MAT195S CALCULUS II

Midterm Test #2

2 April 2013 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

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Family Name: J W Davis.

Given Name: Solutions

Student #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	8	
2	8	
3	8	
4	10	
5	10	
6	10	
7	12	
8	6	
TOTAL	72	/ 65

Tutorial Section:

TA Name:

1) Test the series for convergence or divergence:

a) $\sum_{k=1}^{\infty} \frac{k^2}{e^k}$ b) $\sum_{k=1}^{\infty} \frac{6^k}{5^k - 1}$ c) $\sum_{k=1}^{\infty} (-1)^k (\sqrt{k+1} - \sqrt{k})$

(8 marks)

a) ratio test: $\left| \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} \right| \longrightarrow \frac{1}{e} < 1$ $\therefore$ convergent

b) let $b_n = \frac{6^k}{5^k} = \left(\frac{6}{5}\right)^k$ which diverges $(b_n \not\to 0)$

$\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ $\therefore$ a_n diverges by limit comparison test

c) $|a_k| = \sqrt{k+1} - \sqrt{k} \times \frac{\sqrt{k+1} + \sqrt{k}}{\sqrt{k+1} + \sqrt{k}} = \frac{k+1-k}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{2\sqrt{k}} \xrightarrow{k \to \infty} 0$

$|a_{k+1}| = \sqrt{k+2} - \sqrt{k+1} = \frac{1}{\sqrt{k+2} + \sqrt{k+1}} < \frac{1}{\sqrt{k} + \sqrt{k+1}} = |a_k|$ $\therefore$ decreasing

$\therefore$ convergent by alt series test

2) a) Find the radius and interval of convergence of the following power series: $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

(4 marks)

ratio test: $\left| \frac{x^{2n+2}}{(n+1)(\ln(n+1))^2} \cdot \frac{n(\ln n)^2}{x^{2n}} \right| = \left(\frac{n}{n+1}\right)\underbrace{\left(\frac{\ln n}{\ln(n+1)}\right)^2}\cdot |x|^2 \longrightarrow |x|^2$

$\therefore$ convergent for $|x| < 1$

$\lim_{x\to\infty} \frac{\ln x}{\ln(x+1)} \overset{*}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = 1$

endpoints: $x = \pm 1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Integral test: $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ let $u = \ln x$, $du = dx/x$

$= \int_{\ln 2}^{\infty} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}$ $\therefore$ convergent

$\Rightarrow$ interval of convergence $[-1, 1]$

b) If k is a positive integer, find the radius of convergence of the series: $\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$

(4 marks)

ratio test: $\left| \frac{((n+1)!)^k x^{n+1}}{(kn+k)!} \cdot \frac{(kn)!}{(n!)^k x^n} \right| = |x| (n+1)^k \frac{(kn)!}{(kn+k)!}$

$= |x| \frac{(n+1)(n+1)(n+1)\cdots(n+1)}{(kn+1)(kn+2)(kn+3)\cdots(kn+k)} \longrightarrow \frac{|x|}{k^k}$

$\Rightarrow |x| < k^k$

3) a) Evaluate $\int e^{x^2}\,dx$ as an infinite series. For what values of x is this result valid?

(4 marks)

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{6!}$$

$$\int e^{x^2} dx = \int \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \int \frac{x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!} + C$$

Convergence: As the series for e^x converges for all x, the series for e^{x^2} converges for all x. Thus the series for $\int e^{x^2}$ also converges for all x.

b) Use the binomial series to find the series expansion of $f(x) = \dfrac{1}{\sqrt{1-x^2}}$. Write out the first 5 terms of the series.

(4 marks)

$$f(x) = \frac{1}{\sqrt{1+x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^{2n} (-1)^n$$

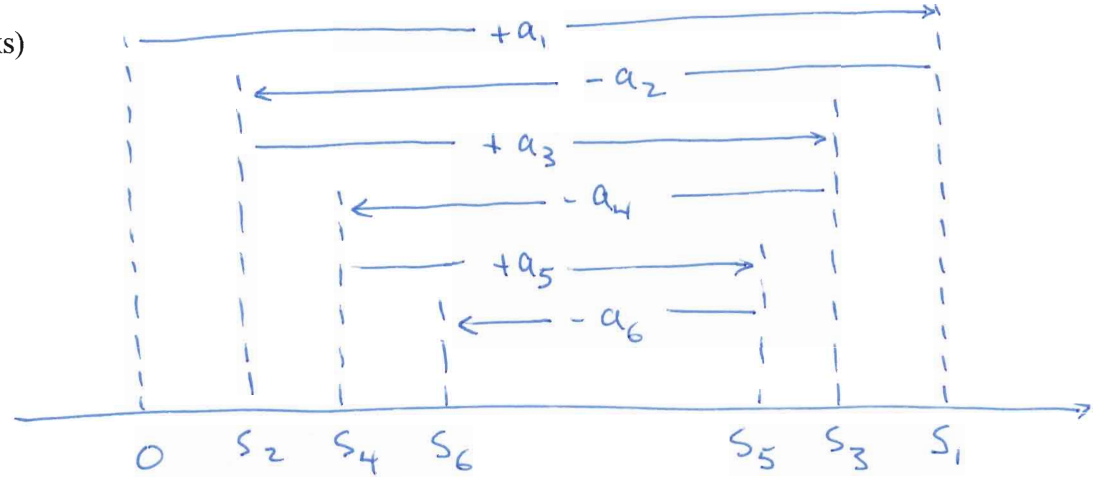
$$= 1 + \frac{1}{2}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})\, x^4}{2!} - \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\, x^6}{3!} + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})\, x^8}{4!} - \cdots$$

