MAT292 - Calculus III - Fall 2014

Solution of Term Test 1 - October 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

Full Name: ______________________________
Last First

Student ID: ______________________________

Email: ______________________________ @mail.utoronto.ca

Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 16 pages (including this title page). Make sure you have all of them.
- You can use pages 14–16 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 14–16.

GOOD LUCK!

PART I No explanation is necessary.

For questions 1–8, consider a constant $a \in \mathbb{R}$ and the differential equation. **(8 marks)**

$$\frac{dy}{dt} = (y+a)(y-a)^2.$$

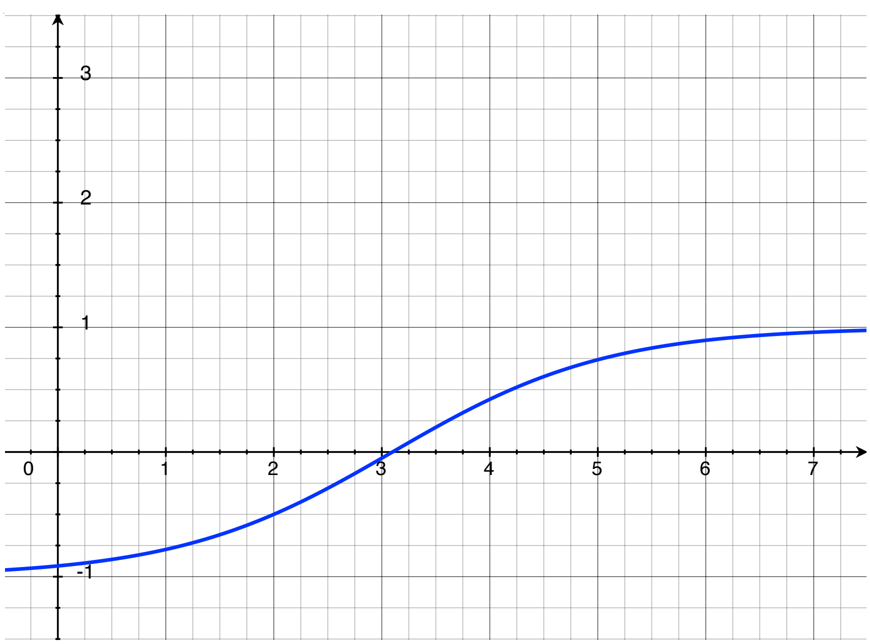
1. If $a > 0$, then the critical point $-a$ is stable / semistable / $\boxed{\text{unstable}}$

2. If $a > 0$, then the critical point a is stable / $\boxed{\text{semistable}}$ / unstable

3. If $a < 0$, then the critical point $-a$ is stable / semistable $\boxed{\text{unstable}}$

4. If $a < 0$, then the critical point a is stable / $\boxed{\text{semistable}}$ / unstable

5. Without solving the differential equation, sketch the solution for $a = 1$ with the initial condition $y(2) = -\frac{1}{2}$.



6. For $a = -1$, the solution has an asymptote at $y = 1$ as $t \to +\infty$ if the initial condition is

(a) $y(42) = 0$

(b) $y(-28) = 2$

(c) $y(0) = -2$

$\boxed{\textbf{(d)}}$ Only the equilibrium solution can have asymptote at $y = 1$.

7. For $a = -1$, the solution has an asymptote at $y = -1$ as $t \to +\infty$ if the initial condition is

$\boxed{\textbf{(a)}}$ $y(10^{10}) = 0$

(a) $y(-4000) = 2$

(b) $y(2014) = -2$

(c) Only the equilibrium solution can have asymptote at $y = -1$.

