MAT195S CALCULUS II

Midterm Test #1

11 February 2014 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

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Family Name: J W Davis

Given Name: Solutions

Student #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	13	
2	11	
3	12	
4	10	
5	10	
6	9	
7	10	
TOTAL	75	/70

Tutorial Section:

TA Name:

1) Evaluate the following integrals.

a) $\int x^{3/2} \ln x \, dx$

b) $\int \frac{dx}{(x+1)(x+2)}$

c) $\int \frac{x^5}{\sqrt{x^2+2}} dx$

(13 marks)

a) $\int x^{3/2} \ln x \, dx \quad : \quad$ let $u = \ln x \quad dv = x^{3/2} dx$

$du = dx/x \quad v = \frac{2}{5} x^{5/2}$

$= \frac{2}{5} x^{5/2} \ln x - \int \frac{2}{5} x^{5/2} \cdot \frac{dx}{x} = \frac{2}{5} x^{5/2} \ln x - \frac{4}{25} x^{5/2} + C$

b) $\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \implies 1 = A(x+2) + B(x+1)$

$x = -2 \implies B = -1$

$x = -1 \implies A = 1$

$\int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$

c) $\int \frac{x^5}{\sqrt{x^2+2}} dx \qquad$ let $x = \sqrt{2} \tan\theta$ for $-\pi/2 < \theta < \pi/2$

$dx = \sqrt{2} \sec^2\theta \, d\theta$

$\sqrt{x^2+2} = \sqrt{2}\sqrt{\tan^2\theta + 1} = \sqrt{2} \sec\theta$

$= \int \frac{4\sqrt{2} \tan^5\theta \cdot \sqrt{2} \sec^2\theta \, d\theta}{\sqrt{2}\sec\theta} = 4\sqrt{2} \int \tan^5\theta \sec\theta \, d\theta$

let $u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$= 4\sqrt{2} \int (\sec^2\theta - 1)^2 \sec\theta \tan\theta \, d\theta$

$= 4\sqrt{2} \int (u^2-1)^2 du = 4\sqrt{2} \int (u^4 - 2u^2 + 1) du = 4\sqrt{2} \left[\frac{u^5}{5} - \frac{2u^3}{3} + u \right] + C$

$= 4\sqrt{2} \left(\frac{\sec^5\theta}{5} - \frac{2\sec^3\theta}{3} + \sec\theta \right) + C$

$= 4\sqrt{2} \frac{(x^2+2)^{5/2}}{4\sqrt{2}} \cdot \frac{1}{5} - \frac{8\sqrt{2}}{3} \frac{(x^2+2)^{3/2}}{2\sqrt{2}} + 4\sqrt{2} \frac{(x^2+2)^{1/2}}{\sqrt{2}} + C$

$= \frac{1}{5} (x^2+2)^{5/2} - \frac{4}{3} (x^2+2)^{3/2} + 4 (x^2+2) + C$

2) a) Show that the surface area of rotation about the x-axis for the function $y = 1/x$ for $1 \le x < \infty$, is infinite.

(5 marks)

$$A = \int 2\pi y \, ds$$

$$= \int_1^\infty 2\pi \cdot \frac{1}{x} \cdot \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} \, dx = \int_1^\infty \frac{2\pi}{x} \sqrt{1 + \frac{1}{x^4}} \, dx$$

$$= \int_1^\infty \frac{2\pi}{x^3} \sqrt{x^4+1} \, dx > \int_1^\infty \frac{2\pi}{x^3} \sqrt{x^4} \, dx = \int_1^\infty \frac{2\pi}{x} \, dx$$

$$= \Big[2\pi \ln x\Big]_1^\infty = 2\pi \,(0 - \ln \infty) = \infty$$

b) Find the value for the constant C for which the integral $\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx$ converges.

Evaluate the integral for this value of C.

