MAT195S CALCULUS II

Midterm Test #1

13 February 2017 9:10 am - 10:55 am

Closed Book, no aid sheets, no calculators

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Given Name: Solutions

Student #: ______

FOR MARKER USE ONLY		
Question	Marks	Earned
1	13	
2	9	
3	7	
4	10	
5	8	
6	12	
7	10	
8	6	
TOTAL	75	/70

Tutorial Section: ______

TA Name: ______

1) Evaluate the following integrals.

a) $\int \sqrt{1-4x^2}\,dx$

b) $\int (2x^2+1)e^{x^2}\,dx$

c) $\int \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx$

(13 marks)

let $2x = \sin\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = \frac{1}{2}\cos\theta\, d\theta \qquad \sqrt{1-4x^2} = \cos\theta$

a) $\int \sqrt{1-4x^2}\,dx$

$= \int \cos\theta\left(\frac{1}{2}\cos\theta\, d\theta\right) = \int \frac{1}{2}\cos^2\theta\, d\theta = \int \frac{1}{4}(1+\cos 2\theta)\,d\theta$

$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{4}(\theta + \sin\theta\cos\theta) + C$

$= \frac{1}{4}\left(\sin^{-1} 2x + 2x\sqrt{1-4x^2}\right) + C$

let $u = x \quad dv = 2xe^{x^2}dx$.

$du = dx \quad v = e^{x^2}$

b) $\int (2x^2+1)e^{x^2}dx = \int 2x^2e^{x^2}dx + \int e^{x^2}dx$

$= xe^{x^2} - \int e^{x^2}dx + \int e^{x^2}dx = xe^{x^2} + C$

c) $\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow x(3-5x) = A(x-1)^2 + B(x-1)(3x-1) + C(3x-1)$

let $x=1 \Rightarrow -2 = C(2) \Rightarrow C = -1$

$x = \frac{1}{3} \Rightarrow \frac{1}{3}\left(3-\frac{5}{3}\right) = A\left(\frac{1}{3}-1\right)^2 \Rightarrow \frac{4}{9} = A\cdot\frac{4}{9} \Rightarrow A = 1$

$x = 0 \Rightarrow 0 = A + B - C \Rightarrow 0 = 1 + B + 1 \Rightarrow B = -2$

$\therefore \int \frac{x(3-5x)}{(3x-1)(x-1)^2}dx = \int \frac{dx}{3x-1} - 2\int\frac{dx}{x-1} - \int\frac{dx}{(x-1)^2}$

$= \frac{1}{3}\ln|3x-1| - 2\ln|x-1| + \frac{1}{x-1} + C$

2) a) Show that $\int_0^\infty x^2 e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$

(5 marks)

let $u = x$ $\quad dv = xe^{-x^2}dx$

$du = dx$ $\quad v = -\frac{1}{2}e^{-x^2}$

$$\therefore \int_0^\infty x^2 e^{-x^2} = \left[-\frac{x}{2}e^{-x^2}\right]_0^\infty - \int_0^\infty -\frac{1}{2}e^{-x^2}dx$$

$$= \underbrace{\lim_{t\to\infty}\left(-\frac{t}{2}e^{-t^2}\right)}_{} + 0 + \frac{1}{2}\int_0^\infty e^{-x^2} = \frac{1}{2}\int_0^\infty e^{-x^2}$$

$$\lim_{t\to 0}\frac{t}{e^{t^2}} \overset{*}{=} \lim_{t\to 0}\frac{1}{2te^{t^2}} = 0$$

b) Use the comparison test to determine if the following integral converges:

$$\int_1^\infty \frac{x}{\sqrt{1+x^5}}\,dx$$

(4 marks)

$$\frac{x}{\sqrt{1+x^5}} < \frac{x}{\sqrt{x^5}} = x^{-3/2}$$

$\int_1^\infty \frac{dx}{x^{3/2}}$ converges $\left(\int_1^\infty \frac{dx}{x^p} \text{ converges for } p>1\right)$

$\therefore \int_1^\infty \frac{x}{\sqrt{1+x^5}}\,dx$ converges

3) For the function $f(x)=\frac{1}{4}e^{x}+e^{-x}$, show that the arclength on any interval has the same value as the area under the curve.

