ESC103F Engineering Mathematics and Computation: Tutorial #7

Question 1: Consider plane #1: $x - 3y - z = 0$ and plane #2: $x - 3y - z = 12$. These planes are parallel because their normal vectors are parallel.

i) Give a vector equation for all points on each plane using the form:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ? \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} ? \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} ? \\ 0 \\ 1 \end{bmatrix}$$

ii) From just looking at the two vector equations derived in part (i), how do you know the two planes are parallel?

Solution:

i) For plane #1,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

For plane #2,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

ii) Starting from $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in the case of plane #1, and from $\begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$ in the case of plane #2, the points on each plane are found by adding combinations of the same two vectors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and therefore the planes are parallel.

Question 2: Consider matrix $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$.

i) Find matrix R by first finding matrix C where $A = CR$.

ii) Use Gaussian elimination to find R by first finding R_0.

Solution:

i) $$C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$\therefore R$ is the following 2x5 matrix,

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

ii) $$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R_0$$

$$\therefore R = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

Question 3: Put as many 1's as possible in a 4x7 R_0 matrix that is in reduced row echelon form where the leading variables correspond to columns 2, 4 and 5.

Solution:

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Question 4: Consider the system $A\vec{x} = \vec{0}$ and matrix A is 3x5. Suppose column 4 of matrix A is all zeros. Then, x_4 is certainly what kind of variable? What is the special solution $\vec{x}$ associated with this variable x_4?

Solution:

Regardless of what operations are performed on matrix A to bring it to its RREF, column 4 will remain all zeros. As a result, x_4 will be a free variable and none of the leading variables will be functions of x_4. Therefore, the special solution associated with x_4 is $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$.

Substituting this special solution for $\vec{x}$ will give $A\vec{x} = \vec{0}$ because column 4 of matrix A is all zeros.

Question 5: Construct a matrix A whose column space contains $\begin{bmatrix}1\\1\\5\end{bmatrix}$ and $\begin{bmatrix}0\\3\\1\end{bmatrix}$ and has $\vec{x} = \begin{bmatrix}1\\1\\2\end{bmatrix}$ as a solution to $A\vec{x} = \vec{0}$. What other $A's$ would have these same properties?

Solution: Given $A\begin{bmatrix}1\\1\\2\end{bmatrix} = \vec{0}$, and that the column space contains vectors in R^3, A is a 3x3 matrix.

Given that matrix A has a column space containing $\begin{bmatrix}1\\1\\5\end{bmatrix}$ and $\begin{bmatrix}0\\3\\1\end{bmatrix}$, let matrix A be given by,

$A = \begin{bmatrix}1 & 0 & x\\1 & 3 & y\\5 & 1 & z\end{bmatrix}$ where x, y, z are unknown

$$\therefore \begin{bmatrix}1 & 0 & x\\1 & 3 & y\\5 & 1 & z\end{bmatrix}\begin{bmatrix}1\\1\\2\end{bmatrix} = \vec{0}$$

$$1 + 2x = 0 \quad \rightarrow \quad x = -\frac{1}{2}$$

$$4 + 2y = 0 \quad \rightarrow \quad y = -2$$

$$6 + 2z = 0 \quad \rightarrow \quad z = -3$$

$\therefore A = \begin{bmatrix}1 & 0 & -\frac{1}{2}\\1 & 3 & -2\\5 & 1 & -3\end{bmatrix}$ and any nonzero scalar multiple of A.