Solution Copy

PHY294 Quantum Term Test (February 9th 2017)

- 75 minutes (closed book, no calculator, one single-sided page of hand-written notes is allowed).
- Note the helpful formulas on the back page.
- The four questions are weighted equally.

1. Consider a particle of mass m in motion:

(a) If the particle is relativistic and free (not bound by any potential), and its deBroglie wavelength is $\pi\hbar/mc$, how fast is this particle moving and what is the phase velocity associated with this wavelength?

(b) If the particle is non-relativistic and bound by an attractive potential $V(r) = -b/r^4$, where b is a constant, what powers of $\hbar$ and of m is the ground-state energy (and thus the quantized energies) proportional to?

2. Consider a particle of mass m in a 1D infinite-potential square well, lying between $x = -2L$ and $x = 0$:

(a) For each of the three lowest energy eigenstates, write down the energy and $\psi(x)$, and sketch $|\psi(x)|^2$.

(b) Initially the system is in the ground state. Suddenly the well is widened, now lying between $x = -4L$ and $x = 0$. Calculate the probability of finding the particle in the first-excited state of the widened well.

3. Consider two 1D Gaussian wave functions, where x_0, p_0, A, B, σ, ε are constants.

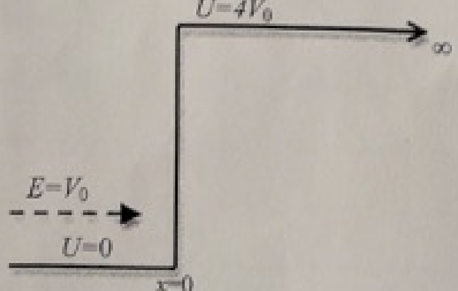
(a) $\psi_a(x) = A\exp[-(x-x_0)^2/2\sigma^2]$, representing the ground state of a quantum harmonic oscillator. Determine if position x and momentum p are well defined. Calculate the expectation value $\langle x \rangle$.

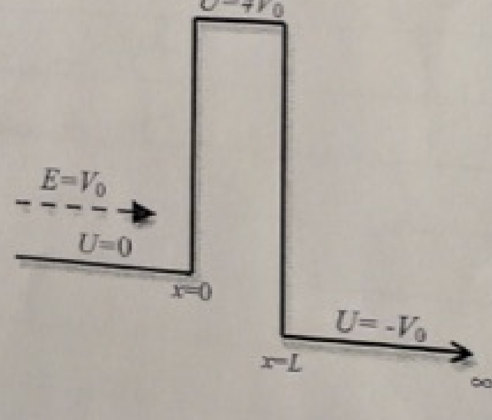
(b) $\psi_b(x) = B\exp[-x^2/2\varepsilon^2 - i\,p_0 x/\hbar]$, representing an unbound wave packet of non-relativistic electrons at $t = 0$. Calculate the expectation value $\langle p \rangle$. Describe the time evolution of this wave packet for $t > 0$.

4. A flux of electrons with energy $E = V_0$ is incident upon each of the potential barriers $U(x)$ shown below.

(a) Left Figure: calculate the reflection probability R, and the probability current $j(x)$ for $x > 0$ and $x < 0$.

(b) Right Figure: write down the proper $\psi(x)$ in each region; state the boundary conditions; and express (need not solve) the transmission probability T in terms of V_0, $\hbar$, m_e and the coefficients in the $\psi(x)$'s. If E is raised from V_0 to $4V_0$, would T rise monotonically? For $E > 4V_0$, how would T vary with E?





$\boxed{1}$

(A) $\lambda = \frac{h}{p} = \frac{h}{\gamma m v}$ $\qquad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}, \quad \beta \equiv \frac{v}{c}$

$$\frac{\pi h}{2\pi m c} = \frac{h\sqrt{1-\beta^2}}{m\beta c} \Rightarrow \beta^2 = 4 - 4\beta^2$$

$$\therefore \; v = \frac{2}{\sqrt{5}} c$$

$$v_\varphi = \frac{\omega}{k} = \frac{E}{p} = \frac{\gamma m c^2}{\gamma m v} = \frac{c^2}{v} = \frac{\sqrt{5}}{2} c$$