(7 marks)

$$\left.\begin{aligned} s &= \int_a^b \sqrt{1+(f'(x))^2}\,dx \\ A &= \int_a^b f(x)\,dx \end{aligned}\right\} \quad s = A$$

For this to be true for all intervals (a, b), the integrands must be equal:

$$f(x) = \sqrt{1+(f'(x))^2}$$

$$f(x) = \frac{1}{4}e^{x} + e^{-x} \qquad \therefore \; f'(x) = \frac{1}{4}e^{x} - e^{-x}$$

$$\begin{aligned} 1+(f'(x))^2 &= 1 + \frac{1}{16}e^{2x} - \frac{1}{2} + e^{-2x} \\ &= \frac{1}{16}e^{2x} + \frac{1}{2} + e^{-2x} \\ &= \left(\frac{1}{4}e^{x} + e^{-x}\right)^2 = (f(x))^2 \end{aligned}$$

$$\therefore \sqrt{1+(f'(x))^2} = f(x)$$

4) Find the area of the surface generated by revolving <u>about the x-axis</u> the curve: $6xy = y^4 + 3$; $y \in (1,3)$.

(10 marks)

We use the parameterization: $y = t \Rightarrow \frac{dy}{dt} = 1$

$x = x(t)$

$\therefore 6xt = t^4 + 3 \Rightarrow x = \frac{t^4+3}{6t}$

$$\frac{dx}{dt} = \frac{1}{6}\left[\frac{4t^3}{t} - \frac{t^4+3}{t^2}\right] = \frac{t^4-1}{2t^2}$$

$$A = \int 2\pi y \sqrt{(x')^2 + (y')^2}\, dt$$

$$= \int_1^3 2\pi t \sqrt{\left(\frac{t^4-1}{2t^2}\right)^2 + 1}\, dt$$

$$= \int_1^3 \frac{2\pi t}{2t^2} \sqrt{t^8 - 2t^4 + 1 + 4t^4}\, dt$$

$$= \int_1^3 \frac{\pi}{t} \sqrt{t^8 + 4t^4 + 1}\, dt = \int_1^3 \frac{\pi}{t}(t^4+1)\, dt$$

$$= \pi\left[\frac{t^4}{4} + \ln t\right]_1^3 = \pi\left(\frac{81}{4} - \frac{1}{4} + \ln 3\right)$$

$$= \pi(20 + \ln 3)$$

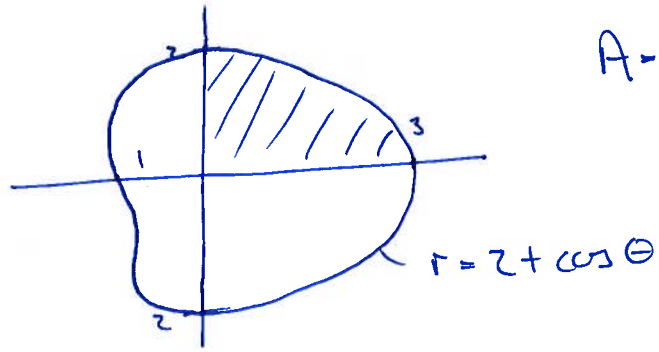
5) Sketch the region indicated, and find an integral representing the area of the region. Do not evaluate the integrals.

a) The region inside the curve $r = 2 + \cos\theta$, that lies in the first quadrant.

b) The region that lies inside the cardioid $r = 1 + \cos\theta$ but outside the circle $r = 3\cos\theta$.

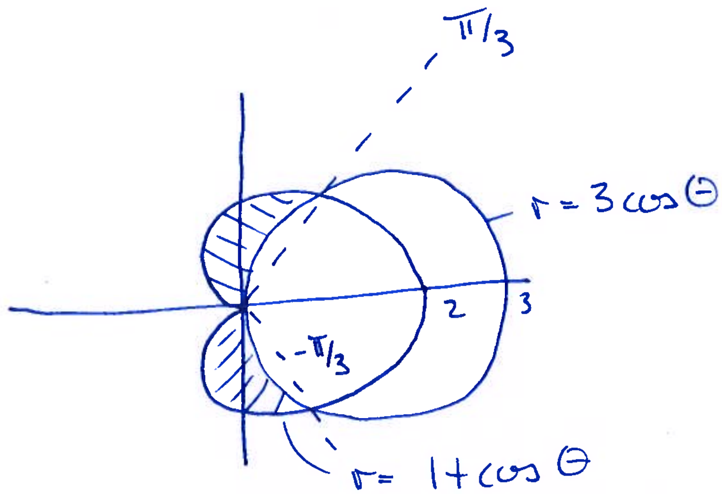
(8 marks)

a)



$$A = \int_0^{\pi/2} \frac{1}{2}(2+\cos\theta)^2\, d\theta$$

b)



