The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Final exam - Monday April 17, 2017
Instructor: Prof. Piero Triverio

Last name: ....

First name: ....

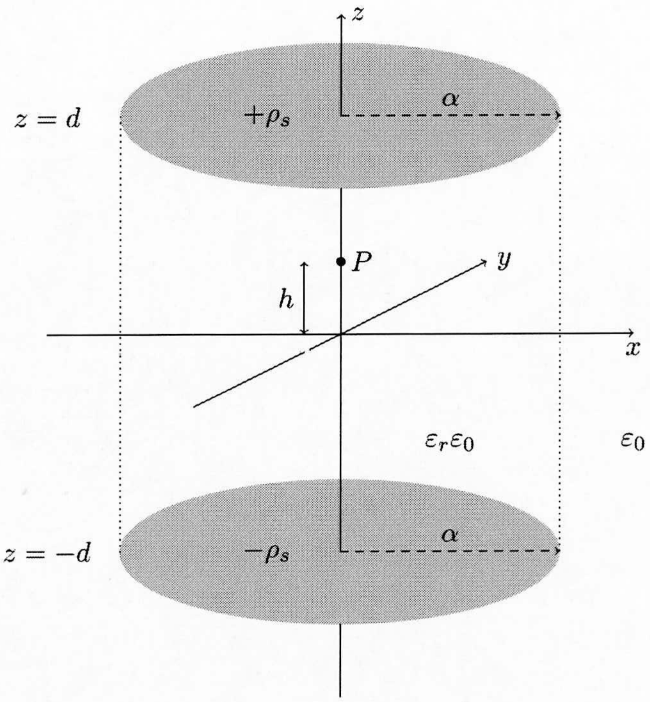
Student number: ....

Instructions

- Duration: 2 hours and 30 minutes (14:00 to 16:30)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full credit!**

Marks:	Q1: /20	Q2: /22	Q3: /20	Q4: /18

Question 1



We have two perfectly conducting circular disks of radius α. The upper disk is in the plane $z = d$ and is uniformly charged with surface charge density $+\rho_s$. The lower disk is in the plane $z = -d$ and is uniformly charged with surface charge density $-\rho_s$. A perfect dielectric fills the volume between the two disks. Permittivity is

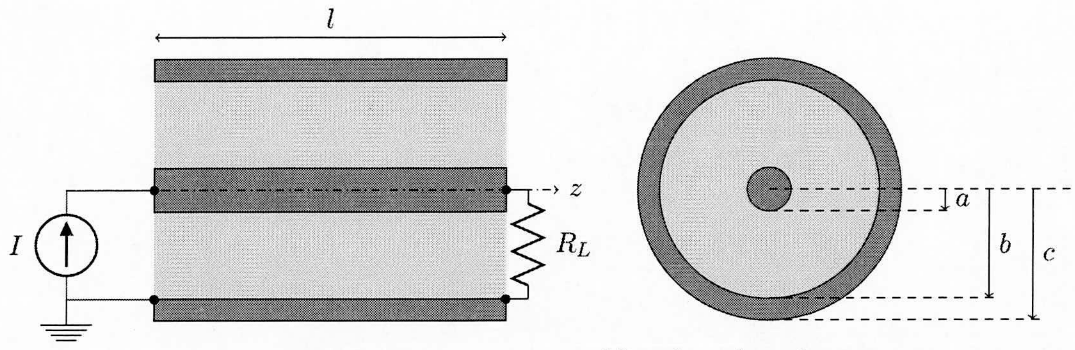
$$\varepsilon(r, z) = \begin{cases} \varepsilon_r\varepsilon_0 & \text{for } r \leq \alpha \text{ and } z \in [-d, d] \\ \varepsilon_0 & \text{elsewhere} \end{cases} \tag{1}$$

(a) Find the electric *flux density* $\mathbf{D}^+$ caused by the **upper** disk at point P. Point P is on the z axis between the two plates, at a distance h from the origin ($h \in [-d, d]$). Remember that $\sqrt{x^2} = |x|$ [8 points].

(b) Find the electric *flux density* $\mathbf{D}^-$ caused by the **lower** disk at point P [8 points].

(c) Find the total *electric field* $\mathbf{E}$ at point P [4 points].

