The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term test 1 - Thursday February 8, 2018
Instructors: Profs. Micah Stickel and Piero Triverio

Last name: Solutions

First name:

Student number:

Tutorial section number:

Section	Day	Time	Room	TA name
TUT0101	Monday	14:00-15:00	BA 3012	Shashwat
TUT0102	Monday	14:00-15:00	RS 310	Gengyu (Paul)
TUT0103	Monday	14:00-15:00	BA 2159	Sameer
TUT0104	Monday	14:00-15:00	BA 3116	Fadime
TUT0105	Wednesday	13:00-14:00	BA 3012	Shashwat
TUT0106	Wednesday	13:00-14:00	WB 144	Gengyu (Paul)
TUT0107	Wednesday	13:00-14:00	BA 2159	Sameer
TUT0108	Wednesday	13:00-14:00	BA 3116	Fadime

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

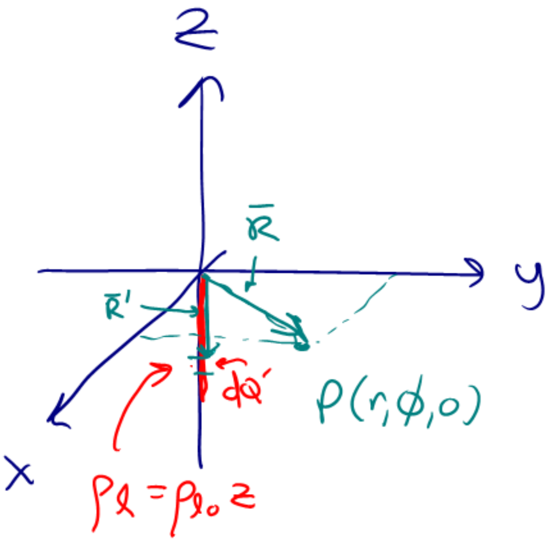
Marks:

Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1

A line charge lies in free space along the z-axis from $z = -h$ to $z = 0$, and is charged with a linear charge density given by $\rho_l(z) = \rho_{l0} z$, where ρ_{l0} is a positive constant.

1. Find the electric scalar potential function at an arbitrary point in the xy-plane, i.e., find $V(r, \phi, 0)$. [9 points]



$*$ Use $V(r,\phi,0) = \int dV = \int_{-h}^{0} \frac{dQ'}{4\pi\varepsilon_0 |\bar{R} - \bar{R}'|}$

$\Rightarrow dQ' = \rho_l \, dz'$
$= \rho_{l0} z' \, dz'$

$\bar{R} = r\hat{a}_r$

$\bar{R}' = z'\hat{a}_z$

$|\bar{R} - \bar{R}'| = \sqrt{r^2 + (z')^2}$

$$\therefore V(r,\phi,0) = \int_{-h}^{0} \frac{\rho_{l0} z' \, dz'}{4\pi\varepsilon_0 \sqrt{r^2 + (z')^2}} = \frac{\rho_{l0}}{4\pi\varepsilon_0} \int_{-h}^{0} \frac{z' \, dz'}{\sqrt{r^2 + (z')^2}}$$

Let $u = r^2 + (z')^2 \rightarrow du = 2z' \, dz'$ $\quad z = -h \rightarrow u = r^2 + h^2$
$\quad z = 0 \rightarrow u = r^2$

$$\therefore V(r,\phi,0) = \frac{\rho_{l0}}{4\pi\varepsilon_0} \int_{r^2+h^2}^{r^2} \frac{1}{2} u^{-\frac{1}{2}} \, du = \frac{\rho_{l0}}{8\pi\varepsilon_0} \left[2u^{\frac{1}{2}} \Big|_{r^2+h^2}^{r^2} \right]$$

$$= \frac{\rho_{l0}}{4\pi\varepsilon_0} \left[\sqrt{r^2} - \sqrt{r^2 + h^2} \right] = \underline{\underline{\frac{\rho_{l0}}{4\pi\varepsilon_0} \left[r - \sqrt{r^2 + h^2} \right]}}$$

$*$ Chose the positive branch, i.e. $+r$, since for large r, $r \rightarrow \infty$, $V = 0$ V.