(6 marks)

$$\int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1} \right) dx = \left[\frac{1}{2} \ln (x^2+1) - \frac{C}{3} \ln (3x+1) \right]_0^\infty = \left[\ln \frac{(x^2+1)^{1/2}}{(3x+1)^{C/3}} \right]_0^\infty$$

For this to have a finite value $\frac{(x^2+1)^{1/2}}{(3x+1)^{C/3}}$ must approach a constant value as $x \to \infty$; approaching either 0 or ∞ would lead to divergence. Thus we must have equal powers of x top and bottom.

$$C = 3 \Rightarrow \frac{(x^2+1)^{1/2}}{3x+1} = \left(\frac{x^2+1}{9x^2+6x+1} \right)^{1/2} = \left(\frac{1 + 1/x^2}{9 + 6/x + 1/x^2} \right)^{1/2} \longrightarrow \frac{1}{3}$$

$$\therefore \left[\ln \frac{(x^2+1)^{1/2}}{(3x+1)^{3/3}} \right]_0^\infty \longrightarrow -0 + \ln \frac{1}{3} = -\ln 3$$

3) Sketch a graph of the parametric curve: $x = t^2 - 2$

$y = t^3 - t$

Show all vertical and horizontal tangents, the tangents at (-1, 0), and identify the asymptotic behaviour.

(10 marks)

$x = t^2 - 2$

$x' = 2t$

$x' = 0 \Rightarrow t = 0 \Rightarrow (-2, 0)$

$y = t^3 - t$

$y' = 3t^2 - 1$

$y' = 0 \Rightarrow t = \pm 1/\sqrt{3} \Rightarrow \left(-1\tfrac{2}{3}, \pm \tfrac{2}{3\sqrt{3}}\right)$

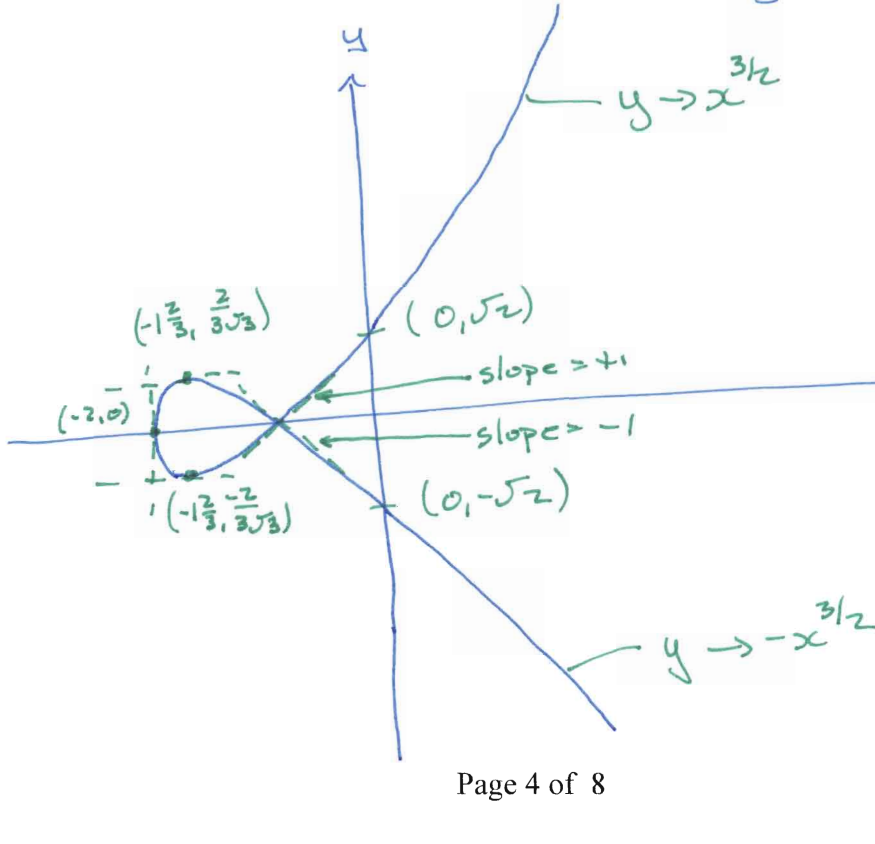
Intercepts: $x = 0 \Rightarrow t = \pm\sqrt{2} \Rightarrow (0 \pm \sqrt{2})$

