ECE259: Electromagnetism

Final exam, April 19th, 2024

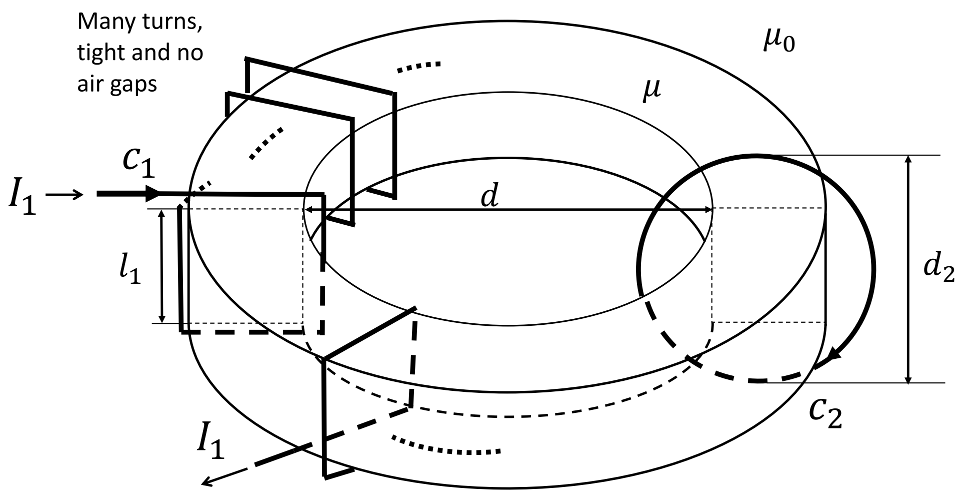
Instructor: Prof. Piero Triverio

SOLUTION

Instructions

- Duration: 2 hours 30 minutes (9:30-12:00)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**
- Please write with a **dark** pen or pencil. This test will be scanned.

Question 1 [20 points]



We have a toroidal core made by a linear material with permeability $\mu \gg \mu_0$. The core has a square cross-section of size $l_1 \times l_1$, and inner diameter d. A coil c_1 made by a conductive wire is wrapped around the core many times, very tightly, leaving no air gaps. This coil makes N_1 turns around the core, and N_1 is very high. A circular circuit c_2 of diameter d_2 is present on one side of the core, and has only one turn. You can assume that $\mathbf{H} = 0$ outside the core.

a) A DC current I_1 is sent through coil c_1. Use Ampere's law to find the magnetic flux density $\mathbf{B}_1$ inside the core [7 points];

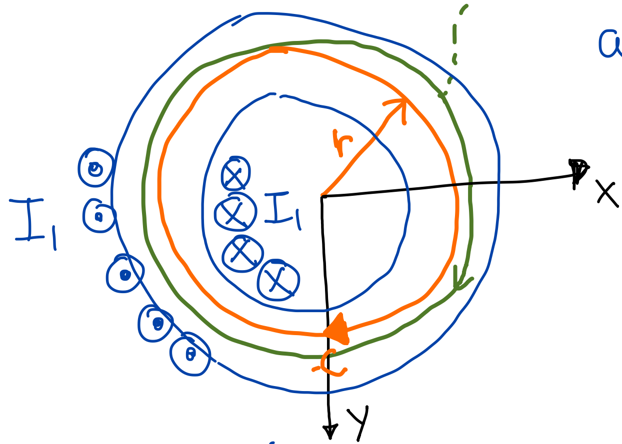
b) Find the self inductance L_{11} of circuit c_1 [7 points];

c) Find the mutual inductance L_{12} between circuit c_1 and circuit c_2 [4 points];

d) If we double the number of turns N_1, by how much the self inductance L_{11} will increase? Explain, from a physical standpoint, why the inductance increases by such factor. [2 points]

Please justify all your answers.

a) Top view



Due to symmetry and that there are many windings, H direction and only function of r [2pt]

$$\overline{H}_1 = \overbrace{H_1(r)}^{[1pt]}\ \overbrace{\overline{a}_\varphi}^{[1pt]}$$

Choose circle of radius r as Amperian path [1 pt]

$$\oint_C \overline{H}_1 \cdot d\overline{\ell} = N_1 \cdot I_1 \qquad \int_{\varphi=0}^{2\pi} H_1(r)\ \underbrace{\overline{a}_\varphi \cdot \overline{a}_\varphi}_{1}\ d\ell_\varphi = N_1 \cdot I_1$$

$$\underbrace{H(r)\ 2\pi r}_{\text{left hand side}} = \underbrace{N_1 \cdot I_1}_{\text{right hand side}}$$

$$\overline{H}_1 = \frac{N_1 I_1}{2\pi r}\ \overline{a}_\varphi$$

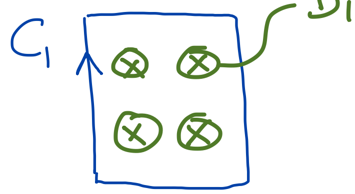
H calculatio
[2pt]
(1pt left hand side
1pt right hand side)

$$\boxed{\overline{B} = \mu\ \overline{H}_1 = \frac{\mu N_1 I_1}{2\pi r}\ \overline{a}_\varphi}$$

