MAT194 Final Review Package

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Peer Assisted Study Sessions - Fall 2014

Instead of just solving past exams for you, we decided to make this review package which goes over the course a bit more systematically. The example problems are all taken from past finals, but we've chosen them to be more canonical – these types of problems have a high chance of coming up on your exam and are tough but doable. Review the study checklist for notes on common variations. Most exams will also include one or two 'off-beat' questions to challenge the strongest students. Our general advice about these is to look at them briefly to push the limits of your analytical abilities, but generally not to worry about them. Like seriously, for the last question from our exam (2012), our instructor told us only one person got the full solution. That said, it's *definitely* a good idea to go over complete past exams in a simulated exam (timed!) setting. A handful of them have solutions posted online; for the rest, check answers with your friends!

By the way, if you are interested in how to make beautifully typeset documents like this, check out LaTeX over the break!

Contents

General tips

- **Calm down.** Stressing won't help you, so why do it? The exam is supposed to be hard, and is one of the harder ones you'll write this semester. Your actual mark will likely be much higher than your raw mark. I've heard Stangeby even does things like:

  $$final\ mark = 10 \times \sqrt{raw\ mark}$$

  If you find it difficult to keep calm, U of T has paid for a therapy dog to help you.

- **Do lots of problems.** Start by making sure you understand everything (read your Stewart; perhaps make study notes) and doing easier textbook problems. Then make sure to attempt past exams. Do them by yourself but review/check answers with friends. The *Problems Plus* sections in the textbook are also a good source of challenging problems from which Stangeby sometimes takes exam questions.

- **Memorize the theorems and their conditions.** This is easiest if you understand them. Do this both because you might be asked to state them, and because having them in your mind might help unlock a tricky proof type problem.

- **Review your trigonometry**, if rusty. Remember you don't have a calculator, so know the important values of *sin* and *cos*. The hyperbolic functions are unlikely to come up, so don't worry about all the identities; just remember the definitions of *sinh* and *cosh* and you can derive anything else on the spot if necessary.

- **Work on getting faster without getting sloppy.** Calc exams can be long (as in, "we don't expect most students to finish"). While studying, practice exactly how you are going to present your answer for certain types of questions you know will appear (δ-ϵ, curve sketching, etc.) so you don't waste time on that in the test. Try timing yourself on problems. Present your answers clearly so it's easier to get part marks.

- **Read the questions carefully.** Don't start a solution until you know what you are doing. Draw a $\boxed{\text{picture}}$ for geometric problems.

- **Do the easy questions first.** You may not have time to finish anyway, so you want to at least spend your time on something you can actually do. That said, try to get at least SOMETHING down for every question. You can't get part marks for a blank answer. My personal advice is this: get something down for each question, then check your work on the easy questions, then try to finish the harder questions.

- **Plan your study time.** Study for each course as appropriate. Plan breaks; avoid marathon studying. In my own experience, studying for two courses at a time has worked well: simultaneously do 'second stage' studying (past exams and overall review) for the next exam and 'first stage' studying (going through lecture notes, making sure you understand each chapter) for the following exam. An oddly spaced exam schedule may call for a different strategy.

- **Sleep well.** The day before and the previous day too. Stick to your normal routine as much as possible. Don't go crazy on coffee if you're not a coffee drinker (you could end up just having a stomach ache or having to pee a lot during your exam. . .)

Study checklist

IMPORTANT
This list is not all-encompassing! Be proactive; you guys are the ones taking this course. However, hopefully this gives you some idea of what might be a worthwhile use of your limited study time.

Bolded items are extra common and are included in the example problems that follow.

