UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

Winter 2024 Term Test 2

ECE 286 H1S

Duration: 100 minutes

Aids Allowed: The exam is Type D. Two two-sided aid sheets (prepared by you). One non-programmable calculator. Distribution tables will be provided in a separate booklet.

Do ***not*** *turn this page until you have received the signal to start.*
In the meantime, write your name, student number, UTORid and email address below (please do this now!) and *carefully* read *all* the information on the rest of this page.

Marking Guide

Nº 1: ______/ 7

Nº 2: ______/ 8

Nº 3: ______/10

Nº 4: ______/ 9

Nº 5: ______/ 9

Nº 6: ______/ 7

TOTAL: ______/50

- This test consists of 6 question on 10 page (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete.*
- Answer each question directly on the test paper, in the space provided.

Question 1. [7 MARKS]

Let the sample X_1, X_2, X_3, X_4 consist of iid versions of a Poisson random variable X with mean $\alpha = 4$. Determine whether they are biased or unbiased, and find the most efficient one(s). You must show your work on any calculations that back your answer.

(a) $\hat{\alpha}_1 = (X_1 + X_2)/2$
(b) $\hat{\alpha}_2 = (3X_3 + 2X_4)/6$
(c) $\hat{\alpha}_3 = (X_1 + 2X_2)/3$
(d) $\hat{\alpha}_4 = (X_1 + X_2 + X_3 + X_4)/4$

Solution:
For the estimator $\hat{\alpha}_1$:

- $E[\hat{\alpha}_1] = E\left[\frac{X_1+X_2}{2}\right] = \frac{E[X_1]+E[X_2]}{2} = \frac{\alpha+\alpha}{2} = \alpha$
- $\mathrm{Var}(\hat{\alpha}_1) = \mathrm{Var}\left[\frac{X_1+X_2}{2}\right] = \frac{\mathrm{Var}(X_1)+\mathrm{Var}(X_2)}{4} = \frac{\alpha+\alpha}{4} = \frac{\alpha}{2}$
- Since $E[\hat{\alpha}_1] = \alpha$, the estimator is unbiased.

For the estimator $\hat{\alpha}_2$:

- $E[\hat{\alpha}_2] = E\left[\frac{3X_3+2X_4}{6}\right] = \frac{3E[X_3]+2E[X_4]}{2} = \frac{3\alpha+2\alpha}{6} = \frac{5}{6}\alpha$
- $\mathrm{Var}(\hat{\alpha}_2) = \mathrm{Var}\left[\frac{3X_3+2X_4}{2}\right] = \frac{9\mathrm{Var}(X_3)+4\mathrm{Var}(X_4)}{4} = \frac{9\alpha+4\alpha}{4} = \frac{13}{4}\alpha$
- Since $E[\hat{\alpha}_2] \neq \alpha$, the estimator is biased.

For the estimator $\hat{\alpha}_3$:

- $E[\hat{\alpha}_3] = E\left[\frac{X_1+2X_2}{3}\right] = \frac{E[X_1]+2E[X_2]}{3} = \frac{\alpha+2\alpha}{3} = \alpha$
- $\mathrm{Var}(\hat{\alpha}_3) = \mathrm{Var}\left[\frac{X_1+2X_2}{3}\right] = \frac{\mathrm{Var}(X_1)+4\mathrm{Var}(X_2)}{9} = \frac{\alpha+4\alpha}{9} = \frac{5\alpha}{9}$
- Since $E[\hat{\alpha}_3] = \alpha$, the estimator is unbiased.

For the estimator $\hat{\alpha}_4$:

- $E[\hat{\alpha}_4] = E\left[\frac{X_1+X_2+X_3+X_4}{4}\right] = \frac{E[X_1]+E[X_2]+E[X_3]+E[X_4]}{4} = \alpha$
- $\mathrm{Var}(\hat{\alpha}_4) = \mathrm{Var}\left[\frac{X_1+X_2+X_3+X_4}{4}\right] = \frac{\mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\mathrm{Var}(X_3)+\mathrm{Var}(X_4)}{16} = \frac{4\alpha}{16} = \frac{\alpha}{4}$
- Since $E[\hat{\alpha}_4] = \alpha$, the estimator is unbiased.

Given these results, $\hat{\alpha}_4$ is the most efficient estimator as it has the lowest variance while still being unbiased.

Question 2. [8 MARKS]

A binary transmission channel introduces bit errors with a probability of 0.15. Estimate the probability that there are 20 or fewer errors in 100-bit transmissions. Draw the normal standard distribution including its mean, the relevant z-score, and the probability found.

