MAT292 - Calculus III - Fall 2015

Term Test 1 - October 19, 2015

Time allotted: 90 minutes. Aids permitted: None.

Solution

Full Name: ______________________________
Last First

Student Number: ______________________________

Email: ______________________________ @mail.utoronto.ca

<u>Instructions</u>

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 12 pages (including this title page). Make sure you have all of them.
- You can use pages 11–12 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGES 11–12.

GOOD LUCK!

PART I   No explanation is necessary.

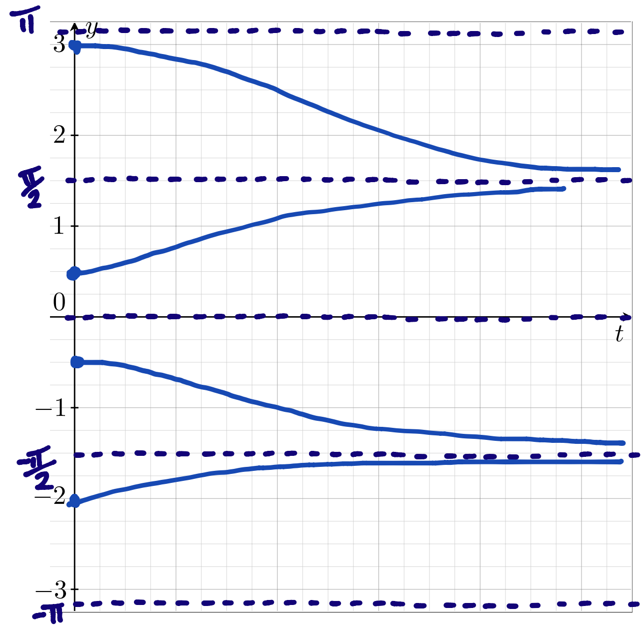
1. (2 marks) Without solving the DE, sketch the graph of the solutions $y_1(t), \ldots, y_4(t)$ of the DE

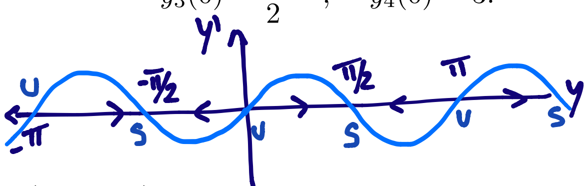
$$\frac{dy}{dt} = \sin(2y)$$

with initial conditions

$$y_1(0) = -2 \quad , \quad y_2(0) = -\frac{1}{2},$$

$$y_3(0) = \frac{1}{2} \quad , \quad y_4(0) = 3.$$





2. (1 mark) What are all the **stable** equilibrium solution for the DE from question **1** ?

$y =$ $\frac{\pi}{2} + k\pi$ for any $k \in \mathbb{Z}$

or

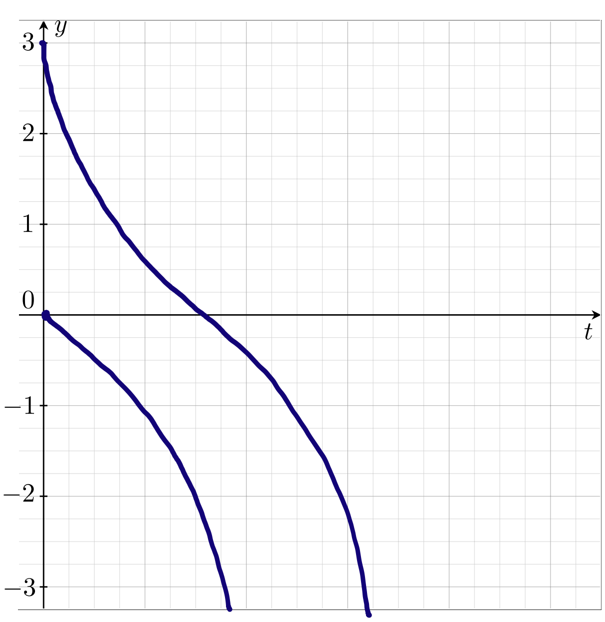
$\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

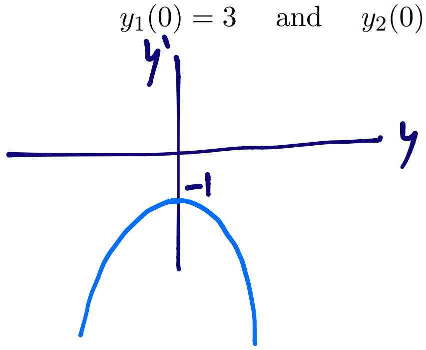
3. (2 marks) Without solving the DE, sketch the graph of the solutions $y_1(t)$, $y_2(t)$ of the DE

$$\frac{dy}{dt} = -1 - y^2$$

with initial conditions

$$y_1(0) = 3 \quad \text{and} \quad y_2(0) = 0.$$





Continued...

