University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test
9:00 – 10:45, 17 October 2018
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

Solutions
JWD

1. Calculate $f'(x)$ for the following functions. Identify the differentiation theorems used.

a) $f(x) = x^{3/2}$ b) $f(x) = sin(2x)$ c) $f(x) = \frac{\sqrt{x}}{2+x}$

d) $f(x) = \frac{1}{\sqrt{x}}$ e) $f(x) = tan^2(sinx)$

a) $f(x) = x^{3/2} \Rightarrow f'(x) = \frac{3}{2} x^{1/2}$ (power DT)

b) $f(x) = \sin(2x) \Rightarrow f'(x) = \cos(2x) \cdot 2$ (chain rule)

c) $f(x) = \frac{\sqrt{x}}{2+x} \Rightarrow f'(x) = \frac{\frac{1}{2}x^{-1/2}(2+x) - x^{1/2}}{(2+x)^2}$ (quotient DT, power DT)

$= \frac{x^{-1/2} + \frac{1}{2}x^{1/2} - x^{1/2}}{(2+x)^2}$

$= \frac{2-x}{2\sqrt{x}(2+x)^2}$

d) $f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}x^{-3/2}$ (power DT)

e) $f(x) = \tan^2(\sin x) \Rightarrow f'(x) = 2\tan(\sin x) \cdot \sec^2(\sin x) \cdot \cos x$

(power DT, chain rule)

2. Prove using $\varepsilon - \delta$ methods:

a) $\lim_{x \to 0} x^3 = 0$

b) $\lim_{x \to 2} \frac{4x+1}{(x+1)^2} = 1$

a) Prove $\lim_{x \to 0} x^3 = 0$

1) Find $\delta > 0$ st for $0 < |x-0| < \delta$, $|x^3 - 0| < \epsilon$

$\Rightarrow |x^3 - 0| = |x|^3 < \epsilon \Rightarrow |x| < \epsilon^{1/3} \Rightarrow$ choose $\delta = \epsilon^{1/3}$

2) Proof: given $\epsilon > 0$, let $\delta = \epsilon^{1/3}$

then $|x^3 - 0| = |x|^3 < (\delta^{1/3})^3 = \epsilon$ for $0 < |x| < \delta = \epsilon^{1/3}$

$\therefore$ by the definition of a limit, $\lim_{x \to 0} x^3 = 0$

b) Prove $\lim_{x \to 2} \frac{4x+1}{(x+1)^2} = 1$

1) Find $\delta > 0$ st. for $|x-2| < \delta$, $\left| \frac{4x+1}{(x+1)^2} - 1 \right| < \epsilon$

$\left| \frac{4x+1}{(x+1)^2} - \frac{(x+1)^2}{(x+1)^2} \right| = \left| \frac{4x+1 - x^2 - 2x - 1}{(x+1)^2} \right| = \left| \frac{-x(x-2)}{(x+1)^2} \right| = \left| \frac{x(x-2)}{(x+1)^2} \right|$

now, take $\delta < 1$ $\therefore |x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$

$\Rightarrow x < 3 \quad \therefore \left| \frac{x(x-2)}{(x+1)^2} \right| < \left| \frac{3(x-2)}{(x+1)^2} \right|$

$\Rightarrow x+1 > 2 \therefore (x+1)^2 > 4 \quad \therefore \frac{1}{(x+1)^2} < \frac{1}{4} \quad \therefore \left| \frac{3(x-2)}{(x+1)^2} \right| < \frac{3}{4} |x-2|$

$\therefore |x-2| < \frac{4}{3} \epsilon \Rightarrow$ choose $\delta^* = \frac{4}{3} \epsilon$

$\therefore$ choose $\delta = \min \{1, \frac{4}{3} \epsilon\}$

2) Proof: given $\epsilon > 0$, let $\delta = \min \{1, \frac{4}{3} \epsilon\}$

then $\left| \frac{4x+1}{(x+1)^2} - 1 \right| = \left| \frac{x(x-2)}{(x+1)^2} \right| < \left| \frac{3 \cdot \frac{4}{3} \epsilon}{4} \right| = \epsilon$

