University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test
9:10 – 10:55, 21 November 2022
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1) Evaluate the integrals:

a) $\int_0^4 (t^2 + t^{3/2})\,dt$ $= \left[\frac{t^3}{3} + \frac{2}{5} t^{5/2} \right]_0^4 = \frac{64}{3} + \frac{64}{5}$

b) $\int_5^5 \sqrt{t^2 + \sin t}\,dt$ $= 0$

c) $\int \frac{\sqrt{2+\sqrt{x}}}{\sqrt{x}}\,\mathrm{dx}$ let $u = 2+\sqrt{x}$ $du = \frac{dx}{2\sqrt{x}}$

$= 2\int \sqrt{u}\,du = \frac{4}{3} u^{3/2} + C = \frac{4}{3}(2+\sqrt{x})^{3/2} + C$

d) $\int_1^2 \frac{dt}{8-3t}$ let $u = 8-3t$ $du = -3\,dt$

$= \int_5^2 \frac{1}{u}\left(-\frac{du}{3}\right) = -\frac{1}{3}\left[\ln u\right]_5^2 = \frac{1}{3}\ln\frac{5}{2}$

e) $\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx$ let $u = 1+e^{-x}$ $du = -e^{-x}\,dx$

$= \int_2^{1+\frac{1}{e}} -\sqrt{u}\,du = \left[\frac{2}{3}u^{3/2}\right]_{1+1/e}^{2} = \frac{4}{3}\sqrt{2} - \frac{2}{3}\left(1+\frac{1}{e}\right)^{3/2}$

2) a) If $f(1) = 12$, f' is continuous, and $\int_1^4 f'(x)dx = 17$, what is the value of $f(4)$?

$$\int_1^4 f'(x)\,dx = f(4) - f(1) = 17 \qquad \therefore f(4) = 17+12 = 29$$

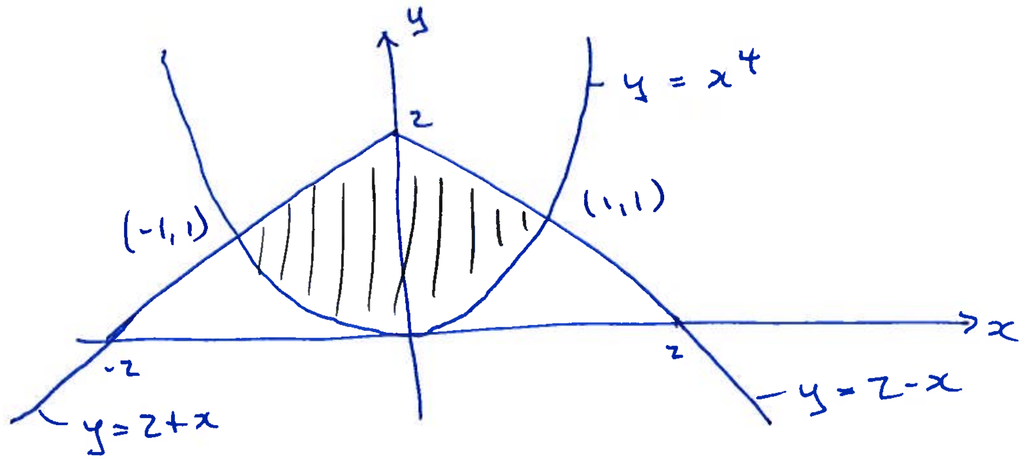
b) If $f(x) = \int_{\cos x}^{\sin x} \sqrt{1+t^2}dt$ and $g(y) = \int_7^y f(x)dx$, find $g''(\frac{\pi}{6})$.

$$\Rightarrow g'(y) = f(y) = \int_{\cos y}^{\sin y} \sqrt{1+t^2}\,dt$$

$$g''(y) = f'(y) = \sqrt{1+\sin^2 y}\cdot \cos y - \sqrt{1+\cos^2 y}\,(-\sin y)$$

$$g''\left(\frac{\pi}{6}\right) = \sqrt{1+\left(\frac{1}{2}\right)^2}\cdot \frac{\sqrt{3}}{2} + \sqrt{1+\left(\frac{\sqrt{3}}{2}\right)^2}\cdot \frac{1}{2} = \frac{\sqrt{15}}{4} + \frac{\sqrt{7}}{4}$$

3) Sketch the region enclosed by the curves $y = x^4$ and $y = 2 - |x|$ and find its area.



