MAT292 - Calculus III - Fall 2014

Term Test 1 - October 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

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Last First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 16 pages (including this title page). Make sure you have all of them.
- You can use pages 14–16 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 14–16.

GOOD LUCK!

PART I No explanation is necessary.

For questions 1–8, consider a constant $a \in \mathbb{R}$ and the differential equation. **(8 marks)**

$$\frac{dy}{dt} = (y+a)(y-a)^2.$$

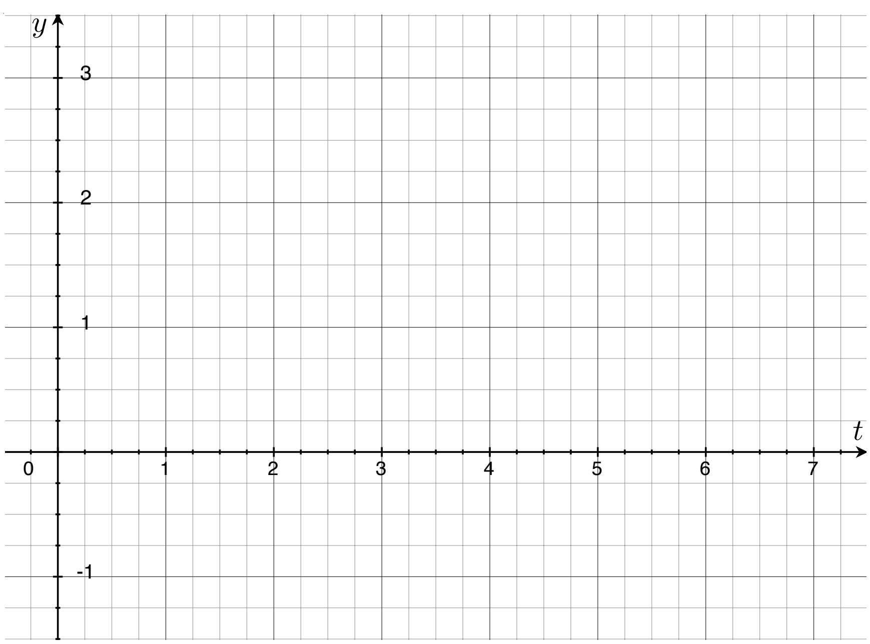
1. If $a > 0$, then the critical point $-a$ is stable / semistable / unstable

2. If $a > 0$, then the critical point a is stable / semistable / unstable

3. If $a < 0$, then the critical point $-a$ is stable / semistable / unstable

4. If $a < 0$, then the critical point a is stable / semistable / unstable

5. Without solving the differential equation, sketch the solution for $a = 1$ with the initial condition $y(2) = -\frac{1}{2}$.



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6. For $a = -1$, the solution has an asymptote at $y = 1$ as $t \to +\infty$ if the initial condition is

(a) $y(42) = 0$

(b) $y(-28) = 2$

(c) $y(0) = -2$

(d) Only the equilibrium solution can have asymptote at $y = 1$.

7. For $a = -1$, the solution has an asymptote at $y = -1$ as $t \to +\infty$ if the initial condition is

(a) $y(10^{10}) = 0$

(b) $y(-4000) = 2$

(c) $y(2014) = -2$

(d) Only the equilibrium solution can have asymptote at $y = -1$.

8. Let $a < 0$ and let $y = \phi(t)$ be the solution with initial condition $y(0) = \dfrac{a}{2}$. Then the maximum of $\phi(t)$ for $t \geqslant 0$ is

$$\max_{t \in [0,\infty)} \phi(t) = \underline{\hspace{6cm}}.$$

PART II **Justify** your answers.

9. Consider the autonomous differential equation $y' = f(y)$, with a critical point c. **(8 marks)**

(a) Assume that $f'(c) > 0$. Graph $z = f(y)$ for values of y near c.

(b) Is c stable or unstable? Justify your answer.

(c) Assume that $f'(c) < 0$. Graph $z = f(y)$ for values of y near c.

(d) Is c stable or unstable? Justify your answer.

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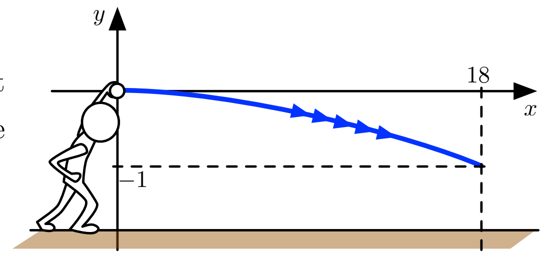
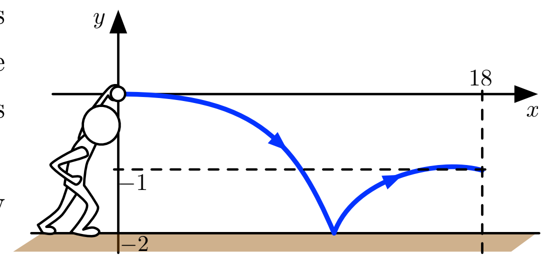
10. You are a baseball pitcher and you want to throw a ball from your position to the catcher 18m away and 1m below your throwing position. Consider gravity only. **(10 marks)**



(a) If the pitcher throws the ball horizontally, how fast should he throw it? And how much time will it take for the ball to reach the catcher?

(b) Assume that the pitcher is used to cricket: he throws the ball horizontally, the ball bounces once on the ground (2m below the throwing position), but loses a quarter of its velocity on the bounce.

With exactly one bounce, how fast should he throw the ball?



(**Hint.** Tricky! Leave for last)

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11. **(a)** Find the general solution of the differential equation **(8 marks)**

$$\left(1 - \cos(y)x^3\right) y'(x) = 3x^2 \sin(y) + \cos(x).$$

(**Hint.** You can leave the solution in implicit form)

(b) The differential equation

$$\left(\frac{1}{x} - \cos(y)x^2\right) y'(x) = 3x\sin(y) \qquad \text{is not exact.}$$

Find an integrating factor $\mu(x, y)$ to make this equation exact. Justify your answer.

12. Consider the following initial value problem: **(8 marks)**

$$\begin{cases} 2y' = y^2 + y \\ y(0) = 1 \end{cases}$$

(a) Using Euler's Method with $h = \dfrac{1}{2}$, approximate the solution at $t = 1$.

(b) Find the solution of the initial value problem and compute the error of the approximation in **(a)** at $t = 1$.

(c) If we need to obtain an error 50 times smaller, which step size h should we choose?

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13. Consider functions $p(t)$ and $g(t)$ continuous for $t \in (a,b)$ and consider the initial value problem **(8 marks)**

$$\begin{cases} y' + p(t)y = g(t) & \text{for } t \in (a,b) \\ y(t_0) = y_0, \end{cases}$$

where $a < t_0 < b$. Let $\phi(t)$ and $\psi(t)$ be two solutions of this initial value problem.

Show that $\phi(t) = \psi(t)$ for $t \in (a,b)$.

Hint. Split the proof in three steps:

(a) Define $F(t) = \phi(t) - \psi(t)$. Show that $F(t)$ is a solution of the initial value problem

$$\begin{cases} F' + p(t)F = 0 & \text{for } t \in (a,b) \\ F(t_0) = 0. \end{cases}$$

(b) Solve this differential equation and find $F(t)$.

(c) Conclusion.

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