$\therefore \left| \frac{4x+1}{(x+1)^2} - 1 \right| < \epsilon$ whenever $0 < |x-2| < \delta = \min \{1, \frac{4}{3} \epsilon\}$

$\therefore$ by the definition of a limit, $\lim_{x \to 2} \frac{4x+1}{(x+1)^2} = 1$

3. Sketch the curve for $y = \frac{x}{x-1}$, indicating all significant features.

$y = \frac{x}{x-1}$ $\Rightarrow$ Domain $x \neq 1$, Range $y \neq 1$

$\Rightarrow$ Intercept: $y(0) = 0$

$\Rightarrow$ No symmetry

$$\left.\begin{array}{l} \lim_{x \to 1^+} \frac{x}{x-1} = +\infty \\ \lim_{x \to 1^-} \frac{x}{x-1} = -\infty \end{array}\right\} \quad x = 1 \text{ is a vertical asymptote}$$

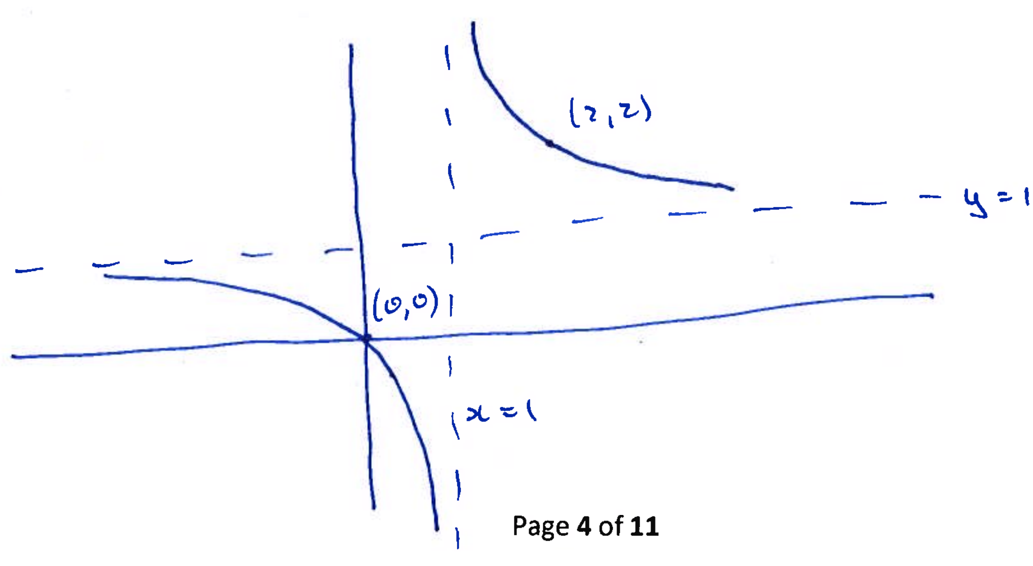
$$\left.\begin{array}{l} \lim_{x \to \infty} \frac{x}{x-1} = 1 \\ \lim_{x \to -\infty} \frac{x}{x-1} = 1 \end{array}\right\} \quad y = 1 \text{ is a horizontal asymptote}$$

$y' = \frac{(x-1) - x}{(x-1)^2} = \frac{-1}{(x-1)^2} < 0, \ x \neq 1 \ \therefore$ decreasing

$y' \neq 0$ for $x \neq 1$; y' DNE at $x = 1$, but $x = 1$ is not part of domain

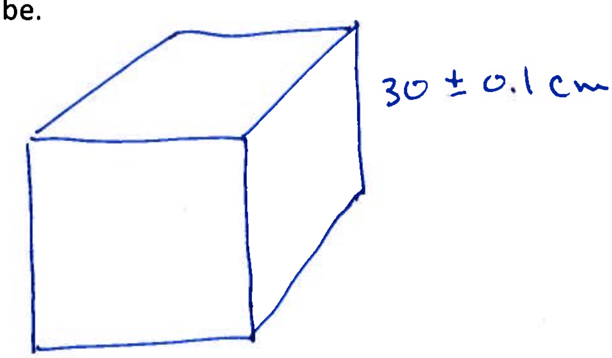
$\therefore$ no critical points, no extreme values

$$y'' = \frac{2}{(x-1)^3} \quad \begin{array}{lll} < 0 & x < 1 & \text{concave down} \\ > 0 & x > 1 & \text{concave up} \end{array}$$



