UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

AER210 Midterm Test #2
Vector Calculus & Fluid Mechanics

Instructor: Alex Bercik November 30, 2023

First name (please write as legibly as possible within the boxes)

S O L U T I O N S

Last name

B E R C I K

Student ID number

:)

This exam contains 22 pages (including this cover page and a 3-page formula sheet at the back) and 8 questions. You may remove the formula sheet. The total number of points is 50.

This exam will be scanned and graded using Crowdmark. Therefore, please write legibly, within the margins, and **make a note to the grader if you need to continue your work on a separate page** (otherwise, your work will not be seen by the grader). Good luck!

Distribution of Marks

Question:	1	2	3	4	5	6	7	8	Total
Points:	6	6	3	4	6	8	12	5	50
Score:									

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1. Provide short answers to the following questions.
   (a) (1 point) What is a Newtonian fluid?
   (b) (1 point) What determines whether the continuum approach is valid or not?
   (c) (1 point) Find the dimensions of dynamic viscosity in an MLT system.
   (d) (1 point) Two spheres of equal diameter are submerged in water. One is made of steel, while the other is made of wood. Assuming the ratio of densities is $\rho_{\text{steel}} = 12\rho_{\text{wood}}$, what is the ratio of the buoyant forces F_{steel} and F_{wood}?
   (e) (2 points) How does the dynamic viscosity of air change with temperature, and why?

a) A Fluid where shear stress is proportional to deformation rate OR normal velocity gradient

i.e. $\tau = \mu \frac{d\gamma}{dt}$ or $\tau_{yx} = \mu \frac{\partial u}{\partial y}$

b) Continuum Approach is valid when ratio of microscopic length scale (mean free path) to macroscopic length scale is << 1.

i.e. Knudsen number $K_n = \frac{\lambda}{L} << 1$

c) $[\mu] = [\tau] / \left[\frac{\partial u}{\partial y}\right] = \left[\frac{\text{Force}}{\text{Area}}\right] / \left[\frac{\text{velocity}}{\text{length}}\right] = \frac{M}{TL}$

d) $\vec{F}_B = W_{\text{Fluid displaced}} = \rho_{\text{Fluid}} \forall_{\text{object}}$. Same radius $\Rightarrow$ Same volum

Same $\forall_{\text{object}} \Rightarrow$ Same $F_B \Rightarrow F_{\text{steel}} = F_{\text{wood}}$

e) μ increases with temperature due to increased particle collisions (1 pt for relationship, 1 pt for reason)

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2. Provide short, but now slightly more detailed answers to the following questions, using mathematical arguments whenever possible.

   (a) (2 points) Will a non-zero pressure gradient always cause a fluid element to accelerate?

   (b) (2 points) Does Archimedes' principle hold only for incompressible fluids?

   (c) (2 points) Why is it possible to use a scaled-down wind tunnel to accurately model the flow of air around an airplane if the pressures, temperatures, densities, length scales, and viscosities are all very different to the actual scenario?

a) No. $\rho \vec{a} = -\vec{\nabla} p + \rho \vec{g}$. If $\vec{a} = 0$ then $\vec{\nabla} p \neq 0$ can be true if $\vec{\nabla} p = \rho \vec{g}$. i.e. a non-zero pressure gradient can exist in a static fluid to balance body forces.

b) No, it holds equally well for compressible fluids. Whenever $\vec{\nabla} p = \rho \vec{g}$, by divergence THM

$$\vec{F}_B = -\int_{\partial\Omega} p \hat{n} \, d\Gamma = -\int_\Omega \vec{\nabla} p \, d\Omega = \underbrace{-\int_\Omega \rho \vec{g} \, d\Omega}_{\text{Weight of the fluid!}}$$

c) Because we can match dimensionless flow parameters to achieve $\underbrace{\text{dynamic}}_{\text{Force ratios}}$ & $\underbrace{\text{Kinematic}}_{\text{Flow patterns}}$ similarity

$\Rightarrow$ Physics can be the same between different scales, and dimensions can be scaled appropriately.

For wind tunnels, we can match Ma & Re.

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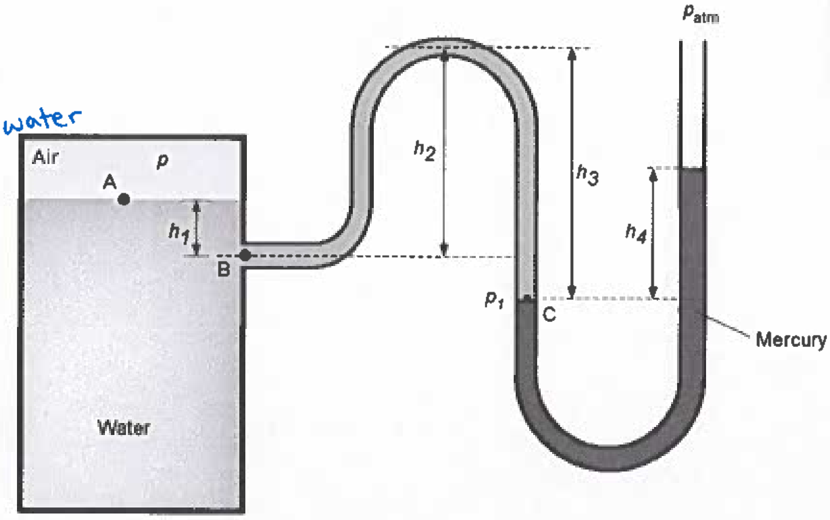
3. (3 points) A manometer containing water with density ρ_w and mercury with density ρ_m is connected to a tank containing air at an internal pressure p, the other end being open to atmospheric pressure p_{atm}. Derive an expression relating p to p_{atm}.



