4.10 Exercises for Chapter 4

4.1) For the incompressible steady flow field

$$\boldsymbol{v}(\boldsymbol{r},t) = x\boldsymbol{i}_x + by\boldsymbol{i}_y + z\boldsymbol{i}_z$$

find the value of b such that the principle of conservation of matter is satisfied. Confirm that, for this field, the resultant volume flux Q out of the right circular cylinder $x^2 + y^2 = R^2$ for $0 \le z \le H$ is zero.

4.2) Verify that the unsteady velocity field

$$\boldsymbol{v}(\boldsymbol{r},t) = x\boldsymbol{i}_x + ty\boldsymbol{i}_y - (1 + t)z\boldsymbol{i}_z$$

satisfies the equation of continuity for incompressible flow, namely $\nabla\cdot\boldsymbol{v} = 0$. Verify also that, for the volume V in the first octant contained by the plane $x + y + z = 1$ and the coordinate planes, the volume flux Q out of V is zero.

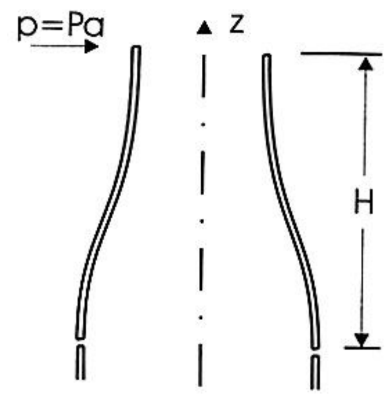
4.3) Although the equation of continuity is an expression of the idea that matter is neither created nor destroyed in fluid flows nevertheless the concept of a source is useful in constructing certain flow fields. The analog of a point charge in electrostatics, if it is located at the origin of co-ordinates, it has a velocity field of the form

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{K}{\rho^2}\boldsymbol{i}_\rho$$

where ρ is the radial distance from the origin and $\boldsymbol{i}_\rho$ is a unit vector pointing along this radius. Show that the volume flux Q out of any surface enclosing the origin is $Q = 4\pi K$.

4.4) Repeat the derivation of Torricelli's Theorem of Example 4.2, this time allowing for the fact that the atmospheric pressure P_a acting on the emerging jet is greater than that acting at the reservoir surface by an amount $\rho_a gH$, where ρ_a is the air density. With $\rho = 1000$ kg/m^3 and $\rho_a = 1.23$ kg/m^3 at $T = 15$ °C, what is the error in ignoring this effect?

4.5) A smooth contoured, gradually converging nozzle has its axis directed vertically upwards as shown in the diagram. It is attached to a pipe of cross-sectional area A_P which delivers water at a volume flow rate Q. The nozzle height is H and exit area is A_N. Taking the nozzle exit pressure as atmospheric, P_a, find the pressure required at the nozzle entrance. Show that it is the sum of the pressures that would be required in the absence of gravity and in the absence of flow.

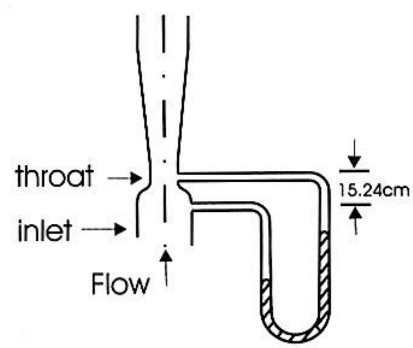


4.6) Air in a reservoir at a temperature of 300 °K and pressure of 120 kPa absolute escapes to atmosphere at pressure 100 kPa through a nozzle which is directed vertically upwards. By assuming that the flow is frictionless, incompressible, and can be modelled as a filament, find the length L that the nozzle has to exceed before omission of gravity from the calculations causes an error of 2.0 percent in the speed of the escaping air.

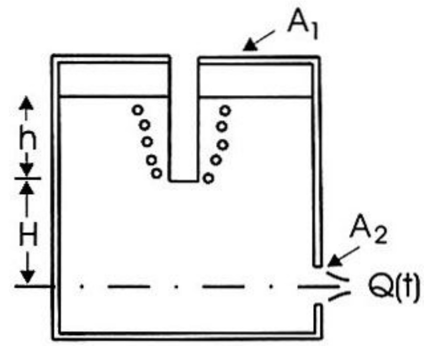
4.7) For the Venturi flow meter of Section 4.3 derive equation 4.3.17 using the stated assumptions.