4. The edge of a cube was found to be 30 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum possible error and relative error in computing:

a) the volume of the cube,
b) the surface area of the cube.



a) Volume: $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

$$\Rightarrow V \simeq V_0 + \frac{dV}{dx} \cdot \Delta x$$

$$= x_0^3 + 3x_0^2 \cdot (x - x_0)$$

$$= 30^3 + 3 \cdot 30^2 \cdot (0.1)$$

absolute error = $270\ cm^3$

relative error = $\frac{0.3}{30} = \frac{1}{100}$

$\left(\text{actual error} = (30.1)^3 - 30^3 = 270.9\ cm^3 \right)$

b) Area: $A = 6x^2$

$$\frac{dA}{dx} = 12x$$

$$\Rightarrow A \simeq A_0 + \frac{dA}{dx} \cdot \Delta x$$

$$= 6x_0^2 + 12x_0 (x - x_0)$$

$$= 6 \cdot 30^2 + 12 \cdot 30 \cdot (0.1)$$

absolute error = $36\ cm^2$

relative error = $\frac{0.2}{30} = \frac{1}{150}$

$\left(\text{actual error} = 6(30.1^2 - 30^2) = 36.06\ cm^2 \right)$

5. Sketch a graph of the region defined by: $x^2 + y^2 \leq |x| + |y|$.
Hint: use symmetry to avoid repetitive calculations.

Consider the first quadrant: $x, y > 0$

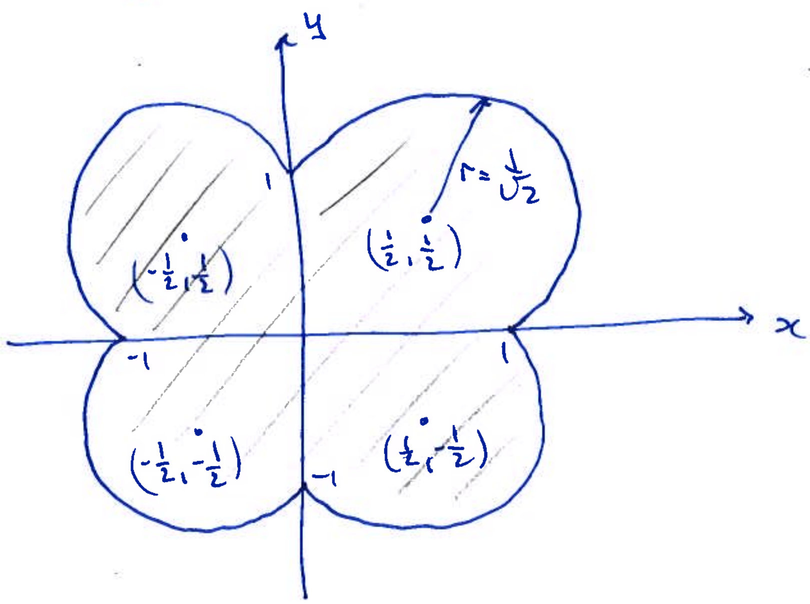
$\Rightarrow \quad x^2 + y^2 \leq x + y \quad \Rightarrow \quad x^2 - x + y^2 - y \leq 0$

$$\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} + \left(y - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} \leq 0$$