$P_{Air} = P_A$ since $\gamma_{air} << \gamma_{water}$

$P_A = P_B - \rho_w g h_1$

$P_B = P_c - \rho_w g (h_3 - h_2)$

$P_c = P_{atm} + \rho_m g h_4$

$\Rightarrow P = P_{Air} = P_{atm} + \rho_m g h_4 - \rho_w g (h_3 - h_2) - \rho_w g h_1$

$\Rightarrow \boxed{P = P_{atm} + \rho_m g h_4 - \rho_w g (h_3 - h_2 + h_1)}$

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4. (4 points) Show that for an incompressible ~~flow~~ Fluid ($\nabla \cdot \boldsymbol{v} = 0$), assuming both the momentum field $\rho \boldsymbol{v}$ and pressure field p are C^1, then

$$\int_\Omega \nabla p \cdot \rho \boldsymbol{v} \, d\Omega = \oint_{\partial\Omega} p \, \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \, d\Gamma$$

whenever Ω is a regular region with piecewise-smooth boundary $\partial\Omega$.

incompressible Fluid $\Rightarrow$ $\vec{\nabla} \cdot \vec{v} = 0$ and ρ constant

Not a Fluids question, just a calculus problem!

product rule (or vector identity) $\quad \nabla p \cdot \rho \vec{v} = \nabla \cdot (p \rho \vec{v}) - p \rho \underbrace{(\nabla \cdot \vec{v})}_{=0}$

possible because all C^1

$$\Rightarrow \int_\Omega (\nabla p \cdot \rho \vec{v}) \, d\Omega = \int_\Omega \nabla \cdot (p \rho \vec{v}) \, d\Omega$$

$$= \oint_{\partial\Omega} p \rho \vec{v} \cdot \hat{n} \, d\Gamma$$

by Divergence THM.

We can apply it since $p \rho \vec{v}$ is C^1 and Ω is a regular region w/ piecewise-smooth bdy.

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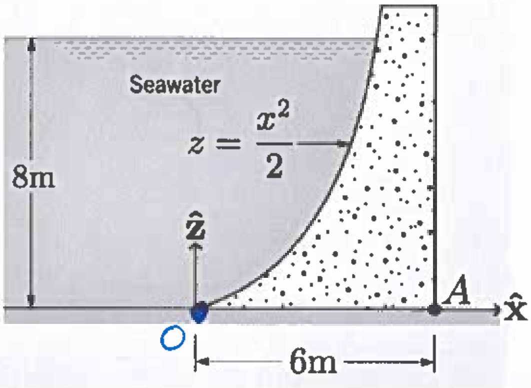
5. (6 points) The concrete seawall in the figure below has a parabolic surface given by $z = \frac{x^2}{2}$, a width (into the page) of $w = 2\text{m}$, and restrains seawater ($\rho = 1000\frac{\text{kg}}{\text{m}^3}$) at a depth of 8m. Determine the moment with respect to the axis through the point A acting on the seawall due to the seawater. Use $\boldsymbol{g} = -9.8\frac{\text{m}}{\text{s}^2}\hat{\boldsymbol{z}}$.



let O be the origin @ base of seawall

parametrize surface $\vec{r}_0 = \begin{pmatrix} x \\ y \\ \frac{x^2}{2} \end{pmatrix}$

From origin O to surface

surface vectors $\frac{\partial \vec{r}_0}{\partial x} = \begin{pmatrix} 1 \\ 0 \\ x \end{pmatrix}$ & $\frac{\partial \vec{r}_0}{\partial y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

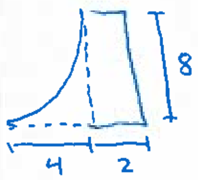
$\Rightarrow$ scaled normal $\vec{n} = \frac{\partial \vec{r}_0}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & x \\ 0 & 1 & 0 \end{vmatrix} = \begin{pmatrix} -x \\ 0 \\ 1 \end{pmatrix}$

Check direction: @ $x=0$ $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{z}$ ✓ points outward to surface

distance from axis to surface $\vec{r}_A = \begin{pmatrix} x-6 \\ 0 \\ x^2/2 \end{pmatrix}$

$$\therefore M_y = \left[\int_\Gamma \vec{r}_a \times d\vec{F}\, d\Gamma\right]_y = \left[\int_\Gamma (\vec{r}_a \times -p\hat{n})\, d\Gamma\right]_y$$

← can easily verify only y-comp. non-zer



$$= \int_\Gamma \left[\begin{pmatrix} x-6 \\ 0 \\ x^2/2 \end{pmatrix} \times \begin{pmatrix} x \\ 0 \\ -1 \end{pmatrix}\right]_y (\rho g(8-z))\, dx\, dy$$

