AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 1

30 September 2013 9:15 am - 10:15 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: JW Davis

Given Name: Solutions

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	8	
3	9	
4	12	
5	11	
6	10	
TOTAL	60	/ 55

Note: The following integrals may be useful.

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

1) a) Evaluate the integrals: i) $\int_0^1 \int_{x^2}^{x} (1+2y)\,dy\,dx$

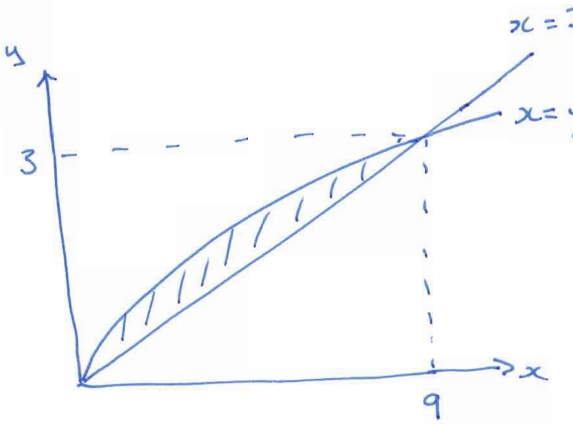
ii) $\int_D (x+2y)\,dR$, D is bounded by $x = y^2$, $x = 3y$, $y \ge 0$

(6 marks)

i) $\int_0^1 dx \int_{x^2}^{x} (1+2y)\,dy = \int_0^1 dx \left[y + y^2 \right]_{x^2}^{x} = \int_0^1 (x + x^2 - x^2 - x^4)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

ii)



$\int_D (x+2y)\,dR = \int_0^3 dy \int_{y^2}^{3y} (x+2y)\,dx$

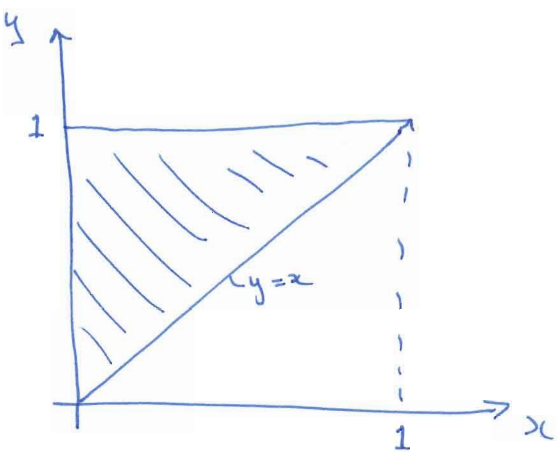
$= \int_0^3 dy \left[\frac{x^2}{2} + 2xy \right]_{y^2}^{3y} = \int_0^3 \left(\frac{9}{2}y^2 + 6y^2 - \frac{y^4}{2} - 2y^3 \right) dy$

$= \int_0^3 \left(\frac{21}{2}y^2 - \frac{y^4}{2} - 2y^3 \right) dy = \left[\frac{7}{2}y^3 - \frac{y^5}{10} - \frac{y^4}{2} \right]_0^3$

$= \frac{189}{2} - \frac{243}{10} - \frac{81}{2} = 54 - 24.3 = 29.7$

b) Evaluate the integral $\int_0^1 \int_x^1 e^{x/y}\,dy\,dx$ by reversing the order of integration. Show a sketch of the region.