B calculation: [2pt]

b) Magnetic flux through one turn of C_1

$$\Phi_t = \int_S \bar{B}_1 \cdot d\bar{S} =$$



Right $d\bar{S}$ [1pt]

1pt/2 1pt/2

$$= \int_{r=\frac{d}{2}}^{\frac{d}{2}+\ell_1} \int_{z=0}^{\ell_1} \frac{\mu N_1 I_1}{2\pi r} \underbrace{\bar{a}_\varphi \cdot \bar{a}_\varphi}_{1} \, dz \, dr = \frac{\mu N_1 I_1}{2\pi} \ell_1 \int_{d/2}^{d/2+\ell_1} \frac{1}{r} dr =$$

calculations: 1pt

$$= \frac{\mu N_1 I_1}{2\pi} \ell_1 \ln\left(\frac{d/2 + \ell_1}{d/2}\right) = \frac{\mu N_1 I_1}{2\pi} \ell_1 \ln\left(1 + \frac{2\ell_1}{d}\right)$$

[3pt]

Flux through the entire circuit C_1

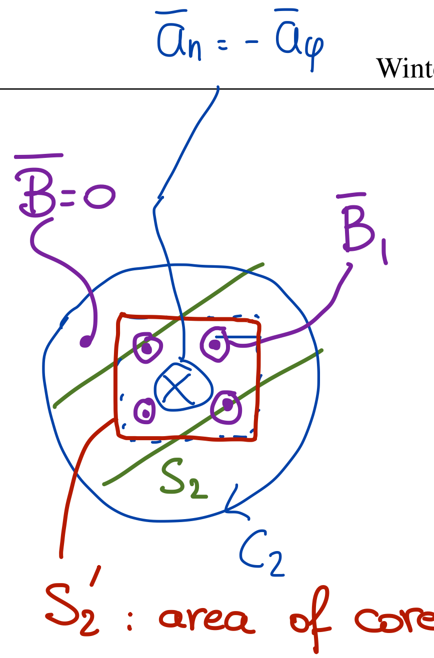
$$\Phi = N_1 \cdot \Phi_t = \frac{\mu N_1^2 I_1}{2\pi} \ell_1 \ln\left(1 + \frac{2\ell_1}{d}\right)$$

Flux entire circuit [2pt]

Inductance of C_1

$$L_{11} = \frac{\Phi_1}{I_1} = \frac{\mu N_1^2}{2\pi} \ell_1 \ln\left(1 + \frac{2\ell_1}{d}\right)$$

[1pt]



Correct direction [1pt]

c) Must calculate flux through S_2, going into the page

Flux calculat. [2pt]

$$\Phi_2 = \int_{S_2} \bar{B} \cdot d\bar{S}_2 = \int_{S_2'} \bar{B}_1 \cdot d\bar{S}_2 =$$

$$= \int_{S_2'} \frac{\mu N_1 I_1}{2\pi r} \underbrace{\bar{a}_\varphi \cdot (-\bar{a}_\varphi)}_{-1} \, dS_2' =$$

$$= - \underbrace{\int_{S_2'} \frac{\mu N_1 I_1}{2\pi r} dS_2'}_{\text{this is the flux through one turn of } C_1 \,:\, \Phi_t} = - \Phi_t =$$

L calc [1pt]

$$L_{12} = \frac{\Phi_2}{I_1} = -\frac{\mu N_1}{2\pi} \ell_1 \ln\left(1 + \frac{2\ell_1}{d}\right)$$

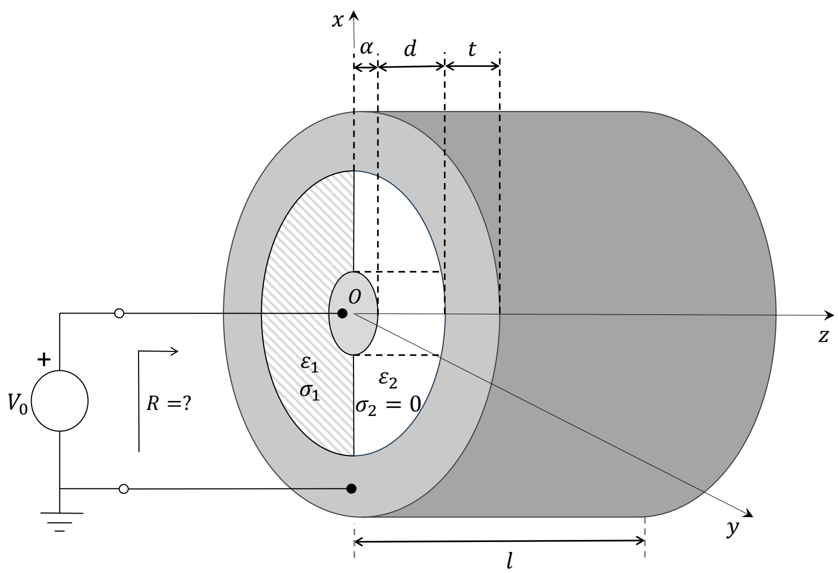
d) $\boxed{\text{If } N_1 \text{ doubles, } L_{11} \text{ will quadruple,}}$ [1pt] for increase factor

Since:

- doubling N_1 doubles the magnetic flux density $\bar{B}_1$ in the core
- doubling N_1 also doubles the area enclosed by $C_1 \Rightarrow$ double flux

[1pt] for explanation

Question 2 [20 points]



Consider the coaxial cable shown in the figure. The cable has length l and is made by:

- an inner cylindrical conductor of radius α;
- an outer cylindrical conductor of thickness t, separated by the inner conductor by a distance d.