$z = \frac{x^2}{2} = 8$ when $x^2 = 4$ $\Rightarrow x = 4 \rightarrow$

$$= \int_0^w dy \int_0^4 \left[\left(\frac{x^2}{2}\right)(x) - (x-6)(-1)\right] \rho g \left(8 - \frac{x^2}{2}\right) dx$$

$$= w\rho g \int_0^4 \left[8\frac{x^3}{2} + 8x - 48 - \frac{x^5}{4} - \frac{x^3}{2} + 3x^2\right] dx$$

$$= w\rho g \left[\cancel{8}\frac{x^4}{\cancel{8}} + 8\frac{x^2}{2} - 48x - \frac{x^6}{24} - \frac{x^4}{8} + \cancel{3}\frac{x^3}{\cancel{3}}\right]_0^4$$

$$= w\rho g\,(-32/3)$$

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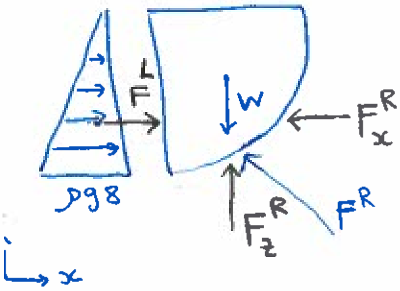
$$\Rightarrow \vec{M} = M_y \hat{y} = w\rho g\left(-32/3\right) = (2)(1000)(9.8)\left(-32/3\right)$$

$$= -209066.\bar{6}\ \hat{y}\ N{\cdot}m \quad \left(\text{or } kg\cdot\frac{m^2}{s^2}\right)$$

Alternatively, use graphical approach. <u>BUT</u> Not very easy!



$F^L = \left(\frac{1}{2}\rho g 8\right)(8)(w) = 32\rho g w$ by pressure prism

$W = -\rho g w A = -\rho g w \int_0^4 \left(8 - \frac{x^2}{2}\right) dx = -\rho g w \left(32 - \frac{4^3}{3}\right)$

$= -\rho g w \frac{64}{3}$

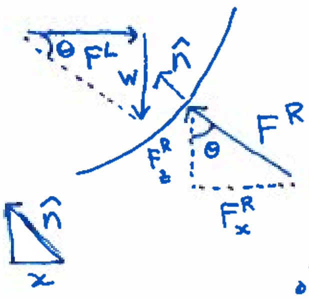
then $F_x^R = -F_L$, $F_z^R = -W$

to balance moments, F_x^R must act at same height as F^L

F_z^R must act at same x as W, x_{CG}

i.e. moments acting on wall will <u>not</u> be same as applying F^R at cp

F^R acts at point x with (scaled) normal $\vec{n} = \begin{pmatrix} -x \\ 0 \\ 1 \end{pmatrix}$ ← from integration approach



by similar triangles, $\tan(\theta) = \frac{1}{x} = \frac{W}{F^L}$

$$\Rightarrow x = \frac{F_L}{W} = \frac{32\rho g w}{64/3\ \rho g w} = \frac{3}{2}$$

$$\therefore \vec{M} = \vec{r} \times F^R = \left(z F_x^R - x \underbrace{F_z^R}_{-F_z^R = W}\right)\hat{y}$$

$$= \rho g w \left(\underbrace{\left(\tfrac{8}{3}\right)}_{}\underbrace{(32)}_{-F_x^R = F^L} - \underbrace{\left(\tfrac{3}{2} - 6\right)}_{x \text{ to axis } = x-6}\left(-\tfrac{64}{3}\right)\right)\hat{y} = -209066.\bar{6}\ \hat{y}$$

Same z as F^L

$\Rightarrow$ centroid of pressure prism

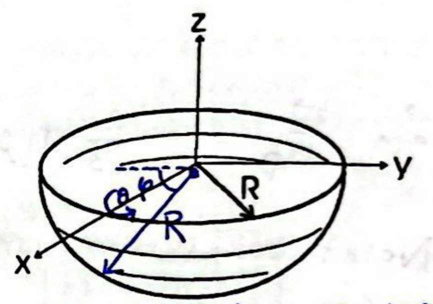
$\Rightarrow z = h/3 = 8/3$

6. An open container in the form of a hemisphere of radius R, as depicted in the diagram below, is fully filled with a liquid of density ρ in hydrostatic equilibrium. Assume $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$.

   (a) (5 points) By direct integration of the pressure forces acting on the surface of the container, find the net resultant pressure force $\boldsymbol{F_p}$ acting on the container.

   Note: the use of any method other than the direct integration of the pressure forces will not receive full credit.

   ~~(b)~~ (3 points) Prove that regardless of the container's shape, $\boldsymbol{F_p}$ depends only on the container's volume and always points in the $-\hat{\boldsymbol{z}}$ direction.



spherical coordinates

$\vec{F}_p = -\int_\Gamma p\,\hat{n}\,d\Gamma$ surface integral of pressure

parametrize surface with $\vec{r} = \begin{pmatrix} R\cos\theta\sin\varphi \\ R\sin\theta\sin\varphi \\ -R\cos\varphi \end{pmatrix}$