4.8) A Venturi flow meter is inserted in an inclined pipe in such a way that the tap measuring the pressure p_2 at the throat is above that measuring the pressure p_1 in the pipe by an amount H. Determine the flow rate Q in terms of the fluid density ρ, the pipe diameter D and the throat diameter D_T. What happens in the limit $D_T/D \equiv \beta \to 0$?

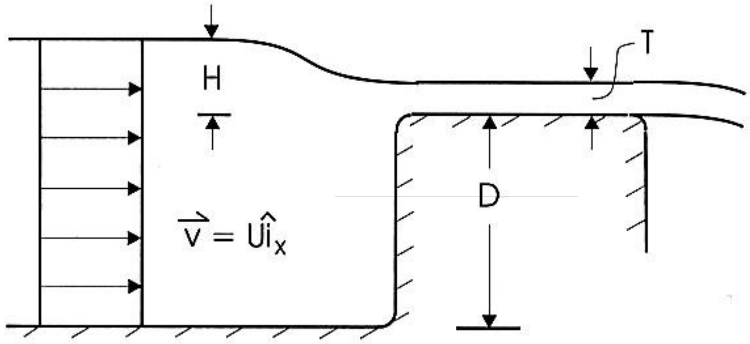
4.9) A Venturi meter having throat diameter 7.62 cm is installed in a vertical pipe of diameter 15.24 cm as shown in the diagram; the pressure tap at the throat is 15.24 cm above that at the inlet. If the volume flow rate of water is $4.08 \times 10^{-2} \mathrm{m}^3/\mathrm{s}$, calculate the difference of levels in a vertical U-tube manometer which uses mercury of relative density 13.56. Hint: Note that the tubes connecting the taps to the manometer mercury contain water up to the mercury interface.



4.10) A tank in the form of a right circular cylinder having its axis vertical and interior cross-sectional area A_1 has a contoured orifice of exit area A_2 in its side near the bottom. As shown in the diagram below the tank is closed at the top except that a large pipe having cross-sectional area A_3 extends vertically downward into the tank interior in such a way that it admits atmospheric air at a height H above the nozzle axis. If the tank is filled with water to a height h above the pipe exit, assuming frictionless flow, calculate the volume flux out of the nozzle.

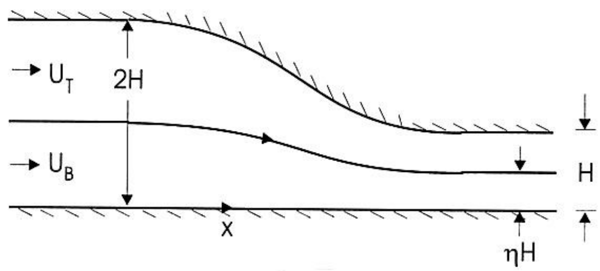


4.11) Water flows in a rectangular channel of width W at some unknown uniform speed U as shown in the diagram. It encounters a broad-crested weir having a crest height of D above the floor of the channel. If the water surface far upstream from the weir is at height H above the crest, and if the stream on the crest has uniform thickness T, determine the volume flux in the channel under the assumption that the flow is frictionless.

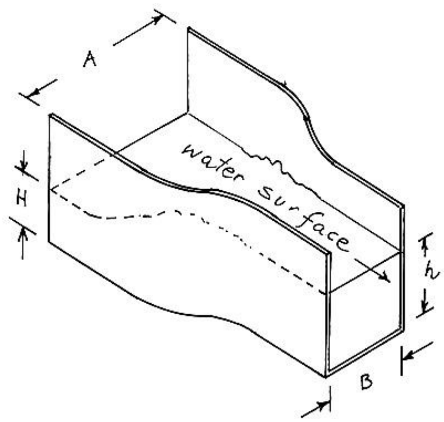


4.12) Two immiscible inviscid liquids flow in a closed horizontal channel of height $2H$ as shown in the diagram. The top liquid has density ρ_T and flows at speed U_T; the bottom liquid has density $\rho_B > \rho_T$ and flows at speed U_B. Each stream occupies half the channel width. The channel contracts to a width H; by ignoring gravitational effects show that, if η is the fraction of the contracted channel width H occupied by the bottom liquid, then

$$\frac{\rho_T U_T^2}{\rho_B U_B^2} = \frac{(1 + \eta)(1 - \eta)^3}{\eta^3 (2 - \eta)}$$