4. **(2 marks)** Consider the DE

$$\frac{dy}{dt} = \sqrt{y-1} + e^t.$$

According to the Theorem of Existence and Uniqueness, for which initial conditions $y(1) = y_0$ are we sure the solution exists and is unique?

$$y_0 \in \left(\underline{\quad 1 \quad} , \underline{\quad \infty \quad} \right).$$

5. **(2 marks)** For the same DE as in question **4**, according to the same Theorem the solution exists for

$$t \in \left(\underline{\quad 1-h \quad} , \underline{\quad 1+h \quad} \right).$$

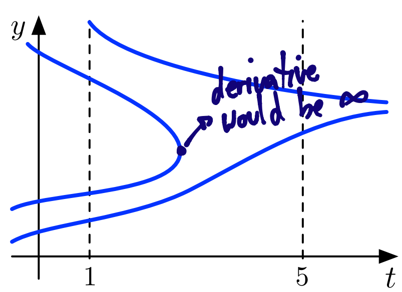
6. **(1 mark)** Consider the problem

$$y' + p(t)y = g(t),$$

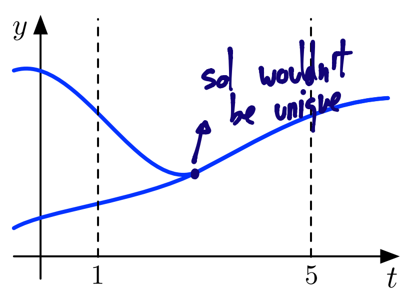
where

- $p(t)$ is continuous for $t \neq 1$.
- $g(t)$ is continuous for $t \neq 5$.

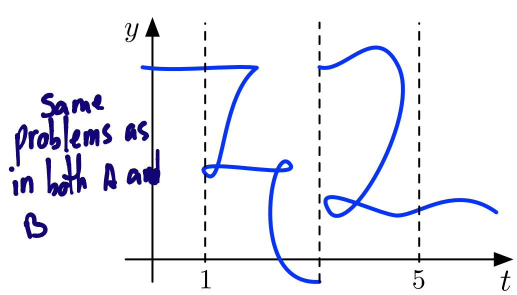
Which one could be a graph of some solutions to this DE? (circle the correct option)



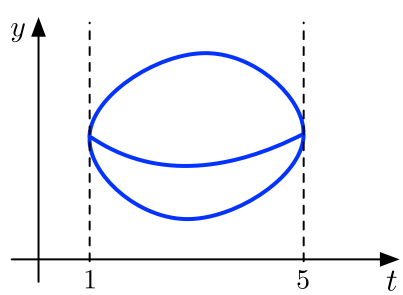
A



B



C



D

Continued...

PART II **Justify** your answers.

7. Two cars are racing on a race track and their velocities (in m/s) are given by the DEs: **(13 marks)**

$$\frac{dv_1}{dt} = \frac{\pi}{30}\sqrt{10^2 - (v_1 - 50)^2} \qquad \text{and} \qquad t\frac{dv_2}{dt} - v_2 = -70.$$

The race has a moving start and $v_1(0) = 50$.

(a) **(1 mark)** What is the initial velocity of car #2?

Set $t=0$ on the DE for v_2 and we get

$-v_2(0) = -70 \iff v_2(0) = 70$ m/s

(b) **(5 marks)** Find an explicit formula for the velocities of both cars.