Intercepts: $x^4 = 2 - x \Rightarrow x = 1$

$x^4 = 2 + x \Rightarrow x = -1$

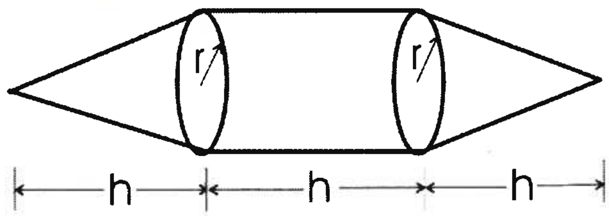
$$A = 2\int_0^1 \left((2-x) - x^4\right) dx$$

$$= 2\left[2x - \frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 2\left(2 - \frac{1}{2} - \frac{1}{5}\right)$$

$$= \frac{13}{5}$$

4) Right circular cones of height h and radius r are attached to each end of a right circular cylinder of height h and radius r, forming a double-pointed object as shown in the figure. For a given surface area A, show that the volume is a maximum when $\frac{r}{h} = \frac{\sqrt{5}}{2}$. (The volume of a cone is: $V = \frac{1}{3}\pi r^2 h$; the surface area is: $SA = \pi r \sqrt{h^2 + r^2}$, not including the area of the circular base.)



$$\Rightarrow A = 2\pi r h + 2\pi r\sqrt{h^2+r^2} = \text{constant}$$

$$\frac{A}{2\pi r} - h = \sqrt{h^2+r^2} \Rightarrow \frac{A^2}{4\pi^2 r^2} - \frac{2hA}{2\pi r} + \cancel{h^2} = \cancel{h^2} + r^2$$

$$\Rightarrow A^2 - 2Ah\cdot 2\pi r = 4\pi^2 r^4 \Rightarrow h = \frac{A^2 - 4\pi^2 r^4}{4A\pi r}$$

$$\Rightarrow V = \pi r^2 h + 2\cdot\frac{1}{3}\pi r^2 h = \frac{5}{3}\pi r^2 h = \frac{5}{3}\cancel{\pi} r^{\cancel{2}}\left(\frac{A - 4\pi^2 r^4}{4A\cancel{\pi}\cancel{r}}\right)$$

$$= \frac{5}{3}\;\frac{Ar - 4\pi^2 r^5}{A} = \frac{5}{12}Ar - \frac{5\pi^2}{3A}r^5$$

$$\frac{dV}{dr} = \frac{5}{12}A - \frac{25\pi^2}{3A}r^4 \qquad \frac{dV}{dr} = 0 \Rightarrow \frac{5A}{12} = \frac{25\pi^2 r^4}{3A}$$

$$\Rightarrow r^4 = \frac{A^2}{20\pi^2}$$

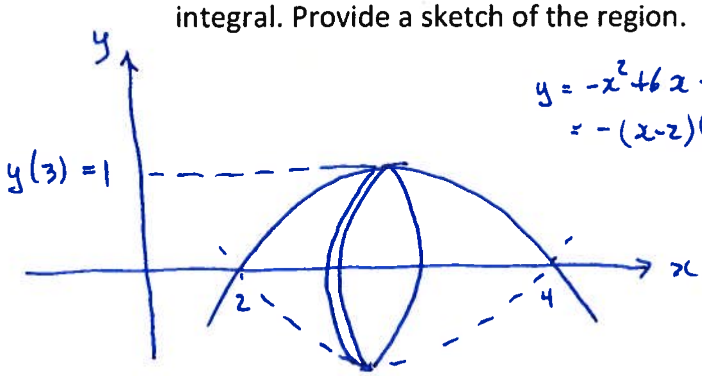
$$\frac{d^2V}{dr^2} = \frac{-100\pi^2 r^3}{3A} < 0 \quad \text{all } r > 0 \quad \therefore \text{ local max}$$

$$\Rightarrow h = \frac{A^2 - 4\pi^2\left(\frac{A^2}{20\pi^2}\right)}{4A\pi\left(A^2/20\pi^2\right)^{1/4}} = \frac{\frac{4}{5}A}{4\pi\sqrt{A}}\sqrt{\pi}(20)^{1/4} = \sqrt{\frac{A}{\pi}}\;\frac{20^{1/4}}{5}$$

$$\Rightarrow r = \sqrt{\frac{A}{\pi}}(20)^{-1/4} \Rightarrow \frac{r}{h} = 5\,\frac{20^{-1/4}}{20^{1/4}} = \frac{5}{\sqrt{20}} = \frac{\sqrt{5}}{2}$$

5) Consider the volume of the solid obtained by rotating the region bounded by the curves $y = -x^2 + 6x - 8,\ \ y = 0$ about the x-axis:

a) Find an integral representing this volume as calculated by the disk method. Do not evaluate the integral. Provide a sketch of the region.