2. Is the electric scalar potential at $P_1(r = h, \phi = 0°, z = 0)$ positive or negative? Briefly explain the physical meaning of the value of $V(r = h, \phi = 0°, z = 0)$. [3 points]

$\rho_{\ell 0} > 0$

* Since the charge density is negative, $\rho_\ell = \rho_{\ell 0} z$ with $z < 0$, the electric scalar potential will be negative at every point in space for $r < \infty$.

* Since V is related to the change in electric potential per charge it can be written as:

$$V(h, 0°, 0) = \frac{\text{Work done by an external agent to move 1C from } r=\infty \text{ to } r=h}{\text{1 C of charge}}$$

or equivalently

$$V(h, 0°, 0) = -\frac{\text{Work done by the electric field to move 1C of charge from } r=\infty \text{ to } r=h}{1C}$$

3. For the line charge described above, find the r-component of the electric field intensity at an arbitrary point in the xy-plane. [4 points]

$$\bar{E} = -\bar{\nabla} V = -\frac{\partial V}{\partial r}\hat{a}_r - \frac{\partial V}{\partial z}\hat{a}_z$$

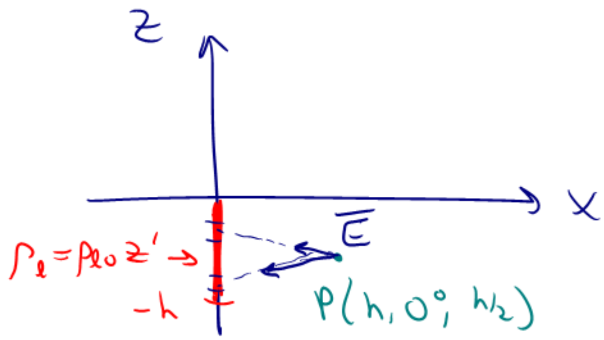
$$\therefore E_r = -\frac{\partial}{\partial r}\left[\frac{\rho_{\ell 0}}{4\pi\varepsilon_0}\left(r - \sqrt{r^2+h^2}\right)\right]$$

$$= \frac{\rho_{\ell 0}}{4\pi\varepsilon_0}\left(-1 + \frac{1}{2}\frac{2r}{\sqrt{r^2+h^2}}\right)$$

$$= \frac{\rho_{\ell 0}}{4\pi\varepsilon_0}\left(-1 + \frac{r}{\sqrt{r^2+h^2}}\right)$$

This makes sense that this is negative for all $r < \infty$, given the negative ρ_ℓ.

4. An electron is introduced to the system at $P_2(r = h, \phi = 0°, z = -h/2)$. Briefly describe *the direction* of the electric force this charge would experience at P_2. Briefly justify your answer. You do NOT need to determine the exact expression for the force at this point, and your answer can be in qualitative terms. [4 points]

