**University of Toronto**
**Faculty of Applied Science and Engineering**

**Quiz 4 – November 27th, 2017**
**9:15 am – 10:15 am**

**SECOND YEAR – ENGINEERING SCIENCE**

**AER210F VECTOR CALCULUS and FLUID MECHANICS**
**Examiner: Philip McCarthy**

Instructions:
(1) Closed book examination; Non-programmable calculator allowed, no other aids are permitted.
(2) Write your name, student number and tutorial group in the space provided below.
(3) Answer as many questions as you can. Parts of questions may be answered.
(4) Use the overleaf side of pages or the extra page at the end for additional work. Indicate clearly if you have continued a question to a second page
(5) Do not separate or remove any pages from this exam booklet.

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Family Name:________________________________

Given Name:________________________________

Student Number:________________________________

TA Name/ Tutorial Session #:__________________________

| FOR MARKER USE ONLY | | |
|---|---|---|
| Question | Mark | Earned |
| 1 | 4 | |
| 2 | 8 | |
| 3 | 7 | |
| 4 | 3 | |
| 5 | 5 | |
| 6 | 12 | |
| 7 | 12 | |
| TOTAL | 51 | |

*Unless otherwise stated in the question, use:*

Gravitational Acceleration – $g = 9.81\ m/s^2$
Density of Water – $\rho = 1000\ kg/m^3$

1. If the differential form of the continuity equation for an infinitesimal *control volume* is given by:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{U}) = 0$$

Find the equivalent differential form for the fluid *system*. **[4 Marks]**

$$\frac{\partial \rho}{\partial t} + \underbrace{\vec{\nabla} \cdot (\rho \vec{U})}_{\text{DIVERGENCE OF THE PRODUCT OF A SCALAR TIMES A VECTOR}} = 0$$

$$\vec{\nabla} \cdot (\rho \vec{U}) = \rho (\vec{\nabla} \cdot \vec{U}) + \vec{U} \cdot \vec{\nabla} \rho$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \rho (\vec{\nabla} \cdot \vec{U}) + \vec{U} \cdot \vec{\nabla} \rho = 0$$

$$\underbrace{\frac{\partial \rho}{\partial t} + \vec{U} \cdot \vec{\nabla} \rho}_{\frac{D\rho}{Dt}} + \rho (\vec{\nabla} \cdot \vec{U}) = 0$$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho \vec{\nabla} \cdot \vec{U} = 0}$$

→ USING AN ALTERNATIVE METHOD TO ARRIVE AT THIS EQUATION IS ALSO VALID

2. A steady, ~~incompressible~~, two-dimensional velocity field is given by

DOES NOT SATISFY CONTINUITY, HOWEVER THIS DOES NOT CHANGE THE SOLUTIONS

$$\vec{U} = (u, v) = (0.43 + 0.84x)\,\hat{\imath} + (1.6 - 0.76\,y)\,\hat{\jmath}$$

where the x- and y- coordinates are in meters and the magnitude of velocity is in m/s.

(a) Determine if there are any stagnation points in this flow field, and if so, where?
*(Hint: Stagnation means no flow)* **[2 Marks]**

(b) For this velocity field, generate an analytical expression for the flow streamlines. **[6 Marks]**

(a) STAGNATION OCCURS WHERE $\vec{U} = 0$

$\hat{\imath}$ − $0.43 + 0.84x = 0$ $\quad x = \frac{-0.43}{0.84} = -0.512\,m$

$\hat{\jmath}$ − $1.6 - 0.76y = 0$ $\quad y = \frac{-1.6}{-0.76} = 2.105\,m$

YES THERE IS ONE STAGNATION POINT AT: $x = -0.512$, $y = 2.105$

(b) STREAMLINES ARE DEFINED AS LINES THAT ARE LOCALLY TANGENT TO THE VELOCITY VECTORS. 2D STREAMLINE EQN:

$$\frac{dy}{v} = \frac{dx}{u} \Rightarrow \underbrace{\int \frac{dy}{(1.6 - 0.76y)}}_{A} = \underbrace{\int \frac{dx}{0.43 + 0.84x}}_{B}$$

$$A = \int \frac{dy}{(1.6 - 0.76y)}$$

let $u = 1.6 - 0.76y$
$du = -0.76\,dy \Rightarrow dy = \frac{du}{-0.76}$

$$\Rightarrow -\frac{1}{0.76} \int \frac{du}{u} = -\frac{1}{0.76} \ln u + C_1 = -\frac{1}{0.76} \ln(1.6 - 0.76y) + C_1$$

$$B = \int \frac{dx}{0.43+0.84x} \qquad u = 0.43+0.84x \quad du = 0.84dx \Rightarrow dx = \frac{du}{0.84}$$

$$\Rightarrow \frac{1}{0.84}\int \frac{du}{u} = \frac{1}{0.84}\ln u + C_2 = \frac{1}{0.84}\ln(0.43+0.84x) + C_2$$

