MAT195S CALCULUS II

Midterm Test #2

29 March 2018 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

Instructors: F. Al Faisal and J. W. Davis

Family Name: J W Davis

Given Name: Solutions

Student #: ____________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	13	
2	8	
3	5	
4	10	
5	10	
6	10	
7	8	
8	8	
TOTAL	72	/ 65

Tutorial Section: ____________

TA Name: ____________

1) Test the series for convergence or divergence:

a) $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$ b) $\sum_{n=1}^{\infty}(-1)^{n+1} ne^{-n}$ c) $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ d) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

(13 marks)

a) $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$ $\quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3-1/n}{2+1/n} = \frac{3}{2} \neq 0$

$\therefore$ diverges by the test for divergence

b) $\sum_{n=1}^{\infty}(-1)^{n+1} ne^{-n}$ $\Rightarrow$ show decreasing: consider $f(x) = xe^{-x}$

$f'(x) = e^{-x}(1-x) < 0$ for $x \geq 1$

$\therefore$ decreasing

$\Rightarrow$ show $a_n \to 0$: $\lim_{x\to\infty} \frac{x}{e^x} \stackrel{*}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

$\therefore \sum_{n=1}^{\infty}(-1)^{n+1} ne^{-n}$ converges by the Alt series test

c) $\sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ $\Rightarrow \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2}{n} \to 0 < 1$

$\therefore$ converges by root test

$\sum_{n=1}^{\infty}\frac{2^n}{n^n}$ converges $\therefore \sum_{n=1}^{\infty}\frac{(-2)^n}{n^n}$ converges

d) $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n} = 2\left(\frac{n}{n+1}\right)^2 \to 2 > 1$

$\therefore \sum_{n=1}^{\infty}\frac{2^n}{n^2}$ diverges

2) a) Find the radius and interval of convergence of the power series: $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

(4 marks)

ratio test: $\left| \frac{x^{2n+2}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^{2n}} \right| = \left(\frac{n}{n+1}\right)\underbrace{\left(\frac{\ln n}{\ln(n+1)}\right)^2}\cdot |x|^2 \longrightarrow |x|^2$

$\therefore$ convergent for $|x| < 1$

$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{*}{=} \lim_{x\to\infty} \frac{1/x}{1/x+1} = \lim_{x\to\infty} \frac{x+1}{x} = 1$

end points : $x = \pm 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Integral test : $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ let $u = \ln x$ $du = dx/x$

$= \int_{\ln u}^{\infty} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}$ $\therefore$ convergent

$\Rightarrow$ Interval of convergence $[-1, 1]$

b) If k is a positive integer, find the radius of convergence of the series: $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$

(4 marks)

ratio test : $\left| \frac{((n+1)!)^k x^{n+1}}{(kn+k)!} \cdot \frac{(kn)!}{(n!)^k x^n} \right| = |x| (n+1)^k \frac{(kn)!}{(kn+k)!}$

$= |x| \frac{(n+1)(n+1)(n+1)\cdots(n+1)}{(kn+1)(kn+2)(kn+3)\cdots(kn+k)} \longrightarrow \frac{|x|}{k^k}$

$\therefore$ convergence for $|x| < k^k$

3) Prove that if $\sum c_n$ converges absolutely, then $\sum c_n^p$ converges absolutely for all integers $p > 1$.

(5 marks)

Given that $\sum |c_n|$ converges, then $|c_n| \to 0$ as $n \to \infty$

$\therefore |c_n| < 1$ for $n > N$

$\therefore |c_n|^p < |c_n|$ for $p > 1$, $n > N$

$\Rightarrow 0 \leq |c_n|^p \leq |c_n|$ for $n > N$

since $\sum |c_n|$ converges, $\sum |c_n|^p$ converges by pinching theorem.

