AER210 VECTOR CALCULUS and FLUID MECHANICS

Midterm Test # 2

Duration: 1 hour, 50 minutes

2 December 2021

Closed Book, no aid sheets, non-programmable calculators are allowed

Instructor: Prof. Alis Ekmekci

Family Name: Alis Ekmekci

Given Name:

Student #: Solutions

TA Name/Tutorial #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	9	
2	15	
3	10	
4	10	
5	8	
6	10	
7	10	
8	20	
9	8	
TOTAL	100	

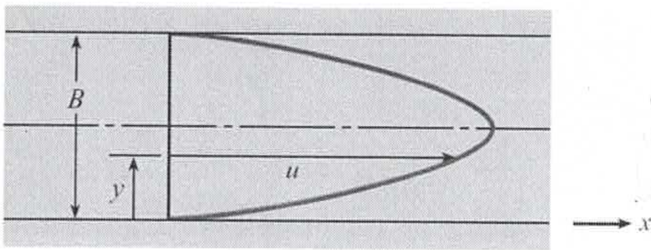
$\tau=\mu\frac{du}{dy}$ $\quad -\nabla p+\rho\vec{g}=\rho\vec{a}$ $\quad \frac{p}{\rho}+\frac{V^2}{2}+gz=constant$ (Bernoulli equation)

$\sum\vec{F}_{CV}=\frac{d}{dt}\iiint_{CV}\vec{V}(\rho d\forall)+\oiint_{CS}\vec{V}(\rho\vec{V}\cdot d\vec{A})$ (general form of the momentum eq. applicable to a CV)

The gravitational acceleration: g = 10 m/s^2

 Name: ____________________

1) a) (4 marks) Suppose that glycerin at $20° C$ is flowing between two stationary parallel walls. The velocity distribution for this viscous flow between the walls is given by $u = -\dfrac{1}{2\mu}\dfrac{dp}{dx}(By - y^2)$ where μ is the viscosity of the glycerin with a constant value of $\mu = 1.41$ Ns/m^2, $\dfrac{dp}{dx}$ is the pressure gradient which has a constant value of $\dfrac{dp}{dx} = -\ 1600\ N/m^3$, and $B = 0.05\ m$ is the space between the walls. Notice that the profile of this velocity distribution is also sketched in the schematic below. What are the velocity and shear stress at the wall (i.e., at $y = 0\ mm$ location)?



$$u(y=0\,mm) = -\frac{1}{2.(1.41)}.(-1600)\left[(0.05).0 - 0^2\right]$$

$$\boxed{u(y=0\,mm) = 0}$$

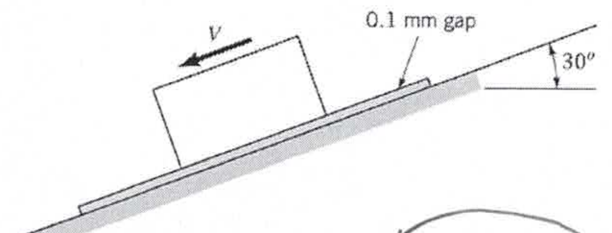
Students can also directly write that $\boxed{u\big|_{y=0\,mm} = 0}$ as it is obvious because of the No-SLIP condition.

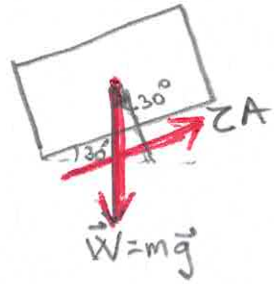
$$\tau = \mu.\frac{du}{dy} = \mu.\frac{d}{dy}\left[-\frac{1}{2\mu}\frac{dp}{dx}(By-y^2)\right] = \cancel{\mu}\left(-\frac{1}{2\cancel{\mu}}\right)\frac{dp}{dx}(B-2y) = -\frac{1}{2}\frac{dp}{dx}(B-2y)$$

$$\tau\big|_{y=0\,mm} = -\frac{1}{2}.(-1600)(0.05-\cancel{2.0}) = \boxed{40\ \frac{N}{m^2}}$$

b) (5 marks) A 10 kg block slides down a smooth 30° inclined surface, as shown in the figure below. Determine the terminal velocity of the block if the 0.1 mm gap between the block and the surface contains oil at 60 °F with a viscosity of $\mu = 0.38\ Ns/m^2$. Assume the velocity distribution in the gap is linear, and the area of the block in contact with the oil is 0.1 m^2.

