ECE259: Electromagnetism

Vector calculus quiz - January 29th, 2024
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This is the first version "A" of the quiz with "vector-calculus-quiz" written near the QR code

SOLUTION

Marking scheme:
0.5 points for choosing the correct answer
0.5 points for the justification

Instructions

- Duration: 15 minutes
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**
- Please write with a **dark** pen or pencil. This test will be scanned.

Question 1 [1 pt]

Consider the closed surface S made by the union of the following two surfaces:

$$S_1 : R = 1 \quad \theta \in [0, \pi/2] \quad \varphi \in [0, 2\pi] \tag{1}$$

$$S_2 : R \in [0, 1] \quad \theta = \pi/2 \quad \varphi \in [0, 2\pi] \tag{2}$$

The normal vector to the surface, pointing **outwards** is:

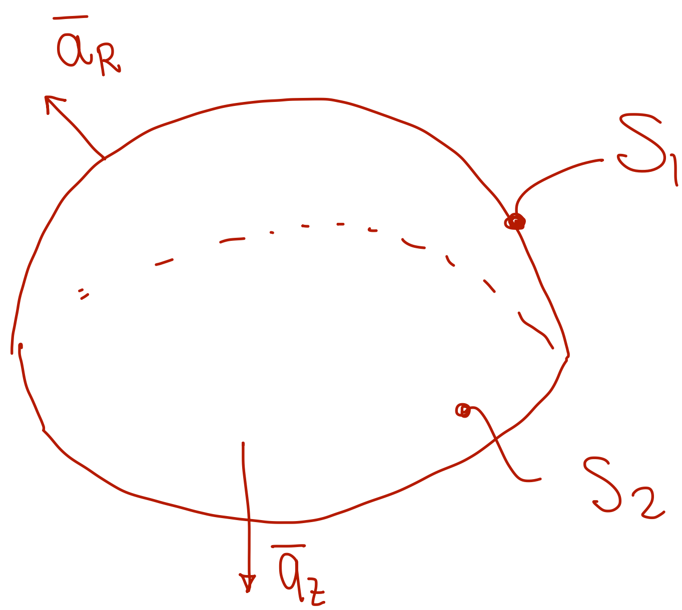
(a) $+\mathbf{a}_R$ on S_1, $+\mathbf{a}_z$ on S_2

(b) $+\mathbf{a}_R$ on S_1, $-\mathbf{a}_z$ on S_2

(c) $-\mathbf{a}_R$ on S_1, $+\mathbf{a}_z$ on S_2

(d) $-\mathbf{a}_R$ on S_1, $-\mathbf{a}_z$ on S_2

Please justify your answer by sketching the two surfaces and the normal vectors on S_1 and S_2.



Question 2 [1 pt]

The unit vector $\mathbf{a}_\theta$ is equal to:

(a) $-\sin\varphi\,\mathbf{a}_x + \cos\phi\mathbf{a}_y$;

(b) $\cos\varphi\,\mathbf{a}_x + \sin\phi\mathbf{a}_y$;

(c) $\cos\theta\cos\varphi\,\mathbf{a}_x + \cos\theta\sin\varphi\,\mathbf{a}_y - \sin\theta\,\mathbf{a}_z$;

(d) $\sin\theta\cos\varphi\,\mathbf{a}_x + \sin\theta\sin\varphi\,\mathbf{a}_y + \cos\theta\,\mathbf{a}_z$;

Please provide a brief justification.

$$\bar{a}_\theta = (\bar{a}_\theta \cdot \bar{a}_x)\,\bar{a}_x + (\bar{a}_\theta \cdot \bar{a}_y)\,\bar{a}_y + (\bar{a}_\theta \cdot \bar{a}_z)\,\bar{a}_z =$$

$$= \cos\theta\cos\varphi\,\bar{a}_x + \cos\theta\sin\varphi\,\bar{a}_y - \sin\theta\,\bar{a}_z$$

Question 3 [1 pt]

Let S be the boundary of the volume $r \in [0, 1]$, $\varphi \in [0, \pi/2]$, $z \in [0, 1]$.