(4 marks)



$\int_0^1 dx \int_x^1 dy\, e^{x/y} = \int_0^1 dy \int_0^y dx\, e^{x/y}$

$= \int_0^1 dy \left[y\, e^{x/y} \right]_0^y = \int_0^1 (ey - y)\,dy$

$= (e-1) \left[\frac{y^2}{2} \right]_0^1 = \frac{e-1}{2}$

2) Solve the integral equation: $f(x) = 7 + 3x + \int_0^x (x-t) f(t)\, dt$

(8 marks)

$$f(x) = 7 + 3x + \int_0^x (x-t)\, f(t)\, dt$$

$$f'(x) = 3 + (x-x) f(x) \cdot \frac{dx}{dx} + \int_0^x \frac{\partial}{\partial x}\left[(x-t) f(t)\right] dt$$

$$= 3 + \int_0^x f(t)\, dt$$

$$f''(x) = f(x) \implies f(x) = Ae^{x} + Be^{-x}$$

now: $f(0) = 7 = A + B$

$f'(0) = 3 = \left[Ae^{x} - Be^{-x}\right]_{x=0} = A - B$

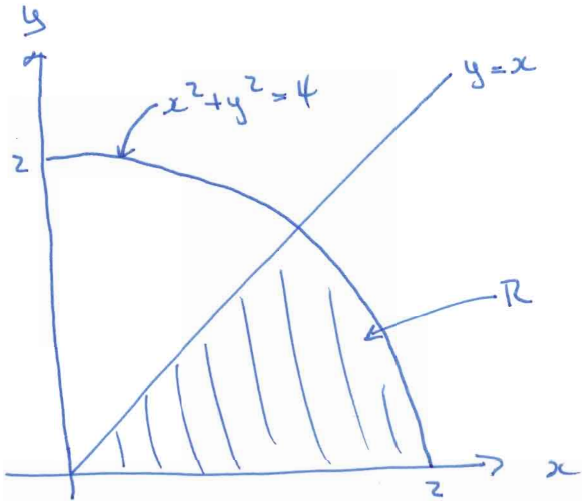
$\therefore\ 7 + 3 = 2A \Rightarrow A = 5$

$7 - 3 = 2B \Rightarrow B = 2$

$$\therefore \boxed{f(x) = 5e^{x} - 2e^{-x}}$$

3) b) Use polar coordinates to evaluate $\int_R \frac{dR}{4 + x^2 + y^2}$, where R is the 1st quadrant section of the circle $x^2 + y^2 = 4$ between $y = 0$ and $y = x$. Sketch the region of integration.

(4 marks)



$$\int_R \frac{dR}{4+x^2+y^2}$$

$$= \int_0^{\pi/4} d\theta \int_0^2 \frac{r\,dr}{4+r^2}$$

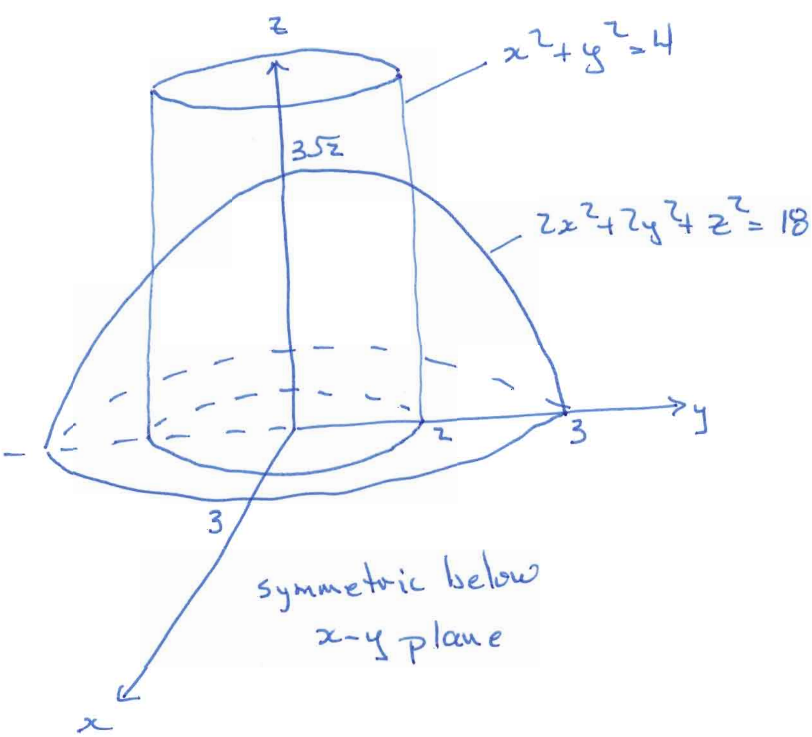
$$= \frac{\pi}{4}\left[\frac{1}{2}\ln(4+r^2)\right]_0^2$$

