_3P_2 = \sqrt{1 + 0.25 + 2.25} = \sqrt{3.5} \approx 1.87$$

MINIMUM DISTANCE BETWEEN THE TWO LINES IS $\sqrt{3.5} \approx 1.87$.

Q2



AREA INSIDE OPQ $= \frac{1}{2}\|\vec{p}\times\vec{q}\|$

AREA INSIDE OQR $= \frac{1}{2}\|\vec{q}\times\vec{r}\|$

AREA INSIDE OPR $= \frac{1}{2}\|\vec{r}\times\vec{p}\|$

BECAUSE P, Q AND R ARE COLLINEAR:

1. $\frac{1}{2}\|\vec{p}\times\vec{q}\| + \frac{1}{2}\|\vec{q}\times\vec{r}\| = \frac{1}{2}\|\vec{r}\times\vec{p}\|$

AND

2. $\vec{p}, \vec{q}$ AND $\vec{r}$ LIE IN THE SAME PLANE.

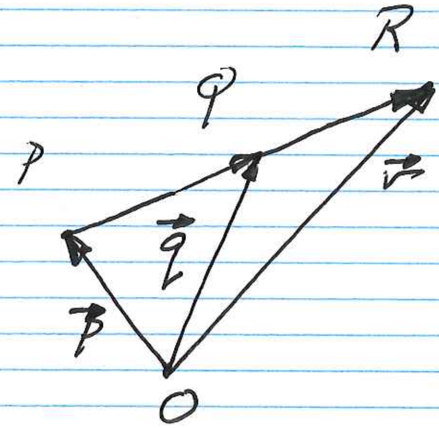
$\therefore$ $\vec{p}\times\vec{q}$, $\vec{q}\times\vec{r}$ AND $\vec{r}\times\vec{p}$ ARE PARALLEL VECTORS BUT BY RIGHT HAND RULE $\vec{p}\times\vec{q}$ AND $\vec{q}\times\vec{r}$ ARE IN THE OPPOSITE (OR NEGATIVE) DIRECTION OF $\vec{r}\times\vec{p}$.

$\therefore$ $\vec{p}\times\vec{q} + \vec{q}\times\vec{r} = -\vec{r}\times\vec{p}$

OR

$$\vec{p}\times\vec{q} + \vec{q}\times\vec{r} + \vec{r}\times\vec{p} = \vec{0}$$

Q2 (ALTERNATE RESULT THANKS TO VALENTIN PERETROUKHIN)



$\vec{PQ} = \vec{q} - \vec{p}$; $\vec{QR} = \vec{r} - \vec{q}$

BECAUSE P, Q AND R ARE COLLINEAR, $\vec{PQ}$ AND $\vec{QR}$ ARE PARALLEL.

$$\therefore \quad (\vec{q} - \vec{p}) \times (\vec{r} - \vec{p}) = \vec{0}$$

$$\vec{q} \times \vec{r} - \vec{q} \times \vec{p} - \vec{p} \times \vec{r} + \overset{0}{\cancel{\vec{p} \times \vec{p}}} = \vec{0}$$

OR $\quad \vec{q} \times \vec{r} + \vec{p} \times \vec{q} + \vec{r} \times \vec{p} = \vec{0}$

Q3

a) GIVEN $\vec{V} = \text{CROSS}_{\vec{w}}\,\vec{u} = \vec{w} \times \vec{u}$ WITH FIXED $\vec{w}$

TO VERIFY THAT $\text{CROSS}_{\vec{w}}\,\vec{u}$ IS A LINEAR TRANSFORMATION, WE NEED TO SHOW THAT:

1. $\text{CROSS}_{\vec{w}}(\vec{u}_1 + \vec{u}_2) = \text{CROSS}_{\vec{w}}\,\vec{u}_1 + \text{CROSS}_{\vec{w}}\,\vec{u}_2$
2. $\text{CROSS}_{\vec{w}}(k\vec{u}) = k\,\text{CROSS}_{\vec{w}}\,\vec{u}$ K SCALAR

1. $\text{CROSS}_{\vec{w}}(\vec{u}_1 + \vec{u}_2) = \vec{w} \times (\vec{u}_1 + \vec{u}_2) = \vec{w} \times \vec{u}_1 + \vec{w} \times \vec{u}_2 = \text{CROSS}_{\vec{w}}\,\vec{u}_1 + \text{CROSS}_{\vec{w}}\,\vec{u}_2$ ✓

2. $\text{CROSS}_{\vec{w}}(k\vec{u}) = \vec{w} \times (k\vec{u}) = k(\vec{w} \times \vec{u}) = k\,\text{CROSS}_{\vec{w}}\,\vec{u}$ ✓

$\therefore$ IT IS A LINEAR TRANSFORMATION.

