MAT292 - Calculus III - Fall 2014

Solution for Term Test 2 - November 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

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Last First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 18 pages (including this title page). Make sure you have all of them.
- You can use pages **??**–18 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES **??**–18.

GOOD LUCK!

PART I

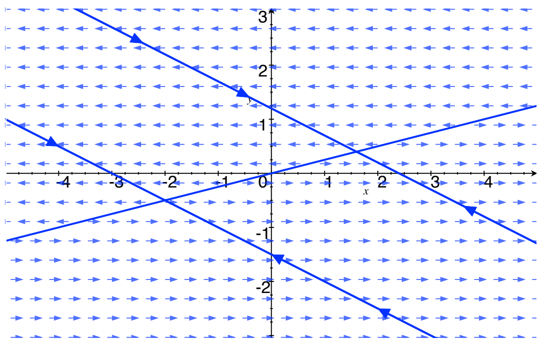
No explanation is necessary.

For questions 1–4, consider the following systems of differential equations: **(4 marks)**

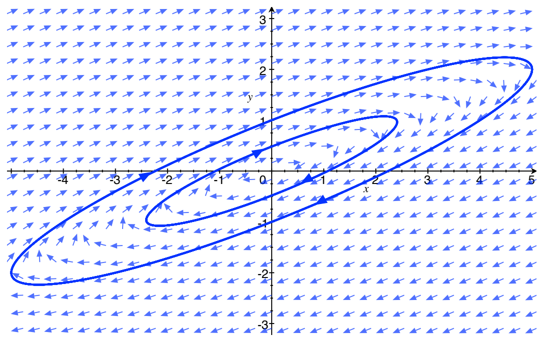
Letter	System Matrix		
a	$\mathbf{A} = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix}$	$\lambda_1 = -3, \vec{\xi_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = 0, \vec{\xi_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
b	$\mathbf{A} = \begin{pmatrix} -1 & 4 \\ \frac{1}{2} & -2 \end{pmatrix}$	$\lambda_1 = 0, \vec{\xi_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$	$\lambda_2 = -3, \vec{\xi_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
c	$\mathbf{A} = \begin{pmatrix} -2 & 5 \\ -1 & 2 \end{pmatrix}$	$\lambda_1 = -i, \vec{\xi_1} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = i, \vec{\xi_2} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
d	$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$	$\lambda_1 = i, \vec{\xi_1} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$	$\lambda_2 = -i, \vec{\xi_2} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$
e	$\mathbf{A} = \begin{pmatrix} -\frac{13}{8} & \frac{3}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{pmatrix}$	$\lambda_1 = -\frac{7}{4}, \vec{\xi_1} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{1}{8}, \vec{\xi_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
f	$\mathbf{A} = \begin{pmatrix} -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{13}{8} \end{pmatrix}$	$\lambda_1 = -\frac{1}{8}, \vec{\xi_1} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$	$\lambda_2 = -\frac{7}{4}, \vec{\xi_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
g	$\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\lambda_1 = -1, \vec{\xi_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\lambda_2 = -1, \vec{\xi_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
h	$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	$\lambda_1 = 2, \vec{\xi_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\lambda_2 = 2, \vec{\xi_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Continued...

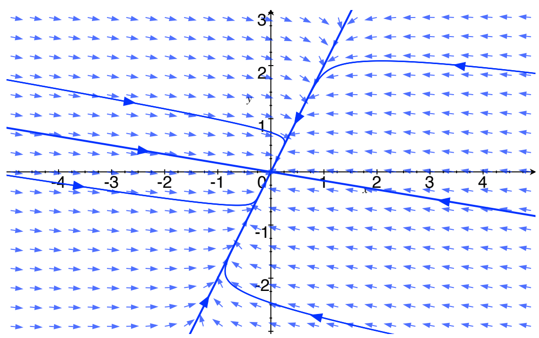
Next to each phase plane diagram, write the letter of the corresponding system of differential equations.