$y = -x^2 + 6x - 8$
$= -(x-2)(x-4)$

$\frac{dy}{dx} = -2x + 6 \Rightarrow \frac{dy}{dx} = 0$ at $x = 3$

$$V = \int_2^4 \pi\left(-x^2 + 6x - 8\right)^2 dx$$

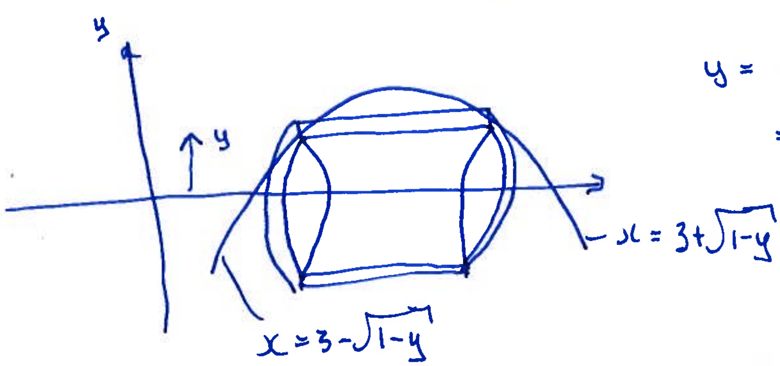
b) Calculate the volume using the shell method.

Note: In order to evaluate this integral you will first need to prove that: $\int x\sqrt{a+bx}\,dx = \frac{2}{15\,b^2}(3bx - 2a)(a+bx)^{\frac{3}{2}} + C$ by taking the derivative of the RHS.

$$\frac{d}{dx}\left[\frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C\right] = \frac{2}{15b^2}\left[3b(a+bx)^{3/2} + (3bx - 2a)\cdot\frac{3}{2}(a+bx)^{1/2}\cdot b\right]$$

$$= \frac{2}{5b}\left[(a+bx)^{1/2}\left((a+bx) + \frac{1}{2}(3bx - 2a)\right)\right] = \frac{2}{5b}(a+bx)^{1/2}\left(\frac{5}{2}bx\right)$$

$$= x\sqrt{a+bx} \Rightarrow \int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C$$



$y = -x^2 + 6x - 8 = -(x-3)^2 + 1$

$\Rightarrow 1 - y = (x-3)^2$

$\pm\sqrt{1-y} = x - 3 \Rightarrow x = 3 \pm \sqrt{1-y}$

$$V = \int_0^1 2\pi y\left(3 + \sqrt{1-y} - \left(3 - \sqrt{1-y}\right)\right)dy = \int_0^1 2\pi y \cdot 2\sqrt{1-y}\,dy$$

$$= 4\pi\left[\frac{2}{15}(-3y - 2)(1-y)^{3/2}\right]_0^1 = \frac{8\pi}{15}(2) = \frac{16\pi}{15}$$

6) For the function: $f(x) = \frac{x^2+1}{x^2-1}$

   i) Determine the domain of f.

   ii) Find the intervals in which f increases or decreases.

   iii) Find the extreme values.

   iv) Determine the concavity of the graph.

   v) Sketch the graph specifying the points of inflection, asymptotes and vertical tangents, if any.

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Sketch $f(x) = \dfrac{x^2+1}{x^2-1} = \dfrac{x^2+1}{(x+1)(x-1)} = \dfrac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$ if $x \neq 0$

(1) Domain all x except $x = \pm 1$

$\lim_{x \to \pm\infty} f(x) = 1 \therefore y=1$ is Horiz. Asym.

