MAT292 – Fall 2021

Term Test – November 1, 2021

Time alotted: 110 minutes

Full Name ____________________________

Student Number ____________________________

Email ____________________________@mail.utoronto.ca

Signature ____________________________

DO NOT OPEN
until instructed to do so

NO CALCULATORS ALLOWED
and no cellphones or other electronic devices

DO NOT DETACH ANY PAGES

This test contains 12 pages (including this title page). Once the test starts, make sure you have all of them.

In Section I, only answers are required. No justification necessary.

In Section II and Section III, you need to justify your answers.
Answers without justification won't be worth points, unless a question says "no justification necessary".

You can use pages 9–12 to complete questions. In such a case, **MARK CLEARLY** that your answer "continues on page X" **AND** indicate on the additional page which questions you are answering.

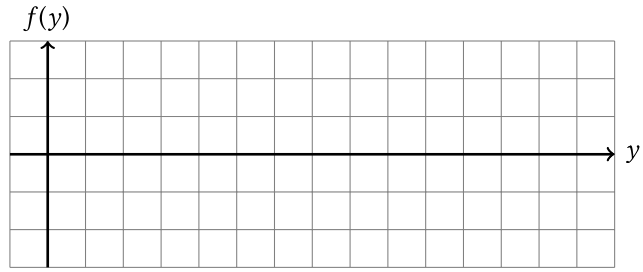
	Short answer	**True/False**	**Long answer**				
Question	Q1-Q5	Q6-Q9	Q10	Q11	Q12	Q13	**Total**
Marks	12	8	14	9	9	9	**61**

GOOD LUCK! YOU GOT THIS!

SECTION I   Provide the final answer. No justification necessary.

1. **(2 marks)** On the right, draw a phase plot for an autonomous ODE $y' = f(y)$ such that:

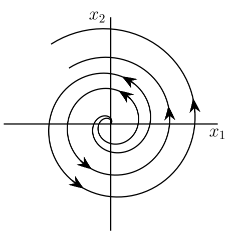
   - The ODE has three equilibria.
   - At least one of the equilibria is stable.
   - **NONE** of the equilibria are unstable.

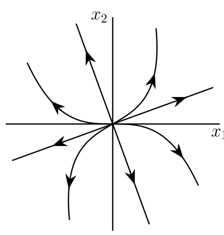


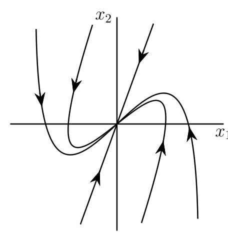
2. **(4 marks)** Consider two-dim. systems of the form $\vec{x}\,' = A\,\vec{x}$ and the following eigenvalue setups for the matrix A.

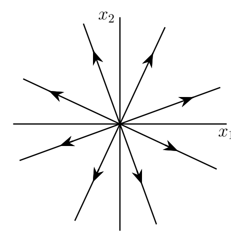
   P: $\lambda_1 = \lambda_2 = -5$ **Q:** $\lambda_1 = \lambda_2 = 5$ **R:** $\lambda_1 = 5 + \pi i$, $\lambda_2 = 5 - \pi i$ **S:** $\lambda_1 = 6$, $\lambda_2 = e$

   Below each phase plot below, **write the letter** of the matching setup.









3. **(2 marks)** Consider the IVP $y' = t + y^2$, $y(1) = 1$. Approximate $y(2)$ using the **Improved** Euler method with a single step.

   $y(2) \approx$

4. **(2 marks)** Consider the IVP $y' = f(t, y)$, $y(0) = y_0$. Approximating $y(1)$ using the Runge-Kutta Method with a fixed step size and 15 steps results in a global truncation error of approximately $\frac{1}{10}$. Give a plausible estimate for the global truncation error if we used 30 steps instead.

   Error$\approx$

5. **(2 marks)** Consider the system of differential equations $\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $\alpha \in \mathbb{R}$ is a parameter.

   Make **exactly one choice** in each box.

   ○ The equilibrium is stable   ○ The equilibrium is unstable   ○ The answer depends on the value of α.

