AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 1

4 October 2018 9:15 am - 10:15 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis.

Given Name: Solutions

Student #: ____________________

Tutorial/TA: ____________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	7	
2	11	
3	6	
4	8	
5	10	
6	6	
7	10	
TOTAL	58	55

Note: The following integrals may be useful.

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

1) a) Evaluate the double integral: $\int_0^1 \int_0^3 e^{x+3y}\, dxdy$

(3 marks)

$$\int_0^1 dy \int_0^3 e^{x+3y}\, dx = \int_0^1 e^{3y}\, dy \int_0^3 e^{x}\, dx$$

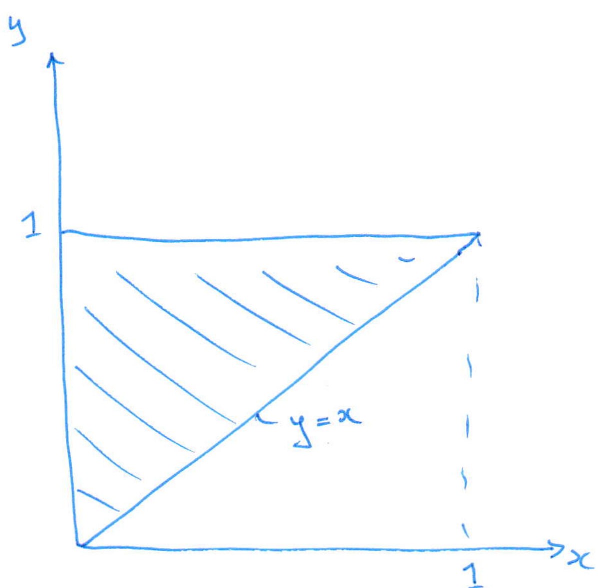
$$= \left[\frac{e^{3y}}{3}\right]_0^1 \left[e^{x}\right]_0^3$$

$$= \frac{1}{3}\left(e^{3}-1\right)\left(e^{3}-1\right)$$

$$= \frac{(e^{3}-1)^2}{3}$$

b) Evaluate the integral $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$ by reversing the order of integration. Show a sketch of the region.

(4 marks)



$$\int_0^1 dx \int_x^1 dy\, e^{x/y}$$

$$= \int_0^1 dy \int_0^y dx\, e^{x/y}$$

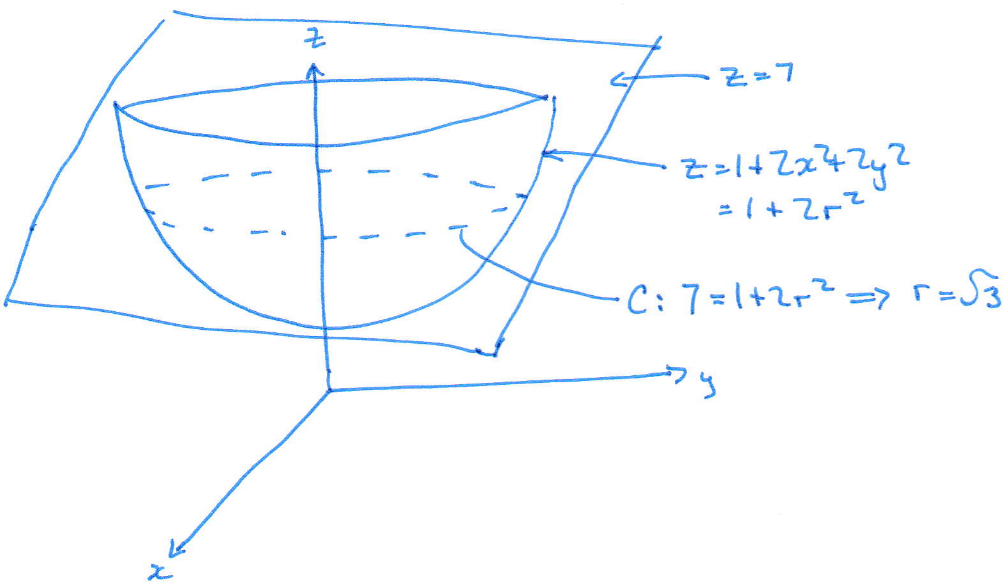
$$= \int_0^1 dy \left[y\, e^{x/y}\right]_0^y$$

$$= \int_0^1 (ey - y)\, dy$$

$$= (e-1)\left[\frac{y^2}{2}\right]_0^1 = \frac{e-1}{2}$$

2) a) Use polar coordinates to find the volume of the solid bounded by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant. Sketch the volume.