$$= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^3 + \frac{15}{48}x^4 + \frac{105}{192}x^6 + \cdots$$

4) Prove the Alternating Series Test for series convergence:

Let $\{a_n\}$ be a sequence of positive numbers. If $a_{k+1} < a_k$, and $a_k \to 0$ as $k \to \infty$, then $\sum_{k=1}^{\infty} (-1)^{k-1} a_k$ converges.

(10 marks)



Consider even partial sums:

$S_2 = a_1 - a_2 > 0$

$S_4 = S_2 + (a_3 - a_4) > S_2$

$\vdots$

$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2}$ $\Leftarrow$ $\{S_{2n}\}$ is monotonic increasing

$\boxed{\text{Note } a_{n+1} < a_n}$

Also: $S_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{+ve} - \underbrace{(a_4 - a_5)}_{+ve} - \dots - \underbrace{(a_{2n-2} - a_{2n-1})}_{+ve} - \underbrace{a_{2n}}_{+ve}$

$\therefore < a_1$ for all n $\Rightarrow$ $\{S_{2n}\}$ is bounded above by a_1

$\Rightarrow$ $\{S_{2n}\}$ is monotonic and bounded, $\therefore \lim_{n\to\infty} S_{2n} = L$ exists

Consider odd partial sums:

$S_{2n+1} = S_{2n} + a_{2n+1}$

$\lim_{n\to\infty} S_{2n+1} = \underbrace{\lim_{n\to\infty} S_{2n}}_{=L} + \underbrace{\lim_{n\to\infty} a_{2n+1}}_{=0} = L$

$\Rightarrow$ Given $a_{n+1} < a_n$ and $\lim_{n\to\infty} a_n = 0$: both odd and even partial sums converge to the same limit, thus we conclude the series converges.

5) Find, from first principles, the Taylor series expansion for $f(x) = 2^x$ about $a = 0$.
Prove that f is equal to the sum of this series by showing that the Taylor remainder, $R_n(x)$, goes to zero as $n \to \infty$. Recall, the Taylor remainder theorem which states that

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \text{ where } |f^{(n+1)}(x)| \le M.$$

(10 marks)

$f(x) = 2^x$ $\quad\quad f(0) = 1$

$f'(x) = 2^x \ln 2$ $\quad\quad f'(0) = \ln 2$

$f''(x) = 2^x (\ln 2)^2$ $\quad\quad f''(0) = (\ln 2)^2$

$\vdots$

$f^{(n)}(x) = 2^x (\ln 2)^n$ $\quad\quad f^{(n)}(0) = (\ln 2)^n$

$$\therefore\ 2^x = 1 + x\ln 2 + \frac{x^2}{2}(\ln 2)^2 + \frac{x^3}{3!}(\ln 2)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(x\ln 2)^n}{n!}$$

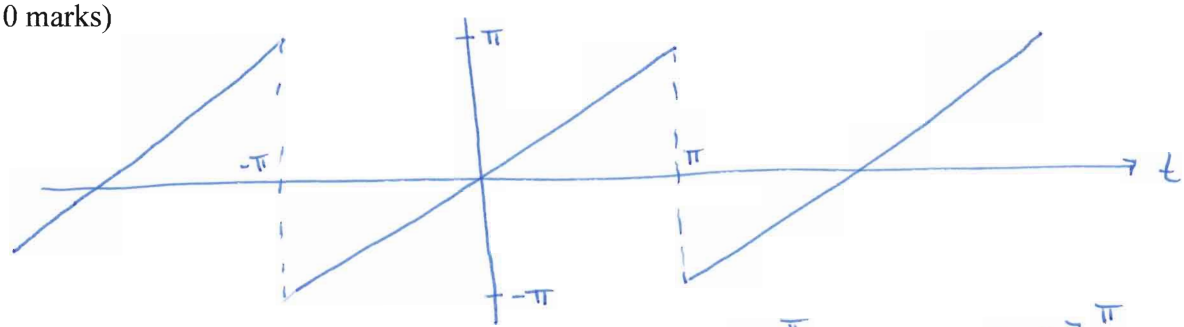
Now $f^{(n+1)}(x) = 2^x (\ln 2)^{n+1}$ $\quad\quad \ln 2 < 1$

$$\therefore\ R_n(x) \le \frac{2^x |x|^{n+1} (\ln 2)^{n+1}}{(n+1)!} < \frac{2^x |x|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$\therefore\ 2^x = \sum_{n=0}^{\infty} \frac{x^n (\ln 2)^n}{n!}$$