Both the inner and outer conductor can be assumed to be perfect conductors. The volume inside the cable is filled by two different materials:

- dielectric 1 in the left half (striped region in the figure). It has permittivity ε_1 and is lossy ($\sigma_1 \neq 0$);
- dielectric 2 in the right half (white region in the figure). It has permittivity ε_2 and is not conductive ($\sigma_2 = 0$).

You can assume that:

- the volume density of free charge is zero in both dielectrics;
- there is no free charge at the interface between them;
- all fringing (edge) effects due to the finite length of the cable are negligible.

A DC voltage source is applied as shown. Answer the following questions, justifying all your answers.

a) solve the Poisson's equation to determine the potential V in both dielectrics [12 points];

b) determine the resistance R of the cable seen by the voltage source [8 points];

a) consider dielectric 1

Poisson's equation $\nabla \cdot (\varepsilon_1 \nabla V) = -\underbrace{\rho_v}_{=0}$ (assumption given)

uniform in this region [1pt]

1pt

Simplifying Poisson equation [2pt]

$\nabla^2 V = 0$

Cylindrical symmetry + neglecting edge effects

$V = V(r, \cancel{\phi}, \cancel{z})$

Form of V [2pt]

$\cancel{\frac{1}{r}} \frac{\partial}{\partial r}\left[r \frac{\partial V}{\partial r}\right] = 0$; $r \frac{\partial V}{\partial r} = c_1$; $\frac{\partial V}{\partial r} = \frac{c_1}{r}$

$V(r) = c_1 \ln r + c_2$

Solving for V [2pt]

Boundary conditions

$$\begin{cases} V(a) = V_0 \\ V(a+d) = 0 \end{cases} ; \quad \begin{cases} C_1 \ln a + C_2 = V_0 \\ C_1 \ln(a+d) + C_2 = 0 \end{cases}$$

Boundary cond's [2pt]

$$C_1 \ln\left(\frac{a}{a+d}\right) = V_0 \; ; \quad C_1 = \frac{V_0}{\ln\left(\frac{a}{a+d}\right)} = -\frac{V_0}{\ln\left(1+\frac{d}{a}\right)}$$

$$C_2 = -C_1 \ln(a+d) = +V_0 \frac{\ln(a+d)}{\ln\left(1+\frac{d}{a}\right)}$$

$$V(r) = \frac{-V_0}{\ln\left(1+\frac{d}{a}\right)} \ln(r) + V_0 \frac{\ln(a+d)}{\ln\left(1+\frac{d}{a}\right)} = \boxed{V_0 \frac{\ln\left(\frac{a+d}{r}\right)}{\ln\left(1+\frac{d}{a}\right)}}$$

V [2pt]

In dielectric 2, the Poisson equation still reads $\nabla^2 V = 0$

$\Rightarrow$ V(r) has the same expression

[2pt]

b) $\bar{E} = -\nabla V = -\dfrac{\partial V}{\partial r}\,\bar{a}_r = +\dfrac{V_0}{\ln\left(1+\dfrac{d}{\alpha}\right)}\dfrac{1}{r}\,\bar{a}_r$] $\bar{E}$ calculat. [2pt]

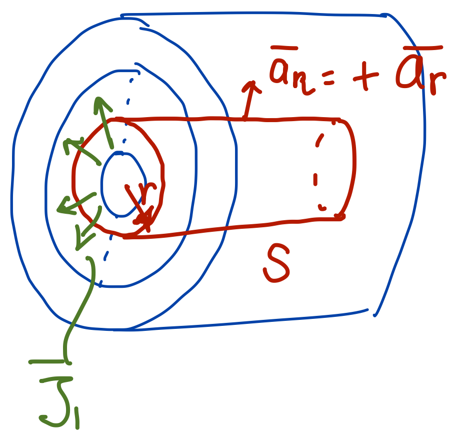
$\bar{J} = 0$ in dielectric 2 since $\sigma_2 = 0$] [1pt] $\bar{J}$ in diel. 2

$\bar{J}_1 = \sigma_1 \bar{E}$ in dielectric 1

$\bar{J}_1 = \dfrac{\sigma_1 V_0}{\ln\left(1+\dfrac{d}{\alpha}\right)}\dfrac{1}{r}\,\bar{a}_r$] [1pt] $\bar{J}$ in diel 1

Current flowing from inner conductor to outer

$I = \int \bar{J}_1 \cdot d\bar{S}$



Integr. limits

[1pt] [1pt] Correct $d\bar{S}$: [1pt]

$$= \int_{\varphi=\pi}^{2\pi}\int_{z=0}^{\ell} \frac{\sigma_1 V_0}{\ln\left(1+\dfrac{d}{\alpha}\right)} \frac{1}{\cancel{r}} \underbrace{\bar{a}_r \cdot \bar{a}_r}_{1} \cancel{r}\, d\varphi\, dz =$$

