University of Toronto – Faculty of Applied Science & Engineering

MAT292 – Deferred Final Examination – January 29, 2022

Examiners: V. Papyan, F. Parsch

Time alotted: 150 minutes

Full Name ______________________________

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Signature ______________________________

You are allowed to access the following **"OK" aids** during the exam. Everything not listed here is **NOT OK**.

- The course textbook
- Your own lecture notes
- Any content that can be found directly on the course's Quercus page, with the exception of external links (i.e. any links outside of Quercus).
  - As an example: You can access the lecture slides available on Quercus.
  - As a non-example: You can't access youtube, even if there is a link to it somewhere on the course page.
- Any calculator listed as Type 3 on the faculty's list: https://undergrad.engineering.utoronto.ca/exams/exam-types-permitted-calculators/

This exam contains 15 pages (including this title page). Once the exam starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages 11–13 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

	Short	**True/False**	**Long answer**						
Question	Q1-Q6	Q7-Q11	Q12/13	Q14	Q15	Q16/Q17	Q18	Q19	**Total**
Marks	19	10	10	10	10	12	8	10	**89**

! ! ! ! THERE IS A TABLE ON PAGE 14 ! ! ! !

GOOD LUCK! YOU GOT THIS!

SECTION I Provide the final answer. No justification necessary.

1. **(3 marks)** For $r > 0$ and $K > T > 0$, identify and classify the equilibrium points of the following autonomous ODE.

$$y' = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

List the equilibria from smallest to largest.

Equilibrium	Classification

2. **(2 marks)** Consider the following IVP

$$y'(t) + \frac{e^t}{(t-1)(t-6)}y(t) = \frac{t^5}{t-3} \qquad y(2) = 5$$

What is the maximal interval of t for which you can guarantee that the solution to this IVP exists?

Interval: $\boxed{}$

3. **(2 marks)** Write a constant in the box such that for ALL solutions except for the trivial solution $y(t) = 0$, we have

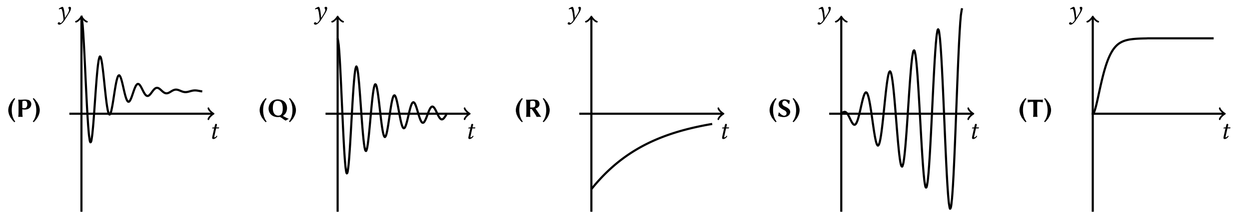
$$|y(t)| \longrightarrow +\infty \quad \text{for} \quad t \longrightarrow +\infty$$

$$y'' - y' + \boxed{}\, y = 0$$

4. **(2 marks)** State the inverse Laplace Transform of $\quad Y(s) = \dfrac{e^{-2s}}{s-1}H(s) \quad$ in terms of the function $h(t)$.

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-1}H(s)\right\} = \boxed{}$$

5. Consider these graphs depicting solutions of ODEs of the form $ay'' + by' + cy = f(t)$.

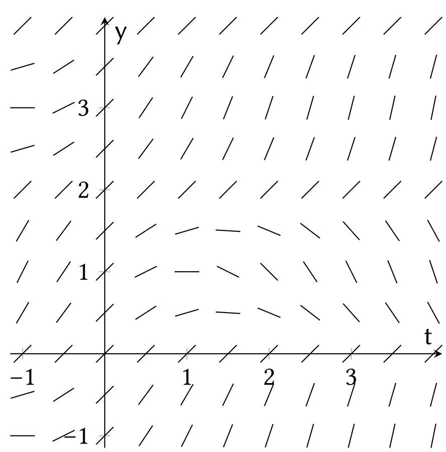


(a) **(2 marks)** Which graph/graphs can describe a free oscillator involving a mass of $2\,kg$, a spring constant of $24\,N/m$ and a damping coefficient of $14\,Ns/m$?

(b) **(2 marks)** Which graph/graphs can describe a forced damped oscillator with **nonzero constant external force**?