Intersection: $1 + \cos\theta = 3\cos\theta$

$1 = 2\cos\theta$

$\frac{1}{2} = \cos\theta \implies \theta = \pm\frac{\pi}{3}$

$$\frac{1}{2}A = \int_{\pi/3}^{\pi/2} \frac{1}{2}\left[(1+\cos\theta)^2 - (3\cos\theta)^2\right] d\theta + \int_{\pi/2}^{\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta$$

6) Determine whether the sequence converges or diverges. If it converges, find the limit:

(i) $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$ (ii) $a_n = \dfrac{3^{n+2}}{5^n}$

(iii) $a_k = \dfrac{\ln k}{\ln 2k}$ (iv) $a_k = \ln(k+1) - \ln k$

(12 marks)

i) $a_n = \dfrac{\sin 2n}{1+\sqrt{n}} \Rightarrow |a_n| \le \dfrac{1}{1+\sqrt{n}} \xrightarrow{n\to\infty} 0$ $\therefore a_n \to 0$ by pinching th'm

ii) $a_n = \dfrac{3^{n+2}}{5^n} = 9\cdot\left(\frac{3}{5}\right)^n$

$\lim\limits_{n\to\infty} a_n = 9 \cdot \lim\limits_{n\to\infty} \left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0$ $\left(x^n \to 0 \text{ for } |x| < 1 \right)$

iii) $a_k = \dfrac{\ln k}{\ln 2k} = \dfrac{\ln k}{\ln 2 + \ln k} = \dfrac{1}{\frac{\ln 2}{\ln k} + 1}$

$\lim\limits_{k\to\infty} \dfrac{\ln 2}{\ln k} = 0$ $\therefore\ a_k \xrightarrow{k\to\infty} 1$

iv) $a_k = \ln(k+1) - \ln k = \ln\left(\frac{k+1}{k}\right)$

now $\lim\limits_{k\to\infty} \dfrac{k+1}{k} = 1$ and $\ln x$ is continuous at $x=1$

$\therefore \lim\limits_{k\to\infty} \ln\left(\frac{k+1}{k}\right) = \ln 1 = 0$

7) The Completeness Axiom states that any non-empty set of real numbers that is bounded below has a greatest lower bound. Given this axiom, prove that a monotonic decreasing sequence that is bounded below converges.

(10 marks)

1) Given $\{a_n\}$ is a monotonic decreasing sequence and is bounded, the Completeness Axiom guarantees that the set of numbers given by $S' = \{a_n \mid n \geq 1\}$ will have a greatest lower bound, L.

2) Now, $L+\epsilon$ cannot be a lower bound for S', since L is the greatest lower bound:

$$\therefore a_N < L+\epsilon \quad \text{for some } N$$

3) But since the sequence is decreasing, $a_n \leq a_N$ for all $n > N$:

$$\therefore a_n < L+\epsilon \quad \text{for } n > N$$

$$\text{or} \quad 0 < a_n - L < \epsilon \quad \text{since } a_n \geq L$$

4) Thus $|L - a_n| < \epsilon$ for $n > N$

$$\therefore \lim_{n\to\infty} a_n = L$$

8) Determine whether the following series converge or diverge. Show your work.

a) $\sum_{n=1}^{\infty} \frac{1}{k^{1/n}}$, for k ≥ 1 b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(6 marks)

a) for $k=1$, $k^{1/n} = 1 \Rightarrow \sum_{n=1}^{\infty} 1$ diverges

for $k>1$, $k^{1/(n+1)} < k^{1/n}$ (eg $\sqrt[3]{2} < \sqrt{2}$) $\therefore \frac{1}{k^{1/(n+1)}} > \frac{1}{k^{1/n}}$

$\therefore a_{n+1} \not< a_n$ $\therefore$ diverges

by the test for divergence

b) $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots (n-1) \cdot n}{n \cdot n \cdot n \cdot n \cdot n \cdots n \cdot n} < \frac{120}{n^5}$

$\Rightarrow \sum \frac{120}{n^5}$ converges (p-series, $p>1$)

$\therefore \sum \frac{n!}{n^n}$ converges by comparison test.