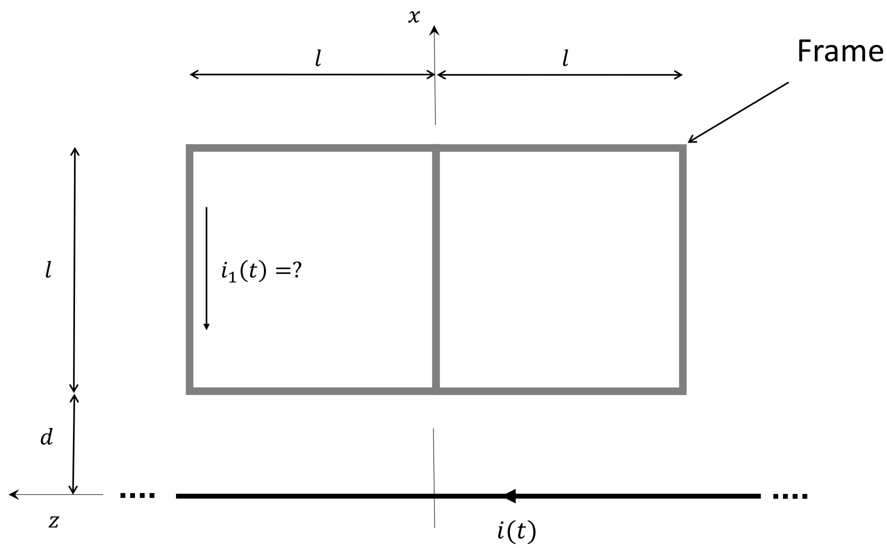
I calculation from $\bar{J}$ [3 pt]

$$= \frac{\sigma_1 V_0}{\ln\left(1+\frac{d}{a}\right)} \pi \ell$$

$$R = \frac{V_0}{I} = \frac{\cancel{V_0}}{\frac{\sigma_1 \cancel{V_0} \pi \ell}{\ln(1+d/a)}} = \boxed{\frac{\ln(1+d/a)}{\sigma_1 \ell \pi}}$$

R calculation [1pt]

Question 3 [20 points]



You are leading the team responsible for the electrical design of a quantum computer. The gray metallic frame in the figure is part of the chassis of the computer, which serves also as a shield from external interference. This frame is metallic and consists of two square loops, each with edge length l. Each edge of the frame (that is, each section of length l) has resistance R and negligible inductance. The frame is in the $x-z$ plane, at a distance d from the z axis. The whole structure is in air. Along the z axis there is a very long straight line carrying an AC current $i(t) = I_0 \cos(\omega_0 t)$. The quantum computer prototype is not functioning properly, and you want to determine if there are unwanted currents flowing in the frame that may cause interference.

a) Calculate the magnetic field $\mathbf{H}(t)$ produced by current $i(t)$ in the whole space. You can neglect any effect due to displacement currents [4 points];

b) Find the electromotive force $\mathcal{V}_L$ induced on the left loop of the frame. Find the electromotive force $\mathcal{V}_R$ induced on the right loop of the frame. Sketch the direction that you consider for the two loops (path orientation) [6 points]

c) Draw an equivalent circuit of the frame. Indicate the value of each circuit element [6 points];

d) Find the current $i_1(t)$ circulating on the left loop of the frame [4 points].

a) Due to cylindrical symmetry of line

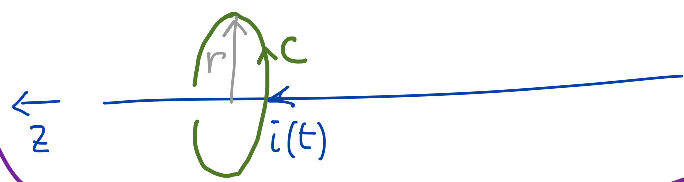
$$\overline{H}(r) = H(r)\,\overline{a}_\varphi$$

[1pt]

Ampere's law

Circular path, radius r

[1pt]



$$\oint_C \overline{H}\cdot d\overline{\ell} = I$$

$$\underbrace{H(r,t)2\pi r}_{[0.5pt]} = \underbrace{i(t)}_{[0.5pt]}$$

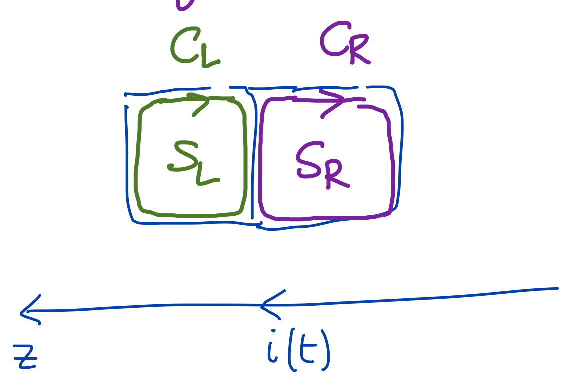
$$\overline{H} = \frac{i(t)}{2\pi r}\,\overline{a}_\varphi = \frac{I_0 \cos(\omega_0 t)}{2\pi r}\,\overline{a}_\varphi$$

