ESC195 - Midterm Test #2

March 22, 2022

9:10 - 10:50 am

Instructor: J. W. Davis

Closed book, no aid sheets, no calculators

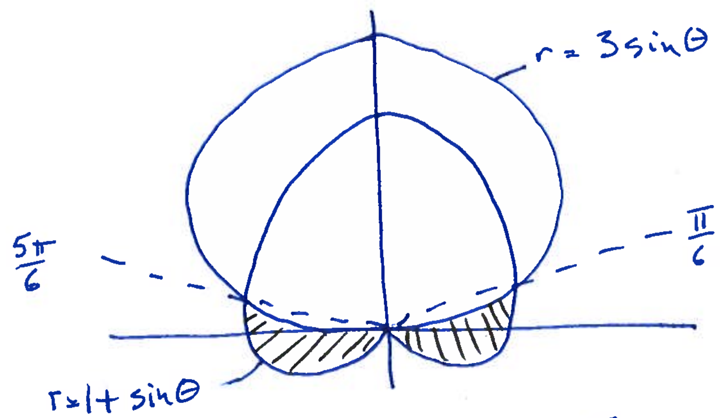
There are 7 questions, each worth 10 marks.

Plus a bonus question worth 5 marks.

JW Davis.

Solutions

1. Find the area of the region that lies inside the cardiod $r = 1 + \sin\theta$, but outside the circle $r = 3\sin\theta$. Provide a sketch.



Intersections:

$3\sin\theta = 1 + \sin\theta$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = 2\left[\int_{-\frac{\pi}{2}}^{0} \frac{1}{2}(1+\sin\theta)^2\, d\theta + \int_{0}^{\frac{\pi}{6}} \frac{1}{2}\left((1+\sin\theta)^2 - (3\sin\theta)^2\right) d\theta \right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (1 + 2\sin\theta + \sin^2\theta)\, d\theta - \int_0^{\pi/2} 9\sin^2\theta\, d\theta$$

$$= \left[\theta - 2\cos\theta + \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}} - 9\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6}$$

$$= \frac{2\pi}{3} - 2\frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{1}{4}\frac{\sqrt{3}}{2} - 9\frac{\pi}{12} + \frac{9}{4}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{4}$$

2. Determine whether the sequence converges or diverges; if it converges, find the limit:

a) $a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}}$ b) $a_n = \left(1+\dfrac{2}{n}\right)^n$ c) $a_n = \displaystyle\int_1^n \frac{dx}{x^p}$, $p>1$, $n \geq 1$

a) $a_n = \sqrt{\dfrac{1+4n^2}{1+n^2}} = \sqrt{\dfrac{\frac{1}{n^2}+4}{\frac{1}{n^2}+1}} \rightarrow \sqrt{\dfrac{4}{1}} = 2$

b) $a_n = \left(1+\frac{2}{n}\right)^n \longrightarrow 1^\infty$ type

consider $\displaystyle\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{2}{x}\right)^x}$

$\Rightarrow \displaystyle\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{\frac{1}{x}} \rightarrow \frac{0}{0}$

$\stackrel{*}{=} \displaystyle\lim_{x\to\infty} \frac{\frac{1}{1+\frac{2}{x}}\left(-\frac{2}{x^2}\right)}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{2}{1+\frac{2}{x}} = 2$

$\therefore \displaystyle\lim_{x\to\infty} e^{\ln\left(1+\frac{2}{x}\right)^x} \longrightarrow e^2$

$\therefore\ a_n = \left(1+\frac{2}{n}\right)^n \longrightarrow e^2$

c) $a_n = \displaystyle\int_1^n \frac{dx}{x^p} = \left[\frac{x^{-p+1}}{-p+1}\right]_1^n = \frac{n^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \frac{1}{p-1} + \frac{1}{1-p}\cdot\frac{n}{n^p}$

$\therefore \displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{p-1} + \frac{1}{1-p}\cdot\frac{n}{n^p}\right) \longrightarrow \frac{1}{p-1}$

3. a) Test the series for convergence or divergence:

i) $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^n$ ii) $\sum_{n=1}^{\infty}\frac{n\,5^{2n}}{10^{n+1}}$

i) $a_n = \left(1+\frac{2}{n}\right)^n \not\to 0$ $\therefore$ diverges by test for divergence

ii) root test : $(a_n)^{1/n} = \frac{n^{1/n}\cdot 5^2}{10^{1/n}\cdot 10} \longrightarrow \frac{1\cdot 25}{1\cdot 10} = \frac{5}{2} > 1$

$\therefore$ diverges by root test

b) For what values of x does the series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n\,4^n}$ converge absolutely? Conditionally? Give the radius and interval of convergence.

ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x+2)^{n+1}}{(n+1)\,4^{n+1}}\cdot\frac{n\,4^n}{(x+2)^n}\right| = \frac{1}{4}\,\frac{n}{n+1}\,|x+2| \to \frac{|x+2|}{4}$

$\Rightarrow$ convergence for $\frac{|x+2|}{4} < 1$ or $|x+2| < 4$ $\Rightarrow R = 4$

$\Rightarrow x \in (-6,2)$ Absolutely convergent

test $x=-6$: $\sum_{n=1}^{\infty}\frac{(-4)^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ conditionall convergent alternating series

test $x=2$: $\sum_{n=1}^{\infty}\frac{(4)^n}{n\,4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ divergent harmonic series

$\therefore \sum_{n=1}^{\infty}\frac{(x+2)^n}{n\,4^n}$ converges for $x \in [-6,2)$

4. Suppose that the series $\sum a_n$ is conditionally convergent.

a) Prove that $\sum n^2 a_n$ is divergent. (Hint: use a proof by contradiction.)

b) Knowing that $\sum a_n$ is conditionally convergent is not sufficient to determine whether $\sum n a_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\sum n a_n$ converges, and an example where $\sum n a_n$ diverges.

a) Assume that $\sum n^2 a_n$ is convergent

$\therefore$ $n^2 |a_n| \to 0$ as $n \to \infty$ (test for divergence)

$\therefore$ for $n > N$, $n^2 |a_n| < \epsilon$ for some N

or $|a_n| < \frac{\epsilon}{n^2}$

But $\sum \frac{\epsilon}{n^2}$ is convergent (p-series, $p > 1$) which means $\sum |a_n|$ must also be convergent, which is a contradiction of the initial statement that $\sum a_n$ is conditionally convergent. $\Rightarrow$ $\therefore$ $\sum n^2 a_n$ must be divergent.

b) i) consider $a_n = (-1)^n \frac{1}{n}$ which is conditionally convergent.

$\Rightarrow \sum \frac{1}{n}$ diverges by integral test: $\int_1^\infty \frac{dx}{x} = [\ln x]_1^\infty \to \infty$

$\Rightarrow \sum \frac{(-1)^n}{n}$ converges by alt series test: $\frac{1}{n+1} < \frac{1}{n}$; $\frac{1}{n} \to 0$

$\Rightarrow \sum n a_n = \sum (-1)^n$ diverges : $|n a_n| \not\to 0$

ii) consider $b_n = (-1)^n \frac{1}{n \ln n}$ which is also conditionally convergent

$\Rightarrow \sum \frac{1}{n \ln n}$ diverges by integral test: $\int_2^\infty \frac{dx}{x \ln x} = [\ln(\ln x)]_2^\infty \to \infty$

$\Rightarrow \sum \frac{(-1)^n}{n \ln n}$ converges by alt series test: $\frac{1}{(n+1)\ln(n+1)} < \frac{1}{n \ln n}$; $\frac{1}{n \ln n} \to 0$

$\Rightarrow \sum n b_n = \sum \frac{(-1)^n}{\ln n}$; $\frac{1}{\ln(n+1)} < \frac{1}{\ln n}$ & $\frac{1}{\ln n} \to 0$

$\therefore$ convergent by alt series test.

5. Find from first principles (that is, by taking derivatives), the Taylor series expansion for $f(x) = \sin x$ about $a = 0$. Prove that f is equal to the sum of this series by showing that the Taylor remainder, $R_n(x)$, goes to zero as $n \to \infty$. Recall, the Taylor remainder theorem which states that : $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$ where $|f^{(n+1)}(x)| \leq M$.

$$f(x) = \sin x \qquad f(0) = 0$$
$$f'(x) = \cos x \qquad f'(0) = 1$$
$$f''(x) = -\sin x \qquad f''(0) = 0$$
$$f'''(x) = -\cos x \qquad f'''(0) = -1$$
$$f^{IV}(x) = \sin x \qquad f^{IV}(0) = 0$$

