PHY293 Modern Physics
Solutions to Midterm Exam # 2

November 30, 2021

Question I The NCC Enterprise, the ship on Star Trek, has a mass of 4×10^9 kg. Its impulse drive allows the ship to travel up to $0.25c$, where c is the speed of light. (The ship travels at $0.999c$ at Warp 1 but lets forget about the these Warp factors).

(a) The Enterprise has to travel from Earth to Pluto. The orbital radii are $R_E = 1.5 \times 10^8$ km and $R_P = 4.5 \times 10^9$ km. Assume that the Enterprise can take the shortest distance, and the inertial dampers on the ship allow it to accelerate to cruise speed very quickly. How long does it take to reach Pluto on impulse drive from the perspective of the Enterprise crew? [5 points]

(b) How long is the journey from the perspective of Starfeet Command on Earth? [5 points]

(c) What would be the increase in the total energy of the Enterprise once it reaches a speed of $0.25c$? [5 points]

(d) If the Enterprise had a way of converting mass into its equivalent energy, how much mass would it take to accelerate the Enterprise to maximum speed using its impulse drive? [5 points]

Solution:

(a) At a speed $\beta = 0.25$, the gamma factor is $\gamma = 1/\sqrt{1-\beta^2} = 1.0328$ [1 point].
The distance from Earth to Pluto is Lorentz-contracted as seen by the Enterprise crew [3 points]. The time they would experience is [1 point]

$$\Delta t = \frac{R_P - R_E}{\gamma \beta c} = \frac{4.35 \times 10^9}{(1.0328)(0.25)(3 \times 10^5)} = 56,158 \text{ s} = 15.6 \text{ hours.} \tag{1}$$

(b) StarFleet would see the distance without Lorentz contraction [4 points].
So [1 point]

$$\Delta t' = \frac{R_P - R_E}{\beta c} = \frac{4.35 \times 10^9}{(0.25)(3 \times 10^5)} = 58,000 \text{ s} = 16.1 \text{ hours.} \tag{2}$$

(c) The total energy at cruise speed of the Enterprise is $\gamma m_{Ent} c^2$, where m_{Ent} is the mass of the Enterprise. So the increase is the total energy minus the energy associated with its rest mass [3 points]. So [2 points]

$$\Delta E = E_{tot} - m_{Ent} c^2 = (\gamma - 1) m_{Ent} c^2 = (0.0328)(4 \times 10^9)(3 \times 10^8)^2 = 1.18 \times 10^{25} \text{ J.} \tag{3}$$

(d) The mass conversion is just the increase in total energy divided by c^2 [3 points]. So [2 points]

$$\Delta m_{Ent} = \left(E_{tot} - m_{Ent} c^2\right)/c^2 = (\gamma - 1) m_{Ent} = (0.0328)(4 \times 10^9) = 1.31 \times 10^8 \text{ kg.} \tag{4}$$

So at least 3% of the Enterprise mass would have to be fuel...

Question II Consider a photon with energy = 30 keV. It strikes an electron and Compton scatters, creating a recoil electron and photon.

(a) What is the wavelength and angular frequency of the initial photon? [5 points]

(b) If the photon scatters through an angle $\theta = 45$ degrees, what is the change in frequency (the Compton shift) of the photon? What is the energy of the recoil photon? Sketch the scattering process in the x-y plane. [10 points]

(c) What is the four-momentum of the recoil electron in that case? [5 points] ?

Solution:

(a) The energy, wavelength and angular frequency are related by $E = \hbar\omega = hc/\lambda$ [3 points]. So [2 points]

$$\lambda = \frac{hc}{E} = \frac{(6.626 \times 10^{-34})(3 \times 10^{8})}{(3 \times 10^{4})(1.602 \times 10^{-19}} = 4.136 \times 10^{-11} \text{ m and} \tag{5}$$

$$\omega = \frac{E}{\hbar} = \frac{(3 \times 10^{4})(1.602 \times 10^{-19})}{(1.055 \times 10^{-34})} = 4.56 \times 10^{19} \text{ s}^{-1}. \tag{6}$$