[1pt]

b) Faraday's law on left loop C_L

Use of Faraday's law: [1pt]

$$V_{emf,L} = -\frac{d}{dt}\int_{S_L} \overline{B}\cdot d\overline{S} =$$



Grader: check that EMF expression is consistent with orientation of path

Correct $\overline{dS}$

[1pt)

[0.5pt] [0.5pt]

$$V_L = -\frac{d}{dt}\left\{ \int_{r=d}^{d+\ell} \int_{z=0}^{\ell} \frac{\mu_0 I_0 \cos(\omega_0 t)}{2\pi r} \underbrace{\bar{a}_\varphi \cdot \bar{a}_\varphi}_{1} \, dr\, dz \right\} =$$

$$= -\frac{d}{dt}\left\{ \frac{\mu_0 I_0}{2\pi} \cos(\omega_0 t)\, \ell \int_{d}^{d+\ell} \frac{1}{r} dr \right\} =$$

$$= -\frac{\mu_0 I_0}{2\pi} \ell \ln\left(\frac{d+\ell}{d}\right) \frac{d}{dt}\left\{\cos(\omega_0 t)\right\} =$$

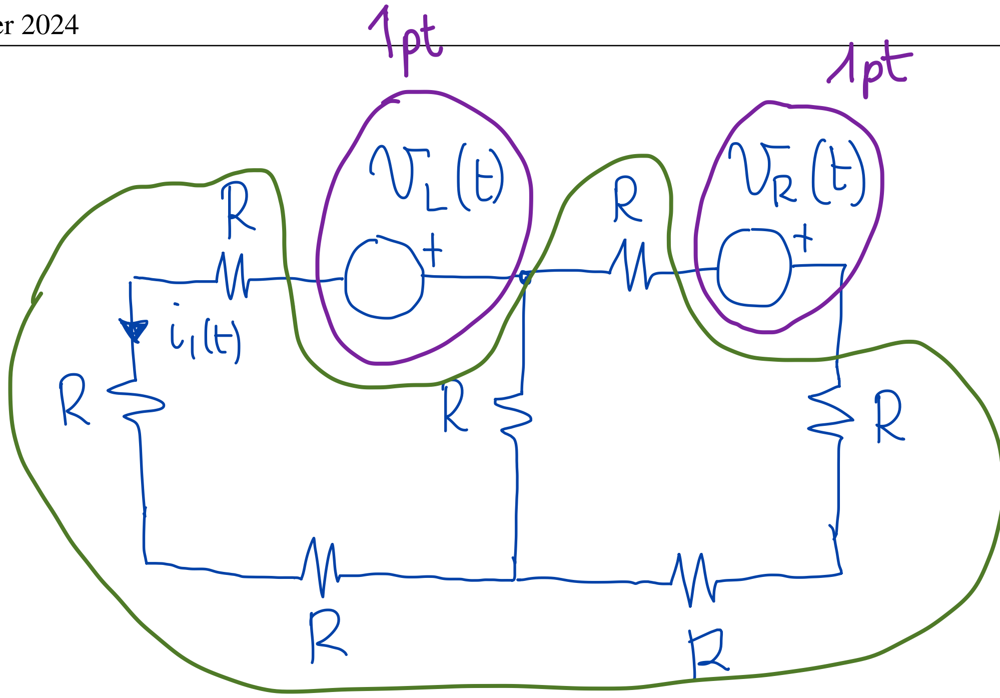
$$= \frac{\mu_0 I_0}{2\pi} \ell \ln\left(1+\frac{\ell}{d}\right) \omega_0 \sin(\omega_0 t)$$

[1pt]

V_R has the same expression since $\bar{B}$ over the right frame is the same [2pt]

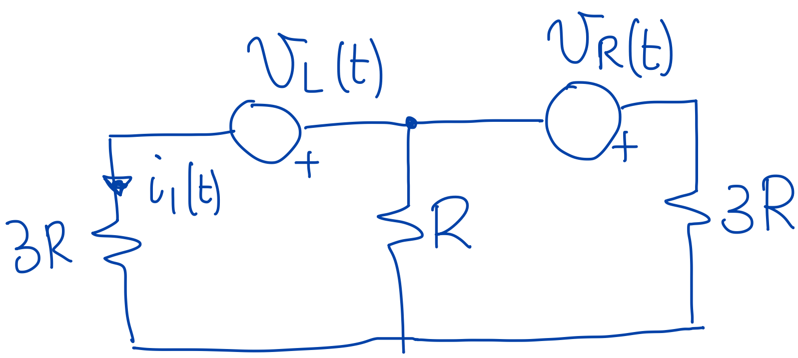
Circuit: right topology : 2 pt (two loops)

c)