Limits

- ☐ $\boldsymbol{\epsilon - \delta}$ **approach to limits**: basic ones, one-sided, at/to infinity, using the min(1, δ) approach, by contradiction
- ☐ Algebraic solutions using change of variables, squeeze theorem, multiplying by the conjugate, etc
- ☐ Limit laws, continuity, IVT (minor, may come up on a proof type question)

Derivatives

- ☐ **All the basic derivatives**! (these will be the easiest marks you get on the exam)
- ☐ Higher derivatives, implicit differentiation, related rates, linear approximations (less likely, may be incorporated into something)
- ☐ **Optimization problems** (finding absolute and local extrema), first and second derivative tests
- ☐ EVT, Fermat's, Rolle's, (Cauchy's) MVT (could come up for proof questions)
- ☐ **Curve sketching**, tangent lines
- ☐ Netwon's method (less likely but possible)

Integrals

- ☐ **All the basic integrals**! (you only know the easy ones plus **substitution** so these should be free marks too as long as you don't forget $+c$)
- ☐ Reimann sums (less likely but possible)
- ☐ **Fundamental Theorem of Calculus** (applications will undoubtedly come up)
- ☐ Area under the curve / between curves: by x or by y
- ☐ **Volumes of revolution**: by cylindrical shells or washers
- ☐ Average value, work, MVT for integrals (less likely, may be incorporated into something)

Advanced functions

- ☐ **Basic properties of *ln* and *e*** including the expression of e as a limit
- ☐ Inverse functions (inverse trig and hyperbolic stuff is less likely but possible)
- ☐ **Logarithmic differentiation**
- ☐ **L'Hôpital's rule**: directly or re-arranged, multiple times, using logarithms

Differential equations

- ☐ Separable (easy-peasy)
- ☐ First order linear (integrating factor)
- ☐ Second order linear homogeneous (three cases)
- ☐ **Second order linear non-homogeneous**: by undetermined coefficients or variation of parameters (most likely to get this one)

Example problems

IMPORTANT
These types of questions are likely to come up, but theoretically *anything* on the syllabus could. Look at several past exams to get a sense for yourself (but remember, yours will still be different).

1. Find the following, or prove that they do not exist.
   (a) $f'(x)$ where $f(x) = \int_{x^2}^{\sin x} 4^{t^2} \mathrm{d}t$
   (b) $f'(x)$ where $f(x) = \frac{10^{\ln x}}{x^{\ln 10}}$
   (c) $\int \sin(e^{2x}+1)e^{2x}\mathrm{d}x$
   (d) $\int_0^1 f(x)\mathrm{d}x$ where
   $$f(x) = \begin{cases} \lim\limits_{t\to 0} x^t + \lim\limits_{t\to 0}(1/x)^t & x \in (0,1) \\ \lim\limits_{t\to 0} t^t + \lim\limits_{t\to 0}(1/t)^t & x = 0,1 \end{cases}$$
   (e) Positive integers a and b such that $\log_2 5 = \frac{a}{b}$

2. (a) Provide a δ–ϵ proof that $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.
   (b) Without using the fact that $\lim\limits_{x\to 1} x^2 = 1$, prove that $\lim\limits_{x\to 1} x^2 \neq 1 + 10^{-10}$.

3. Sketch the curve $y = x + \ln(x^2+1)$ indicating all important features.

4. (a) State L'Hôspital's rule including the conditions that the functions have to satisfy.
   (b) Evaluate $\lim\limits_{x\to 0} \frac{\sin 4x}{\tan 5x}$
   (c) Evaluate $\lim\limits_{x\to 0^+} (\cos x)^{x^{-2}}$
   (d) Find $f'(0)$ if $f(x) = \begin{cases} \frac{g(x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$
   where $g(0) = g'(0) = 0,\ g''(0) = 17$ and $g''(x)$ is continuous at 0.

5. Let $f(x) = 2 - x^{1/3}$ for $0 \leq x \leq 8$.
   (a) Suppose that $0 \leq \alpha \leq 8$. Show that the equation of the tangent to $y = f(x)$ at the point$(\alpha, f(\alpha))$ is
   $$y = -\frac{1}{3}\alpha^{-2/3}x + (2 - \frac{2}{3}\alpha^{1/3})$$
   (b) The tangent found in (a) intersects with the the two axes to form a triangle. Determine the value of $\alpha \in [0,8]$ for which the area of this triangle is maximized.

6. The region between $y = \sqrt{x}$ and $y = x^2,\ 0 \leq x \leq 1$, is revolved about the line $x = -3$. Find the volume of the solid which is generated.

7. Solve
   $$y'' + 4y' + 4y = 2\mathrm{e}^{-2x} + \sin 2x$$
   subject to the boundary conditions $y'(0) = y(0) = 0$

Solutions to example problems

IMPORTANT
These are our own solutions. We do not guarantee their accuracy.

