ESC195 - Midterm Test #2
March 30, 2023
9:10 - 10:50 am
Instructor: J. W. Davis

Closed book, no aid sheets, no calculators
There are 7 questions, each worth 10 marks.
Plus a bonus question worth 5 marks.

1. Determine whether the sequence converges or diverges; if it converges, find the limit:

a) $a_n = 1 + \frac{10^n}{9^n}$ b) $a_n = \frac{\ln n}{\ln 2n}$ c) $a_n = \sqrt[n]{2^{1+3n}}$ d) $a_n = n \sin \frac{1}{n}$

a) $\lim_{n\to\infty} \left(\frac{10}{9}\right)^n \to \infty$ $\therefore$ a_n diverges

b) $a_n = \frac{\ln n}{\ln 2 + \ln n} = \frac{1}{\frac{\ln 2}{\ln n} + 1} \longrightarrow \frac{1}{0+1} = 1$ converges

c) $a_n = 2^{\frac{1+3n}{n}} \Rightarrow \lim_{n\to\infty} 2^{\frac{1+3n}{n}} = 2^{\left(\lim_{n\to\infty} \frac{1+3n}{n}\right)} = 2^3 = 8$ converges

d) $a_n = \frac{\sin \frac{1}{n}}{\frac{1}{n}} \Rightarrow \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$ converges

2. Determine whether the series converges or diverges:

a) $\sum_{n=1}^{\infty} \frac{n^2+1}{5^n}$ b) $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ c) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$

a) ratio test: $\left| \frac{(n+1)^2+1}{5^{n+1}} \cdot \frac{5^n}{n^2+1} \right| = \frac{1}{5}\left(\frac{n^2+2n+2}{n^2+1}\right) = \frac{1}{5}\left(\frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+1/n^2}\right) \to \frac{1}{5}$

$\frac{1}{5} < 1$ $\therefore$ convergent

b) root test: $(a_n)^{1/n} = \frac{n!}{n^4} = \frac{n(n-1)(n-2)(n-3)\cdot(n-4)!}{n\cdot n\cdot n\cdot n}$

$= \left(1-\frac{1}{n}\right)\left(n-\frac{2}{n}\right)\left(n-\frac{3}{n}\right)\cdot(n-4)! \to \infty$ diverges

$\therefore \sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$ diverges

c) $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} = \sum \frac{1}{n}\left(\frac{1}{n}\right)^{1/n}$ $\Rightarrow$ limit comparison with $\sum \frac{1}{n}$

$\Rightarrow \frac{\frac{1}{n}\left(\frac{1}{n}\right)^{1/n}}{1/n} = \left(\frac{1}{n}\right)^{1/n} \to 1$

$\therefore$ since $\sum \frac{1}{n}$ diverges $\therefore \sum \frac{1}{n^{1+1/n}}$ diverges

Alternate: consider $x < e^x$, $x \geq 0$ (eg. $\frac{d}{dx}(e^x - x) > 0$)

$\therefore x^{1/x} < e \Rightarrow x \cdot x^{1/x} < x \cdot e$

$\therefore n \cdot n^{1/n} < n \cdot e \Rightarrow \frac{1}{n \cdot n^{1/n}} = \frac{1}{n^{1+1/n}} > \frac{1}{ne}$

$\Rightarrow \frac{1}{e}\sum \frac{1}{n}$ diverges $\therefore \sum \frac{1}{n^{1+1/n}}$ diverges

by comparison test

3. (a) Determine the radius of convergence for the power series: $\sum_{k=1}^{\infty} \frac{k!x^k}{k^k}$

ratio test: $\left| \frac{(k+1)!\, x^{k+1}}{(k+1)^{k+1}} \cdot \frac{k^k}{k!\, x^k} \right| = \left| \frac{(k+1)\, x}{k+1} \cdot \left(\frac{k}{k+1}\right)^k \right| = |x| \left(\frac{k}{k+1}\right)^k$

$\Rightarrow \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k \Longrightarrow \lim_{k\to\infty} k \ln\left(\frac{k}{k+1}\right) = \lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+1}\right)}{1/k} \longrightarrow \frac{0}{0}$

