Question 1

1. The path shown in figure 1 consists of four segments:
   - a semicircle γ_1 of radius a;
   - a straight segment γ_2;
   - a semicircle γ_3 of radius b;
   - a straight segment γ_4.

   Charge has been placed on the whole path with uniform density $\rho_l > 0$. We want to calculate the electric field $\mathbf{E}$ at the origin.

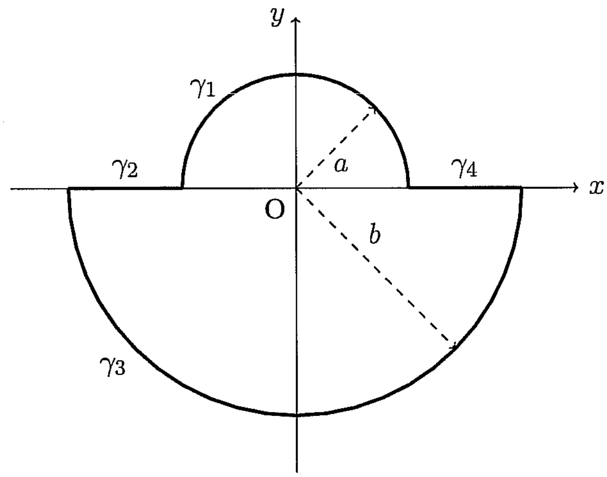
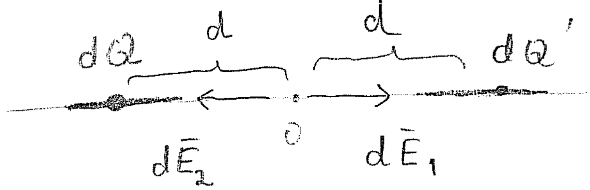


Figure 1

a) Let $\mathbf{E}_2$ be the contribution to $\mathbf{E}$ due to segment γ_2. Let $\mathbf{E}_4$ be the contribution to $\mathbf{E}$ due to segment γ_4. Show that $\mathbf{E}_2 + \mathbf{E}_4$ is zero **(4 pts)**.

Solution 1: By symmetry, any dQ on γ_2 has a dQ' on γ_4 that contributes to O the exactly opposite $d\bar{E}$



$$|d\bar{E}_1| = \frac{dQ}{4\pi\epsilon_0 |\bar{R}-\bar{R}'|} = \frac{dQ}{4\pi\epsilon_0 d} = |d\bar{E}_2| \Rightarrow \bar{E}_2 + \bar{E}_4 = 0.$$

Solution 2:

- Use cartesian coordinates.

- $dQ = \rho_e dx'$, $\bar{R} = 0$, $\bar{R}' = x\bar{a}_x$, $|\bar{R} - \bar{R}'| = |x'| = \begin{cases} -x' & \text{for } \gamma_2 \\ x' & \text{for } \gamma_4 \end{cases}$

$\bar{R} - \bar{R}' = -x\bar{a}_x$ [3pts]

$$\Rightarrow \quad d\bar{E}_{\gamma_2} = \frac{\rho_e dx'}{4\pi\epsilon_0(-x')}(-x'\bar{a}_x) = \frac{\rho_e dx'}{4\pi\epsilon_0}\bar{a}_x \Rightarrow$$

$$\bar{E}_{\gamma_2} = \frac{\rho_e}{4\pi\epsilon_0}(\text{length } \gamma_2)\,\bar{a}_x$$

Similarly: $$\bar{E}_{\gamma_4} = -\int_{\gamma_4} \frac{\rho_e dx'}{4\pi\epsilon_0}\bar{a}_x$$

$$= -\frac{\rho_e}{4\pi\epsilon_0}(\text{length } \gamma_4)\,\bar{a}_x$$

Lengths are equal $\Rightarrow$ $\bar{E}_{\gamma_2} + \bar{E}_{\gamma_4} = 0$.

b) Using superposition, calculate the contribution $\mathbf{E}_1$ due to the semicircle γ_1. **(7 pts)**

Choose cylindrical system [0.5 pt]

$dQ' = \rho_e \alpha \, d\varphi'$ (1 pt)

$\bar{R} = 0$ (0.5 pt), $\bar{R}' = \alpha \bar{a}_{r'} = \alpha(\bar{a}_x \cos\varphi' + \bar{a}_y \sin\varphi')$ (1 pt)