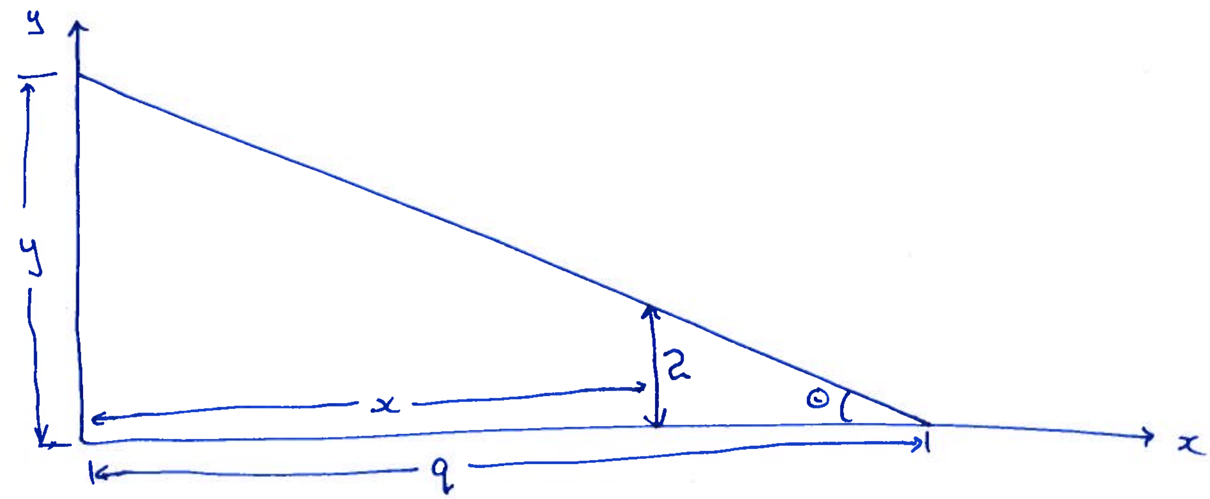
$$\left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 \leq \tfrac{1}{2}$$

$\Rightarrow$ this gives the region inside the circle, radius $\frac{1}{\sqrt{2}}$, centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$

$\Rightarrow$ replacing x with $-x$, or y with $-y$ does not change the original equation, thus there is symmetry about both axes:



6. A balloon carrying a light is released 3 m above the ground and rises upward at a constant speed of 10 m/s. The person (2 m tall) who releases the balloon runs away from directly under the balloon at a constant speed of 5 m/s. Two seconds later, how fast is the tip of the person's shadow, cast by the light on the balloon, moving?



$y = 3 + 10t \qquad \frac{dy}{dt} = 10$

$x = 5t \qquad \frac{dx}{dt} = 5$

Similar triangles: $\frac{q}{y} = \frac{q-x}{2} \qquad \therefore 2q = qy - xy$

$q(2-y) = -xy \quad \Rightarrow \quad q = \frac{xy}{y-2} = xy(y-2)^{-1}$

$$\frac{dq}{dt} = \frac{dx}{dt}\, y(y-2)^{-1} + \frac{dy}{dt}\, x(y-2)^{-1} - xy(y-2)^{-2} \cdot \frac{dy}{dt}$$

At $t = 2$, $x = 10$, $y = 23$

$$\therefore \frac{dq}{dt} = \frac{5 \cdot 23}{21} + \frac{10 \cdot 10}{21} - \frac{10 \cdot 23 \cdot 10}{(21)^2} = \frac{215}{21} - \frac{2300}{(21)^2} = \frac{2215}{441}$$

$[= 5.023 \text{ m/s}]$

7. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & if\ x < 2 \\ ax^2 - bx + 3 & if\ 2 \le x < 3 \\ 2x - a + b & if\ x \ge 3 \end{cases}$$

$$\left.\begin{aligned} &\lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} x+2 = 4 \\ &\lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3 \end{aligned}\right\} \quad \begin{aligned} &4a - 2b + 3 = 4 \\ &\Rightarrow 4a - 2b = 1 \end{aligned}$$

$$\left.\begin{aligned} &\lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3 \\ &\lim_{x \to 3^+} 2x - a + b = 6 - a + b \end{aligned}\right\} \quad \begin{aligned} &9a - 3b + 3 = 6 - a + b \\ &\Rightarrow 4b = 10a - 3 \end{aligned}$$

$$4a - 2b = 1 \Rightarrow 8a - 4b = 2 \Rightarrow 8a - (10a - 3) = 2$$

$$\Rightarrow -2a = -1$$

$$\text{or } a = 1/2$$

$$\therefore b = \frac{5-3}{4} = 1/2$$

8. Evaluate the following limits, if they exist (do not use l'Hospital's rule): Indicate the limit laws or theorems used.