Calculate the **outward** flux through S of the vector field $\mathbf{F} = \varphi \mathbf{a}_\varphi$.

The flux is equal to:

(a) $\int_S \mathbf{F} \cdot \mathbf{dS} = \pi/2$

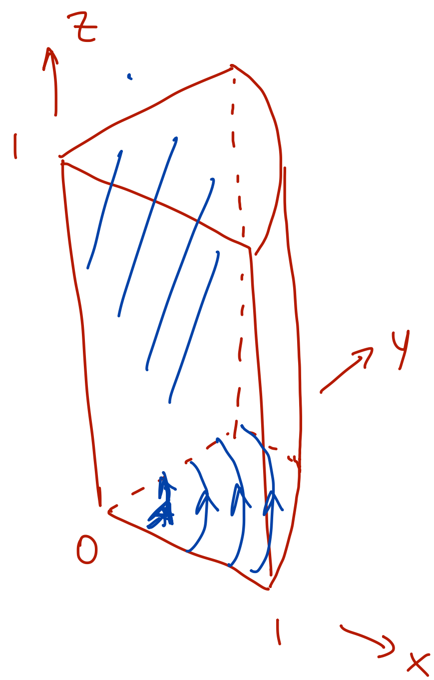
(b) $\int_S \mathbf{F} \cdot \mathbf{dS} = -\pi/2$

(c) $\int_S \mathbf{F} \cdot \mathbf{dS} = 0$

(d) $\int_S \mathbf{F} \cdot \mathbf{dS} = \pi$

(e) $\int_S \mathbf{F} \cdot \mathbf{dS} = -\pi$

Please justify your answer.



- No flux on upper and lower bases, and on side face since $\bar{F} \cdot \bar{dS} = 0$
- No flux on face on $y = 0$ since $\bar{F} = 0$ for $\varphi = 0$
- On face on $x = 0$ $\Rightarrow$ $\bar{F} = \frac{\pi}{2} \bar{a}_\varphi$ $\qquad \bar{dS} = \bar{a}_\varphi \, dr\, dz$

$$\int \bar{F} \cdot \bar{dS} = \int_{r=0}^{1} \int_{z=0}^{1} \frac{\pi}{2} \bar{a}_\varphi \cdot \bar{a}_\varphi \, dr\, dz = \frac{\pi}{2}$$

Question 4 [1 pt]

Consider two points P_1 and P_2 with coordinates

$$P_1 = (r = 5, \varphi = +\pi/2, z = 1) \quad (3)$$
$$P_2 = (x = 2, y = +2, z = 2) \quad (4)$$

Which expression is correct for the distance vector $\mathbf{d}$ which goes from P_1 to P_2?

(a) $\mathbf{d} = (2\sqrt{2} - 5)\mathbf{a}_r + \mathbf{a}_z$

(b) $\mathbf{d} = (5 - 2\sqrt{2})\mathbf{a}_r - \mathbf{a}_z$

(c) $\mathbf{d} = +2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$

(d) $\mathbf{d} = -2\mathbf{a}_x + 3\mathbf{a}_y - \mathbf{a}_z$

Please provide a brief justification.

since $\varphi = \frac{\pi}{2}$

$$\bar{R}_1 = 5\bar{a}_r + \bar{a}_z = 5\bar{a}_y + \bar{a}_z$$

$$\bar{R}_2 = 2\bar{a}_x + 2\bar{a}_y + 2\bar{a}_z$$

$$\bar{d} = \bar{R}_2 - \bar{R}_1 = 2\bar{a}_x - 3\bar{a}_y + \bar{a}_z$$

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Differential operators

4.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial R}\vec{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

4.2 Divergence

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

4.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

4.4 Curl

$$\begin{aligned}
\nabla\times\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\vec{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\vec{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\vec{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\vec{a}_\phi
\end{aligned}$$