UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test II

First Year — Program 5

MAT185H1S — Linear Algebra

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Student Name:	Fair Copy
	Last Name / First Names

Student Number:		Tutorial Section:	TUT

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 7 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). State the *Fundamental Theorem of Linear Algebra*.

Let $\mathcal{V}$ be a vector space spanned by n vectors. If a set of m vectors from $\mathcal{V}$ is linearly independent, then $m \leq n$.

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1(b). A *basis* for a vector space $\mathcal{V}$ is defined as

a set of vectors $E = \{\mathbf{e}_1, \mathbf{e}_2 \ldots \mathbf{e}_n\} \in \mathcal{V}$ if and only if (i) E is linearly independent and (ii) E spans $\mathcal{V}$.

/2

1(c). The *dimension* of a vector space $\mathcal{V}$ is defined as

the number of vectors in any of its bases.

/2

1(d). The *column space* of $\mathbf{A} \in {}^m\mathbb{R}^n$ is defined as

$$\operatorname{col} \mathbf{A} = \left\{ \mathbf{y} \,\middle|\, \mathbf{y} = \sum_{i=1}^{n} x_i \mathbf{c}_i, \forall x_i \in \mathbb{R} \right\}$$

where $\mathbf{A} = [\mathbf{c}_1 \; \mathbf{c}_2 \; \cdots \; \mathbf{c}_n]$, $\mathbf{c}_i \in {}^n\mathbb{R}$.

/2

1(e). Give two equivalent statements to "$\mathbf{A} \in {}^n\mathbb{R}^n$ is not invertible."

[Any two statements in Chapter 7, Theorem III, other than statement 1.]

/2

B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). Let p, q be polynomials. Then $\{p, pq\}$ is linearly independent if and only $\deg q \geq 1$.

F

2(b). If $1 < k < n$ and $\{\mathbf{x}_1, \mathbf{x}_2 \ldots \mathbf{x}_k\} \subset {}^n\mathbb{R}$ is linearly independent and $\mathbf{A} \in {}^n\mathbb{R}^n$ not invertible then $\{\mathbf{A}\mathbf{x}_1, \mathbf{A}\mathbf{x}_2 \ldots \mathbf{A}\mathbf{x}_k\}$ is linearly dependent.

F

2(c). If, for a given $\mathbf{A} \in {}^2\mathbb{R}^2, \mathcal{U} = \{\mathbf{X} \in {}^2\mathbb{R}^2 \mid \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{X}\}$ then $\dim \mathcal{U} \geq 2$.

T

2(d). If $\mathbf{A} \in {}^m\mathbb{R}^n$ then $\dim \operatorname{null} \mathbf{A}^T = m - \operatorname{rank} \mathbf{A}$.

T

2(e). If $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\dim \operatorname{null} \mathbf{A} \leq m$ then $m \geq \frac{1}{2}n$.

T

3. Let $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\mathbf{V} \in {}^n\mathbb{R}^n$. Show that $\operatorname{col}\mathbf{AV} \subseteq \operatorname{col}\mathbf{A}$ with the equality holding if $\mathbf{V}$ is invertible.

To show that $\operatorname{col}\mathbf{AV} \subseteq \operatorname{col}\mathbf{A}$, *let* $\mathbf{V}$ *have columns* $\mathbf{v}_k$ *and* $\mathbf{A}$ *columns* $\mathbf{c}_k$, $k = 1 \cdots n$. *By block multiplication,*

$$\mathbf{AV} = \mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{Av}_1 & \cdots & \mathbf{Av}_n \end{bmatrix}$$

and so the k*th column of* $\mathbf{AV}$ *is*

$$\mathbf{c}'_k = \mathbf{Av}_k = \mathbf{A}\begin{bmatrix} v_{1k} \\ v_{2k} \\ \vdots \\ v_{nk} \end{bmatrix} = \sum_{j=1}^{n} v_{jk}\mathbf{c}_j$$

which is a linear combination of the columns of $\mathbf{A}$. *It follows that all the columns of* $\mathbf{AV}$ *are in* $\operatorname{col}\mathbf{A}$. *Thus*

$$\operatorname{col}\mathbf{AV} = \operatorname{span}\{\mathbf{v}'_1, \mathbf{v}'_2 \cdots \mathbf{v}'_n\} \subseteq \operatorname{col}\mathbf{A}$$

If $\mathbf{V}$ *is invertible, then by the result we have just proven*

$$\operatorname{col}\mathbf{A} = \operatorname{col}\mathbf{AVV}^{-1} \subseteq \operatorname{col}\mathbf{AV}$$

and so $\operatorname{col}\mathbf{AV} = \operatorname{col}\mathbf{A}$.

