AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 1

29 September 2014 9:15 am - 10:15 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis

Given Name: Solutions

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	8	
2	8	
3	8	
4	8	
5	8	
6	8	
7	10	
TOTAL	58	/ 54

Note: The following integrals may be useful.

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\int \cos^4\theta\, d\theta = \frac{1}{4}\cos^3\theta \sin\theta + \frac{3}{4}\int \cos^2\theta\, d\theta$$

1) a) Given: $R = \{(x, y) | 0 \le x \le 3, 0 \le y \le 2\}$

Calculate: $\int_R (3xy^2 + 4x^2 y) dR$

(3 marks)

$$= \int_0^3 dx \int_0^2 dy \, (3xy^2 + 4x^2y)$$

$$= \int_0^3 dx \left[xy^3 + 2x^2y^2 \right]_0^2$$

$$= \int_0^3 (8x + 8x^2) \, dx$$

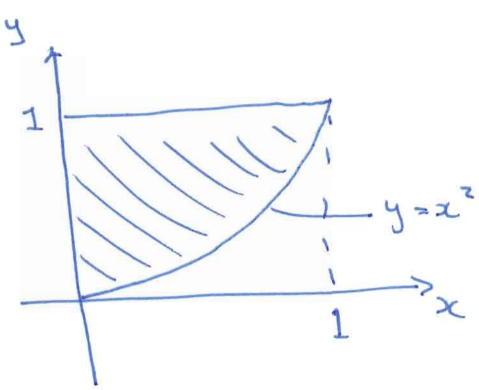
$$= \left[4x^2 + \frac{8x^3}{3} \right]_0^3$$

$$= 36 + 72 = 108$$

b) Evaluate the integral by reversing the order of integration, show a sketch of the region:

$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3) \, dy \, dx$$

(5 marks)



$$= \int_0^1 dy \int_0^{\sqrt{y}} x^3 \sin y^3 \, dx$$

$$= \int_0^1 dy \left[\frac{x^4}{4} \sin y^3 \right]_0^{\sqrt{y}}$$

$$= \int_0^1 \frac{y^2}{4} \sin y^3 \, dy$$

$$= \left[-\frac{1}{12} \cos y^3 \right]_0^1$$

$$= \frac{1}{12} (1 - \cos 1)$$

2) Given $\int \frac{1}{y^2+x^2}\,dx = \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$ find a formula for $\int \frac{1}{(y^2+x^2)^2}\,dx$.

(8 marks)

let $F(y) = \int \frac{dx}{y^2+x^2}$

$\therefore F'(y) = \frac{d}{dy}\int \frac{dx}{y^2+x^2} = \int \frac{\partial}{\partial y}\left(\frac{1}{y^2+x^2}\right)dx = \int \frac{-2y}{(y^2+x^2)^2}\,dx$

$\Rightarrow F'(y) = -2y \int \frac{dx}{(y^2+x^2)}$

also $F(y) = \frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$

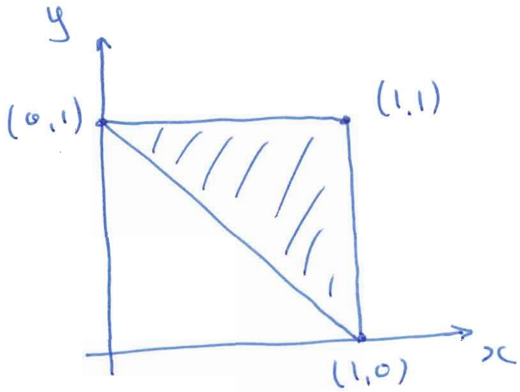
$\therefore F'(y) = \frac{1}{y}\cdot\frac{1}{\left(\frac{x}{y}\right)^2+1}\cdot\frac{-x}{y^2} + \frac{-1}{y^2}\tan^{-1}\left(\frac{x}{y}\right)$

$= -\frac{x}{y}\,\frac{1}{x^2+y^2} - \frac{1}{y^2}\tan^{-1}\left(\frac{x}{y}\right.$

$\Rightarrow \int \frac{dx}{(y^2+x^2)^2} = \frac{F'(y)}{-2y} = \frac{x}{2y^2}\,\frac{1}{x^2+y^2} + \frac{1}{2y^3}\tan^{-1}\left(\frac{x}{y}\right) + C$

3) Find the area of the part of the surface $z = \frac{2}{3}(x^{\frac{3}{2}} + y^{\frac{3}{2}})$ that lies above the triangle with vertices: (0, 1), (1, 0), (1, 1).