(Note that $\sin(30°) = \dfrac{1}{2}$, $\cos(30°) = \dfrac{\sqrt{3}}{2}$)





$\Sigma F_x = 0 \Rightarrow \boxed{mg\sin 30° = \tau A}$; $\tau = \mu\dfrac{V}{gap}$

$$mg\sin 30° = \mu\frac{V}{gap}.A$$

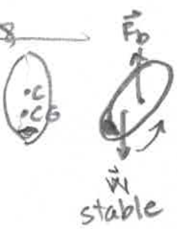
$$V = \frac{m.g.\sin 30°.(gap)}{\mu.A} = \frac{10.10.(0.5).(0.1\times10^{-3})}{(0.38).(0.1)} = 0.13\ m/s.$$

Name: ____________________

2) a) (6 marks) Indicate whether the statement is True (T) or False (F).

__T__ A fluid at rest is at a state of zero shear force.

__F__ Surface forces are forces that are proportional to the mass of a fluid element.



__T__ If the center of gravity of a fully submerged body is below its center of buoyancy, then the body is stable.

__T__ Viscosity of gases increases with increasing temperature.

__T__ Dye or smoke injection into a fluid flow at a particular point gives a streakline.

__F__ $\vec{F} = m\vec{a}$ is the Eulerian form of the Newton's second law.

b) (4 marks) Indicate the four major assumptions used in the derivation of the Bernoulli equation.

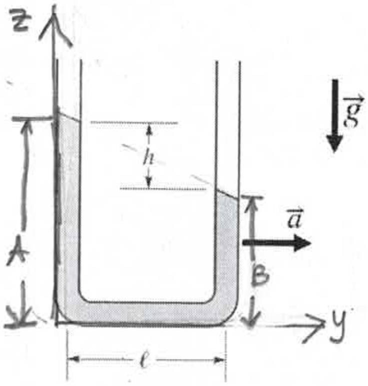
- steady flow
- incompressible flow
- frictionless (inviscid) flow
- flow along a streamline

c) (5 marks) Complete the statements below by adding the missing words on the dashed lines:

- In a __Newtonian__ fluid, the shear stress varies linearly with the rate of deformation, and the constant of proportionality in this variation is called __viscosity__.
- A __streamline__ is a line that is tangent to the local velocity vector at every point along the line at the instant in time it is drawn.
- In a __steady__ flow, fluid properties (such as velocity, pressure, density etc.) at any given point are time independent.
- If the velocity distribution and the density are uniform over an area, density times velocity component normal to the area times area gives the __mass flow rate__ through that area.

Name: ____________________

3) (10 marks) The U-tube manometer in the figure below is used to measure the constant horizontal acceleration of the cart on which it sits. Develop an expression for the acceleration of the cart in terms of the liquid height difference between the manometer arms h, the liquid density ρ, the local acceleration of gravity g, and the length l. In this question, you are required to develop the expression starting from the equation $-\nabla p+\rho\vec{g}=\rho\vec{a}$ and showing how you arrive at the result.



$$-\vec{\nabla} p + \rho\vec{g} = \rho\vec{a}$$

$$-\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right) - \rho g\vec{k} = \rho \underbrace{a_x}_{a_x=0}\vec{i} + \rho \underbrace{a_y}_{a_y=a}\vec{j} + \rho \underbrace{a_z}_{a_z=0}\vec{k}$$

$$-\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right) - \rho g\vec{k} = \rho a\vec{j}$$

$\vec{i}$: $\boxed{\frac{\partial p}{\partial x} = 0}$

$\vec{j}$: $\boxed{\frac{\partial p}{\partial y} = -\rho a}$

$\vec{k}$: $-\frac{\partial p}{\partial z} - \rho g = 0 \Rightarrow \boxed{\frac{\partial p}{\partial z} = -\rho g}$

$\Big\}\ p = p(y, z)$

$\frac{\partial p}{\partial y} = -\rho a \Rightarrow \int dp = -\int \rho a\, dy \Rightarrow p = -\rho a y + f(z)$