Alternate: limit comparison test:

$$\lim_{n\to\infty} \frac{c_n^p}{c_n} = \lim_{n\to\infty} c_n^{p-1}$$

$\Rightarrow$ For $p = 2$: $\lim_{n\to\infty} c_n^{p-1} = \lim_{n\to\infty} c_n = 0$ since $\sum c_n$ converges

$\Rightarrow$ For $p > 2$: $|c_n|^{p-1} < |c_n|$ for $n > N$

$\therefore 0 \leq |c_n|^{p-1} < |c_n|$ for $n > N$

$\lim_{n\to\infty} |c_n| = 0 \quad \therefore \lim_{n\to\infty} |c_n|^{p-1} = 0$

$\therefore \lim_{n\to\infty} c_n^{p-1} = 0$

4) Find, from first principles, the Taylor series expansion for $f(x) = 3^x$ about $a = 0$.
Prove that f is equal to the sum of this series by showing that the Taylor remainder, $R_n(x)$, goes to zero as $n \to \infty$. Recall, the Taylor remainder theorem which states that

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ where } |f^{(n+1)}(x)| \le M.$$

(10 marks)

$f(x) = 3^x$ $\qquad f(0) = 1$

$f'(x) = 3^x \ln 3$ $\qquad f'(0) = \ln 3$

$f''(x) = 3^x (\ln 3)^2$ $\qquad f''(0) = (\ln 3)^2$

$\vdots$

$f^{(n)}(x) = 3^x (\ln 3)^n$ $\qquad f^{(n)}(0) = (\ln 3)^n$

$$\therefore 3^x = 1 + x\ln 3 + \frac{x^2}{2}(\ln 3)^2 + \frac{x^3}{3!}(\ln 3)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(x \ln 3)^n}{n!}$$

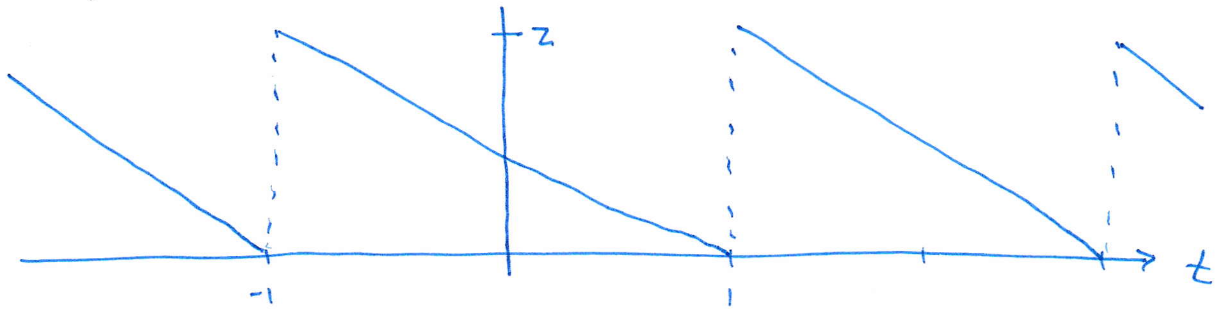
Now $f^{(n+1)}(x) = 3^x (\ln 3)^{n+1}$ $\quad \Rightarrow \quad M = 3^x (\ln 3)^{n+1}$

$$\therefore R_n(x) \le \frac{3^x |x \ln 3|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$\therefore 3^x = \sum_{n=0}^{\infty} \frac{x^n (\ln 3)^n}{n!}$$

5) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function:
$f(t) = 1 - t, -1 < t \leq 1$.
Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.

