Email ______________________________ @mail.utoronto.ca

Signature ______________________________

- Time allotted: 110 minutes
- **DO NOT OPEN** until instructed to do so.
- **NO CALCULATORS ALLOWED**, and no cellphones or other electronic devices.
- **DO NOT DETACH ANY PAGES**. This test contains 9 pages (including this title page). Once the test starts, make sure you have all of them.
- You can use pages ??–9 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

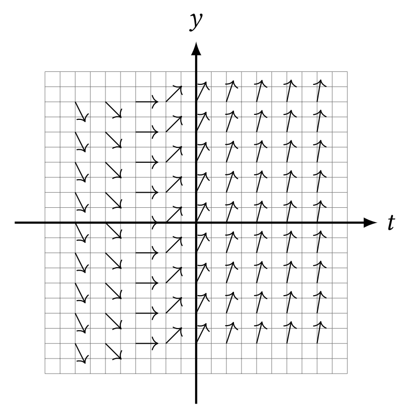
SECTION I **No justification necessary.**

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

1. Each of the following plots shows the direction field of an ODE of the form $\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y)$. Classify the ODEs based on its corresponding plot.

(a) (8 marks)

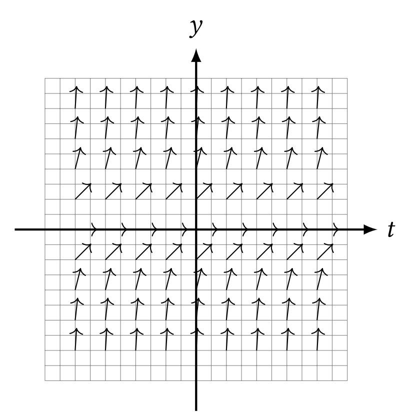
☐ autonomous	☐ nonautonomous	☐ can't deduce
☐ linear	☐ nonlinear	☐ can't deduce
☐ homogeneous	☐ nonhomogeneous	☐ can't deduce
☐ separable	☐ non separable	☐ can't deduce



Solution: nonautonomous, linear, nonhomogeneous, separable.

(b) (8 marks)

☐ autonomous	☐ nonautonomous	☐ can't deduce
☐ linear	☐ nonlinear	☐ can't deduce
☐ homogeneous	☐ nonhomogeneous	☐ can't deduce
☐ separable	☐ non separable	☐ can't deduce



Solution: autonomous, nonlinear, homogeneous, separable.

2. **(2 marks)** Let $A(t)$ be a 2×2 matrix with continuous entries and $F(t)$ a 2×1 matrix with discontinuous entries on a common interval I containing t_0. The initial value problem

$$X' = AX + F, \quad X(t_0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

may have a unique solution on I.
Solution: True, because the continuity condition is only sufficient not necessary.

3. **(2 marks)** All solutions of the system $X' = AX$ converge to the origin $(0, 0)$ as $t \to \infty$. The matrix A must have a repeated negative eigenvalue.
Solution: False. If both eigenvalues are negative, then the solution decays to zero at infinity.

4. **(2 marks)** Euler's method is also known as first-order Runge-Kutta method. If h is the size step, then the associated local truncation error is proportional to h^5.
Solution: False. The local truncation error for Euler's method is proportional to h^2.

5. **(2 marks)** The following three vectors form a fundamental set on the interval $(-\infty, \infty)$.

$$X_1 = \begin{pmatrix} 6 \\ -1 \\ -5 \end{pmatrix} e^{-t}, \quad X_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} e^{-2t}, \quad X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} e^{3t}.$$

Solution: True. The Wronskian of the three vectors is nonzero for all values of t.

SECTION II **Justify all your answers.**

6. In a remote desert region, a dedicated team of scientists has undertaken a perplexing investigation involving an enigmatic extraterrestrial craft. This peculiar aircraft, commonly referred to as a "tic tac" due to its distinctive shape, possesses a remarkable capability: it can deploy advanced alien technology enabling it to split into two separate entities.