a) $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$

b) $\lim_{y\to 0} \left(\frac{1}{y\sqrt{y+1}} - \frac{1}{y}\right)$

c) $\lim_{x\to -1} \frac{x^2-x-2}{|x+1|}$

a) $\lim_{x\to 0} \frac{(2+x)^3-8}{x} = \lim_{x\to 0} \frac{(8+12x+6x^2+x^3)-8}{x}$ sum, product, power laws

$= \lim_{x\to 0} 12+6x+x^2 = 12$

b) $\lim_{y\to 0} \left(\frac{1}{y\sqrt{y+1}} - \frac{1}{y}\right) = \lim_{y\to 0}\left(\frac{1-\sqrt{y+1}}{y\sqrt{y+1}} \times \frac{1+\sqrt{y+1}}{1+\sqrt{y+1}}\right)$ sum, product, quotient, square root laws.

$= \lim_{y\to 0}\left(\frac{1-y-1}{y\sqrt{y+1}(1+\sqrt{y+1})}\right)$

$= \lim_{y\to 0}\left(\frac{-1}{\sqrt{y+1}(1+\sqrt{y+1})}\right) = -\frac{1}{2}$

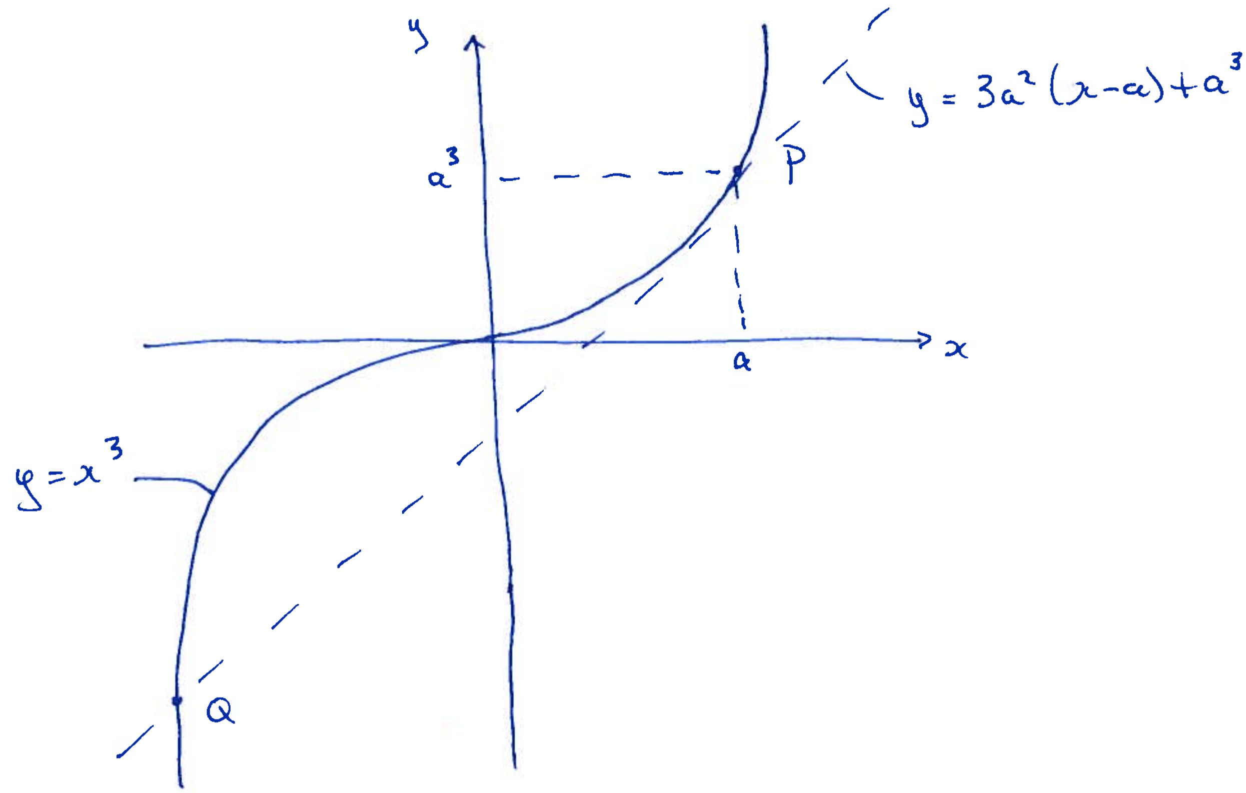
c) $\lim_{x\to -1} \frac{x^2-x-2}{|x+1|} = \lim_{x\to -1} \frac{(x+1)(x-2)}{|x+1|}$