4.13) A uniform horizontal channel having a rectangular cross section of width A contracts gradually to a width B as depicted in the diagram. Water flows in the channel at uniform speed U and depth H upstream of the contraction and, after passing through the contraction to the section of width B, eventually attains a new uniform height h. With $\psi \equiv h/H$ being the depth ratio, and $E_k = U^2/(2gH)$ as in Example 4.5, show that



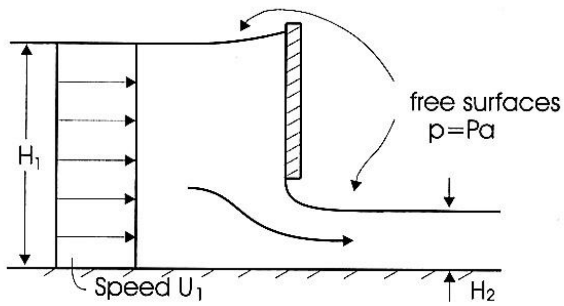
$$\psi^3 - (1 + \bar{E}_k)\psi^2 + \frac{A^2}{B^2}\bar{E}_k = 0$$

Explore what happens as $B/A \to 1$. That is, obtain an approximate solution by putting $B = A(1 - b)$, where $b \ll 1$, and discarding second and higher order quantities. Hint: Assume (1) inviscid flow; and (2), that vertical accelerations of fluid particles are zero both upstream of the contraction and in the region downstream where the stream has the uniform height h. Is h greater or less than H?

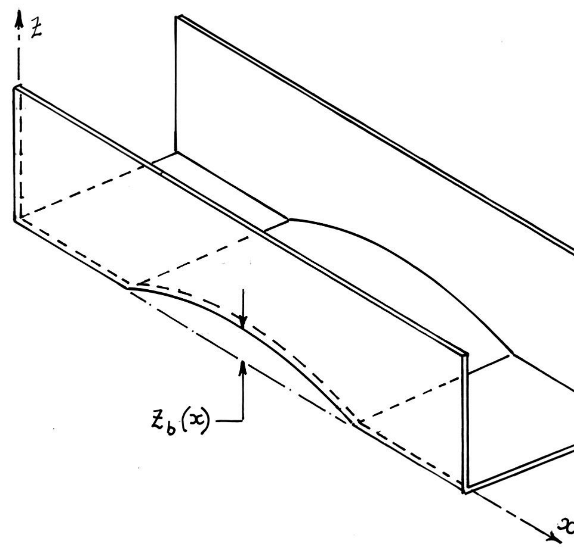
4.14) A uniform horizontal stream of speed U_1, depth H_1 and width W approaches a vertical sluice gate as depicted in the diagram below, flow under it, and some distance again becomes a horizontal uniform stream of depth H_2. By assuming inviscid flow show that, with $\psi \equiv H_2/H_1$ and $E_k = U^2/(2gH)$,

$$\psi^3 - (1 + \bar{E}_k)\psi^2 + \bar{E}_k = 0$$

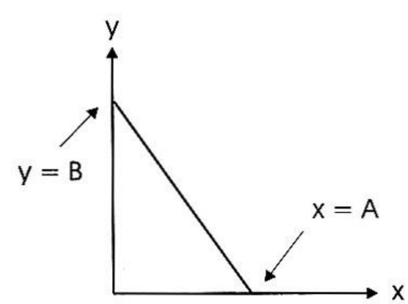
By noting that $\psi = 1$ is a solution to this cubic equation, show how this expression can be used to calculate the volume flux Q under the gate if U_1 is unknown, but H_1 and H_2 are measured.