6) Find the Fourier series, ie., evaluate the Fourier coefficients, for the function $f(t) = t, \ -\pi \le t \le \pi$.

(10 marks)



$T = 2\pi$

$\omega = 1$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt \quad \Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos(n\omega t)\,dt \qquad \text{let } u = t \quad dv = \cos nt$$
$$du = dt \quad v = \frac{1}{n}\sin nt$$

$$= \frac{1}{\pi}\left[\frac{t}{n}\underbrace{\sin nt}_{0}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{n}\underbrace{\sin nt\,dt}_{0} = 0$$

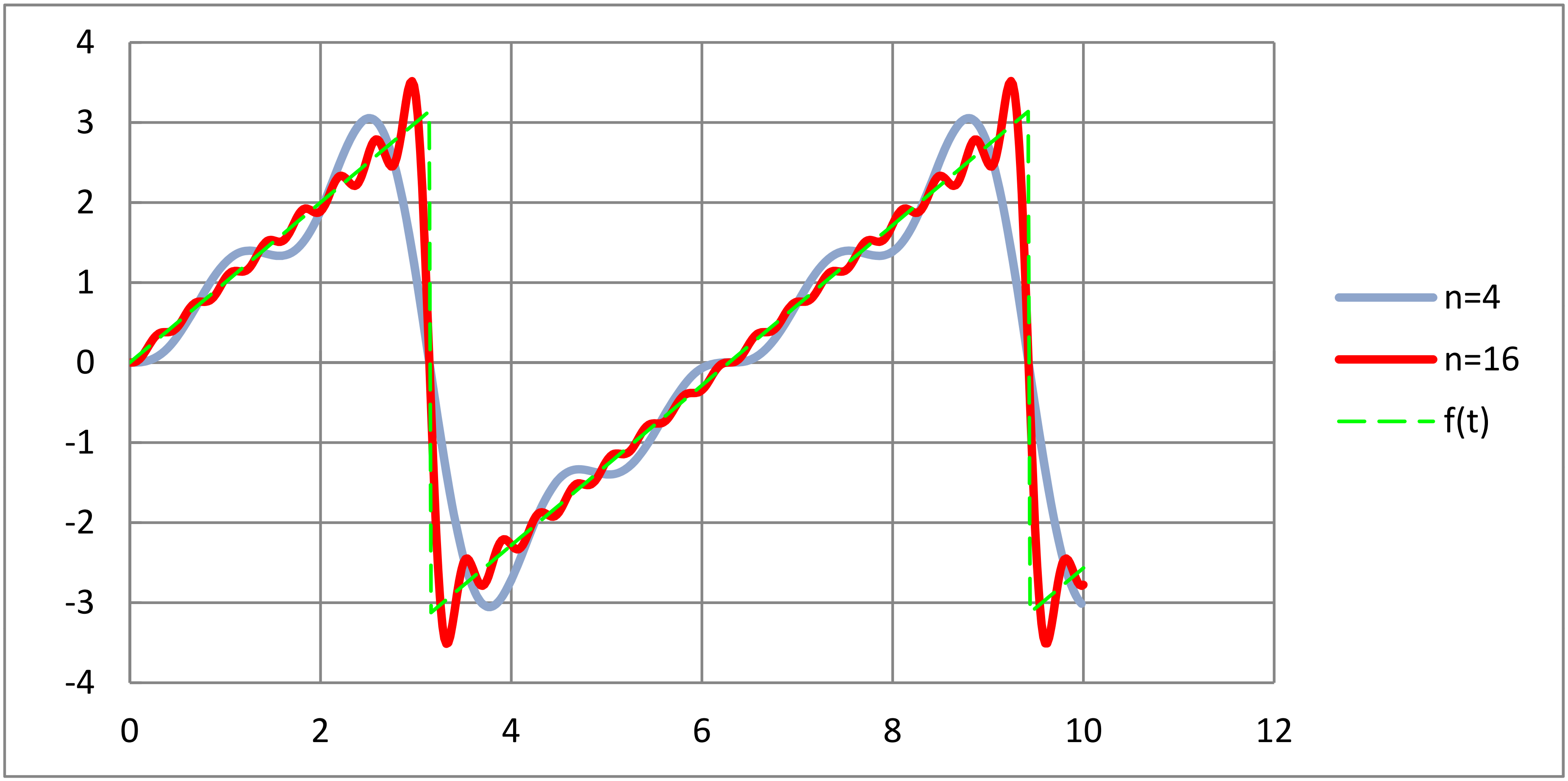
(Alternately, one can note that $f(t)$ is an odd function and thus all $a_n = 0$)