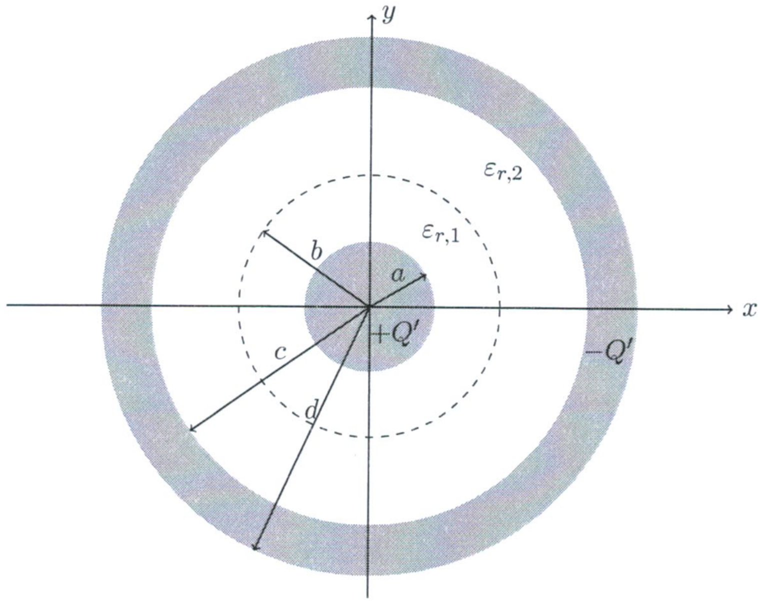


* Due to the non-uniform nature of ρ_ℓ, the sections of the distribution below $z = -h/2$, will dominate over the contributions from the sections above $z = -h/2$.

$\therefore$ The net electric field intensity at P will have negative x and negative z components.

* Since $\bar{F}_e = -e\bar{E}_{net} \rightarrow \bar{F}_e$ will then have $\underline{\underline{+x}}$ and $\underline{\underline{+z}}$ components

Question 2



1. Consider the structure shown in the figure above, which is infinitely long along the z axis. The structure consists of:

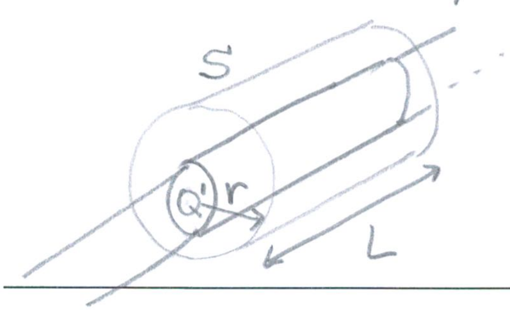
   - an inner solid cylinder of radius a. This cylinder is made of a perfect electric conductor, and is positively charged. The charge per unit length is $+Q'$;
   - a first layer of a perfect dielectric with relative permittivity $\varepsilon_{r,1}$;
   - a ~~first~~ second layer of a perfect dielectric with relative permittivity $\varepsilon_{r,2}$;
   - an outer hollow cylinder, with inner radius c and outer radius d. This cylinder is also made of a perfect electric conductor, and is negatively charged. The charge per unit length is $-Q'$;

   Use Gauss' law to find the electric field $\mathbf{E}_1(r)$ in the first dielectric layer ($r \in [a, b]$) and the electric field $\mathbf{E}_2(r)$ in the second dielectric layer ($r \in [b, c]$). [10 points]

[1pt] Cylindrical symmetry $\Rightarrow$ $\bar{D} = D(r)\bar{a}_r$ [1pt]

Use Gauss' law

Gaussian surface: cylinder, radius r, length L } [1pt]



$$\int_S \bar{D} \cdot \overline{dS} = Q_{enc}$$

$$\underbrace{\underbrace{\int_{\substack{\text{top}\\\text{base}}} \bar{D} \cdot \overline{dS}}_{=0} + \underbrace{\int_{\substack{\text{bottom}\\\text{base}}} \bar{D} \cdot \overline{dS}}_{=0}}_{[1pt]} + \int_{\text{side}} \bar{D} \cdot \overline{dS} = Q' \cdot L \quad [1pt]$$

$$D(r)\, 2\pi r \cdot \cancel{L} = Q' \cancel{L}$$

$$\bar{D} = \frac{Q'}{2\pi r} \bar{a}_r \quad \text{for } r \in [a, c] \quad [1pt]$$

[1] [1]