$\frac{\partial p}{\partial z} = -\rho g \Rightarrow \int dp = -\int \rho g\, dz \Rightarrow p = -\rho g z + f(y)$

$\Big\} \Rightarrow \boxed{p = -\rho a y - \rho g z + C}$

@ $y=0,\ z=A \Rightarrow p(y=0, z=A) = P_{atm} \rightarrow P_{atm} = -\rho a 0 - \rho g A + C$ (1)

@ $y=\ell,\ z=B \Rightarrow p(y=\ell, z=B) = P_{atm} \rightarrow P_{atm} = -\rho a \ell - \rho g B + C$ (2)

Equating eqns. (1) & (2) $\Rightarrow$ $\cancel{-\rho a 0} - \rho g A + \cancel{C} = -\rho a \ell - \rho g B + \cancel{C}$

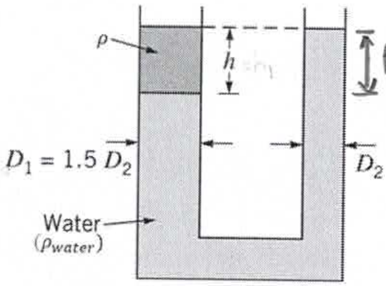
$\cancel{\rho} a \ell = \cancel{\rho} g \underbrace{(A-B)}_{h}$

$$\boxed{a = g\frac{h}{\ell}}$$

Name: ____________________

4) a) (5 marks) The U-shaped tube shown in the figure below initially contains water only. A second liquid with a density lesser than water is placed on top of the water with no mixing occurring. Can the height, h, of the second liquid be adjusted so that the left and right levels are at the same height? Provide proof of your answer.

$\rho < \rho_{water}$



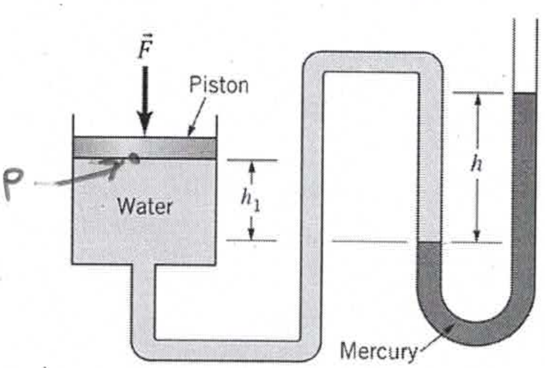
$P_{atm} + \rho_2 g h - \rho_{water} g h_w = P_{atm}$

$\rho_2 g h = \rho_{water} g h_w$

If $h = h_w$ then $\boxed{\rho_2 = \rho_{water}}$

* For left and right levels to be at the same height, the density of the second liquid should be equal to water's density. Since $\rho \neq \rho_{water}$, the configuration shown in the figure is not possible. No!

b) (5 marks) A piston having a cross-sectional area of 0.07 m^2 is located in a cylinder containing water, as shown in the figure below. An open U-tube manometer is connected to the cylinder as shown. For $h_1 = 60\ mm$ and $h = 100\ mm$, what is the value of the applied force, F, acting on the piston? The weight of the piston is negligible. The density of water is $\rho_{water} = 1000\ kg/m^3$ and the density of mercury is $\rho_{Hg} = 13300 kg/m^3$.



$P + \rho_{water} g h_1 - \rho_{Hg} g h = 0$ ← atmospheric gage pressure

$\boxed{P = \rho_{Hg} g h - \rho_{water} g h_1}$

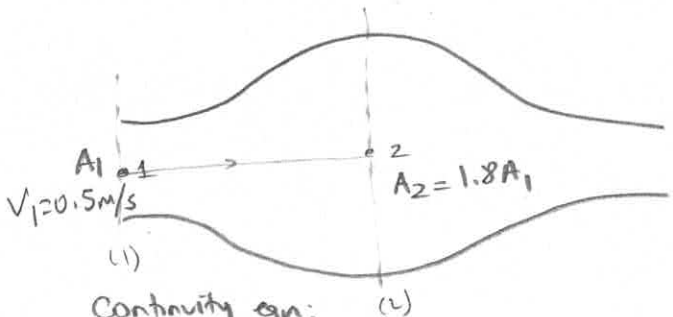
$F = P \cdot A \Rightarrow \boxed{F = (\rho_{Hg} g h - \rho_{water} g h_1) \cdot A}$