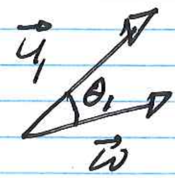
b) $$\vec{V} = \text{CROSS}_{\vec{w}}\,\vec{u} = \vec{w} \times \vec{u} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} w_2 u_3 - w_3 u_2 \\ w_3 u_1 - w_1 u_3 \\ w_1 u_2 - w_2 u_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

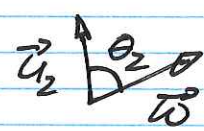
$$= M\vec{u}$$

c)



WE KNOW THAT $\|\vec{\omega} \times \vec{u}_1\| = \|\vec{\omega}\| \|\vec{u}_1\| \sin\theta_1$ AND BY RIGHT HAND RULE, $\vec{\omega} \times \vec{u}_1$ IS POINTING ORTHOGONALLY OUT OF THE PAGE.

ONE CAN IMAGINE ANOTHER VECTOR $\vec{u}_2$



WHERE $\sin\theta_2 > \sin\theta_1$

AND

$$\|\vec{u}_2\| < \|\vec{u}_1\|$$

AND

$$\|\vec{u}_1\| \sin\theta_1 = \|\vec{u}_2\| \sin\theta_2$$

IN THIS CASE, $\vec{\omega} \times \vec{u}_1 = \vec{\omega} \times \vec{u}_2$

BUT $\vec{u}_1 \neq \vec{u}_2$.

$\therefore$ THIS MATRIX WOULD NOT BE EXPECTED TO HAVE AN INVERSE.

Q4

a) $\begin{bmatrix} -3 & 1 & -6 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 16 \\ 14 \end{bmatrix}$ WHICH IS NOT PARALLEL TO $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ IS NOT AN EIGENVECTOR

$$\begin{bmatrix} -3 & 1 & -6 \\ 8 & -3 & 8 \\ 8 & -1 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ IS AN EIGENVECTOR WITH EIGENVALUE $= -2$

b) GIVEN $B\vec{u} = \lambda\vec{u}$, $C\vec{u} = \alpha\vec{u}$, $\vec{u} \neq \vec{0}$

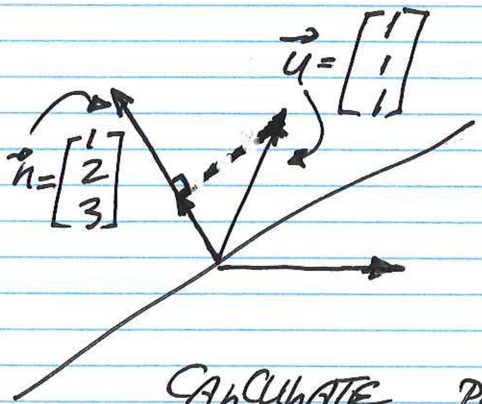
$\therefore B\vec{u} + C\vec{u} = (B+C)\vec{u} = \lambda\vec{u} + \alpha\vec{u} = (\lambda+\alpha)\vec{u}$

$\therefore \vec{u}$ IS AN EIGENVECTOR OF $B+C$ WITH AN EIGENVALUE $= \lambda + \alpha$

$\therefore (BC)\vec{u} = B(C\vec{u}) = B(\alpha\vec{u}) = \alpha(B\vec{u}) = (\alpha\lambda)\vec{u}$

$\therefore \vec{u}$ IS AN EIGENVECTOR OF BC WITH AN EIGENVALUE $= \alpha\lambda$

Q5



CALCULATE $\text{proj}_{\vec{n}}\vec{u} = \dfrac{\vec{u}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n}$

$$\vec{u}\cdot\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix} = 6$$

$$\|\vec{n}\|^2 = 1+4+9 = 14$$

$$\therefore \text{proj}_{\vec{n}}\vec{u} = \frac{6}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{3}{7}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

CALCULATE $\vec{u} - \text{proj}_{\vec{n}}\vec{u} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{3}{7}\begin{bmatrix}1\\2\\3\end{bmatrix}$

$$= \begin{bmatrix}4/7\\1/7\\-2/7\end{bmatrix}$$

$\therefore$ REFLECTION OF $\vec{u}$ THROUGH THE PLANE IS GIVEN BY

$$-\frac{3}{7}\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}4/7\\1/7\\-2/7\end{bmatrix} = \begin{bmatrix}1/7\\-5/7\\-11/7\end{bmatrix}$$

Q6

a)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \begin{matrix} \\ R2-R1 \\ R3-R1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} R2 \leftrightarrow R3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R3 \times (-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1-R3 \\ R2+R3 \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R2 \times (-1)$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R1-R2$$

RNF
$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

RANK = 3

b)

SOLVE: $A\vec{x} = \vec{0}$

AFTER APPLYING GAUSSIAN ELIMINATION:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

LEADING VARIABLES: x_1, x_3, x_4

FREE VARIABLES: x_2

SOLUTION:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

c) SOLUTION IS A LINE THROUGH THE ORIGIN IN R^4.