Question 2



Side view View from the voltage source

We have a coaxial cable of length l. The cable is made by an inner cylindrical conductor (for $r < a$), a perfect dielectric ($a < r < b$), and an outer hollow cylindrical conductor ($b < r < c$). The two conductors have permittivity ε_0, permeability $\mu_r\mu_0$, and conductivity σ_c. The dielectric has permittivity $\varepsilon_r\varepsilon_0$, permeability μ_0, and conductivity $\sigma_d = 0$. A current source is connected at one end of the cable, and a resistor R_L is connected at the other end.

(a) Find the magnetic field $\mathbf{H}$ in the regions $r < a$, $a < r < b$ and $b < r < c$. When finding the magnetic field, you can neglect edge effects, as if the cable were infinitely long [8 points].

(b) Find the magnetic energy $W_{m,1}$ stored inside the inner conductor (ie in the region $r < a$) [4 points].

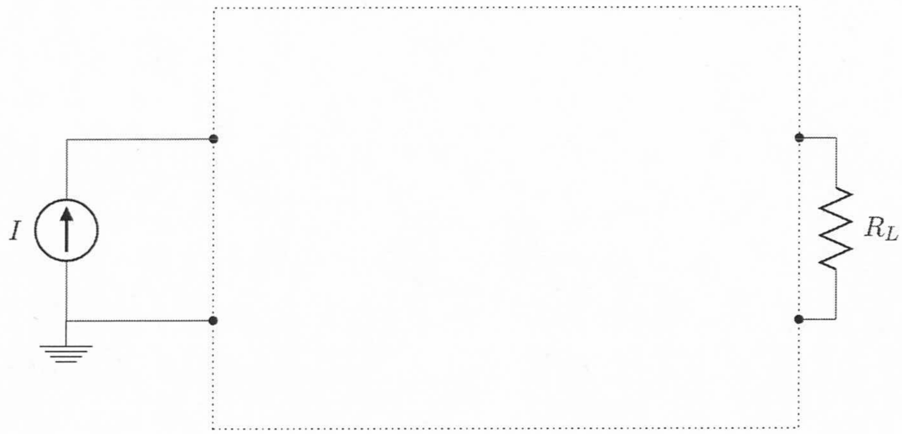
(c) Find the magnetic energy $W_{m,2}$ stored in the dielectric (ie in the region $a < r < b$) [4 points].

(d) Knowing that the energy stored inside the outer conductor (region $b < r < c$) is

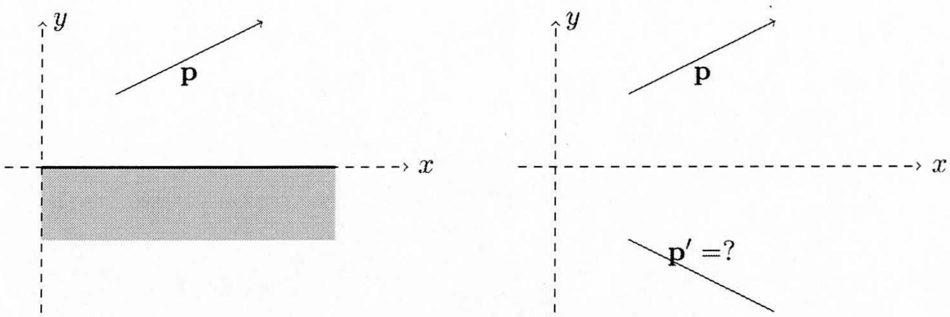
$$W_{m,3} = \frac{\mu_0 \mu_r l I^2}{4\pi}\left[\frac{c^4}{(c^2-b^2)^2}\ln\left(\frac{c}{b}\right) + \frac{b^2-3c^2}{4(c^2-b^2)}\right], \tag{2}$$

find the inductance L of the cable [2 points].

(e) In the dotted box in the following diagram, draw a **complete** equivalent circuit of the coaxial cable. Explain what each component models. You do **not** need to give component values [4 points].



Question 3.1



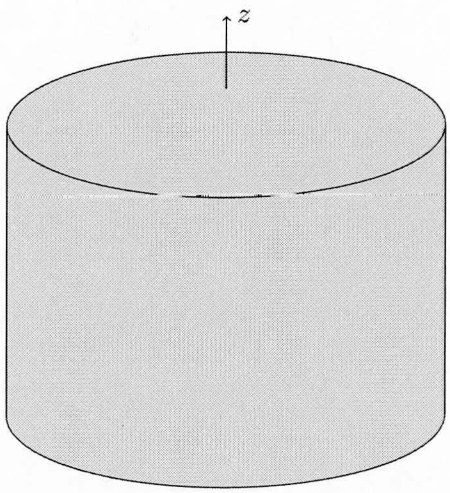
We have an electric dipole above an infinite perfectly-conducting plane, as shown in the left panel of the figure above. The electric dipole moment is $\mathbf{p} = 2\mathbf{a}_x + \mathbf{a}_y \, \mathrm{C \cdot m}$. We apply image theory, obtaining the configuration shown in the right panel. What is the electric dipole moment $\mathbf{p}'$ of the image dipole?