   ○ $\lim_{t\to\infty} |x(t)| = 0$ for all solutions.   ○ We must know the initial value to determine $\lim_{t\to\infty} |x(t)|$.

   ○ $\lim_{t\to\infty} |x(t)| = \infty$ for all solutions.   ○ We must know α to determine $\lim_{t\to\infty} |x(t)|$.

   ○ We must know the initial value **AND** α to determine $\lim_{t\to\infty} |x(t)|$.

SECTION II

For each of the following statements, decide if it is true or false. **Then justify your choice.**

Remember: A statement is only true if you can guarantee it is ALWAYS true given the information.
In other words: If something is "only true under certain circumstances", it is still false.

6. **(2 marks)** Consider the solution $y(t)$ to the initial value problem $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -r\left(1 - \dfrac{y}{T}\right)y, \qquad y(0) = y_0.$

   Given any y_1, there exists a time t_1 at which $y(t_1) = y_1$.

   Choose true or false, then justify: ○ TRUE ○ FALSE

7. **(2 marks)** Consider a two-dimensional system $\vec{x}\,' = A\vec{x}$ where A has two complex eigenvalues. If there is at least one nonzero solution such that $\lim_{x\to\infty} \vec{x}(t) = [0,0]$, then $\lim_{x\to\infty} \vec{x}(t) = [0,0]$ is true for all solutions.

   Choose true or false, then justify: ○ TRUE ○ FALSE

8. **(2 marks)** Euler's Method (**not** improved Euler) is used to approximate the two IVPs below.

   (A) $y' = \sin y, \qquad y(0) = 100$ (B) $y' = 5y, \qquad y(0) = 100$

   The local truncation error should be larger when approximating the solution of (A).

   Choose true or false, then justify: ○ TRUE ○ FALSE

9. **(2 marks)** Assume $\vec{\phi}(t)$ and $\vec{\psi}(t)$ solve the system $\vec{x}\,' = A\vec{x} + \vec{b}$ where $A \in \mathbb{R}^{n\times n}$ and $\vec{b} \in \mathbb{R}^n$.
   For any $a \in \mathbb{R}$, $\vec{\phi}(t) + a\vec{\psi}(t)$ also solves the system.

   Choose true or false, then justify: ○ TRUE ○ FALSE

SECTION III **Justify** all your answers.

10. Consider the temperature of two adjacent rooms in a house.

Denote by $A(t)$ the temperature in room A and by $B(t)$ the temperature in room B.

We measure temperature in degrees Celsius and time in hours.

There are two effects, coming from the fact that the air in one room heats/cools the air in the other room.

- The temperature in room A changes at a rate proportional to the difference in temperature between room A and room B. The proportionality constant is 2.
- The temperature in room B changes at a rate proportional to the difference in temperature between room B and room A. The proportionality constant is 3.

(a) **(3 marks)** Find two ODEs involving $A(t)$ and $B(t)$. **Explain.**

First ODE:	Second ODE:

(b) **(1 mark)** The ODEs that you found in the previous part produce a system of two ODEs. Fill in the matrix on the right.

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix}' = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix}$$

(c) **(2 marks)** Explain **from a physical perspective** why it's **impossible** for the matrix in part (b) to have positive eigenvalues.

(d) **(6 marks)** Now find the general solution to the system and draw a phase portrait on the next page.
Make sure to label the phase portrait appropriately.
Put your final answer in the box on the next page.

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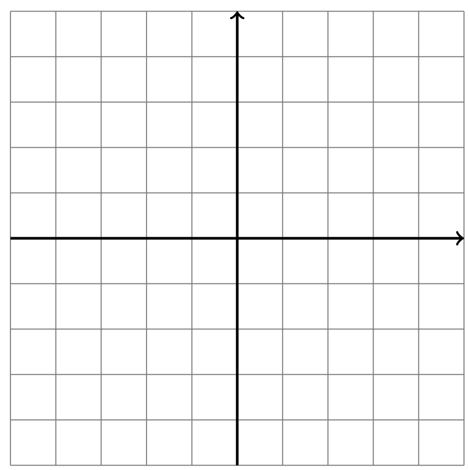
Continue question 10(d) here...