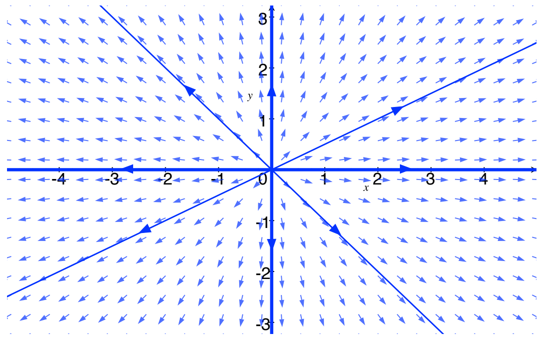
1. This is system **b**



2. This is system **c**



3. This is system **e**



4. This is system **h**

5. Write a differential equation whose complementary solution is **2 Marks**

$$y_c(t) = c_1 e^{-2t} + c_2 t e^{-2t} + c_3 t^2 e^{-2t} + c_4$$

$y^{(4)} + 6y^{(3)} + 12y'' + 8y' = e^{t^2}$

Continued...

6. Consider the ODE $y^{(6)} + 2y^{(4)} + y^{(2)} = \cos(t) + t^2$. When using the Method of Undetermined Coefficients, we assume that the terms in the *particular solution* that are *not in the complementary solution* have the form (**select all that apply**): **2 Marks**

(a) $A\cos t$	**(d)** $D\sin t$	**(g)** G	$\boxed{\textbf{(j)}\ Jt^3}$	**(m)** Me^t	**(p)** Pe^{-t}
(b) $Bt\cos t$	**(e)** $Et\sin t$	**(h)** Ht	$\boxed{\textbf{(k)}\ Kt^4}$	**(n)** Nte^t	**(q)** Qte^{-t}
$\boxed{\textbf{(c)}\ Ct^2\cos t}$	$\boxed{\textbf{(f)}\ Ft^2\sin t}$	$\boxed{\textbf{(i)}\ It^2}$	**(l)** Lt^5	**(o)** Ot^2e^t	**(r)** Rt^2e^{-t}

For questions 7 and 8, consider the ODE: **2 Marks**

$$ay'' + by' + cy = 0,$$

with $a, b, c \in \mathbb{R}$ and $b^2 - 4ac < 0$.

7. The solutions decay while oscillating if $\frac{b}{2a} > 0$ or b and a have the same sign.

8. The solutions grow while oscillating if $\frac{b}{2a} < 0$ or b and a have opposite signs.

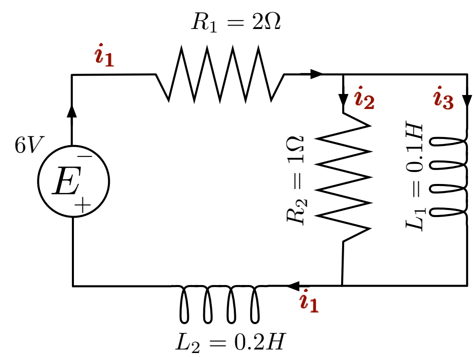
Continued...

PART II **Justify** your answers.

9. Consider the following parallel circuit. **(10 marks)**

Using Kirchhoff's First Law, we deduce that $i_1 = i_2 + i_3$, so we consider only the currents i_1 and i_2.
Using Kirchhoff's Second Law, we can show that this parallel circuit is modelled by

$$\begin{cases} \dfrac{di_1}{dt} = -10i_1 - 5i_2 + 30 \\ \dfrac{di_2}{dt} = -10i_1 - 15i_2 + 30 \end{cases}$$



(a) The system of differential equations above in *non-homogeneous*, so we can "change variables" to transform the system into a *homogeneous* system of differential equations. **2 Marks**

Consider a vector $\vec{x} = \vec{i} + \vec{b}$, with $\vec{i} = \begin{pmatrix} i_1 \\ i_2 \end{pmatrix}$.

Find $\vec{b}$ so that $\vec{x}$ is the solution of *homogeneous* system of differential equations.

Solution. By changing variables, the new unknown $\vec{x}$ satisfies the system:

$$\vec{x}' = \vec{i}' = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix} (\vec{x} - \vec{b}) + \begin{pmatrix} 30 \\ 30 \end{pmatrix} = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix} \vec{x} + \begin{pmatrix} 30 + 10b_1 + 5b_2 \\ 30 + 10b_1 + 15b_2 \end{pmatrix}$$

So this system is homogeneous, if the last vector is $\vec{0}$, which is equivalent to:

$$\begin{cases} 10b_1 + 5b_2 = -30 \\ 10b_1 + 15b_2 = -30 \end{cases} \Rightarrow \begin{cases} b_1 = -3 \\ b_2 = 0 \end{cases}$$

The vector $\vec{b}$ is $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$. $\square$

(b) The new system is **5 Marks**

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -10 & -5 \\ -10 & -15 \end{pmatrix} \vec{x}.$$

Find the general solution $\vec{x}$.

