University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test

9:10 – 10:55, 23 November 2023

105 minutes

No calculators or aids

There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1) Find $\frac{dy}{dx}$ for: a) $y = \ln(2x)$ b) $y = e^{-x^2}$ c) $y = \ln[x + (x^2 - 1)^{1/2}]$

d) $y = \frac{1}{x\,(x+1)(x+2)\cdots(x+n)}$ where $n > 2$ is a positive integer

a) $\frac{dy}{dx} = \frac{1}{2x} \cdot 2 = \frac{1}{x}$

b) $\frac{dy}{dx} = e^{-x^2}(-2x) = -2x e^{-x^2}$

c) $\frac{dy}{dx} = \frac{1}{x + (x^2-1)^{1/2}} \cdot \left(1 + \frac{1}{2}(x^2-1)^{-1/2}(2x)\right) = \frac{1}{x+(x^2-1)^{1/2}}\left(1 + \frac{x}{(x^2-1)^{1/2}}\right)$

$= \frac{1}{x + (x^2-1)^{1/2}}\left(\frac{(x^2-1)^{1/2} + x}{(x^2-1)^{1/2}}\right) = (x^2-1)^{-1/2}$

d) $\frac{dy}{dx} = \frac{-1}{x(x+1)\cdots(x+n)}\left[\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+n}\right]$

2) Evaluate the integrals:

a) $\int_0^{\pi} \sin 5x \, dx$ $= \left[-\frac{1}{5} \cos 5x \right]_0^{\pi} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

b) $\int_1^8 x^{-2/3} \, dx$ $= \left[3x^{1/3} \right]_1^8 = 6 - 3 = 3$

c) $\int \frac{dx}{ax+b} \cancel{dx}$ $= \frac{1}{a} \ln|ax+b| + C$

$(a \neq 0)$

d) $\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx$ $= 0$ $(x^4 \sin x$ is an odd function$)$

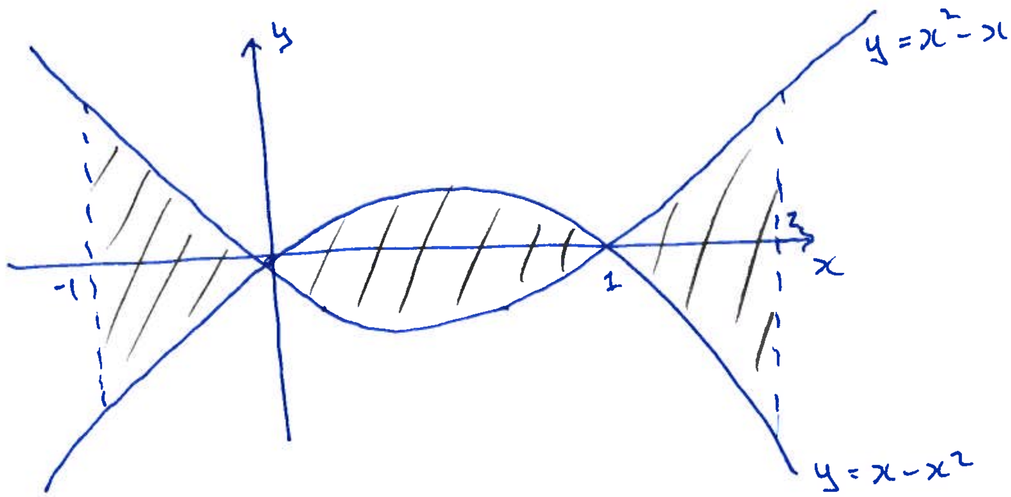
e) $\int_0^4 |\sqrt{x} - 1| \, dx$ $= \int_0^1 (1 - \sqrt{x}) \, dx + \int_1^4 (\sqrt{x} - 1) \, dx$

$= \left[x - \frac{2}{3} x^{3/2} \right]_0^1 + \left[\frac{2}{3} x^{3/2} - x \right]_1^4$

$= 1 - \frac{2}{3} + \frac{16}{3} - 4 - \frac{2}{3} + 1$

$= 2$

3) Find the area of the region that lies between the curves $f(x) = x^2 - x$ and $g(x) = x - x^2$ on the interval $[-1,2]$. Provide a sketch of the region.