General solution of the system:



(e) (2 marks) Find the equlibrium/equlibria of the system. Explain the result from a physical perspective.

11. A MAT292 student is trying to to approximate the solution of the IVP $y' = \sin t \cos y$, $y(0) = 1$ at the point $t = 5$ using a numerical method implemented in MATLAB, having fixed step size $h = 0.5$.

The code they produced has an error and is inefficient.

(a) **(1 mark)** Before we talk about errors or inefficiencies: Which numerical method is the student *trying* to implement? No justification necessary.

Method:

(b) **(2 marks)** Consider the method you just identified in part (a). In an *ideal* implementation of it, how many times would the function $f(t, y) = \sin t \cos y$ need to be evaluated to get to the desired approximation of $y(5)$?

Number of evaluations:

```
1  t=0;h=0.5;y=zeros(11,1);y(1)=1;
2  for i=1:10
3     slope = eval(t,y(i),1,h)+2*eval(t,y(i),2,h)...
4             +2*eval(t,y(i),3,h)+eval(t,y(i),4,h);
5     slope = slope/6;
6     y(i+1) = y(i)+slope*h;    t=t+h;
7  end
8
9  disp('The value y(5) is approximately:');
10 disp(y(11));
11
12 function a=eval(t,y,n,h)
13    f=@(t,y) sin(t)*cos(y);
14    if n == 1
15       a = f(t,y);
16    else
17       a = f(t+h,y+h*eval(t,y,n-1,h));
18    end
19 end
```

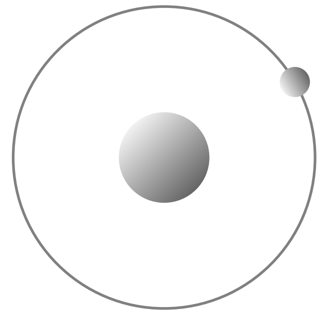
(c) **(3 marks)** Now let's have a look at the MATLAB code above. How many times is the function $f(t, y) = \sin t \cos y$ being evaluated if this MATLAB script is run? Explain. You are encouraged to reference line numbers.

Number of evaluations:

(d) **(3 marks)** The code is not only inefficient, but it is also an *incorrect* implementation of the method you identified in part (a). Explain what is wrong. You are encouraged to reference line numbers.

We are not looking for a clerical error like "they forgot a comma" or "the parentheses don't match". There is an actual semantic mistake in the code.

12. A SpaceX aircraft is orbiting around Mars in a perfect circle (see figure). The aircraft's location, represented by three coordinates $[x_1, x_2, x_3]$, satisfies the following linear system of differential equations:



$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{x} = A\vec{x} \qquad \text{with} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

The centre of Mars is located at the origin $[0, 0, 0]$.

(a) (2 marks) Looking at the physical model, why should the following be true? Explain briefly.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2 + x_2^2 + x_3^2\right) = 0$$

(b) (2 marks) Using the linear system, show that $\frac{\mathrm{d}}{\mathrm{d}t}\left(x_1^2 + x_2^2 + x_3^2\right) = 0$ is in fact true.

(c) (3 marks) Show that $\left(e^{At}\right)^T = e^{-At}$. Hint: notice that $A^T = -A$.

For the next part, recall: a matrix U is orthogonal if and only if $U^T U = U U^T = I$, where I is the identity matrix.

(d) (2 marks) Using part (c), show that the solution of the system is $\vec{x}(t) = Q\,\vec{x}(0)$ for some orthogonal matrix Q.

13. Consider the differential equation $ty' = y^\alpha$ where $\alpha \in \mathbb{R}$ is a scalar parameter. We only consider $t \geq 0$.

(a) **(3 marks)** Consider $\alpha = 1$ and $y(1) = 0$. How many solutions does this IVP have? Justify.

How many solutions:

(b) **(3 marks)** Consider $\alpha = 1$ and $y(0) = 0$. How many solutions does this IVP have? Justify.

How many solutions:

(c) **(3 marks)** Consider $\alpha = 2$ and $y(1) = 0$. How many solutions does this IVP have? Justify.

How many solutions:

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