(c) **(2 marks)** For what value of p is **(S)** a plot of a solution to $y'' + 3y = \cos(pt)$? $p =$

6. **(4 marks)** Consider the ODE $y' = f(t, y)$ whose direction field is plotted below.



With initial value $y(1) = 2$, approximate $y(2)$ using Euler's Method with a single step.
Choose the value closest to your estimate.

○ 0.00 ○ 0.25 ○ 0.50 ○ 0.75 ○ 1.00 ○ 1.25 ○ 1.50 ○ 1.75

○ 2.00 ○ 2.25 ○ 2.50 ○ 2.75 ○ 3.00 ○ 3.25 ○ 3.50 ○ 3.75

With initial value $y(1) = 1$, approximate $y(2)$ using **Improved** Euler's Method with a single step.
Choose the value closest to your estimate.

○ 0.00 ○ 0.25 ○ 0.50 ○ 0.75 ○ 1.00 ○ 1.25 ○ 1.50 ○ 1.75

○ 2.00 ○ 2.25 ○ 2.50 ○ 2.75 ○ 3.00 ○ 3.25 ○ 3.50 ○ 3.75

SECTION II For each of the following statements, decide if it is true or false. Then justify your choice.

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

7. **(2 marks)** If $f(t, y)$ is continuous for all t and all y and never zero, then $y' = f(t, y)$ can be solved using separation of variables.

   Choose true or false, then justify: ○ TRUE ○ FALSE

8. **(2 marks)** Consider the IVP $y' = f(t, y)$ with $y(0) = 0$. When using Euler's method to approximate $y(10)$, a smaller step size will always lead to a smaller error in our approximation.

   Choose true or false, then justify: ○ TRUE ○ FALSE

9. **(2 marks)** For all square matrices A, we have $\lim_{t\to\infty} e^{-tA} = 0$

   Choose true or false, then justify: ○ TRUE ○ FALSE

10. **(2 marks)** If the constant-coefficient ODE $ay'' + by' + cy = 0$ has ONE periodic solution (not the zero solution), then ALL solutions are periodic.

    Choose true or false, then justify: ○ TRUE ○ FALSE

11. **(2 marks)** Consider $\vec{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]$ and the system $\vec{x}\,'(t) = P(t)\vec{x}(t) + \vec{g}(t)$ where $P(t)$ is an $n \times n$ matrix and $\vec{g}(t) = [g_1(t), g_2(t), \dots, g_n(t)]$.

    There must be at least one solution to this system such that $\vec{x}(0) = [0, \dots, 0]$.

    Choose true or false, then justify: ○ TRUE ○ FALSE

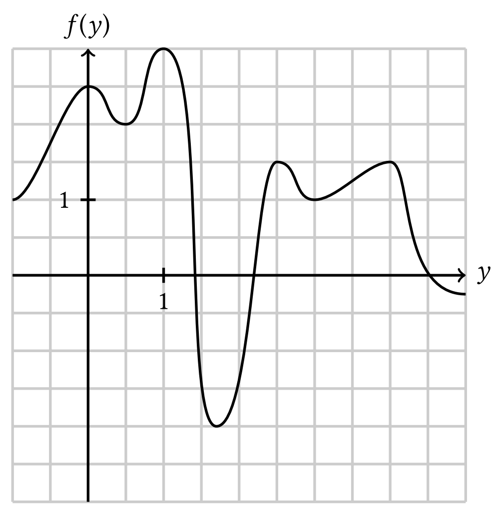
SECTION III **Justify** all your answers, unless it **specifically** says that you do not need to justify.

12. **(4 marks)** Find the maximum of the solution to this IVP:

$$y' + ty = 0 \qquad y\left(\sqrt{2}\right) = \frac{1}{e}$$

Maximum: $\boxed{}$

13. **(6 marks)** Consider the ODE $y' = f(y)$ where $f(y)$ is given by the **phase plot** on the right.



(a) You do NOT need to justify your answer to this part.
If $y(t)$ is a solution to this ODE with $y(0) = 2.5$, then

$$\lim_{t\to\infty} y(t) = \boxed{}$$

(b) Explain and justify your answer to this part.

Assume $y(t)$ is a solution to this ODE with $y(0.5) = 2$. Is y concave up, concave down, or neither at the point $(t, y) = (0.5, 2)$?