Intersections: $x^2 - x = x - x^2 \Rightarrow 2x = 2x^2 \Rightarrow x = 0, 1$

$$A = \int_{-1}^{0} [(x^2-x)-(x-x^2)]\,dx + \int_0^1 [(x-x^2)-(x^2-x)]\,dx + \int_1^2 [(x^2-x)-(x-x^2)]\,dx$$

$$= \int_{-1}^{0} (2x^2-2x)\,dx + \int_0^1 (2x-2x^2)\,dx + \int_1^2 (2x^2-2x)\,dx$$

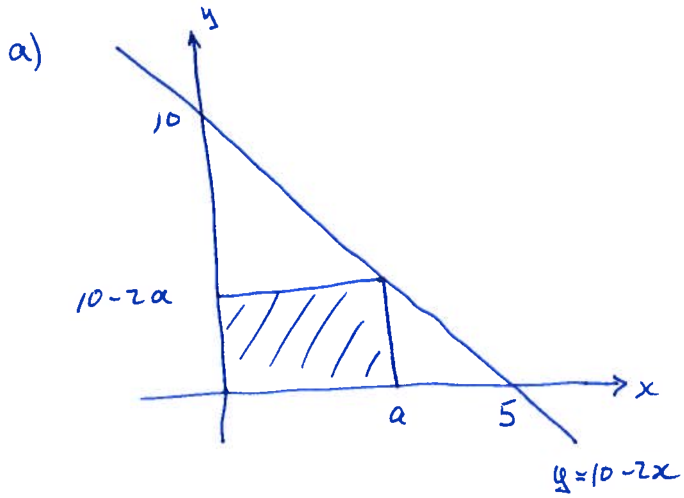
$$= \left[\frac{2x^3}{3} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{2x^3}{3}\right]_0^1 + \left[\frac{2x^3}{3} - x^2\right]_1^2$$

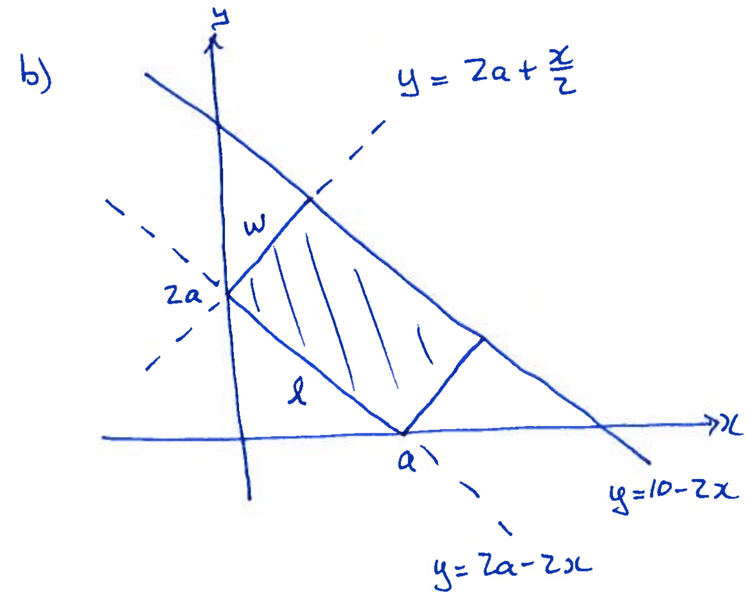
$$= \left(\frac{2}{3}+1\right) + \left(1 - \frac{2}{3}\right) + \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 1\right)$$

$$= \frac{5}{3} + \frac{1}{3} + \frac{4}{3} + \frac{1}{3} = \frac{11}{3}$$

4) a) A rectangle is constructed with one side on the x-axis, one side on the positive y-axis, and the vertex opposite the origin on the line $y = 10 - 2x$. What dimensions maximize the area of the rectangle? What is the maximum area?

b) Is it possible to construct a rectangle with a greater area than that found in part (a) by placing one side of the rectangle on the line $y = 10 - 2x$ and the two vertices not on that line on the positive x- and y-axes? Find the dimensions of the rectangle of maximum area that can be constructed this way.





a) $A = a(10-2a) \Rightarrow A' = 10-4a \Rightarrow A' = 0 \Rightarrow a = \frac{5}{2} \quad \therefore A = \frac{25}{2}$

b) $l = \sqrt{(2a)^2 + a^2} = \sqrt{5}\, a$

to find w, set $2a + \frac{x}{2} = 10 - 2x \Rightarrow \frac{5}{2}x = 10 - 2a \Rightarrow x = \frac{20-4a}{5}$

$\therefore y = 2 + \frac{8a}{5}$

$\therefore w = \sqrt{\left(2 + \frac{8a}{5} - 2a\right)^2 + \left(\frac{20-4a}{5}\right)^2} = \frac{1}{5}\sqrt{(10 + 8a - 10a)^2 + (20-4a)^2}$