$0 \le \theta < 2\pi$, $0 \le \varphi \le \pi/2$

note: could aternatively use $z = R\cos\varphi$ if $\frac{\pi}{2} \le \varphi \le \pi$

$$\frac{\partial\vec{r}}{\partial\theta} = \begin{pmatrix} -R\sin\theta\sin\varphi \\ R\cos\theta\sin\varphi \\ 0 \end{pmatrix} \qquad \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix} R\cos\theta\cos\varphi \\ R\sin\theta\cos\varphi \\ R\sin\varphi \end{pmatrix} \qquad \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\varphi} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -R\sin\theta\sin\varphi & R\cos\theta\sin\varphi & 0 \\ R\cos\theta\cos\varphi & R\sin\theta\cos\varphi & R\sin\varphi \end{vmatrix}$$

$$= \underbrace{R^2\sin\varphi}_{} \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ -\cos\varphi \end{pmatrix}$$

Consistent with what we know about spherical coordinates!

check sign: $\hat{n} = \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ -\cos\varphi \end{pmatrix}$

@ $\varphi = \pi$ $\hat{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ -ve of what we want!

take $-\vec{n}$

and $p(z) = \rho g(-z) = \rho g(R\cos\varphi)$ * ignore p_{atm} or use $p_{atm} = 0$

$$\therefore \vec{F}_p = -\int \rho g R\cos\varphi\,\hat{n}\,d\Gamma$$

$$= -\int_0^{\pi/2}\int_0^{2\pi} \rho g R\cos\varphi \begin{pmatrix} -\cos\theta\sin\varphi \\ -\sin\theta\sin\varphi \\ +\cos\varphi \end{pmatrix} R^2\sin\varphi\,d\theta\,d\varphi$$

$-\vec{n}$

do each component:

$$F_p^x = +\rho g R^3 \underbrace{\int_0^{\pi/2} \cos\varphi\sin^2\varphi\,d\varphi}_{=1/3} \underbrace{\int_0^{2\pi} \cos\theta\,d\theta}_{=0} = 0$$

$$F_p^y = +\rho g R^3 \underbrace{\int_0^{\pi/2} \cos\varphi\sin^2\varphi\,d\varphi}_{=1/3} \underbrace{\int_0^{2\pi} \sin\theta\,d\theta}_{=0} = 0$$

cont'd...

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and

$$F_p^z = -\rho g R^3 \underbrace{\int_0^{\pi/2} \cos^2\varphi \sin\varphi \, d\varphi}_{=1/3} \underbrace{\int_0^{2\pi} d\theta}_{=2\pi} = -\frac{2}{3}\pi \rho g R^3$$

if used $P_{atm} \neq 0$

$$\therefore \vec{F}_p = -\frac{2}{3}\pi \rho g R^3 \hat{z} \quad \left(-\pi R^2 P_{atm} \hat{z}\right)$$

Note: consistent with idea that Force = Weight = $\overbrace{\frac{1}{2} \mathrm{Vol(sphere)}}^{\frac{1}{2}\cdot\frac{4}{3}\pi R^3} \cdot \rho g$

Only 1/5 marks to be awarded if answered with this method.

If Divergence theorem is used, i.e. $\int_{\Gamma} p\hat{n}\, d\Gamma = \int_{\Omega} \nabla p \, d\Omega$
(and then hydrostatics)

$$= \int_{\Omega} \rho \vec{g} \, d\Omega$$
$$= \rho \vec{g} \int_{\Omega} d\Omega$$
$$= \rho \vec{g} \, \frac{2}{3} \rho g R^3$$

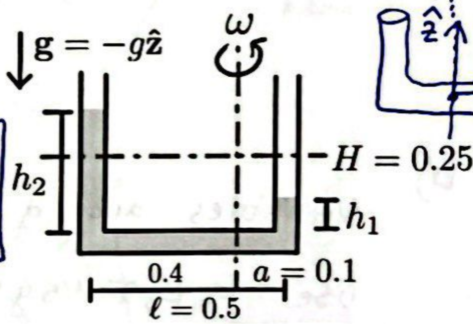
then only award 3/5 marks.

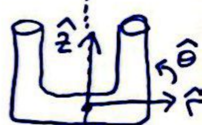
(this was the originally intended purpose of part b, but was removed for time's sake)

7. A U-tube of bottom width $\ell = 0.5$m is filled with water such that the liquid reaches an initial height $H = 0.25$m in both tubes. The tube is then rotated with a constant angular velocity ω about the offset vertical axis shown in the figure, 0.1m away from the inner tube. The water in the tubes eventually reach different heights h_1 and h_2 in the inner and outer tubes, respectively.

   (a) (7 points) Assuming that no spillage occurs, find the angular velocity ω required to make the fluid in the inner tube drop to a height of $h_1 = 0$m.
   Hint: The magnitude of centripetal acceleration is $\cancel{r^2\omega}$ $r\omega^2$. Find an expression for the free surface, then use a conservation of mass argument relating h_1 and h_2 to $H = 0.25$ to eliminate any unknown constants. Use $\boldsymbol{g} = -9.81\hat{\boldsymbol{z}}$.