(a) **(2 marks)** **First Contact**: During their initial encounter with the tic tac, scientists meticulously scrutinized its movements and found it was governed by the following ODE:

$$x'(t) + \exp\left\{t^2\right\}x(t) = \sin\left(t^3\right).$$

Here, the function $x(t)$ represents the altitude of the aircraft over time. Given precise altitude data at a specific moment, can the scientists confidently predict whether the tic tac will undergo a split a minute later?

Solution: By linear existence and uniqueness theorem (need to check conditions), there exists a unique trajectory and therefore the scientists can predict the tic tac does not split.

(b) **(2 marks)** **Second Contact**: In a subsequent observation of the tic tac, researchers closely monitored its flight path, which followed this time the ODE:

$$x'(t) = x^2\sin(t) + \exp\{t\ x(t)\}.$$

Given precise altitude data at a specific moment, for how long can the scientists confidently guarantee that the tic tac will not undergo a split?

Solution: By nonlinear existence and uniqueness theorem (need to check conditions), we can guarantee that the tic tac does not split from that specific time t_s up to $t_s + h$ for some $h > 0$.

(c) **(2 marks)** **Third Contact**: A third sighting occurred right after the tic tac split into two entities. The behavior of these two resulting entities appeared to be described by a system of coupled differential equations:

$$\frac{dx_1}{dt} = -x_1 + 2x_2 + t$$
$$\frac{dx_2}{dt} = 3x_1 - x_2 + t^2.$$

Here, $x_1(t)$ and $x_2(t)$ represent the altitudes of the two entities. Given precise altitude measurements for both entities at a specific moment, can the scientific team confidently predict whether the tic tacs will undergo a split a minute later?

Solution: By existence and uniqueness theorem for linear system of ODEs (need to check conditions), there exists a unique trajectory and therefore the scientists can predict the tic tac will not split.

7. An agricultural research station recruits a team of engineers to support farmers in improving their food crops production. Farmers report plants injuries in some part of the field due to a colony of insects (weevils, maggots, etc.). The following mathematical model describing the evolution of the area A (in *ha*) occupied by the insects over time t (in months) is proposed,

$$\frac{dA}{dt} = A(2 - A).$$

Assume the initial area is $A_0 = 0.5$ *ha*. The team would like to approximate the area at certain times.

(a) **(2 marks)** Using Euler's method with step size $h = 1$, find an approximation of the occupied area after a period of 2 months (the answer can be given as a fraction).
Solution: After two steps approximations, we find $A_2 = \frac{35}{16}$ *ha*.

(b) **(2 marks)** How can one choose the step size h to make sure the local truncation error is not exceeding $\varepsilon = 10^{-6}$ if $M = 1$ is an upper-bound for A''?
Solution: Since the local truncation error satisfies $|e_n| \leq \frac{Mh^2}{2}$, it does not exceed 10^{-6} if $h < \sqrt{2 \times 10^{-6}} = \sqrt{2} \times 10^{-3}$.

(c) **(2 marks)** Approximate the occupied area after a period of one month using the *improved* Euler's method with step size $h = 1$.
Solution: One step approximation gives $A_1 = \frac{43}{32}$ *ha*.

(d) **(2 marks)** From the model, can the engineering team predict accurately whether or not the occupied area will expand or shrink over time? Justify your answer.
Solution: The system has two equilibrium points $A = 0$ and $A = 2$. Since the initial area is $0.5 \in (0, 2)$, the area will expand but will not exceed 2 *ha* over time. This can be seen from the phase portrait of the ODE.