$\lim_{x \to +1^+} f(x) = +\infty$ $\lim_{x \to +1^-} f(x) = -\infty \therefore x=+1$ Vert. Asym

$\lim_{x \to -1^-} f(x) = +\infty$ $\lim_{x \to -1^+} f(x) = -\infty \therefore x=-1$ Vert. Asy.

(2) ~~Range~~: Intercepts $f(0) = -1$

(3) Symmetry $f(x)$ is even

(4) $f'(x) = \dfrac{-4x}{(x^2-1)^2}$ Sign of f': $f' < 0$ for $x > 0 \therefore f$ dec.
$f' > 0$ for $x < 0 \therefore f$ inc.

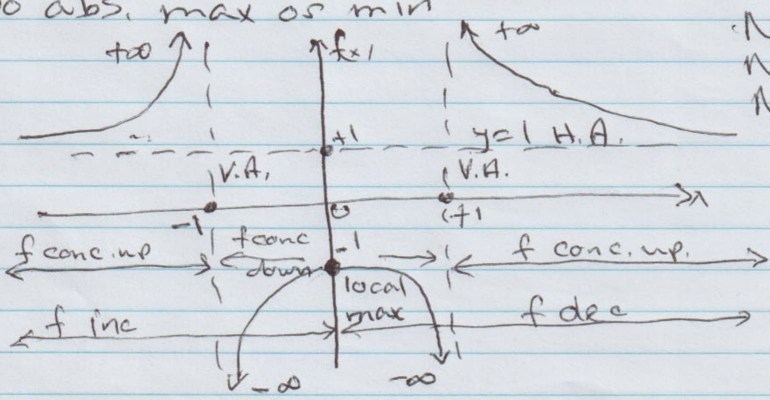
crit pts: 1. $f' = 0$? yes, at $x = 0$
2. f' DNE? yes, but these are V.A.'s not /Vert. Tang.
from 1st Deriv. T. $f(0) = -1 \therefore$ local max.

(5) $f''(x) = \dfrac{4(3x^2+1)}{(x^2-1)^3}$ sign of f'': $f'' > 0$ for $x > +1, < -1$
$\therefore f$ conc. up.
$f'' < 0$ for $-1 < x < +1$
$\therefore f$ conc. down.

$\therefore$ by 2nd DT $f(0) = -1$ is a local max
no Point of Inflection

(6) No abs. max or min

Range: all x except $[-1, +1]$



No P of I
No Vert Tang
No abs max, min

7) For the function: $f(x) = \sqrt{\frac{x}{x-2}}$

i) Determine the domain of *f*.
ii) Find the intervals in which *f* increases or decreases.
iii) Find the extreme values.
iv) Determine the concavity of the graph.
v) Sketch the graph specifying the points of inflection, asymptotes and vertical tangents, if any.

Sketch $f(x) = \sqrt{\frac{x}{x-2}}$ indicating all important features

1. Domain: one is tempted to 'simplify' to get $f(x) = \frac{\sqrt{x}}{\sqrt{x-2}}$ from which one would conclude that f(x) DNE for $x < 0$ (from the numerator) or for $x < 2$ (from the denominator), thus for $x < 2$.

   Wrong! Consider, for example, $f(-1) = \sqrt{\frac{-1}{-1-2}} = \sqrt{\frac{1}{3}}$ which *does* exist.

   Where did we go wrong? Answer: $\boxed{\sqrt{ab} = \begin{cases} (\sqrt{a})(\sqrt{b}), & \text{if } a \geq 0 \text{ and } b \geq 0 \\ (\sqrt{-a})(\sqrt{-b}), & \text{if } a < 0 \text{ and } b < 0 \\ \text{DNE, if } a, b \text{ have different signs} \end{cases}}$

   Be careful with square roots and all even roots! It can be safer not to 'simplify' them.

   Here the actual domain is all x except (0, 2].

   Note: $f(x) = \sqrt{1/(1 - 2/x)}$ $\therefore$ $\lim_{x \to \mp\infty} f(x) = 1$ $\therefore$ y = 1 is a Horizontal Asymptote and

   $\lim_{x \to 2^+} f(x) = +\infty$ $\therefore$ x = 2 is a Vertical Asymptote.