$\Rightarrow \lim_{x\to -1^+} \frac{(x+1)(x-2)}{+(x+1)} = \lim_{x\to -1^+} (x-2) = -3$

$\Rightarrow \lim_{x\to -1^-} \frac{(x+1)(x-2)}{-(x+1)} = \lim_{x\to -1^-} -(x-2) = 3$

$\lim_{x\to -1} \frac{x^2-x-2}{|x+1|}$ DNE

9. Let P be a point on the curve $y = x^3$ and suppose that the tangent line at P intersects the curve again at Q. What is the slope of the curve at Q?



slope of tangent line at P: $y' = 3x^2$; $y'(a) = 3a^2$

equ'n of tangent line: $\dfrac{(y-a^3)}{x-a} = 3a^2$

to find Q, solve $3a^2(x-a) + a^3 = x^3$

$3a^2x - 2a^3 = x^3 \Rightarrow x^3 - 3a^2x + 2a^3 = 0$

$\Rightarrow$ we know that $x = a$ is a sol'n: $(x-a)(x^2+ax-2a^2) = 0$

$(x-a)(x-a)(x+2a) = 0$

$\therefore$ the other point of intersection is $x = -2a$

$y'(-2a) = 3(-2a)^2 = 12a^2 = 4y'(a)$

10. Consider the function: $f(x) = x(x-3)^{2/3}$
   a) Determine the domain of f and find the asymptotes, if any.
   b) Find the intervals in which f increases or decreases. Find the extreme values, if any.
   c) Find the intervals in which the graph of f is concave up or down. Find the inflection points, if any.
   d) Sketch the graph of f, identifying the important features.

a) Domain : $x \in \mathbb{R}$; no horizontal or vertical asymptotes

b) $f(x) = x(x-3)^{2/3} \Rightarrow f(0) = 0$; $f(x) = 0 \Rightarrow x = 0,\ x = 3$

$\therefore$ intercepts $(0,0)$, $(3,0)$

$$f'(x) = (x-3)^{2/3} + x\left(\tfrac{2}{3}\right)(x-3)^{-1/3} = \frac{x-3+\frac{2}{3}x}{(x-3)^{1/3}} = \frac{5x-9}{3(x-3)^{1/3}}$$

$f'(x) = 0 \Rightarrow x = \frac{9}{5}$ $\qquad f'(x) > 0 \quad x \in (-\infty, 9/5)$ incr