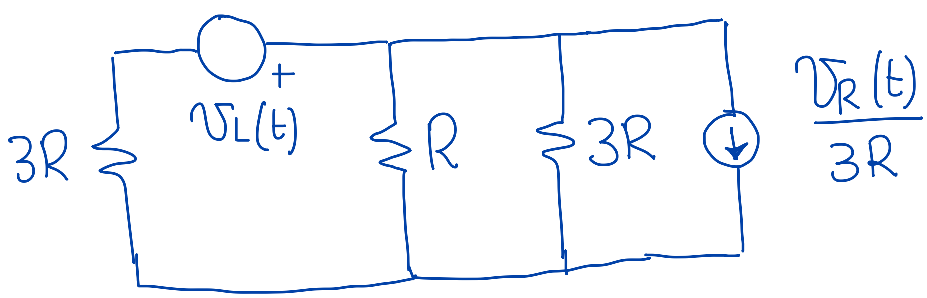


resistors: [2pt]

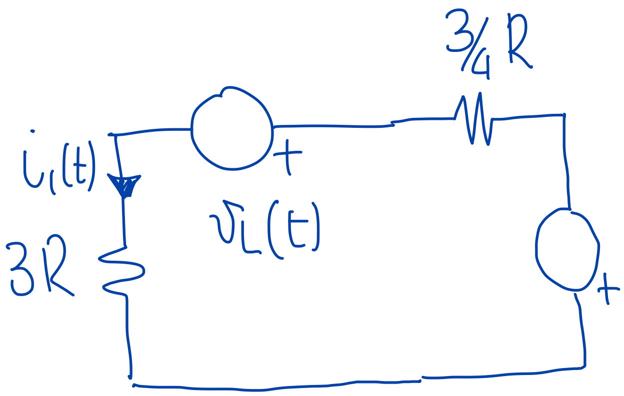
d)



Solving for $i_1(t)$: [4pts]



$$3R // R = \frac{3R^2}{4R} = \frac{3}{4}R$$



$$\frac{v_R(t)}{\cancel{3R}} \cdot \frac{\cancel{3}}{4}\cancel{R} = \frac{v_R(t)}{4}$$

$$i_1(t) = -\frac{v_L(t) + \frac{v_R(t)}{4}}{\left(3+\frac{3}{4}\right)R} = \frac{\frac{\overset{1}{\cancel{5}}}{\cancel{4}}}{\frac{\underset{3}{\cancel{15}}}{\cancel{4}}R} \cdot \cancel{2}\,\frac{\mu_0 I_0}{\cancel{2}\pi}\,\ell \ln\left(1+\frac{\ell}{d}\right)\omega_0 \sin(\omega_0 t)$$

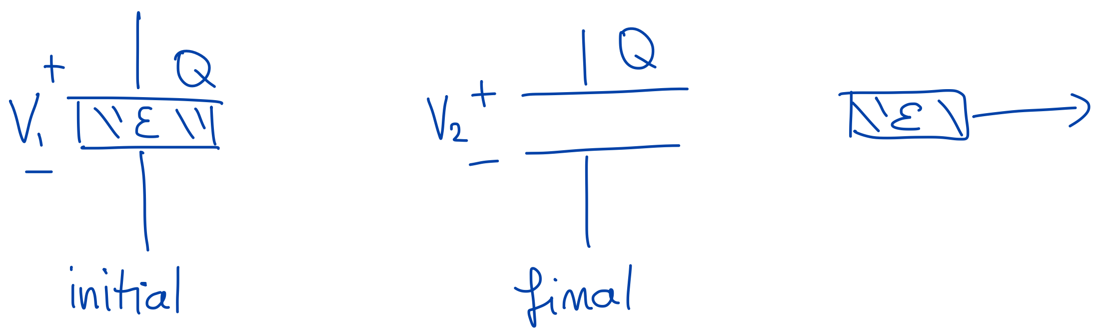
$$= \frac{1}{3R}\,\frac{\mu_0 I_0}{\pi}\,\ell \ln\left(1+\frac{\ell}{d}\right)\omega_0 \sin(\omega_0 t)$$

Question 4.1 [4 points]

We have a capacitor made by two parallel metallic plates and a block of perfect dielectric in between. Initially, the capacitor is connected to a voltage source. Then, the voltage source is disconnected. Finally, the dielectric block is removed. Is the energy stored in the capacitor in its final state

- lower than
- the same as
- higher than — right answer: [1pt]

the energy stored in the capacitor in the initial state? [4 points]. Justify your answer.



Explanation: [3pt] in total

Q remains the same [1pt]

C decreases [1pt]

$C = \frac{Q}{V}$ $\quad V = \frac{Q}{C}$

$$W_e = \frac{1}{2} C V^2 = \frac{1}{2} \cancel{C} \frac{Q^2}{C^{\cancel{2}}} = \frac{1}{2} \frac{Q^2}{C}$$

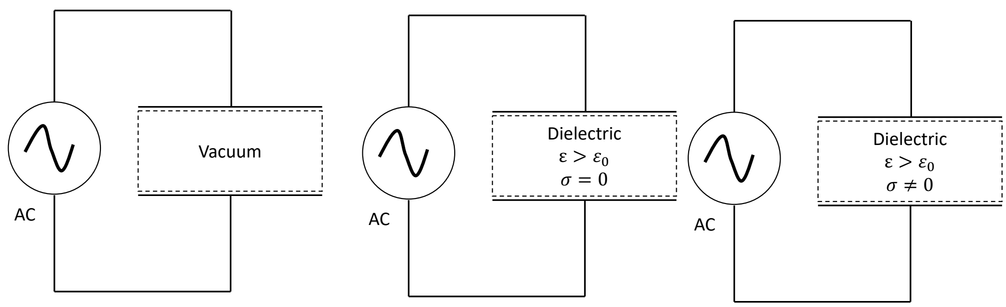
Q^2 unchanged

C decreases

[1pt]