$F = (13300 \cdot 10 \cdot 0.1 - 1000 \cdot 10 \cdot 0.06) \cdot 0.07$

$F = 889\,N$

Name: ____________________

5) (8 marks) Blood flows with a velocity of 0.5 m/s in an artery. It then enters an aneurysm in the artery (an area with weakened and stretched artery walls that cause the ballooning of the vessel) where cross-sectional area is 1.8 times that of the artery. Determine the pressure difference between the blood in the aneurysm and that in the normal artery. Assume that the flow is steady and inviscid, and the gravitational effects are negligible. $(\rho_{blood} = 1000\ kg/m^3)$



Continuity eqn:

$$V_1 A_1 = V_2 A_2 \Rightarrow V_2 = V_1 \cdot \frac{A_1}{A_2} = (0.5) \cdot \frac{1}{1.8} = 0.278\ m/s$$

Bernoulli eqn. between point (1) & (2)

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + \cancel{g z_1} = \frac{P_2}{\rho} + \frac{V_2^2}{2} + \cancel{g z_2}$$

$$P_2 - P_1 = \rho \frac{(V_1^2 - V_2^2)}{2} = \frac{1000}{2} \cdot \left(0.5^2 - \left(\frac{0.5}{1.8}\right)^2\right) = \underline{\underline{86.42\ Pa}}$$

Name: ____________________

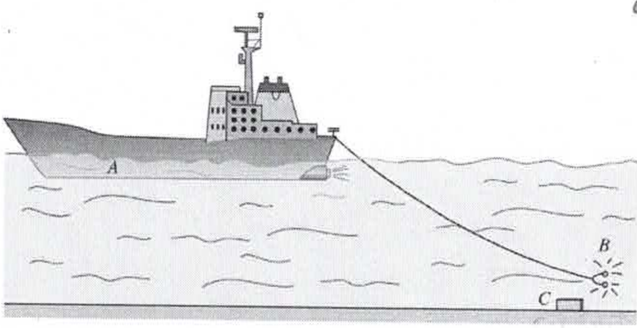
6) As shown in the figure below, a noisemaker "B" is towed behind a minesweeper ship "A" to set off enemy acoustic mines such as at "C". The drag force (F_D) is assumed to be a function of the speed (V) of the ship, the density (ρ) and viscosity (μ) of the fluid, and the diameter (D) the noisemaker.

a) (6 marks) Using the method of repeating variables in dimensional analysis and **choosing the repeating variables as ρ, V and D,** find the dimensionless parameters (π terms) describing the relationship between the variables. Write the non-dimensional form of the functional relationship between the variables for this problem.

b) (2 marks) The drag force (F_D) of the noisemaker is to be studied in a water tunnel with a $1/4$ scale model (i.e., the size of the model is $1/4$ of the size of the prototype). If the prototype towing speed by the ship is 3 m/s (which is the speed for the full-scale, real noisemaker used in the ocean), determine what the velocity of the water tunnel should be for the model tests.

c) (2 marks) If the water tunnel experiments on the 1/4-scale model resulted in a drag force of 900 N for the model, determine the drag force on the prototype (the full-scale, real noisemaker in the ocean).

Note: The density and viscosity of the water in the water tunnel can be taken to be identical to those in the ocean.



a) $F_D = f_1(V, \rho, \mu, D)$

$[F_D] = [\text{Mass}]\,[\text{acceleration}] = M \cdot \frac{L}{T^2}$

$[V] = \frac{L}{T}$

$[\rho] = \frac{M}{L^3}$

$[\mu] = \left[\frac{\text{Force}}{\text{Area}}\right]\left[\frac{\text{Length}}{\text{Velocity}}\right] = \frac{M\cancel{L}}{T^2} \cdot \frac{1}{L^{\cancel{2}}} \cdot \frac{\cancel{L}}{\frac{\cancel{L}}{T}} = \frac{M}{LT}$