Car #1. Separable DE:

$$\int \frac{1}{\sqrt{10^2-(v_1-50)^2}}\,dv_1 = \frac{\pi}{30}t + C_1 \quad \overset{\boxed{v_1-50=10\sin(u)}}{\iff} \quad \int \frac{10\cancel{\cos(u)}}{10\cancel{\cos(u)}}\,du = \frac{\pi}{30}t + C_1$$

$$\iff u = \frac{\pi}{30}t + C_1 \iff \arcsin\left(\frac{v_1-50}{10}\right) = \frac{\pi}{30}t + C_1$$

$$\iff v_1 = 50 + 10\sin\left(\frac{\pi}{30}t + C_1\right)$$

Using the initial condition, we find C_1: $\cancel{50} = v_1(0) = \cancel{50} + 10\sin(C_1) \iff \sin(C_1) = 0$

So we can take $C_1 = 0$.

The solution for Car #1 is

$$\boxed{v_1 = 50 + 10\sin\left(\frac{\pi}{30}t\right)}$$

Car #2. Linear DE: $v_2' - \frac{1}{t}v_2 = -\frac{70}{t}$

Multiply by integrating factor $\mu = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = t^{-1} = \frac{1}{t}$.

We get $\left(\frac{1}{t}v_2\right)' = -\frac{70}{t^2}$

$$\iff \frac{v_2}{t} = -\int \frac{70}{t^2}\,dt = \frac{70}{t} + C_2$$

$$\iff \boxed{v_2 = 70 + C_2 t}$$

We can't find the constant C_2 using the initial condition $v_2(0)=70$. There are multiple solutions

Continued...

(c) **(5 marks)** Find an explicit formula for the distance of the cars from the starting line.

We only need to integrate the solutions from (b).

Car #1. $d_1 = \int v_1\, dt = \int \left[50 + 10\sin\left(\frac{\pi}{30}t\right)\right] dt = 50t - \frac{300}{\pi}\cos\left(\frac{\pi}{30}t\right) + B_1$

The initial condition is $d_1(0) = 0$ since the race starts at the starting line, so

$0 = d_1(0) = -\frac{300}{\pi} + B_1 \iff B_1 = \frac{300}{\pi}$

and we get $\boxed{d_1(t) = 50t + \frac{300}{\pi}\left(1 - \cos\left(\frac{\pi}{30}t\right)\right)}$

Car #2. $d_2 = \int (70 - C_2 t)\, dt = 70t - C_2\frac{t^2}{2} + B_2$

$0 = d_2(0) = B_2$

So $\boxed{d_2(t) = 70t - C_2\frac{t^2}{2}}$

(d) **(2 marks)** Assume that the second car has velocity $v_2(10) = 60$ and that the race is 2000 m long. Which car won the race?

If $v_2(10) = 60$, then we can find C_2 : $60 = v_2(10) = 70 - 10C_2 \iff C_2 = 1$

So $\boxed{d_2(t) = 70t - \frac{t^2}{2}}$.

The distance for car #2 is simpler, so let us find when car #2 arrived at the finish line: $d_2(T_2) = 2000 \iff 70T_2 - \frac{T_2^2}{2} = 2000 \iff T_2^2 - 140T_2 + 4000 = 0$

$\iff T_2 = 70 \pm \sqrt{70^2 - 4000} = 70 \pm \sqrt{4900 - 4000} = 70 \pm \sqrt{900} = 70 \pm 30$

So Car #2 finished the race in 40s.

Where was Car #1 at $t = 40$s? $d_1(40) = 50 \cdot 40 + \frac{300}{\pi}\underbrace{\left[1 - \cos\left(\frac{\pi}{30} \cdot 40\right)\right]}_{>0} > 2000$ past the finish line

$\boxed{\text{Car \#1 won the race}}$

Continued...

8. Solve the initial-value problem **(7 marks)**

$$\frac{x}{(x^2+y^2)^{\frac{3}{2}}} + \frac{y}{(x^2+y^2)^{\frac{3}{2}}}\frac{dy}{dx} = 0$$

$$y(3) = -4.$$

Solution #1 Multiply the DE by $(x^2+y^2)^{3/2}$ to get a Separable DE.

