University of Toronto
Faculty of Applied Science and Engineering
FINAL EXAMINATION – April, 2017

FIRST YEAR – ENGINEERING SCIENCE
Program 5

MAT195S CALCULUS II

Examiners: F. Al Faisal and J. W. Davis

Instructions:
(1) Closed book examination; no calculators, no aids are permitted
(2) Write your name and student number in the space provided below.
(3) Answer as many questions as you can. Parts of questions may be answered.
(4) Use the overleaf side of pages for additional or preliminary work.
(5) Do not separate or remove any pages from this exam booklet.

Family Name: ______________________

Given Name: ______________________

Student #: ______________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	12	
2	10	
3	13	
4	10	
5	8	
6	11	
7	8	
8	5	
9	8	
10	10	
11	13	
12	10	
TOTAL	118	

1) Evaluate the integrals: a) $\int \frac{\sin\phi}{\cos^3\phi} d\phi$

b) $\int e^{-\theta} \cos 2\theta \, d\theta$

c) $\int_0^1 \frac{x-4}{x^2-5x+6} dx$

(12 marks)

2) Evaluate the integral $\int_0^\infty \frac{\sqrt{x}\ln x}{(1+x)^2}dx$ by taking the following steps:

a) Integrate by parts with $u = \sqrt{x}\ln x$

b) Show that $\int_0^1 \frac{\ln x}{\sqrt{x}(x+1)}dx = -\int_1^\infty \frac{\ln x}{\sqrt{x}(x+1)}dx$. Hint: Change variables by letting $y = 1/x$

c) Evaluate the remaining integral using the change of variables $z = \sqrt{x}$

(10 marks)

3) a) Sketch the polar curve: $r = 1 + 2\sin(3\theta)$.

b) Find the area enclosed by the three big leaves and outside the three small leaves.

(13 marks)

4) a) Find the arc length of the curve $x = \frac{1}{2}\ln(t^2+1)$, $y = \arctan t$, from $t = 0$ to $t = 1$.

(5 marks)

b) Show that the radius of curvature of the curve $y = e^x$ is smallest at the point $x = (-\ln 2)/2$.

(5 marks)

5) Show that the sequence defined by $a_1 = 2$, $a_{n+1} = \dfrac{1}{3 - a_n}$ satisfies $0 < a_n \le 2$ and is decreasing.
Deduce that the sequence is convergent, and find its limit.

(8 marks)

6) a) Find the interval of convergence of the power series: $\sum_{n=1}^{\infty} \frac{(n+2)(x-1)^n}{(n+3)2^n}$

(5 marks)

b) Determine if the following series converge or diverge: i) $\sum_{n=1}^{\infty} \frac{n^n}{(3n)!}$

ii) $\sum_{k=1}^{\infty} \left(\frac{\ln k}{k}\right)^k$

(6 marks)

7) Derive, from first principles (that is, by finding the derivatives), the Taylor series centered at $x = \frac{\pi}{2}$ for the function $f(x) = \cos x$. For what values of x does the series converge?

(8 marks)

8) Suppose $f(x)$ is integrable on $[a,b]$ and let $g(x)$ be a function with the property that the set {x in $[a,b] \mid f(x) \neq g(x)$} is finite. Must $g(x)$ be integrable? If yes, prove your claim; if not, provide an example.

(5 marks)

9) Consider a particle moving in the plane with displacement vector: $\vec{r}(t) = (t^2, t^3)$

a) Find the velocity and acceleration vectors, $\vec{v}(t)$ and $\vec{a}(t)$, for this particle.

b) Find the magnitudes of the normal and tangential components of the acceleration vector, a_N and a_T.

c) Evaluate the components of the acceleration vector tangential, $a_T\vec{T}$, and normal, $a_N\vec{N}$, to the particle path at $t = 1$.

(8 marks)

10) a) Let $f(x,y)$ be differentiable function, and (x_0, y_0) be a point on the level curve $f(x,y) = k$. Show that the gradient vector $\nabla f(x_0, y_0)$ is perpendicular to the level curve.

(5 marks)

b) Find the point on the surface $z = 16 - 4x^2 - y^2$ at which the tangent plane is perpendicular to the line $x = 3 + 4t$, $y = 2t$, $z = 2 - t$.

(5 marks)

11) a) For what value(s) of k, if any, does the function $f(x,y) = x^2 + kxy + 4y^2$ have a local minimum at (0,0)?

(7 marks)

11) b) If α, β, and γ are the angles of a triangle, use Lagrange multipliers to show: $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leq \frac{1}{8}$. For what triangle does equality hold?

(6 marks)

12) Let F_n be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$.

a) Show that $F_{n+1}/F_n \leq 2$ for all n.

b) Show that the power series $f(x) = \sum_{n=1}^{\infty} F_n x^n$ converges for all $|x| < \frac{1}{2}$, and in fact $f(x) = x/(1 - x - x^2)$ for $|x| < \frac{1}{2}$.

c) Show that $F_n = \dfrac{\left(\left(1+\sqrt{5}\right)/2\right)^n - \left(\left(1-\sqrt{5}\right)/2\right)^n}{\sqrt{5}}$

Hint: Use the result from part (b) to obtain another power series expansion for *f(x)*.

(10 marks)