(b) The shift in wavelength is given by [3 points]

$$\Delta\omega = \omega' - \omega = \frac{\omega}{1 + \frac{\hbar\omega}{m_e c^2}(1 - \cos\theta)} - \omega \tag{7}$$

$$= \omega\left(\frac{1}{1 + \frac{\hbar\omega}{m_e c^2}(1 - \cos\theta)} - 1\right) \tag{8}$$

$$= \omega\left(\frac{1}{1 + \frac{\hbar\omega}{m_e c^2}(1 - \cos\theta)} - 1\right). \tag{9}$$

We can now calculate [2 points]

$$\Delta\omega = (4.56 \times 10^{19})\left(\frac{1}{1 + \frac{(1.055\times10^{-34})(4.56\times10^{19})}{(9.109\times10^{-31})(3\times10^{8})^2})(1 - 1\sqrt{2})} - 1\right) \tag{10}$$

$$= (4.56 \times 10^{19})(-0.01690) = -7.7 \times 10^{17} \text{ s}^{-1}. \tag{11}$$

So, we can now calculate the energy of the recoil photon [1 point]:

$$E' = (3 \times 10^{4})(1 - 0.01690) = 29.5 \text{ keV}. \tag{12}$$

A sketch of the scattering process is shown in Figure 1 [2 points].

(c) The three-momentum of the recoil electron is [2 points]

$$\vec{p_e} = \vec{p_\gamma} - \vec{p_{\gamma'}} \tag{13}$$

where the two vectors on the R.H.S. are the 3-momenta of the incoming and outgoing photons. These two vectors are

$$\vec{p_\gamma} = \hbar\omega c\,(1, 0, 0) \tag{14}$$

$$\vec{p_{\gamma'}} = \hbar\omega' c\left(1\sqrt{2}, -1\sqrt{2}, 0\right), \tag{15}$$

where I've chosen the recoil photon to scatter in the $-\hat{y}$ direction. The difference of these two 3-vectors is

$$\begin{aligned}
\vec{p_e} &= c\left(\hbar\omega - \hbar\omega'/\sqrt{2}, \hbar\omega'/\sqrt{2}, 0\right) && (16)\\
|\vec{p_e}| &= c\hbar\sqrt{\left(\omega - \omega'/\sqrt{2}\right)^2 + \left(\omega'/\sqrt{2}\right)^2} = 22.8 \text{ keV/c.} && (17)
\end{aligned}$$

The energy of the electron is given by

$$E_e = \sqrt{m_e c^2 + (pc)^2} \approx m_e c^2, \tag{18}$$

since the energy associated with the electron mass is much greater than the kinetic energy of the electron. Hence

$$\begin{aligned}
\tilde{p_e} &= \left(m_e c, -\hbar\omega + \hbar\omega'/\sqrt{2}, -\hbar\omega'/\sqrt{2}, 0\right) && (19)\\
&= \left(511, -30 + 29.5/\sqrt{2}, -29.5/\sqrt{2}, 0\right) = (511, -9.14, -20.85, 0) \text{ keV/c.} && (20)
\end{aligned}$$

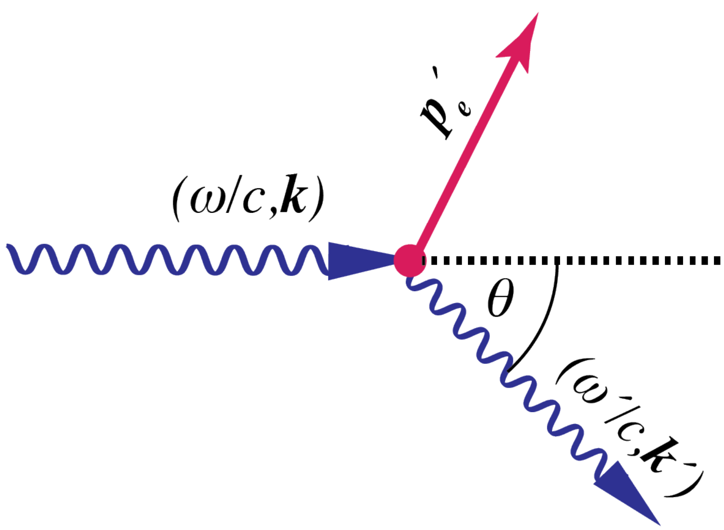

Figure 1: Sketch of Compton scattering.