4.15) A uniform steady stream of water flows at depth H in a horizontal channel of rectangular cross-section at a depth H. It encounters a long shallow bump having the shape $z_b(x) \ll H$ as depicted in the diagram. This causes a disturbance in the elevation of the water surface. Assume that the flow is frictionless, and that, as a consequence of the bump geometry, the velocity field can be approximated as $\boldsymbol{v}(\boldsymbol{r}, t) = u(x, z)\,\boldsymbol{i}_x + w(x, z)\,\boldsymbol{i}_z$. If the elevation of the water surface is also assumed to be a function of x alone, $z_{surface} \equiv z_s(x)$, find an equation for the disturbance $d(x) = z_s(x) - H$. Because the bump is gradual and shallow, assume both $a_z \ll g$ and $w \ll u$ to show that, if the stream is initially uniform, then $u = u(x)$ only. By assuming that $d(x) \ll H$ make an appropriate approximation to show that, with $Fr = U/\sqrt{(gH)}$, if $Fr < 1$, $d < 0$, and if $Fr > 1$, $d > 0$.



4.16) From theorem ℑ12.3, the resultant volume flux δQ out of a small volume δV_C is given by $\delta Q = \nabla\cdot\boldsymbol{v}\,\delta V$. Demonstrate this for a two-dimensional flow field by calculating the volume flux out of a cylinder of radius R located with its axis at the origin of the (x,y) plane which is small enough to allow the



velocity field to be locally represented by

$$\boldsymbol{v}(\boldsymbol{r},t) = (a + bx + cy)\boldsymbol{i}_x + (d + ex + fy)\boldsymbol{i}_y$$

where the coefficients a, b, ... may be taken as constant as in steady flow. Repeat the calculation for the right triangular volume depicted in the diagram.

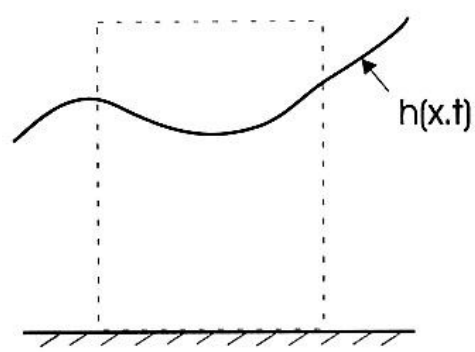
4.17) Of fundamental importance to classical physics is the *wave equation* which, in its simplest form is,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Rewrite the equation in terms of the new independent variables $r = x - ct$ and $s = x + ct$ and thus obtain the general solution $y = F(x - ct) + G(x + ct)$, where F and G are arbitrary functions of integration.

4.18) Unsteady frictionless incompressible flow in an open rectangular horizontal channel is assumed to have a velocity field of the form $\boldsymbol{v}(\boldsymbol{r}, t) = u(x, t)\,\boldsymbol{i}_x + w(x, z, t)\,\boldsymbol{i}_z$ with $w \ll u$ as in Russell's solitary waves. The water surface profile is given by $z_{surface} = h(x, t)$. By considering both volume accumulation in and fluxes out of the control volume in the diagram, show that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0$$



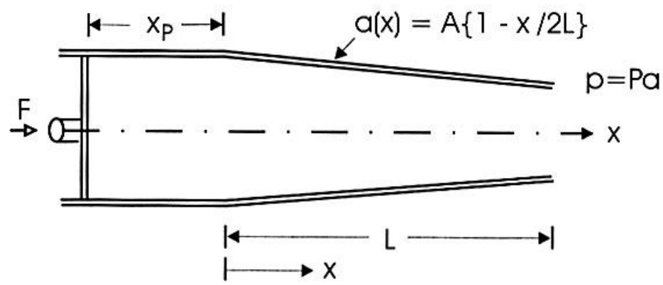
Hint: Follow Example 4.4.1

4.19) For flows of the type discussed in Exercise 4.18, show that, under the assumption that vertical accelerations are negligible, that is $|a_z| \ll g$, the momentum equations reduce to

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\,g\frac{\partial h}{\partial x}$$

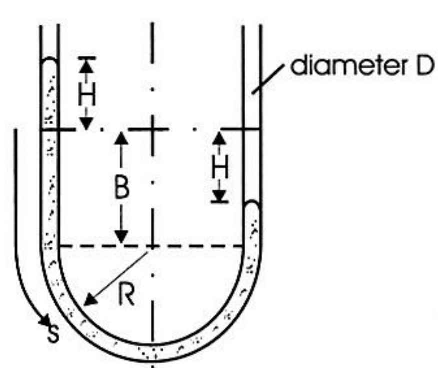
Show that, for small amplitude waves propagating into water at rest having a depth H, the speed c of propagation is given by $(gH)^{1/2}$.

