AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 2

21 October 2013 9:15 am - 10:15 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis

Given Name: Solutions.

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	7	
3	10	
4	8	
5	8	
6	12	
TOTAL	55	/ 50

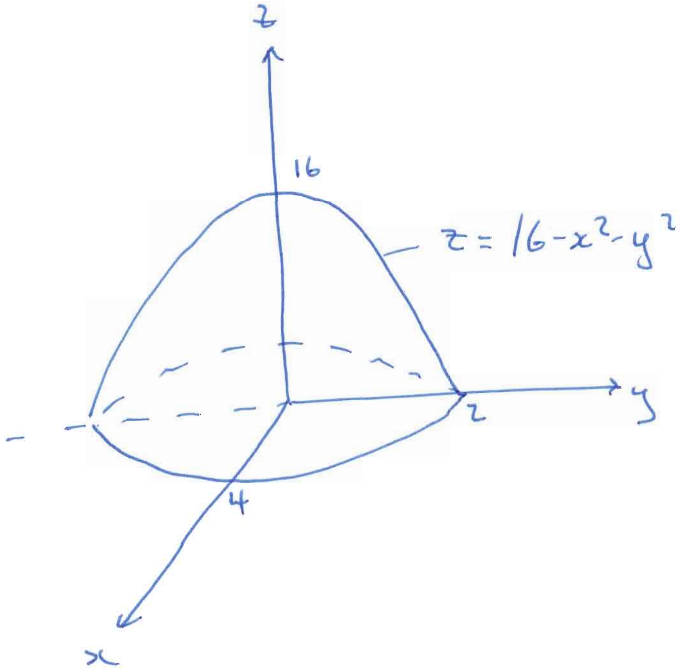
Note: The following integrals may be useful.

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \qquad \oiint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV; \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

1) Use the coordinate transformation: $x = 4u\cos v$, $y = 2u\sin v$, $z = w$, to evaluate the triple integral $I = \int_V z\,dV$, where V is the volume bounded by the paraboloid: $z = 16 - x^2 - 4y^2$, and the x-y plane. Provide a sketch of the volume.

(10 marks)



let $x = 4u\cos v$ $\quad 0 \le u \le 1$

$y = 2u\sin v$ $\quad 0 \le v \le 2\pi$

$z = w$

$0 \le w \le 16 - x^2 - y^2 = 16 - 16u^2$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 4\cos v & 2\sin v & 0 \\ -4u\sin v & 2u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 8u\cos^2 v + 8u\sin^2 v + 0 = 8u$$

$$I = \int_0^{2\pi} dv \int_0^1 du \int_0^{16-16u^2} dw \cdot w \cdot |8u| = 2\pi \int_0^1 8u\,du \left[\frac{w^2}{2}\right]_0^{16-16u^2}$$

$$= 8\pi \int_0^1 256\left(u - 2u^3 + u^5\right) du = 2048\pi \left[\frac{u^2}{2} - \frac{u^4}{2} + \frac{u^6}{6}\right]_0^1$$

$$= 2048\pi\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1024\pi}{3}$$

2) a) Find the work done by the force $\vec{F}(x,y,z) = x^2\,\hat{i} + xy\,\hat{j} + z^2\,\hat{k}$ applied to an object that moves along the circular helix $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k},\ 0 \le t \le 2\pi$.

(3 marks)

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\cos^2 t,\ \cos t\sin t,\ t^2)\cdot(-\sin t,\ \cos t,\ 1)\,dt$$

$$= \int_0^{2\pi} (-\sin t\cos^2 t + \sin t\cos^2 t + t^2)\,dt$$

$$= \int_0^{2\pi} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{8\pi^3}{3}$$

b) Let $\vec{F} = \nabla f$, where $f(x,y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equations:

i) $\int_{C_1} \vec{F}\cdot d\vec{r} = 0$ ii) $\int_{C_2} \vec{F}\cdot d\vec{r} = 1$

(4 marks)

Given $\vec{F} = \nabla f \longrightarrow \int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$

i) $f(0,0) = \sin(0) = 0$
$f(2,1) = \sin(0) = 0$

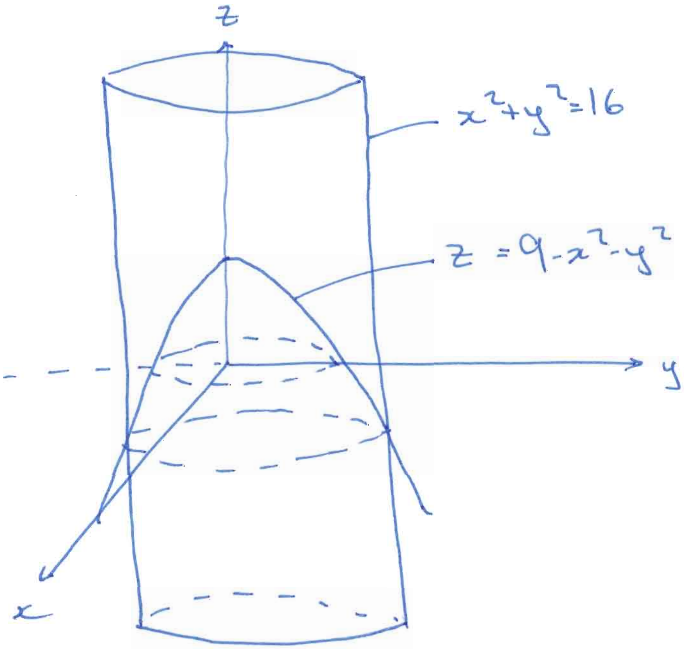
$\therefore$ a straight line or any curve connecting $(0,0)$ to $(2,1)$ would work:
$\Rightarrow \vec{r}_1(t) = 2t\,\hat{i} + t\,\hat{j}\quad 0 \le t \le 1$