(5 marks)



1st octant

$$V = \int_0^{\pi/2} d\theta \int_0^{\sqrt{3}} r\,dr\,(7-(1+2r^2))$$

$$= \frac{\pi}{2}\int_0^{\sqrt{3}} (6r - 2r^3)\,dr$$

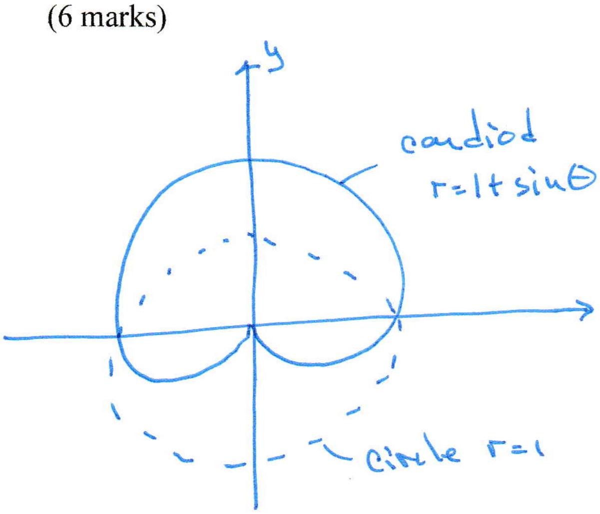
$$= \frac{\pi}{2}\left[3r^2 - \frac{r^4}{2}\right]_0^{\sqrt{3}}$$

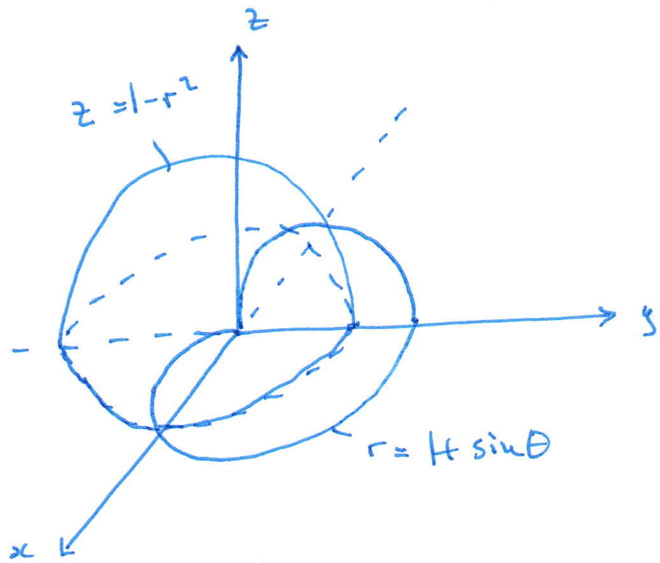
$$= \frac{\pi}{2}\left(9 - \frac{9}{2}\right)$$

$$= \frac{9\pi}{4}$$

b) Set up, but do not solve, a double integral(s) in polar coordinates which gives the volume of the solid that lies below the paraboloid $z = 1 - r^2$, and above the cardioid $r = 1 + \sin\theta$. Provide a sketch of the region in the x–y plane, and a 3-D sketch of the volume.