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ even} \\ (-1)^{\frac{n-1}{2}} & n \text{ odd} \end{cases}$$

$$\therefore f(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Now $f^{(n+1)}(x) = \pm \sin x$ or $\pm \cos x \Rightarrow |f^{(n+1)}(x)| \leq 1 = M$

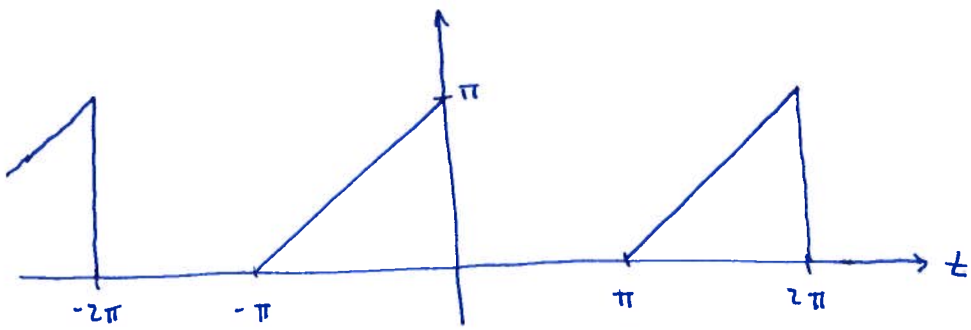
$$\therefore |R_n(x)| \leq \frac{|x-a|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$\therefore \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad \text{all } x \in \mathbb{R}$$

6. Find the Fourier series; ie., evaluate the Fourier coefficients, for the function:

$$f(t) = \begin{cases} \pi + t & -\pi \le t \le 0 \\ 0 & 0 < t < \pi \end{cases}$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.



$T = 2\pi$

$\omega = \frac{2\pi}{2\pi} = 1$

$a_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt = \frac{1}{\pi} \int_{-\pi}^{0} (\pi + t) \cos nt \, dt$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{0} (\pi + t) dt = \frac{1}{\pi} \left[\pi t + \frac{t^2}{2} \right]_{-\pi}^{0} = \frac{1}{\pi} \left(\pi^2 - \frac{\pi^2}{2} \right) = \frac{\pi}{2} \Rightarrow \frac{a_0}{2} = \frac{\pi}{4}$

$a_n = \frac{1}{\pi} \int_{-\pi}^{0} \pi \cos nt \, dt + \frac{1}{\pi} \int_{-\pi}^{0} t \cos nt \, dt$ let $u = t$, $du = dt$; $dv = \cos nt \, dt$, $v = \frac{1}{n} \sin nt$

$= \left[\frac{1}{n} \cancel{\sin nt} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[\frac{t}{n} \cancel{\sin nt} \right]_{-\pi}^{0} - \frac{1}{\pi} \int_{-\pi}^{0} \frac{1}{n} \sin nt \, dt = \frac{-1}{n\pi} \left[\frac{-1}{n} \cos nt \right]_{-\pi}^{0}$

$= \frac{1}{n^2 \pi} (1 \pm 1) = \frac{2}{n^2 \pi}$ (n odd) ; $= 0$ (n even)

