Feb 19, 2015

Term Test #1 — Soln (J.W.)

1 Ⓐ $E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad \Delta p \Delta x \gtrsim \frac{\hbar}{2} \Rightarrow p \cdot x \sim \frac{\hbar}{2}$

$$= \frac{\hbar^2}{8mx^2} + \frac{1}{2}kx^2$$

$$\frac{\partial E}{\partial x} = 0 = -\frac{\hbar^2}{4mx^3} + kx \quad \Rightarrow \quad x^2 = \frac{\hbar}{2\sqrt{mk}}$$

$$\Rightarrow \quad E_{min} = \frac{\hbar^2}{8m}\frac{2\sqrt{mk}}{\hbar} + \frac{1}{2}\frac{k}{2}\frac{\hbar}{\sqrt{mk}} = \left(\frac{1}{4}+\frac{1}{4}\right)\hbar\sqrt{\frac{k}{m}} = \frac{1}{2}\hbar\omega_0$$

Ⓑ $\psi = Ae^{-\frac{ax^2}{2}}, \quad a = \frac{\sqrt{mk}}{\hbar}$

$$\frac{d\psi}{dx} = -ax\psi, \quad \frac{d^2\psi}{dx^2} = -a\left[\psi + x\frac{d\psi}{dx}\right] = -a\psi - a^2x^2\psi$$

Let $H = \frac{p^2}{2m} + \frac{kx^2}{2}$; $p = -i\hbar\frac{\partial}{\partial x} \Rightarrow p^2 = -\hbar^2\frac{\partial^2}{\partial x^2}$

$$H\psi = -\frac{\hbar^2}{2m}\left(-a - a^2x^2\right)\psi + \frac{1}{2}kx^2\psi$$

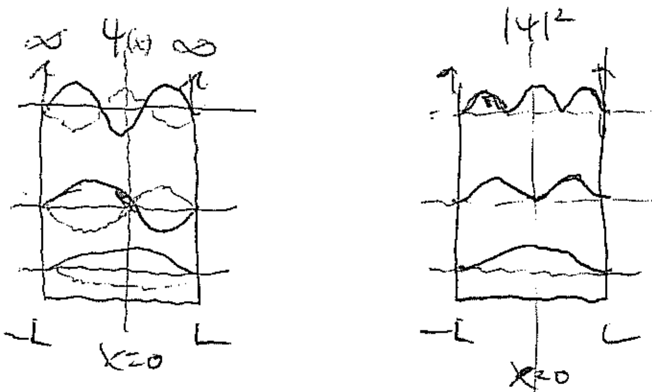
$$= -\frac{\hbar^2}{2m}\left[-\frac{\sqrt{mk}}{\hbar} + \frac{mk}{\hbar^2}x^2 + \frac{1}{2}kx^2\right]\psi$$

$$= \left[\frac{\hbar}{2}\sqrt{\frac{k}{m}} - \frac{kx^2}{2} + \frac{kx^2}{2}\right]\psi$$

$$\therefore H\psi = \underbrace{\left(\frac{\hbar}{2}\omega_0\right)}_{E_0}\psi$$

a constant

$\boxed{2}$ (A)



$\frac{d^2\psi}{dx^2} = -k^2\psi$, $k = \sqrt{2mE}/\hbar$ inside the well

ψ must be $A\sin kx$ or $A\cos kx$, depending on parity

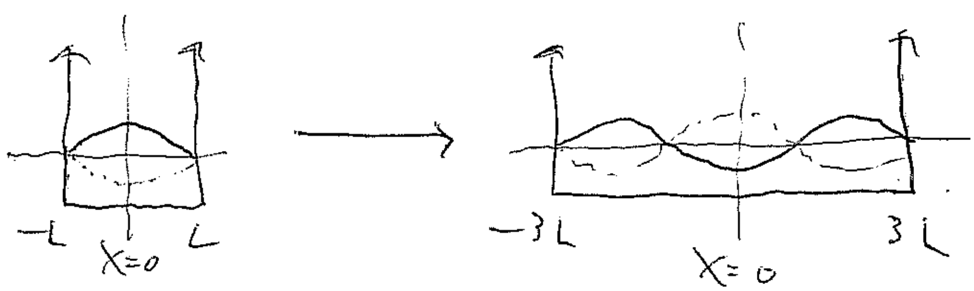
$\int|\psi|^2dx = 1 \Rightarrow A$; $\underset{\psi(x=\pm L)=0}{\overset{\Uparrow}{\pm kL = n\pi}} \Rightarrow k, E$

