# AER210F VECTOR CALCULUS AND FLUID MECHANICS

## Quiz 2

22 October 2018 9:15 am - 10:15 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis.

Given Name: solutions

Student #: ____

Tutorial/TA: ____

| FOR MARKER USE ONLY | | |
|---|---|---|
| Question | Marks | Earned |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 8 | |
| 4 | 17 | |
| 5 | 8 | |
| | | |
| | | |
| | | |
| TOTAL | 53 | / 50 |

Note: The following integrals may be useful.

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta \, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \qquad \oiint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \nabla \cdot \vec{F}\, dV; \qquad \oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$$

1) a) The force at a point (x, y) in the coordinate plane is given by: $\bar{F}(x,y) = (x^2 + y^2)\hat{i} + xy\,\hat{j}$ .
Find the work done by $\bar{F}(x,y)$ as its point of application moves along the path $y = x^2$ from (0, 0) to (2, 4).

(4 marks)

$W = \int_C \bar{F} \cdot d\bar{r}$ where $\bar{r}(t) = t\,\hat{i} + t^2\,\hat{j}$ $\quad 0 \le t \le 2$

$\therefore d\bar{r} = \bar{r}'(t)\,dt = dt\,\hat{i} + 2t\,dt\,\hat{j}$

$W = \int_0^2 (t^2 + t^4)(dt) + (t \cdot t^2)(2t\,dt)$

$= \int_0^2 (t^2 + 3t^4)\,dt = \left[\frac{t^3}{3} + \frac{3t^5}{5}\right]_0^2 = \frac{8}{3} + \frac{96}{5} = \frac{328}{15}$

b) Show by the equality of mixed partials that the integral is independent of path, and find its value: $\int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)\,dx + 9x^2y^2\,dy + (4xz + 1)\,dz$

(6 marks)

$P = 6xy^3 + 2z^2 \qquad Q = 9x^2y^2 \qquad R = 4xz + 1$

$\frac{\partial P}{\partial y} = 18xy^2 = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial z} = 4z = \frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$

$\therefore$ $\bar{F}$ is a gradient and the integral is independent of path

let $f = 3x^2y^3 + 2xz^2 + z \Rightarrow$

$f_x = 6xy^3 + 2z^2$

$f_y = 9x^2y^2$

$f_z = 4xz + 1$

$\therefore \int_{(1,0,2)}^{(-2,1,3)} (6xy^3 + 2z^2)\,dx + 9x^2y^2\,dy + (4xz+1)\,dz$

$= \left[3x^2y^3 + 2xz^2 + z\right]_{(1,0,2)}^{(-2,1,3)} = (12 - 36 + 3) - (8 + 2) = -31$

2) Find the moment of inertia about the $z$-axis of the parametric surface: $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, where $u^2 + v^2 \leq 1$.

(10 marks)

We are asked to find $\int_S (x^2+y^2)\lambda \, dS$ $\qquad \lambda = \text{constant} \, [\text{kg}/\text{m}^2]$

$\vec{r}(u,v) = (2uv, u^2-v^2, u^2+v^2)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 2v & 2u & 2u \\ 2u & -2v & 2v \end{vmatrix} = (4uv + 4uv)\hat{i} + (4u^2-4v^2)\hat{j} + (-4v^2-4u^2)\hat{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = 4\sqrt{4u^2v^2 + (u^2-v^2)^2 + (u^2+v^2)^2}$$
$$= 4\sqrt{4u^2v^2 + u^4 - 2u^2v^2 + v^4 + u^4 + 2u^2v^2 + v^4}$$
$$= 4\sqrt{2(u^4 + 2u^2v^2 + v^4)} = 4\sqrt{2}(u^2+v^2)$$

$$\Rightarrow dS = \|\vec{r}_u \times \vec{r}_v\| \, du \, dv = 4\sqrt{2}(u^2+v^2) \, du \, dv$$

$$\int_S (x^2+y^2) \, dS = \int_S (4u^2v^2 + u^4 - 2u^2v^2 + v^4) \cdot 4\sqrt{2}(u^2+v^2) \, du \, dv$$
$$= \int_{S: u^2+v^2 \leq 1} (u^2+v^2)^2 \cdot 4\sqrt{2}(u^2+v^2) \, du \, dv$$

use polar coordinates: $r = \sqrt{u^2+v^2}$ ; $0 \leq r \leq 1$, $0 \leq \Theta \leq 2\pi$

$$= \int_0^{2\pi} d\Theta \int_0^1 r \, dr \cdot 4\sqrt{2} \, r^6$$
$$= 2\pi \cdot 4\sqrt{2} \left[ \frac{r^8}{8} \right]_0^1 = \sqrt{2}\pi$$