(10 marks)



$$T = 2 \Rightarrow \omega = 2\pi / T = \pi$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt \Rightarrow a_0 = \int_{-1}^{1}(1-t)\,dt = \left[t - \frac{t^2}{2}\right]_{-1}^{1} = \left(1 - \frac{1}{2} + 1 + \frac{1}{2}\right) = 2$$

$$a_n = \underbrace{\int_{-1}^{1}\cos n\pi t\,dt}_{0} - \int_{-1}^{1} t\cos n\pi t\,dt \qquad \begin{array}{ll} \text{let } u = t & dv = \cos n\pi t \\ du = dt & v = \frac{1}{n\pi}\sin n\pi t \end{array}$$

$$= -\left[\frac{t}{n\pi}\underbrace{\sin n\pi t}_{0}\right]_{-1}^{1} + \int_{-1}^{1}\frac{1}{n\pi}\underbrace{\sin n\pi t}_{0}\,dt = 0$$

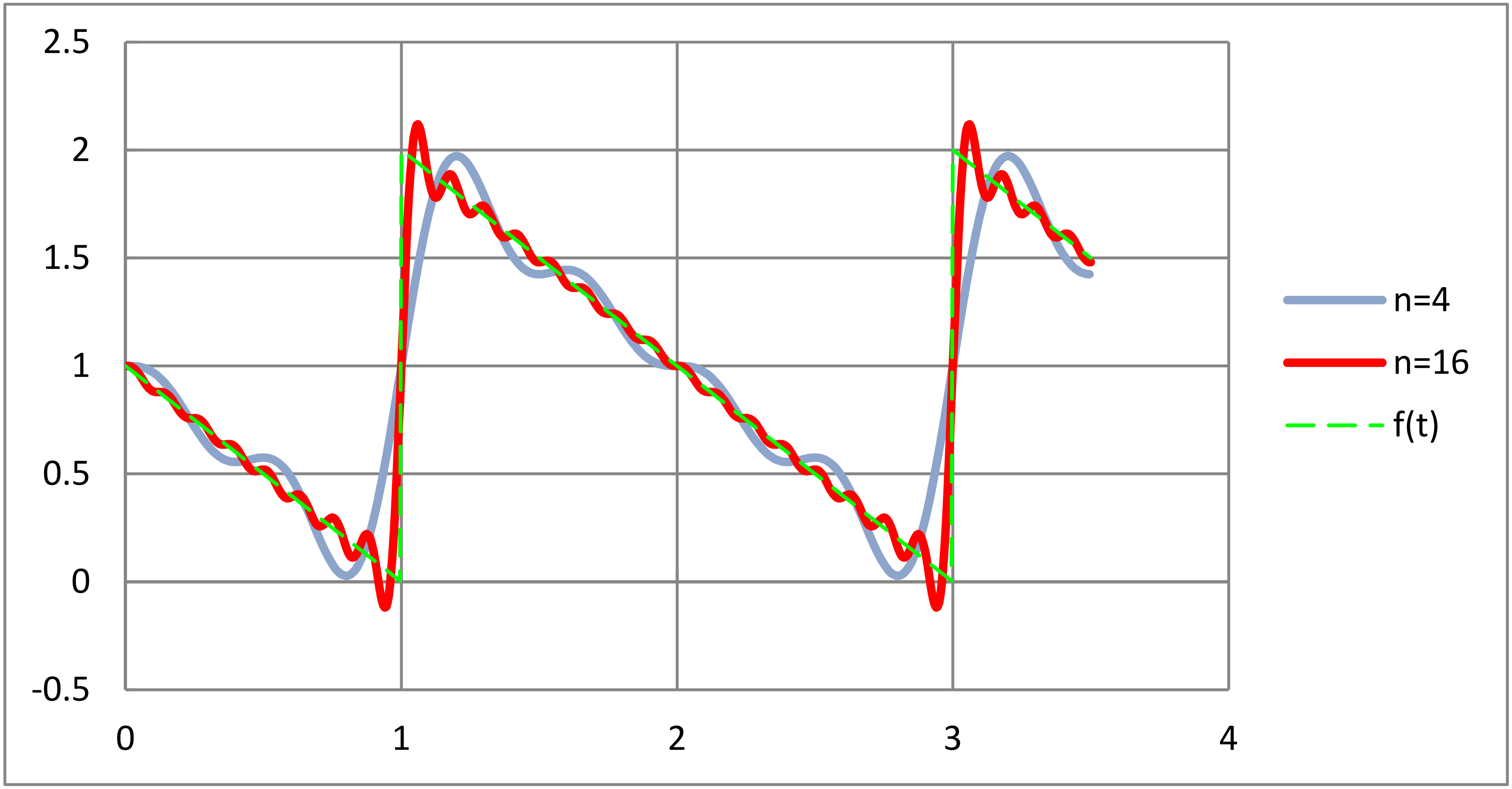
(Alternately, one can note that $f(t) - 1$ is an odd function, and thus all $a_n = 0$, $n \geq 1$.)