$\psi_1(x) = \sqrt{\frac{2}{2L}}\cos\frac{\pi x}{2L}$ $\quad E_1 = \frac{h^2}{8m(2L)^2}$

$\psi_2(x) = \sqrt{\frac{2}{2L}}\sin\frac{2\pi x}{2L}$ $\quad E_2 = \frac{4h^2}{8m(2L)^2}$

$\psi_3(x) = \sqrt{\frac{2}{2L}}\cos\frac{3\pi x}{2L}$ $\quad E_3 = \frac{9h^2}{8m(2L)^2}$

(B)



$\psi_{i=1}(x) = \sqrt{\frac{2}{2L}}\cos\frac{\pi x}{2L}$ $\quad \psi_{f=3}(x) = \sqrt{\frac{2}{6L}}\cos\frac{3\pi x}{6L}$

$\psi_i(x) = \sum_{f=1}^{\infty} b_f\psi_f(x)$, $\underset{\overset{\|}{3}}{b_f} = \int_{-L}^{L}\underset{\overset{\|}{1}}{\psi_i(x)}\,\underset{\overset{\|}{3}}{\psi_f(x)}\,dx$

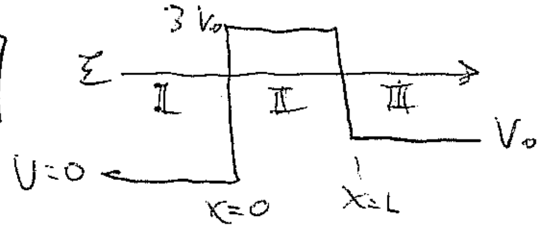
$b_{f=3} = \sqrt{\frac{1}{L}}\sqrt{\frac{1}{3L}}\int_{-L}^{L}\left[\cos\frac{\pi x}{2L}\cos\frac{\pi x}{2L}\right]dx$

$= \frac{1}{\sqrt{3}L}\left[\int_{-L}^{L}\frac{1}{2}dx + \int_{-L}^{L}\frac{1}{2}\cos\frac{2\pi x}{2L}dx\right]$

$= \frac{1}{\sqrt{3}L}\left[\left(\frac{L}{2}+\frac{L}{2}\right)+0\right] = \frac{1}{\sqrt{3}}$

$\therefore\ P = |b_{f=3}|^2 = \frac{1}{3}$

$\boxed{3}$



$$\psi'' = \frac{d^2\psi}{dx^2}$$
$$\psi' = \frac{d\psi}{dx}$$

(A)

$$\text{I:}\quad \psi'' = -k_1\psi \;,\quad k_1 = \frac{\sqrt{2mE}}{\hbar} \;\Rightarrow\; \psi_I(x) = Ae^{+ik_1x} + Be^{-ik_1x}$$

$$\text{II:}\quad \psi'' = +\alpha\psi \;,\quad \alpha = \frac{\sqrt{2m(3V_0 - E)}}{\hbar} \;\Rightarrow\; \psi_{II}(x) = Ce^{+\alpha x} + De^{-\alpha x}$$

$$\text{III:}\quad \psi'' = -k_3\psi \;,\quad k_3 = \frac{\sqrt{2m(E - V_0)}}{\hbar} \;\Rightarrow\; \psi_{III}(x) = Fe^{+ik_3x}$$

(B)

$$\psi_I(0) = \psi_{II}(0)$$
$$\psi_I'(0) = \psi_{II}'(0)$$
$$\psi_{II}(L) = \psi_{III}(L)$$
$$\psi_{II}'(L) = \psi_{III}'(L)$$

$$T = \frac{J_{transmitted}}{J_{incident}} = \frac{|\psi|^2_{trans}\, v_{transmitted}}{|\psi|^2_{incid}\, v_{incident}}$$

$$= \frac{|F|^2\, \hbar k_3/m}{|A|^2\, \hbar k_1/m} \qquad \frac{k_3}{k_1} = \frac{\sqrt{E - V_0}}{\sqrt{E}}$$

$$\therefore\quad T = \frac{|F|^2}{|A|^2}\left(1 - \frac{V_0}{E}\right)^{\frac{1}{2}}$$