$A = B$

$$\frac{-1}{0.76}\ln(1.6-0.76y)+C_1 = \frac{1}{0.84}\ln(0.43+0.84x)+C_2$$

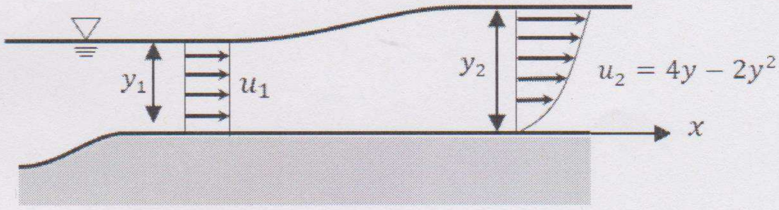
$$\Rightarrow \ln(1.6-0.76y) = \frac{-0.76}{0.84}\ln(0.43+0.84x) + \ln C'$$

$$\Rightarrow \ln(1.6-0.76y) = \ln \frac{C'}{(0.43+0.84x)^{19/21}}$$

$$\Rightarrow -0.76y = \frac{C'}{(0.43+0.84x)^{19/21}} - 1.6$$

$$\Rightarrow \boxed{y = \frac{C}{0.76(0.43+0.84x)^{19/21}} + \frac{40}{19}}$$

3. (a) At the entrance to a 3m wide rectangular channel, the velocity distribution is uniform with a velocity $u_1$. Further downstream, the velocity profile is given by $u_2 = 4y - 2y^2$. The height at $y_1$=0.75m and the height at $y_2$ = 1.1 m. Determine the value of $u_1$. **[4 Marks]**




$b = 3m$

$y_1 = 0.75m$

$y_2 = 1.1m$

$$\dot{V}_1 = \dot{V}_2$$

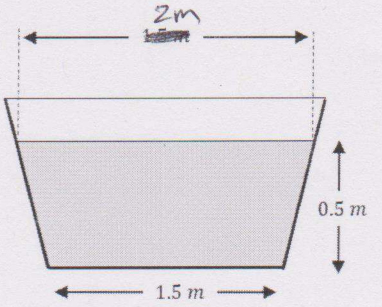
$$u_1 A_1 = \int_{A_2} u\, dA = \int_0^{1.1} (4y-2y^2)\, b\, dy$$

$$u_1\left(\frac{3}{4}b\right) = \left[2y^2 - \frac{2}{3}y^3\right]_0^{1.1} b$$

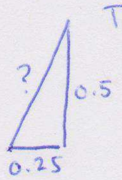
$$\frac{3}{4}u_1 = 1.533$$

$$u_1 = 2.044 \text{ m/s}$$

3.(b) An open channel with the dimensions shown in the figure is used to transport water in a waste treatment plant. Find
(i) the wetted perimeter of the channel **[1 Mark]**
(ii) the equivalent depth of a rectangular channel. **[2 Marks]**




(i) WETTED PERIMETER IS THE LENGTH OF THE CHANNEL IN CONTACT WITH THE LIQUID, SO:




$$P_w = 1.5 + 2\sqrt{0.25^2 + 0.5^2} = \underline{\underline{2.618\,m}}$$

(ii) EQUIVALENT DEPTH IS GIVEN BY THE HYDRAULIC DEPTH $y_H = \frac{A}{b_{TOP}} = \frac{\text{AREA}}{\text{WIDTH AT WATER FREE SURFACE}}$

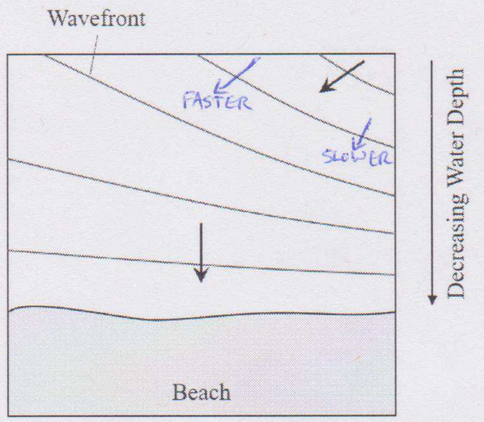
$$y_H = \frac{\left(\frac{1.5+2}{2}\,0.5\right)}{2} = \underline{\underline{0.438\,m}}$$

