TERM TEST 2015 — SOLUTIONS

Q1: a) 2

$$M = \left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

b) 5

⋮ (GAUSSIAN ELIMINATION)

$$M' = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/4 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/2 \\ 1/4 \end{bmatrix}$$

c) 3

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3/4 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + 1/2 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + 1/4 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Q2: a) 4 $x = 1 \quad y = 2 \quad z = -3 \quad x+y+z=0$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} = -7$$

$$\|\vec{u}\| = \|\vec{v}\| = \sqrt{14}$$

$$\cos\theta = \frac{-7}{14} = -\frac{1}{2}$$

$$\theta = 120^\circ \text{ OR } \frac{2\pi}{3}$$

b) 6 $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \vec{v} = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$

$$\vec{u}\cdot\vec{v} = xz + xy + yz$$

$$\|\vec{u}\| = \|\vec{v}\| = \sqrt{x^2+y^2+z^2}$$

$$\therefore \cos\theta = \frac{xz+xy+yz}{x^2+y^2+z^2}$$

WE KNOW $x+y+z=0$

$$\therefore (x+y+z)^2=0$$

$$\therefore x^2+y^2+z^2+xy+xz+xy+yz+xz+yz=0$$

$$\therefore x^2+y^2+z^2=-2xy-2xz-2yz$$

$$=-2(xy+xz+yz)$$

$$\therefore \cos\theta=-\frac{1}{2}=\frac{xy+xz+yz}{x^2+y^2+z^2}$$

Q3: a) 5 PLANE CONTAINS THE TWO VECTORS:

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \text{ AND } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A NORMAL TO THIS PLANE:

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

FIRST PROJECT $\vec{c}$ ONTO THE NORMAL:

$$\mathrm{proj}_{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = \frac{0+4-8}{\sqrt{5}^{\,2}} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \frac{-4}{5} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

THEN SOLVE FOR THE PROJECTION OF $\vec{c}$ ONTO THE PLANE:

$$\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ -4/5 \\ 8/5 \end{bmatrix} = \begin{bmatrix} 3 \\ 24/5 \\ 12/5 \end{bmatrix}$$

b) 2 THIS IS THE NORMAL VECTOR $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

OR ANY VECTOR THAT IS A SCALAR MULTIPLE OF THE NORMAL.

c) 3 FIRST PROJECT $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ ONTO THE NORMAL:

$$\operatorname{proj}_{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{0 + y - 2z}{\sqrt{5}^{\,2}} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \frac{y - 2z}{5} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

THEN SOLVE FOR PROJECTION ONTO THE PLANE:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{y-2z}{5} \\ \frac{-2y+4z}{5} \end{bmatrix} = \begin{bmatrix} x \\ 4/5\, y + 2/5\, z \\ 2/5\, y + 1/5\, z \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 2/5 \\ 0 & 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Q4: a) TO FIND EIGENVALUES, NEED TO SOLVE:

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$$\det\left(\begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} -\lambda & 1 \\ c & d-\lambda \end{bmatrix} = 0$$

$$(-\lambda)(d-\lambda) - c = 0$$
$$\lambda^2 - d\lambda - c = 0$$

$$\lambda = \frac{d \pm \sqrt{d^2+4c}}{2}$$

LET $\displaystyle 7 = \frac{d + \sqrt{d^2+4c}}{2}$

$$4 = \frac{d - \sqrt{d^2+4c}}{2}$$

$(14-d)^2 = d^2+4c$

$196 - 28d + \cancel{d^2} = \cancel{d^2} + 4c$

$4c = 196 - 28d$

$(d-8)^2 = d^2+4c$

$\cancel{d^2} - 16d + 64 = \cancel{d^2} + 4c$

$4c = 64 - 16d$

$$\therefore\ 196 - 28d = 64 - 16d$$
$$132 = 12d$$
$$d = 11$$

$c = -28$
$d = 11$

$$c = \frac{64 - 16(11)}{4} = -28$$

b) 4 TO FIND EIGENVECTORS ASSOCIATED WITH $\lambda = 4$,
NEED TO SOLVE:

$$\begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$y = 4x$$
$$-28x + 11y = 4y \Rightarrow 7y = 28x \text{ OR } y = 4x$$