$[D] = L$

# of reference dimensions $= 3$ (M, L, T)

# of variables $= 5$ (F_D, V, ρ, μ, D)

From Buckingham Pi Thrm:

(# of π terms) = (# of variables) − (min # of reference dimensions)

$= 5 - 3 = 2$ $\begin{cases}\pi_1 \\ \pi_2\end{cases}$

Repeating variables: ρ, V, D

$\pi_1 = F_D \cdot \rho^a V^b D^c$

$M^0 L^0 T^0 = \frac{ML}{T^2} \cdot \frac{M^a}{L^{3a}} \cdot \frac{L^b}{T^b} \cdot L^c$

$M^0 L^0 T^0 = M^{1+a}\, L^{1-3a+b+c}\, T^{-2-b}$

$1 + a = 0 \Rightarrow \boxed{a = -1}$

$1 - 3a + b + c = 0 \Rightarrow 1 + 3 - 2 + c = 0 \Rightarrow \boxed{c = -2}$

$-2 - b = 0 \Rightarrow \boxed{b = -2}$

$$\boxed{\pi_1 = \frac{F_D}{\rho V^2 D^2}}$$

Name: ____________________

EXTRA PAGE

$$\Pi_2 = \mu \rho^a V^b D^c$$

$$M^0 L^0 T^0 = \frac{M}{LT} \frac{M^a}{L^{3a}} \frac{L^b}{T^b} L^c$$

$$M^0 L^0 T^0 = M^{1+a} L^{-1-3a+b+c} T^{-1-b}$$

$$1 + a = 0 \Rightarrow \boxed{a = -1}$$

$$-1 - 3a + b + c = 0 \Rightarrow -1 + 3 - 1 + c = 0 \Rightarrow \boxed{c = -1}$$

$$-1 - b = 0 \Rightarrow \boxed{b = -1}$$

$$\boxed{\Pi_2 = \frac{\mu}{\rho V D}}$$

$$\Pi_1 = f_2(\Pi_2) \Rightarrow \boxed{\frac{F_D}{\rho V^2 D^2} = f_2\left(\frac{\mu}{\rho V D}\right)}$$

b) $D_m = \frac{1}{4} D_p$

$V_m = ?$ $V_p = 3 \, m/s$

$\mu_m = \mu_p$

$\rho_m = \rho_p$

$$\Pi_{2,m} = \Pi_{2,p}$$

$$\frac{\cancel{\mu_m}}{\cancel{\rho_m} V_m D_m} = \frac{\cancel{\mu_p}}{\cancel{\rho_p} V_p D_p}$$

$$V_m = V_p \cdot \frac{D_p}{D_m} \Rightarrow V_m = 3 \cdot 4 = 12 \, m/s$$

c) $F_{D,m} = 900 \, N$

$F_{D,p} = ?$

$$\Pi_{1,m} = \Pi_{1,p}$$

$$\frac{F_{D,m}}{\cancel{\rho_m} V_m^2 D_m^2} = \frac{F_{D,p}}{\cancel{\rho_p} V_p^2 D_p^2}$$

$$F_{D,p} = F_{D,m} \cdot \left(\frac{V_p}{V_m}\right)^2 \left(\frac{D_p}{D_m}\right)^2$$

$$= 900 \left(\frac{\cancel{3}}{\cancel{12}_{4}}\right)^2 \cdot (4)^2$$

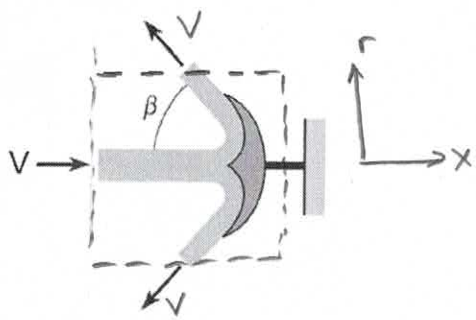
$$\boxed{F_{D,p} = 900 \, N}$$

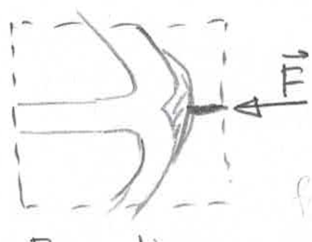
Note: the prototype has the same drag as the model.