$= \frac{1}{5}\sqrt{(100 - 40a + 4a^2) + (400 - 160a + 16a^2)} = \frac{1}{5}\sqrt{20a^2 - 200a + 500}$

$= \frac{2}{\sqrt{5}}\sqrt{a^2 - 10a + 25} = \frac{2}{\sqrt{5}}(5-a)$ note: $0 < a < 5$

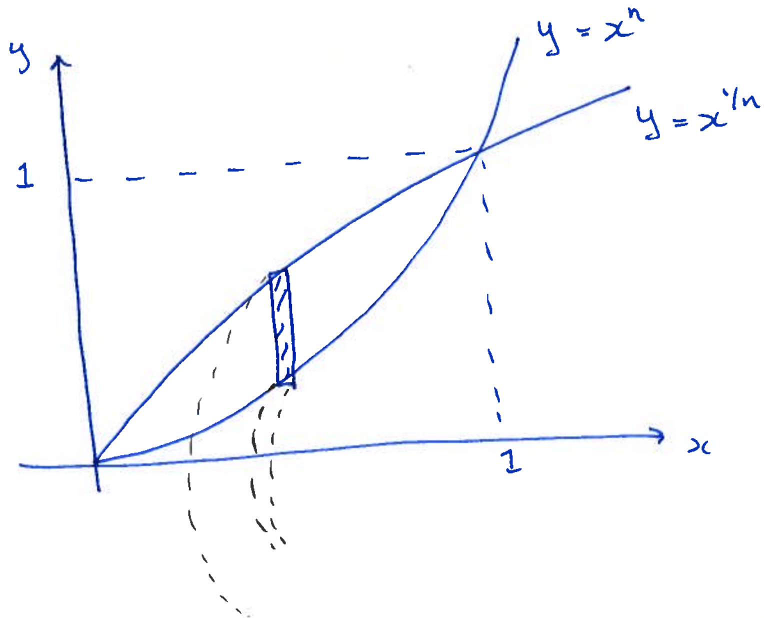
$\therefore A = l \cdot w = 10a - 2a^2 \quad \therefore A' = 10 - 4a \Rightarrow A' = 0 \Rightarrow a = \frac{5}{2}$

$\therefore A = 10\left(\frac{5}{2}\right) - 2\left(\frac{5}{2}\right)^2 = \frac{25}{2}$

5) Consider the region R in the first quadrant bounded by $y = x^{1/n}$ and $y = x^n$, where $n > 1$ is a positive number.

a) Find the volume $V(n)$ of the solid generated when R is revolved about the x-axis. Express your answer in terms of n. Provide a sketch of the region.

b) Evaluate $\lim_{n\to\infty} V(n)$. Interpret this limit geometrically.



a) $$V = \int_0^1 \pi \left(\left(x^{1/n} \right)^2 - x^{2n} \right) dx = \pi \left[\frac{x^{\frac{2}{n}+1}}{\frac{2}{n}+1} - \frac{x^{2n+1}}{2n+1} \right]_0^1$$

$$= \pi \left(\frac{1}{\frac{2}{n}+1} - \frac{1}{2n+1} \right) = \pi \left(\frac{n}{2+n} - \frac{1}{2n+1} \right)$$

$$= \pi \left(\frac{2n^2+n-2-n}{(2+n)(2n+1)} \right) = \pi \, \frac{2(n^2-1)}{2n^2+5n+2}$$

b) $\lim_{n\to\infty} V = \pi$ = volume of a solid disk radius = 1, thickness = 1.

6) Show that the function $g(x) = \sqrt{x-2},\ x \geq 2$, is one-to-one and find its inverse. Provide a simple sketch of $g(x)$ and $g^{-1}(x)$.