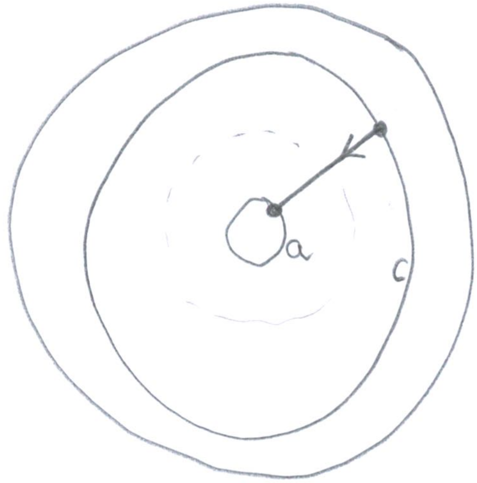
$$\bar{E}_1 = \frac{Q'}{2\pi r \, \varepsilon_0 \, \varepsilon_{r,1}} \bar{a}_r \qquad r \in [a, b]$$

[1]

$$\bar{E}_2 = \frac{Q'}{2\pi r \, \varepsilon_0 \, \varepsilon_{r,2}} \bar{a}_r \qquad r \in [b, c]$$

[1]

2. Find the voltage ΔV between the inner conductor and the outer conductor, i.e., find $\Delta V = V_{inner} - V_{outer}$. Express ΔV in terms of Q'. [4 points]



[1] $$\Delta V = -\int_{r=c}^{r=a} \bar{E}\cdot d\bar{l} = +\int_{r=a}^{c} \bar{E}\cdot d\bar{l} =$$

[2] $$= +\int_{a}^{b} \frac{Q'}{2\pi r \varepsilon_0 \varepsilon_{r,1}} \bar{a}_r \cdot \bar{a}_r \, dr + \int_{b}^{c} \frac{Q'}{2\pi r \varepsilon_0 \varepsilon_{r,2}} \bar{a}_r \cdot \bar{a}_r \, dr =$$

$$= +\frac{Q'}{2\pi \varepsilon_0 \varepsilon_{r,1}} \int_{a}^{b} \frac{dr}{r} + \frac{Q'}{2\pi \varepsilon_0 \varepsilon_{r,2}} \int_{b}^{c} \frac{dr}{r} =$$

$$= \boxed{\frac{Q'}{2\pi \varepsilon_0 \varepsilon_{r,1}} \ln\frac{b}{a} + \frac{Q'}{2\pi \varepsilon_0 \varepsilon_{r,2}} \ln\frac{c}{b}}$$

Correct final answer [1]

3. Now, assume that dimensions are: $a = 1\,\text{mm}$, $b = 2\,\text{mm}$, $c = 4\,\text{mm}$, $d = 6\,\text{mm}$. Dielectrics have the following characteristics:

   - first dielectric layer: $\varepsilon_{r,1} = 2$ and dielectric strength $E_{br,1} = 40\,\text{MV/m}$;
   - second dielectric layer: $\varepsilon_{r,2} = 6$ and dielectric strength $E_{br,2} = 20\,\text{MV/m}$.

   Remember that $\varepsilon_0 = 8.85 \times 10^{-12}\,\text{F/m}$. Find the maximum charge per unit length Q' that can be placed on the conductors without causing any damage to the dielectrics. Express Q' in μC/m. [4 points]

We must ensure that

1) $|\bar{E}_1(r)| \leq E_{br,1} \quad \forall\, r \in [a, b]$

$$\frac{Q'}{2\pi r \varepsilon_0 \varepsilon_{r,1}} \leq E_{br,1} \; ; \quad Q' \leq E_{br,1}\, 2\pi r\, \varepsilon_0 \varepsilon_{r,1}$$

worst case: $r = a$

$$Q' \leq E_{br,1}\, 2\pi a\, \varepsilon_0 \varepsilon_{r,1} = 4.45\,\mu C/m$$

[1.5]

2) $|\bar{E}_2(r)| \leq E_{br,2} \quad \forall\, r \in [b, c]$

$$\frac{Q'}{2\pi r \varepsilon_0 \varepsilon_{r,2}} \leq E_{br,2} \; ; \quad Q' \leq E_{br,2}\, 2\pi r\, \varepsilon_0 \varepsilon_{r,2}$$

worst case: $r = b$

$$Q' \leq E_{br,2}\, 2\pi b\, \varepsilon_0 \varepsilon_{r,2} = 13.3\,\mu C/m$$

[1.5]