(8 marks)



$$z = g(x) = \frac{2}{3}\left(x^{3/2} + y^{3/2}\right)$$

$$\Rightarrow g_x = x^{1/2}$$

$$g_y = y^{1/2}$$

$$S = \int_R \sqrt{1 + g_x^2 + g_y^2}\, dR$$

$$\Rightarrow S = \int_0^1 dx \int_{1-x}^1 dy \sqrt{1 + x + y}$$

$$= \int_0^1 dx \left[\frac{2}{3}(1+x+y)^{3/2}\right]_{1-x}^1 = \frac{2}{3}\int_0^1 \left((2+x)^{3/2} - 2^{3/2}\right) dx$$

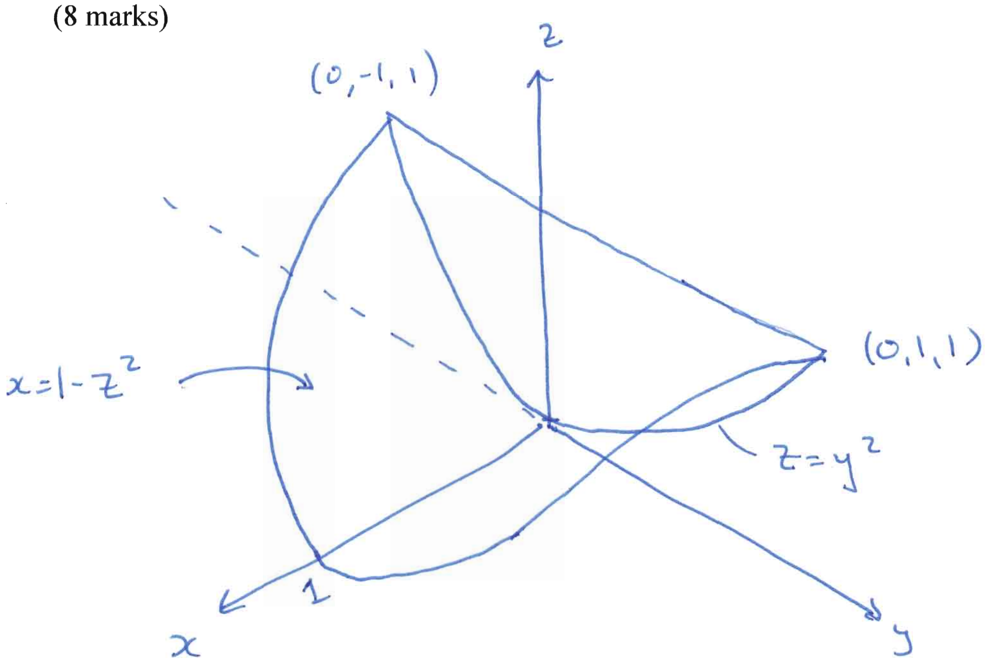
$$= \frac{2}{3}\left[\frac{2}{5}(2+x)^{5/2} - 2^{3/2}x\right]_0^1 = \frac{2}{3}\left(\frac{2}{5}\left(3^{5/2} - 2^{5/2}\right) - 2^{3/2}\right)$$

$$= \frac{2 \cdot 2 \cdot 3 \cdot 3 \cdot \sqrt{3}}{3 \cdot 5} - \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot \sqrt{2}}{3 \cdot 5} - \frac{2 \cdot 2 \cdot \sqrt{2}}{3}$$

$$= \frac{12\sqrt{3}}{5} - \frac{36\sqrt{2}}{15}$$

4) Use a triple integral to determine the volume of the solid bounded by the cylindrical surfaces: $z = y^2$, $x = 0$ and $z = \sqrt{1-x}$. You may wish to integrate in the x-direction first. Provide a sketch of the volume.

