

PHY294, Winter 2023, Midterm Exam

Duration: 90 minutes.

In this file, the prof went through the effort to explicitly outline the connection of the Midterm problems to the homework and tutorial problems, as well as to the class material. The point of this effort is to help you prepare for the final exam in a similar manner (I will post a file with all problems assigned past-Midterm, scanned and clearly outlined). The point of the exam is not to test your ability to take formula A and plug into formula B to obtain some result, but to show that you understand the connections between doing these operations and the behaviour of physical systems. Then, saying the appropriate words and arguing about what physical assumption is valid under what circumstances becomes crucial... hence, simply memorizing expressions is not sufficient, as some of you have undoubtedly found out.

I. Consider a single-atomic ideal classical gas of N atoms. It is somehow engineered to undergo an isobaric quasistatic expansion to twice the volume (recall that *isobaric* means constant pressure).

Connection to class material, homework, or tutorial: This is simply one-quarter part (the contour B) of the cycle studied in Homework Problem 1.34. There, you were asked to compute the work, heat, change of energy of the system for each of the four legs. I only considered one leg to make the problem shorter.

1. Is there work done in the process? Is there heat transferred to the gas?

The gas does work in this isobaric expansion, equal to $W = pV$. No calculation is needed, this is just the area under the isobar on the $p - V$ diagram. The temperature of the gas, as per the ideal gas law, also increases by a factor of 2, and so does the energy of the gas. The initial energy is $U_{\text{initial}} = \frac{3}{2}NkT = \frac{3}{2}pV$ while the final energy is $U_{\text{final}} = 3pV$. Thus, $\Delta U = \frac{3}{2}pV$. Thus, heat must be supplied in this quasistatic process to compensate for the work being done by the gas and to increase the temperature of the gas.

12 points

2. Determine their values (of the work and the heat, that is) in terms of p and V . V is the initial volume of the gas and p is its pressure.

The work was given above. From the first law, we have $\Delta U = Q - W$, with W the work done by the gas. Hence, $Q = \Delta U + W = \frac{5}{2}pV$, the amount of heat the gas absorbs from its surroundings in this quasistatic process.

8 points

3. Do you expect the entropy of the gas to change? (Qualitative argument suffices.)

Yes, we expect the entropy to increase because the energy of the gas (or its temperature) increases in this process.

8 points

Total number of points for **I.**: 28 points

II. Consider two ideal monatomic gases. They occupy volumes V_1 and V_2 , initially taken to be equal $V_1 = V_2$. The atoms in gas 1 are roughly 16 times heavier than the ones in the other, i.e. $m_1 = 16m_2$. In addition, gas 1 consists of N_1 atoms and gas 2 of N_2 atoms, with equal numbers $N_1 = N_2$. Thus, the initial number densities of the gases are equal. The gases are kept in separate vessels but at the same temperature, which is kept fixed throughout. We have learned that at sufficiently low density, both gases are classical and obey the ideal gas law and the equipartition theorem. We assume that at the initial values of the volumes and temperature (given above), the gases obey the classical gas laws.

Suppose now the gases get compressed keeping T fixed (assume that each gas can be compressed independent of the other). As we also learned, upon increasing the density, classical equipartition and the ideal gas law will fail. Estimate the ratio of volumes V'_1 (for gas 1) and V'_2 (for gas 2) where the two gases are expected to cease obeying the classical ideal gas law, i.e. estimate the ratio V'_1/V'_2 . Simply put, which gas must be compressed more, and by how much, to make it non-classical?

Connection to class material, homework, or tutorial: The comparison of the thermal de Broglie wavelength and the average distance between particles, of order, $(\frac{V}{N})^{1/3}$, as a criterion of determining when ideal gas law and classical equipartition would cease to apply was the subject of an extensive in-class discussion on March 24 (see entry 1. in the corresponding lecture announcement!). This was also the subject of Tutorial Problem 2.35. In this problem, it is asked when the gas would cease being classical upon lowering T (which increases the thermal de Broglie wavelength), while in the problem at hand, we instead compress it, which brings the average distance between the particles close to the thermal de Broglie wavelength.

For fixed temperature, the volume per particle below which a gas ceases to be classical is, roughly, estimated by demanding that the average distance between atoms be smaller than the thermal de Broglie wavelength, as given in the formula sheet. Thus, for this critical value of the volume per particle, we have the estimate $\frac{V^{\frac{2}{3}}}{N^{\frac{2}{3}}} \sim \frac{h^2}{mkT}$. Since $N_1 = N_2$ and T is fixed, we have that $\frac{(V'_1)^{\frac{2}{3}}}{(V'_2)^{\frac{2}{3}}} \sim \frac{m_2}{m_1}$ or $\frac{V'_1}{V'_2} \sim (\frac{m_2}{m_1})^{\frac{3}{2}} = \frac{1}{16^{3/2}} = \frac{1}{4^3} = \frac{1}{64}$. Thus, gas 1 has to be compressed 64 times more.

Total number of points for **II.**: 24 points

III. The multiplicity function of an Einstein solid of N oscillators can, sometimes, be approximated as $\Omega(q, N) \simeq (\frac{Ne}{q})^q$. As per the discussion in class and in the homework, this expression is only valid when the number of quanta per oscillator is smaller than unity, $q/N \ll 1$, while both N and q are taken large. Let the frequency of each oscillator be ω .