1. $\mathbf{p}' = 2\mathbf{a}_x + \mathbf{a}_y \, \mathrm{C \cdot m}$
2. $\mathbf{p}' = 2\mathbf{a}_x - \mathbf{a}_y \, \mathrm{C \cdot m}$
3. $\mathbf{p}' = -2\mathbf{a}_x - \mathbf{a}_y \, \mathrm{C \cdot m}$
4. $\mathbf{p}' = -2\mathbf{a}_x + \mathbf{a}_y \, \mathrm{C \cdot m}$

Briefly justify your answer [4 points].

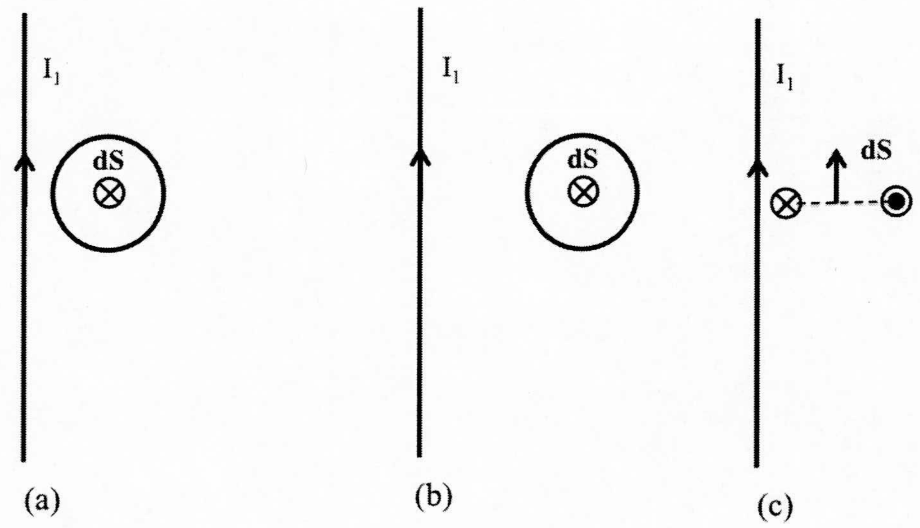
Question 3.2

Consider a cylindrical bar magnet with uniform magnetization $\mathbf{M} = M_0\mathbf{a}_z$ where M_0 is a positive constant. Draw any volume or surface magnetization current densities that may exist and indicate the "North" and "South" pole of the magnet. Justify your answer [4 points].



Question 3.3

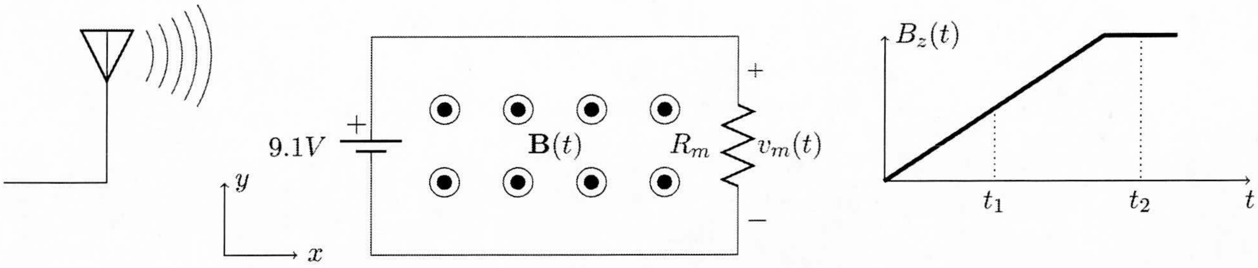
Rank the mutual inductances $L_{12}^{(a)}$, $L_{12}^{(b)}$, $L_{12}^{(c)}$ between a wire along the entire $z-$axis and a circular loop in the following three cases: in (a) and (b) the normal vector to the surface of the loop $\mathbf{dS}$ is perpendicular to the page, however, the distance of the center of the loop from the $z-$axis is larger in case (b). In case (c), the distance of the center of the loop from the $z-$axis is the same as in (a), but $\mathbf{dS}$ is parallel to the current (i.e. in the direction of the $z-$axis). Briefly explain [4 points].