4.20) A piston in a cylinder having cross-sectional area A moves with time t according to the law $x = X_P(t) = -D + Kt^2$, where the origin of coordinates is depicted in the diagram. At $x = 0$, the cylinder is connected to a slowly converging nozzle having cross-sectional area $a(x) = A\{1 - x/(2L)\}$. The assembly is filled with an incompressible frictionless fluid which is expelled to atmosphere. Assuming that the flow is one-dimensional and unsteady, that is $\boldsymbol{v}(\boldsymbol{r}, t) = u(x, t)\, \boldsymbol{i}_x$, find the force that must be applied to the piston to maintain the specified motion.



4.21) A one-dimensional unsteady inviscid compressible flow field, such as might occur in the piston problem of Section 4.6, is described in Lagrangian terms by $y = x_0(1 + t^3)$, where y is the position at time t of the particle initally at position x_0. Calculate the acceleration a_x of a particle. Then find the Eulerian or velocity field form $u(x, t)$, and verify that the acceleration calculated using Du/Dt has the same value. Finally, if the initial density and pressure is ρ_0 and p_0 respectively, find the pressure field needed to sustain the motion.

4.22) An incompressible inviscid liquid having density ρ in a U-tube manometer has suction applied to one side, as a result of which the height differential is 2H as shown in the diagram below. At time $t = 0$ the suction is suddenly removed. By treating the liquid in the tube as a constant area one-dimensional filament, find a differential equation for the motion, and hence show that it is an oscillation. Find the period of the oscillation.



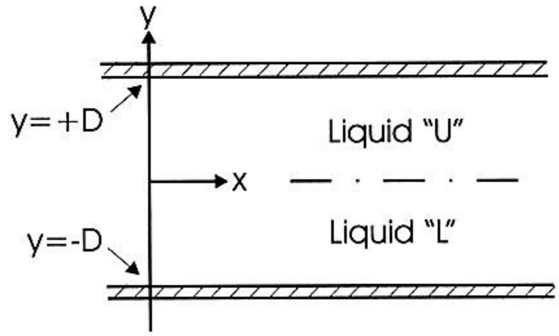
4.23) Two very long circular cylinders, having radii R_1 and $R_2 > R_1$ are aligned concentrically with their axes on the horizontal x-axis. The space between them is filled with an incompressible fluid having viscosity μ and, with the outer tube fixed, the inner tube moves at constant velocity $\boldsymbol{v} = U\boldsymbol{i}_x$. With r being the radial distance from the x-axis, the fluid responds by moving with velocity $\boldsymbol{v} = u\,(r)\boldsymbol{i}_x$; that is to say, the motion may be visualized as concentric laminae sliding one over another at constant speed as in the analysis leading to equation 4.7.13. Find the force per unit axial length of cylinder required to maintain the motion.

4.24) For incompressible viscous laminar flow in a long tube, derive the equation 4.7.19 for the volume flux in the presence of slip at the wall as given by equation 4.7.18b.

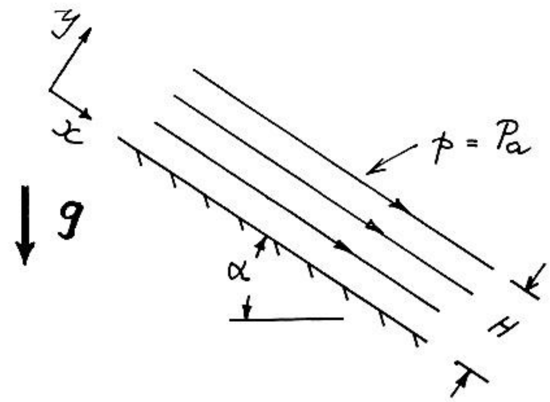
4.25) In 1880 the German hydraulics engineer Gotthilf Hagen attempted to relate the volume flux Q for laminar incompressible flow through a long tube to the pressure gradient $\Delta p/\Delta L$ by assuming that the velocity distribution $u(r)$ was not given by equation 4.7.12, but increased linearly from the wall to the centre. That is, he assumed $u(r) = u_0(1 - r/R)$ where u_0 is the centreline speed (Rouse and Ince 1963, p. 158). We now know that the discontinuity in du/dr that occurs at $r = 0$ is physically unrealistic because it would correspond to infinite shearing stress. However, in the absence of such an understanding, one can use this expression to obtain an equivalent of equation 4.7.13. Derive this equivalent. Given that all quantities could be measured to say $\pm$ 2%, would an experiment have been able to discriminate between the two formulae?