Question III We are going to use slow neutron scattering to study the structure of a crystal, with the geometry shown in Figure 2.

N.B. This may require relativistic kinematics, at least in parts. If you choose to do a non-relativistic calculation, confirm after the fact that it is a reasonable assumption.

(a) The crystal has a lattice spacing of order $d = 5$ nm. If we want to resolve the atom locations to a precision of about 1 nm (meaning that we will need to use wavelengths of that size), what is the momentum of the slow neutrons in eV/c? What would be the velocity of the neutrons? [8 marks]

(b) In the diagram, under what conditions will the diffracted beam show constructive interference? What would be the minimum angle θ where we would see constructive interference? [8 marks]

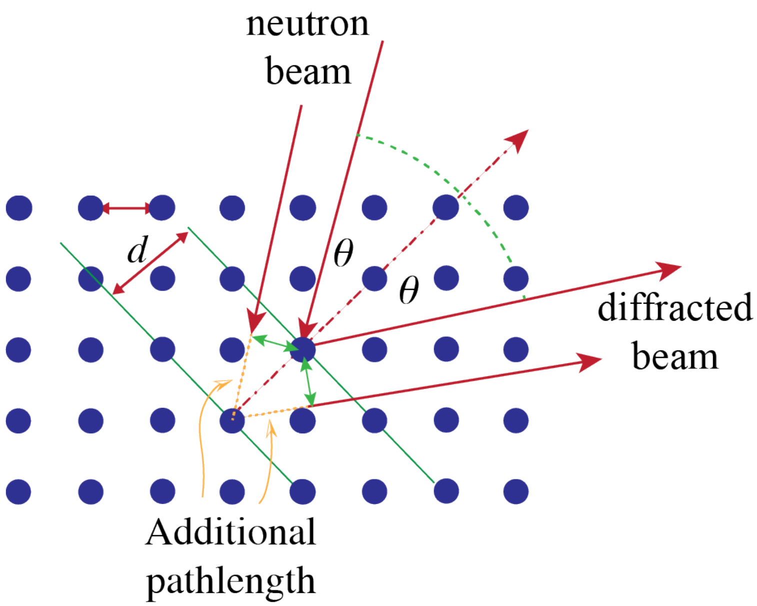

Figure 2: Crystal Bragg scattering.

(c) What would be the minimum angle if we doubled the velocity of the neutrons? [4 marks]

Solution:

(a) The de Broglie wavelength for the neutron is $\lambda = h/p$ [2 marks].
So, we can first calculate the momentum in kg·m/s and then convert: [4 marks]

$$p = \frac{h}{\lambda} = \frac{(6.626 \times 10^{-34})}{(1 \times 10^{-9})} = 6.63 \times 10^{-25} \text{ kg m/s} \quad (21)$$

$$= (6.63 \times 10^{-25})(5.344 \times 10^{-28}) = 1,240 \text{ eV/c}. \quad (22)$$

The velocity would then be $v = p/m_n = (6.63 \times 10^{-25})/(1.6749 \times 10^{-27}) = 396$ m/s [2 marks].

(b) The difference in path length would be $2d\cos\theta$ so constructive interference would occur when that difference is an integer number of neutron wavelengths [3 marks]. Hence [3 marks],

$$n\lambda = 2d\cos\theta_n \Rightarrow \theta_n = \cos^{-1}(n\lambda/(2d)), \quad (23)$$

where θ_n is the angle of the n^{th} interference peak. As n increases, the angle θ decreases. However the value of the $n\lambda$ cannot exceed $2d$, so $n_{max} = 2d/\lambda = 10$ [2 marks]. Since $2d$ is an integer number of wavelengths, the minimum angle is $\cos^{-1}(1) = 0$!

(c) If we doubled the velocity, the momentum would double and the de Broglie wavelength would halve. Since $2d$ would still be an integer number of wavelengths ($2d/\lambda = 20$), the minimum angle would still be 0 [4 marks].