ESC103F Engineering Mathematics and Computation: Tutorial #2

Question 1: Consider the points located at A(1,1,1), B(2,2,3) and C(6,1,10). Find the angle ABC where B is the vertex.

Solution:

Begin by finding $\overrightarrow{BA}$ and $\overrightarrow{BC}$:

$$\overrightarrow{BA} = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}, \|\overrightarrow{BA}\| = \sqrt{6}$$

$$\overrightarrow{BC} = \begin{bmatrix} 4 \\ -1 \\ 7 \end{bmatrix}, \|\overrightarrow{BC}\| = \sqrt{66}$$

$$cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\|\overrightarrow{BA}\|\|\overrightarrow{BC}\|} = \frac{-17}{\sqrt{396}}$$

$$\therefore \theta = 2.59\, radians\ (149\ degrees)$$

Question 2: Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in 2-D or 3-D and let $k = ||\vec{u}||$ and $l = ||\vec{v}||$. Prove that the vector $\vec{w} = l\vec{u} + k\vec{v}$ bisects the angle between $\vec{u}$ and $\vec{v}$.

Solution:

Let θ_1 denote the angle between $\vec{u}$ and $\vec{w}$, and let θ_2 denote the angle between $\vec{v}$ and $\vec{w}$.

Given $\vec{w} = \|\vec{v}\|\vec{u} + \|\vec{u}\|\vec{v}$:

$$cos\theta_1 = \frac{\vec{u} \cdot \vec{w}}{||\vec{u}||||\vec{w}||} = \frac{\vec{u} \cdot (\|\vec{v}\|\vec{u} + \|\vec{u}\|\vec{v})}{||\vec{u}||||\vec{w}||} = \frac{||\vec{v}||||\vec{u}|| + \vec{u} \cdot \vec{v}}{||\vec{w}||}$$

Similarly:

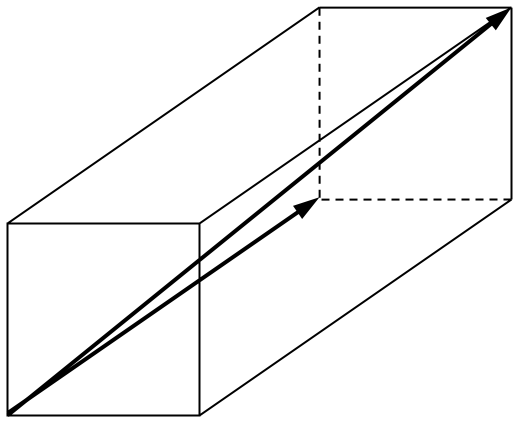
$$cos\theta_2 = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||||\vec{w}||} = \frac{\vec{v} \cdot (\|\vec{v}\|\vec{u} + \|\vec{u}\|\vec{v})}{||\vec{v}||||\vec{w}||} = \frac{||\vec{v}||||\vec{u}|| + \vec{u} \cdot \vec{v}}{||\vec{w}||}$$

$$\therefore cos\theta_1 = cos\theta_2$$

$$\therefore \theta_1 = \theta_2$$

$\therefore \vec{w}$ bisects $\vec{u}$ and $\vec{v}$.

Question 3: Given a rectangular solid with sides of lengths 1, 1, and $\sqrt{2}$, use a vector approach to find the angle between a body diagonal and one of the longest sides.



Solution:

Let us begin by establishing a coordinate system. We will treat the point in the diagram at the tail ends of the two vectors as the origin. From the origin, we will treat the bottom of the square end as the x-axis going from left to right, the left side of the square end as the y-axis going from bottom to top, and longest side as the z-axis going from front to back.

Then, the vector on the body diagonal, $\vec{u}$, is given by:

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}$$

and the vector on the longest side, $\vec{v}$, is given by:

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

$$\therefore cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \theta = \frac{\pi}{4}\ radians\ (45\ degrees)$$

Question 4: Using cross product, find the area of the triangle having vertices A(1, 2), B(7, -2) and C(7, 20/3).

