First name (please write as legibly as possible within the boxes)

Last name

Student ID number

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

December 17, 2022

Instructor – Professor W.R. Cluett

This is a closed book test. No calculators are permitted.

All six questions are of equal value.

This test contains 30 pages including the cover page 1, this information page 2, and page 30 that is for rough work only. The test is printed two-sided.

Do <u>not</u> tear any pages from this test.

Present complete solutions in the space provided.

Given information:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

The formula for the inverse of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by:

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The normal system of equations corresponding to $A\vec{x} = \vec{b}$ is given by:

$$(A^T A)\vec{x}_{LS} = A^T \vec{b}$$

Euler's method for solving a first order differential equation $y'(t) = f(t, y(t))$ is given by:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

For higher order systems that can be expressed in the vector-matrix form $Z' = AZ$, Euler's method is given by:

$$Z_{n+1} = Z_n + \Delta t A Z_n$$

Q1: Consider matrix A given below,

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix}$$

a) Express $\begin{bmatrix} 6 \\ 7 \\ 5 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$.

b) Express $\begin{bmatrix} 4 \\ 6 \\ 2 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$.

c) Using Gaussian elimination, find the reduced row echelon form (R_0) of matrix A.

d) For the homogeneous system $A\vec{x} = \vec{0}$, find the special solution(s). (As a check on your solution, remember that $n - r$ corresponds to the number of special solutions where n is the number of unknowns and r is the rank of matrix A.)

e) Factor matrix $A = CR$ by determining matrices C and R.

Q2: Consider matrix A given below,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

a) By inspection, can you determine a nonzero vector $\vec{x}$ such that $A\vec{x} = \vec{0}$?

b) Using Gaussian elimination, determine if the inverse of matrix A (A^{-1}) exists.

c) Using Gaussian elimination, find the vector solution to $A\vec{x} = \vec{0}$ and give a geometric interpretation of the solution.

d) Note: this part is not related to parts a-c.

Suppose matrix B is square ($n \times n$) and invertible. Let us say that you replace row 2 of matrix B by row 2 plus row 1 and this produces matrix C. Is matrix C invertible? Explain your answer. If matrix C is invertible, how could you find C^{-1} from B^{-1}? Explain your answer.

Q3: Consider the following three (x, y) data points that have been collected, $(-1,7), (1,7), (2,21)$. We wish to fit a straight line to this data of the form $y = cx + d$.

a) Formulate this problem in the form $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} c \\ d \end{bmatrix}$.

b) Determine a scalar equation that describes the column space of matrix A.

c) Determine if the vector $\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$ lies in the column space of matrix A.

d) Find the projection of the vector $\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$ in the column space of matrix A, i.e. what we will refer to as $A\vec{x}_{LS}$, by making use of the scalar equation that describes the column space of matrix A as determined in part b.

e) From the vector $A\vec{x}_{LS}$ found in part d, find the values for $\vec{x}_{LS} = \begin{bmatrix} c \\ d \end{bmatrix}$.

Q4: Consider the system of linear equations,

$$x + 4y - 2z = 1$$
$$x + 7y - 6z = 6$$
$$0x + 3y + qz = t$$

The goal is to find solutions to this system.

a) Begin by formulating and defining the problem in the form $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

b) Determine values for q and t that result in matrix A not being invertible.

c) Determine values for q and t that result in infinite solutions.

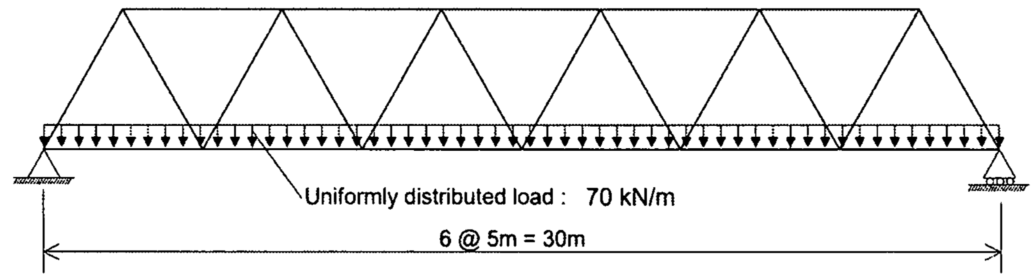
d) Give the vector equation representing the solution for the case with infinite solutions as determined in part c.

e) Give the row picture of the infinite solutions to the system of linear equations from part c.

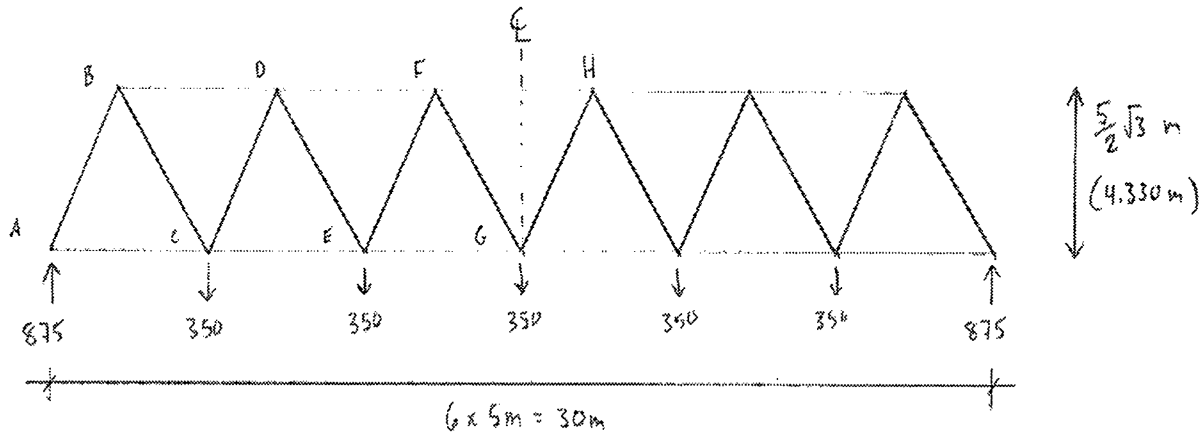
f) Find the solution when $z = 1$.