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt$$

$$= \underbrace{\int_{-1}^{1}\sin n\pi t\,dt}_{0} - \int_{-1}^{1} t\sin n\pi t\,dt \qquad \begin{array}{ll} \text{let } u = t & dv = \sin n\pi t \\ du = dt & v = \frac{1}{n\pi}\cos n\pi t \end{array}$$

$$= -\left[\frac{-t}{n\pi}\cos n\pi t\right]_{-1}^{1} - \underbrace{\int_{-1}^{1}\frac{1}{n\pi}\cos n\pi t\,dt}_{0} = \begin{cases} \frac{-2}{n\pi} & n \text{ odd} \\ \frac{2}{n\pi} & n \text{ even} \end{cases}$$

$$\therefore f(t) = 1 + \sum_{n=1}^{\infty}(-1)^n \frac{2}{n\pi}\sin n\pi t$$


2.5
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
n=4
n=16
f(t)

6) a) Give an ϵ-δ definition of uniform continuity of a function f defined for real numbers.

b) Using the ϵ-δ definition, prove that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.
HINT: Use the mean value theorem.

(10 marks)

a) We say that a function f is uniformly continuous on a set Ω, if for any $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x_1 - x_2| < \delta$, for any $x_1, x_2 \in \Omega$, then $|f(x_1) - f(x_2)| < \epsilon$.

b) Given $|x_2 - x_1| < \delta \rightarrow$ show $|\sin x_2 - \sin x_1| < \epsilon$

Mean Value Theorem: $f'(z) = \dfrac{f(b) - f(a)}{b - a}$; $z \in (b, a)$

or $\cos z = \dfrac{\sin x_2 - \sin x_1}{x_2 - x_1}$; $z \in (x_1, x_2)$

$\Rightarrow |\sin x_2 - \sin x_1| = |\cos z \cdot (x_2 - x_1)|$

since $|\cos z| \leq 1 \Rightarrow |\sin x_2 - \sin x_1| \leq |x_2 - x_1|$

$\therefore$ let $\delta = \epsilon$

$\therefore$ For $|x_2 - x_1| < \delta = \epsilon$ then $|\sin x_2 - \sin x_1| < \epsilon$

$\therefore$ $\sin x$ is uniformly continuous.

7) Find the curvature at a general point on the curve: $r(t) = t\,\hat{i} + \sqrt{2}\,\ln t\,\hat{j} + \frac{1}{t}\hat{k}$.

(8 marks)

$$\vec{r}(t) = \left(t,\ \sqrt{2}\ln t,\ 1/t\right)$$

$$\vec{r}\,'(t) = \left(1,\ \frac{\sqrt{2}}{t},\ \frac{-1}{t^2}\right)$$

$$\|\vec{r}\,'(t)\| = \sqrt{1 + \frac{2}{t^2} + \frac{1}{t^4}} = \frac{1}{t^2}\sqrt{t^4 + 2t^2 + 1} = \frac{t^2+1}{t^2}$$