$|\bar{R} - \bar{R}'| = \alpha$ (0.5 pt)

$$d\bar{E}_1 = \frac{\rho_e \alpha \, d\varphi'}{4\pi \epsilon_0 \alpha^3}(-\alpha \bar{a}_{r'}) \Rightarrow \bar{E} = -\frac{\rho_e}{4\pi\epsilon_0 \alpha} \underbrace{\int_0^{\pi} (\bar{a}_x \cos\varphi' + \bar{a}_y \sin\varphi') \, d\varphi'}_{\text{integral w.r.t. } \varphi \text{, (1) correct limits (0.5)}}$$

$$\left.\begin{aligned} &\int_0^{\pi} \cos\varphi' \, d\varphi' = 0, \quad \int_0^{\pi} \sin\varphi' \, d\varphi' = -\cos\pi + \cos 0 = 2 \\ &\Rightarrow \bar{E}_{\gamma_1} = -\frac{2\rho_e}{4\pi\epsilon_0 \alpha} \cdot \bar{a}_y = -\frac{\rho_e}{2\pi\epsilon_0 \alpha} \bar{a}_y \end{aligned}\right\} [2 \text{ pts}]$$

c) Find the contribution $\mathbf{E}_3$ due to semicircle γ_3. **(7 pts)**

Same steps as (c) $\Rightarrow$

$$\bar{E}_{\gamma_3} = + \frac{\rho_e}{2\pi\epsilon_0 b} \bar{a}_y$$

d) Find the total field **E** at the origin. **(4 pts)**

Superposition:

$$\bar{E} = \bar{E}_{\gamma_1} + \bar{E}_{\gamma_2} + \bar{E}_{\gamma_3} + \bar{E}_{\gamma_4}$$

$$= \bar{E}_{\gamma_1} + \bar{E}_{\gamma_3}$$

$$= \bar{a}_y \frac{\rho_\ell}{2\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)$$

2. Consider an infinitely-long cylinder parallel to the z-axis. The cylinder cross-section is shown in figure 2. On the cylinder, there is a surface charge density $\rho_s(\phi) = 2\cos(\phi)$ nC/m^2. A point charge $q = 1$ nC is placed at the origin.

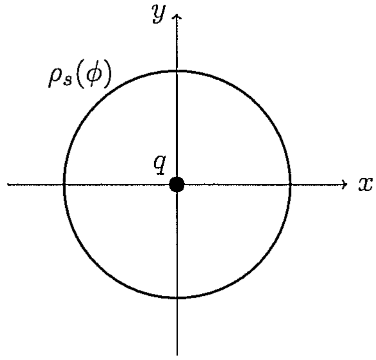


Figure 2

The charge will:

a) move in the $\mathbf{a}_x$ direction

b) move in the $-\mathbf{a}_x$ direction   1 pt

c) move in the $\mathbf{a}_y$ direction

d) move in the $-\mathbf{a}_y$ direction

e) will remain stationary

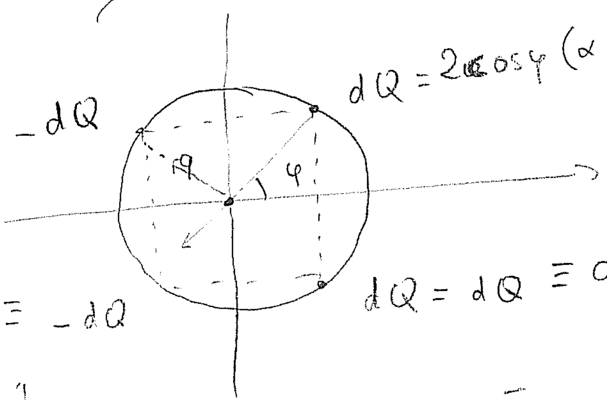
Justify your answer. **(3 pts)**

symmetry in charges (1pt)

By symmetry:

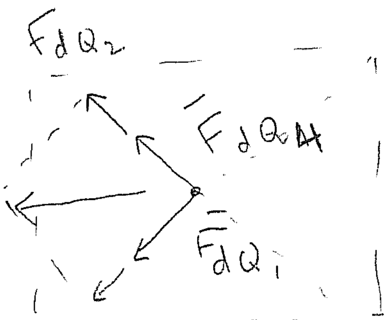
$dQ_2 \equiv -dQ$

$dQ = 2\cos\varphi\,(\alpha\, d\varphi') \equiv dQ_1$



$dQ_3 \equiv -dQ$

$dQ = dQ \equiv dQ_4$



Total force in $-\bar{a}_x$ direction by superposition (1pt)