$$= \frac{\pi}{8}(\ln 8 - \ln 4)$$

$$= \frac{\pi}{8}\ln 2$$

b) Use a double integral in polar coordinates to find the volume of the solid that lies inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 18$. Provide a sketch of the volume.

(5 marks)



$$V = \int_0^{2\pi} d\theta \int_0^2 r\,dr \cdot 2\cdot\sqrt{18-2x^2-2y^2}$$

$$= 2\int_0^{2\pi} d\theta \int_0^2 r\sqrt{18-2r^2}\,dr$$

$$= 4\pi\left[(18-2r^2)^{3/2}\cdot\frac{2}{3}\cdot\left(-\frac{1}{4}\right)\right]_0^2$$

$$= 4\pi\cdot\frac{1}{6}\cdot\left(18^{3/2} - 10^{3/2}\right)$$

4) a) Evaluate $\int_V xy\,dV$, where V is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = x + y$. Provide a sketch of the volume.

(6 marks)

$$\int_V xy\,dV = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{x+y} dz\,(xy) = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y\,dy\,\left[z\right]_0^{x+y}$$

sketch on following page

$$= \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} (xy + y^2)\,dy = \int_0^1 x\,dx \left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{x^2}^{\sqrt{x}}$$

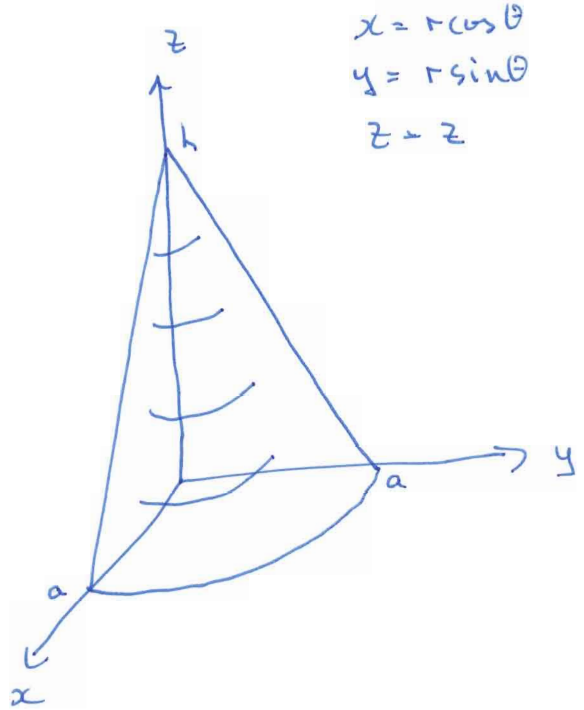
$$= \int_0^1 x\,dx\left(\frac{x^2}{2} + \frac{x^{3/2}}{3} - \frac{x^5}{2} - \frac{x^6}{3}\right) = \int_0^1 \left(\frac{x^3}{2} + \frac{x^{5/2}}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right)dx$$

$$= \left[\frac{x^4}{8} + \frac{2x^{7/2}}{21} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{1}{12} + \frac{1}{42} = \frac{9}{84} = \frac{3}{28}$$

b) Find $\int_V x^2\,dV$, where V is the part of the solid cone $0 \le z \le h\left(1 - \frac{\sqrt{x^2+y^2}}{a}\right)$ that lies in the first octant. Provide a sketch of the volume V.