Connection to class material, homework, or tutorial: This regime of the Einstein solid was the subject of Homework Problem 2.17. The problem was also solved in class on March 17 (see entries 2 and 3 there). There, we derived the expression for the energy in the low- T limit (which, in any case, is a straightforward application of general formulae given in the formula sheet). In

addition, we also explained the relation of this problem to the decoupling (or freezing out) of the vibrational degrees of freedom at low- T in a diatomic molecular gas (which prompted the question in part 4. below), since, as pointed out many times, the Einstein solid can be used to describe the collection of N vibrational degrees of freedom of N diatomic molecules.

1. Determine the entropy of the gas as a function of the total energy and the number of the oscillators of the gas.

We have $\Omega(E, N) = (\frac{N\epsilon}{q})^q$ with q replaced by $E/(\hbar\omega)$. Then, doing some simple algebra, we find $S(E, N) = k \ln \Omega(E, N) = k \frac{E}{\hbar\omega} \ln \frac{eN\hbar\omega}{E}$.

7 points

2. Find the total energy of the gas as a function of the temperature. *Determine* (i.e. argue why this is so, don't just state it!) the temperature range that the $q/N \ll 1$ regime corresponds to. Is this a regime where the oscillators behave classically? Does classical equipartition hold?

Then, $\frac{1}{T} = (\partial S / \partial E)_N = \frac{k}{\hbar\omega} (\ln \frac{eN\hbar\omega}{E} - 1) = \frac{k}{\hbar\omega} \ln \frac{\hbar\omega N}{E}$. Rearranging, we find $-\frac{\hbar\omega}{kT} = \ln \frac{E}{N\hbar\omega}$ and re-exponentiating, $E = N\hbar\omega e^{-\frac{\hbar\omega}{kT}}$. We note that $q/N = E/(N\hbar\omega) \ll 1$ corresponds to the regime when $e^{-\frac{\hbar\omega}{kT}} \ll 1$, thus the factor in the exponent has to be large. So, the temperature regime is $kT \ll \hbar\omega$. This is a very quantum regime, since there are only a few quanta per oscillator. Classical equipartition $E/(N\hbar\omega) \simeq kT$ does not hold at such low temperatures.

12 points

3. Find the heat capacity of the gas. (Normally this would be the constant volume heat capacity, but in this example the notion of the volume is not defined, so we have simply $C = (\frac{\partial E}{\partial T})_N$.) How does the heat capacity behave as a function of T ? Determine its behaviour as $T \rightarrow 0$.

The heat capacity is, taking the derivative as given, seen to be $C = kN (\frac{\hbar\omega}{kT})^2 e^{-\frac{\hbar\omega}{kT}}$. The function $e^{-1/x}$ vanishes, as $x \rightarrow 0$, faster than any power of x , hence the limiting behaviour of C is $C \rightarrow 0$ as $T \rightarrow 0$.

10 points

4. Does this behaviour of C have anything to do with the behaviour of heat capacity for molecular gases as a function of temperature? (A qualitative but clear answer is all that's needed!)

Very much so. The decrease of C towards zero when $kT \ll \hbar\omega$ is responsible for the "freezing out" of the vibrational degrees of freedom of gases seen in the temperature dependence of heat capacity measurements.

11 points

5. Find the entropy as a function of temperature and determine its $T \rightarrow 0$ behaviour. (We have not yet discussed this in class, but you should find a result which is consistent with what is often called the "third law of TD" or "Nernst's theorem").

Above we found $S = k \frac{E}{\hbar\omega} \ln \frac{eN\hbar\omega}{E}$ and $E = N\hbar\omega e^{-\frac{\hbar\omega}{kT}}$. Combining, we find $S(T, N) = Nk e^{-\frac{\hbar\omega}{kT}} (\frac{\hbar\omega}{kT} + 1)$ (the factor of 1 inside the brackets can be safely omitted in the small- kT regime). The entropy vanishes as $kT \rightarrow 0$.

8 points

Total number of points for **III.**: 48 points

Total number of points: 28 + 24 + 48 = 100.

Useful formulae are to be found below.

Some useful mathematical expressions are: $\frac{d \ln x}{dx} = 1/x$, $\ln x^a = a \ln x$, $\ln e = 1$, $e^{\ln x} = x$, $4^{1/2} = 2$, $4^3 = 64$, $\frac{de^{a(x)}}{dx} = e^{a(x)} \frac{da(x)}{dx}$, $\frac{dx^k}{dx} = kx^{k-1}$, $\forall k \in R$, the function $e^{-\frac{1}{x}} \rightarrow 0$ as $x \rightarrow 0$ and is positive for $x > 0$; also $\frac{d^n e^{-\frac{1}{x}}}{dx^n} \big|_{x \rightarrow 0} = 0$, for all $n > 0$. The ideal gas law is $pV = NkT$. The energy of a single atomic classical ideal gas of N atoms is $U = \frac{3}{2}NkT$. The work done by the gas in a quasistatic process along some curve $p(V)$ in the $p - V$ plane is $W = \int_{V_1}^{V_2} p(V) dV$. The entropy and multiplicity function are related by $S(E, N, V) = k \ln \Omega(E, N, V)$, while the temperature is $T = 1/(\partial S / \partial E)_{N, V}$. The thermal de Broglie wavelength of a gas of particles of mass m at temperature T is $\lambda_{th.} \sim \frac{h}{\sqrt{mkT}}$. The ratio between the thermal de Broglie wavelength and the typical distance between particles in the gas (as determined by the number density of the gas) determines, roughly, under what conditions a gas can be considered classical and obeying the ideal gas law. The energy levels of an oscillator, ignoring the zero-point energy, are $\hbar \omega n$, where $n \geq 0$ is the (integer) number of quanta. The first law of TD expresses energy conservation: the energy of a system of fixed number of particles can change because of the heat transfer and work performed.