4. Far from the beach where the sea is deep, surface water waves approach at some angle towards the shore, as shown in the image. As the water depth decreases towards the beach, the wave fronts eventually become almost parallel to the beach. Explain why this occurs? *(Note: The wind direction is constant throughout)* **[3 Marks]**

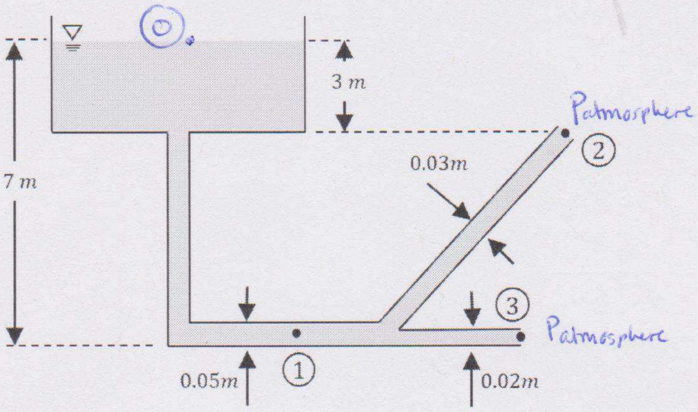
WAVESPEED IS GIVEN BY $c = \sqrt{gy}$

SO DEEPER WAVES MOVE FASTER.

WHEN WAVES APPROACH AT AN ANGLE, AS SHOWN, THE WAVES CLOSER TO THE SHORE MOVE MORE SLOWLY THAN THOSE DEEPER PARTS OF THE WAVE THAT IS FURTHER OUT. THIS CAUSES THE WAVE FRONT TO ROTATE SUCH THAT IT IS ALMOST PARALLEL TO THE BEACH.



5. Water flows from a large tank through a large pipe that splits into two smaller pipes, as shown in the diagram. Determine the flow rate from the tank and the pressure at point (1).
**[5 marks]**




FIND VELOCITY AT 2

$P_0 + \frac{1}{2}\rho U_0^2 + \rho g Z_0 = P_2 + \frac{1}{2}\rho U_2^2 + \rho g Z_2$

Tank is large, so $U_0 = 0$

$g Z_0 = \frac{1}{2} U_2^2 + g Z_2$

$U_2 = \sqrt{2g(Z_0 - Z_2)} = \sqrt{2 \times 9.81(7-4)} = 7.67\,m/s$

$P_0 = 0$ GAUGE $\quad Z_0 = 7m$

$P_2 = 0$ GAUGE $\quad Z_2 = 4m$

$P_3 = 0$ GAUGE

FIND VELOCITY AT 3

$U_3 = \sqrt{2g(Z_0 - Z_3)} = \sqrt{2 \times 9.81(7 - 0.01)} = 11.71\,m/s$

USING CONTINUITY

$\dot{V}_1 = \dot{V}_2 + \dot{V}_3 = \frac{\pi}{4} D_2^2 U_2 + \frac{\pi}{4} D_3^2 = \frac{\pi}{4}(0.03^2 \times 7.67 + 0.02^2 \times 11.71) = 9.10 \times 10^{-3}\,m^3/s$

PRESSURE AT POINT 1

$P_0 + \frac{1}{2}\rho U_0^2 + \rho g Z_0 = P_1 + \frac{1}{2}\rho U_1^2 + \rho g Z_1$

$P_1 = \rho g(Z_0 - Z_1) - \frac{1}{2}\rho U_1^2$

$U_1 = \frac{\dot{V}}{A_1} = \frac{9.10 \times 10^{-3}}{0.05^2 \frac{\pi}{4}} = 4.63\,m/s$

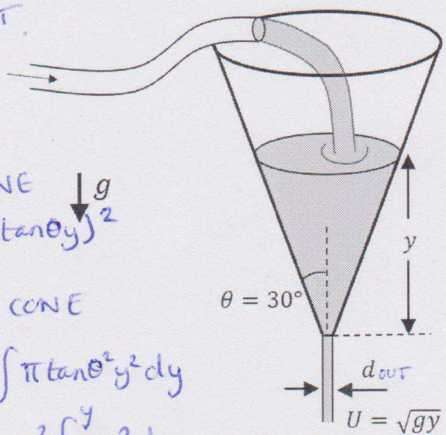
$P_1 = 1000 \times 9.81(7 - 0.025) - \frac{1}{2} \times 1000 \times 4.63^2 = 57\,706\ N/m^2$