Or: $v_\varphi = \frac{(p^2c^2 + m^2c^4)^{1/2}}{p} = c\left(1 + \frac{m^2c^2}{h^2/\lambda^2}\right)^{1/2} = \cdots = \frac{\sqrt{5}}{2}$

Or: $v_\varphi = \frac{c^2}{v_g} = \frac{c^2}{v} = \frac{\sqrt{5}}{2} c$ [But must state that particle velocity is the same as v_g]

(B) $E = ap^2 - br^{-4}$ $\qquad a \equiv \frac{1}{m}$

$= a\hbar^2 r^{-2} - br^{-4}$ $\qquad p \cdot r \sim \hbar$

$$\left.\frac{\partial E}{\partial r}\right|_{r_0} = -2a\hbar^2 r^{-3} + 4br^{-5} = 0$$

$$r_0^{-2} = 2a\hbar^2 / 4b$$

$$\Rightarrow E_0 = a\hbar^2\left(\frac{2a\hbar^2}{4b}\right) - b\left(\frac{2a\hbar^2}{4b}\right)^2$$

$$\therefore \; E_0 \sim \hbar^4$$

$$E_0 \sim m^{-2}$$

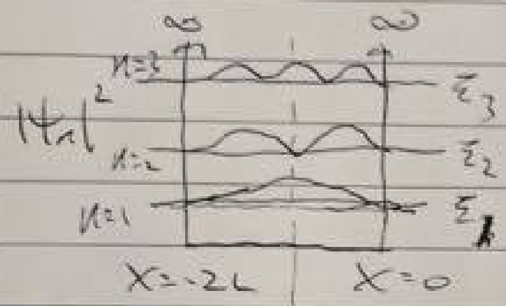
$\boxed{2}$

Ⓐ

$$\psi_1 = \sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L} \qquad E_1 = \frac{1^2h^2}{8m(2L)^2}$$

$$\psi_2 = \sqrt{\frac{2}{2L}}\sin\frac{2\pi x}{2L} \qquad E_2 = \frac{2^2h^2}{8m(2L)^2}$$

$$\psi_3 = \sqrt{\frac{2}{2L}}\sin\frac{3\pi x}{2L} \qquad E_3 = \frac{3^2h^2}{8m(2L)^2}$$



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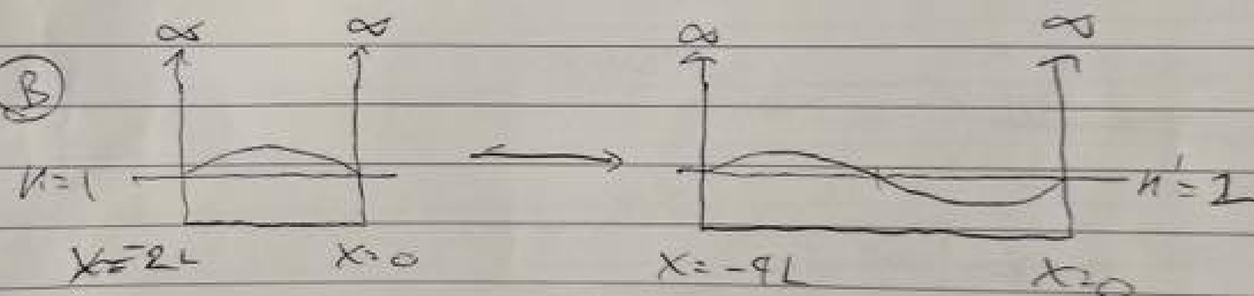

$$\psi_{n=1} = \sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L} \qquad \psi_{n=1} = \sum_{n'=1}^{\infty} b_{n'}\varphi_{n'} \qquad \varphi_{n'=2} = \sqrt{\frac{2}{4L}}\sin\frac{2\pi x}{4L}$$

$$\Rightarrow b_{n'=1} = \int_{-2L}^{0} dL\, \psi_{n=1}\,\varphi_{n'=2} =$$

$$= \sqrt{\frac{1}{L}}\sqrt{\frac{1}{2L}}\underbrace{\int_{-2L}^{0}\sin\frac{\pi x}{2L}\sin\frac{2\pi x}{4L}\,dx}_{2L/2}$$