Q5: One of the problems EngSci students get asked in another first year course, CIV102F, is to solve for the forces in the members of a Warren truss, like the one shown below:

2. Calculate the joint loads and solve for the forces in the members of the Warren truss shown below. All members have the same length. Select the lightest square HSS which can be used for all of the truss members. Use E = 200,000 MPa and σ_y = 350 MPa.



After converting the 70 kN/m load to discrete "joint loads", we get:



Let $\overrightarrow{AB}$ denote the force in kN carried in the member which connects joints A and B, with tension denoted as positive and compression denoted as negative. Treat the leftmost joint as letter A and the rightmost joint as letter M. There are 13 joints that produce two equations each from the x and y static force balances at each joint for a total of 26 equations.

For the purpose of this question on the ESC103F final exam, we want to express the system of equations that need to be solved in the form $A\vec{x} = \vec{b}$. In the vector of unknowns $\vec{x}$, if we start with $\overrightarrow{AB}$ and end with $\overrightarrow{LM}$, you end up with a vector of 23 unknowns:

$$\vec{x} = \begin{bmatrix} \overrightarrow{AB} \\ \overrightarrow{AC} \\ \overrightarrow{BC} \\ \overrightarrow{BD} \\ \overrightarrow{CD} \\ \vdots \\ \overrightarrow{LM} \end{bmatrix}$$

a) What is the size of matrix A? What is the shape of matrix A (square, tall and thin, short and wide)? What is the size of vector $\vec{b}$?

b) After applying Gaussian elimination to the augmented matrix $[A|\vec{b}]$ in order to obtain $[R_0|\vec{d}]$, the reduced row echelon form of matrix A is of this form,

$$R_0 = \begin{bmatrix} I_{23\times 23} \\ row\ of\ zeros \\ row\ of\ zeros \\ row\ of\ zeros \end{bmatrix}$$

In other words, a 23×23 identity matrix appears on the top of R_0, followed by three rows of zeros. What is the rank of matrix A? Explain your answer.

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c) Based on the specific form of R_0 given in part b, what are the possible outcomes for solutions to $A\vec{x} = \vec{b}$, i.e., no solution, one solution, infinite solutions? Explain your answer.

d) After applying Gaussian elimination to the augmented matrix $[A|\vec{b}]$ to obtain $[R_0|\vec{d}]$, vector $\vec{d}$ is given by,

$$\vec{d} = \begin{bmatrix} -1010 \\ 505 \\ 1010 \\ -1010 \\ -606 \\ 1313 \\ 606 \\ -1617 \\ -202 \\ 1718 \\ 202 \\ -1819 \\ 202 \\ 1718 \\ -202 \\ -1617 \\ 606 \\ 1313 \\ -606 \\ -1010 \\ 1010 \\ 505 \\ -1010 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Given the vector $\vec{d}$ shown above, how many solutions are there to $A\vec{x} = \vec{b}$? Explain your answer.

e) What is the solution for $\overrightarrow{LM}$, the force carried in the member which connects joints L and M?

Q6: From physics, we know that a spring-mass system can be described by the following second order differential equation,

$$mx'' + kx = 0$$

with mass m, spring constant k, and $x(t)$ representing the displacement of the mass from its equilibrium position.

For a particular spring-mass system, $m = \frac{1}{16}$ and $k = 4$. Therefore,

$$\frac{1}{16}x'' + 4x = 0$$

or,

$$x'' + 64x = 0$$

The initial conditions for this particular spring-mass system are given as follows,

$$x(0) = 0 \text{ and } x'(0) = -16$$

The exact analytical solution for this initial value problem (IVP) is given by,

$$x(t) = -2\sin(8t)$$

a) Show that this analytical solution satisfies both the second order differential equation and the initial conditions.

b) Formulate this problem by defining $Z(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix}$ and $Z' = AZ$. Determine matrix A and $Z(0)$.

c) If you were to solve this IVP numerically using Euler's Method (EM), determine two separate difference equations, one for x_{n+1} and one for x'_{n+1}, assuming a step size of Δt.

d) Consider solving this problem using EM with two different step sizes, $\Delta t = 0.005$ and $\Delta t = 0.01$. Calculate two numerical estimates of the displacement $x(t)$ at $t = 0.01$, one using $\Delta t = 0.005$ and one using $\Delta t = 0.01$.

e) Consider a different version of the Improved Euler's method studied in ESC103F. Write the difference equation(s) for updating vector Z_{n+1} for a method that uses the slope estimate at the midpoint of the interval instead of the average of the left and right slope estimates.

Rough work only on this page.