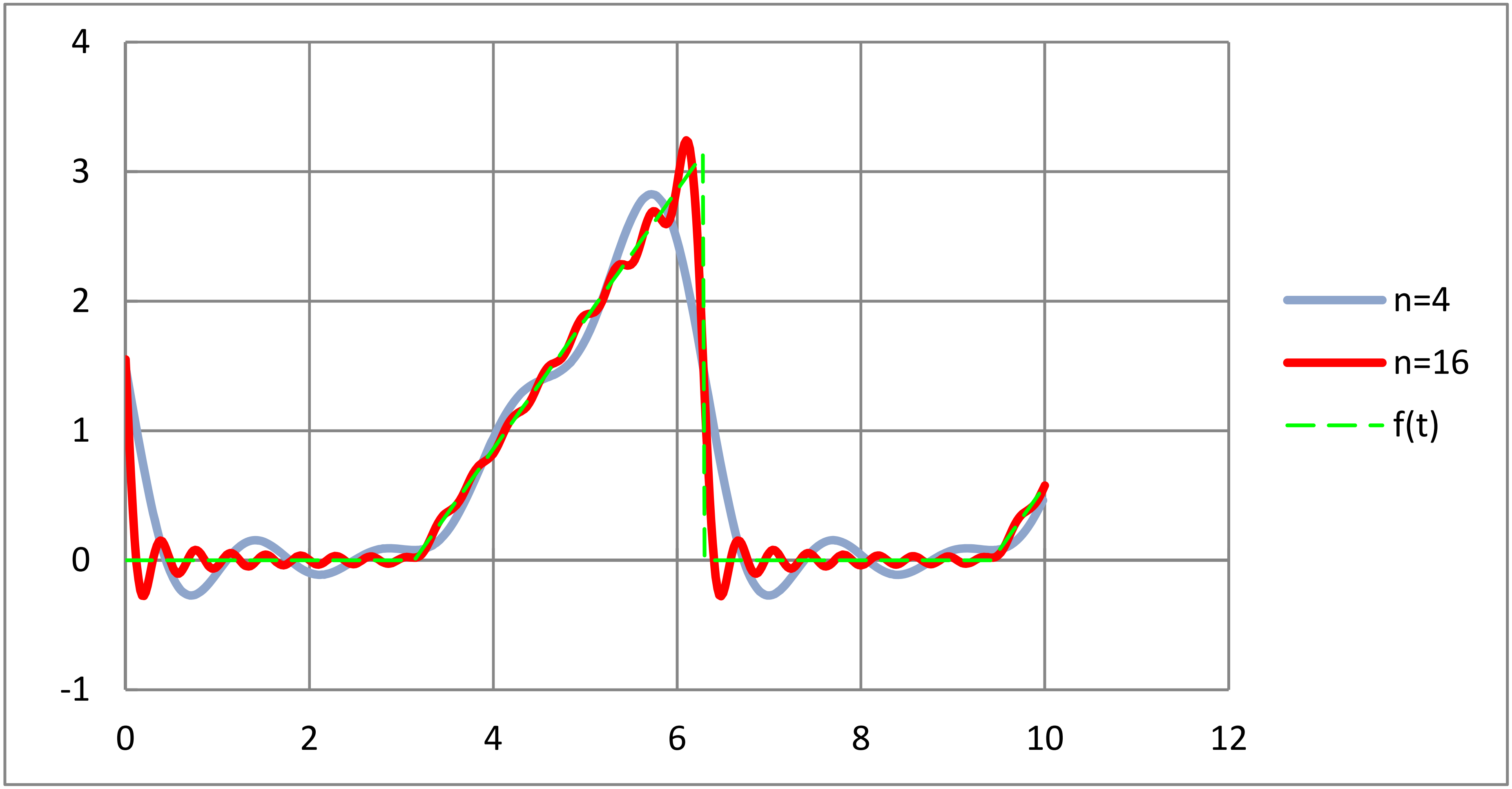
$b_n = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt = \frac{1}{\pi} \int_{-\pi}^{0} (\pi + t) \sin nt \, dt$

$= \frac{1}{\pi} \int_{-\pi}^{0} \pi \sin nt \, dt + \frac{1}{\pi} \int_{-\pi}^{0} t \sin nt \, dt$ let $u = t$, $du = dt$; $dv = \sin nt \, dt$, $v = -\frac{1}{n} \cos nt$

$= \left[\frac{-1}{n} \cos nt \right]_{-\pi}^{0} - \frac{1}{\pi} \left[\frac{t}{n} \cos nt \right]_{-\pi}^{0} + \frac{1}{\pi} \int_{-\pi}^{0} \frac{1}{n} \cos nt \, dt$

$= \left(-\frac{1}{n} \right) \left(1 \overset{\text{odd}}{\underset{\text{even}}{\pm}} 1 \right) - \frac{\pi}{\pi n} \left(\overset{\text{odd}}{\underset{\text{even}}{\mp}} 1 \right) + \frac{1}{\pi n^2} \left[\cancel{\sin nt} \right]_{-\pi}^{0} = \begin{cases} -\frac{1}{n} & n \text{ even} \\ -\frac{1}{n} & n \text{ odd} \end{cases}$

$\therefore f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{\pi (2n-1)^2} \cos (2n-1)t + \sum_{n=1}^{\infty} \frac{-1}{n} \sin nt$


4
3
2
1
0
-1
0
2
4
6
8
10
12
n=4
n=16
f(t)

7. Find the unit tangent vector, the principle normal vector and an equation in x, y, z for the osculating plane at the point $(1, 2, 2)$ on the curve: $\vec{r} = t^2\,\hat{\imath} + (t+1)\,\hat{\jmath} + 2t\,\hat{k}$.

