SOLUTIONS

Q1. a)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

LEADING VARIABLES: x_1, x_2, x_4

FREE VARIABLES: x_3

$$x_1 = -2x_3$$
$$x_2 = 5x_3 \qquad \therefore \ \vec{x} = x_3\begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \end{bmatrix}$$
$$x_3 = x_3$$
$$x_4 = 0$$

GEOMETRIC INTERPRETATION:

$\vec{x}$ IS A LINE THROUGH THE ORIGIN IN R^4.

b) $$A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 1 & c-3a & -3 \end{bmatrix} \begin{matrix} \\ R2+R1 \\ R3-3R1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & -1 & a+b & -1 \\ 0 & 0 & b+c-2a & -4 \end{bmatrix} \begin{matrix} \\ \\ R3+R2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & -(a+b) & 1 \\ 0 & 0 & b+c-2a & -4 \end{bmatrix} \ R2\times(-1)$$

COMPARE TO R

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad b+c-2a=0$

$$A \sim \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & -(a+b) & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & a & 1 \\ 0 & 1 & -(a+b) & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} R3 \times \left(-\frac{1}{4}\right)$$

$$\sim \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & -(a+b) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R1-R3 \\ R2-R3 \end{matrix}$$

COMPARE AGAIN TO R

$$\therefore \quad -(a+b) = -5$$
$$a = 2$$

SOLVING FOR a, b, c

$$a = 2$$
$$b = 5-2 = 3$$
$$c = 2(2)-3 = 1$$

Q2. a)

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c+b \\ b+a+c \\ c+b+a \end{bmatrix}$$

$$= (a+b+c)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ IS AN EIGENVECTOR WITH $\lambda = a+b+c$

b)

$$A\vec{v} = \begin{bmatrix} a_{11}(1) + a_{12}(1) + \cdots + a_{1n}(1) \\ a_{21}(1) + a_{22}(1) + \cdots + a_{2n}(1) \\ \vdots \\ a_{n1}(1) + a_{n2}(1) + \cdots + a_{nn}(1) \end{bmatrix}$$

$$= \begin{bmatrix} c \\ c \\ \vdots \\ c \end{bmatrix} = c\vec{v}$$

$\therefore$ $\vec{v}$ IS AN EIGENVECTOR WITH $\lambda = c$.

c)

$$A\vec{v} = 5\vec{v}$$

$$A(A\vec{v}) = A(5\vec{v})$$

$$A^2\vec{v} = 5A\vec{v} = 5^2\vec{v}$$

$\therefore$ $A^4\vec{v} = 5^4\vec{v}$

$\therefore$ $\vec{v}$ IS AN EIGENVECTOR OF A^4 WITH $\lambda = 5^4$.

Q3. i) $(\vec{v}\vec{v}^T)\vec{v} = \vec{v}(\vec{v}^T\vec{v}) = \vec{v}(1) = \vec{v}$

ii) $A = I - 2\vec{v}\vec{v}^T$

$$(I - 2\vec{v}\vec{v}^T)\vec{v} = I\vec{v} - 2(\vec{v}\vec{v}^T)\vec{v}$$
$$= \vec{v} - 2\vec{v} \qquad \text{(FROM PART i)}$$
$$= -\vec{v}$$

$\therefore$ $\vec{v}$ IS AN EIGENVECTOR OF A WITH $\lambda = -1$

iii) $$A^T = (I - 2\vec{v}\vec{v}^T)^T$$
$$= I^T - (2\vec{v}\vec{v}^T)^T$$
$$= I - 2(\vec{v}^T)^T\vec{v}^T$$
$$= I - 2\vec{v}\vec{v}^T$$
$$= A$$
$\therefore$ A IS SYMMETRIC

iv) $$AA = (I - 2\vec{v}\vec{v}^T)(I - 2\vec{v}\vec{v}^T)$$
$$= I - 2\vec{v}\vec{v}^T - 2\vec{v}\vec{v}^T + 4\vec{v}\vec{v}^T\vec{v}\vec{v}^T$$
$$= I - 4\vec{v}\vec{v}^T + 4\vec{v}\vec{v}^T \qquad \text{(FROM PART i)}$$
$$= I$$
$\therefore$ A IS THE INVERSE OF A

Q4.

$$A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \qquad \begin{bmatrix} \cdot & 0 & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} R1 \times (\tfrac{1}{2}) \qquad \begin{bmatrix} 2 & 0 & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} R2 + (0)R1 \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 4 & 1 \end{bmatrix} R3 + (1)R1 \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & \cdot & 0 \\ -1 & \cdot & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} R2 \times (\tfrac{-1}{2}) \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & \cdot & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix} R3 - 4R2 \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & \cdot \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} R3 \times \tfrac{1}{5} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix}$$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Check

$$LU = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix} \checkmark$$

$$= A$$

Q5. a)

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix}$$

CORRESPONDING TO

$$C_1 + C_2 V_1 = V_2$$

SYSTEM WILL BE INCONSISTENT BECAUSE NO STRAIGHT LINE WILL FIT EXACTLY THROUGH ALL 5 DATA POINTS.