Must take most restrictive condition $\boxed{Q' \leq 4.45\,\mu C/m}$ [1pt]

4. Find the maximum voltage ΔV_{max} that the structure can withstand without damaging the dielectrics. Express ΔV_{max} in kV. [2 points]

$\Delta V_{max} = (\Delta V$ for $Q' = 4.45 \mu C/m)$

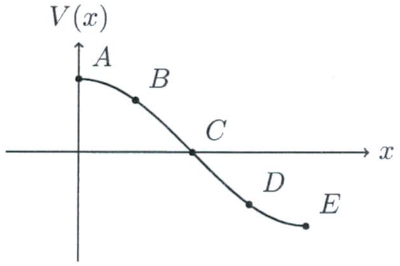
$\boxed{\Delta V_{max} = 37 \text{ kV}}$

← [2pt]

0 pt wrong

2pt correct

Question 3.1



The electrostatic potential V in a region depends only on the x coordinate, and is given in the graph above. The electric field intensity $|\mathbf{E}|$ is:

(a) maximal at point A, minimal at point E;

(b) minimal at point A, maximal at point E;

(c) minimal at points A and E, maximal at point C; [2pt]

(d) maximal at points A and E, minimal at point C;

(e) none of the above.

Briefly justify your answer. [4 points]

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial x}$$

$$|\bar{E}| = \left|\frac{\partial V}{\partial x}\right|$$

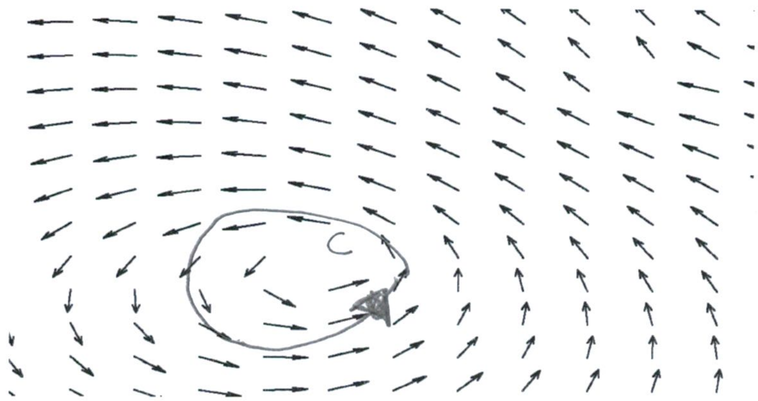
at A and E, $\frac{\partial V}{\partial x} = 0 \Rightarrow |\bar{E}| = 0$

at C $\frac{\partial V}{\partial x} < 0$, and $V(x)$ has the highest slope

$\Rightarrow \left|\frac{\partial V}{\partial x}\right|$ maximal

[2pt]

Question 3.2



Consider the vector field **F** depicted in the figure above. Can **F** be the electric field produced by a static distribution of charge in vacuum (ie, there is anything else apart from the charges)?

(a) yes;

(b) no; [1pt]

(c) more information is needed to answer this question.

Briefly justify your answer. [2 points]

The $\bar{E}$ field produced by a static charge distribution must be conservative