   (b) (5 points) Suppose the relevant variables for this problem are g, ω, H, ℓ, and the distance from the inner tube to the axis, a (in this case $a = 0.1$m). If our dependent variable is ω, find appropriate Pi terms required to non-dimensionalize this problem (Note: radians are treated as dimensionless).





a) Very similar to Newton's bucket!

$$\rho\vec{a} = -\nabla p - \rho g\hat{z} \xrightarrow{\text{cylindrical coords}} \rho\begin{bmatrix}-r\omega^2\\0\\0\end{bmatrix} = -\begin{bmatrix}\frac{\partial p}{\partial r}\\ \frac{1}{r}\frac{\partial p}{\partial \theta}\\ \frac{\partial p}{\partial z}\end{bmatrix} - \rho g\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\Rightarrow \frac{\partial p}{\partial r} = \rho r\omega^2, \quad \frac{\partial p}{\partial \theta} = 0, \quad \frac{\partial p}{\partial z} = -\rho g$$

integrate in z: $p(r,z) = \int -\rho g\, dz = -\rho g z + f(r)$

differentiate in r: $\frac{\partial p}{\partial r} = f'(r) = \rho r\omega^2 \Rightarrow f(r) = \frac{\rho}{2}r^2\omega^2 + C$

remove one point if total differential $dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$ is used. <u>NOT</u> correct, as explained on PS2

$$\therefore\ p(r,z) = \frac{1}{2}\rho r^2\omega^2 - \rho g z + C$$

To find C, argue that $H + H = h_1 + h_2$ (conservation of mass!)

$\Rightarrow$ if z_s is s.t. $p(r, z_s) = P_{atm} = \frac{1}{2}\rho r^2\omega^2 - \rho g z_s + C$

$$\Rightarrow z_s = \underbrace{\frac{C - P_{atm}}{\rho g}}_{= \tilde{C}\ \leftarrow \text{constant!}} + \frac{1}{2}\frac{1}{g}r^2\omega^2$$

$$h_1 = z_s(r=0.1) = \tilde{C} + \frac{1}{2}\frac{1}{g}(0.1)^2\omega^2$$

$$h_2 = z_s(r=0.4) = \tilde{C} + \frac{1}{2}\frac{1}{g}(0.4)^2\omega^2$$

$$\Rightarrow 2H = h_1 + h_2 = 2\tilde{C} + \frac{1}{2}\frac{1}{g}\omega^2(0.1^2 + 0.4^2)$$

$$\Rightarrow \tilde{C} = \underbrace{H}_{0.25} - \frac{1}{4}\frac{1}{g}\omega^2\underbrace{(0.1^2 + 0.4^2)}_{= 0.17}$$

Cont'd...

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$$\therefore h_1 = 0 = z_s(r=0.1) = H - \frac{1}{4g}\omega^2(0.1^2+0.4^2) + \frac{1}{2}\frac{1}{g}\omega^2(0.1)^2$$

$$= H + \frac{1}{4}\frac{1}{g}\omega^2(0.1)^2 - \frac{1}{4g}\omega^2(0.4^2)$$

$$\Rightarrow \underbrace{H}_{0.25} = \frac{1}{4}\frac{1}{g}\omega^2(0.4^2-0.1^2) \Rightarrow \omega = \underbrace{\sqrt{\frac{4g(0.25)}{(0.4)^2-(0.1)^2}}} = 8.087\ s^{-1}$$

general form $\omega^2 = \frac{4gH}{(\ell-a)^2-a^2} = \frac{4gH}{\ell^2-2\ell a}$

b) Variables are g, ω, H, ℓ, a

use M, L, T system : $[g] = \frac{L^2}{T}$, $[\omega] = \frac{1}{T}$, $[H] = L$

$[\ell] = L$, $[a] = L$

3 primary dimensions, but only 2 reference dimensions, L & T

$\Rightarrow$ by Buckingham Pi Theorem,

\# of Pi terms = \# of variables − \# of reference dimensions

$= 5 - 2 = 3$

Choose 2 repeating variables : g, ℓ

ω is dependent variable, thus we need g to get T

(could choose H or a instead but can <u>not</u> choose ω & <u>must</u> choose g)

Can form Π terms with

$[\Pi_1] = \cancel{M^0} L^0 T^0 = [\omega][g]^a[\ell]^b \rightarrow$

$[\Pi_2] = \cancel{M^0} L^0 T^0 = [H][g]^a[\ell]^b \rightarrow$ solve for a, b

$[\Pi_3] = \cancel{M^0} L^0 T^0 = [a][g]^a[\ell]^b \rightarrow$

OR... cont'd...

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Q7 cont'd:

OR from visual inspection can easily see

$$\Pi_1 = \omega\sqrt{\frac{\ell}{g}}, \quad \Pi_2 = \frac{H}{\ell}, \quad \Pi_3 = \frac{a}{\ell}$$

(swap ℓ with H & a to get all other possible combinations)

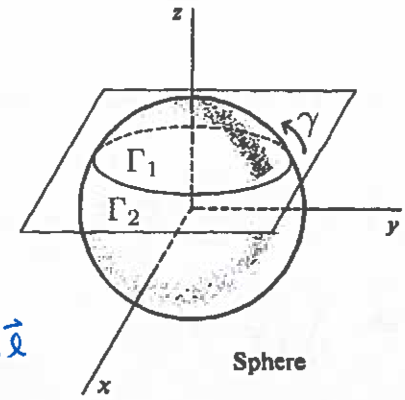
(not part of Question)

Note: we can use this to write a non-dimensional form of our answer

$$\omega^2 = \frac{4gH}{\ell^2 - 2\ell a} \longrightarrow \omega^2 \frac{\ell}{g} = \frac{4(H/\ell)}{1-2(a/\ell)}$$