$\lim_{x\to\infty} \frac{\ln\left(\frac{x}{x+1}\right)}{1/x} \stackrel{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \left(\frac{x^2}{x+1} - x\right) = \lim_{x\to\infty} \frac{x^2 - x^2 - x}{x+1}$

$= \lim_{x\to\infty} \frac{-x}{x+1} = \lim_{x\to\infty} \frac{-1}{1+1/x} \longrightarrow -1 \quad \therefore \lim_{k\to\infty} \frac{\ln\left(\frac{k}{k+1}\right)}{1/k} \to -1$

$\therefore \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^k \longrightarrow e^{-1} = \frac{1}{e}$

$\therefore \left|\frac{a_{k+1}}{a_k}\right| \longrightarrow \frac{1}{e}|x| \quad \therefore$ convergence for $|x| < e$ or $R = e$

(b) From first principles (that is, by finding derivatives) determine the first four terms ($n = 3$) of the Maclaurin series for $f(x) = \ln(b + x)$. (Here, b is some fixed number, with $b > 0$.)

$f(x) = \ln(b+x) \qquad f(0) = \ln b$

$f'(x) = \frac{1}{b+x} \qquad f'(0) = \frac{1}{b}$

$f''(x) = \frac{-1}{(b+x)^2} \qquad f''(0) = \frac{-1}{b^2}$

$f'''(x) = \frac{2}{(b+x)^3} \qquad f'''(0) = \frac{2}{b^3}$

$\Rightarrow \ln(b+x) \approx \ln b + \frac{1}{b}\frac{x}{1!} - \frac{1}{b^2}\frac{x^2}{2!} + \frac{2}{b^3}\frac{x^3}{3!} - \cdots$

4. (a) Prove that if both $\sum a_n$ and $\sum b_n$ are convergent series with positive terms, then $\sum a_n b_n$ is convergent.

Given that Σa_n converges, then $a_n \to 0$

$\therefore$ for $n > N$, $a_n < 1$

$\therefore$ For $n > N$, $a_n \cdot b_n < b_n$

Since all terms are $+^{ve}$: $0 < a_n b_n < b_n$

$\Rightarrow 0 < \Sigma a_n b_n < \Sigma b_n$

since Σb_n converges, $\Sigma a_n b_n$ converges by pinching theorem

(b) The sequence a_n is monotonic with positive terms and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges.

$\Rightarrow$ The sum of the first n terms is:

$(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + \dots + (n-1)(a_{n-1} - a_n) + n(a_n - a_{n+1})$

$= a_1 + a_2 + a_3 + \dots + a_{n-1} + a_n - n a_{n+1} = \underbrace{\sum_{k=1}^{n} a_k}_{\text{converges as } n \to \infty} - n a_{n+1}$

$\Rightarrow$ Since a_n is monotonic decreasing: $a_{n+1} < a_n < a_{n-1} \dots < a_3 < a_2 < a_1$

$\therefore n a_{n+1} < \sum_{k=1}^{n} a_k$

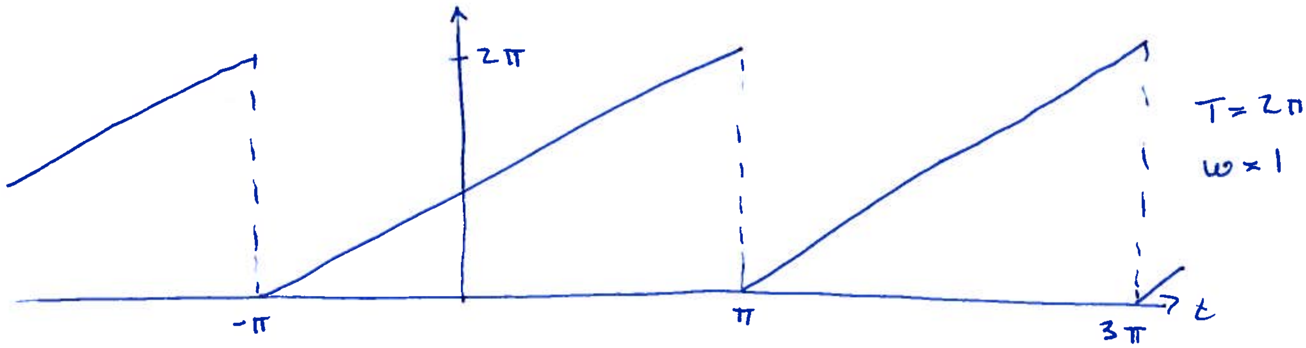
$\therefore \sum_{n=1}^{\infty} n(a_n - a_{n+1}) \leq \sum_{n=1}^{\infty} a_n$ which converges