Question 3.4

A point charge q moves with constant velocity $\mathbf{u} = u_0\mathbf{a}_x$ in a region where there is a uniform magnetic flux density $\mathbf{B} = B_0\mathbf{a}_y$. Calculate the electric field $\mathbf{E}$ in the region (in terms of constants u_0 and B_0) [4 points].

Question 3.5



A robot is used to measure chemical batteries. The measurement circuit is shown in the figure, and consists of the battery under test and the measurement device. The input resistance of the measurement device is R_m. The emf of the specific battery under test is 9.1 V. The system tests the battery by measuring voltage $v_m(t)$. Because of a transmitting antenna nearby, a magnetic flux density $\mathbf{B}(t) = B_z(t)\mathbf{a}_z$ is present throughout the circuit as shown in the figure (the z axis points out from the page, towards you). The system performs the measurement twice, at $t = t_1$ and $t = t_2$.

Which statement is correct about the first measured voltage?

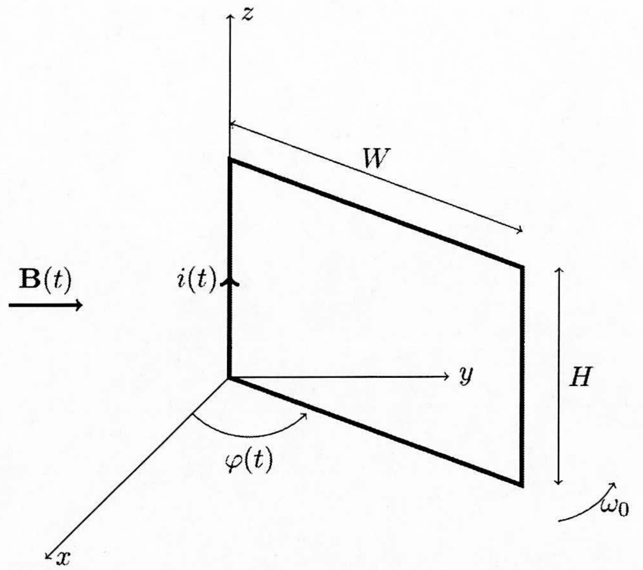
1. $v_m(t_1) = 9.1\,\mathrm{V}$
2. $v_m(t_1) > 9.1\,\mathrm{V}$
3. $v_m(t_1) < 9.1\,\mathrm{V}$

Which statement is correct about the second measured voltage?

1. $v_m(t_2) = 9.1\,\mathrm{V}$
2. $v_m(t_2) > 9.1\,\mathrm{V}$
3. $v_m(t_2) < 9.1\,\mathrm{V}$

Briefly justify your answer [4 points].

Question 4



A rectangular conductive loop rotates about the z axis with constant angular velocity ω_0. A magnetic flux density $\mathbf{B}(t) = B_0 \sin(\omega_0 t)\, \mathbf{a}_y$ is present everywhere. At $t = 0$, the loop makes an angle $\varphi(0) = 0$ with the x axis. The loop has total resistance R and negligible inductance.

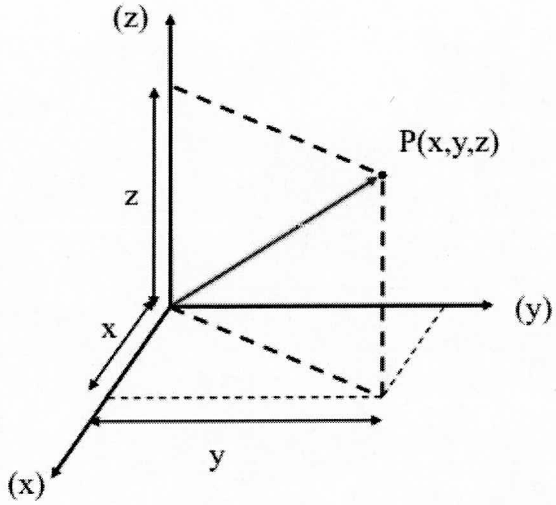
(a) Find the transformer emf induced in the loop (the transformer emf is the emf caused by the time variation of $\mathbf{B}(t)$) [6 points].