$x + y\frac{dy}{dx} = 0 \iff \int y\,dy = -\int x\,dx \iff \frac{y^2}{2} = -\frac{x^2}{2} + \bar{C} \iff \boxed{x^2+y^2 = C}$

Solution #2. $M = \frac{x}{(x^2+y^2)^{3/2}} \Rightarrow M_y = -\frac{3xy}{(x^2+y^2)^{5/2}}$ The DE is exact.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \|$

$N = \frac{y}{(x^2+y^2)^{3/2}} \Rightarrow N_x = -\frac{3xy}{(x^2+y^2)^{5/2}}$

$u = x^2+y^2$
$du = 2x\,dx$

Then $\frac{\partial \psi}{\partial x} = M = \frac{x}{(x^2+y^2)^{3/2}} \Rightarrow \psi = \int M\,dx = \int \frac{x}{(x^2+y^2)^{3/2}}dx \overset{\downarrow}{=} \frac{1}{2}\int u^{-3/2}\,du = -u^{-1/2} + h(y)$

$= -\frac{1}{\sqrt{x^2+y^2}} + h(y)$

Also $\frac{\partial \psi}{\partial y} = N \iff \cancel{\frac{y}{(x^2+y^2)^{3/2}}} + h'(y) = \cancel{\frac{y}{(x^2+y^2)^{3/2}}} \iff h'(y) = 0 \iff h(y) = C.$

Let $\boxed{\psi(x,y) = -\frac{1}{\sqrt{x^2+y^2}}}$

The solution satisfies $-\frac{1}{\sqrt{x^2+y^2}} = \psi = \bar{C}$

$\iff \boxed{x^2+y^2 = C}$

Continuation of the answer for both Solutions #1 and #2.

Using the initial condition: $3^2+(-4)^2 = C \iff C = 25$

The solution satisfies $x^2+y^2 = 25 \iff y = \pm\sqrt{25-x^2}$

The solution is the negative because the initial condition is $y(3) = -4 < 0$

So $\boxed{y = -\sqrt{25-x^2}}$

Continued...

9. Consider the Differential Equation **(7 marks)**

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0.$$

(a) **(4 marks)** Assume that this differential equation is not Exact, but can be transformed into an Exact Differential Equation by multiplying it by an integrating factor $\boldsymbol{\mu(y)}$.

Show that μ is a solution of the differential equation $M\mu' = (N_x - M_y)\mu$.

Multiply the DE by $\mu(y)$: $\qquad \underbrace{\mu M}_{\bar{M}} + \underbrace{\mu N}_{\bar{N}}\frac{dy}{dx} = 0$

This DE is exact, so :

$\bar{M}_y = \bar{N}_x$

$\bar{M}_y = \mu' M + \mu M_y = \bar{N}_x = \mu N_x \qquad (\mu_x = 0)$

So $\quad \mu' M + \mu M_y = \mu N_x \iff \boxed{M\mu' = (N_x - M_y)\mu}$

(b) **(3 marks)** Write a differential equation that satisfies the conditions described in **(a)**. Justify your answer.

Want to find M, N such that $\begin{cases} M_y \neq N_x \iff M + N\frac{dy}{dx} = 0 \text{ is not exact} \\ (\mu M)_y = (\mu N)_x \iff \mu M + \mu N \frac{dy}{dx} = 0 \text{ is exact} \\ \mu = \mu(y) \end{cases}$

start with $\mu = y$ and $M = y$.

Then $\quad (\mu M)_y = (\mu N)_x = yN_x$

$(\mu M)_y = (y^2)' = 2y$

$\Rightarrow 2y = yN_x \iff N_x = 2 \iff N = 2x$

Check that $M_y \neq N_x = 2$ ✓, $M_y = 1$

The DE $\quad \boxed{y + 2x\frac{dy}{dx} = 0} \quad$ satisfies the conditions of (a).

Continued...

10. In a small animal reserve in Africa, there are lions and antelopes. **(13 marks)**

The park rangers want to model these populations as they depend on each other.

They hire an Eng. Sci. student and she considers the Lotka-Volterra model for a predator-prey population:

$$\frac{dx}{dt} = x - \frac{1}{2}xy \tag{A}$$

$$\frac{dy}{dt} = -y + \frac{1}{2}xy \tag{L}$$

where

- $x(t)$ = population (**in hundreds**) of antelopes (prey) at time t (in years),
- $y(t)$ = population (**in tens**) of lions (predator) at time t .