4.26) Two solid discs of diameter D are placed coaxially at a very small distance H apart. The space between them is filled with a fluid having viscosity μ, and the top disc is rotated about the common axis at an angular speed Ω rad/s. Using cylindrical coordinates, and with the axis of disc rotation being the $\boldsymbol{i}_z$-axis, by assuming that the velocity field has the form $\boldsymbol{v}(\boldsymbol{r}) = q(r,z)\,\boldsymbol{i}_\theta$, calculate the torque needed to maintain the motion.

4.27) Two immiscible incompressible viscous fluids flow between two infinite horizontal flat plates located parallel to the (x, z) plane and a distance $2D$ apart as shown in the diagram below. The heavier fluid, of density ρ_L and viscosity μ_L, occupies the space $-D \le y \le 0$, and the lighter fluid, having density ρ_U and viscosity μ_U occupies the space $0 \le y \le +D$. The plates are at rest, but laminar flow occurs as a result of an imposed constant pressure gradient $dp/dx = -P$. Locate the value of y at which the maximum speed occurs, and find the total volume flux.



4.28) A viscous incompressible liquid of density ρ and μ flows down a plane inclined at an angle α to the horizontal, forming a sheet of uniform thickness H. This suggests that all the particles travel parallel to the plane. Thus, using the co-ordinate system depicted in the diagram, assume $\boldsymbol{v}(\boldsymbol{r}, t) = u(x, y)\,\boldsymbol{i}_x$. Show that $\partial u/\partial x = 0$, implying that the accelerations $a_x = a_y = 0$. By considering the force balances acting in the x and y directions on a particle having the form of a rectangular volume such as that used to derive equation 4.7.6, show that the equation of motion is



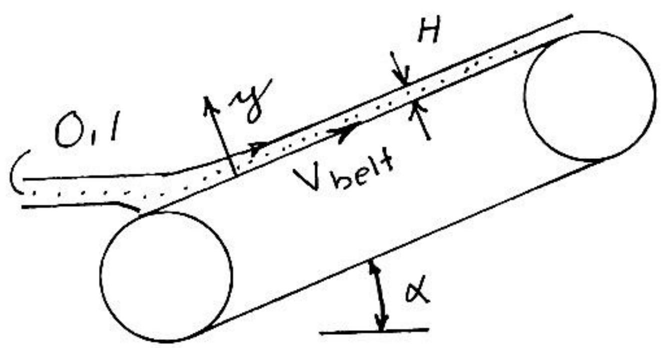
$$\rho g \sin\alpha \; + \; \mu \frac{\partial^2 u}{\partial y^2} \; = \; 0$$

4.29) For Exercise 4.28, integrate the equation of motion to show that the volume flux down the plane per unit width $Q^* = ⅓\,(\rho g/\mu)\,\sin\alpha\, H^3$. Is this result consistent with the dimensional analysis of Exercise 3.11? Hint: The boundary condition at the air-liquid interface is continuity of shear stress; and we may take the shear stress exerted by the air above the liquid to be zero.

4.30) An oil slick removal apparatus consists of an endless elastomer-fabric belt mounted on two drums as depicted in the diagram below, with the lower drum immersed in the body of water from which the slick is to be removed, and the upper drum driven so that the belt moves with constant speed V_{belt}. This apparatus drags a film of oil up the belt as depicted in the diagram, and a scraping knife adjacent to the upper drum removes the oil. Estimate the volume flux Q^* per unit belt width dragged up the belt by using the ideas in Exercises 4.28 and 4.29 and assuming that the oil film has a uniform thickness H over most of the belt. Show that this estimate is

$$Q^* = V_{belt} H - \frac{\rho g \sin\alpha}{\mu} H^3$$

Show that, with other quantities being assumed given, there is a value of H for which Q^* is a maximum. Estimate this maximum flux for $V_{belt} = 5$ m/s, $\alpha = 30°$, and for SAE 30 oil at 15°C having $\rho = 912$ kg/m^3, and $\mu = 0.38$ Ns/m^2.