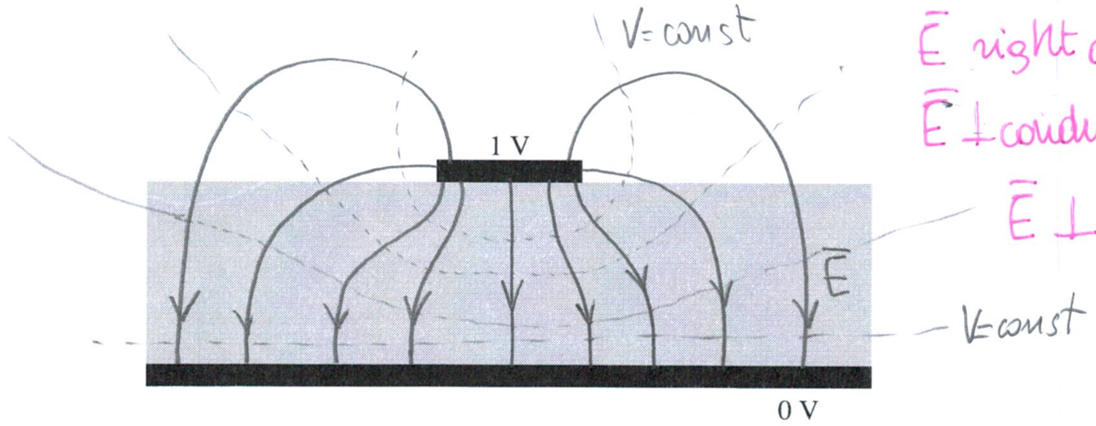
$$\Downarrow$$

$$\oint_C \bar{E} \cdot \bar{d\ell} = 0$$

Clearly not the case for $\bar{F}$ (see path in diagram) [1]

Question 3.3

A very common element in printed circuit boards is the microstrip line, which consists of a dielectric substrate with a thin rectangular conductor on top and a wide conductive plane at the bottom which serves as ground plane. Both conductors are made of a highly conductive material. The cross section is shown in the figure below.



$\bar{E}$ right direction [0.5]
$\bar{E} \perp$ conductors [1]
$\bar{E} \perp$ equipot [1]

Assuming that the top conductor is at a potential of 1 V with respect to the ground plane, sketch in the figure:

- the electric field lines, indicating their direction; → solid lines
- the equipotential lines. → dashed lines

Briefly justify your answer. [4 points]

There will be positive charges on top conductor and negative charges on bottom conductor ⇒ $\bar{E}$ lines will source at top conductor & sink at bottom conductor

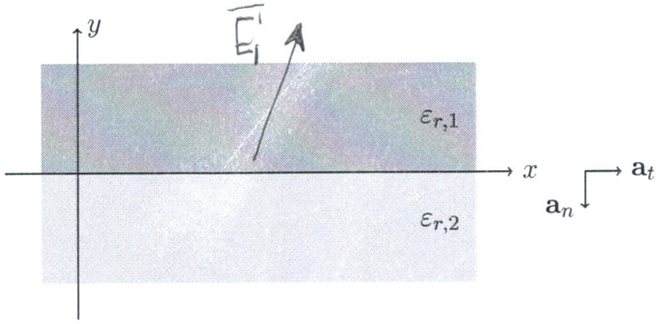
$\bar{E}$ must be normal to the interface of good conductors because of boundary conditions

$\bar{E} = -\nabla V$ ⇒ equipotential lines must be normal to the $\bar{E}$ field lines

[1.5] for justification

Question 3.4

The plane $y = 0$ is the interface between two perfect dielectrics with $\varepsilon_{r,1} = 3$ and $\varepsilon_{r,2} = 6$, as shown in the figure below. Given that the field right above the interface ($y = 0^+$) is $\mathbf{E}_1 = 2\mathbf{a}_x + 4\mathbf{a}_y$ V/m, complete the table below, and briefly explain your results. [6 points]



These points include answer in table & explanation

	Tangential component (measured along $+\mathbf{a}_t$)	Normal component (measured along $+\mathbf{a}_n$)	Unit	
Electric field $\mathbf{E}_1$ in medium 1	2	−4	V/m	[2 pt]
Electric field $\mathbf{E}_2$ in medium 2	2	−2		
Electric flux density $\mathbf{D}_1$ in medium 1	$2\varepsilon_0\varepsilon_{r,1} = 6\varepsilon_0$	$-4\varepsilon_0\varepsilon_{r,1} = -12\varepsilon_0$	C/m²	[2 pt]
Electric flux density $\mathbf{D}_2$ in medium 2	$2\varepsilon_0\varepsilon_{r,2} = 12\varepsilon_0$	$-12\varepsilon_0$		
Net **bound** surface charge density at the interface		$2\varepsilon_0$	C/m²	[2 pt]