(6 marks)



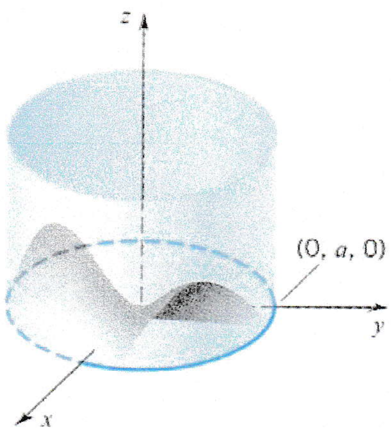


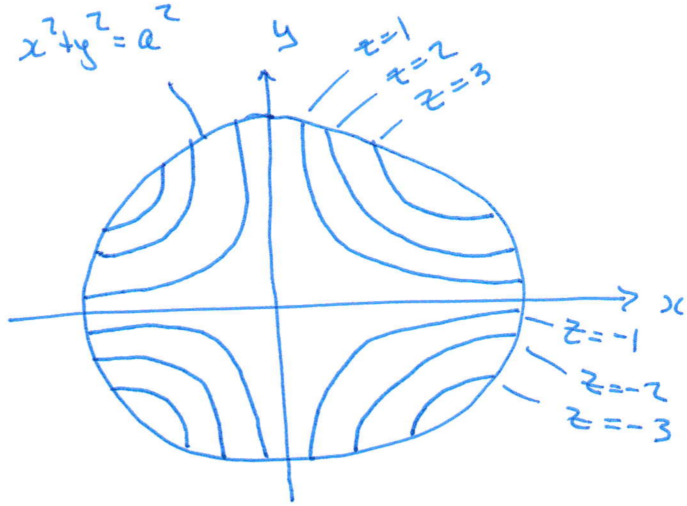
- For +ve y, the boundary of the region is formed by the intersection of the paraboloid and the x-y plane : r=1
- For -ve y, the cardiod forms the boundary

$$V = \int_0^{\pi} d\theta \int_0^{1} r\,dr\,(1-r^2) + \int_{\pi}^{2\pi} d\theta \int_0^{1+\sin\theta} r\,dr\,(1-r^2)$$

3) Find the surface area of that part of the hyperbolic paraboloid $z = xy$ that lies inside the cylinder $x^2 + y^2 = a^2$. Provide a sketch of the area. (Bonus marks will be given for any sketches that actually look like the surface.)

(6 marks)





$\Rightarrow \quad g(x,y) = xy \qquad \therefore \quad g_x = y \quad , \quad g_y = x$

$$\therefore \quad S = \int_R \sqrt{1 + y^2 + x^2} \, dR \qquad \rightarrow \text{use polar coordinates}$$

$$= \int_0^{2\pi} d\theta \int_0^a r\,dr \sqrt{1+r^2} = 2\pi \left[\frac{2}{3} (1+r^2)^{3/2} \cdot \frac{1}{2} \right]_0^a$$

$$= \frac{2\pi}{3} \left((1+a^2)^{3/2} - 1 \right)$$

4) Solve the integral equation: $f(x) = 2 + 4x + \int_0^x (x-t) f(t)\,dt$

(8 marks)

$$f(x) = 2 + 4x + \int_0^x (x-t) f(t)\,dt$$

$$\Rightarrow f'(x) = 4 + (x-x) f(x) \cdot \frac{dx}{dx} + \int_0^x \frac{\partial}{\partial x}\left[(x-t) f(t)\right] dt$$

$$= 4 + \int_0^x f(t)\,dt$$

$$f''(x) = f(x) \qquad \Rightarrow \qquad f(x) = Ae^{x} + Be^{-x}$$

now $f(0) = 2 = A + B$

$$f'(0) = 4 = \left[Ae^{x} - Be^{-x}\right]_{x=0} = A - B$$

$$\therefore \quad 2+4 = 2A \quad \Rightarrow \quad A = 3$$

$$2 - 4 = 2B \quad \Rightarrow \quad B = -1$$

$$\therefore \quad \boxed{f(x) = 3e^{x} - e^{-x}}$$

5) By directly calculating the partial derivatives, find the second degree polynomial approximation to the function $f(x,y)=\ln(x+y^2)$ near the point $(0,1)$.