SECTION III **Justify all your answers.**

8. Scientists have developed a new deep sea exploration submarine called "Abyss Explorer." The exploration starts at the surface of the ocean, $x = 0$, with velocity v_0. As the submarine dives deeper, x increases. As the submarine descends, its acceleration is influenced by the temperature difference between its internal systems and the surrounding ocean water. The acceleration $x''(t)$ is equal to the rate of change of the external temperature with respect to the depth $x(t)$ so that:

$$x''(t) = \frac{\mathrm{d}T_{\mathrm{ext}}}{\mathrm{d}x},$$

where

$$T_{\mathrm{ext}}(x) = T_0 + 0.5\exp(-x(t)),$$

and T_0 is a constant.

(a) **(1 mark)** Derive an expression for $x''(t)$ in terms of $x(t)$.
Solution: Differentiating T_{ext} with respect to x, we get:

$$\frac{\mathrm{d}T_{\mathrm{ext}}}{\mathrm{d}x} = -0.5\exp(-x)$$

So,

$$x''(t) = -0.5\exp(-x(t))$$

(b) **(2 marks)** Reduce the second order ODE in the previous item to a first order ODE in terms of the velocity of the submarine v and its depth x.
Solution: Using the chain rule, differentiate $x'(t)$ with respect to t:

$$x''(t) = \frac{\mathrm{d}x'(t)}{\mathrm{d}t} = x'(t)\frac{\mathrm{d}x'(t)}{\mathrm{d}x}$$

Equating the two expressions for $x''(t)$ and rearranging, we get:

$$x'(t)\frac{\mathrm{d}x'(t)}{\mathrm{d}x} = -0.5\exp(-x(t)).$$

(c) **(2 marks)** Derive the solution to the ODE in the previous item. Hint: it is equal to

$$v(x) = \pm\sqrt{C + \exp\{-x\}}$$

for some constant C
Solution:

(d) (2 marks) Explain what does the plus minus correspond to physically in terms of the submarine.
Solution: The submarine first descends and then ascends.

(e) (2 marks) Find the constant C and substitute it into the solution of the ODE.
Solution: Using the initial conditions,

$$v(0) = v_0 = \sqrt{C + \exp\{-0\}} = \sqrt{C + 1}$$

we get

$$C = v_0^2 - 1.$$

Therefore,

$$v(x) = \pm\sqrt{v_0^2 - 1 + \exp\{-x\}}$$

(f) (2 marks) What is the maximal depth that the submarine will reach?
Solution: At the maximal depth, the velocity is zero:

$$0 = \pm\sqrt{v_0^2 - 1 + \exp\{-x_{\max}\}}$$

and therefore

$$0 = v_0^2 - 1 + \exp\{-x_{\max}\}.$$

Rearranging the equation, we get

$$-\ln\left(1 - v_0^2\right) = x_{\max}.$$

(g) (2 marks) Under what conditions will the submarine reach the bottom of the ocean?
Solution: The maximal depth $x_{\max}$ tends to infinity as v_0 tends to one.

9. An automotive manufacturer is producing a series of cars designed for motorsports (off-road racing). The engineering team would like to build sophisticated suspensions to prevent under- or overreaction when cars encounter obstacles. They decide to model the problem as a spring-mass system. Newton's second law gives the equation for the directed distance $x(t)$ of the mass (car) beyond the equilibrium position when the system is set in motion,

$$x'' + 2\omega d x' + \omega^2 x = 0 \tag{E}$$

where the physical parameters $\omega > 0$ and $d \geq 0$ represent the frequency and the damping ratio, respectively.

(a) **(2 marks)** Convert System (E) into a first-order homogeneous linear system (Write your answer in matrix form).
Solution: The system is of the form

$$X' = AX \text{ where } A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2d\omega \end{pmatrix}.$$

(b) **(2 marks)** For which values of ω, d does the system found in **(a)** have real and distinct eigenvalues? complex eigenvalues? repeated eigenvalues?
Solution: The characteristic equation of the matrix is $\lambda^2 + 2d\omega\lambda + \omega^2 = 0$. Thus, the matrix A has

- distinct eigenvalues if $4\omega^2(d^2 - 1) > 0$ or $w > 0$ and $d > 1$,
- complex eigenvalues if $\omega > 0$ and $d < 1$,
- repeated eigenvalues if $\omega > 0$ and $d = 1$.