(6 marks)



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\int_V x^2\,dV = \int_0^{\pi/2} d\theta \int_0^a r\,dr \int_0^{h(1-r/a)} dz\,(r\cos\theta)^2$$

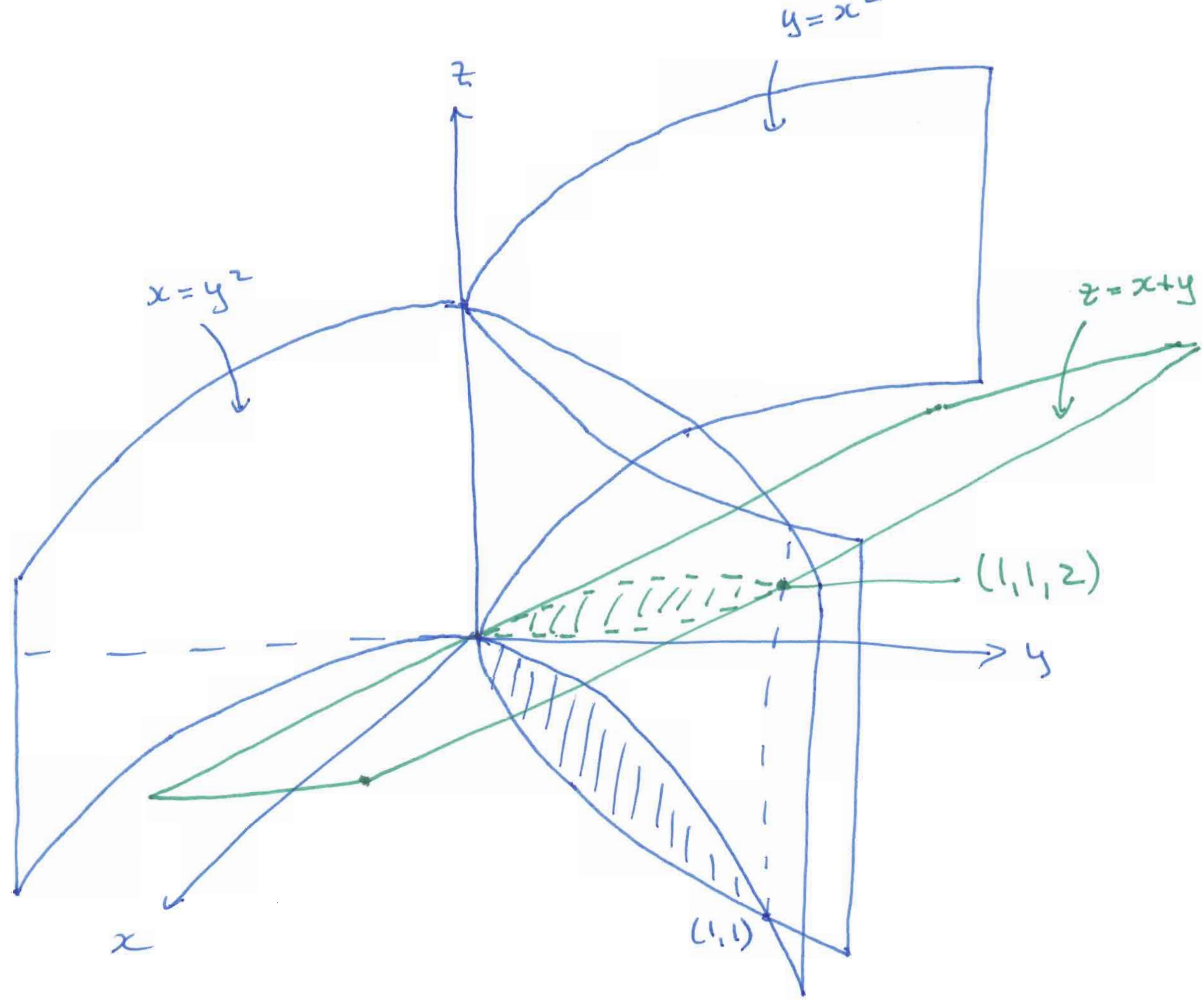
$$= \int_0^{\pi/2} \cos^2\theta\,d\theta \int_0^a r^3\,dr\,\left[z\right]_0^{h(1-r/a)}$$