1. The first question (or two) on your real exam will the be easy "find the derivative/anti-derivative" type. You may also see a couple of less trivial derivatives/integrals like the ones here, but with a little studying these should basically be free marks as well. These questions are taken from 2007 #4 and 2011 #1 and #4.

   (a) First we rewrite the function so it is easier to analyze, then we apply the FTOC in conjunction with the chain rule (see Stewart §4.3 Example 4 on page 314).

   $$\begin{aligned} f(x) &= \int_0^{\sin x} 4^{t^2}\,\mathrm{d}t - \int_0^{x^2} 4^{t^2}\,\mathrm{d}t \\ f'(x) &= 4^{\sin^2 x} \cdot \cos x - 4^{x^4} \cdot 2x \end{aligned}$$

   (b) This is kind of a troll question because it turns out $\frac{10^{\ln x}}{x^{\ln 10}} = 1$. Regardless of whether you can see this immediately (I couldn't!), we can proceed with logarithmic differentiation, like you should for anything that has a lot of multiplication/division/exponents (see Stewart §6.2*, page 427).

   $$\begin{aligned} \ln f(x) &= \ln x \ln 10 - \ln 10 \ln x \\ \frac{1}{f(x)} f'(x) &= 0 \\ f'(x) &= 0 \end{aligned}$$

   (c) You only know one method of integrating (substitution), so obviously you use substitution. Remember the exponential function is its own derivative and adding a constant doesn't change anything. Let $u = e^{2x} + 1$ such that $\mathrm{d}u = 2e^{2x}\mathrm{d}x$:

   $$\int \sin(e^{2x}+1)e^{2x}\mathrm{d}x = \int \frac{1}{2}\sin(u)\mathrm{d}u = -\frac{1}{2}\cos(e^{2x}+1) + c$$

   (d) It should be obvious that for $x \in (0,1)$, $f(x) = 1 + 1 = 2$, so the integral is just the area of a rectangle $= 2 \times 1 = 2$. (Using L'Hôspital's rule we can see that $f(0) = f(1) = 2$ as well; however, it doesn't actually matter what the value of the function is at the endpoints, or even if it's defined there. You'll learn a bit more about this next semester.)

   (e) We don't expect $\log_2 5$ to be rational, so we need to prove that these numbers don't exist. We proceed with a proof by contradiction. Suppose there *are* positive integers a and b such that $\log_2 5 = \frac{a}{b}$. Then:

   $$\begin{aligned} \log_2 5 &= \frac{a}{b} \\ 5 &= 2^{a/b} \\ 5^b &= 2^a \end{aligned}$$

But since a and b are both positive integers, the LHS is odd and the RHS is even, so this is impossible. Therefore our supposition must be false, and there do **not** exist positive integers a and b such that $\log_2 5 = \frac{a}{b}$.

2. This question taken from 2008 #2. Pretty standard δ-ϵ stuff.

   (a) Like always with a δ-ϵ proof, we start by stating the condition we must satisfy (small detail: no absolute values on the LHS since this is a one-sided limited):

   $$\forall\, \epsilon > 0\ \exists\, \delta > 0 \ \text{ such that } 0 < x - 0 < \delta \implies |\sqrt{x} - 0| < \epsilon$$

   We need to find a δ for ANY ϵ, so we probably want to define the former in terms of the latter. We reverse-engineer the problem by taking the square root of both sides of the inequality involving δ:

   $$x < \delta \iff \sqrt{x} < \sqrt{\delta}$$

   But previously, we saw that we wanted $|\sqrt{x}| < \epsilon$! Thus we can always choose $\delta = \epsilon^2$. Plugging this in, we see that we have now satisfied our condition:

   $$0 < x - 0 < \delta \implies 0 < x < \epsilon^2 \implies |\sqrt{x} - 0| < \epsilon$$

   (b) As with most of our experience trying to disprove something, a useful starting point is to try and approach this proof by contradiction. This means assuming that the limit would be true, or more precisely:

   $$\begin{gathered}\forall\, \epsilon > 0\ \exists\, \delta > 0 \ \text{ such that}\\ 0 < |x - 1| < \delta \implies |x^2 - 1 - 10^{-10}| < \epsilon\end{gathered}$$

