UNIVERSITY OF TORONTO

Faculty of Applied Science and Engineering

Term Test III

First Year — Program 5

MAT185S — Linear Algebra

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31 March 2011

Student Name: ____________________
Last Name First Names

Student Number: ____________________ Tutorial Section: ________

Instructions:

1. Attempt *all* questions.
2. The value of each question is indicated at the end of the space provided for its solution; a summary is given in the table opposite.
3. Write the final answers *only* in the boxed space provided for each question.
4. *No* aid is permitted.
5. The duration of this test is 90 minutes.
6. There are 10 pages and 5 questions in this test paper.

For Markers Only		
Question	**Value**	**Mark**
A		
1	10	
B		
2	10	
C		
3	10	
4	10	
5	10	
Total	50	

A. Definitions and Statements

Fill in the blanks.

1(a). The *coordinates* of a vector $\boldsymbol{v} \in \mathcal{V}$ in terms of the basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2 \cdots \boldsymbol{b}_n\}$ are

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/2

1(b). The function $\Delta : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is a *determinant function* if and only if

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1(c). The (i, j)-*minor matrix* of $\mathbf{A} \in {}^n\mathbb{R}^n$ is

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/2

1(d). The cross product of two vectors $\underset{\rightarrow}{u} = (u_1, u_2, u_3)$ and $\underset{\rightarrow}{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ can be expressed using the determinant as

/2

1(e). State the *Cauchy-Binet product theorem* for determinants.

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B. True or False

Determine if the following statements are true *or* false *and indicate by "T" (for true) and "F" (for false) in the box beside the question. The value of each question is 2 marks.*

2(a). If $\mathbf{P}$ is the transition (transformation) matrix between two bases for a vector space, then $\mathbf{Px} = \mathbf{0}$ has a nontrivial solution.

2(b). If $\mathbf{A}, \mathbf{E} \in {}^n\mathbb{R}^n$ and $\mathbf{E}$ is an elementary matrix of Type III, then $\det \mathbf{AE} = \det \mathbf{A}$.

2(c). The determinant function $\det : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ is injective.

2(d). If $\mathbf{A} \in {}^m\mathbb{R}^n$ and $\operatorname{rank} \mathbf{A} = m$, then $\det \mathbf{AA}^T \neq 0$.

2(e). There is only one determinant function $\Delta_n : {}^n\mathbb{R}^n \mapsto \mathbb{R}$.

C. Problems

3. Let $F = \{\mathbf{f}_1, \mathbf{f}_2 \cdots \mathbf{f}_n\}$ and $G = \{\mathbf{g}_1, \mathbf{g}_2 \cdots \mathbf{g}_n\}$ be two bases for ${}^n\mathbb{R}$. Also let $\mathbf{Q}_1$ be the transition (transformation) matrix from F to the standard basis E and $\mathbf{Q}_2$ be the transition (transformation) matrix from G to E.

   (a) Show that $\mathbf{Q} = \mathbf{Q}_1^{-1}\mathbf{Q}_2$ is the transition (transformation) matrix from G to F.

   (b) Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}\mathbf{f}_j = \mathbf{g}_j$, $j = 1 \cdots n$.

3(a). Show that $\mathbf{Q} = \mathbf{Q}_1^{-1}\mathbf{Q}_2$ is the transition (transformation) matrix from G to F.

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3(b). Find an invertible matrix $\mathbf{P}$ such that $\mathbf{P}\mathbf{f}_j = \mathbf{g}_j$, $j = 1 \cdots n$.

/5

4. Let $\{\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_{n-1}\} \subset {}^n\mathbb{R}$ be linearly independent and $\mathbf{x} \in {}^n\mathbb{R}$. Show that $\mathbf{x} \in \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2 \cdots \mathbf{x}_{n-1}\}$ if and only if $\det \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{n-1} & \mathbf{x} \end{bmatrix} = 0$.

. . . cont'd

4. . . . cont'd

/10

5. Let $\Delta_1 : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ and $\Delta_2 : {}^n\mathbb{R}^n \mapsto \mathbb{R}$ be two determinant functions where $\Delta_1 \neq 0$, *i.e.*, $\Delta_1(\mathbf{A})$ is not zero for all $\mathbf{A} \in {}^n\mathbb{R}^n$.

(a) Show that $\Delta_1(\mathbf{1}) \neq 0$.

(b) Show that

$$\Delta_2(\mathbf{A}) = \frac{\Delta_2(\mathbf{1})}{\Delta_1(\mathbf{1})}\Delta_1(\mathbf{A})$$

for any $\mathbf{A} \in {}^n\mathbb{R}^n$.

5(a). Show that $\Delta_1(\mathbf{1}) \neq 0$.

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5(b). Show that

$$\Delta_2(\mathbf{A}) = \frac{\Delta_2(\mathbf{1})}{\Delta_1(\mathbf{1})}\Delta_1(\mathbf{A})$$

for any $\mathbf{A} \in {}^n\mathbb{R}^n$.

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5(b). . . . *cont'd*

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