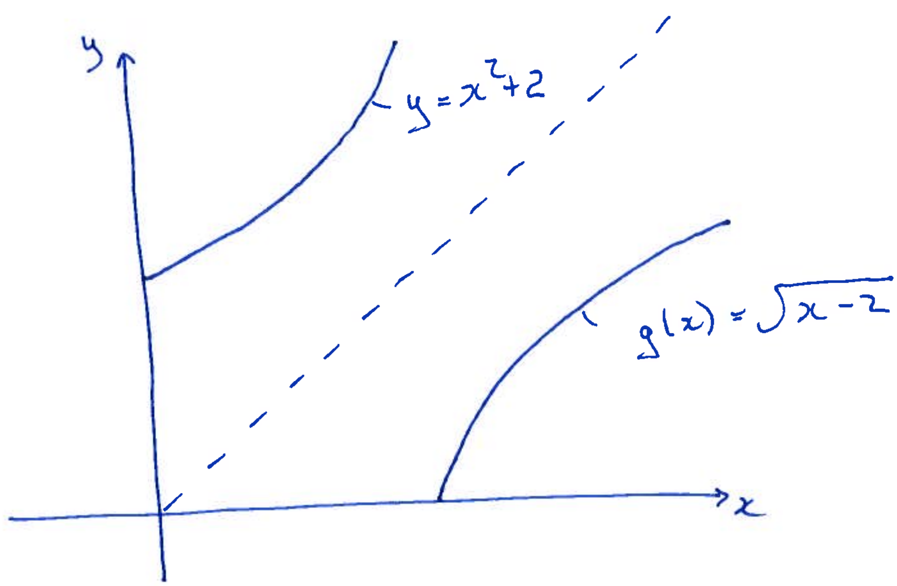
For $x_1 \neq x_2 \Rightarrow x_1 - 2 \neq x_2 - 2 \Rightarrow \sqrt{x_1 - 2} \neq \sqrt{x_2 - 2}$

$\therefore g(x) = \sqrt{x-2}\ ,\ x \geq 2$ is 1-1.

let $y = g^{-1}(x) \Rightarrow x = g(y) = \sqrt{y-2} \Rightarrow y = x^2 + 2$

$y \geq 2 \ \therefore\ x \geq 0$

$\therefore g^{-1}(x) = x^2 + 2\ ,\ x \geq 0$



7) For the function: $f(x) = \ln(1 + x^3)$

i) Determine the domain of f, the x and y intercepts, and identify any symmetry.

ii) Find the intervals in which f increases or decreases.

iii) Find the extreme values.

iv) Determine the concavity of the graph.

v) Sketch the graph specifying the points of inflection, asymptotes and vertical tangents, if any.

i) $1 + x^3 > 0 \Rightarrow x^3 > -1 \Rightarrow x > -1$: Domain $(-1, \infty)$

$f(0) = \ln 1 = 0$ $\therefore$ intercept $(0,0)$

no symmetry

ii) $f'(x) = \dfrac{3x^2}{1+x^3}$ $\quad f' > 0$ on $(-1,0)$ & $(0,\infty)$ $\therefore$ increasing on $(-1,\infty)$

iii) $f'(x) = 0 \Rightarrow x = 0$ but $f(x)$ is increasing on $(-1,\infty)$

$\therefore$ no extreme values

iv) $f''(x) = \dfrac{(1+x^3)(6x) - 3x^2(3x^2)}{(1+x^3)^2} = \dfrac{3x[2(1+x^3) - 3x^3]}{(1+x^3)^2} = \dfrac{3x(2-x^3)}{(1+x^3)^2}$

$f'' = 0$ at $x = 0$, $x = 2^{1/3}$: $(-1,0) \Rightarrow f'' < 0 \therefore$ concave down

$(0, 2^{1/3}) \Rightarrow f'' > 0 \therefore$ " up

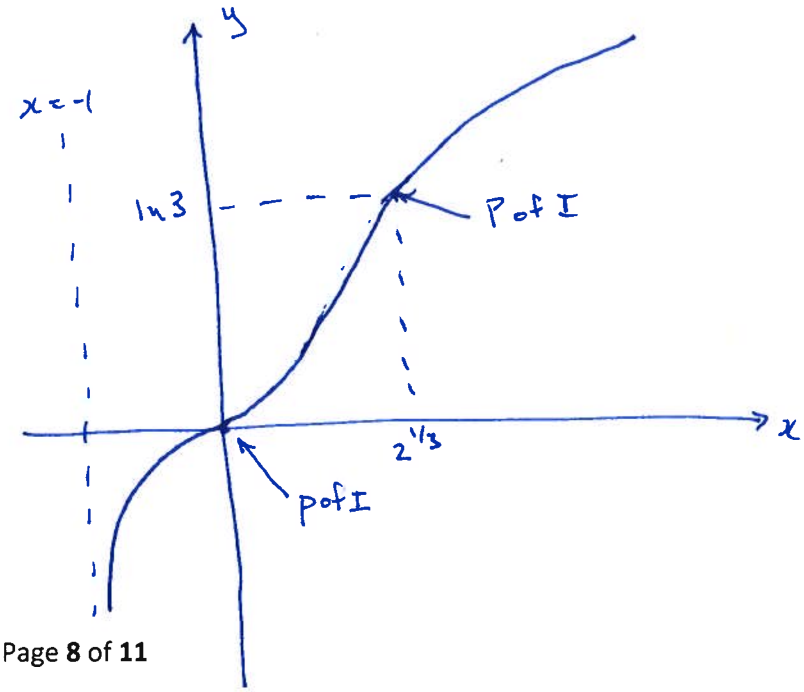
$(2^{1/3}, \infty) \Rightarrow f'' < 0 \therefore$ " down