(10 marks)

$f(x,y) = \ln(x+y^2)$ $\qquad$ $f(0,1) = \ln(1) = 0$

$f_x = \dfrac{1}{x+y^2}$ $\qquad$ $f_x(0,1) = 1$

$f_y = \dfrac{2y}{x+y^2}$ $\qquad$ $f_y(0,1) = 2$

$f_{xx} = \dfrac{-1}{(x+y^2)^2}$ $\qquad$ $f_{xx}(0,1) = -1$

$f_{yy} = \dfrac{2}{x+y^2} - \dfrac{4y^2}{(x+y^2)^2}$ $\qquad$ $f_{yy}(0,1) = 2-4 = -2$

$f_{xy} = \dfrac{-2y}{(x+y^2)^2}$ $\qquad$ $f_{xy}(0,1) = -2$

$$\therefore f(x,y) \simeq f(0,1) + f_x(x) + f_y(y-1) + \frac{1}{2!}\left(f_{xx}x^2 + 2f_{xy}x(y-1) + f_{yy}(y-1)^2\right) + \cdots$$

$$= 0 + x + 2(y-1) + \frac{-x^2}{2} - 2x(y-1) - (y-1)^2 + \cdots$$

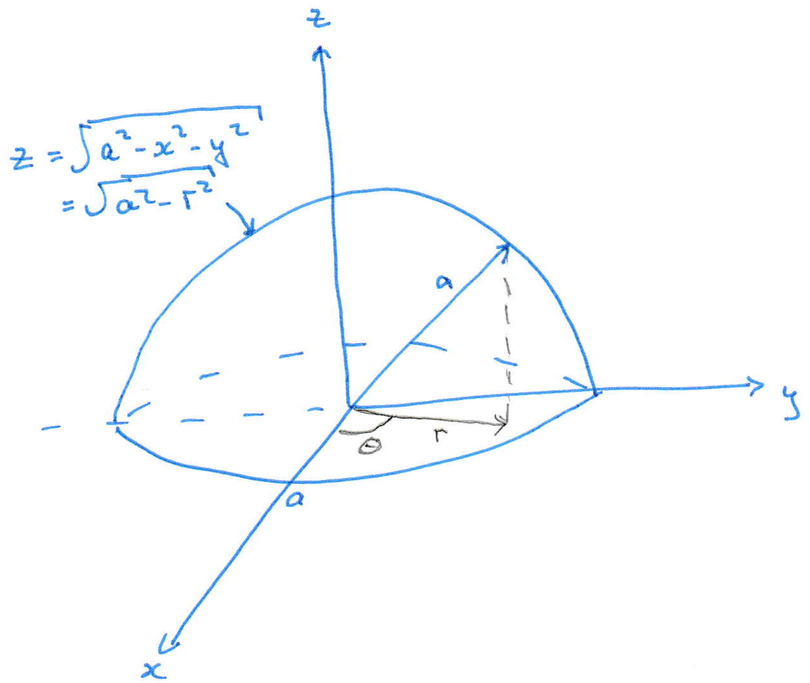
$$\text{or } \ln(x+y^2) \approx x + 2(y-1) - \frac{x^2}{2} - 2x(y-1) - (y-1)^2$$

test: $\ln(0.1+1.05^2) = 0.1844$

$$\simeq 0.1 + 2(.05) - \frac{0.1^2}{2} - 2(.1)(.05) - (.05)^2 = 0.1825$$

6) A solid half-ball H of radius a has a density depending on the distance ρ from the centre of the base disk. The density is given by $\lambda = k(2a-\rho)$, where k is a constant. Set up, but do not solve, triple iterated integrals for the mass of the ball in spherical, cylindrical and Cartesian coordinates.