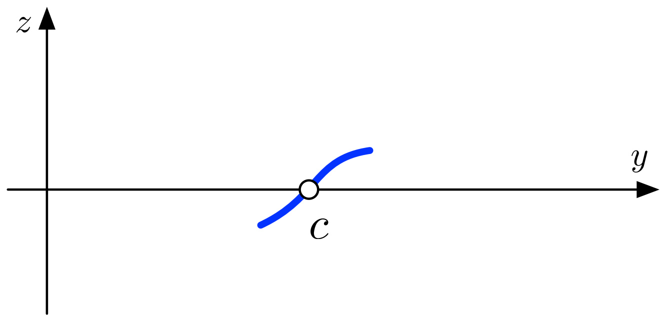
8. Let $a < 0$ and let $y = \phi(t)$ be the solution with initial condition $y(0) = \dfrac{a}{2}$. Then the maximum of $\phi(t)$ for $t \geqslant 0$ is

$$\max_{t\in[0,\infty)} \phi(t) = \underline{\quad a/2. \qquad\qquad\qquad\qquad}$$

PART II **Justify** your answers.

9. Consider the autonomous differential equation $y' = f(y)$, with a critical point c. **(8 marks)**

(a) Assume that $f'(c) > 0$. Graph $z = f(y)$ for values of y near c. $\boxed{\textbf{2 marks}}$



(b) Is c stable or unstable? Justify your answer. $\boxed{\textbf{2 marks}}$

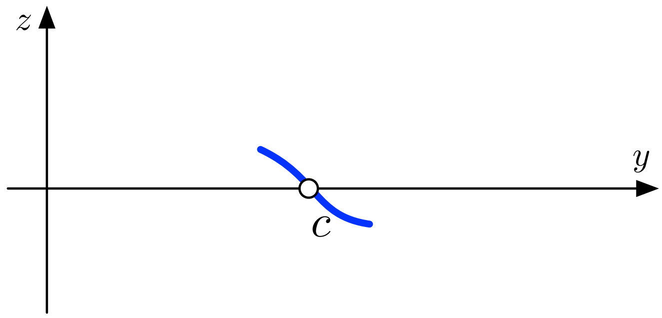
Solution. The critical point c is **unstable**, because in the graph from part **(a)**, we have the following

- If $y > c$, then $y' > 0$. This implies that y is becoming larger: moving away from c
- If $y < c$, then $y' < 0$. This implies that y is becoming smaller: moving away from c

So solutions with initial condition $y(t_0) = y_0$ for y_0 near c, will move away from c. Thus the critical point is unstable.

□

(c) Assume that $f'(c) < 0$. Graph $z = f(y)$ for values of y near c. **2 marks**



(d) Is c stable or unstable? Justify your answer. **2 marks**

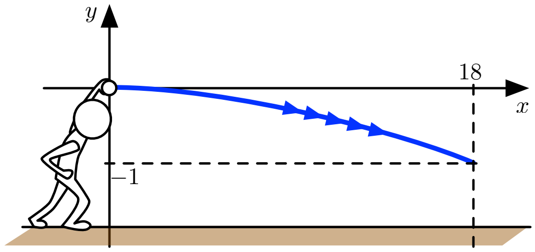
Solution. The critical point c is **stable**, because in the graph from part **(a)**, we have the following

- If $y > c$, then $y' < 0$. This implies that y is becoming smaller: moving towards c
- If $y < c$, then $y' > 0$. This implies that y is becoming larger: moving towards c

So solutions with initial condition $y(t_0) = y_0$ for y_0 near c, will converge to c. Thus the critical point is stable.

□

10. You are a baseball pitcher and you want to throw a ball from your position to the catcher 18m away and 1m below your throwing position. Consider gravity only. **(10 marks)**



5 marks **(a)** If the pitcher throws the ball horizontally, how fast should he throw it? And how much time will it take for the ball to reach the catcher?

Solution. Using Newton's 2nd Law of motion, we have

$$\vec{F} = m\vec{a}.$$

Define $\big(x(t), y(t)\big)$ as the position of the ball at time t. Then

$$\vec{a} = \big(x''(t), y''(t)\big) \qquad \text{and} \qquad \vec{F} = (0, -mg).$$

So we have the differential equations:

$$x''(t) = 0 \qquad \text{and} \qquad y''(t) = -g. \tag{$\star$}$$

The solution is

$$x(t) = u_0 t + x_0 \qquad \text{and} \qquad y(t) = -\frac{g}{2}t^2 + v_0 t + y_0,$$

assuming the initial conditions

$$\big(x(0), y(0)\big) = (0,0) \qquad \text{and} \qquad \big(x'(0), y'(0)\big) = (u_0, v_0).$$

The ball is thrown horizontally, so $v_0 = 0$.