From BC $E_{t,1} = E_{t,2}$

From BC $D_{2,n} - D_{1,n} = \rho_s$ $= 0$ since perfect dielectric

$D_{2,n} = D_{1,n} = -12\varepsilon_0$

$E_{2,n} = \dfrac{D_{2,n}}{6\varepsilon_0} = -2$ V/m

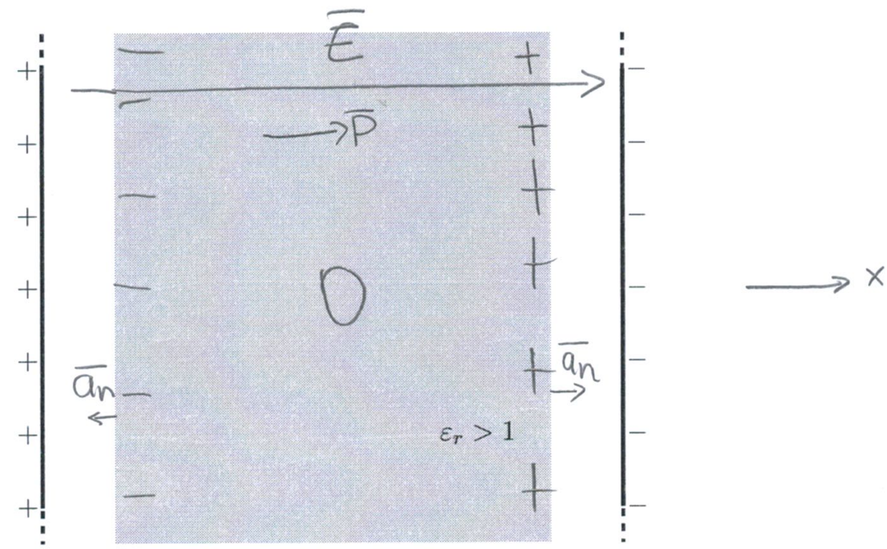
Unit of D: $D = \underset{F/m}{\varepsilon}\ \underset{V/m}{E} = \dfrac{F \cdot V}{m^2} = \dfrac{C\,\cancel{V}}{\cancel{V}\,m^2} = \dfrac{C}{m^2}$

Net bound charge at interface: $\rho_{p,s} = -\bar{a}_n \cdot (\bar{P}_2 - \bar{P}_1) = -P_{2,n} + P_{1,n} =$

$= -(D_{2,n} - \varepsilon_0 E_{2,n}) + (D_{1,n} - \varepsilon_0 E_{1,n}) = \varepsilon_0(E_{2,n} - E_{1,n}) = \varepsilon_0(-2+4) = 2\varepsilon_0$

Question 3.5

We have two infinitely-wide charged planes, one positively charged and the other negatively charged, as shown in the figure. The gap between the planes is partially filled with a dielectric with relative permittivity $\varepsilon_r > 1$. Draw the densities of polarization charge ρ_p and $\rho_{p,s}$ that exist in the dielectric, using a "+" sign to indicate a positive charge density, a "-" sign to indicate a negative charge density, and a "0" to indicate a vanishing charge density. Justify your answer. [4 points]



$\bar{E}$ uniform and directed towards $+\bar{a}_x$

$\bar{P} = \bar{D} - \varepsilon_0 \bar{E} = (\varepsilon - \varepsilon_0)\bar{E} = \varepsilon_0 \underbrace{(\varepsilon_r - 1)}_{>0} \bar{E} \Rightarrow$ also $\bar{P}$ uniform and directed along $+\bar{a}_x$

$\rho_{p,v} = -\nabla \cdot \bar{P} = 0$ since $\bar{P}$ uniform [1]

$\rho_{p,s} = \bar{P} \cdot \bar{a}_n$
- > 0 on right side [1]
- < 0 on left side [1]
- $= 0$ on top/bottom sides [1]