Name: ____________________

7) (10 marks) As shown in the figure below, an incident steady jet of water with density ρ, speed V, and cross-sectional area A is deflected through an angle β by a stationary, <u>axisymmetric vane</u>. Find the horizontal force required to hold the vane stationary. Express your answer in terms of ρ, V, A and β. Neglect the influence of gravity. Notice that the outflow of fluid from the vane is axisymmetric.

Choose: - stationary CV
- stationary reference frame





Force diagram.

From Bernoulli: $\boxed{V_{in} = V_{out} = V}$

($P_{in} = P_{out} = P_{atm}$ & gravitational effects are negligible)

$\frac{P_{in}}{\rho}$

<u>Continuity:</u> $\dot{m}_{in} = \dot{m}_{out} = \dot{m}$

<u>Momentum:</u> $\Sigma \vec{F} = \dot{m}_{out} \vec{V}_{out} - \dot{m}_{in} \vec{V}_{in} = \dot{m} (\vec{V}_{out} - \vec{V}_{in})$

(Note that the r-direction momentum flow will be zero because the vane is axisymmetric)

x-direction: $\Sigma F_x = -F = \underbrace{\dot{m}}_{\rho A V} (-V \cos\beta - V)$

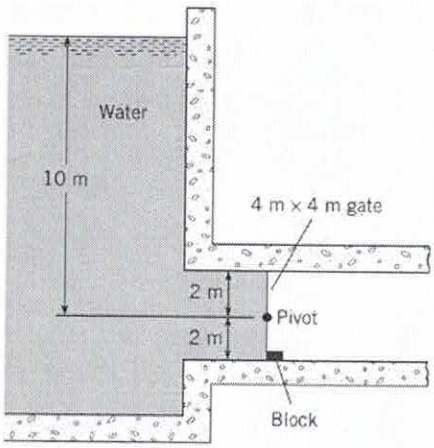
$$\boxed{F = \rho A V^2 (\cos\beta + 1)}$$

Name: ____________________

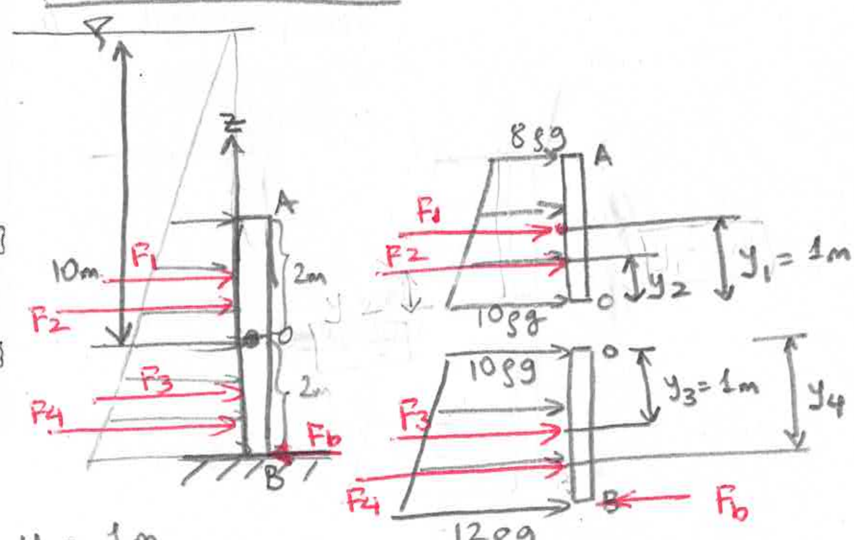
8) The 4 $m \times 4m$ square gate shown in the figure is pivoted as shown below. Density of the water is given as $\rho = 1000\ kg/m^3$. Find the force of the block on the gate using the following two methods:

a) (10 marks) The pressure prism method. Show the forces acting on the gate clearly on a sketch.

b) (10 marks) The integration method.



a) Pressure prism method



$F_1 = 8\rho g.(2).(4) = 64\rho g \quad , \quad y_1 = 1m.$

$F_2 = \dfrac{2\rho g.(2).(4)}{2} = 8\rho g \quad , \quad y_2 = 2.\dfrac{1}{3} = \dfrac{2}{3}m.$