$\Rightarrow$ We increases $\Rightarrow$ "higher than"

Question 4.2 [6 points]



Capacitor 1 Capacitor 2 Capacitor 3

We have three identical parallel plate capacitors, all connected to a sinusoidal voltage source. In capacitor 1, there is vacuum between the plates. In capacitor 2, there is an ideal dielectric with $\varepsilon > \varepsilon_0$ and $\sigma = 0$. In capacitor 3, there is a lossy dielectric with $\varepsilon > \varepsilon_0$ and $\sigma \neq 0$.

Consider the region between the plates of each capacitor, and answer these questions (with justification):

a) in which capacitor (or capacitors) a current density due to free charges **J** is present in that region? [2 points]

b) in which capacitor (or capacitors) a displacement current density $\mathbf{J}_d$ is present in that region? [2 points]

c) in which capacitor (or capacitors) there is a motion of polarization charges in that region? With motion, we refer to any motion, even by a small distance [2 points]

Right Answer [1pt]

a) only in capacitor 3, since free charges are present only in conductive media where $\sigma \neq 0$

Explanation: [1pt]

b) $\bar{J}_d = \frac{\partial \bar{D}}{\partial t} = \varepsilon \frac{\partial \bar{E}}{\partial t}$

Explanation: [1pt]

Right answer: [1pt]

In all three capacitors there is a time-varying $\bar{E}$ $\Rightarrow$ displacement current density

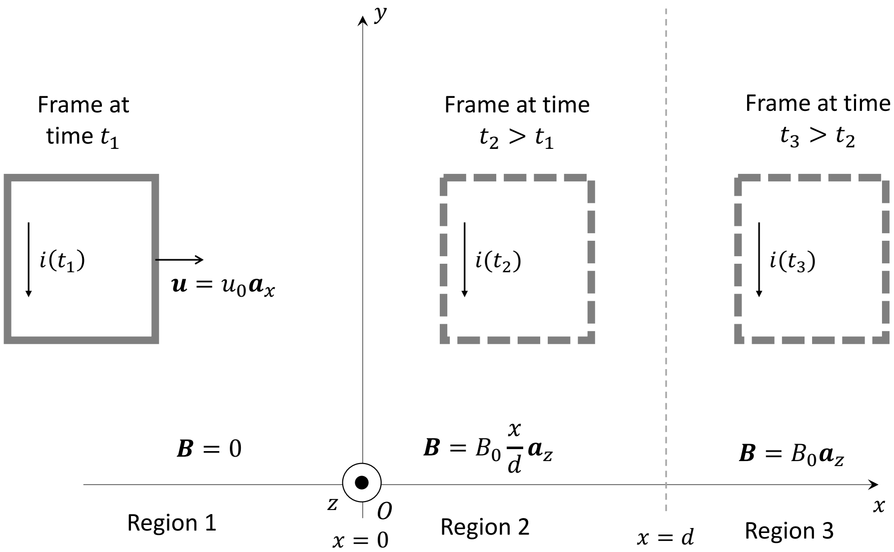
Explanation: [1pt]

Right answer: [1pt]

c) In capacitors 2 and 3, since there is a material that polarizes in the $\bar{E}(t)$ direction $\Rightarrow$ motion of polarization charges.

Not in capacitor 1, where there are no charges (it's vacuum)

Question 4.3 [10 points]



A metallic frame is traveling with uniform velocity $\mathbf{u} = u_0\mathbf{a}_x$ in the direction of the positive x axis ($u_0 > 0$).

- Initially, at time t_1, the frame is entirely in region 1 ($x < 0$). In this region, $\mathbf{B} = 0$;
- At time $t_2 > t_1$, the whole frame is in region 2 ($0 < x < d$). In this region, $\mathbf{B} = B_0\frac{x}{d}\mathbf{a}_z$, where B_0 is a positive constant;
- Finally, at time $t_3 > t_2$, the whole frame is in region 3 ($x > d$). In this region, $\mathbf{B} = B_0\mathbf{a}_z$.

Let $i(t)$ be the current induced in the frame. Please answer the following questions, and justify each answer:

a) Is $i(t_2)$ zero, positive, or negative? [2 points]

b) Is $i(t_3)$ zero, positive, or negative? [2 points]

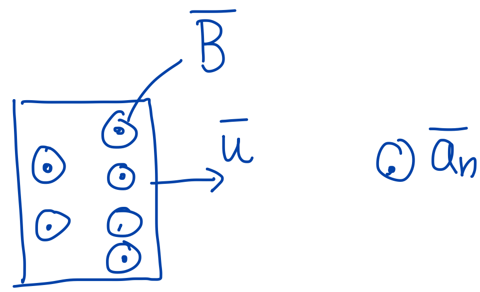
c) At time t_2, is the frame accelerating, decelerating, or still moving with constant velocity? Why? [3 points]

d) At time t_3, is the frame accelerating, decelerating, or moving with constant velocity? Why? [3 points]