other possibilities: $(0,0)$ to (π,π)
or any line with $x - 2y = $ constant

ii) $f(\frac{\pi}{2}, 0) = \sin(\frac{\pi}{2}) = 1$ $\therefore$ a straight line or any curve connecting $(0,0)$ to $(\frac{\pi}{2}, 0)$ would work:
$\Rightarrow \vec{r}_2(t) = \frac{\pi}{2}t\,\hat{i}\quad 0 \le t \le 1$

3) Find a parametric representation of the surface, and use this to find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies inside the cylinder $x^2 + y^2 = 16$. Provide a sketch of the surface.

(10 marks)



parameterize surface:

$x = r\cos\theta \quad y = r\sin\theta \quad z = 9 - r^2$

$0 \le r \le 4 \quad 0 \le \theta \le 2\pi$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (2r^2\cos\theta,\ 2r^2\sin\theta,\ r\cos^2\theta + r\sin^2\theta)$$

$$= (2r^2\cos\theta,\ 2r^2\sin\theta,\ r)$$

$$S = \int_S \|\vec{r}_r \times \vec{r}_\theta\|\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 dr \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \int_0^{2\pi} d\theta \int_0^4 dr \sqrt{4r^4 + r^2} = 2\pi \int_0^4 r\sqrt{4r^2+1}\, dr$$

$$= 2\pi \left[(4r^2+1)^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{8} \right]_0^4 = \frac{\pi}{6}\left(65^{3/2} - 1\right)$$

4) Let S be the surface given in cylindrical coordinates by $z = f(r, \theta)$, where $(r, \theta) \in \Omega$. Show that if f is continuously differentiable then the surface area of S is given by:

$$S = \iint_{\Omega} \sqrt{r^2\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 + r^2}\, dr d\theta$$

(8 marks)

parameterize the surface: $x = r\cos\theta \quad y = r\sin\theta \quad z = f(r,\theta)$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & f_r \\ -r\sin\theta & r\cos\theta & f_\theta \end{vmatrix} = \begin{array}{l} (\sin\theta f_\theta - r\cos\theta f_r)\hat{i} \\ + (-r\sin\theta f_r - \cos\theta f_\theta)\hat{j} \\ + (r\cos^2\theta + r\sin^2\theta)\hat{k} \end{array}$$

$$= (f_\theta \sin\theta - r f_r \cos\theta,\ -r f_r \sin\theta - f_\theta \cos\theta,\ r)$$

$$S = \int_S \| \vec{r}_r \times \vec{r}_\theta | \, dr d\theta$$

$$= \int_\Omega \sqrt{(f_\theta \sin\theta - r f_r \cos\theta)^2 + (r f_r \sin\theta + f_\theta \cos\theta)^2 + r^2}\, dr d\theta$$

$$= \int_\Omega \sqrt{f_\theta^2 \sin^2\theta - \cancel{2r f_\theta f_r \sin\theta\cos\theta} + r^2 f_r^2 \cos^2\theta + r^2 f_r^2 \sin^2\theta + \cancel{2r f_\theta f_r \sin\theta \cos\theta} + f_\theta^2 \cos^2\theta + r^2}\, dr d\theta$$

$$= \int_\Omega \sqrt{f_\theta^2 + r^2 f_r^2 + r^2}\, dr d\theta$$

5) Calculate the flux of the vector field $\vec{F} = e^{-y}\,\hat{i} + 2z\,\hat{j} + xy\,\hat{k}$ across the curved sides of the surface: $z = \cos y,\ 0 \le x \le 4,\ 0 \le y \le \pi/2$, where the normal vectors point upward.