   Then this definition says we can pick ANY ϵ and we should be able to find a δ such that these inequalities hold true. Then trying $\epsilon = 10^{-11}$ we need to ensure that

   $$|x^2 - 1 - 10^{-10}| < 10^{-11}$$

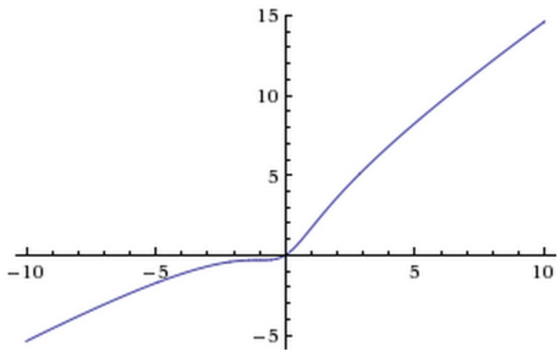
   If the limit really is $1 + 10^{-10}$, then by the definition of a limit we can ensure this by choosing x to be any number (except perhaps for 1 itself) in the open interval $(1-\delta, 1+\delta)$ where δ is some positive number. However, we can easily see a contradiction for $x \in (1-\delta, 1)$ – crucially, for x arbitrarily close to 1 but still less than 1 – as this implies:

   $$\begin{aligned}|x^2 - 1 - 10^{-10}| &< 10^{-11}\\ |(\text{perhaps small but still negative number}) - 10^{-10}| &< 10^{-11}\\ (\text{positive number at least as great as } 10^{-10}) &< 10^{-11}\end{aligned}$$

   But this is obviously false. Therefore, the limit does not equal $1 + 10^{-10}$.

3. This question taken from 2008 #3. It's pretty standard curve sketching, but work on your presentation and practise to make it quick. I think my solution is a pretty good template, if I do say so myself. Make sure you've understood the first and second derivative tests (for example, remember that it is wrong to say there is a POI because $f''(x) = 0$; you only have a POI when $f''(x)$ changes sign).

- The function is defined for all $x \in \mathbb{R}$.
- $\lim_{x\to\infty} y(x) = \lim_{x\to-\infty} y(x) = x$; the function has slant asymptote $y = x$ on both ends
- $y' = 1 + \frac{2x}{x^2+1}$; the only critical point is $(-1, \ln 2 - 1)$, but the derivative is positive (function is increasing) elsewhere. Therefore there are no local extrema.
- $y'' = -\frac{2(x^2-1)}{(x^2+1)^2}$; the function is concave down for $|x| > 1$ and concave up for $|x| < 1$. Points of inflection are thus $(-1, \ln 2 - 1)$ and $(1, \ln 2 + 1)$.



4. This question taken from 2010 #7. Content is all straight from Stewart §6.8. It's possible you will also be asked to prove L'Hôspital's rule, as is done on page 476.

   (a) If f and g are differentiable with $g'(x) \neq 0$ on an open interval I containing a (except possibly at a itself), and either

   $$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0 \qquad \text{or}$$
   $$\lim_{x\to a} g(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty \qquad \text{, then}$$

   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

   assuming the limit on the RHS exists (or is $\pm\infty$).

   (b) This is a $\frac{0}{0}$ type limit, so we can apply L'Hôspital's rule directly:

   $$\lim_{x\to 0} \frac{\sin 4x}{\tan 5x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{4\cos 4x}{5\sec^2 5x} = \frac{4}{5}$$

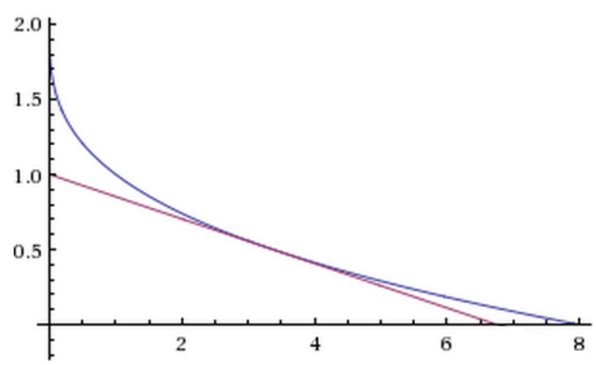
   (c) This is a 1^∞ type limit, so first we have to use logarithms to get it in an acceptable form. Then we apply L'Hôspital's rule twice.