Also, we want the ball to reach the catcher, so the solution must satisfy

$$x(T) = 18 \qquad \text{and} \qquad y(T) = -1.$$

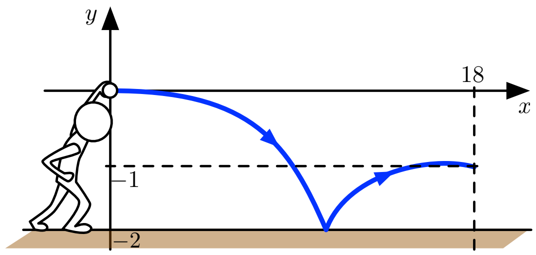
This implies that

$$\begin{cases} u_0 T = 18 \\ -\frac{g}{2}T^2 = -1 \end{cases} \Leftrightarrow \begin{cases} u_0 = \frac{18}{T} \\ T = \sqrt{\frac{2}{g}} \end{cases} \Leftrightarrow \begin{cases} u_0 = 18\sqrt{\frac{g}{2}} \\ T = \sqrt{\frac{2}{g}} \end{cases}$$

The pitcher should throw the ball at $18\sqrt{\dfrac{g}{2}}$ m/s and it will take $\sqrt{\dfrac{2}{g}}$ s to reach the catcher.

□

5 marks **(b)** Assume that the pitcher is used to cricket: he throws the ball horizontally, the ball bounces once on the ground (2m below the throwing position), but loses a quarter of its velocity on the bounce.
With exactly one bounce, how fast should he throw the ball?



Solution. We need to split the calculations in two parts: before and after the bounce.

Before the bounce. We have

$$x(t) = u_0\, t \quad \text{and} \quad y(t) = -\frac{g}{2}t^2.$$

Define t_b = time when the ball touches the ground. Then $y(t_b) = -2$, which implies that

$$t_b^2 = \frac{4}{g} \quad \Leftrightarrow \quad t_b = \frac{2}{\sqrt{g}}.$$

This means that $x(t_b) = \dfrac{2u_0}{\sqrt{g}}$ and the velocity at the time of the bounce is

$$x'(t_b) = u_0 \quad \text{and} \quad y'(t_b) = -gt_b = -2\sqrt{g}.$$

After the bounce. The differential equation after the bounce is the same as before, so its solution is the same

$$x(t) = u_b(t - t_b) + x(t_b) \qquad \text{and} \qquad y(t) = -\frac{g}{2}(t - t_b)^2 + v_b(t - t_b) + y(t_b),$$

where

$$u_b = \frac{3}{4}u_0 \quad \text{and} \quad v_b = \frac{3}{4}2\sqrt{g} = \frac{3}{2}\sqrt{g}.$$

We have

$$x(t) = \frac{3}{4}u_0(t - t_b) + \frac{2u_0}{\sqrt{g}} \qquad \text{and} \qquad y(t) = -\frac{g}{2}(t - t_b)^2 + \frac{3}{2}\sqrt{g}(t - t_b) - 2.$$

We now need to find T such that

$$x(T) = 18 \quad \text{and} \quad y(T) = -1,$$

which implies

$$-\frac{g}{2}(T-t_b)^2+\frac{3}{2}\sqrt{g}(T-t_b)-2=-1$$
$$\frac{g}{2}(T-t_b)^2-\frac{3}{2}\sqrt{g}(T-t_b)+1=0$$
$$T-t_b=\frac{\frac{3}{2}\sqrt{g}\pm\sqrt{\frac{9}{4}g-2g}}{g}$$
$$T-t_b=\frac{3\pm 1}{2\sqrt{g}}$$

So we have two solutions

$$T=t_b+\frac{2}{\sqrt{g}}\quad\text{or}\quad T=t_b+\frac{1}{\sqrt{g}}.$$

The initial speed of the ball u_0 which is the solution of

$$u_0\left(\frac{3}{4}(T-t_b)+\frac{2}{\sqrt{g}}\right)=18$$
$$u_0=\frac{18\sqrt{g}}{\frac{3}{2}+2}=\frac{36}{7}\sqrt{g}\quad\text{m/s.}$$

or

$$u_0=\frac{18\sqrt{g}}{\frac{3}{4}+2}=\frac{72}{11}\sqrt{g}\quad\text{m/s.}$$

□

Continued...