/10

4. Let

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \\ 3 & 0 & 0 & 4 & 5 \\ 4 & -1 & 1 & 3 & 6 \end{bmatrix}$$

(a) Find a basis for col $\mathbf{A}$ from among the columns of $\mathbf{A}$. What is rank $\mathbf{A}$?

(b) Find a basis for null $\mathbf{A}$. What is dim null $\mathbf{A}$?

4(a). Find a basis for col $\mathbf{A}$ from among the columns of $\mathbf{A}$. What is rank $\mathbf{A}$?

Row-reduce $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \\ 3 & 0 & 0 & 4 & 5 \\ 4 & -1 & 1 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 3 & -3 & 7 & 2 \\ 0 & 3 & -3 & 7 & 2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & \frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & -1 & \frac{7}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{\mathbf{A}}$$

The first two columns are the only ones with leading "1"s; accordingly a basis for col $\mathbf{A}$ *is the first two columns of* $\mathbf{A}$.

Also

$$\operatorname{rank} \mathbf{A} = 2$$

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4(b). Find a basis for null $\mathbf{A}$. What is $\dim \operatorname{null} \mathbf{A}$?

Solving

$$\mathbf{Ax} = \mathbf{0} \quad \Rightarrow \quad \widetilde{\mathbf{A}}\mathbf{x} = \mathbf{0}$$

where $\widetilde{\mathbf{A}}$ *was computed above. The free variables are* x_3, x_4, x_5 *and*

$$x_1 = -\frac{4}{3}x_4 - \frac{5}{3}x_5$$
$$x_2 = x_3 - \frac{7}{3}x_4 - \frac{2}{3}x_5$$

or

$$\mathbf{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{4}{3} \\ -\frac{7}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence a basis for null $\mathbf{A}$ *is*

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{3} \\ -\frac{7}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ -\frac{2}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

As expected,

$$\dim \operatorname{null} \mathbf{A} = n - \operatorname{rank} \mathbf{A} = 5 - 2 = 3$$

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5. Let $\mathbf{A} \in {}^m\mathbb{R}^n$. Show that the following statements are equivalent:

1. $\operatorname{null} \mathbf{A} = {}^n\mathbb{R}$
2. $\operatorname{col} \mathbf{A} = \{\mathbf{0}\}$
3. $\mathbf{A} = \mathbf{O}$

[1 $\Rightarrow$ 2] If $\operatorname{null} \mathbf{A} = {}^n\mathbb{R}$, *then* $\dim \operatorname{null} \mathbf{A} = n$. *By the Dimension Formula,* $\operatorname{rank} \mathbf{A} = n - \dim \operatorname{null} \mathbf{A} = 0$, *which means that* $\dim \operatorname{col} \mathbf{A} = 0$. *Thus* $\operatorname{col} \mathbf{A} = \{\mathbf{0}\}$.

[2 $\Rightarrow$ 3] If $\operatorname{col} \mathbf{A} = \{\mathbf{0}\}$, *then* $\mathbf{A}$ *cannot have a nonzero column; otherwise, its column space would contain a nonzero column. Hence* $\mathbf{A} = \mathbf{O}$.

[3 $\Rightarrow$ 1] If $\mathbf{A} = \mathbf{O}$, *then* $\operatorname{rank} \mathbf{A} = 0$. *By the Dimension Formula,* $\dim \operatorname{null} \mathbf{A} = n - \operatorname{rank} \mathbf{A} = n$. *Now* $\operatorname{null} \mathbf{A} \subseteq {}^n\mathbb{R}$ *and* $\dim {}^n\mathbb{R} = n$. *Thus, by Chapter 6, Theorem VI (not necessary to cite explicitly),* $\operatorname{null} \mathbf{A} = {}^n\mathbb{R}$.

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