Solution. First we find the eigenvalues:

$$\begin{vmatrix} -10-r & -5 \\ -10 & -15-r \end{vmatrix} = 0 \quad \Leftrightarrow \quad (10+r)(15+r) - 50 = 0 \quad \Leftrightarrow \quad r^2 + 25r + 100 = 0$$

$$\Leftrightarrow \quad r = \frac{-25 \pm \sqrt{25^2 - 400}}{2} \quad \Leftrightarrow \quad r = \frac{-25 \pm 15}{2}$$

$$\Leftrightarrow \quad r = -20 \quad \text{or} \quad r = -5$$

Now we find the eigenvectors for each eigenvalue.

$r_1 = -20$**.** We have to solve

$$\begin{pmatrix} 10 & -5 \\ -10 & 5 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad 2\xi_1 = \xi_2$$

So the eigenvector is $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$r_2 = -5$**.** We have to solve

$$\begin{pmatrix} -5 & -5 \\ -10 & -10 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \xi_1 = -\xi_2$$

So the eigenvector is $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

This means that the general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-20t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}.$$

□

Continued...

(c) Given the initial conditions $\vec{i}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, what is the solution $\vec{i}$ of the original system? **2 Marks**

Solution. Since we have $\vec{i} = \vec{x} - \vec{b}$, we know that

$$\vec{i} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-20t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}.$$

We now find the constants c_1 and c_2 from the initial condition:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{i}(0) = \begin{pmatrix} c_1 + c_2 - 3 \\ 2c_1 - c_2 \end{pmatrix}$$

This means that $c_1 = 1$ and $c_2 = 2$, so the solution is

$$\vec{i} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-20t} + 2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-20t} + 2e^{-5t} - 3 \\ 2e^{-20t} - 2e^{-5t} \end{pmatrix}.$$

□

(d) What is i_3? **1 Mark**

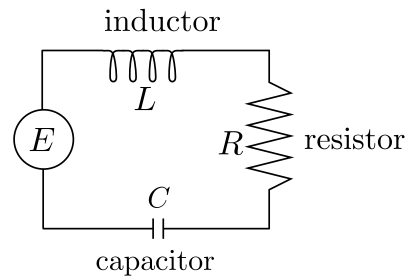
Solution. Using the formula $i_1 = i_2 + i_3$ we have

$$i_3 = i_1 - i_2 = -e^{-20t} + 4e^{-5t} - 3.$$

□

Continued...

10. Consider the following electrical circuit. **(10 marks)**



The charge on the capacitor $q(t)$ is modelled by

$$Lq'' + Rq' + \frac{1}{C}q = E(t),$$

(a) Give a condition on the constants L, R, C that guarantees that the solution oscillates. Justify your answer. **3 Marks**

Solution. The solution will oscillate if it contains sines and/or cosines, which means that the roots of the characteristic equation must be complex. The characteristic equation is

$$Lr^2 + Rr + \frac{1}{C} = 0 \quad \Leftrightarrow \quad r = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L},$$

The condition is

$$R^2 - \frac{4L}{C} < 0 \quad \Leftrightarrow \quad CR^2 < 4L.$$

□

(b) Let $L = 1$, $R = 0$, and $C = \frac{1}{4}$, and $E(t) = \sin(2t)$. Also assume that the capacitor starts with no charge and the circuit starts with no current. Find the solution of this initial-value problem. $\boxed{\textbf{5 Marks}}$

(**Hint.** Recall that current $i(t) = q'(t)$)

Solution. First, we find the complementary solution. From part (a), we know that the roots of the characteristic equation are $r = \pm 2i$, so the general solution is

$$q_c(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Then the particular solution is of the form:

$$q_p = At \cos 2t + Bt \sin 2t,$$

so we need to compute the derivatives

$$\begin{aligned} q_p' &= A \cos 2t - 2At \sin 2t + B \sin 2t + 2Bt \cos 2t = (A + 2Bt) \cos 2t + (B - 2At) \sin 2t \\ q_p'' &= (4B - 4At) \cos 2t + (-4A - 4Bt) \sin 2t \end{aligned}$$

so we use these formulas in the DE to obtain

$$(4B - 4At + 4At) \cos 2t + (-4A - 4Bt + 4Bt) \sin 2t = \sin 2t \quad \Leftrightarrow \quad \begin{cases} 4B = 0 \\ -4A = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} B = 0 \\ A = -\frac{1}{4} \end{cases}$$