Question 2

1. A total amount of charge $Q = -1.6 \times 10^{-19}$ C is distributed within a sphere of radius $\alpha = 10^{-10}$ m, with a volume charge density $\rho_v = kR$, where k is a constant.

   a) Show that $k = \frac{Q}{\pi\alpha^4}$. You may use the integrals: $\int R^p dR = R^{p+1}/(p+1)$ and $\int \sin\theta d\theta = -\cos\theta$. **(2 pts)**

   $$Q = \underbrace{\int kR \cdot \underbrace{R^2 \sin\theta \, d\theta \, d\varphi}_{1}}_{1} = k \int_0^{\alpha} R^3 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi$$

   $$= \frac{k\alpha^4}{4} \cdot 4\pi = \pi \cdot k\alpha^4$$

   $$\Rightarrow k = Q/\pi\alpha^4$$

   b) Determine the electric field **E** inside and outside the sphere. Based on that, find the electric potential difference between any point on the surface of the sphere and the center of the sphere. **(10 pts)**

   Gauss Law applied to spheres with radii $\underbrace{R < \alpha \text{ (for inside)}}_{(1\text{ pt})}$, $\underbrace{R > \alpha \text{ (for outside)}}_{(2\text{ pts})}$

   Spherical symmetry $\Rightarrow \bar{E} = E_R(R)\, \bar{a}_R$ (1pt)

   Inside: $E_R \cdot 4\pi R^2 = \frac{Q_{enclosed}}{\epsilon_0}$ with

   $$Q_{enclosed} = \int_0^R k \cdot R' \cdot (R')^2 \sin\theta \, d\theta \, d\varphi \, dR' = \pi \cdot k \cdot R^4 \quad (2\text{pts})$$

   Outside: $Q_{enclosed} = Q$ (1pt)

   $$\Rightarrow E_R = \begin{cases} \frac{\pi k R^4}{4\pi\epsilon_0 R^2} = \frac{k \cdot R^2}{4\epsilon_0} & \text{(inside)} \\ \frac{Q}{4\pi\epsilon_0 R^2} & \text{outside} \end{cases}$$

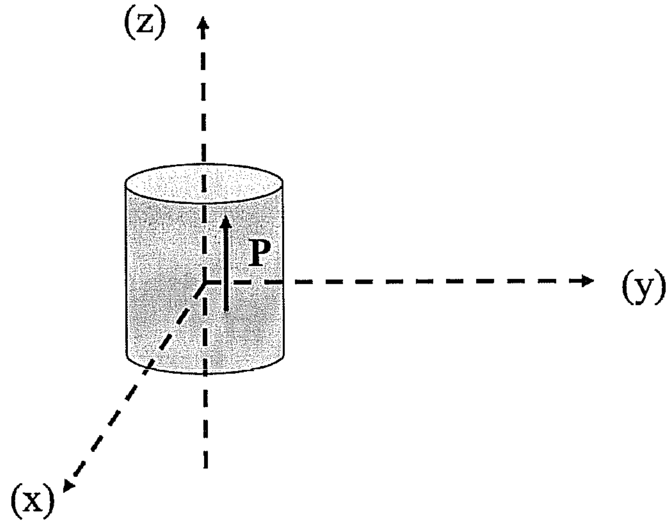
$$\underbrace{V(R=\alpha) - V(R=0)}_{\text{Def. (1pt)}} = -\int_0^{\alpha} \underbrace{\frac{k \cdot R^2}{4\epsilon_0}}_{\text{inside field} \Rightarrow \text{(1pt)}} \cdot dR \text{ (1pt)} = -\frac{k\alpha^3}{12\epsilon_0}$$

c) What is the electric potential at any point on the surface of the sphere with respect to infinity? **(3 pts)**

Analogy w. point charge: (2pts)

$$V = \frac{Q}{4\pi\epsilon_0 \alpha} \quad \text{(1pt)}$$

2. A cylinder of cross-section A and length d is made of a medium with permanent polarization $\mathbf{P} = P_0\mathbf{a}_z$, as shown in the figure. Can this cylinder be modeled as an electric dipole ? If yes, what is its equivalent electric dipole moment $\mathbf{p}$? If not, why? **(5 pts)**



Yes, because there are surface polarization charges (2pts).