(8 marks)



$$V = \int_{-1}^{1} dy \int_{y^2}^{1} dz \int_{0}^{1-z^2} dx \cdot 1$$

$$= \int_{-1}^{1} dy \int_{y^2}^{1} dz \, (1-z^2)$$

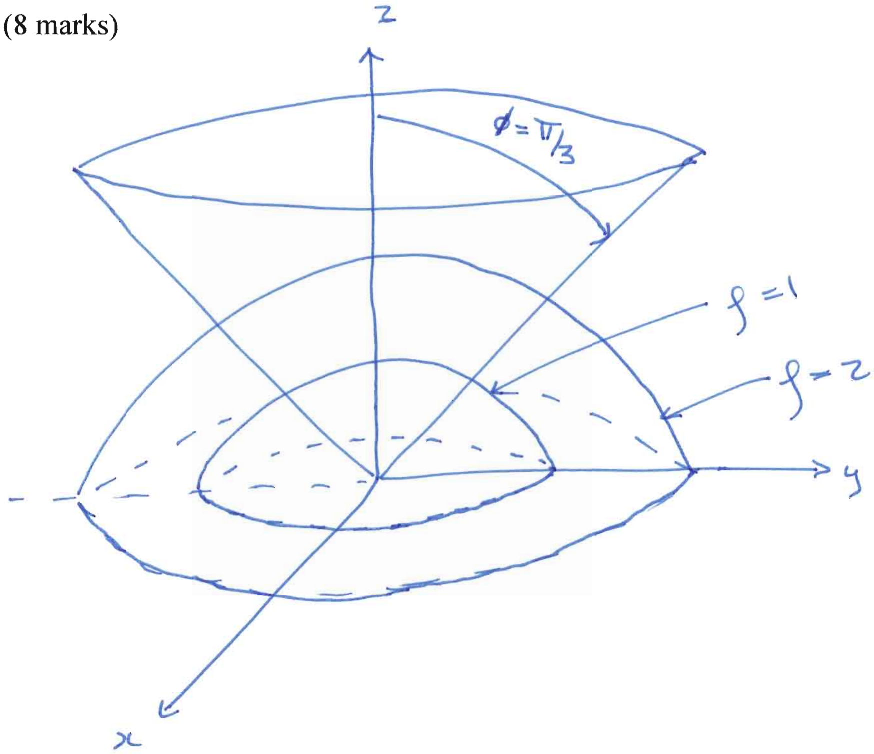
$$= \int_{-1}^{1} dy \left[z - \frac{z^3}{3} \right]_{y^2}^{1}$$

$$= \int_{-1}^{1} \left(\frac{2}{3} - y^2 + \frac{y^6}{3} \right) dy$$

$$= \left[\frac{2}{3}y - \frac{y^3}{3} + \frac{y^7}{21} \right]_{-1}^{1} = \frac{4}{3} - \frac{2}{3} + \frac{2}{21} = \frac{16}{21}$$

5) Evaluate $\int_V x^2 z\, dV$ where V lies between the spheres $\rho = 1$ and $\rho = 2$ and above the cone $\phi = \pi/3$. Provide a sketch of the volume.

(8 marks)



$x = \rho \sin\phi \cos\theta$

$z = \rho \cos\phi$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/3} d\phi \int_1^2 d\rho \cdot \rho^2 \sin\phi \cdot x^2 z$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/3} d\phi \int_1^2 d\rho \cdot \rho^2 \sin\phi \cdot \rho^2 \sin^2\phi \cos^2\theta \cdot \rho \cos\phi$$

$$= \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi/3} \sin^3\phi \cos\phi \, d\phi \int_1^2 \rho^5 \, d\rho$$

$$= \left[\frac{\theta}{2} + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \left[\frac{\sin^4\phi}{4} \right]_0^{\pi/3} \left[\frac{\rho^6}{6} \right]_1^2$$

$$= \pi \cdot \frac{1}{4} \left(\frac{\sqrt{3}}{2} \right)^4 \cdot \left(\frac{64-1}{6} \right) = \pi \cdot \frac{9}{64} \cdot \frac{63}{6} = \pi \cdot \frac{189}{128}$$

6) By directly calculating the partial derivatives, find the second degree polynomial approximation to the function $f(x,y)=\sqrt{x^2+y^3}$ near the point $(1,2)$.

(8 marks)

$$f(x,y) = \sqrt{x^2+y^3} \qquad f(1,2) = 3$$

$$f_x = \frac{x}{\sqrt{x^2+y^3}} \qquad f_x(1,2) = \frac{1}{3}$$

$$f_y = \frac{3}{2}\frac{y^2}{\sqrt{x^2+y^3}} \qquad f_y(1,2) = 2$$

$$f_{xx} = (x^2+y^3)^{-1/2} + x\left(-\tfrac{1}{2}\right)(x^2+y^3)^{-3/2}\cdot 2x$$

$$= \frac{x^2+y^3-x^2}{(x^2+y^3)^{3/2}} = \frac{y^3}{(x^2+y^3)^{3/2}} \qquad f_{xx}(1,2) = \frac{8}{27}$$