8. (5 points) Consider the vector field $\boldsymbol{F} = x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}} + 2x\hat{\boldsymbol{z}}$ and the two surfaces Γ_1 and Γ_2 formed by the intersection of the sphere $x^2 + y^2 + z^2 = R^2$ with the horizontal plane $z = k$ at some height $-R < k < R$. The curve at which they intersect is γ and goes in the counterclockwise direction as looking from above. Show that the surface integral of $\boldsymbol{F}$ through Γ_1 is the opposite of that through Γ_2, i.e.

$$\int_{\Gamma_1} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, d\Gamma = -\int_{\Gamma_2} \boldsymbol{F} \cdot \hat{\boldsymbol{n}}\, d\Gamma$$



Can solve either with Stokes' THM
or Divergence THM

Stokes THM : $\int_\Gamma (\nabla \times \vec{A}) \cdot \hat{n}\, d\Gamma = \oint_{\gamma = \partial\Gamma} \vec{A} \cdot d\vec{l}$

idea: if $\nabla \times \vec{A} = \vec{F}$ we can apply Stokes' THM.

Does such an $\vec{A}$ exist? Yes! Infinitely many.

e.g. let $\vec{A} = \begin{bmatrix} 0 \\ x^2 \\ xy \end{bmatrix}$, then $\nabla \times \vec{A} = \begin{bmatrix} x \\ -y \\ 2x \end{bmatrix} = \vec{F}$ ✓

$\vec{A}$ is C^1, Γ_1 & Γ_2 are bounded smooth surfaces w/ boundaries γ & $-\gamma$

$\Rightarrow \int_{\Gamma_1} \vec{F} \cdot \hat{n}\, d\Gamma = \int_{\Gamma_1} (\nabla \times \vec{A}) \cdot \hat{n}\, d\Gamma = \oint_\gamma \vec{A} \cdot d\vec{l}$ -ve's!

$\int_{\Gamma_2} \vec{F} \cdot \hat{n}\, d\Gamma = \int_{\Gamma_2} (\nabla \times \vec{A}) \cdot \hat{n}\, d\Gamma = \oint_{-\gamma} \vec{A} \cdot d\vec{l} = -\oint_\gamma \vec{A} \cdot d\vec{l}$

$\therefore \int_{\Gamma_1} \vec{F} \cdot \hat{n}\, d\Gamma = -\int_{\Gamma_2} \vec{F} \cdot \hat{n}\, d\Gamma$ as desired.

Divergence THM : $\int_\Omega (\nabla \cdot \vec{F})\, d\Omega = \oint_{\partial\Omega} \vec{F} \cdot \hat{n}\, d\Gamma$

idea: the divergence inside the sphere is equal to the flux out of $\Gamma_1 + \Gamma_2$

if $\nabla \cdot \vec{F} = 0$ then, those fluxes must be opposite

cont'd...

This page is intentionally left blank for continued work. Please make a note to the grader on the main question page if you wish them to grade any work on this page.

check: $\nabla \cdot \vec{F} = \nabla \cdot \begin{bmatrix} x \\ -y \\ 2x \end{bmatrix} = 1 - 1 + 0 = 0$ ✓

$\vec{F}$ is C^1, the sphere Ω is a regular region with smooth boundary $\partial\Omega = \Gamma_1 + \Gamma_2$

$$\Rightarrow 0 = \int_\Omega \nabla \cdot \vec{F} \, d\Omega = \oint_{\partial\Omega} \vec{F} \cdot \hat{n} \, d\Gamma = \int_{\Gamma_1} \vec{F} \cdot \hat{n} \, d\Gamma + \int_{\Gamma_2} \vec{F} \cdot \hat{n} \, d\Gamma$$

$$\Rightarrow \int_{\Gamma_1} \vec{F} \cdot \hat{n} \, d\Gamma = - \int_{\Gamma_2} \vec{F} \cdot \hat{n} \, d\Gamma$$ as desired.

Alternative proof w/ Divergence THM:

instead pick regions Ω_{top} = top of sphere

Ω_{bot} = bottom of sphere

Γ_{mid} = intersection surface w/ plane

$$\Rightarrow \int_{\Omega_{top}} \nabla \cdot \vec{F} \, d\Omega = \int_{\Gamma_1} \vec{F} \cdot \hat{n} \, d\Gamma + \int_{\Gamma_{mid}} \vec{F} \cdot \hat{n} \, d\Gamma$$

points down

same $\hat{n}$ (different "out" direction)

$$\int_{\Omega_{bot}} \nabla \cdot \vec{F} \, d\Omega = \int_{\Gamma_2} \vec{F} \cdot \hat{n} \, d\Gamma + \int_{\Gamma_{mid}} \vec{F} \cdot (-\hat{n}) \, d\Gamma$$

since $\nabla \cdot \vec{F} = 0$ as before, $-\int_{\Gamma_{mid}} \vec{F} \cdot \hat{n} \, d\Gamma = \int_{\Gamma_1} \vec{F} \cdot \hat{n} \, d\Gamma$

$$\int_{\Gamma_{mid}} \vec{F} \cdot \hat{n} \, d\Gamma = \int_{\Gamma_2} \vec{F} \cdot \hat{n} \, d\Gamma$$

$$\Rightarrow \int_{\Gamma_1} \vec{F} \cdot \hat{n} \, d\Gamma = - \int_{\Gamma_2} \vec{F} \cdot \hat{n} \, d\Gamma$$ as desired.