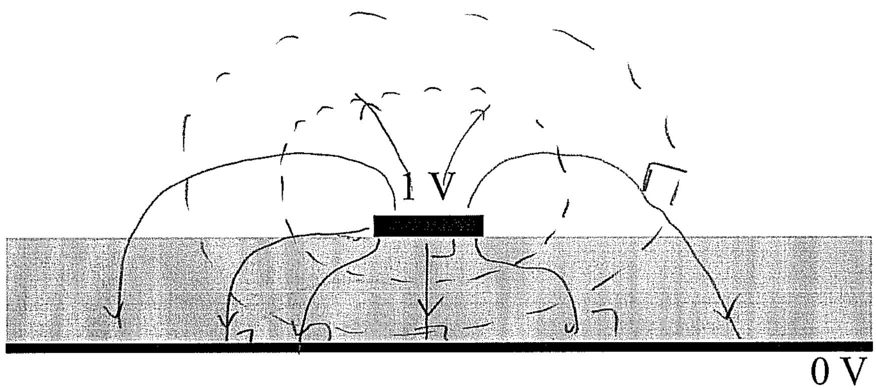
On top: $\rho_{s,p} = P_0$ (1pt)

On bottom: $\rho_{s,p} = -P_0$ (1pt)

$p = Q\bar{d} = (P_0 \cdot A)\, d\, \bar{a}_z$ (1pt)

3. A very common printed circuit element (used for digital interconnects) is the microstrip line, consisting of a thin conductor above a ground plane, with a dielectric substrate in between. The cross-section of this line is shown in the figure.

   a) Assuming that the microstrip is at an electric potential 1 V with respect to the ground, sketch the electric field lines, clearly indicating their direction, and the equipotential lines. **(3 pts)**



Field lines : Normal to conductor 1pt

Going from high to low potential 1pt

Equi-potentials $\perp$ field 1pt

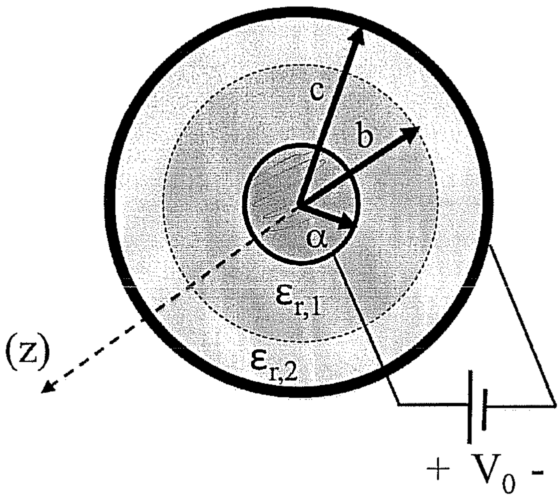
   b) Do you expect the electric field magnitude to be stronger near the center or near the edges of the microstrip? **(2 pts)**

$\underbrace{|\bar{E}| \sim \rho_s}_{1\,pt} \Rightarrow \underbrace{\rho_s \text{ higher near edges}}_{1\,pt} \Rightarrow$

field stronger near edges.

Question 3

1. A cylindrical capacitor consists of an inner and an outer perfect conductor of radius α and c, respectively. In between the conductors, there are two cylindrical dielectric layers with relative dielectric permittivities $\epsilon_{r,1}$ and $\epsilon_{r,2}$, respectively, as shown in the figure. The voltage between the two conductors is V_0.



a) Show that the electric flux density for $\alpha < r < c$ is given as:

$$\mathbf{D} = \frac{\rho_s \alpha}{r}\mathbf{a}_r,$$

where $\rho_s = \dfrac{\epsilon_0 V_0}{\alpha\left(\dfrac{1}{\epsilon_{r,1}}\ln\dfrac{b}{\alpha} + \dfrac{1}{\epsilon_{r,2}}\ln\dfrac{c}{b}\right)}$. **(10 pts)**

Note: you can take this expression for granted for the next questions.