$$= \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{\pi/2} \cdot \int_0^a r^3 h\left(1 - \frac{r}{a}\right)dr$$

$$= \frac{\pi}{4} \cdot h\left[\frac{r^4}{4} - \frac{r^5}{5a}\right]_0^a$$

$$= \frac{\pi h}{4}\left(\frac{a^4}{4} - \frac{a^4}{5}\right) = \frac{\pi h a^4}{80}$$

4 a)



5) Find, but DO NOT evaluate, the integrals giving:

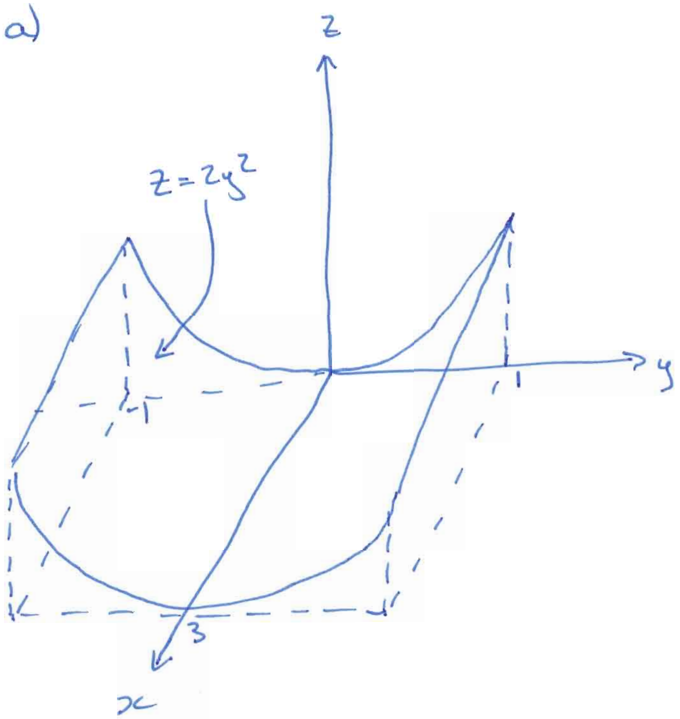
a) The surface area of the parabolic cylinder $z = 2y^2$ lying above the region: $0 \le x \le 3, \quad -1 \le y \le 1$.

b) The area of the surface $z^2 = 2 - x - y$ in the first octant.

Provide a sketch of both surfaces.

(11 marks)

a)



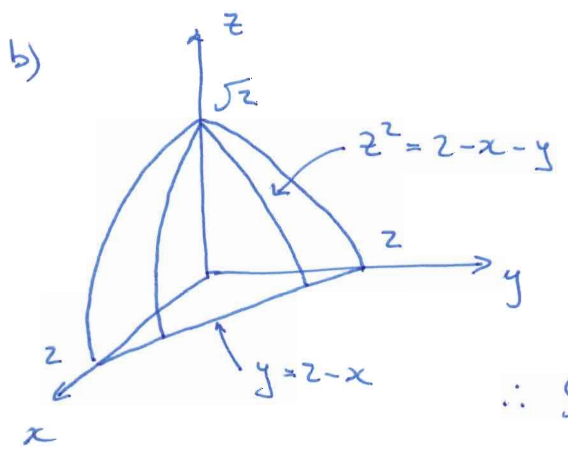
$$S = \int_R \sqrt{1 + (\partial g/\partial x)^2 + (\partial g/\partial y)^2}\, dR$$

$$z = g(x,y) = 2y^2 \Rightarrow \frac{\partial g}{\partial x} = 0, \quad \frac{\partial g}{\partial y} = 4y$$