$$
\begin{aligned}
\text{Let } L &= \lim_{x\to 0^+} (\cos x)^{x^{-2}} \\
\ln L &= \lim_{x\to 0^+} \frac{\ln(\cos x)}{x^2} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{-\tan x}{2x} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{-\sec^2 x}{2} \\
&= -\frac{1}{2} \\
L &= \frac{1}{\sqrt{e}}
\end{aligned}
$$

(d) When it's not super obvious what to do, a good idea is to start from the definition of what you're trying to find and see what happens:

$$
\begin{aligned}
f'(0) &\triangleq \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} \\
&= \lim_{x\to 0} \frac{\frac{g(x)}{x}-0}{x-0} \\
&= \lim_{x\to 0} \frac{g(x)}{x^2} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{g'(x)}{2x} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{g''(x)}{2} \\
&= \frac{17}{2}
\end{aligned}
$$

5. A relatively common type of optimization problem. This question is taken from 2002 #2 and is good algebra practice (not gunna lie, I used Wolfram Alpha when solving this. . .)



(a) In general the equation for the tangent line of $f(x)$ at α is

$$y = f'(\alpha)x + (f(\alpha) - f'(\alpha)\alpha)$$

Here $f'(x) = -\frac{1}{3}x^{-2/3}$ so we have:

$$\begin{aligned} y &= -\frac{1}{3}\alpha^{-2/3}x + ((2 - \alpha^{1/3}) - (-\frac{1}{3}\alpha^{-2/3}\alpha)) \\ &= -\frac{1}{3}\alpha^{-2/3}x + (2 - \frac{2}{3}\alpha^{1/3}) \end{aligned}$$

(b) The area of the triangle is given by:

$$\begin{aligned} A(\alpha) &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times \frac{-(2 - \frac{2}{3}\alpha^{1/3})}{-\frac{1}{3}\alpha^{-2/3}} \times (2 - \frac{2}{3}\alpha^{1/3}) \\ &= \frac{2}{3}\alpha^{4/3} + 6\alpha^{2/3} - 4\alpha \end{aligned}$$

To find the maximum area, we set the derivative of this function to zero:

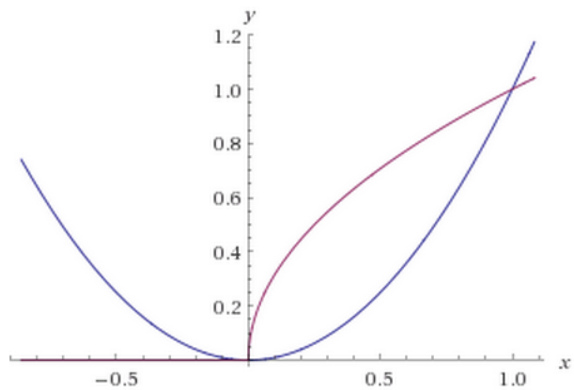
$$\begin{aligned} A'(\alpha) &= \frac{8}{9}\alpha^{1/3} + 4\alpha^{-1/3} - 4 = 0 \\ &\text{Let } \beta = \alpha^{1/3} \\ &\frac{8}{9}\beta + \frac{4}{\beta} - 4 = 0 \\ &\frac{8}{9}\beta^2 - 4\beta + 4 = 0 \\ \beta &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot \frac{8}{9} \cdot 4}}{2 \cdot \frac{8}{9}} \\ &\beta = 3 \text{ or } \frac{3}{2} \\ &\alpha = 27 \text{ or } \frac{27}{8} \end{aligned}$$

Since A is continuous, $\frac{27}{8}$ is the only critical point in [0,8] and

$$A''\left(\frac{27}{8}\right) = \frac{8}{27}\left(\frac{27}{8}\right)^{-2/3} - \frac{4}{3}\left(\frac{27}{8}\right)^{-4/3} = -\frac{32}{243} < 0$$

the area is maximized at $\alpha = \frac{27}{8}$. (Note: if you don't want to take the second derivative, you can also say that since $\frac{27}{8}$ is the only critical point in (0,8) and $A(\alpha)$ is continuous, the maximum must be the largest of $A(0)$, $A(8)$ and $A(\frac{27}{8})$, which is the latter.)