Cylindrical symmetry $\Rightarrow$ $\bar{D} = D_r(r)\,\bar{a}_r$ (1pt)

From boundary conditions:

$D_{r_1} = D_{r_2}$ in regions 1, 2

$\equiv D_r$ (2pts)

If Q is the charge on inner conductor: $D_r \cdot 2\pi r L = Q$

$\Rightarrow D_r = \dfrac{Q}{2\pi r L}$ from Gauss Law on cylinder of radius R, length L. (2pts)

$$\underbrace{E_{r_1} = D_r / \epsilon_0 \epsilon_{r_1} \qquad E_{r_2} = D_r / \epsilon_0 \epsilon_{r_2}}_{2\text{pts}}$$

$$V_0 = V(r=a) - V(r=c) \quad \} \; 1\text{pt}$$

$$= -\int_c^a \bar{E} \cdot d\bar{l} = \int_a^c \bar{E} \cdot d\bar{l}$$

$$= \int_a^b \frac{D_r}{\epsilon_0 \epsilon_{r_1}} dr + \int_b^c \frac{D_r}{\epsilon_0 \epsilon_{r_2}} dr$$

$$= \frac{Q}{2\pi L \epsilon_0} \left\{ \frac{1}{\epsilon_{r_1}} \int_a^b \frac{dr}{r} + \frac{1}{\epsilon_{r_2}} \int_b^c \frac{dr}{r} \right\}$$

$$= \frac{Q}{2\pi L \epsilon_0} \left\{ \frac{1}{\epsilon_{r_1}} \ln \frac{b}{a} + \frac{1}{\epsilon_{r_2}} \ln \frac{c}{b} \right\} \Rightarrow$$

$$\frac{Q}{2\pi L} = \frac{\epsilon_0 V_0}{\frac{1}{\epsilon_{r_1}} \ln \frac{b}{a} + \frac{1}{\epsilon_{r_2}} \ln \frac{b}{a}} \Rightarrow$$

2pts

$$D = \underbrace{\frac{\epsilon_0 V_0}{\left(\frac{1}{\epsilon_{r_1}} \ln \frac{b}{a} + \frac{1}{\epsilon_{r_2}} \ln \frac{b}{a} \right) a}}_{\rho_s} \cdot \frac{a}{r} \bar{a}_r$$

b) Show that the expression for **D** satisfies the boundary conditions at $r = \alpha$ and $r = b$. **(5 pts)**

Recognizing ρ_s as surface charge density, at $r=\alpha$: $\bar{a}_r \cdot \bar{D}(r=\alpha) = \rho_s$ } 3pts

$$\Rightarrow \frac{\rho_s \alpha}{\alpha} = \rho_s \checkmark$$

At $r=b$: $\underbrace{\bar{a}_r \cdot \left(\bar{D}(r=b^+) - \bar{D}(r=b^-)\right)}_{2\text{pts}} = 0$

↑ perfect dielectrics (1/2 pts)

c) Find the capacitance per unit length of this capacitor and the total energy that is stored in the capacitor per unit length. Is this a case of a "series" or a "parallel" connection? **(10 pts)**

Solution 1: $C = \frac{Q}{V_0}$, $Q = 2\pi\alpha \cdot L \cdot \rho_s$ } $\underline{2\text{pts}}$

$$= 2\pi\alpha \cdot L \cdot \frac{\epsilon_0 V_0}{\alpha\left(\frac{1}{\epsilon_{r_1}}\ln\frac{b}{\alpha} + \frac{1}{\epsilon_{r_2}}\ln\frac{c}{b}\right)} \Rightarrow \frac{Q}{V_0 \cdot L} = \overbrace{C_{\text{per unit length}}}^{\underline{2\text{pts}}\ (Q\text{ on cylinder})}$$

result $\underline{2\text{pts}}$

$$= \frac{2\pi\epsilon_0}{\frac{1}{\epsilon_{r_1}}\ln\frac{b}{\alpha} + \frac{1}{\epsilon_{r_2}}\ln\frac{c}{b}}$$

, $W = \frac{1}{2} C V_0^2$ ($\underline{2\text{pts}}$)

"SERIES" (voltage is not common at 2 sections)
($\underline{2\text{pts}}$)

Solution 2: $w_e = \frac{1}{2}\epsilon E^2$ (2pts) $W_e = \int w_e \, dv$ (2pts)

result 2pts $C = \frac{2W_e}{V_0^2}$ (2pts) Series (2pts)