$F_3 = 10\rho g.(2).(4) = 80\rho g \quad , \quad y_3 = 1m.$

$F_4 = \dfrac{2\rho g.(2).(4)}{2} = 8\rho g, \quad y_4 = 2.\dfrac{2}{3} = \dfrac{4}{3}m$

$\Sigma M_{pivot} = 0$

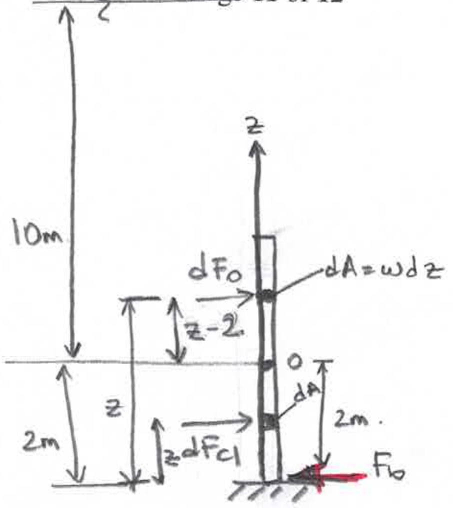
$F_1 \cdot y_1 + F_2 \cdot y_2 + F_b.(2) = F_3\, y_3 + F_4 \cdot y_4$

$64\rho g.(1) + 8\rho g\left(\dfrac{2}{3}\right) + F_b(2) = 80\rho g\,(1) + 8\rho g\left(\dfrac{4}{3}\right)$

$F_b = \dfrac{1}{2}\left[80\rho g - 64\rho g + \dfrac{32\rho g - 16\rho g}{3}\right] = \dfrac{32}{3}\rho g = \dfrac{32.1000.10}{3}$

$F_b = 106,666.7\ N$ (this force acts on the gate to the left)

Name: ____________________



dF =

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Above the pivot:

$$dF_o = p\,dA = \underbrace{\rho g(12-z)}_{p}\,.\,\underbrace{w\,dz}_{dA} \quad \text{where } 2 \le z \le 4$$

$$dM_o = (z-2).dF_o = (z-2).\rho g(12-z)\,w\,dz$$

$$(M_o)_{water} = \int_{z=2}^{4} (z-2)\,\rho g(12-z)\,w\,dz$$

$$(M_o)_{water} = \rho g w \int_{z=2}^{z=4} (-z^2 + 14z - 24)\,dz$$

$$= \rho g w \left[-\frac{z^3}{3} + 7z^2 - 24z \right]_{z=2}^{z=4}$$

$$(M_o)_{water} = \rho g w \left[-\frac{4^3}{3} + 7.4^2 - 24.(4) + \frac{2^3}{3} - 7.2^2 + 24.(2) \right]$$

1000, 10, w=4m.

$$\boxed{(M_o)_{water} = 693{,}333.3\ Nm.}$$ (opening moment applied by the water)

Below the pivot:

$$dF_{cl} = p\,dA = \rho g(12-z)\,w\,dz \quad \text{where } 0 \le z \le 2$$

$$dM_{cl} = (2-z)\,dF_{cl} = (2-z).\rho g(12-z)\,w\,dz$$

$$(M_{cl})_{water} = \int_{z=0}^{z=2} (2-z)\,\rho g\,(12-z)\,w\,dz$$

$$= \rho g w \int_{z=0}^{2} (z^2 - 14z + 24)\,dz$$

$$= \rho g w \left[\frac{z^3}{3} - 7z^2 + 24z \right]_{z=0}^{2} = \rho g w \left[\frac{2^3}{3} - 7.2^2 + 24.2 \right]$$

1000, 10, w=4m.