$f'(x)$ DNE $\Rightarrow x = 3$ $\qquad f'(x) < 0 \quad x \in (9/5, 3)$ decr

$\qquad f'(x) > 0 \quad x \in (3, \infty)$ incr

$$\left.\begin{aligned}\lim_{x\to 3^-} \frac{5x-9}{3(x-3)^{1/3}} &= -\infty \\ \lim_{x\to 3^+} \frac{5x-9}{3(x-3)^{1/3}} &= +\infty\end{aligned}\right\} (3,0) \text{ is a cusp}$$

c) $$f''(x) = \frac{1}{3}5(x-3)^{-1/3} + \frac{5x-9}{3}\left(-\frac{1}{3}\right)(x-3)^{-4/3} = \frac{1}{9}\left[\frac{3\cdot 5(x-3)-(5x-9)}{(x-3)^{4/3}}\right] = \frac{1}{9}\left(\frac{10x-36}{(x-3)^{4/3}}\right)$$

critical points: $\left(\frac{9}{5}, \frac{9}{5}\left(\frac{6}{5}\right)^{2/3}\right)$: $f''\left(\frac{9}{5}\right) = -ve$ $\therefore$ local max

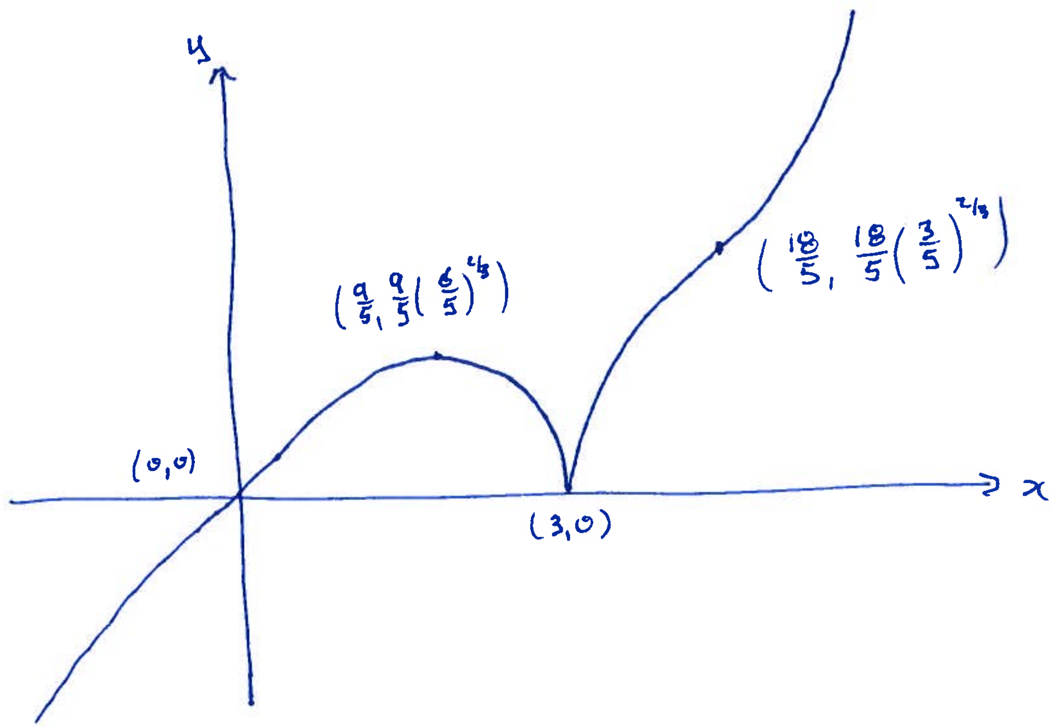
$(3,0)$: $f''(3)$ DNE

also $f''(x) = 0 \Rightarrow x = \frac{18}{5} \rightarrow$ pt. of inflection

$f'' < 0$ on $(-\infty, 3)$ $\therefore$ concave down

$f'' < 0$ on $\left(3, \frac{18}{5}\right)$ $\therefore$ concave down

$f'' > 0$ on $\left(\frac{18}{5}, \infty\right)$ $\therefore$ concave up



y
x
(0,0)
(3,0)
$\left(\frac{9}{5}, \frac{9}{5}\left(\frac{6}{5}\right)^{2/3}\right)$
$\left(\frac{18}{5}, \frac{18}{5}\left(\frac{3}{5}\right)^{2/3}\right)$