3) Find the flux of $\bar{F} = \dfrac{2x\hat{i} + 2y\hat{j}}{x^2+y^2} + \hat{k}$ downward through the surface $S$ defined parametrically by: $\bar{r}(u,v) = u\cos v\,\hat{i} + u\sin v\,\hat{j} + u^2\,\hat{k}$, $\quad (0 \le u \le 1, 0 \le v \le 2\pi)$

(8 marks)

Find $\int_S \bar{F} \cdot \vec{N}\, du\, dv$ for $\bar{F} = \dfrac{2x}{x^2+y^2}\,\hat{i} + \dfrac{2y}{x^2+y^2}\,\hat{j} + 1\,\hat{k}$

$\bar{r}(u,v) = (u\cos v,\ u\sin v,\ u^2)$

$$\bar{r}_u \times \bar{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{array}{l} -2u^2\cos v\,\hat{i} \\ -2u^2\sin v\,\hat{j} \\ +(u\cos^2 v + u\sin^2 v)\,\hat{k} \end{array}$$

Note: since "u" is always +ve, this is the upward normal.

$$\therefore \int_S \bar{F}\cdot(\bar{r}_u \times \bar{r}_v)\,du\,dv = \int_S \left(\frac{2u\cos v}{u^2}, \frac{2u\sin v}{u^2}, 1\right)\cdot\left(2u^2\cos v, 2u^2\sin v, -u\right) du\,dv$$

$$= \int_0^1 du \int_0^{2\pi} dv\ \left(4u\cos^2 v + 4u\sin^2 v - u\right)$$

$$= 2\pi \int_0^1 3u\,du$$

$$= 2\pi \left[\frac{3u^2}{2}\right]_0^1$$

$$= 3\pi$$

4) Given the vector field $\bar{F}(x,y,z) = 2y\hat{i} + e^z\hat{j} + \tan^{-1}x\hat{k}$, confirm Stokes' theorem where $S$ is the part of the paraboloid $z = 4 - x^2 - y^2$ above the $x$-$y$ plane. .

(17 marks)

Stokes' Th'm: $\oint_C \bar{F}\cdot d\bar{r} = \int_S \nabla\times\bar{F}\cdot d\vec{S}$

a) $\int_C \bar{F}\cdot d\vec{r}$ when C is the intersection of the paraboloid with the plane $z=0$ $\Rightarrow$ $x^2+y^2=4$

$\therefore$ let $x = 2\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = 2\sin\theta$

$\therefore$ $\bar{r}(\theta) = (2\cos\theta, 2\sin\theta, 0)$

$d\vec{r} = (-2\sin\theta, 2\cos\theta, 0)\,d\theta$

$$= \int_0^{2\pi} \left[2\cdot 2\sin\theta(-2\sin\theta\,d\theta) + e^0\cdot(2\cos\theta\,d\theta) + \tan^{-1}(2\cos\theta)\cdot 0\,d\theta\right]$$

$$= \int_0^{2\pi} (-8\sin^2\theta + 2\cos\theta)\,d\theta = -8\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$= -8\pi$

4) Continued ...

b) $\int_S \nabla \times \vec{F} \cdot d\vec{S}$

$S: \quad z = 4 - x^2 - y^2 \quad \Rightarrow \quad x = r\cos\theta \quad y = r\sin\theta \quad z = 4 - r^2 \qquad 0 \le \theta \le 2\pi \quad 0 \le r \le 2$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta, \ +2r^2\sin\theta, \ r\cos^2\theta + r\sin^2\theta)$$

$$= (2r^2\cos\theta, \ +2r^2\sin\theta, \ r)$$

$\vec{F} = (2y, \ e^z, \ \tan^{-1}x)$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & e^z & \tan^{-1}x \end{vmatrix} = (0 - e^z)\hat{i} + \left(0 - \frac{1}{1+x^2}\right)\hat{j} + (0-2)\hat{k}$$

$$= \left(-e^{4-r^2}, \ \frac{-1}{1+r^2\cos^2\theta}, \ -2\right)$$

$$\therefore \int_S \nabla \times \vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta)\, dr\, d\theta = \iint_S \left[\underbrace{(2r^2\cos\theta)(-e^{4-r^2})}_{0} - \frac{2r^2\sin\theta}{1+r^2\cos^2\theta} - 2r\right] dr\, d\theta$$

now $\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$$\Rightarrow \int_0^{2\pi} \frac{\sin\theta}{1+r^2\cos^2\theta}\, d\theta = \int_0^{\pi} \frac{\sin\theta}{1+r^2\cos^2\theta}\, d\theta + \int_{\pi}^{2\pi} \frac{\sin\theta}{1+r^2\cos^2\theta}\, d\theta$$

$$= \int_0^{\pi} \frac{\sin\theta}{1+r^2\cos^2\theta}\, d\theta - \int_0^{\pi} \frac{\sin\theta\, d\theta}{1+r^2\cos^2\theta} = 0$$

$$\therefore \int_S \nabla \times F \cdot (\vec{r}_r \times \vec{r}_\theta)\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 dr\, (-2r) = -2\pi\left[\frac{2r^2}{2}\right]_0^2 = -8\pi$$