2. Intercepts: f (0) = 0

3. Symmetric, periodic? No

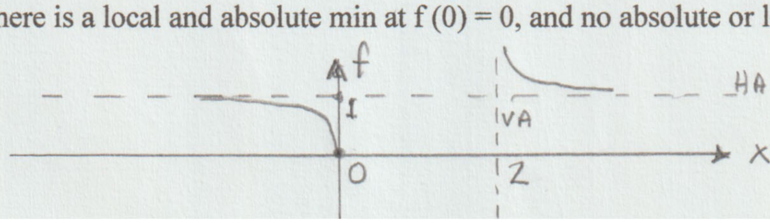
4. $f'(x) = -\left(x/(x-2)\right)^{-1/2}(x-2)^{-2}$ **Don't 'simplify'!** $\therefore$ $\lim_{x \to 0^-} f'(x) = -\infty$ $\therefore$ there is a one-sided Vertical Tangent at x = 0. Sign of f': < 0 for all x in domain of f (x) $\therefore$ f is decreasing.

   Critical Points: $f' = 0$ or DNE? Yes, f' DNE at x = 0.

5. $f''(x) = (x-2)^3\left((x-2)/2\right)^{1/2}(2 - 1/x)$

   $\therefore$ for $x > 2$, $f'' > 0$ $\therefore$ f is concave up and for $x < 0$, $f'' < 0$ $\therefore$ f is concave down

6. Max, Min: there is a local and absolute min at f (0) = 0, and no absolute or local max.



8) Show that the function $g(x) = \frac{1}{x-1}$, $x > 1$, is one-to-one and find its inverse. Provide a simple sketch of $g(x)$ and $g^{-1}(x)$.

$g(x) = \frac{1}{x-1} \Rightarrow g(x_1) = g(x_2) \rightarrow \frac{1}{x_1-1} = \frac{1}{x_2-1}$

$x_2 - 1 = x_1 - 1 \qquad x_1, x_2 \neq 1$

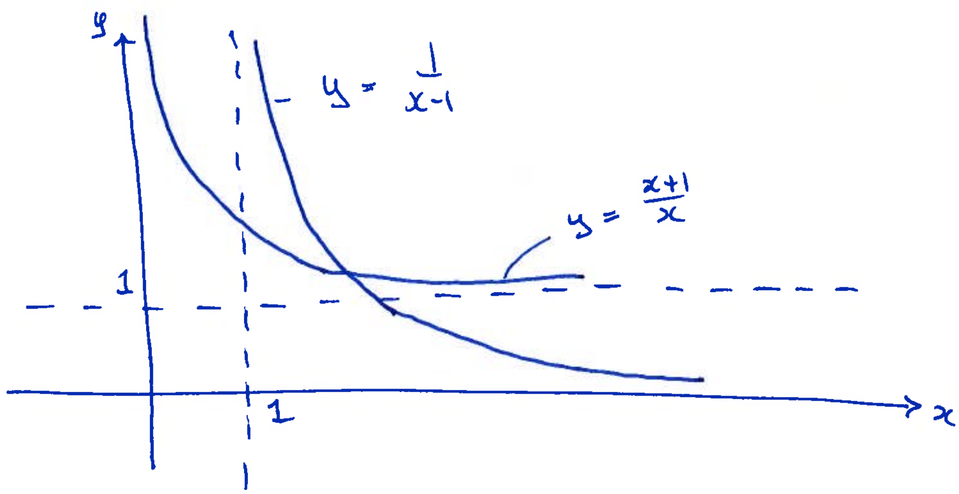
$x_2 = x_1$

$\therefore$ one to one

let $y = g^{-1}(x) \Rightarrow x = g(y) = \frac{1}{y-1} \Rightarrow y = 1 + \frac{1}{x} = \frac{x+1}{x} \quad x > 0$

since $y > 1$

$\therefore g^{-1}(x) = \frac{x+1}{x}, \quad x > 0$



9) a) Show that, for all positive values of x and y, $\frac{e^{x+y}}{xy} \geq e^2$

b) For what values of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

a) $\frac{e^{x+y}}{xy} = \frac{e^x}{x} \cdot \frac{e^y}{y}$ $\therefore$ show $\frac{e^x}{x} \geq e$, $x > 0$