4. (A) $\left.\begin{array}{l}\lambda_D = h/p \\ \lambda_c = h/m_e c\end{array}\right\} \Rightarrow \left.\begin{array}{l}p = m_e c \\ p = \gamma m_e v\end{array}\right\} \Rightarrow \gamma = \left(\frac{v}{c}\right)^{-1}$

$\left.\begin{array}{l}\beta \equiv v/c \\ \gamma \equiv (1-\beta^2)^{-1/2}\end{array}\right\} \Rightarrow 1-\beta^2 = \beta^2 \Rightarrow \beta = \sqrt{\frac{1}{2}} \quad \boxed{v = \frac{\sqrt{2}}{2} c}$

$\left.\begin{array}{l}\gamma = 1/\beta = \sqrt{2} \\ KE = (\gamma - 1) m_e c^2\end{array}\right\} \Rightarrow \boxed{KE = (\sqrt{2}-1) m_e c^2}$

(B) $\psi = A e^{-\frac{x^2}{2\sigma^2} + i\frac{p_0}{\hbar}x} \qquad \psi^* = A e^{-\frac{x^2}{2\sigma^2} - i\frac{p_0}{\hbar}x}$

$\frac{d\psi}{dx} = \left(-\frac{2x}{2\sigma^2} + i\frac{p_0}{\hbar}\right)\psi \qquad \frac{d\psi^*}{dx} = \left(-\frac{2x}{2\sigma^2} - i\frac{p_0}{\hbar}\right)\psi^*$

$\psi \equiv \psi(x)$

$\Rightarrow j(x) = -\frac{i\hbar}{2m}\left[\psi^*\left(-\frac{x}{\sigma^2} + \frac{ip_0}{\hbar}\right)\psi - \psi\left(-\frac{x}{\sigma^2} - \frac{ip_0}{\hbar}\right)\psi^*\right]$

$= -\frac{i\hbar}{2m}\left[\psi^*\psi\,\frac{ip_0}{\hbar} + \psi\psi^*\,\frac{ip_0}{\hbar}\right]$ Note: $\psi^*\psi = \psi\psi^*$ $\because e^{i\theta}e^{-i\theta} = 1$

$= -\frac{i^2}{2m}\, 2|\psi|^2 p_0 \quad \therefore \boxed{j(x) = |\psi|^2 \frac{p_0}{m}}$

Method #1: $\langle p \rangle = A^2 \int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma^2} - i\frac{p_0}{\hbar}x} (i\hbar)\left[-\frac{2x}{2\sigma^2} + i\frac{p_0}{\hbar}\right] e^{-\frac{x^2}{2\sigma^2} + i\frac{p_0}{\hbar}x}$

$= A^2\left[-\int_{-\infty}^{\infty} dx\, \underbrace{\frac{x}{\sigma^2}}_{\text{odd}}\, \underbrace{e^{-\frac{x^2}{\sigma^2}}}_{\text{even}} - i\hbar \int_{-\infty}^{\infty} dx\, \frac{ip_0}{\hbar} e^{-\frac{x^2}{\sigma^2}}\right]$

$\Rightarrow$ 1st $\int_{-\infty}^{\infty}$ vanishes

$= A^2[-(i)^2]\, p_0 \underbrace{\int_{-\infty}^{\infty} e^{-\frac{x^2}{\sigma^2}} dx}_{= A^{-2}\ \because \text{Normalization}} \Rightarrow \boxed{\langle p \rangle = p_0}$

Method #2: $j(x) = -\frac{i\hbar}{2m}\left[\psi^* \frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right]$ $\because p_{op} \equiv -i\hbar\frac{\partial}{\partial x}$

$= +\frac{1}{2m}\left[\psi^* p_{op}\psi - \psi p_{op}\psi^*\right]$

$\int_{-\infty}^{\infty} j(x) = \frac{1}{2m}\int_{-\infty}^{\infty} \psi^* p_{op}\psi\, dx - \frac{1}{2m}\int_{-\infty}^{\infty}\psi p_{op}\psi^*\, dx$ $dx \to -dx'$

From 1st part of (B)

$= \quad " \quad + \frac{1}{2m}\int_{-\infty}^{\infty}\psi^* p_{op}\psi\, dx$

$\int_{-\infty}^{\infty} |\psi|^2 \frac{p_0}{m} dx = \frac{1}{2m}\langle p \rangle \cdot 2 \Rightarrow \boxed{\langle p \rangle = p_0}$