11. (a) Find the general solution of the differential equation $\boxed{\textbf{5 marks}}$

$$\left(1-\cos(y)x^3\right)y'(x) = 3x^2\sin(y)+\cos(x).$$

(**Hint.** You can leave the solution in implicit form)

Solution. This equation is exact:

$$\underbrace{-\left(3x^2\sin(y)+\cos(x)\right)}_{M(x,y)} + \underbrace{\left(1-\cos(y)x^3\right)}_{N(x,y)}y'(x) = 0,$$

and

$$M_y = -3x^2\cos(y) = N_x.$$

This means that we can solve it by finding $\psi(x,y)$ such that

$$\psi_x = M \quad \Leftrightarrow \quad \psi = \int M(x,y)\,dx = -x^3\sin(y)+\sin(x)+h(y).$$

We now find $h(y)$ using $\psi_y = N$:

$$\psi_y = -x^3\cos(y)+h'(y) = 1-x^3\cos(y) = N$$

so

$$h'(y) = 1 \quad \Leftrightarrow \quad h(y) = y + C.$$

We can then take

$$\psi(x,y) = -x^3\sin(y)+\sin(x)+y,$$

and the general solution is given by

$$-x^3\sin(y)+\sin(x)+y = C.$$

□

(b) The differential equation **3 marks**

$$\left(\frac{1}{x} - \cos(y)x^2\right) y'(x) = 3x\sin(y) \qquad \text{is not exact.}$$

Find an integrating factor $\mu(x, y)$ to make this equation exact. Justify your answer.

Solution #1. This differential equation is very similar to the one is part **(a)**. If we multiply it by $\mu(x, y) = x$, we get

$$\underbrace{\left(1 - \cos(y)x^3\right)}_{N(x,y)} y'(x) = \underbrace{3x^2\sin(y)}_{-M(x,y)},$$

which is exact:

$$M_y = -3x^2\cos(y) = N_x.$$

□

Solution #2. If we multiply the DE by $x\mu(x, y)$, we get

$$\underbrace{-3x\sin(y)\mu}_{\bar{M}} + \underbrace{\left(\frac{1}{x} - \cos(y)x^2\right)\mu}_{\bar{N}} y'(x) = 0,$$

Then

$$\begin{aligned} M_y &= -3x\cos(y)\mu - 3x\sin(y)\mu_y \\ N_x &= -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x, \end{aligned}$$

so we want to choose μ such that

$$-3x\cos(y)\mu - 3x\sin(y)\mu_y = -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x$$

If we choose $\mu = \mu(x)$, then we get

$$\begin{aligned} -3x\cos(y)\mu &= -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x \\ x\cos(y)\left(x\mu_x - \mu\right) &= \frac{1}{x^2}\left(x\mu_x - \mu\right) \\ \left(x\cos(y) - \frac{1}{x^2}\right)\left(x\mu_x - \mu\right) &= 0 \end{aligned}$$

So we can choose $\mu(x)$ which satisfies

$$\begin{aligned} x\mu_x - \mu &= 0 \\ \frac{1}{\mu}\mu_x &= \frac{1}{x} \\ \mu &= x \end{aligned}$$

The integrating factor is $\mu(x) = x$. □

12. Consider the following initial value problem: **(8 marks)**

$$\begin{cases} 2y' = y^2 + y \\ y(0) = 1 \end{cases}$$

$\boxed{\textbf{2 marks}}$ **(a)** Using Euler's Method with $h = \frac{1}{2}$, approximate the solution at $t = 1$.

Solution. First we write the differential equation as

$$y' = \frac{y(y+1)}{2}.$$

Euler's method yields:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= 1 + \frac{1}{2} f(0,1) = 1 + \frac{1}{2} \cdot \frac{1 \cdot 2}{2} = \frac{3}{2} \\ y_2 &= \frac{3}{2} + \frac{1}{2} f\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{3}{2} + \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \frac{5}{2}}{2} = \frac{3}{2} + \frac{15}{16} = \frac{39}{16} \end{aligned}$$

$\square$

$\boxed{\textbf{4 marks}}$ **(b)** Find the solution of the initial value problem and compute the error of the approximation in **(a)** at $t = 1$.