Solution:

- $$\mu = np = 100 \times 0.15 = 15$$
- $$\sigma^2 = np(1-p) = 100 \times 0.15 \times (1 - 0.15) = 12.75$$
- $$\sigma = \sqrt{12.75} \approx 3.57$$
- $$P(X \leq x) \approx P\left(Z \leq \frac{x + 0.5 - np}{\sqrt{npq}}\right),$$
- $$\begin{aligned} z &= \frac{20.5 - \mu}{\sigma} \\ &= \frac{20.5 - 15}{3.57} \\ &= \frac{5.5}{3.57} \end{aligned}$$
- $$z = 1.54$$
- $$P(Z \leq 1.54) = 0.9382$$

(Alternative solution:) The probability that there are 20 or fewer errors is given by the cumulative probability:

$$P(X \leq 20) = \sum_{k=0}^{20} \binom{100}{k} p^k (1-p)^{n-k}$$

(Drawing:)

Question 3. [10 marks]

Let T be a random variable for the manufacturing time, in hours, for a specialized component in an aerospace factory. The random variable M represents the actual processing time on the machine for the component. This manufacturing time T includes a fixed preparation time of 4 hours for setting up the machinery. Thus, we can model it as $T = M + 4$, where M has the density function

$$f(m) = \begin{cases} \frac{32}{(m+4)^3}, & m > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Part (a) [3 marks] Find the probability density function of the random variable T.

Solution:

- The inverse of $t = m + 4$ is $m = t - 4$, for $t > 4$
- from which we obtain $|J| = 1$. Therefore,
- $$g(t) = f(t-4)|J| = \frac{32}{t^3},$$
- for $t > 4$

Part (b) [3 marks] Using the density function of T, find the probability that the manufacturing time for a component will exceed 8 days.

Solution: The probability $P(T > 8)$ is given by

- $$P(T > 8) = 32 \int_8^\infty t^{-3}\, dt$$
- $$= -16t^{-2}\Big|_8^\infty = \frac{1}{4}$$
- $$= \frac{1}{4}$$

Part (c) [1 MARK] Calculate the moment generating function (MGF) of T.

Solution: The moment generating function $M_T(m)$ of T is defined as

$$M_T(x) = E[e^{xT}] = \int_4^{\infty} e^{xt} g(t)\, dt = \int_4^{\infty} e^{xt} \frac{32}{t^3}\, dt$$

Part (d) [3 MARKS] Find the first moment of T.

Solution: To calculate the first moment (mean) of T, we use the density function $g(y)$:

$$\mu_1' = \int_4^{\infty} t g(t)\, dt = \int_4^{\infty} t \frac{32}{t^3}\, dt = 32 \int_4^{\infty} t^{-2}\, dt.$$

Solving the integral, we get:

$$\begin{aligned} \mu_1' = E[T] &= 32 \left[-\frac{1}{t} \right]_4^{\infty} \\ &= 32 \left[0 - \left(-\frac{1}{4} \right) \right] \\ &= 32 \times \frac{1}{4} \end{aligned}$$

$$E[T] = 8$$

Question 4. [9 MARKS]

The production line of a smartphone factory consistently measures the battery life of its devices. Historically, the battery life is approximately normally distributed with an average operational time of 52.5 hours and a standard deviation of 3.3 hours. Quality control engineers conduct tests on 150 random batches of 25 smartphones each to assess the consistency of battery life across different production runs. The average battery life for each batch is recorded to the nearest tenth of an hour.

Part (a) [2 MARKS] Determine the mean and standard deviation of the sampling distribution of $\bar{X}$.

Solution:

- Given the population mean
- $\mu_{\bar{X}} = \mu = 52.5$
  and the population standard deviation $\sigma = 3.3$, the standard deviation of the sampling distribution of $\bar{X}$ is
- $\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.3}{5} = 0.66$

Part (b) [4 MARKS] Determine the number of sample means that fall between 51.8 and 53.7 centimeters inclusive

Solution:

- For
$$z_1 = \frac{(51.8 - 52.5)}{0.66} = -1.06$$
- and
$$z_2 = \frac{(53.7 - 52.5)}{0.66} = 1.82$$

  the probability that $\bar{X}$ falls between 51.8 and 53.7 is
- $$P(51.8 < \bar{X} < 53.7) = P(-1.06 < Z < 1.82) = 0.9656 - 0.1446 = 0.821$$

  Therefore, the number of sample means between 51.8 and 53.7 inclusive is
- approximately $(150)(0.821) = 123$

Part (c) [3 marks] Calculate the number of sample means falling above 53 hours.