$$f_{xy} = x\left(-\tfrac{1}{2}\right)(x^2+y^3)^{-3/2}\cdot 3y^2$$

$$= -\frac{3}{2}\frac{xy^2}{(x^2+y^3)^{3/2}} \qquad f_{xy}(1,2) = \frac{-2}{9}$$

$$f_{yy} = 3y(x^2+y^3)^{-1/2} + \frac{3}{2}y^2\left(-\tfrac{1}{2}\right)(x^2+y^3)^{-3/2}\cdot 3y^2$$

$$= \frac{3y(x^2+y^2) - \frac{9}{4}y^4}{(x^2+y^3)^{3/2}} = \frac{12x^2y+3y^4}{4(x^2+y^3)^{3/2}} \qquad f_{yy}(1,2) = \frac{2}{3}$$

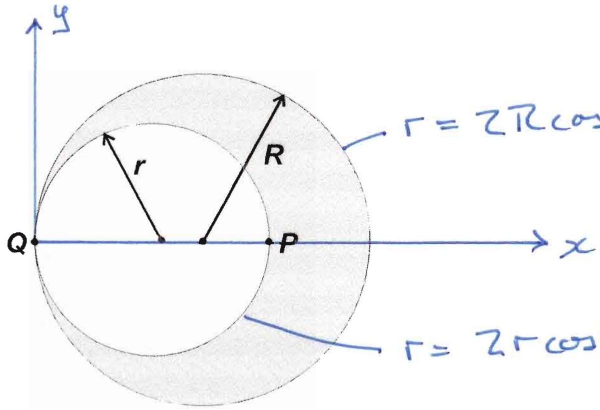
$$\therefore f(x,y) \simeq 3 + \frac{1}{3}(x-1) + 2(y-2) + \frac{4}{27}(x-1)^2 - \frac{2}{9}(x-1)(y-2) + \frac{1}{3}(y-2)^2$$

7) A disk of radius r is removed from a larger disk of radius R to form an earring (see figure). Assume the earring is a thin plate of uniform density.

a) Find the centre of mass of the earring in terms of r and R. Hint: set the origin at the point Q.

b) Show that the ratio R/r such that the centre of mass lies at point P (on the edge of the inner disk) is the golden mean: $(1+\sqrt{5})/2$.

(10 marks)



$r = 2R\cos\theta = A\cos\theta$

$r = 2r\cos\theta = a\cos\theta$

a) $\bar{y} = 0$ by symmetry

$$M = \pi\left(\left(\frac{A}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right)\lambda = \frac{\pi\lambda}{4}\left(A^2 - a^2\right)$$

$$\bar{x}M = \int_R x\lambda\, dR = \lambda\int_{-\pi/2}^{\pi/2} d\theta \int_{a\cos\theta}^{A\cos\theta} r\cos\theta \cdot r\,dr$$

$$= \lambda\int_{-\pi/2}^{\pi/2} d\theta \left[\frac{r^3}{3}\cos\theta\right]_{a\cos\theta}^{A\cos\theta} = \frac{\lambda}{3}\int_{-\pi/2}^{\pi/2} \cos\theta\left(A^3 - a^3\right)\cos^3\theta\, d\theta$$

$$= \frac{\lambda}{3}\left(A^3 - a^3\right)\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta = \frac{\lambda}{3}\left(A^3 - a^3\right)\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{\lambda}{3}\left(A^3 - a^3\right)\cdot\frac{3\pi}{8} = \frac{\lambda\pi}{8}\left(A^3 - a^3\right)$$

$$\therefore \bar{x} = \frac{\lambda\pi\left(A^3 - a^3\right)4}{\lambda\pi\left(A^2 - a^2\right)8} = \frac{1}{2}\frac{(A-a)\left(A^2 + Aa + a^2\right)}{(A-a)(A+a)} = \frac{1}{2}\frac{A^2 + Aa + a^2}{A+a}$$

$$= \frac{R^2 + Rr + r^2}{R+r}$$

b) point P is located at $x = 2r$

$$\Rightarrow \quad \frac{R^2 + Rr + r^2}{R + r} = 2r$$

$$R^2 + Rr + r^2 = 2rR + 2r^2$$

$$\therefore \quad R^2 - Rr - r^2 = 0$$

$$\left(\frac{R}{r}\right)^2 - \left(\frac{R}{r}\right) - 1 = 0$$

$$\Rightarrow \quad \frac{R}{r} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 + \sqrt{5}}{2}$$