ESC195 - Final Examination
April 2020

In submitting this assessment, I confirm that my conduct during this exam adheres to the Code of Behaviour on Academic Matters. I confirm that I did NOT act in such a way that would constitute cheating, misrepresentation, or unfairness, including but not limited to, using unauthorized aids and assistance, impersonating another person, and committing plagiarism. *I pledge upon my honour that I have not violated the Faculty of Applied Science & Engineerings Honour Code during this assessment.*

The following materials are considered to be acceptable aids during the writing of this exam:

- The Stewart textbook and the student solution manuals
- Any course notes or problem solutions prepared by the student
- Any handouts, past tests or other materials posed on the ESC195 course website

1. Evaluate the integrals:

$$\text{a)} \int xe^{2x}dx \qquad \text{b)} \int \sqrt{3-2x-x^2}dx \qquad \text{c)} \int \frac{1-\sqrt{x}}{1+\sqrt{x}}dx$$

(10 marks)

2. a) Find the length of the curve: $x = e^t - t,\ y = 4e^{t/2}, \quad 0 \le t \le 2$

(4 marks)

b) Find the area of the surface obtained by rotating the curve $y = 2\sqrt{1-x}, \quad x \in [-1, 0]$ about the x-axis. Provide a sketch.

(5 marks)

3. Evaluate the integral

$$I = \int_0^\infty \frac{dx}{(1+x^a)(1+x^2)}$$

where a is a real number.

(10 marks)

4. a) Plot the polar curve $r = \frac{1}{2} - \cos 4\theta$; find and show the locations of maximum and minimum values.

b) Find the area of the region that lies inside a small petal.

c) Find the area of the region that lies inside a large petal.

(11 marks)

5. Test the series for convergence or divergence:

$$\text{a)} \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+5} \qquad \text{b)} \sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n} \qquad \text{c)} \sum_{n=1}^{\infty} \frac{1}{n(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n})}$$

(12 marks)

6. a) Prove the Extended Ratio Test: If $a_n \neq 0$ for $n = 1, 2, \ldots$ and there exists a number r such that $0 < r < 1$ and an integer N such that $|\frac{a_{n+1}}{a_n}| \leq r$ for $n > N$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

   b) Use this test to show that the following series is convergent:

$$1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{12} + \frac{1}{16} + \frac{1}{48} + \cdots + \frac{5 - (-1)^n}{3 \cdot 2^n} + \cdots$$

(10 marks)

7. a) Use direct differentiation to derive the Maclaurin series for $f(x) = 1/(b+x)$, where b is some fixed number with $b > 0$. For what values of x does the series converge?

(8 marks)

   b) Assume that a function f has at least two continuous derivatives on an interval containing a, with $f'(a) = 0$. Use Taylors Theorem to prove the following version of the second derivative test:

      i) If $f''(x) > 0$ on some interval containing a, then f has a local minimum at a.

      ii) If $f''(x) < 0$ on some interval containing a, then f has a local maximum at a.

(5 marks)

8. Find the Fourier series, ie., evaluate the Fourier coefficients, for the function

$$f(x) = \begin{cases} 0 & -\pi \leq t \leq 0 \\ \sin t & 0 < t \leq \pi \end{cases}$$

   Provide a sketch of the function, and a sketch of what you imagine the sum of the first few terms of the series would look like.

(10 marks)

9. Find the unit tangent vector and the principal unit normal vector at a general point on the curve: $\vec{r}(t) = \cos t\,\hat{\imath} + \sin t\,\hat{\jmath} + \ln(\cos t)\,\hat{k}$. Find an equation in x, y, z for the osculating plane at the point corresponding to $t = 0$.

(11 marks)

10. a) Find the directional derivative of $f(x, y, z) = x^2 + yz$ at $(1, -3, 2)$ in the direction of the path: $\vec{r}(t) = t^2\,\hat{\imath} + 3t\,\hat{\jmath} + (1 - t^3)\,\hat{k}$

(4 marks)

b) Show that the curve: $x^2 - y^2 + z^2 = 1 \qquad xy + xz = 2$

is tangent to the surface: $xyz - x^2 - 6y = -6$

at the point $(1, 1, 1)$.

(6 marks)

11. Use the formal definition for the derivative of a multivariable function (the $o(h)$ formulation) to find the gradient of: $f(x, y) = 3x^2 - xy + y$. Show that all remainder terms are $o(h)$.

(10 marks)

12. Find the absolute maximum and minimum values of $f(x, y) = (x + y - 2)^2$ on the set: $D = \{(x, y) | 0 \leq x \leq 3, x \leq y \leq 3\}$. Provide a sketch of the region, and identify and show the locations of all critical points.

(10 marks)

13. Use Lagrange multipliers to find the extreme values of the function $f(x, y, z) = x^2 + y^2 - z$ subject to the constraint: $z = 2x^2y^2 + 1$.

(9 marks)