So the particular solution is

$$q_p = -\frac{1}{4} t \cos 2t,$$

and the general solution is

$$q = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} t \cos 2t.$$

We now need to find the constants c_1 and c_2 from the initial conditions. We have that $q(0) = 0$ and $q'(0) = i(0) = 0$, so

$$\begin{cases} 0 = q(0) = c_1 \\ 0 = q'(0) = 2c_2 - \frac{1}{4} \end{cases} \quad \Leftrightarrow \quad \begin{cases} c_1 = 0 \\ c_2 = \frac{1}{8} \end{cases}$$

This means that the solution is

$$q = \frac{1}{8} \sin 2t - \frac{1}{4} t \cos 2t.$$

□

Continued...

(c) How does the solution to **(b)** behave (grow / decay / oscillate) as t becomes larger and larger? Justify your answer. $\boxed{\textbf{2 Marks}}$

(**Hint.** You don't need to have solved **(b)** to answer this question)

Solution. The solution grows while oscillating. The circuit is resonating! □

11. Consider the ODE **(10 marks)**

$$y'' - (3+2t)y' + (6t-2)y = 0. \qquad (\star)$$

(a) Show that $y_1(t) = e^{t^2}$ is a solution of this differential equation. $\boxed{\textbf{2 Marks}}$

Solution. For this, we need to compute the derivatives of $y_1(t)$:

$$y_1' = 2te^{t^2}$$
$$y_1'' = 2(2t^2+1)e^{t^2}$$

so

$$y_1''-(3+2t)y_1'+(6t-2)y_1 = 2(2t^2+1)e^{t^2}-(3+2t)2te^{t^2}+(6t-2)e^{t^2} = (4t^2+2-6t+4t^2+6t-2)e^{t^2} = 0,$$

so y_1 is a solution of the DE. $\square$

(b) Using reduction of order, consider a second solution of the form **3 Marks**

$$y_2(t) = u(t)y_1(t).$$

Deduce a differential equation for $u(t)$.

Solution. First we need to compute the derivatives of $y_2 = u(t)e^{t^2}$:

$$\begin{aligned} y_2' &= u'e^{t^2} + 2tue^{t^2} \\ y_2'' &= u''e^{t^2} + 4tu'e^{t^2} + u(2+4t^2)e^{t^2} \end{aligned}$$

Then we use these formulas in the DE to obtain a DE for $u(t)$:

$$u''e^{t^2} + 4tu'e^{t^2} + u(2+4t^2)e^{t^2} - (3+2t)(u'e^{t^2} + 2tue^{t^2}) + (6t-2)ue^{t^2} = 0$$

We can simplify this DE by factoring the different derivatives of u:

$$\left[u'' + (2t-3)u' + u\underbrace{(2+4t^2-2t(3+2t)+6t-2)}_{=0}\right]e^{t^2} = 0$$

So $u(t)$ satisfies the differential equation:

$$u'' + (2t-3)u' = 0.$$

$\square$

Continued...

(c) Find $u(t)$. $\boxed{\textbf{2 Marks}}$

(**Hint.** You can leave u in the form of an integral)

Solution. First let $v = u'$, which satisfies

$$v' + (2t - 3)v = 0.$$

We can use the integrating factor to solve this DE:

$$e^{t^2-3t}v = C \quad \Leftrightarrow \quad v = Ce^{3t-t^2}.$$

This means that

$$u = \int v(t)\,dt = C\int e^{3t-t^2}\,dt.$$

$\square$

Continued...

(d) Write the second solution $y_2(t)$ of $(\star)$ using a definite integral between 0 and t. $\boxed{\textbf{2 Marks}}$

Show that y_1 and y_2 form a fundamental set of solutions.

