PHY293 Oscillations FALL 2014 Midterm Solutions

October 15, 2014

1 Question 1

Answer: B (10 marks)

I. False $\omega^2 = \omega_o^2 - \frac{\gamma^2}{4}$ (section 2.2.1 in text)

II. True (chapter 3 in text)

III. False $Q = \frac{\omega_o}{\gamma}$ (section 2.3.1 in text)

2 Question 2

Answer: D (10 marks)

I. True (section 1.4.1 in text)

II. False $\omega = 1/\sqrt{LC}$ (section 1.4.1 in text)

III. True (see lectures for chapter 3)

3 Question 3

a. We can deduce $kh = mg$ from the first observation and $bu = mg$ from the second. Our usual definition of γ is b/m so the second observation gives $\gamma = g/u$. This allows us to write down the un-driven oscillator the equation of motion (8 marks):

$$\ddot{x} + \dot{x}g/u + xg/h = 0$$

b. In the case of $u = 4\sqrt{gh}$, $\gamma = g/u = \frac{1}{4}\sqrt{g/h}$ and $\omega_o = \sqrt{g/h}$. For damped oscillations $\omega^2 = \omega_o^2 - \frac{\gamma^2}{4} = \frac{g}{h} - \frac{g}{64h} = \frac{63g}{64h}$ or $\omega = \frac{3}{8}\sqrt{\frac{7g}{h}}$ (4 marks).

c. $E(t) = e^{-\gamma t}E_0$ (2 marks), so the energy will fall to $1/e$ of the initial value when $\gamma t = 1$ or $t = 1/\gamma = 4\sqrt{h/g}$ (6 marks). This is 4 multiples of $\sqrt{h/g}$. The question can also be interpreted as asking when the energy decreases by $1/e$ of the initial value. This occurs when $1 - 1/e = e^{-\gamma t}$ or $t = -\frac{ln(1-1/e)}{\gamma} = -4ln(1-1/e)\sqrt{h/g} \approx 1.83\sqrt{h/g}$

d. $Q = \omega_o/\gamma = \sqrt{g/h}/(\sqrt{g/h}/4) = 4$ (8 marks)

e. We know:

$$a(\omega) = \frac{a_o\omega_o^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + (\gamma\omega)^2}}$$

Since the amplitude of the driving force is mg (the force at which the displacement of the spring is h), we can conclude that $a_o = h$ (4 marks). Substituting $\omega = \sqrt{2g/h}$, $\omega_o = \sqrt{g/h}$, $\gamma = \frac{1}{4}\sqrt{g/h}$ (4 marks), we get $a(\omega) = h\sqrt{8/9} \approx 0.94h$ (4 marks).

4 Question 4

a. Equations of motion (4 marks):

$$m\ddot{x}_A + (mg/l + k)x_A - kx_B = 0$$

$$m\ddot{x}_b + (mg/l + k)x_B - kx_A = 0$$

The equations of motion can be rewritten in the following matrix form (4 marks):

$$M\ddot{\vec{x}} + K\vec{x} = 0$$

where

$$M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$$K = \begin{bmatrix} mg/l + k & -k \\ -k & mg/l + k \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$$

Assuming a harmonic response $M\ddot{\vec{x}} = -\omega^2 M\vec{x}$ makes the equations of motion $(-\omega^2 M + K)\vec{x} = 0$ (4 marks). To solve for ω we need to find the determinant: $|-\omega^2 M + K| = 0$

$$\begin{vmatrix} -\omega^2 m + mg/l + k & -k \\ -k & -\omega^2 m + mg/l + k \end{vmatrix} = 0$$

$$(-\omega^2 m + mg/l + k)^2 - k^2 = 0 \text{ (4 marks)}$$

The solutions are $\omega_1^2 = g/l$ and $\omega_2^2 = g/l + 2k/m$ (4 marks). Using $g = 9.8\text{m/s}^2$, $\omega_1 \approx 5.72\text{rad/s}$ and $\omega_2 \approx 6.24\text{rad/s}$ (4 marks).

b. The complete solution to the equation of motion using the initial condition of one mass held a distance A away from equilibrium and the other at equilibrium is (4 marks):

$$\vec{x} = \frac{A}{2}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\cos(\omega_1 t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\cos(\omega_2 t)\right)$$

We can rewrite the equation of the first mass as (4 marks):

$$\begin{aligned} x_A &= \frac{A}{2}(\cos(\omega_1 t) + \cos(\omega_2 t))) \\ &= A\cos([\omega_1 + \omega_2]t/2)\cos([\omega_2 - \omega_1]t/2) \end{aligned}$$

The first mass comes to rest when the lower frequency cos becomes 0 (4 marks). $\cos([\omega_2 - \omega_1]t/2) = 0$ when $(\omega_2 - \omega_1)t/2 = \pi/2$ or $t = \pi/(\omega_2 - \omega_1) \approx 6.04\text{s}$ (4 marks).