$$= \frac{1}{\sqrt{2}}$$

$$\text{Prob}_{n=1\to n'=2} = |b_{n'=2}|^2 = \frac{1}{2}$$

$\boxed{3}$

(A) $\psi_a(x) = A e^{-(x-x_0)^2/2\sigma^2}$

$\hat{x}\psi_a = x\psi_a \neq \text{const.}\ \psi_a$

$\hat{p}\psi_a = -i\hbar\frac{\partial}{\partial x}\psi_a = -i\hbar\left(\frac{2(x-x_0)}{2\sigma^2}\right)\psi_a \neq \text{const.}\ \psi_a$

$\Rightarrow$ $\psi_a(x)$ is not an eigenstate of either $\hat{x}$ or $\hat{p}$

$\therefore$ Neither position nor momentum is well defined

$$\langle x\rangle = \int_{-\infty}^{\infty}\psi_a^* x\,\psi_a\,dx$$

$$= A^2\int_{-\infty}^{\infty} x\,e^{-(x-x_0)^2/\sigma^2}\,dx$$

$x' = x - x_0$

$dx' = dx$

$$= A^2\left[\underbrace{\int_{-\infty}^{\infty}dx'\,e^{-x'^2/\sigma^2}\,x'}_{=0} + x_0\underbrace{\int_{-\infty}^{\infty}dx'\,e^{-x'^2/\sigma^2}}_{=A^{-2}}\right]$$

$\because$ integrand is odd about $x'=0$

$\because$ A is normalizing constant

i.e. $1 = A^2\int_{-\infty}^{\infty}dx'\,e^{-x'^2/\sigma^2}$

$\therefore$ $\langle x\rangle = x_0$

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(B) $\psi_b(x) = B e^{-\frac{x^2}{2\varepsilon^2} - i \cancel{P_0 x/\hbar}}$

$$\frac{\partial \psi_b}{\partial x} = \psi_b \left(-\frac{2x}{2\varepsilon^2} - i\frac{P_0}{\hbar} \right)$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}, \quad 1 = \int_{-\infty}^{\infty} \psi_b^* \psi \, dx \Rightarrow B$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi_b^* (\hat{p}\, \psi_b)\, dx$$

$$= (-i\hbar) \left[- \underbrace{\int_{-\infty}^{\infty} \frac{x}{\varepsilon^2} \psi_b^* \psi_b \, dx}_{=0} - \frac{iP_0}{\hbar} \underbrace{\int \psi_b^* \psi_b \, d_b}_{1} \right]$$

∵ integrand is odd about $x=0$

$$\therefore \quad \langle p \rangle = -P_0$$

$t>0$: wave packet moves from $x=0$ with velocity $-\frac{P_0}{m}$ and spreads out ~~(or momentum ... $\hbar k_0$)~~ as it moves, $\therefore\ E = \frac{p^2}{2m}$

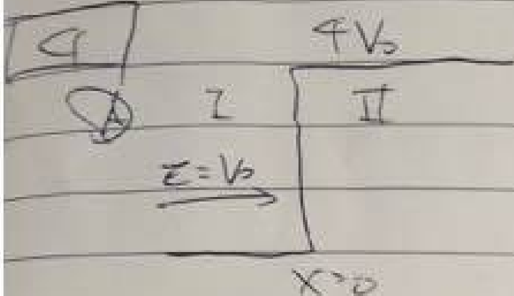
$\Downarrow$ ~~for free~~ electron

$\omega \propto k^2$

i.e. component plane waves disperse

$\boxed{9}$

a)