Solution:

Here we will make use of the fact that the area of triangle ABC is equal to ½ of the area of the parallelogram formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ that is given by:

$$\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\|$$

$$\overrightarrow{AB} = \begin{bmatrix} 6 \\ -4 \end{bmatrix} \; and \; \overrightarrow{AC} = \begin{bmatrix} 6 \\ 14/3 \end{bmatrix}$$

In R^3:

$$\overrightarrow{AB} = \begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix} \; and \; \overrightarrow{AC} = \begin{bmatrix} 6 \\ 14/3 \\ 0 \end{bmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ 0 \\ 52 \end{bmatrix}$$

∴ area of the triangle ABC= $\frac{1}{2}52 = 26$

Question 5: Suppose you start with the standard Cartesian coordinate system for 2-D in terms of the unit vectors:

$$\vec{\imath} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \vec{\jmath} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We are now going switch to a new 2-D coordinate system where the new x-axis, denoted by x', has been rotated 30 degrees counterclockwise from the original x-axis, and the new y-axis, denoted by y', has been rotated 15 degrees clockwise from the original y-axis.

i) Make a sketch of the old and new coordinate systems.

ii) Show that the new unit vector $\vec{\imath'}$ along x' can be expressed in terms of the original coordinate system as:

$$\vec{\imath'} = \begin{bmatrix} cos30^o \\ sin30^o \end{bmatrix}$$

iii) Derive a similar expression for the new unit vector $\vec{\jmath'}$ along y' in terms of the original coordinate system.

iv) Express the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in the original coordinate system as a linear combination of the new unit vectors associated with the new coordinate system.

Solution:

ii) Given that $\|\vec{\imath'}\| = 1$ as a result of rotating $\vec{\imath}$ by 30 degrees counterclockwise.

∴ $\vec{\imath'} = \begin{bmatrix} cos30^o \\ sin30^o \end{bmatrix}$ from right angle trigonometry.

iii) Given that $\|\vec{\jmath'}\| = 1$ as a result of rotating $\vec{\jmath}$ by 15 degrees clockwise.

∴ $\vec{\jmath'} = \begin{bmatrix} sin15^o \\ cos15^o \end{bmatrix}$ from right angle trigonometry.

iv) To answer this part of the question, we need to find a combination of the new unit vectors that produce $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$:

$$\begin{bmatrix}2\\1\end{bmatrix} = c\vec{i'} + d\vec{j'} = c\begin{bmatrix}cos30^o\\sin30^o\end{bmatrix} + d\begin{bmatrix}sin15^o\\cos15^o\end{bmatrix}$$

Solving the corresponding two equations in two unknowns:

$c \approx 2.37$

$d \approx -0.190$

$\therefore \begin{bmatrix}2\\1\end{bmatrix}$ in the original coordinate system is equivalent to $\begin{bmatrix}2.37\\-0.190\end{bmatrix}$ in the new coordinate system.

Question 6: Using projections, show that the sum of the squares of the distances from a point $P = (x_1, y_1)$ to the perpendicular lines $ax + by = 0$ and $bx - ay = 0$ is equal to the square of the length of the vector $\overrightarrow{OP} = \begin{bmatrix}x_1\\y_1\end{bmatrix}$.

Solution:

Let's take $\overrightarrow{d_1}$ as the direction vector associated with the line $ax + by = 0$ and $\overrightarrow{d_2}$ as the direction vector associated with the line $bx - ay = 0$. To obtain the distances of P to the two lines, we will project $\overrightarrow{OP}$ onto the two direction vectors.

The slope of the line $ax + by = 0$ is given by $\frac{-a}{b}$. Hence, we will take $\overrightarrow{d_1} = \begin{bmatrix}b\\-a\end{bmatrix}$.

The slope of the line $bx - ay = 0$ is given by $\frac{b}{a}$. Hence, we will take $\overrightarrow{d_2} = \begin{bmatrix}a\\b\end{bmatrix}$.

$$proj_{\overrightarrow{d_1}}\overrightarrow{OP} = \frac{\overrightarrow{OP} \cdot \overrightarrow{d_1}}{\|\vec{d}_1\|} \frac{\vec{d}_1}{\|\vec{d}_1\|}$$

$$\therefore \left\|proj_{\overrightarrow{d_1}}\overrightarrow{OP}\right\| = \frac{|\overrightarrow{OP} \cdot \overrightarrow{d_1}|}{\|\vec{d}_1\|} = \frac{|bx_1 - ay_1|}{\sqrt{a^2 + b^2}}$$

Similarly:

$$\left\|proj_{\overrightarrow{d_2}}\overrightarrow{OP}\right\| = \frac{|\overrightarrow{OP} \cdot \overrightarrow{d_2}|}{\|\vec{d}_2\|} = \frac{|ax_1 + by_1|}{\sqrt{a^2 + b^2}}$$