$y = 0 \Rightarrow t = 0,\ t = \pm 1 \Rightarrow (-2, 0)$ & $(-1, 0)$

slope at $x = -1$: $t = \pm 1$: $t = 1 \Rightarrow \frac{y'}{x'} = \frac{2}{2} = 1$

$t = -1 \Rightarrow \frac{y'}{x'} = \frac{2}{-2} = -1$

Asymptotic behaviour: As $t \to \pm\infty$: $\left.\begin{array}{l} x \to t^2 \\ y \to t^3 \end{array}\right\} y = \pm x^{3/2}$

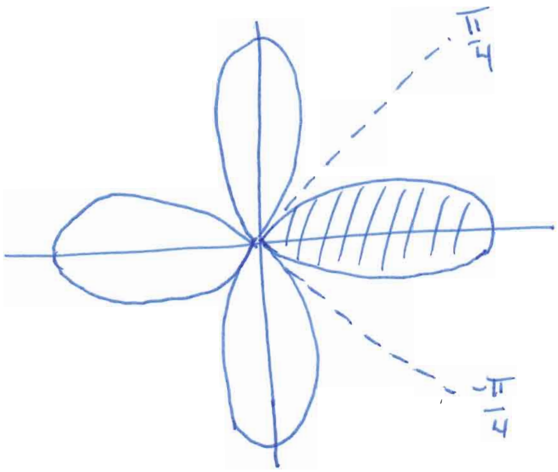


4) Sketch the region indicated, and find an integral representing the area of the region. Do not evaluate the integrals.

   a) The region enclosed by one petal of the curve $r = \cos 2\theta$.
   b) The region that lies inside both $r = \sin 2\theta$ and $r = \cos 2\theta$.
   c) The region that lies inside $r = 2 - \cos\theta$ but outside $r = 1 + \cos\theta$.

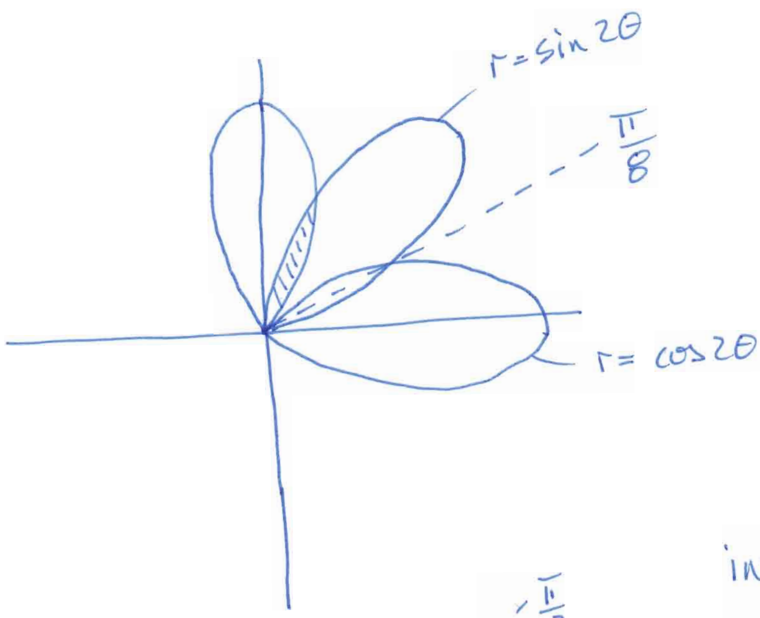
(12 marks)

a)