(6 marks)



$$M = \int_V \lambda \, dV$$
$$= \int_V k(2a-\rho)\, dV$$

Spheical: $M = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \, d\phi \int_0^{a} \rho^2 d\rho \cdot k(2a-\rho)$

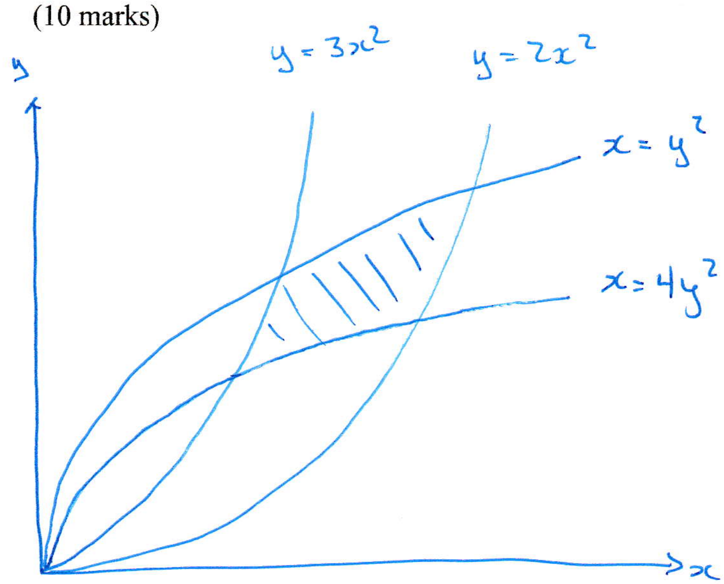
Cylindrical: $M = \int_0^{2\pi} d\theta \int_0^{a} r\,dr \int_0^{\sqrt{a^2-r^2}} dz \left(k\left(2a-\sqrt{z^2+r^2}\right)\right)$

Cartesian: $M = \int_0^{a} dx \int_0^{\sqrt{a^2-x^2}} dy \int_0^{\sqrt{a^2-x^2-y^2}} dz \cdot k\left(2a-\sqrt{x^2+y^2+z^2}\right)$

7) Use the coordinate transformation: $u = y/x^2$, $v = x/y^2$ to evaluate the double integral:

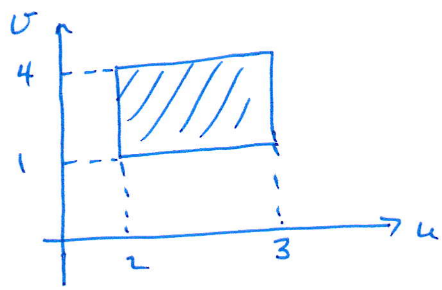
$I = \int_R \frac{dR}{xy}$, where R is the region bounded by the four parabolas: $y = 2x^2$, $y = 3x^2$, $x = y^2$ and $x = 4y^2$. Provide a sketch of the region in both the x-y plane and the u-v plane.

(10 marks)



$u = \frac{y}{x^2}$ $\quad 2 \le u \le 3$

$v = \frac{x}{y^2}$ $\quad 1 \le v \le 4$



$$\left.\begin{aligned} y = ux^2 &= uv^2y^4 \Rightarrow y^3 = \frac{1}{uv^2} \\ x = vy^2 &= vu^2x^4 \Rightarrow x^3 = \frac{1}{u^2v} \end{aligned}\right\} \quad x^3y^3 = \frac{1}{u^3v^3}$$

$$\Rightarrow xy = \frac{1}{uv}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{-2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & \frac{-2x}{y^3} \end{vmatrix} = \frac{4xy}{x^3y^3} - \frac{1}{x^2y^2} = \frac{3}{x^2y^2}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2y^2}{3} = \frac{1}{3u^2v^2}$$

$$\therefore I = \int_R \frac{dR}{xy} = \int_2^3 du \int_1^4 dv \left| \frac{1}{3u^2v^2} \right| \cdot uv$$

$$= \frac{1}{3}\int_2^3 \frac{du}{u} \int_1^4 \frac{dv}{v} = \frac{1}{3}\left[\ln u\right]_2^3 \left[\ln v\right]_1^4 = \frac{1}{3}\ln\frac{3}{2} \cdot \ln 4$$