v) $\lim\limits_{x \to -1^+} \ln(1+x^3) = -\infty$

$\therefore x = -1$ is a vertical asymptote

As $x \to \infty$

$\ln(1+x^3) \sim \ln x^3 = 3\ln x$



8) Given a a constant, find $f'(x)$ for: a) $f(x) = x^{a^a}$ b) $f(x) = a^{x^a}$ c) $f(x) = a^{a^x}$ d) $f(x) = x^{x^x}$

a) let $y = x^{a^a} \Rightarrow \ln y = a^a \ln x \Rightarrow \frac{y'}{y} = \frac{a^a}{x} \Rightarrow y' = a^a \frac{x^{a^a}}{x}$

$\Rightarrow y' = a^a x^{a^a - 1}$

b) let $y = a^{x^a} \Rightarrow \ln y = x^a \ln a \Rightarrow \frac{y'}{y} = a x^{a-1} \ln a$

$\Rightarrow y' = a \ln a \left(x^{a-1} \cdot a^{+x^a} \right)$

$= x^{a-1} a^{1+x^a} \ln a$

c) let $y = a^{a^x} \Rightarrow \ln y = a^x \ln a \Rightarrow \frac{y'}{y} = \ln a \cdot a^x \cdot \ln a$

$\Rightarrow y' = a^{a^x} \cdot a^x \cdot (\ln a)^2$

$= a^{x + a^x} (\ln a)^2$

d) let $y = x^{x^x} \Rightarrow \ln y = x^x \ln x \Rightarrow \frac{y'}{y} = \frac{x^x}{x} + (x^x)' \ln x$

let $z = x^x \Rightarrow \ln z = x \ln x \Rightarrow \frac{z'}{z} = \ln x + \frac{x}{x} = 1 + \ln x$

$\therefore z' = x^x (1 + \ln x) = (x^x)'$

$\Rightarrow y' = x^{x^x} \left(x^{x-1} + x^x (1 + \ln x) \ln x \right)$

$= x^{(x^x + x - 1)} + x^{(x^x + x)} (1 + \ln x) \ln x$

9) A function F is defined by the following integral: $F(x) = \int_1^x \frac{e^t}{t}\, dt$ for $x > 0$.
For what values of x is $\ln x \leq F(x)$?

$$F(x) = \int_1^x \frac{e^t}{t}\,dt \quad , \quad \ln x = \int_1^x \frac{1}{t}\,dt$$

$$\Rightarrow \text{ for } x = 1 \quad , \quad F(1) = \ln 1 = 0$$

$$\text{for } t > 1 \ , \ e^t > 1 \quad \therefore \ \frac{e^t - 1}{t} > 0$$

$$\therefore \ F(x) - \ln x = \int_1^x \frac{e^t - 1}{t}\,dt > 0 \quad \text{for } x > 1$$

$$\therefore \ F(x) > \ln x$$

$$\text{for } 0 < x < 1 \ : \ \ln x = -\int_x^1 \frac{1}{t}\,dt$$

$$F(x) = -\int_x^1 \frac{e^t}{t}\,dt$$

$$\text{for } 0 < t < 1 \ , \ e^t > 1 \quad \therefore \ \frac{e^t - 1}{t} > 0$$

$$\therefore \ F(x) - \ln x = -\int_x^1 \frac{e^t - 1}{t}\,dt < 0$$

$$\therefore \ F(x) < \ln x$$

$$\Rightarrow \ \ln x \leq F(x) \quad \text{for } x \geq 1$$

10) Evaluate: $\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n}\sqrt{n+i}} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{\sqrt{1+\frac{i}{n}}}$$

Consider a uniform partition of $[1,2]$

$\therefore \Delta x = \frac{1}{n}$ & RH end pt. $x_i^* = 1+\frac{i}{n}$

$\therefore$ for $f(x) = \frac{1}{\sqrt{x}} \Rightarrow f(x_i^*) = \frac{1}{\sqrt{1+\frac{i}{n}}}$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{\sqrt{1+\frac{i}{n}}} = \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x\cdot f(x_i^*) = \int_1^2 f(x)\,dx$$

$$=\int_1^2\frac{dx}{\sqrt{x}} = \left[2\sqrt{x}\right]_1^2 = 2\sqrt{2}-2$$