$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}(\cos 2\theta)^2\, d\theta$$

b)



intersections:

$\sin 2\theta = \cos 2\theta \Rightarrow \tan 2\theta = 1$

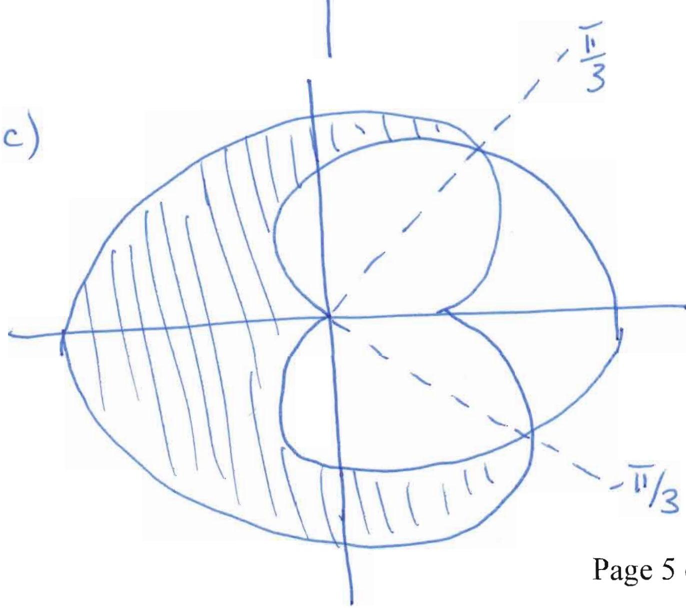
$\Rightarrow 2\theta = \pi/4 \ldots$

$\theta = \pi/8 \ldots$

$$A = 16\int_0^{\pi/8} \frac{1}{2}(\sin 2\theta)^2\, d\theta$$

$$= 16\int_{\pi/8}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2\, d\theta$$

c)



intersections:

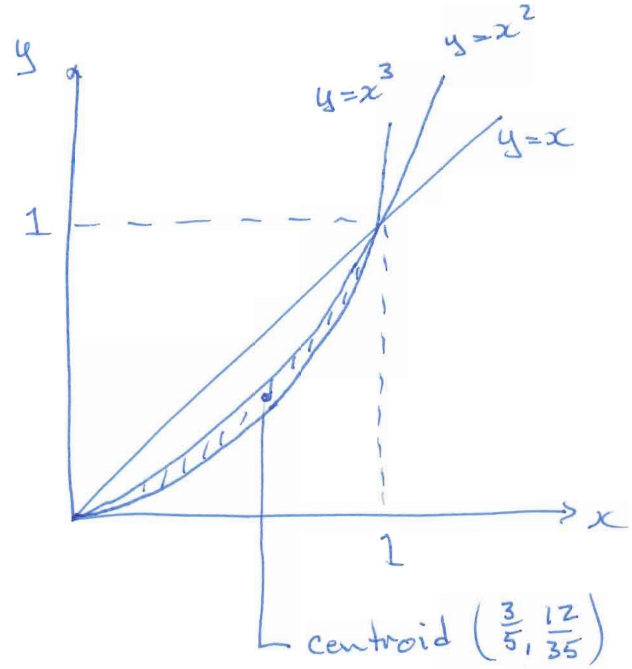
$2 - \cos\theta = 1 + \cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \pm\frac{\pi}{3}, \frac{5\pi}{3}$

$$A = \int_{\frac{\pi}{3}}^{5\pi/3} \frac{1}{2}\left[(2-\cos\theta)^2 - (1+\cos\theta)^2\right] d\theta$$

5) Find the centroid of the region trapped between the curves $y = x^2$ and $y = x^3$. Use Pappus's theorem to find the volume formed by rotating this region about the line $y = x$.