CHECK BY VISUAL INSPECTION OF A PLOT OF THE DATA IN THE V_1-V_2 PLANE OR CHECK THAT $\frac{\Delta V_2}{\Delta V_1}$ IS NOT CONSTANT FROM POINT TO POINT.

b)

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 \\ 20 & 90 \end{bmatrix}$$

$$A^T B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 43 \\ 190 \end{bmatrix}$$

b) CONT'D

NORMAL EQUATIONS: $A^TA\vec{x} = A^T\vec{b}$

$\therefore \quad \vec{x} = (A^TA)^{-1}A^T\vec{b}$

$$(A^TA)^{-1} = \frac{1}{(5)(90)-20^2}\begin{bmatrix} 90 & -20 \\ -20 & 5 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 90 & -20 \\ -20 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9/5 & -2/5 \\ -2/5 & 1/10 \end{bmatrix}$$

$$(A^TA)^{-1}A^TB = \begin{bmatrix} 9/5 & -2/5 \\ -2/5 & 1/10 \end{bmatrix}\begin{bmatrix} 43 \\ 190 \end{bmatrix}$$

$$= \begin{bmatrix} 7/5 \\ 9/5 \end{bmatrix}$$

$$\therefore \quad \vec{x}_{LS} = \begin{bmatrix} 1.4 \\ 1.8 \end{bmatrix}$$

$$1.4 + 1.8V_1 = V_2$$

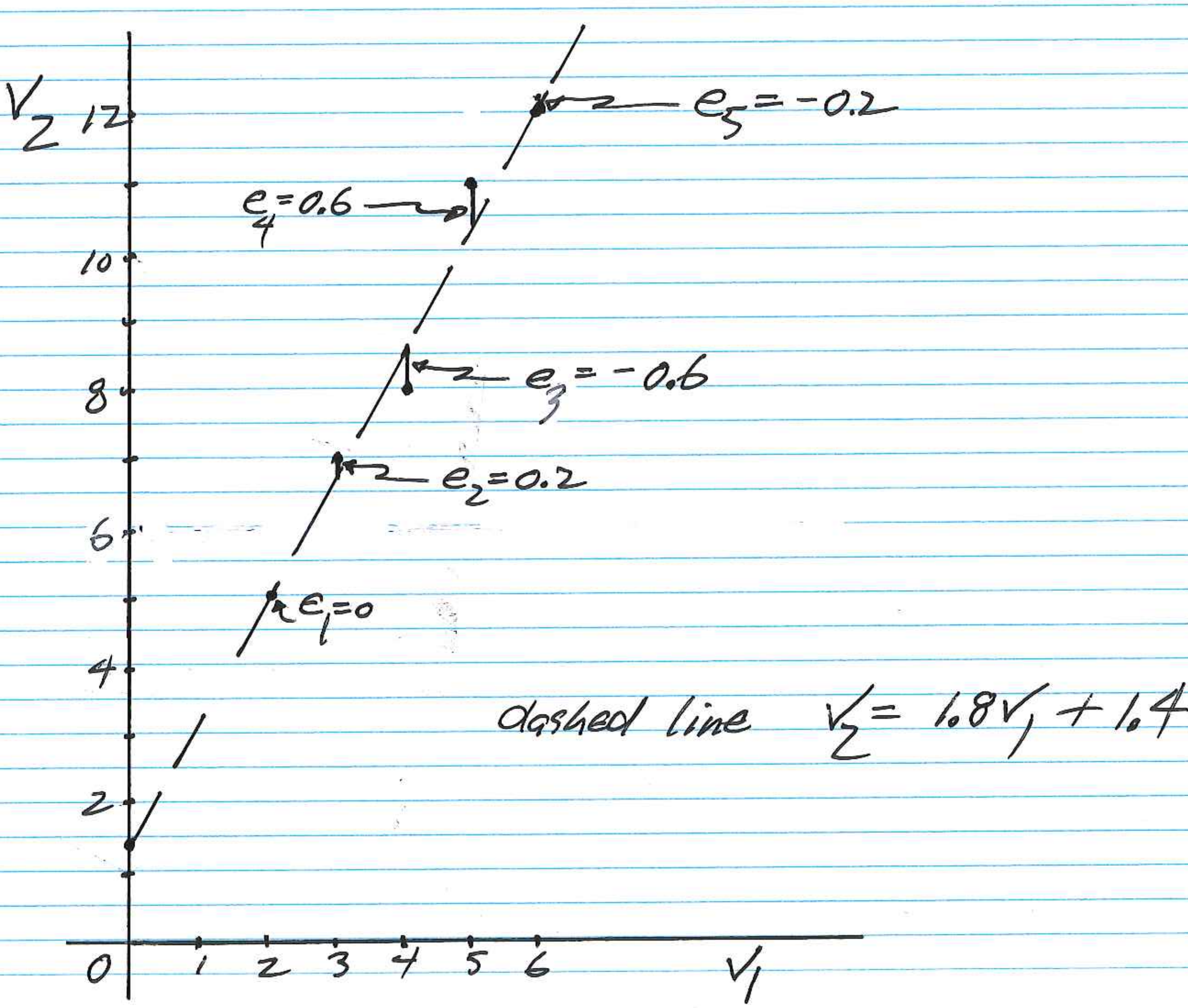
c)

$$\vec{E} = \vec{b} - A\vec{x}_{ls}$$

$$= \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1.4 \\ 1.8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 7 \\ 8 \\ 11 \\ 12 \end{bmatrix} - \begin{bmatrix} 5 \\ 6.8 \\ 8.6 \\ 10.4 \\ 12.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0.2 \\ -0.6 \\ 0.6 \\ -0.2 \end{bmatrix}$$