$\Rightarrow e^x \geq ex$ or $e^x - ex \geq 0$

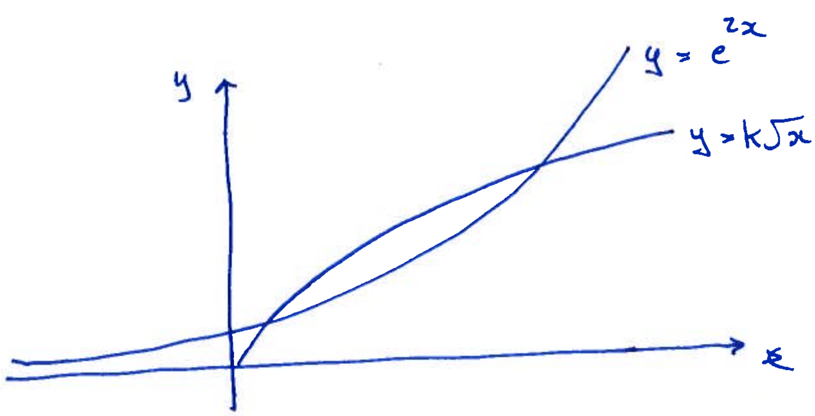
let $f(x) = e^x - ex$

$f'(x) = e^x - e$ $\quad f'(x) = 0 \Rightarrow e^x = e$ or $x = 1$

$f''(x) = e^x$ $\quad f''(1) = e > 0$ $\therefore$ local min

$f(1) = e - e = 0$ $\quad \therefore f(x) \geq 0$ $\quad \therefore \frac{e^x}{x} \geq e$ as required

b)



when the two curves touch at one point, they will have a common tangent line:

$\Rightarrow 2e^{2x} = \frac{k}{2\sqrt{x}} \Rightarrow 2k\sqrt{x} = \frac{k}{2\sqrt{x}} \Rightarrow x = \frac{1}{4}$

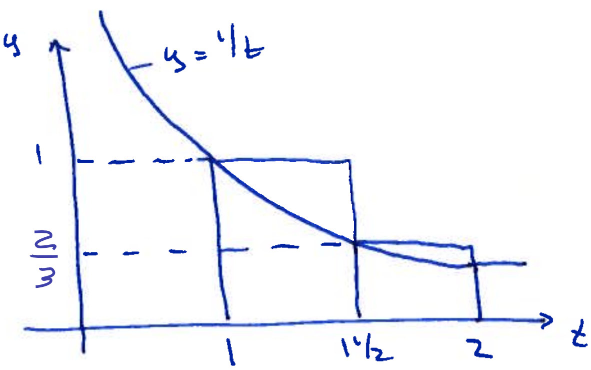
$\Rightarrow k = \frac{e^{1/2}}{\sqrt{1/4}} = 2\sqrt{e}$

10) a) Use a left Reimann sum with at least $n = 2$ subintervals of equal length to approximate $\ln 2 = \int_1^2 \frac{dt}{t}$ and show that $\ln 2 < 1$.

b) Use a right Reimann sum with $n = 3$ subintervals of equal length to approximate $\ln 4 = \int_1^4 \frac{dt}{t}$ and show that $\ln 4 > 1$.

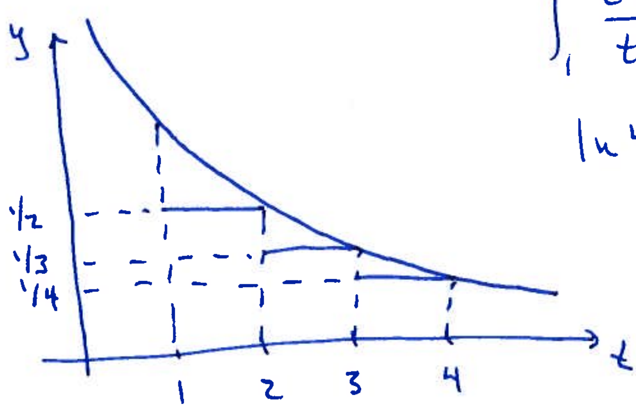
c) What bounds does this place on the value of e?

a)



$$\underbrace{\int_1^2 \frac{dt}{t}}_{\ln 2} < 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{2}{3} = \frac{5}{6} < 1$$

b)



$$\underbrace{\int_1^4 \frac{dt}{t}}_{\ln 4} > \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \frac{1}{4} \cdot 1 = \frac{13}{12} > 1$$

c) $\therefore \ln 2 < 1 = \ln e < \ln 4$

or $2 < e < 4$