6. Estimate the time required for the water level in a cone shaped container to rise from 130mm to 250mm, given that a hose is filling the container at 180 m$^3$/hour, but a hole of diameter $d$ = 8 mm is leaking water from the apex of the container. The speed of the water exiting the container is given by $U = \sqrt{gy}$, where g is gravitational acceleration and $y$ is the height from the hole to the liquid free surface. The cone half angle is $\theta = 30°$. **[12 Marks]**

*NOTE: For calculations, treat the cone as a perfect cone that extends to a point, i.e. the hole in the tip does not change the height of the cone)*

INCOMPRESSIBLE CONTINUITY WITH ONE INLET & ONE OUTLET

$$\frac{d}{dt}\iiint dV + U_{OUT}A_{OUT} - \dot{V}_{IN} = 0$$




AREA OF CONE

$$A = \pi r^2 = \pi(\tan\theta y)^2$$

VOLUME OF CONE

$$V = \int A\,dy = \int \pi \tan\theta^2 y^2\,dy$$

$$V = \pi\tan\theta^2\int_0^y y^2\,dy = \pi\tan\theta^2\left[\frac{y^3}{3}\right]_0^y = \pi\tan\theta^2\frac{y^3}{3}$$

$$\frac{dV}{dt} + \sqrt{gy}\,A_{OUT} - \dot{V}_{IN} = 0$$

INSERT CONE VOLUME

$$\Rightarrow \frac{d\left(\pi\tan\theta^2\frac{y^3}{3}\right)}{dt} + \sqrt{gy}\,A_{OUT} - \dot{V}_{IN} = 0$$

$$\frac{d\left(\frac{y^3}{3}\right)}{dt} = \frac{\dot{V}_{IN}}{\pi\tan\theta^2} - \frac{\sqrt{gy}\,d_{OUT}^2}{4\tan\theta^2}$$

SEPARATE VARIABLES

$$\Rightarrow y^2\,dy = \left(\frac{\dot{V}_{IN}}{\pi\tan\theta^2} - \frac{\sqrt{g}\,d_{OUT}^2}{4\tan\theta^2}\sqrt{y}\right)dt$$

$$\frac{y^2}{\frac{\dot{V}_{IN}}{\pi\tan\theta^2} - \frac{\sqrt{g}\,d_{OUT}^2\sqrt{y}}{4\tan\theta^2}}\,dy = dt$$

WHERE:

$$\frac{\dot{V}_{IN}}{\pi\tan\theta^2} = \frac{0.05}{\pi(\tan 30)^2} = 0.04775$$

$$\frac{\sqrt{g}\,d_{OUT}^2}{4\tan\theta^2} = \frac{\sqrt{9.81}\,0.008^2}{4(\tan 30)^2} = 0.00015$$

INTEGRATE

$$\Rightarrow \int_{0.13}^{0.25}\frac{y^2}{0.04775 - 0.00015\sqrt{y}}\,dy = \int_0^t dt$$

$$t = 0.09387 \text{ Seconds}$$

NOTE:

THE INTEGRAL IS DIFFICULT TO SOLVE IN THE EXAM, SO GETTING TO THERE, WITHOUT ARRIVING AT THE FINAL OUTPUT SHOULD NOT RESULT IN LOSS OF MARKS IF ALL ELSE IS COMPLETE.

SOLUTION CAN ALSO BE ACCEPTED IF THE FOLLOWING REASONING IS TAKEN:

$\dot{V}_{OUT} = \sqrt{gy}\,A_{OUT}$ RANGES FROM $7.87\times10^{-5}$ m$^3$/s FOR $y = 0.25$ to $5.68\times10^{-5}$ m$^3$/s FOR $y = 0.13$.