$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II}(x) = \underbrace{Ce^{\alpha x}}_{=0} + De^{-\alpha x}$$

to prevent ψ_{II} from blowing up as $x \to \infty$

$$k = \frac{\sqrt{2mV_0}}{\hbar}$$

$$\alpha = \frac{\sqrt{2m(3V_0)}}{\hbar}$$

B.C. $\begin{cases} \psi_I(0) = \psi_{II}(0) & : \quad A + B = D \\ \psi_I'(0) = \psi_{II}'(0) & : \quad ik(A-B) = -D\alpha \end{cases}$

$$2A = D\left(1 - \frac{\alpha}{ik}\right)$$
$$2B = D\left(1 + \frac{\alpha}{ik}\right)$$

$$R = \frac{|B|^2 \frac{\hbar k}{m}}{|A|^2 \frac{\hbar k}{m}} = \frac{|1 + i\alpha/k|^2}{|1 - i\alpha/k|^2} = 1$$

$$j(x) = -\frac{i\hbar}{2m}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$D^*D = DD^*$

Region II:

$$j(x) = -\frac{i\hbar}{2m}\left[D^*D(-\alpha)e^{-2\alpha x} - DD^*(-\alpha)e^{-2\alpha x}\right] = 0$$

Region I: $\psi = Ae^{i\theta} + Be^{-i\theta}$ $\quad \theta = kx$
$\psi^* = A^*e^{-i\theta} + B^*e^{i\theta}$

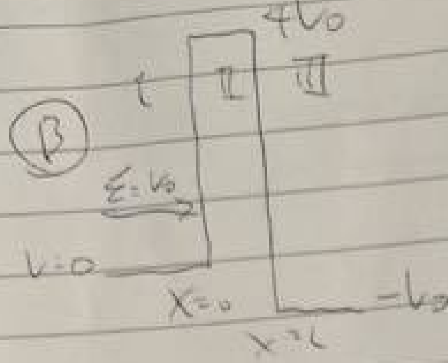
$$\frac{\partial \psi}{\partial x} = ik\left[Ae^{i\theta} - Be^{-i\theta}\right]$$
$$\frac{\partial \psi^*}{\partial x} = -ik\left[A^*e^{-i\theta} - B^*e^{i\theta}\right]$$

$$j(x) = -\frac{i\hbar}{2m}\Big[ik\left[(A^*A - B^*B) + A^*Be^{-2i\theta} - B^*Ae^{2i\theta}\right] + ik\left[(AA^* - BB^*) + AB^*e^{2i\theta} - BA^*e^{-2i\theta}\right]\Big]$$

$= 0$

note: $A^*A = B^*B$ $\quad$ same for complex conjugates
$A^*B = BA^*$

$\boxed{4}$ (B)

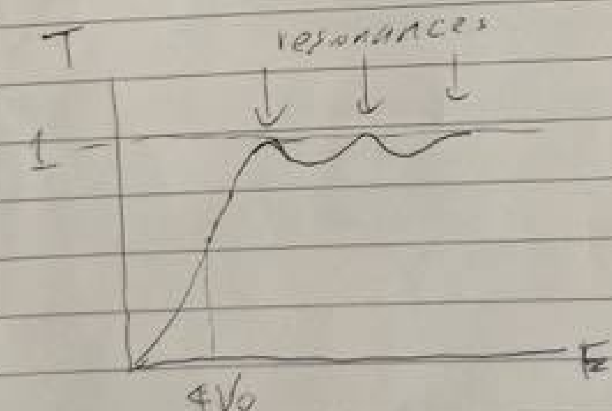


$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}$$
$$\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$$
$$\psi_{III}(x) = Fe^{ik_3x}$$
$$k_1 = \frac{\sqrt{2mV_0}}{\hbar}$$
$$\alpha = \frac{\sqrt{2m(3V_0)}}{\hbar}$$
$$k_3 = \frac{\sqrt{2m(2V_0)}}{\hbar}$$

$$\text{B.C.s}\begin{cases} \psi_I(0) = \psi_{II}(0) \\ \psi_I'(0) = \psi_{II}'(0) \\ \psi_{II}(L) = \psi_{III}(L) \\ \psi_{II}'(L) = \psi_{III}'(L) \end{cases}$$

$$T = \frac{|F|^2 \frac{\hbar k_3}{m}}{|A|^2 \frac{\hbar k_1}{m}}$$

Note: $\frac{k_3}{k_1} = \sqrt{2}$



$T \to 1$ as $E \to \infty$

$T(E)$ is Monotonically increasing for $E < 4V_0$

$T(E)$ is Non monotonic above $4V_0$ (Note resonances)