Quiz 1- Solution

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1.

The deBroglie wavelength λ is defined as

$$\lambda = \frac{h}{p}, \tag{1}$$

where h is the Planck constant, and p is the momentum for the electron, which has the form $p = \gamma m v$. Here the speed of electron $v = \frac{1}{3}c$, and the γ is

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{3}{2\sqrt{2}}. \tag{2}$$

Hence the deBroglie wavelength λ is

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m v} = \frac{h}{3/2\sqrt{2} \cdot m \cdot \frac{1}{3}c} = \frac{2\sqrt{2}h}{m}. \tag{3}$$

2.

The total energy for the unconventional 1D quantum harmonic oscillator is given by

$$E = ap^2 + bx^4. \tag{4}$$

The uncertainty principle is $\Delta x \cdot \Delta p \sim \hbar$. Δx can be approximated by x and Δp is approximated by p, hence the position and momentum has the relation: $x \cdot p = \hbar$. Therefore, the total energy E is

$$E = ap^2 + bx^4 = a\left(\frac{\hbar}{x}\right)^2 + bx^4. \tag{5}$$

The local minimum of total energy E is given by requiring the first derivate of E respect to x vanishes.

$$\frac{\partial E}{\partial x} = -2a\hbar^2 x^{-3} + 4bx^3 = 0, \tag{6}$$

which implies that the minimum position x_0 is

$$x_0^2 = 2^{-1/3} \left(\frac{a}{b}\right)^{1/3} \hbar^{2/3}. \tag{7}$$

Plugging back Eq. (7) to the expression of total energy, we obtain the minimum energy E_0

$$E_0 = \hbar^{4/3} a^{2/3} b^{1/3} \left(2^{1/3} + 2^{-2/3}\right). \tag{8}$$

Case 2. The relation between x and p can also be approximated by $x \cdot p = \frac{\hbar}{2}$ from uncertainty principle. In this case, the minimum position x_0 is

$$x_0^6 = \frac{a\hbar^2}{8b}. \tag{9}$$

And the minimum energy E_0 is

$$E_0 = \frac{3a^{2/3}b^{1/3}\hbar^{4/3}}{4}. \tag{10}$$

3.

The first excited state $\phi_2(x)$ of the infinite well potential, which has center at $x = L/2$ is

$$\phi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right). \tag{11}$$

For part (a), recall that the momentum operator $\hat{p}$ in position x space is $\hat{p} = -i\hbar\frac{\partial}{\partial x}$. Hence the kinetic energy operator $\hat{H}_{KE} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$. The expectation value of kinetic energy is

$$\langle \hat{H}_{KE} \rangle = \int_0^L \phi_2^*(x) \left(\frac{\hat{p}^2}{2m}\phi_2(x)\right) dx = \left(\frac{2}{L}\right) \int_0^L \sin\frac{2\pi x}{L} \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \sin\frac{2\pi x}{L}\right) dx. \tag{12}$$

For part (b), check the consequence of momentum operator $\hat{p}$ acting on $\phi_2(x)$

$$\hat{p}\phi_2(x) = -i\hbar\frac{\partial}{\partial x}\phi_2(x) = -i\hbar\sqrt{\frac{2}{L}} \cos\frac{2\pi x}{L} \neq \text{const.} \cdot \phi_2(x). \tag{13}$$

Therefore, the wavefunction $\phi_2(x)$ is not the eigenstate of the momentum operator $\hat{p}$, which implies $\phi_2(x)$ doesn't have well-defined momentum.

4.

The wavefunction for the ground state of 1D quantum harmonic oscillator is

$$\psi(x) = Ae^{-ax^2}. \tag{14}$$

The Hamiltonian operator $\hat{H}$, representing the total energy of harmonic oscillator, is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}k\hat{x}^2. \tag{15}$$

The first and second derivative on $\psi(x)$ gives

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= -ax\psi(x), \\ \frac{\partial^2 \psi}{\partial x^2} &= (a^2x^2 - a)\psi(x). \end{aligned} \tag{16}$$

Therefore, we have

$$\hat{H}\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + \frac{1}{2}k\hat{x}^2\psi(x) = -\frac{\hbar^2}{2m}(ax^2 - a)\psi(x) + \frac{1}{2}k\hat{x}^2\psi(x) = \frac{\hbar}{2}\sqrt{\frac{k}{m}}\psi(x). \tag{17}$$

The energy of the ground state is $E_0 = \frac{\hbar}{2}\sqrt{\frac{k}{m}}$.