Q3: (all parts are related)

Consider the following initial value problem (IVP):

$y''(t) + 4y'(t) + 3y(t) = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

a) Set up the state space representation of this IVP, i.e., $Y' = AY$.

b) Using Euler's Method with $\Delta t = 0.5$, calculate the estimated values for $y(1)$ and $y'(1)$.

c) Using Improved Euler's Method with $\Delta t = 1.0$, calculate the estimated values for $y(1)$ and $y'(1)$.

Solutions:

a) $y''(t) = -4y'(t) - 3y(t)$

Let $Y = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$; $Y' = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$

$\therefore Y' = AY = \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} Y$

b) $Y_1 = Y_0 + 0.5 \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} Y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & +0.5 \\ -1.5 & -2.0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$

$Y_2 = Y_1 + 0.5 \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} Y_1 = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} + \begin{bmatrix} 0 & +0.5 \\ -1.5 & -2.0 \end{bmatrix} \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix} + \begin{bmatrix} -0.75 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix}$

Therefore, the EM estimate for $y(1) = 0.25$ and the EM estimate for $y'(1) = 0$.

c) $Y_{1,EM} = Y_0 + 1.0 \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} Y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$$Y_{1,IEM} = Y_0 + \frac{1}{2}\left(\begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & +1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right)$$
$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.5 \left(\begin{bmatrix} 0 \\ -3 \end{bmatrix} + \begin{bmatrix} -3 \\ 9 \end{bmatrix} \right) = \begin{bmatrix} -0.5 \\ 3 \end{bmatrix}$$

Therefore, the IEM estimate for $y(1) = -0.5$ and the IEM estimate for $y'(1) = 3$.