$\therefore \Sigma n(a_n - a_{n+1})$ converges

5. Find the Fourier series; ie., evaluate the Fourier coefficients, for the function:

$$f(t) = \pi + t, \quad -\pi \leq t \leq \pi$$

Provide a sketch of the function, and a sketch of what you **imagine** the sum of the first few terms of the series would look like.



$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + t)\,dt = \frac{1}{\pi}\left[\pi t + \frac{t^2}{2}\right]_{-\pi}^{\pi} = 2\pi$$

$$\therefore \frac{a_0}{2} = \pi$$

$$\underset{n\geq 1}{a_n} = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt$$

let $u = t$ $\quad dv = \cos nt$
$du = dt$ $\quad v = \frac{1}{n}\sin nt$

$$= \frac{1}{\pi}\left[\frac{t}{n}\underbrace{\sin nt}_{0}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{n}\underbrace{\sin nt}_{0}\,dt = 0$$

(Alternately, one can note that $f(t) - \pi = t$ is an odd function, thus all $a_n = 0$, $n \geq 1$)

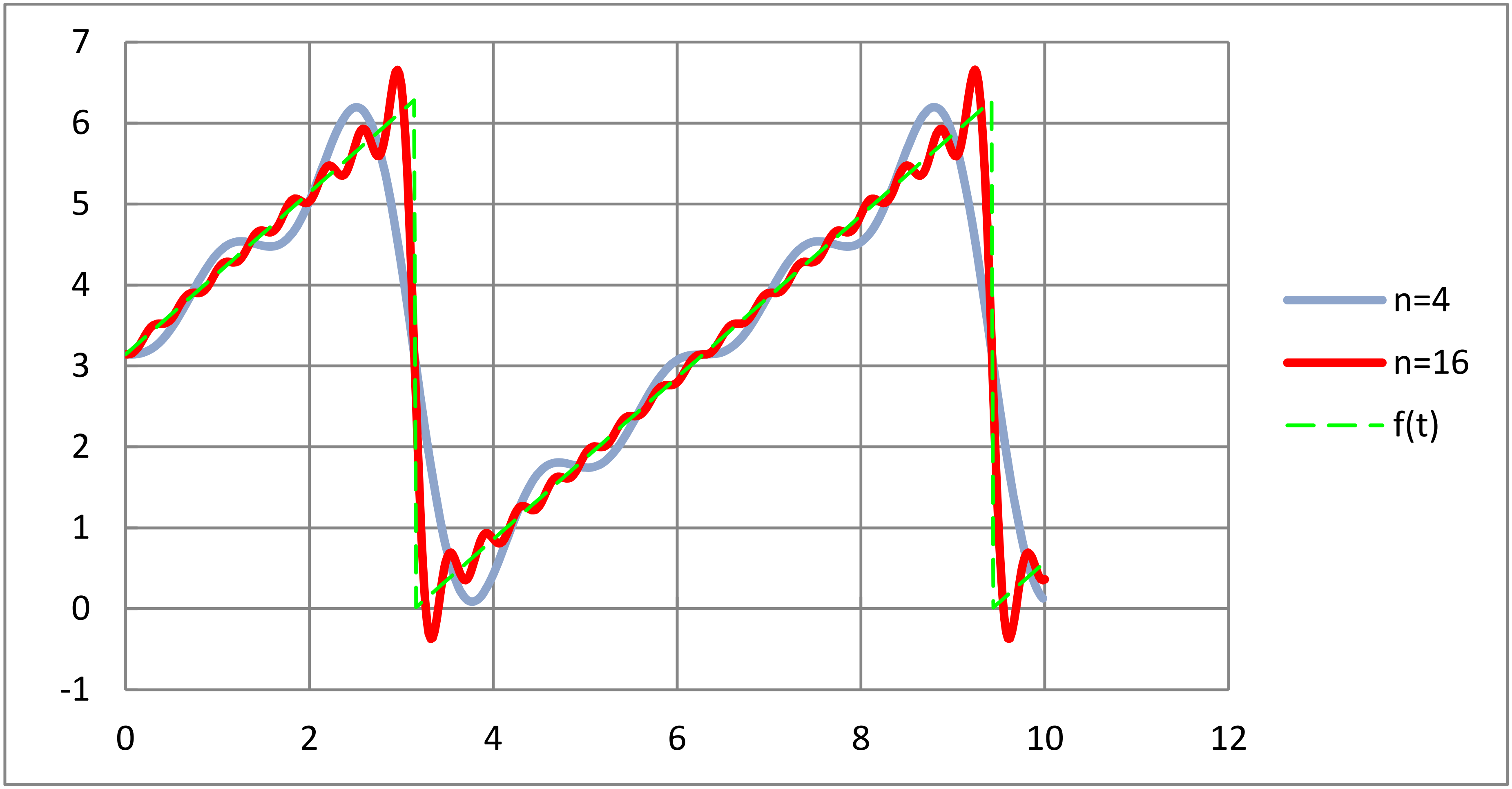
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi + t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt$$