6. Taken from 2004 #5.



It looks like cylindrical shells will be easiest. We start from the general formula

$$V = \int 2\pi r h \mathrm{d}r$$

and identify:

$$r = x + 3 \qquad\qquad h = \sqrt{x} - x^2 \qquad\qquad \mathrm{d}r = \mathrm{d}x$$

where x varies from 0 to 1. Thus:

$$\begin{aligned} V &= \int_0^1 2\pi(x+3)(\sqrt{x} - x^2)\mathrm{d}x \\ &= 2\pi \int_0^1 x^{3/2} - x^3 + 3x^{1/2} - 3x^2 \ \mathrm{d}x \\ &= 2\pi \left[\frac{2x^{5/2}}{5} - \frac{x^4}{4} + 2x^{3/2} - x^3 \right]_0^1 \\ &= 2\pi \left[\frac{2}{5} - \frac{1}{4} + 2 - 1 \right] \\ &= \frac{23\pi}{10} \end{aligned}$$

7. This question taken from 2002 #9. Like the ODE question that will probably appear on your exam, this is a second order linear non-homogeneous equation (see Stewart §17.2). The general solution will be of the form $y = y_c + y_p$. First we find the complementary solution y_c by solving the associated quadratic:

$$r^2 + 4r + 4 = 0 \implies r = -2 \text{ (one root)}$$
$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$$

Now we work on the particular solution y_p. Based on the form of the equation it is obvious the solution is of the form

$$y_p = Ax^2 e^{-2x} + B \sin 2x + C \cos 2x$$

Review the pink box in Stewart page 1177 if this is not obvious to you. Note we have to use a x^2e^{-2x} term because both e^{-2x} and xe^{-2x} are part of the complementary solution. To find the values of the coefficients, we find y_p' and y_p'' and plug them into the original equation:

$$y_p' = 2Axe^{-2x} - 2Ax^2e^{-2x} + 2B\cos 2x - 2C\sin 2x$$
$$y_p'' = 2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2e^{-2x} - 4B\sin 2x - 4C\cos 2x$$

Rather than waste time writing out the whole equation with substituted values, let's just look at each term individually, since they all have to equal:

$$\begin{aligned}
e^{-2x} \text{ term}: &\qquad 2A = 2 &&\implies A = 1\\
\sin 2x \text{ term}: &\qquad -4B - 8C + 4B = 1 &&\implies C = -\frac{1}{8}\\
\cos 2x \text{ term}: &\qquad -4B - 8C + 4B = 0 &&\implies B = 0
\end{aligned}$$

We can also quickly confirm that the xe^{-2x} and x^2e^{-2x} terms equal as well (to zero).

The general solution is thus:

$$y = c_1e^{-2x} + c_2xe^{-2x} + x^2e^{-2x} - \frac{1}{8}\cos 2x$$

Now we have just one more step: to solve for the coefficients c_1 and c_2 (because we are looking for the unique solution to the boundary value problem). We are given:

$$y(0) = c_1e^{-2\cdot 0} + c_2\cdot 0\cdot e^{-2\cdot 0} + 0^2\cdot e^{-2\cdot 0} - \frac{1}{8}\cos(2\cdot 0) = c_1 - \frac{1}{8} = 0$$
$$y'(0) = -2c_1e^{-2\cdot 0} + c_2e^{-2\cdot 0} - 2c_2\cdot 0\cdot e^{-2\cdot 0} + 2\cdot 0\cdot e^{-2\cdot 0} - 2\cdot 0^2\cdot e^{-2\cdot 0} + \frac{1}{4}\sin(2\cdot 0) = -2c_1 + c_2 = 0$$

From this is it trivial to see that $c_1 = \frac{1}{8}$ and $c_2 = \frac{1}{4}$. Thus our final solution is:

$$y = \frac{1}{8}e^{-2x} + \frac{1}{4}xe^{-2x} + x^2e^{-2x} - \frac{1}{8}\cos 2x$$

GOOD LUCK!!!