4 b) Alternate

$$S: \; z = 4 - x^2 - y^2 \implies \begin{aligned} x &= u \\ y &= v \\ z &= 4 - u^2 - v^2 \end{aligned} \qquad x^2 + y^2 \le 4$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -2u \\ 0 & 1 & -2v \end{vmatrix} = (2u, 2v, 1)$$

$$\vec{F} = (2y, e^z, \tan^{-1} x)$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & e^z & \tan^{-1} x \end{vmatrix} = (0 - e^z)\hat{i} + \left(0 - \frac{1}{1+x^2}\right)\hat{j} + (0 - 2)\hat{k}$$

$$= \left(-e^{4-u^2-v^2}, \frac{-1}{1+u^2}, -2\right)$$

$$\therefore \int_S \nabla \times \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv = \int_S \left[-2u e^{4-u^2-v^2} - \frac{2v}{1+u^2} - 2\right] du\, dv$$

polar coordinates: $u = r\cos\theta$, $v = r\sin\theta$ $\qquad 0 \le \theta \le 2\pi$, $0 \le r \le 2$

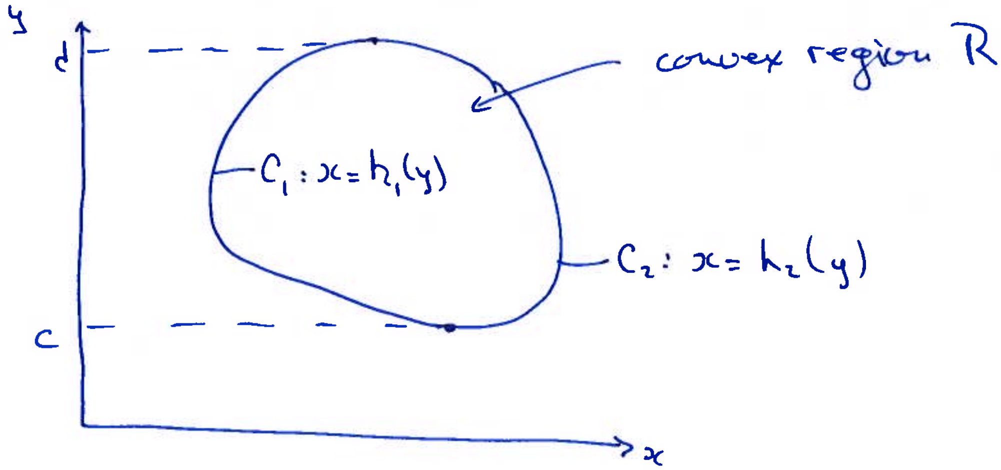
$$= \int_0^{2\pi} d\theta \int_0^2 r\, dr \left[-2r\underbrace{\cos\theta}_{\to 0} e^{4-r^2} - \underbrace{\frac{2r\sin\theta}{1+r^2\cos^2\theta}}_{\to 0} - 2r\right]$$

$$= 2\pi \left[-2\frac{r^2}{2}\right]_0^2 = -8\pi$$

5) Prove the second half of Green's Theorem for a simple, convex region; that is, prove:

$$\int_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dR$$

(8 marks)




$$\Rightarrow \int_R \frac{\partial Q}{\partial x}\,dR = \int_c^d dy \int_{h_1(y)}^{h_2(y)} \frac{\partial Q}{\partial x}\,dx = \int_c^d dy\,\big(Q(h_2(y),y) - Q(h_1(y),y)\big)$$

parameterize curve: $C_1 : \vec{r}_1(t) = h_1(t)\,\hat{i} + t\,\hat{j}$ $\quad c \le t \le d$

$C_2 : \vec{r}_2(t) = h_2(t)\,\hat{i} + t\,\hat{j}$

$$\therefore \int_C Q\,dy = -\int_c^d Q(h_1(t),t)\,dt + \int_c^d Q(h_2(t),t)\,dt$$

let $t = y \Rightarrow \int_C Q\,dy = \int_c^d \big(Q(h_2(y),y) - Q(h_1(y),y)\big)\,dy$

$$\therefore \int_C Q\,dy = \int_R \frac{\partial Q}{\partial x}\,dR$$