Q6.

a) $n=2$

$$T_2 = \left[\sum_{i=1}^{2} \frac{f(x_{i-1})+f(x_i)}{2}\right]\Delta X$$

$$= \left[\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2}\right]\Delta X$$

$$= \left[\frac{f(x_0)}{2} + f(x_1) + \frac{f(x_2)}{2}\right]\Delta X$$

$$= \left[\frac{0}{2} + 8.7025 + \frac{.3537}{2}\right](15)$$

$$= 133.19 \text{ ft}\cdot\text{lb}$$

$n = 6$

$$T_6 = \left[\sum_{i=1}^{6} \frac{f(x_{i-1})+f(x_i)}{2}\right]\Delta X$$

$$= \left[\frac{f(x_0)}{2} + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + \frac{f(x_6)}{2}\right]\Delta X$$

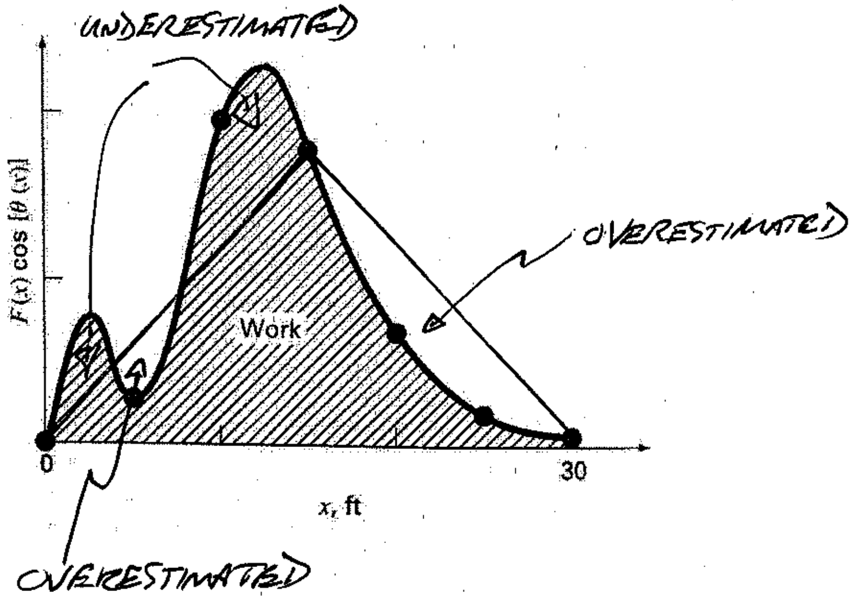
$$= \left[\frac{0}{2} + 1.5297 + 9.5120 + 8.7025 + 2.8087 + 1.0881 + \frac{.3537}{2}\right](5)$$

$$= 119.09 \text{ ft}\cdot\text{lb}$$

b) ALTHOUGH THE OVERESTIMATED AND UNDERESTIMATED AREA REGIONS ARE SIGNIFICANT, THEY APPROXIMATELY BALANCE EACH OTHER OFF GIVING A REASONABLY ACCURATE ESTIMATE OF WORK.

c) ALTHOUGH MORE SUBINTERVALS GENERALLY LEADS TO MORE ACCURATE AREA ESTIMATES, THE FIRST AND THIRD SUBINTERVALS PRODUCE SIGNIFICANT UNDERESTIMATES RESULTING IN A LESS ACCURATE ESTIMATE OF WORK AS COMPARED TO USING ONLY TWO SUBINTERVALS.

b) A continuous plot of $F(x)\cos[\theta(x)]$ versus position along with the seven discrete points given in the table are shown in the figure below. A "true" value of the work has been estimated as $129.52\ ft \cdot lb$ based on values collected at 1 foot intervals. Directly on the figure below, show your trapezoidal approximation with 2 subintervals and use this to explain why the trapezoidal approximation is reasonably accurate with only 2 subintervals.



c) Again, the same continuous plot of $F(x)\cos[\theta(x)]$ versus position along with the seven discrete points given in the table are shown in the figure below. As stated above, a "true" value of the work has been estimated as $129.52\ ft \cdot lb$ based on values collected at 1 foot intervals. Directly on the figure below, show your trapezoidal approximation with 6 subintervals and use this to explain why the trapezoidal approximation is less accurate with 6 subintervals as compared to the trapezoidal approximation obtained using only 2 subintervals.