○ concave UP ○ concave DOWN ○ neither (inflection point)

14. There is a yet another zombie outbreak in a remote town in Northern Saskatchewan. We denote by $H(t)$ the number of humans and by $Z(t)$ the number of zombies in town. Every individual in town is either a human or a zombie. t is measured in days. The following effects are given:

- Every day, each human kills n zombies, where $n \geq 1$ is an unknown parameter.
- Every day, each zombie bites 2 humans, and as a result transforms them into zombies.
- Every day, 3 humans arrive from out of town.
- Every day, 5 zombies get lost in the forest and never come back (i.e. they leave town).

(a) **(4 marks)** Set up a system of ODEs. **You must explain your system to receive full points.**

$$\begin{bmatrix} H(t) \\ Z(t) \end{bmatrix}' = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} H(t) \\ Z(t) \end{bmatrix} + \begin{bmatrix} \\ \end{bmatrix}$$

(b) **(2 marks)** Find the two eigenvalues of the matrix that you just defined. Your answers should depend on the parameter n. **You do not need to justify, just state the two eigenvalues.**

$\lambda_1 =$

$\lambda_2 =$

(c) **(4 marks)** For each of the following four scenarios, discuss if it is possible or not.

Assume that the initial population $[H(0), Z(0)]$ **is NOT proportional to one of the eigenvectors.**

The number of humans and zombies will converge to a fixed, finite limit.

The zombies will win eventually, i.e. humans will go extinct.

The humans will win eventually, i.e. zombies will go extinct.

The number of humans and zombies will cycle endlessly, without either of them becoming extinct.

15. Consider the IVP

$$y'' + 3y' + 2y = \delta(t-5) - u_{10}(t) \qquad y(0) = 0 \qquad y'(0) = \frac{1}{2}$$

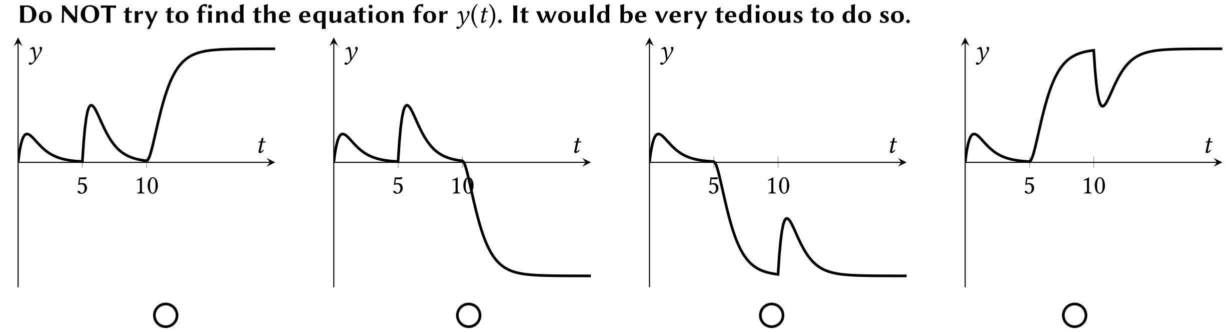
(a) (5 points) Find the Laplace Transform $Y(s)$ of the solution to this IVP. Simplify as much as possible.

$Y(s) =$

(b) (2 points) Describe how you would now find $y(t)$. You do NOT need to actually find $y(t)$, just describe the general steps that are necessary.

(c) (3 points) Even without finding the actual equation for $y(t)$ you should be able to tell which of these four plots shows the solution to this IVP. Make a choice and justify.

Do NOT try to find the equation for $y(t)$. It would be very tedious to do so.



16. Consider a continuous function $g(t)$ that has Laplace Transform $G(s)$ and define $f(t) = \int_0^t g(u)\,du$.

(a) **(2 marks)** Using the Fundamental Thm. of Calculus, we know that $f'(t) = \boxed{}$ and $f(0) = \boxed{}$.
You don't need to justify your answer for part (a).

(b) **(3 marks)** Use your answer from part (a) and a formula from the table to express the Laplace Transform of $f(t)$ in terms of $G(s)$.

$\mathcal{L}\{f(t)\} = \boxed{}$

17. In this question, we want to verify that the following equation holds:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0 \quad \text{for all integers } n \geq 0$$

In this question you are not allowed to use the table of Laplace transforms.