(b) Find the motional emf induced in the loop (the motional emf is the emf caused by the motion of the circuit) [8 points].

(c) Find the current $i(t)$ that flows in the loop (in the direction shown in the figure) [4 points].

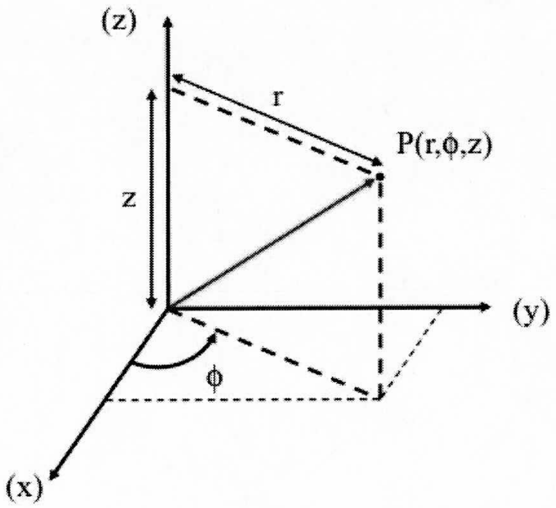
1 Coordinate Systems

1.1 Cartesian coordinates



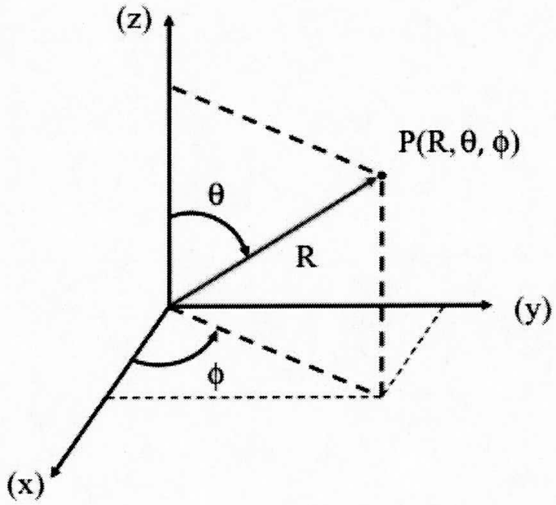
Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$
Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates



Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$
Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$
Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$
Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates



Position vector: $\mathbf{R} = R\mathbf{a}_R$
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2 Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\dfrac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\dfrac{r}{\sin\theta}$	R
θ	$\cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\dfrac{r}{z}$	θ
ϕ	$\tan^{-1}\dfrac{y}{x}$	ϕ	ϕ

3 Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4 Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r \cos\phi - A_\phi \sin\phi$	$A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$
A_y	A_y	$A_r \sin\phi + A_\phi \cos\phi$	$A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi - A_\phi \cos\phi$
A_z	A_z	A_z	$A_R \cos\theta - A_\theta \sin\theta$
A_r	$A_x \cos\phi + A_y \sin\phi$	A_r	$A_R \sin\theta + A_\theta \cos\theta$
A_ϕ	-$A_x \sin\phi + A_y \cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R \cos\theta - A_\theta \sin\theta$
A_R	$A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$	$A_r \sin\theta + A_z \cos\theta$	A_R
A_θ	$A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$	$A_r \cos\theta - A_z \sin\theta$	A_θ
A_ϕ	$-A_x \sin\phi + A_y \cos\phi$	A_ϕ	A_ϕ

5 Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

6 Electromagnetic formulas

Table 1 Electrostatics

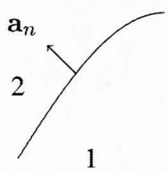
$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R}-\mathbf{R}')}{\lvert\mathbf{R}-\mathbf{R}'\rvert^3}dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0$	$\oint_S \mathbf{D}\cdot d\mathbf{S} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R}-\mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2}\mathbf{E}\cdot d\mathbf{l}$	
$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E}$	$\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$
$\rho_{p,v} = -\nabla\cdot\mathbf{P}$ $\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2 - \mathbf{P}_1)$ $\mathbf{a}_n\cdot(\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$ $E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2}QV$
$W_e = \frac{1}{2}\int_{vol}\rho_v V dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}dv$	
$\nabla\cdot(\varepsilon\nabla V) = -\rho_v$	$\nabla\cdot(\varepsilon\nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu\mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$
$\mathbf{B} - \mu_0 (\mathbf{H} + \mathbf{M}) - \mu\mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho\mathbf{u} = \sigma\mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$