Solution. The DE is separable:

$$\frac{y'}{y(y+1)} = \frac{1}{2}.$$

The solution satisfies

$$\int \frac{1}{y(y+1)}\, dy = \int \frac{1}{2}\, dt.$$

Using partial fractions, we write

$$\begin{aligned} \int \left(\frac{1}{y} - \frac{1}{y+1} \right) dy &= \frac{t}{2} + k \\ \ln|y| - \ln|y+1| &= \frac{t}{2} + k \\ \left| \frac{y}{y+1} \right| &= ce^{\frac{t}{2}} \\ \frac{y}{y+1} &= ce^{\frac{t}{2}} \end{aligned}$$

We can find c using the initial condition:

$$\frac{1}{2} = c.$$

So we can find y explicitly:

$$\begin{aligned} \frac{y}{y+1} &= \frac{1}{2}e^{\frac{t}{2}} \\ 2y &= (y+1)e^{\frac{t}{2}} \\ \left(2 - e^{\frac{t}{2}}\right) &= e^{\frac{t}{2}} \\ y &= \frac{e^{\frac{t}{2}}}{2 - e^{\frac{t}{2}}}. \end{aligned}$$

This gives

$$y(1) = \frac{e^{\frac{1}{2}}}{2 - e^{\frac{1}{2}}}.$$

So the error of the previous approximation is

$$\text{error} = |y(1) - y_2| = \left| \frac{e^{\frac{1}{2}}}{2 - e^{\frac{1}{2}}} - \frac{39}{16} \right|.$$

$\square$

2 marks **(c)** If we need to obtain an error 50 times smaller, which step size h should we choose?

Solution. Since Euler's method has an error of the order of h, to obtain an error 50 times smaller, we need h to be 50 times smaller. So we need

$$h = \frac{1}{2}\frac{1}{50} = \frac{1}{100}.$$

$\square$

13. Consider functions $p(t)$ and $g(t)$ continuous for $t \in (a,b)$ and consider the initial value problem **(8 marks)**

$$\begin{cases} y' + p(t)y = g(t) & \text{for } t \in (a,b) \\ y(t_0) = y_0, \end{cases}$$

where $a < t_0 < b$. Let $\phi(t)$ and $\psi(t)$ be two solutions of this initial value problem. Show that $\phi(t) = \psi(t)$ for $t \in (a,b)$.

Hint. Split the proof in three steps:

(a) Define $F(t) = \phi(t) - \psi(t)$. Show that $F(t)$ is a solution of the initial value problem

$$\begin{cases} F' + p(t)F = 0 & \text{for } t \in (a,b) \\ F(t_0) = 0. \end{cases}$$

(b) Solve this differential equation and find $F(t)$.

(c) Conclusion.

$\boxed{\textbf{3 marks}}$ *Solution.* **(a)** First define $F(t) = \phi(t) - \psi(t)$. Then

$$\begin{aligned} F' + p(t)F &= \phi'(t) - \psi'(t) + p(t)\big(\phi(t) - \psi(t)\big) \\ &= \phi'(t) + p(t)\phi(t) - \big(\psi'(t) + p(t)\psi(t)\big) \\ &= g(t) - g(t) = 0. \end{aligned}$$

and

$$F(t_0) = \phi(t_0) - \psi(t_0) = y_0 - y_0 = 0.$$

$\boxed{\textbf{4 marks}}$ **(b)** The equation is separable: we can write it as

$$\begin{aligned} \int \frac{1}{F}\, dF &= -\int p(t)\, dt \\ \ln|F| &= -\int p(t)\, dt + k \\ |F| &= c\, e^{-\int p(t)\, dt} \end{aligned}$$

We can use the initial condition to find $c = 0$ and we obtain

$$F(t) = 0.$$

$\boxed{\textbf{1 mark}}$ **(c)** This implies that $\phi(t) = \psi(t)$.

□