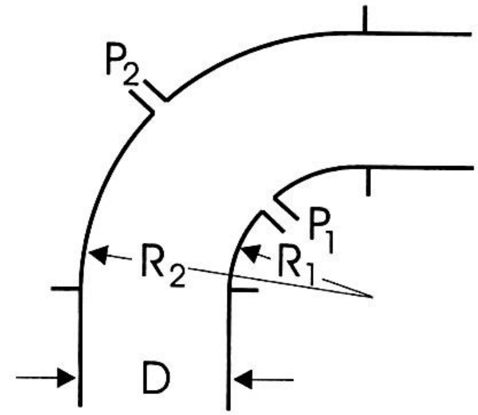
4.31) For the unsteady parallel flow problem specified by equation 4.7.22, the equation governing the plate motion is specified as $U_p(t) = At^m$ for $t \geq 0$, where A and m are positive constants. Use the affine transformation technique, equation 4.7.24, to show that, with $\nu = \mu/\rho$, the variables

$$\eta = \frac{y}{\sqrt{\nu t}} \quad \text{and} \quad \phi = \frac{u(y,t)}{At^m} = \frac{u}{U_p}$$

reduce the equation governing the motion to

$$\frac{d^2\phi}{d\eta^2} + \frac{\eta}{2}\frac{d\phi}{d\eta} - m\phi = 0 \quad \text{subject to} \quad \phi(0) = 1 \text{ and } \phi(\infty)=0$$

4.32) Air flows in a square duct having width D. This duct is connected to a square-sectioned right-angled bend having inner radius $r = R_1$ and outer radius $r = R_2 = R_1 + D$. Pressure taps are installed on the bend sides along the 45° radius. By assuming that the flow is incompressible and inviscid and that, along the 45° radius, the streamlines are circular arcs concentric width the bend centre, relate the pressure difference $\Delta p = p_2 - p_1$ to the volume flux Q through the bend, the air density ρ and the bend geometry. For this problem, omit body forces, assume that the flow upstream of the bend is uniform at speed $U = Q/D^2$ and at some constant pressure p_∞, and that the pressure gradient along the 45° radius balances the centrifugal acceleration. With $q = q(r)$, this last assumption imposes $\partial p/\partial r = \rho q^2/r$.



4.33) For compressible flow through a nozzle as described in Section 4.9, derive the expression for mass flux corresponding to isothermal flow, namely equation 4.9.6. Also obtain the value of p_T/p_0 at which the mass flux m_f is a maximum.

4.34) For compressible one-dimensional frictionless adiabatic steady channel flow of a prefect gas with $p = \rho RT$ and constant specific heat ratio γ show that, in the presence of terrestrial gravity, the energy equation takes the form

$$1 + \frac{\gamma - 1}{2} M^2 = \left[1 - \frac{\gamma - 1}{\gamma} \frac{gz}{RT_0} \right] \frac{T_0}{T}$$

For air flowing from a reservoir at $T_0 = 300$ °K through a 10 metre long nozzle which has its axis vertical, find the error in T at the nozzle exit caused by omitting gravity if the flow Mach number $M = 1$ at the exit.

4.35) Section 4.9 defines a Reynolds Number for compressible flow through a choked nozzle based on the reservoir conditions and the throat diameter, thus:

$$Re_0 \equiv \frac{d_T V_{ref} \rho_0}{\mu_0} = \frac{d_T \sqrt{p_0 \rho_0}}{\mu_0} = \frac{d_T p_0}{\mu_0 \sqrt{R T_0}}$$

This choice reflected the exploratory nature of the development. With the flow being established as frictionless and adiabatic, it might seem more appropriate to base the Reynolds number on choked flow conditions at the throat. Assuming inviscid adiabatic flow of a perfect gas with constant specific heats as described in Example 4.9.1, and assuming that $\mu_{air} \propto T^{0.73}$ as in Table 4.9.1, define a Reynolds Number $Re_T = d_T V_T \rho_T / \mu_T$ corresponding to choked conditions, and express it in terms of Re_0, γ and any other parameters involved. For the data in Table 4.9.1, use Re_T to test the validity of using Re_0.