let $u = t$ $\quad dv = \sin nt$
$du = dt$ $\quad v = -\frac{1}{n}\cos nt$

$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos nt\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\underbrace{\cos nt}_{0} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd}\end{cases}$$

$$\therefore f(t) = \pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nt)$$


7
6
5
4
3
2
1
0
-1
0
2
4
6
8
10
12
n=4
n=16
f(t)

6. The motion of a particle is given by $\vec{r}(t) = e^t\,\hat{\imath} - e^t \sin t\,\hat{\jmath} + e^t \cos t\,\hat{k}$. Determine the unit tangent vector, the unit normal vector and the tangential and normal components of acceleration of this particle at time $t = 0$. Also find the curvature of its path at $t = 0$.

$\vec{r}(t) = (e^t, -e^t \sin t, e^t \cos t)$ $\qquad \vec{r}(0) = (1, 0, 1)$

$\vec{r}\,'(t) = (e^t, -e^t(\sin t + \cos t), e^t(\cos t - \sin t))$ $\qquad \vec{r}\,'(0) = (1, -1, 1)$

$\vec{r}\,''(t) = (e^t, -e^t(\sin t + \cos t + \cos t - \sin t), e^t(\cos t - \sin t - \sin t - \cos t))$

$= (e^t, -2e^t \cos t, -2e^t \sin t)$ $\qquad \vec{r}\,''(0) = (1, -2, 0)$

$\frac{ds}{dt} = \|\vec{r}\,'(t)\| = e^t \sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2}$

$= e^t \sqrt{1 + \cos^2 t + 2\sin t \cos t + \sin^2 t + \cos^2 t - 2\sin t \cos t + \sin^2 t} = e^t\sqrt{3}$

$\Rightarrow \vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\sqrt{3}}(1, -\sin t - \cos t, \cos t - \sin t)$ $\qquad \vec{T}(0) = \frac{1}{\sqrt{3}}(1, -1, 1)$

$\Rightarrow \frac{d^2s}{dt^2} = \frac{d}{dt}(e^t\sqrt{3}) = e^t\sqrt{3}$

$\vec{T}\,'(t) = \frac{1}{\sqrt{3}}(0, -\cos t + \sin t, -\sin t - \cos t)$ $\qquad \vec{T}\,'(0) = \frac{1}{\sqrt{3}}(0, -1, -1)$