(c) **(2 marks)** Assume $\omega = d = 1$. Write down the general solution of the system obtained in **(a)**.
Solution: When $d = \omega = 1$, we have a repeated eigenvalue $\lambda = -1$ and its associated eigenvector is $V = (1, -1)^T$. Find the generalized eigenvalue V' by solving

$$(A + I)V' = V \implies \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies V' = (0, 1)^T$$

The general solution is given by

$$X(t) = \alpha_1 e^{-t}(1, -1)^T + \alpha_2 t e^{-t}(1, -1)^T + \alpha_2 e^{-t}(1, 0)^T.$$

(d) **(2 marks)** In the previous item, does the system have an equilibrium point? If yes what is its type and stability?
Solution: The origin $(0,0)^T$ is the only equilibrium point. It is an improper asymptotically stable node.

(e) **(2 marks)** Compute the exponential matrix e^{At} where A is the matrix defining the system in **(a)** with the parameters in item **(c)**.
Solution: The eigenspace has dimension 1 and whence, the matrix A not diagonalizable. However, A can be decomposed as follows:

- $A = D + N$, D is diagonal 2×2 matrix and N is such that $N^2 = \mathbf{0}$.

- $ND = DN$

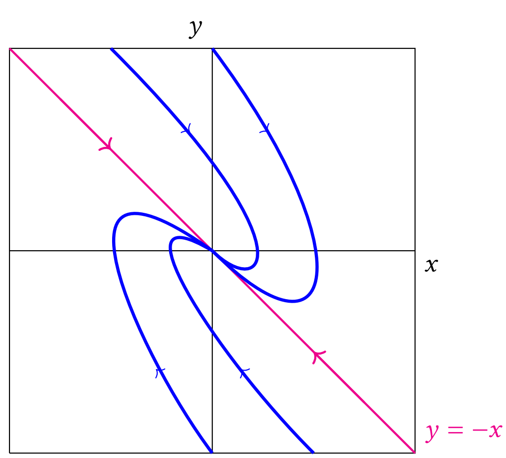
This follows from a standard result in linear algebra. In our case, we observe that

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} = D + N; \quad D = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$

We have $e^{Dt} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}$ since D is diagonal and $e^{Nt} = \sum_{k=0}^{\infty} \frac{t^k N^k}{k!} = I + tN = \begin{pmatrix} t+1 & t \\ -t & -t+1 \end{pmatrix}$. Moreover, N and D commutes so that

$$e^{At} = e^{Dt}e^{Nt} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} t+1 & t \\ -1 & -t+1 \end{pmatrix} = \begin{pmatrix} (t+1)e^{-t} & te^{-t} \\ -e^{-t} & e^{-t}(1-t) \end{pmatrix}.$$

(f) **(2 marks)** Draw the phase portrait of the system with the parameters in item **(c)**. Explain why the choice of these parameters is reasonable for the engineering team.
Solution: From item **(c)**, we see that when $\alpha_1 \neq 0$ and $\alpha_2 = 0$, the solution is $\alpha_1 e^{-t} V$ which is the line $y = -x$ in the phase plane. All solutions approach the origin $(0,0)$ as $t \to \infty$ while being tangent to this line because $\lim_{t\to\infty} \frac{dy}{dx} = -1$. The limit as $t \to -\infty$ is also the same meaning that the slope of the curve far away from the origin tends to -1. Moreover, if $\alpha_2 > 0$ and $t > 0$ (resp. $\alpha_2 < 0$), the curve is always above (resp. below) the line $y = -x$.



The choice of the parameters is reasonable because the equilibrium point of the dynamical system associated to (E) is an attractor.