(8 marks)

let $x = u \qquad y = v \qquad z = \cos v$

$0 \le u \le 4 \qquad 0 \le v \le \pi/2$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -\sin v \end{vmatrix} = (0, \sin v, 1) \quad \text{(direction OK)}$$

$$\text{Flux} = \int_S \vec{F} \cdot d\vec{S} = \int_S \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, du\, dv$$

$$= \int_0^4 du \int_0^{\pi/2} dv \left(e^{-v}, 2\cos v, uv\right) \cdot (0, \sin v, 1)$$

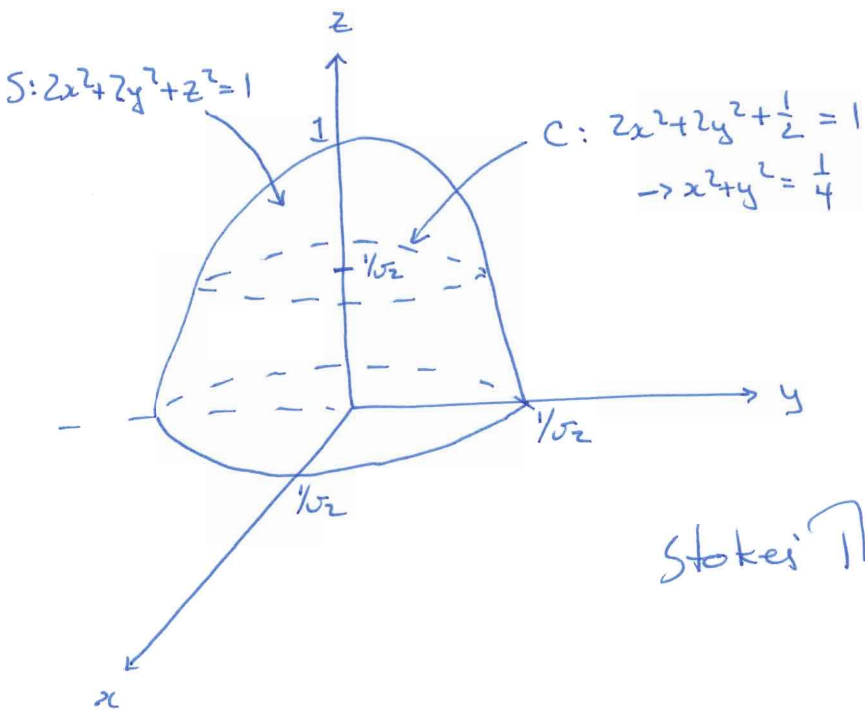
$$= \int_0^4 du \int_0^{\pi/2} (2 \sin v \cos v + uv)\, dv$$

$$= \int_0^4 du \left[\sin^2 v + \frac{uv^2}{2}\right]_0^{\pi/2}$$

$$= \int_0^4 \left(1 + \frac{\pi^2}{8} u\right) du = \left[u + \frac{\pi^2}{16} u^2\right]_0^4 = 4 + \pi^2$$

6) Verify Stokes' Theorem for $\vec{F} = -3y\hat{i} + 3x\hat{j} + z^4\hat{k}$ taking S as the portion of the ellipsoid $2x^2 + 2y^2 + z^2 = 1$ that lies above the plane $z = 1/\sqrt{2}$. Provide a sketch of the region.

(12 marks)



Stokes' Theorem: $\oint_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\cdot d\vec{S}$

LHS: parameterize curve: $\vec{r}(t) = \frac{1}{2}\cos t\,\hat{i} + \frac{1}{2}\sin t\,\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$

$0 \le t \le 2\pi$

$\Rightarrow \vec{r}'(t) = -\frac{1}{2}\sin t\,\hat{i} + \frac{1}{2}\cos t\,\hat{j} + 0\,\hat{k}$

$\Rightarrow \vec{F}(\vec{r}(t)) = -\frac{3}{2}\sin t\,\hat{i} + \frac{3}{2}\cos t\,\hat{j} + \frac{1}{4}\hat{k}$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(-\frac{3}{2}\sin t, \frac{3}{2}\cos t, \frac{1}{4}\right)\cdot\left(-\frac{1}{2}\sin t, \frac{1}{2}\cos t, 0\right) dt$$

$$= \int_0^{2\pi} \left(\frac{3}{4}\sin^2 t + \frac{3}{4}\cos^2 t\right) dt = \frac{3}{4}\int_0^{2\pi} dt = \frac{3\pi}{2}$$

RHS: $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3y & 3x & z^4 \end{vmatrix} = (0-0,\ 0-0,\ 3+3) = (0,0,6)$

parameterize surface: $x = r\cos\theta \qquad 0 \le r \le \frac{1}{2}$

$y = r\sin\theta \qquad 0 \le \theta \le 2\pi$

$z = \sqrt{1-2x^2-2y^2} = \sqrt{1-2r^2}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \frac{1}{2}(1-2r^2)^{-1/2}(-4r) \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= 2r^2\cos\theta\,(1-2r^2)^{-1/2}\,\hat{i} + 2r^2\sin\theta\,(1-2r^2)^{-1/2}\,\hat{j} + r\,\hat{k}$$

$$\therefore \int_S \nabla \times \vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta)\, dS = \int_0^{1/2} dr \int_0^{2\pi} d\theta \cdot 6r$$

$$= 2\pi \left[3r^2\right]_0^{1/2} = \frac{3}{2}\pi$$

Alternate parameterization:

$$\vec{r}(u,v) = \frac{1}{\sqrt{2}}\cos u \sin v\,\hat{i} + \frac{1}{2}\sin u \sin v\,\hat{j} + \cos v\,\hat{j}$$

$$0 \le u \le 2\pi \qquad 0 \le v \le \pi/4$$