$\|\vec{T}\,'(0)\| = \sqrt{\frac{2}{3}}$ $\qquad \Rightarrow \vec{N}(0) = \frac{\vec{T}\,'(0)}{\|\vec{T}\,'(0)\|} = \frac{1}{\sqrt{2}}(0, -1, -1)$

$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{(ds/dt)^3} = \frac{\|(1,-1,1) \times (1,-2,0)\|}{(\sqrt{3})^3}$ $\qquad \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{vmatrix} = (2, 1, -1)$

$= \frac{\sqrt{4+1+1}}{3^{3/2}} = \frac{\sqrt{6}}{3\sqrt{3}} = \frac{\sqrt{2}}{3}$

$\vec{a} = \frac{d^2s}{dt^2}\vec{T} + \kappa\left(\frac{ds}{dt}\right)^2 \vec{N} = \sqrt{3} \cdot \frac{1}{\sqrt{3}}(1, -1, 1) + \frac{\sqrt{2}}{3}(\sqrt{3})^2 \frac{1}{\sqrt{2}}(0, -1, -1)$

$= \underbrace{(1, -1, 1)}_{\vec{a}_T} + \underbrace{(0, -1, -1)}_{\vec{a}_N} = (1, -2, 0) = \vec{r}\,''(0)$

7. (a) Find the limit if it exists, or show that it does not exist: $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$

$0 \le \frac{x^2 \sin^2 y}{x^2+2y^2} \le \sin^2 y$ since $\frac{x^2}{x^2+2y^2} \le 1$

$\lim_{(x,y)\to(0,0)} \sin^2 y = 0$ $\therefore \lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+y^2} = 0$

by pinching theorem

(b) Find the first partial derivatives of the function: $f(x,t) = \sqrt{3x+4t}$

$f(x,t) = (3x+4t)^{1/2}$

$f_x = \frac{1}{2}(3x+4t)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+4t}}$

$f_t = \frac{1}{2}(3x+4t)^{-1/2} \cdot 4 = \frac{4}{2\sqrt{3x+4t}} = \frac{2}{\sqrt{3x+4t}}$

8. Bonus Question

Let a and b be positive numbers and consider the series:

$$a-\frac{b}{2}+\frac{a}{3}-\frac{b}{4}+\frac{a}{5}-\frac{b}{6}+\cdots$$

(a) Express this series in $\sum$ notation.

(b) For what values of a and b is the series absolutely convergent? Conditionally convergent?

a) $$\sum_{n=1}^{\infty}\left(\frac{a}{2n-1}-\frac{b}{2n}\right)=\sum_{n=1}^{\infty}\left(\frac{2na-2nb+b}{2n(2n-1)}\right)=\sum_{n=1}^{\infty}\left(\frac{2n\left(a-b+\frac{b}{2n}\right)}{2n(2n-1)}\right)$$

$$=\sum_{n=1}^{\infty}\frac{a-b+\frac{b}{2n}}{2n-1}$$

b) case ①: $a=b=0$: $0-0+0-0+\cdots=0$ $\therefore$ absolutely convergent.

case ②: $a=b\neq 0$: $a\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)=a\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

$\Rightarrow$ Conditionally convergent alternating harmonic series

case ③: $a\neq b$ $\Rightarrow$ limit comparison test with $\frac{a-b}{2n-1}$

$$\lim_{n\to\infty}\frac{a-b+\frac{b}{2n}}{2n-1}\cdot\frac{2n-1}{a-b}=\lim_{n\to\infty}\frac{a-b+\frac{b}{2n}}{a-b}\to 1$$

$\Rightarrow \sum\frac{a-b}{2n-1}$ diverges by integral test

$\therefore \sum\frac{a-b+\frac{b}{2n}}{2n-1}$ diverges

Alternate:

a) $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(a+b)+(a-b)}{2n}=\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(a+b)}{2n}}_{\text{convergent}}+\sum\frac{a-b}{2n}$$

b) case ③: $a\neq b$ $\Rightarrow \sum\frac{a-b}{2n}=\frac{a-b}{2}\sum\frac{1}{n}$ diverges