THIS IS 3 ORDERS OF MAGNITUDE LESS THAN $\dot{V}_{IN} = 5\times10^{-2}$ m$^3$/s

THEREFORE $(\dot{V}_{IN} - \sqrt{gy}\,A_{OUT}) \approx \dot{V}_{IN}$

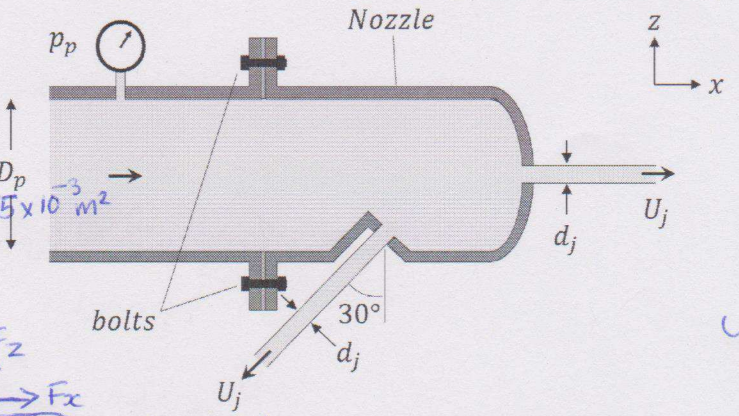
$$\Rightarrow \frac{dV}{dt} - \dot{V}_{IN} = 0 \Rightarrow \frac{\pi\tan\theta^2}{\dot{V}_{IN}}\int_{0.13}^{0.25} y^2\,dy = \int_0^t dt$$

$$t = \frac{\pi(\tan 30)^2}{0.05}\left(\frac{0.25^3 - 0.13^3}{3}\right) = 0.09375 \text{ seconds}$$

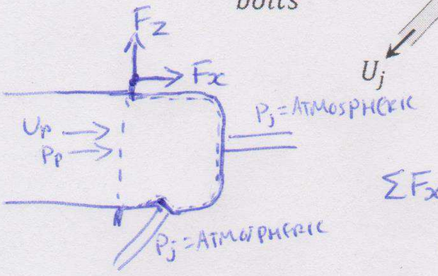
7. Two circular water jets, $d_j$ = 30 mm in diameter are produced by the nozzle. When the pressure in the pipe, with diameter $D_p$ = 10 cm, is 300 kPa (gauge), the jet speed for the two jets is equal at $U_j$=25 m/s. The nozzle itself weighs 200N and the volume of water inside the nozzle is 1414 cm$^3$. Find the force required at the flange to hold the nozzle in place.
**[12 Marks]**

$U_j = 25$ m/s

$A_j = \frac{0.03^2 \pi}{4} = 0.707 \times 10^{-3} \text{ m}^2$




$A_p = \frac{0.1^2 \pi}{4} = 7.85 \times 10^{-3} \text{ m}^2$




CONTINUITY

$U_p A_p = \sum U_j A_j = 2 U_j A_j$

$U_p (7.85 \times 10^{-3}) = 2 \times 25 \times (0.707 \times 10^{-3})$

$U_p = \frac{0.03534}{7.85 \times 10^{-3}} = 4.50 \text{ m/s}$

$P_j$ = ATMOSPHERIC = 0 Pa (GAUGE)

$P_p$ = 300 kPa (GAUGE)

CONSERVATION OF MOMENTUM

$\sum F_x = P_p A_p + F_x = \rho U_p (-U_p) A_p + \rho U_j (U_j) A_j + \rho (-U_j \sin 30)(U_j) A_j$

$P_p A_p + F_x = \rho U_j^2 A_j (1 - \sin 30) - \rho U_p^2 A_p$

$F_x = \rho U_j A_j (U_j (1 - \sin 30) - 2 U_p) - P_p A_p$

$F_x = 1000 \times 25 \times [0.707 \times 10^{-3}] (25 (1 - \sin 30) - 2 \times 4.5) - 300000 \times [7.85 \times 10^{-3}]$

$F_x = -2293 \text{ N}$

$\sum F_z = \rho (-U_j \cos 30)(U_j) A_j$

WATER VOLUME

$V = 1414 \text{ cm}^3 = 0.001414 \text{ m}^3$

WEIGHT OF WATER $= \rho V g$

WEIGHT OF NOZZLE = 200 N

$-200 - 0.001414 \rho g + F_z = -\rho U_j^2 \cos 30 A_j$

$F_z = 200 + 0.001414 \rho g - \rho U_j^2 \cos 30 A_j$

$F_z = 200 + 0.001414 \times 1000 \times 9.81 - 1000 \times 25^2 \times \cos 30 \times [0.707 \times 10^{-3}]$

$F_z = -168.8$

$\vec{F} = -2293 \hat{i} - 168.8 \hat{k}$ N    ON CONTROL VOLUME

FORCES IN FLANGE (IN BOLTS)

$\vec{F}_F = -\vec{F} = 2293 \hat{i} + 168.8 \hat{k}$ N

NOTE: AWARD FULL MARKS EVEN IF ONLY $\vec{F}$ IS GIVEN.