$$\vec{r}\,''(t) = \left(0,\ \frac{-\sqrt{2}}{t^2},\ \frac{2}{t^3}\right)$$

$$\vec{r}\,' \times \vec{r}\,'' = \begin{vmatrix} i & j & k \\ 1 & \frac{\sqrt{2}}{t} & \frac{-1}{t^2} \\ 0 & \frac{-\sqrt{2}}{t^2} & \frac{2}{t^3} \end{vmatrix} = \overbrace{\left(\frac{2\sqrt{2}}{t^4} - \frac{\sqrt{2}}{t^4}\right)}^{\frac{\sqrt{2}}{t^4}}\hat{i} + \left(-\frac{2}{t^3}\right)\hat{j} + \left(\frac{-\sqrt{2}}{t^2}\right)\hat{k}$$

$$\|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{\frac{2}{t^8} + \frac{4}{t^6} + \frac{2}{t^4}} = \frac{\sqrt{2}}{t^4}\sqrt{1 + 2t^2 + t^4} = \frac{\sqrt{2}(1+t^2)}{t^4}$$

$$\therefore\ \kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{\sqrt{2}(1+t^2)}{t^4} \cdot \frac{t^6}{(1+t^2)^3} = \frac{\sqrt{2}\,t^2}{(1+t^2)^2}$$

8) a) Let $f(x.y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2} & for \quad (x,y) \neq (0,0) \\ 1 & for \quad (x,y) = (0,0) \end{cases}$

Show that f is continuous on $\mathbb{R}^2$.

(4 marks)

$\Rightarrow$ Show $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = 1$

let $x^2+y^2 = r^2$ (polar coordinates)

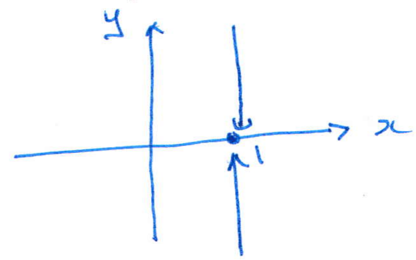
$\lim\limits_{r\to 0} \dfrac{\sin r^2}{r^2} \overset{*}{=} \lim\limits_{r\to 0} \dfrac{2r\cos r^2}{2r} = \lim\limits_{r\to 0} \cos r^2 = 1$

$\therefore f(x,y)$ is continuous at $(0,0)$

(There are no other locations where $\dfrac{\sin(x^2+y^2)}{x^2+y^2}$ DNE)

b) Does $\lim\limits_{(x,y)\to(1,0)} \tan^{-1}\dfrac{x}{y}$ exist? If so, find its value; if not, explain.

(4 marks)



$\Rightarrow$ consider the line $x=1$

$\Rightarrow$ we can approach $y=0$ from above or below:

$\lim\limits_{y\to 0^-} \tan^{-1}\dfrac{1}{y} = -\dfrac{\pi}{2} = \lim\limits_{z\to -\infty} \tan^{-1} z$

$\lim\limits_{y\to 0^+} \tan^{-1}\dfrac{1}{y} = +\dfrac{\pi}{2} = \lim\limits_{z\to +\infty} \tan^{-1} z$

$\therefore \lim\limits_{(x,y)\to(1,0)} \tan^{-1}\dfrac{x}{y}$ DNE

8a) Somewhat different approach

Given $\lim_{t \to 0} \frac{\sin t}{t} = 1$

$\therefore$ given $\epsilon > 0$ we can find a $\delta > 0$ such that

for $0 < |t| < \delta \Rightarrow \left| \frac{\sin t}{t} - 1 \right| < \epsilon$

Now, for this same δ, we can set:

$$0 < \| (x,y) - (0,0) \| < \sqrt{\delta}$$

or $0 < \sqrt{x^2+y^2} < \sqrt{\delta}$

or $0 < x^2+y^2 < \delta$

So, by the result above, with $t = x^2+y^2$,

we have $\left| \frac{\sin(x^2+y^2)}{x^2+y^2} - 1 \right| < \epsilon$