$(M_{cl})_{water} = 906{,}666.7\ Nm.$ (closing moment applied by the water)

$$\Sigma M_{opening} = \Sigma M_{closing}$$

$$F_b.(2) + (M_o)_{water} = (M_{cl})_{water} \Rightarrow F_b = \frac{\left[(M_{cl})_{water} - (M_o)_{water}\right]}{2}$$

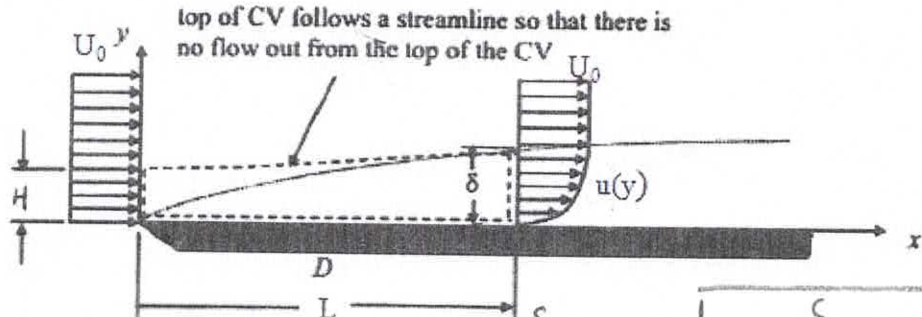
$$F_b = \frac{1}{2}(906{,}666.7 - 693{,}333.3) = 106{,}666.7\ N$$

Name: ____________________

9) (8 marks) Air at standard conditions flows along a flat plate. The undisturbed uniform freestream speed is U_o (as shown in the figure below). At L distance downstream from the leading edge of the plate, the boundary layer height is δ. Assuming the flow is incompressible, the pressure everywhere is atmospheric pressure, find the horizontal force per unit width required to hold the plate stationary in terms of density ρ, freestream velocity U_o and θ. While solving this problem, use the control volume shown by the dashed lines in the figure below and name the following integral over the control surface at $x = L$ as θ:

$$\theta = \int_0^\delta \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right)dy$$

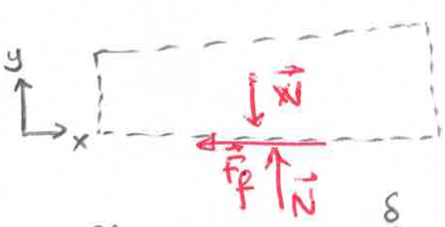
Note that H (which is the height of the inlet of the control volume) is not equal to δ (which is the height of the outlet of the control volume in order to account for the velocity deficit within the boundary layer. Also, as marked on the figure, the top of the control volume follows a streamline so that there is no flow out from the top surface of the control volume.



From continuity: (per unit depth) $\rho U_o H = \rho \int_{y=0}^{\delta} u\, dy \Rightarrow \boxed{H = \int_{y=0}^{\delta} \frac{u}{U_o} dy}$

Momentum eqn: (per unit depth) $\sum \vec{F}_{cv} = \cancel{\frac{d}{dt}\iiint_{CV} \vec{V}(\rho d\forall)} + \iint_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A})$

steady



$\sum \vec{F} = -F_f \vec{i} + (N - W)\vec{j}$ (right hand side of mom. eqn.)

$\iint_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A}) = \int_0^\delta u \rho u\, dy \vec{i} - U_o(\rho U_o H)\vec{i} = \int_0^\delta \rho u^2 dy \vec{i} - \rho U_o^2 H \vec{i}$ (left-hand side of mom. eqn)

Name: ____________________

EXTRA PAGE

<u>Momentum eqn. in x-direction:</u>

$$-F_f = \int_0^\delta \rho u^2 \, dy - \rho U_o^2 H$$

from continuity: $H = \int_0^\delta \frac{u}{U_o} \, dy$

$$-F_f = \int_0^\delta \rho u^2 \, dy - \rho U_o^2 \int_0^\delta \frac{u}{U_o} \, dy$$

$$-F_f = \rho \int_0^\delta \left(u^2 \frac{U_o^2}{U_o^2} - U_o^2 \frac{u}{U_o} \right) dy$$

$$= \rho U_o^2 \underbrace{\int_0^\delta \frac{u}{U_o} \left(\frac{u}{U_o} - 1 \right) dy}_{-\theta}$$

where θ is defined as: $\theta = \int_0^\delta \frac{u}{U_o} \left(1 - \frac{u}{U_o} \right) dy$

$$\boxed{F_f = \rho U_o^2 \theta}$$

F_f to the left acts on the CV, F_f to the right acts on the plate. To hold the plate stationary, a force equal to F_f <u>to the left</u> should be applied to the plate.