Solution. We write

$$u = \int_0^t e^{3\tau - \tau^2}\, d\tau,$$

so

$$y_2 = e^{t^2} \int_0^t e^{3\tau - \tau^2}\, d\tau.$$

Then

$$y_1' = 2te^{t^2}$$
$$y_2' = e^{3t} + 2te^{t^2} \int_0^t e^{3\tau - \tau^2}\, d\tau$$

This implies that

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2 == e^{t^2}\left[e^{3t} + 2te^{t^2} \int_0^t e^{3\tau - \tau^2}\, d\tau\right] - 2te^{2t^2} \int_0^t e^{3\tau - \tau^2}\, d\tau = e^{3t + t^2} \neq 0,$$

so these solutions form a fundamental set of solutions. □

(e) What is the general solution of the differential equation $(\star)$? $\boxed{\textbf{1 Mark}}$

Solution. From the part (d), the general solution is

$$y = c_1 e^{t^2} + c_2 e^{t^2} \int_0^t e^{3\tau - \tau^2}\, d\tau.$$

□

12. Consider the system of differential equations: **(10 marks)**

$$\vec{x}' = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \vec{x}.$$

(a) Find two linearly independent solutions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$. $\boxed{\textbf{3 Marks}}$

Solution. First we find the eigenvalues:

$$\begin{vmatrix} -2-r & -4 \\ -\frac{1}{2} & -1-r \end{vmatrix} = 0 \quad \Leftrightarrow \quad (2+r)(1+r) - 2 = 0 \quad \Leftrightarrow \quad r^2 + 3r = 0$$

$$\Leftrightarrow \quad r = 0 \quad \text{or} \quad r = -3$$

Now we find the eigenvectors for each eigenvalue.

$r_1 = 0$**.** We have to solve

$$\begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \xi_1 = -2\xi_2$$

So the eigenvector is $\vec{\xi}^{(1)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

We obtain one solution

$$\vec{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$r_2 = -3$**.** We have to solve

$$\begin{pmatrix} 1 & -4 \\ -\frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad \xi_1 = 4\xi_2$$

So the eigenvector is $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

We obtain a second solution

$$\vec{x}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{-3t}.$$

These two solutions are linearly independent, because

$$W[\vec{x}_1, \vec{x}_2](0) = \begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} = -2 - 4 = -6 \neq 0.$$

□

Continued...

(b) Consider the eigenvectors found in **(a)**. Construct a matrix $\mathbf{T}$ by putting each eigenvector as a column. **1 Mark**

Find the matrix $\mathbf{T}^{-1}$.

$\left(\textbf{Hint.}\quad \text{For the forgetful ones, } \mathbf{A}^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)$

Solution. We have

$$\mathbf{T} = \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix}$$

So

$$\mathbf{T}^{-1} = \frac{1}{-6}\begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix}.$$

□

(c) Consider a new variable $\vec{y}$ such that $\vec{x} = \mathbf{T}\,\vec{y}$. Which system of differential equations does it satisfy? $\boxed{\textbf{2 Marks}}$

Solution. We know that

$$\vec{x}' = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \vec{x}.$$

We can use the formula above to obtain

$$\mathbf{T}\vec{y}' = \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \mathbf{T}\vec{y}$$

We can simplify this system of DEs by multiplying both sides by $\mathbf{T}^{-1}$ on the left:

$$\vec{y}' = \mathbf{T}^{-1} \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \mathbf{T}\vec{y} = -\frac{1}{6} \begin{pmatrix} 1 & -4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} -2 & -4 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & 1 \end{pmatrix} \vec{y}$$

Multiply these matrices to get

$$\vec{y}' = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix} \vec{y}$$

□

(d) Find $\vec{y}$. **2 Marks**

Solution 1. The eigenvalues are $r_1 = 0$ and $r_2 = -3$ and the eigenvectors are

$$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{\xi}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the general solution for this problem is

$$\vec{y} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

□

Solution 2. The system of DEs for $\vec{y}$ is

$$\begin{cases} y_1' = 0 \\ y_2' = -3y_2 \end{cases} \quad \Rightarrow \quad \begin{cases} y_1 = c_1 \\ y_2 = c_2 e^{-3t} \end{cases}$$

Thus, the general solution for this problem is

$$\vec{y} = \begin{pmatrix} c_1 \\ c_2 e^{-3t} \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

□

(e) What is the special fundamental matrix $\mathbf{\Phi}$ for the system of differential equations in **(c)**? **2 Marks**

Solution. The special fundamental matrix is

$$\mathbf{\Phi} = e^{\mathbf{A}t} \quad \text{for} \quad \mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}.$$

Since the matrix $\mathbf{A}$ is diagonal, its exponential is

$$\mathbf{\Phi} = e^{\mathbf{A}t} = \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-3t} \end{pmatrix}.$$

□

The end.