The Eng. Sci. student observes that this is a **nonlinear system** of DEs. She didn't learn about these kinds of systems of DEs in MAT292, so she cannot solve it. She asks the park rangers for some data on these populations and they tell her the population of lions oscillates periodically with a period of 2π. They also give her the table below (she doctored the figures a little to make them look nicer for your test!):

time (in years)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
lions (in tens)	3	$2+\frac{\sqrt{3}}{2}$	$2+\frac{\sqrt{2}}{2}$	$2+\frac{1}{2}$	2

Using this data, help her successfully approximate the DE and help the park rangers.

You don't need to know anything about systems of DEs.

(a) **(3 marks)** Using the table above, give a formula for the lion population $y(t)$ that matches the observations.

From the table above and the fact that $y(t)$ is periodic with period 2π, we have

$$\boxed{y(t) = 2 + \cos(t)}$$

Continued...

(b) (2 marks) Use your formula for $y(t)$ in equation (A) to obtain a simpler DE for $x(t)$.

Then (A) $\Leftrightarrow$ $\boxed{x'(t) = x - \frac{1}{2}x\left(2+\cos(t)\right) = -\frac{1}{2}\cos(t)\,x}$

(c) (4 marks) The park rangers counted the number of antelopes and figured that there were about 100 antelopes at $t = \pi$. Find $x(t)$ by solving the DE found in **(b)**.

We now have an initial condition $x(\pi) = 1$ ($x(t)$ is given in hundreds)

Separable DE $\quad \frac{1}{x}\frac{dx}{dt} = -\frac{1}{2}\cos(t)$

$$\Leftrightarrow \int \frac{1}{x}\,dx = -\frac{1}{2}\int \cos(t)\,dt \Leftrightarrow \ln x = -\frac{1}{2}\sin(t) + \bar{C}$$

$$\Leftrightarrow x = C e^{-\frac{1}{2}\sin(t)}$$

Use the initial condition to find C: $\quad 1 = x(\pi) = C$

Thus $\boxed{x = e^{-\frac{1}{2}\sin(t)}}$

(d) (2 marks) Show that the solutions $x(t)$ and $y(t)$ are periodic. What is the period?

$y(t) = 2+\cos(t)$ is periodic with period 2π: $\quad y(t+2\pi) = 2+\cos(t+2\pi)$
$= 2+\cos(t)$
$= y(t)$

$x(t) = e^{-\frac{1}{2}\sin(t)}$ is also periodic with period 2π:

$x(t+2\pi) = x(t)$.

Continued...

(e) (2 marks) What are the maximum and minimum values for $x(t)$?

$\max \sin(t) = 1$ when $t = \frac{\pi}{2}$ $(+2k\pi)$

$\min \sin(t) = -1$ when $t = \frac{-\pi}{2}$ $(+2k\pi)$

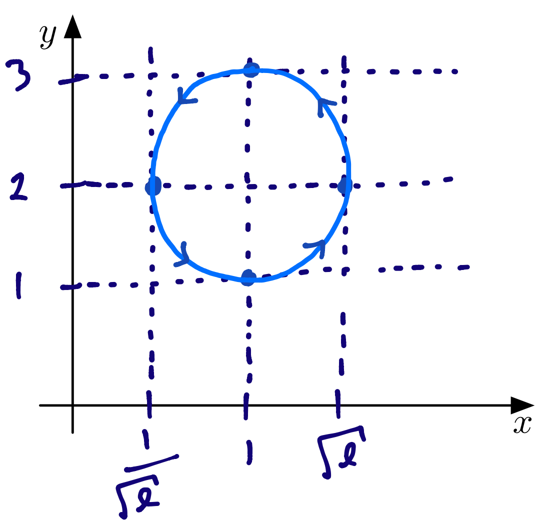
So $\max -\frac{1}{2}\sin(t) = -\frac{1}{2}\min \sin(t) = \frac{1}{2}$ at $t = -\pi/2$

$\min -\frac{1}{2}\sin(t) = -\frac{1}{2}\max \sin(t) = -\frac{1}{2}$ at $t = \pi/2$

Thus $\boxed{\max x(t) = e^{\max -\frac{1}{2}\sin(t)} = e^{1/2} = \sqrt{e}}$ at $t = -\pi/2$

$\boxed{\min x(t) = e^{\min -\frac{1}{2}\sin(t)} = e^{-1/2} = \frac{1}{\sqrt{e}}}$ at $t = \pi/2$

(bonus) (3 marks) Show how the two populations are related, by sketching a graph of y vs x.



Continued...