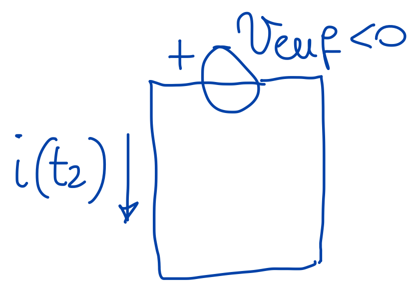
For each point: right answer: [1pt]

explanation : [1pt]

a) when the frame is in region 2, the magnetic flux through it is increasing (measured out of the page)



$$\Rightarrow V_{emf} = -\frac{d\Phi_m}{dt} < 0$$



$$\Rightarrow i(t_2) < 0$$

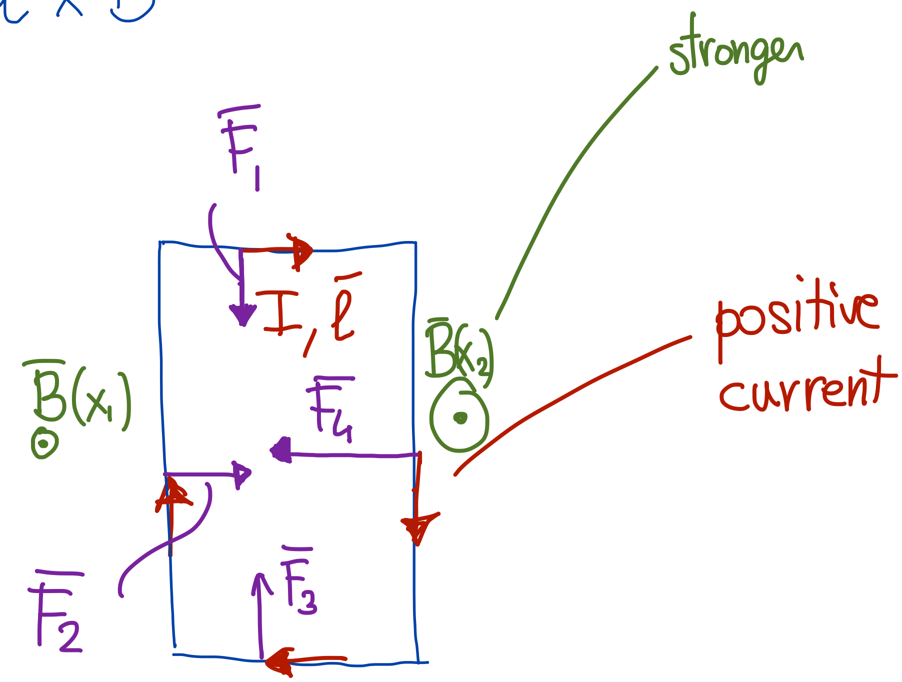
Can also be determined using Lenz's law

b) $i(t_3) = 0$ since Φ_m remains constant when the frame is fully in region 3

explanation : [1pt]

c) Forces acting on each edge of the frame

$\bar{F} = I\,\bar{\ell} \times \bar{B}$



$\bar{F}_1 + \bar{F}_3 = 0$

$|\bar{F}_4| > |\bar{F}_2|$ since $|B|$ higher on right edge

$\Rightarrow$ deceleration explanation : [2pt]

Alternative justification:

- induced currents will dissipate energy by Joule's law
- such energy is taken away from kinetic energy (it cannot come from anywhere else)

d) There is no current induced

$\Rightarrow$ no net magnetic force

$\Rightarrow$ constant velocity

explanation : [2pt]

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Differential operators

4.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial R}\vec{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

4.2 Divergence

$$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial(R^2A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

4.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

4.4 Curl

$$\begin{aligned}
\nabla\times\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a}_\phi + \frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\vec{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\vec{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\vec{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\vec{a}_\phi
\end{aligned}$$

5. Electromagnetic formulas

Table 1 Electrostatics

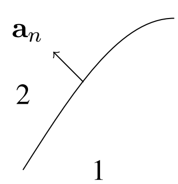
$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3} dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$	$\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}$	
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$	$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$
$\rho_{p,v} = -\nabla \cdot \mathbf{P}$ $\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$ $\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$ $E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2} QV$
$W_e = \frac{1}{2}\int_v \rho_v V dv = \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{D} \cdot \mathbf{E} dv$	
$\nabla \cdot (\varepsilon \nabla V) = -\rho_v$	$\nabla \cdot (\varepsilon \nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$	
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{B} = \nabla \times \mathbf{A}$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$	$\nabla \cdot \mathbf{A} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert \mathbf{R} - \mathbf{R}' \rvert^3}$	$\mathbf{A} = \frac{\mu I}{4\pi} \int_{c'} \frac{\mathbf{dl}'}{\lvert \mathbf{R} - \mathbf{R}' \rvert}$	$\mathbf{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}}{\lvert \mathbf{R} - \mathbf{R}' \rvert} dv'$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$	
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$	
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$	
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$	
$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$	
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$	
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$	

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$