Useful Formulas

Vector Identities

$$
\begin{aligned}
\boldsymbol{f}\cdot(\boldsymbol{g}\times\boldsymbol{h}) &= \boldsymbol{g}\cdot(\boldsymbol{h}\times\boldsymbol{f}) = \boldsymbol{h}\cdot(\boldsymbol{f}\times\boldsymbol{g})\\
\boldsymbol{f}\times(\boldsymbol{g}\times\boldsymbol{h}) &= \boldsymbol{g}\,(\boldsymbol{f}\cdot\boldsymbol{h}) - \boldsymbol{h}\,(\boldsymbol{f}\cdot\boldsymbol{g})\\
\nabla(fg) &= f\,(\nabla g) + g\,(\nabla f)\\
\nabla(\boldsymbol{f}\cdot\boldsymbol{g}) &= \boldsymbol{f}\times(\nabla\times\boldsymbol{g}) + \boldsymbol{g}\times(\nabla\times\boldsymbol{f}) + (\boldsymbol{f}\cdot\nabla)\,\boldsymbol{g} + (\boldsymbol{g}\cdot\nabla)\,\boldsymbol{f}\\
\nabla\cdot(f\boldsymbol{g}) &= f\,(\nabla\cdot\boldsymbol{g}) + \boldsymbol{g}\cdot(\nabla f)\\
\nabla\cdot(\boldsymbol{f}\times\boldsymbol{g}) &= \boldsymbol{g}\cdot(\nabla\times\boldsymbol{f}) - \boldsymbol{f}\cdot(\nabla\times\boldsymbol{g})\\
\nabla\times(f\boldsymbol{g}) &= f\,(\nabla\times\boldsymbol{g}) - \boldsymbol{g}\times(\nabla f)\\
\nabla\times(\boldsymbol{f}\times\boldsymbol{g}) &= (\boldsymbol{g}\cdot\nabla)\,\boldsymbol{f} - (\boldsymbol{f}\cdot\nabla)\,\boldsymbol{g} + \boldsymbol{f}\,(\nabla\cdot\boldsymbol{g}) - \boldsymbol{g}\,(\nabla\cdot\boldsymbol{f})\\
\nabla\cdot(\nabla\times\boldsymbol{f}) &= 0\\
\nabla\times(\nabla f) &= 0\,, \quad \text{where } f \text{ is the } \textit{scalar potential} \text{ of the } \textit{conservative} \text{ vector field } \nabla f\\
\nabla\times(\nabla\times\boldsymbol{f}) &= \nabla(\nabla\cdot\boldsymbol{f}) - \nabla^2\boldsymbol{f}\,, \quad \text{where } \nabla^2\boldsymbol{f} \text{ is the } \textit{vector Laplacian}
\end{aligned}
$$

(Note: the following are *not* rigorous mathematical statements!)

Change of Variables
Let $\boldsymbol{T}:\widetilde{\Omega}\to\Omega$ be a diffeomorphism (invertible & differentiable bijection). Then

$$\int_\Omega f(\boldsymbol{x})\,d\Omega = \int_{\widetilde{\Omega}=\boldsymbol{T}^{-1}(\Omega)} f(\boldsymbol{T}(\boldsymbol{u}))\,|\det(D_{\boldsymbol{u}}\boldsymbol{T})|\,d\widetilde{\Omega}$$

where $D_{\boldsymbol{u}}\boldsymbol{T}$ is the Jacobian matrix of $\boldsymbol{T}$ with respect to variables $\boldsymbol{u}=\boldsymbol{T}^{-1}(\boldsymbol{x})$.

Gradient Theorem (Fundamental Theorem of Calculus for line integrals)
Let γ be a continuous curve which starts at $\boldsymbol{a}$ and ends at point $\boldsymbol{b}$. Then

$$\int_\gamma (\nabla f)\cdot d\boldsymbol{\ell} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

Divergence Theorem (Gauss' Theorem)
Let Ω be a regular region with piecewise-smooth boundary $\Gamma=\partial\Omega$ and outward normal $\hat{\boldsymbol{n}}$. Then

$$\int_\Omega \nabla\cdot\boldsymbol{f}\,d\Omega = \oint_{\partial\Omega} (\boldsymbol{f}\cdot\hat{\boldsymbol{n}})\,d\Gamma$$

Stokes' Theorem (Curl Theorem, Baby Stokes' Theorem)
Let Γ be a bounded piecewise-smooth surface (2D area) with closed boundary $\partial\Gamma=\gamma$. Then

$$\int_\Gamma \underbrace{(\nabla\times\boldsymbol{f})\cdot d\vec{\Gamma}}_{(\nabla\times\vec{F})\cdot\hat{n}\,d\Gamma} = \oint_{\partial\Gamma} \boldsymbol{f}\cdot d\boldsymbol{\ell}$$

Buckingham Pi Theorem
(# of Pi terms) = (# of variables) − (min. # of reference dimensions)

Fluids Equations

$$\rho \boldsymbol{a} = -\nabla p + \rho \boldsymbol{g} \quad , \quad \tau_{yx} = \mu \frac{\partial v_x}{\partial y} \quad , \quad \Delta p = \rho g \Delta z$$

$$Re = \frac{\rho\,|v|\,L}{\mu} \quad , \quad Ma = \frac{|v|}{c} \quad , \quad c = \sqrt{\gamma RT} \quad , \quad p = \rho RT$$