$$\therefore S = \int_{-1}^{1} dy \int_0^3 dx \sqrt{1 + 0 + (4y)^2}$$

$$= \int_{-1}^{1} dy \int_0^3 dx \sqrt{1 + 16y^2}$$

b)



$$S = \int_R \frac{\sqrt{(\partial f/\partial x)^2 + (\partial f/\partial y)^2 + (\partial f/\partial z)^2}}{|\partial f/\partial z|}\, dR$$

$$f(x,y,z) = z^2 + x + y - 2 = 0$$
$$\rightarrow f_x = 1, \quad f_y = 1, \quad f_z = 2z$$

$$\therefore S = \int_0^2 dx \int_0^{2-x} dy \frac{\sqrt{1 + 1 + 4z^2}}{|2z|}$$

$$= \int_0^2 dx \int_0^{2-x} dy \frac{\sqrt{2 + 4z^2}}{\underbrace{2z}_{\text{in 1st octant, } z > 0}} = \int_0^2 dx \int_0^{2-x} dy \frac{\sqrt{2 + 4(2 - x - y)}}{2\sqrt{2 - x - y}}$$

$$= \int_0^2 dx \int_0^{2-x} dy \frac{\sqrt{10 - 4x - 4y}}{2\sqrt{2 - x - y}}$$

6) By directly calculating the partial derivatives, find the second degree polynomial approximation to the function $f(x,y) = (x+y^2)^{1/3}$ near the point $(4,2)$.

(10 marks)

$$f(x,y) = (x+y^2)^{1/3} \quad \Rightarrow f(4,2) = 8^{1/3} = 2$$

$$f_x = \frac{1}{3}\frac{1}{(x+y^2)^{2/3}} \quad \Rightarrow f_x(4,2) = \frac{1}{3}\frac{1}{8^{2/3}} = \frac{1}{12}$$

$$f_y = \frac{1}{3}\frac{2y}{(x+y^2)^{2/3}} \quad \Rightarrow f_y(4,2) = \frac{1}{3}\frac{4}{8^{2/3}} = \frac{1}{3}$$

$$f_{xx} = \frac{1}{3}\left(-\frac{2}{3}\right)\frac{1}{(x+y^2)^{5/3}} \quad \Rightarrow f_{xx}(4,2) = -\frac{2}{9}\cdot\frac{1}{8^{5/3}} = \frac{-2}{9\cdot 32} = \frac{-1}{144}$$

$$f_{yy} = \frac{2}{3}\frac{1}{(x+y^2)^{2/3}} + \frac{2y}{3}\left(-\frac{2}{3}\right)\frac{2y}{(x+y^2)^{5/2}}$$

$$= \frac{2}{3}\frac{1}{(x+y^2)^{2/3}} - \frac{8y^2}{9}\frac{1}{(x+y^2)^{5/2}} \quad \Rightarrow f_{yy}(4,2) = \frac{2}{3}\cdot\frac{1}{8^{2/3}} - \frac{8\cdot 4}{9}\cdot\frac{1}{8^{5/3}}$$

$$= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$f_{xy} = \frac{1}{3}\left(-\frac{2}{3}\right)\frac{2y}{(x+y^2)^{5/3}} \quad \Rightarrow f_{xy}(4,2) = -\frac{4}{9}\cdot\frac{2}{8^{5/3}} = -\frac{1}{36}$$

$$\therefore (x+y^2)^{1/3} \simeq 2 + \frac{x-4}{12} + \frac{y-2}{3} + \frac{1}{2!}\left(-\frac{(x-4)^2}{144} - 2\frac{(x-4)(y-2)}{36} + \frac{(y-2)^2}{18}\right)$$

$$= 2 + \frac{x-4}{12} + \frac{y-2}{3} - \frac{(x-4)^2}{288} - \frac{(x-4)(y-2)}{36} + \frac{(y-2)^2}{36}$$