Solution:

- For
$$z = \frac{(53 - 52.5)}{0.66} = 0.7576$$
the probability that $\bar{X}$ falls above 53 is
- $$P(\bar{X} > 53) = P(Z > 0.7576) = 1 - P(Z < 0.7576) = 0.2236$$
Therefore,
- approximately $(150)(0.2236) = 34$ sample means fall above 53 hours.

Question 5. [9 MARKS]

An artisanal bakery specializes in making gourmet loaves of bread. The weight of each loaf is critical and is measured precisely. A sample of loaves produced in a week yielded the following weights in kilograms: 0.98, 1.01, 1.03, 0.99, 1.00, 1.00, 0.99, 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 1.01, 1.02, 0.99. Assume the weights are normally distributed for all computations.

Part (a) [3 MARKS] Compute a 99% confidence interval on the mean weight of the bread loaves.

Solution:

- $n = 16$
- sample mean $\bar{x} = 1.0025$ kg
- sample standard deviation $s = 0.0202$ kg
- With a confidence level of $1 - \alpha = 0.99$ and a t-value for 15 degrees of freedom of $t_{0.005} = 2.947$,
- the 99% confidence interval for the mean weight is:

$$1.0025 \pm \frac{(2.947)(0.0202)}{\sqrt{16}}$$

- $(0.9876, 1.0174)$ kilograms.

Part (b) [3 MARKS] Compute a 99% prediction interval on the weight of the next loaf to be produced.

Solution:

- The variance for the predicted new loaf is

$$s'^2 = s^2/n + s^2 = s^2(1 + 1/n) = (0.0202)\sqrt{1 + \frac{1}{16}}$$

- The 99% prediction interval for the weight of a new loaf is:

$$1.0025 \pm (2.947)(0.0202)\sqrt{1 + \frac{1}{16}}$$

- $(0.9411, 1.0639)$ kilograms.

Part (c) [3 MARKS] Compute a 99% tolerance interval for coverage of the central 95% of the distribution of loaf weights.

Solution:

- For $n = 16$, $1 - \gamma = 0.99$, and $1 - \alpha = 0.95$, we find the critical value

$$k = 3.421$$

- The tolerance limits for the central 95% of the bread weights distribution are:

$$1.0025 \pm (3.421)(0.0202)$$

- $(0.9334, 1.0716)$ kilograms.

Question 6. [7 MARKS]

Consider the case of two competing tech startups that have developed new smartphone apps. To compare user engagement, the average daily usage times for their apps are recorded in minutes over a week.

Startup	Average Daily Usage Time (minutes)
A	94, 98, 103, 87, 110
B	123, 82, 118, 92, 88, 97, 175

Compute a 90% confidence interval for the difference in the average daily usage times between the two startup apps. Assume that the differences in daily usage times are approximately normally distributed with unequal variances.

Solution: Given the sample sizes $n_A = 5$ and $n_B = 7$, the sample means

- $\bar{x}_A = 98.4$ and
- $\bar{x}_B = 110.7$, and the sample standard deviations
- $s_A = 8.375$ and
- $s_B = 32.185$

the degrees of freedom for the confidence interval are calculated by:

- $$v = \left(\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}\right)^2 \Big/ \left(\frac{(s_A^2/n_A)^2}{n_A - 1} + \frac{(s_B^2/n_B)^2}{n_B - 1}\right)$$
- $$v = \left(\frac{8.375^2}{5} + \frac{32.185^2}{7}\right)^2 \Big/ \left(\frac{(8.375^2/5)^2}{4} + \frac{(32.185^2/7)^2}{6}\right) \approx 7$$

Using the t-distribution, the critical value for t with 7 degrees of freedom at a 90% confidence level is:

- $t_{0.05,7} = 1.895$

The 90% confidence interval for the difference between the average daily usage times is:

- $$(\bar{x}_A - \bar{x}_B) \pm t_{0.05,7}\sqrt{\frac{s_A^2}{n_A} + \frac{s_B^2}{n_B}},$$
- $$(110.7 - 98.4) \pm 1.895\sqrt{\frac{8.375^2}{5} + \frac{32.185^2}{7}} = 12.3 \pm 24.2,$$

which gives us a range of:

- $-11.9 < \mu_B - \mu_A < 36.5$

[Use the space below for rough work. This page will not be marked unless you clearly indicate the part of your work that you want us to mark.]