(a) **(3 marks)** Assume $n = 0$. Show, using the definition of Laplace Transform, that $\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$

(b) **(3 marks)** Use int. by parts and the definition of the Laplace Transform to show $\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \quad s > 0$

(c) **(1 mark)** Explain briefly how the results from part (a) and part (b) give you the formula stated at the top.

18. Consider the following differential equation: $x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$

Assume $x > 0$ and denote $t = \ln x$. Using the chain rule for $y = y(x(t))$, we can observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x}$ and therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}$$

(a) (2 marks) Use chain rule again to show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x^2}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t}\right)$$

(b) (2 marks) Substitute our results into the original ODE to show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} + y = 0$$

(c) (3 mark) Find the general solution to the ODE in part (b) to obtain $y(t)$.

$y(t) =$

(d) (1 mark) Use the $y(t)$ that you just found to state $y(x)$.

$y(x) =$

19. Consider the following partial differential equation with initial and boundary conditions:

$$\begin{aligned} u_{xx} &= u_{tt}, \qquad 0 < x < \pi, \quad t \geq 0 \\ u(x,0) &= \sin x, \quad u_t(x,0) = 0 \\ u(0,t) &= u(\pi,t) = 0. \end{aligned}$$

Assume the solution factorizes as follows: $u(x,t) = X(x)T(t)$.

(a) **(2 marks)** Using the PDE, fill in the three blanks.

$$\frac{T''(t)}{\boxed{}} = \frac{\boxed{}}{\boxed{}} = -\lambda \qquad \text{for some constant } \lambda > 0$$

(b) **(2 marks)** Find two ODEs, as well as two boundary conditions for $X(x)$ and an initial condition for $T(t)$.

$X''(x) = \boxed{}$ boundary conditions: $\boxed{}$ $\boxed{}$

$T''(t) = \boxed{}$ initial condition: $\boxed{}$

(c) **(4 marks)** Show that nontrivial solutions to these boundary and initial value problems are given by

$$X_n(x) = \sin nx \qquad T_n(t) = \cos nt \qquad \text{where } n = 1, 2, \ldots$$

(d) **(1 mark)** Explain briefly why the following function solves the PDE: $u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$

(e) **(1 mark)** Consider the initial condition $u(x,0) = \sin x$. State the solution $u(x,t)$ matching this initial condition.

$u(x,t) =$

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TABLE 5.3.1 Elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\dfrac{1}{s}, \quad s > 0$
2.	e^{at}	$\dfrac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\dfrac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$
6.	$\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$
7.	$\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \lvert a\rvert$
8.	$\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \lvert a\rvert$
9.	$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$
10.	$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
14.	$e^{ct}f(t)$	$F(s-c)$
15.	$\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
16.	$\delta(t-c)$	e^{-cs}
17.	$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
18.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$

You can wait with filling in this page until you are back on the Zoom call with us.
We will then let you know when to fill in the page.

We at U of T want you to feel proud of what you accomplish as a student. Please respect all of the hard work you've done by making sure that the work you do is your own.

We don't expect you to score perfectly on the assessments and there will be some things that you may not know. Using an unauthorized resource or asking someone else for the answer robs you of the chance later to feel proud of how well you did because you'll know that it wasn't really your work that got you there.

Success in university isn't about getting a certain mark, it's about becoming the very best person you can by enriching yourself with knowledge, strengthening yourself with skills, and building a healthy self-esteem based on how much you've grown and achieved. No one assessment captures that but your conscience will stay with you forever.

Make yourself and your loved ones proud of the student that you are by conducting yourself honestly at all times. Hold each other accountable to these standards.

In submitting this assessment ...	**Short sentences**
... I confirm that my conduct regarding this test adheres to the Code of Behaviour on Academic Matters.	I know the Code.
... I confirm that I have not acted in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism.	I didn't cheat.
... I confirm that the work I am submitting in my name is the work of no one but myself.	This is only my work.
... I confirm that all pages have been handwritten by myself.	I wrote all pages.
... I confirm that I have not received help from others, whether directly or indirectly.	I didn't receive help.
... I confirm that I have not provided help to others, whether directly or indirectly.	I didn't provide help.
... I confirm that I have only used the aids marked as "OK" on the cover page.	I only used "OK" aids.
... I am aware that not disclosing another student's misconduct despite my knowledge is an academic offence.	I know I must report cheating.

In this box, handwrite the sequence of short sentences (starting with "I know the Code. I didn't cheat...").

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