$\vec{r}(t) = (t^2, t+1, 2t) \Rightarrow \vec{r}(1) = (1,2,2) \Rightarrow t=1$

$\vec{r}\,'(t) = (2t, 1, 2) \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{4t^2+1+4} = \sqrt{5+4t^2}$

$\therefore \vec{T} = \frac{\vec{r}\,'}{\|\vec{r}\,'\|} = \left(\frac{2t}{\sqrt{5+4t^2}}, \frac{1}{\sqrt{5+4t^2}}, \frac{2}{\sqrt{5+4t^2}}\right) \Rightarrow \boxed{\vec{T}(1) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)}$

$\vec{T}\,' = \left(-\frac{1}{2}(5+4t^2)^{-3/2}\cdot 8t\cdot 2t + 2(5+4t^2)^{-1/2},\ -\frac{1}{2}(5+4t^2)^{-3/2}\,8t,\ -\frac{1}{2}(5+4t^2)^{-3/2}\cdot 8t\cdot 2\right)$

$\vec{T}\,'(t=1) = \left(-\frac{8}{27}+\frac{2}{3}, \frac{-4}{27}, -\frac{8}{27}\right) = \frac{1}{27}(10, -4, -8)$

$\|\vec{T}\,'(1)\| = \frac{2}{27}\sqrt{25+4+16} = \frac{2\sqrt{45}}{27} = \frac{2\sqrt{5}}{9}$

$\therefore \vec{N}(1) = \frac{\vec{T}\,'(1)}{\|\vec{T}\,'(1)\|} = \frac{2}{27}(5,-2,-4)\cdot\frac{9}{2\sqrt{5}} \Rightarrow \boxed{\vec{N}(1) = \frac{1}{3\sqrt{5}}(5,-2,-4)}$

$\vec{T}(1)\times\vec{N}(1) = \frac{1}{3}(2,1,2)\times\frac{1}{3\sqrt{5}}(5,-2,-4) = \frac{1}{9\sqrt{5}}(0, 18, -9)$

$= \frac{1}{\sqrt{5}}(0, 2, -1)$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 5 & -2 & -4 \end{vmatrix} = (-4+4,\ 10+8,\ -4-5) = (0, 18, -9)$$

Osculating plane: $\left.\begin{array}{l}\text{point: } (1,2,2) \\ \text{normal: } (0,2,-1)\end{array}\right\}$ $0(x-1)+2(y-2)-(z-2)=0$

or $2y - z = 2$

8. Bonus Question

On page 1 of a book, there is one circle of radius 1. On page 2, there are two circles of radius $\frac{1}{2}$. On page n, there are 2^{n-1} circles of radius 2^{-n+1}.

a) What is the sum of the areas of the circles on page n of the book?

b) Assuming the book continues indefinitely ($n \to \infty$), what is the sum of the areas of all the circles in the book?

p 1 1 circle radius 1 area π

p 2 2 circles radius $\frac{1}{2}$ area $2\pi\left(\frac{1}{2}\right)^2$

p 3 4 circles radius $\frac{1}{4}$ area $4\pi\left(\frac{1}{4}\right)^2$

p n 2^{n-1} circles radius $\frac{1}{2^{n-1}}$ area $2^{n-1}\pi\left(\frac{1}{2^{n-1}}\right)^2 = \frac{\pi}{2^{n-1}}$

$$\text{total area} = \sum_{i=1}^{n} \frac{\pi}{2^{n-1}} = \pi\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}\right)$$

$$= \sum_{j=0}^{n-1} \frac{\pi}{2^j} = \pi \sum_{j=0}^{n-1}\left(\frac{1}{2}\right)^j = \pi \sum_{j=0}^{n}\left(\frac{1}{2}\right)^j - \pi\left(\frac{1}{2}\right)^n$$

$$= \pi\left[\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} - \left(\frac{1}{2}\right)^n\right] = \pi\left(2-\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^n\right)$$

$$= \pi\, 2\left(1-\left(\frac{1}{2}\right)^n\right)$$

as $n \to \infty$, total area $\longrightarrow 2\pi$