Therefore, the sum of the squares of these two distances is given by:

$$\frac{(bx_1 - ay_1)^2}{a^2 + b^2} + \frac{(ax_1 + by_1)^2}{a^2 + b^2} = \frac{(a^2 + b^2)x_1^2 + (a^2 + b^2)y_1^2}{a^2 + b^2} = x_1^2 + y_1^2 = \left\|\overrightarrow{OP}\right\|^2$$

Question 7: Prove that if $\vec{u}$ and $\vec{v}$ are vectors in R^3, then $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

Solution:

Take $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. Then, by using the definitions of both dot product and cross product:

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= x_1(y_1z_2 - y_2z_1) - y_1(x_1z_2 - x_2z_1) + z_1(x_1y_2 - x_2y_1) \\ &= 0\end{aligned}$$

by cancellation of terms.

Question 8: Prove the Lagrange Identity, that is if $\vec{u}$ and $\vec{v}$ are vectors in R^3, then

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2.$$

Solution:

Take $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. From the definition of cross product:

$$\begin{aligned}\|\vec{u} \times \vec{v}\|^2 &= (y_1z_2 - y_2z_1)^2 + (x_1z_2 - x_2z_1)^2 + (x_1y_2 - x_2y_1)^2 \\ &= y_1^2z_2^2 - 2y_1y_2z_1z_2 + y_2^2z_1^2 + x_1^2z_2^2 - 2x_1x_2z_1z_2 + x_2^2z_1^2 + x_1^2y_2^2 \\ &\quad - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + z_1^2z_2^2 + (x_1^2y_2^2 + x_1^2z_2^2 + x_2^2y_1^2 + y_1^2z_2^2 + x_2^2z_1^2 + y_2^2z_1^2) \\ &\quad - (x_1^2x_2^2 + y_1^2y_2^2 + z_1^2z_2^2) - (2x_1x_2y_1y_2 + 2x_1x_2z_1z_2 + 2y_1y_2z_1z_2) \\ &= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1x_2 + y_1y_2 + z_1z_2)^2 \\ &= \|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2\end{aligned}$$

Where in the 3rd line we have added and then subtracted the same three terms to maintain the equality.

Question 9: True or false? If $\vec{u}$ is orthogonal to $\vec{v} + \vec{w}$, then $\vec{u}$ is orthogonal to $\vec{v}$ and $\vec{w}$. Justify your answer with a proof (for true) or a counter example (for false).

Solution:

The statement is **false**.

If $\vec{u}$ is orthogonal to $\vec{v} + \vec{w}$, then:

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} = 0$

Therefore:

$\vec{u} \cdot \vec{v} = -\vec{u} \cdot \vec{w}$

However, this equality can be satisfied without requiring that $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w} = 0$.

For one counter example, let $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v} + \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{u} \cdot (\vec{v} + \vec{w}) = 0$ ($\vec{u}$ is orthogonal to $\vec{v} + \vec{w}$)

$\vec{u} \cdot \vec{v} = 1$ and $\vec{u} \cdot \vec{w} = -1$ ($\vec{u}$ is not orthogonal to $\vec{v}$ and $\vec{u}$ is not orthogonal to $\vec{w}$)

Question 10: True or false? If $\vec{u}$ and $\vec{v}$ are nonzero vectors such that

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

then $\vec{u}$ and $\vec{v}$ are orthogonal. Justify your answer with a proof (for true) or a counter example (for false).

Solution:

The statement is **true**.

$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\vec{u} \cdot \vec{v} \end{aligned}$$

However, we were given the following information:

$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\therefore \vec{u} \cdot \vec{v} = 0$ ($\vec{u}$ and $\vec{v}$ are orthogonal)

Question 11: True or false? If $\vec{u}$ and $\vec{v}$ are nonzero vectors, then $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$ if and only if $\vec{u}$ and $\vec{v}$ are parallel vectors. Justify your answer with a proof (for true) or a counter example (for false).

Solution:

The statement is **false**.

For one counter example, let's choose $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$, where $\vec{u}$ and $\vec{v}$ are parallel.

Is it true that $\|\vec{u} + \vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$?

Let's check:

$\|\vec{u} + \vec{v}\| = 0$

$\|\vec{u}\| + \|\vec{v}\| = 1 + 1 = 2$

$\therefore \|\vec{u} + \vec{v}\| \neq \|\vec{u}\| + \|\vec{v}\|$