(10 marks)



$$A = \int_0^1 (x^2 - x^3)\,dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}$$

$$\bar{x}A = \int_0^1 x(x^2 - x^3)\,dx = \left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = \frac{1}{20}$$

$$\therefore \bar{x} = \frac{12}{20} = \frac{3}{5}$$

$$\bar{y}A = \int_0^1 \frac{1}{2}(x^4 - x^6)\,dx = \frac{1}{2}\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 = \frac{1}{35}$$

$$\therefore \bar{y} = \frac{12}{35}$$

Distance to line $y = x$:

1) find $\perp$ line through centroid:

slope $= -1 \Rightarrow \frac{12}{35} = (-1)\left(\frac{3}{5}\right) + b \Rightarrow b = \frac{33}{35} \Rightarrow \boxed{y = \frac{33}{35} - x}$

2) find intersection with line $y = x$:

$y = \frac{33}{35} - y \Rightarrow y = \frac{33}{70} \quad \therefore x = \frac{33}{70}$

3) distance from centroid to line $y = x$:

$$\bar{R} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{42}{70} - \frac{33}{70}\right)^2 + \left(\frac{33}{70} - \frac{24}{70}\right)^2} = \frac{9\sqrt{2}}{70}$$

Pappus's theorem: $V = 2\pi \bar{R} A = 2\pi \cdot \frac{9\sqrt{2}}{70} \cdot \frac{1}{12} = \frac{3\sqrt{2}\pi}{140}$

6) Determine whether the following sequence converges or diverges; if it converges, find the limit:

a) $a_n = \dfrac{(\ln n)^2}{n}$

b) $a_n = \sqrt[n]{2^{1+3n}}$

c) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(9 marks)

a) $a_n = \dfrac{(\ln n)^2}{n} \Rightarrow f(x) = \dfrac{(\ln x)^2}{x}$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{*}{=} \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln x}{x}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

$\therefore \lim_{n\to\infty} \dfrac{(\ln n)^2}{n} = 0$ converges

b) $a_n = \left(2^{1+3n}\right)^{1/n} = 2^{\frac{1+3n}{n}}$ converges.

now $\lim_{n\to\infty} \dfrac{1+3n}{n} = 3 \Rightarrow \lim_{n\to\infty} 2^{\frac{1+3n}{n}} = 2^3 = 8$

($f(x) = 2^x$ is continuous at $x = 3$)

c) $a_n = n - \sqrt{n+1}\sqrt{n+3} \times \dfrac{n + \sqrt{n+1}\sqrt{n+3}}{n+\sqrt{n+1}\sqrt{n+3}}$

$$= \frac{n^2 - n^2 - 4n - 3}{n + \sqrt{n+1}\sqrt{n+3}} = -\frac{4 + 3/n}{1 + \left(1 + \frac{4}{n} + \frac{3}{n^2}\right)^{1/2}} \longrightarrow -\frac{4}{2} = -2$$

converges.

7) The Completeness Axiom states that any non-empty set of real numbers that is bounded below has a greatest lower bound. Given this axiom, prove that a monotonic decreasing sequence that is bounded below converges.

(10 marks)

1) Given $\{a_n\}$ is a monotonic decreasing sequence and is bounded, the Completeness Axiom guarantees that the set of numbers given by $S = \{a_n \mid n \geq 1\}$ will have a greatest lower bound, L.

2) Now $L + \epsilon$ cannot be a lower bound for S since L is the greatest lower bound:

$$\therefore \quad a_N < L + \epsilon \quad \text{for some } N$$

3) But since the sequence is decreasing, $a_n \leq a_N$ for all $n > N$

$$\therefore \quad a_n < L + \epsilon \quad \text{for } n > N$$

$$\text{or} \quad 0 < a_n - L < \epsilon \quad \text{since } a_n \geq L$$

4) Thus $|L - a_n| < \epsilon$ for $n > N$

$$\therefore \lim_{n \to \infty} a_n = L$$