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,dt$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(nt)\,dt \qquad \text{let } u = t \quad dv = \sin nt$$
$$du = dt \quad v = \frac{-1}{n}\cos nt$$

$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos nt\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{n}\underbrace{\cos nt\,dt}_{0} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd}\end{cases}$$

$$\therefore f(t) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nt)$$


4
3
2
1
0
-1
-2
-3
-4
0
2
4
6
8
10
12
n=4
n=16
f(t)

7) Find the unit tangent vector, the principal normal vector and an equation in x, y, z for the osculating plane at the point (1,2,2) on the curve: $\vec{r}(t) = t^2\,\hat{i} + (t+1)\,\hat{j} + 2t\,\hat{k}$

(12 marks)

$\vec{r}(t) = (t^2, t+1, 2t) \Rightarrow \vec{r}(1) = (1,2,2) \Rightarrow t = 1$

$\vec{r}'(t) = (2t, 1, 2) \Rightarrow \|\vec{r}'(t)\| = \sqrt{4t^2+1+4} = \sqrt{5+4t^2}$

$\therefore \vec{T} = \dfrac{\vec{r}'}{\|\vec{r}'\|} = \left(\dfrac{2t}{\sqrt{5+4t^2}}, \dfrac{1}{\sqrt{5+4t^2}}, \dfrac{2}{\sqrt{5+4t^2}}\right) \Rightarrow \boxed{\vec{T}(1) = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)}$

$\vec{T}' = \left(-\frac{1}{2}(5+4t^2)^{-3/2}\cdot 8t\cdot 2t + 2(5+4t^2)^{-1/2}, -\frac{1}{2}(5+4t^2)^{-3/2}\cdot 8t, -\frac{1}{2}(5+4t^2)^{-3/2}\cdot 8t\cdot 2\right)$

$\vec{T}'(t=1) = \left(-\frac{8}{27}+\frac{2}{3}, \frac{-4}{27}, \frac{-8}{27}\right) = \frac{1}{27}(10, -4, -8)$

$\|\vec{T}'(1)\| = \frac{2}{27}\sqrt{25+4+16} = \frac{2\sqrt{45}}{27} = \frac{2\sqrt{5}}{9}$

$\therefore \vec{N}(1) = \dfrac{\vec{T}'(1)}{\|\vec{T}'(1)\|} = \frac{2}{27}(5,-2,-4)\cdot\frac{9}{2\sqrt{5}} = \boxed{\frac{1}{3\sqrt{5}}(5,-2,-4)}$

$\vec{T}(1)\times\vec{N}(1) = \frac{1}{3}(2,1,2)\times\frac{1}{3\sqrt{5}}(5,-2,-4) = \frac{1}{9\sqrt{5}}(0, 18, -9)$

$= \frac{1}{\sqrt{5}}(0,2,-1)$

$\begin{vmatrix} i & j & k \\ 2 & 1 & 2 \\ 5 & -2 & -4 \end{vmatrix} = (-4+4, 10+8, -4-5)$

$= (0, 18, -9)$

Osculating Plane: $\left.\begin{array}{l}\text{point: } (1,2,2) \\ \text{normal: } (0,2,-1)\end{array}\right\}$ $0(x-1) + 2(y-2) - (z-2) = 0$

or $2y - z = 2$

8) a) Show that the following limit does not exist: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^8 y^2}{x^{16} + y^4}$

(3 marks)

let $y=0 \Rightarrow \lim\limits_{x\to 0} \dfrac{0}{x^{16}} = 0$

$y=x^4 \Rightarrow \lim\limits_{x\to 0} \dfrac{x^8 x^8}{x^{16}+x^{16}} = \dfrac{1}{2}$

Approaching $(0,0)$ along different paths gives a different result: $\therefore \lim\limits_{(x,y)\to(0,0)} \dfrac{x^8 y^2}{x^{16}+y^4}$ DNE

b) Find the limit: $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 16} - 4}$

(3 marks)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+16}-4} \times \frac{\sqrt{x^2+y^2+16}+4}{\sqrt{x^2+y^2+16}+4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\cdot\left(\sqrt{x^2+y^2+16}+4\right)}{x^2+y^2+16-16}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+16}+4 = 8$$