Useful Integrals

$$\int_0^{2\pi} \cos\theta \sin\theta \; d\theta = 0 \quad , \quad \int_0^{\pi} \cos\theta \sin\theta \; d\theta = 0 \quad , \quad \int_0^{\pi/2} \cos\theta \sin\theta \; d\theta = \frac{1}{2}$$

$$\int_0^{2\pi} \cos\theta \sin^2\theta \; d\theta = 0 \quad , \quad \int_0^{\pi} \cos\theta \sin^2\theta \; d\theta = 0 \quad , \quad \int_0^{\pi/2} \cos\theta \sin^2\theta \; d\theta = \frac{1}{3}$$

$$\int_0^{2\pi} \cos^2\theta \sin\theta \; d\theta = 0 \quad , \quad \int_0^{\pi} \cos^2\theta \sin\theta \; d\theta = 0 \quad , \quad \int_0^{\pi/2} \cos^2\theta \sin\theta \; d\theta = \frac{1}{3}$$

$$\int \cos^2\theta \; d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \quad , \quad \int \sin^2\theta \; d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

Trigonometric Identities

$$\sin(\theta \pm \tfrac{\pi}{2}) = \pm\cos\theta \quad , \quad \sin(\theta + \pi) = -\sin\theta \quad , \quad \cos(\theta + \pi) = -\cos\theta$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad , \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \quad , \quad \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Cartesian Coordinates (x, y, z)

Line element: $d\boldsymbol{\ell} = dx\;\hat{\boldsymbol{x}} + dy\;\hat{\boldsymbol{y}} + dz\;\hat{\boldsymbol{z}}$

Volume element: $d\Omega = dx\;dy\;dz$

Gradient: $\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

Divergence: $\nabla \cdot \boldsymbol{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$

Curl: $\nabla \times \boldsymbol{f} = \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\hat{\boldsymbol{z}}$

Cross Product: $\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y)\hat{\boldsymbol{x}} + (a_z b_x - a_x b_z)\hat{\boldsymbol{y}} + (a_x b_y - a_y b_x)\hat{\boldsymbol{z}}$

Spherical Coordinates (r, θ, ϕ)

$x = r\cos\theta\sin\phi \quad \hat{\boldsymbol{x}} = \cos\theta\sin\phi\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\phi}}$

$y = r\sin\theta\sin\phi \quad \hat{\boldsymbol{y}} = \sin\theta\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\cos\phi\,\hat{\boldsymbol{\phi}}$

$z = r\cos\phi \quad \hat{\boldsymbol{z}} = \cos\phi\,\hat{\boldsymbol{r}} - \sin\phi\,\hat{\boldsymbol{\phi}}$

$r = \sqrt{x^2+y^2+z^2} \quad \hat{\boldsymbol{r}} = \cos\theta\sin\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\phi\,\hat{\boldsymbol{z}}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}$

$\phi = \cos^{-1}\left(\frac{z}{r}\right) \quad \hat{\boldsymbol{\phi}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\cos\phi\,\hat{\boldsymbol{y}} - \sin\phi\,\hat{\boldsymbol{z}}$

Line element: $d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\sin\phi\,d\theta\,\hat{\boldsymbol{\theta}} + r\,d\phi\,\hat{\boldsymbol{\phi}}$

Volume element: $d\Omega = r^2\sin\phi\,dr\,d\theta\,d\phi$

Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r\sin\phi}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$

Divergence: $\nabla\cdot\boldsymbol{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial\theta} + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi f_\phi\right)$

Curl:
$$\nabla\times\boldsymbol{f} = \frac{1}{r\sin\phi}\left(\frac{\partial}{\partial\phi}\left(\sin\phi f_\theta\right) - \frac{\partial f_\phi}{\partial\theta}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r f_\phi\right) - \frac{\partial f_r}{\partial\phi}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{1}{\sin\phi}\frac{\partial f_r}{\partial\theta} - \frac{\partial}{\partial r}\left(r f_\theta\right)\right)\hat{\boldsymbol{\phi}}$$

Cylindrical Coordinates (r, θ, z)

$x = r\cos\theta \quad \hat{\boldsymbol{x}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$

$y = r\sin\theta \quad \hat{\boldsymbol{y}} = \sin\theta\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}$

$z = z \quad \hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$

$r = \sqrt{x^2+y^2} \quad \hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{x}} + \sin\theta\,\hat{\boldsymbol{y}}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\boldsymbol{x}} + \cos\theta\,\hat{\boldsymbol{y}}$

Line element: $d\boldsymbol{\ell} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + dz\,\hat{\boldsymbol{z}}$

Volume element: $d\Omega = r\,dr\,d\theta\,dz$

Gradient: $\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

Divergence: $\nabla\cdot\boldsymbol{f} = \frac{1}{r}\frac{\partial}{\partial r}\left(r f_r\right) + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_z}{\partial z}$

Curl: $\nabla\times\boldsymbol{f} = \left(\frac{1}{r}\frac{\partial f_z}{\partial\theta} - \frac{\partial f_\theta}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r f_\theta\right) - \frac{\partial f_r}{\partial\theta}\right)\hat{\boldsymbol{z}}$