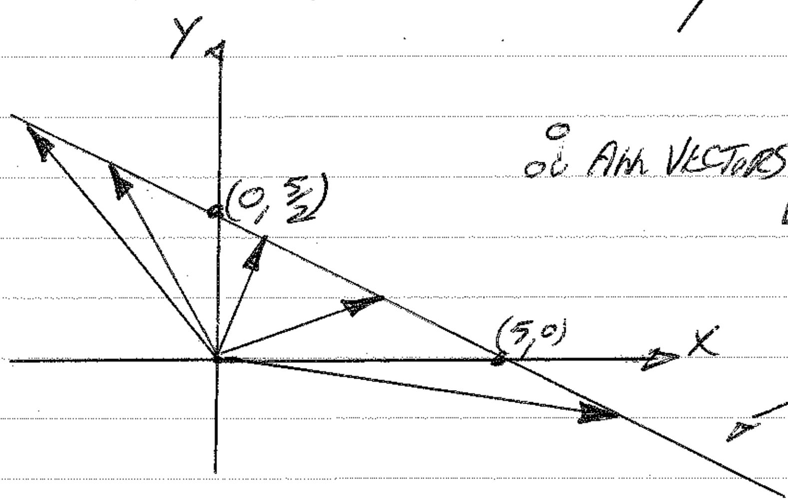
$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 4x \end{bmatrix} = x \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

EIGENVECTORS ASSOCIATED WITH $\lambda = 4$
ARE ALL SCALAR MULTIPLES OF $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Q5: a) 5 $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

LOOKING FOR ALL VECTORS $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ WITH $\vec{u} \cdot \vec{v} = 5$

$$\vec{u} \cdot \vec{v} = x + 2y = 5$$



∴ ALL VECTORS $\begin{bmatrix} x \\ y \end{bmatrix}$ WHERE $x + 2y = 5$

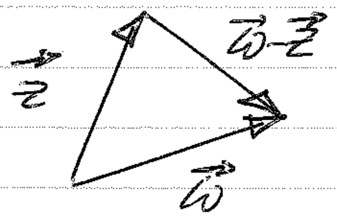
LINE $y = -\frac{1}{2}x + \frac{5}{2}$

b) 5 $\|\vec{w}\| = 5 \quad \|\vec{z}\| = 3$

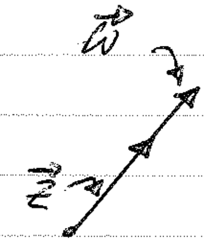
$$\vec{w} \cdot \vec{z} = \|\vec{w}\| \|\vec{z}\| \cos\theta$$

SINCE $-1 \leq \cos\theta \leq 1$, THEN $-1 \leq \dfrac{\vec{w} \cdot \vec{z}}{\|\vec{w}\| \|\vec{z}\|} \leq 1$

$$-15 \leq \vec{w} \cdot \vec{z} \leq 15$$

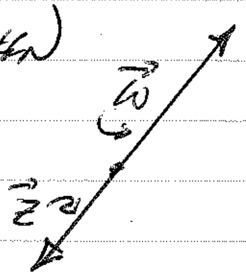


$\|\vec{w}-\vec{z}\|$ SMALLEST WHEN



$$\|\vec{w}-\vec{z}\| = 5-3 = 2$$

$\|\vec{w}-\vec{z}\|$ LARGEST WHEN



$$\|\vec{w}-\vec{z}\| = 5+3 = 8$$

$$2 \leq \|\vec{w}-\vec{z}\| \leq 8$$

Q6: a) 2

$$\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

b) 3

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow{T} \begin{bmatrix} y \\ z \\ x \end{bmatrix} \xrightarrow{T} \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

$$\begin{bmatrix} z \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$ OBTAINED DIRECTLY

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$ OBTAINED BY MATRIX MULTIPLICATION

c) APPLYING T THREE TIMES WILL RETURN

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$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ TO $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. THEREFORE APPLYING T

100 TIMES WOULD BE EQUIVALENT TO APPLYING IT ONE TIME $(100 = 3 \times 33 + 1)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xrightarrow[\text{TIMES}]{T\ 100} \begin{bmatrix} y \\ z \\ x \end{bmatrix}$$