AER210 VECTOR CALCULUS and FLUID MECHANICS

Midterm Test # 1

Duration: 1 hour, 50 minutes

28 October 2021

Closed Book, no aid sheets, no calculators

Instructor: Prof. Alis Ekmekci

Family Name: Prof Alis Ekmekci

Given Name:

Student #: Solutions

TA Name/Tutorial #:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	17	
2	8	
3	10	
4	13	
5	10	
6	12	
7	12	
8	18	
TOTAL	100	

Note the following integrals may be useful:

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \quad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C; \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$$

$$\oiint_S \vec{F}\cdot\vec{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$$

Name: ____________________

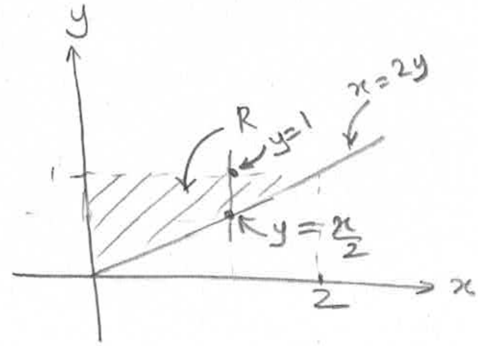
1) a) (4 marks) Evaluate the following double integral:

$$\int_0^1\int_0^{2x}(x+2y)\,dy\,dx = \int_0^1 \left(xy+y^2\right)\Big|_{y=0}^{2x} dx = \int_0^1 \left(2x^2+4x^2\right)dx = \int_0^1 6x^2\,dx = 2x^3\Big|_0^1$$

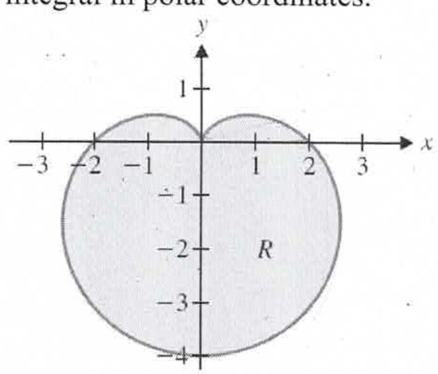
$$= 2$$

b) (5 marks) Sketch the region over which the integration is defined and change the order of integration for the following double integral:

$$\int_0^1\int_0^{2y} f(x,y)\,dx\,dy = \int_{x=0}^{2}\int_{y=\frac{x}{2}}^{1} f(x,y)\,dy\,dx$$



c) (8 marks) Find the area inside the curve defined by $r = 2 - 2sin\theta$ by forming a double integral in polar coordinates.



$$A = \iint_R dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{r=2-2\sin\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Bigg|_{r=0}^{r=2-2\sin\theta} d\theta$$

$$= \int_0^{2\pi} \frac{(2-2\sin\theta)^2}{2}\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4-8\sin\theta+4\sin^2\theta\right)d\theta$$

$$= \frac{1}{2}\left(4\theta + 8\cos\theta + 2\theta - \sin 2\theta\right)\Big|_{\theta=0}^{2\pi}$$

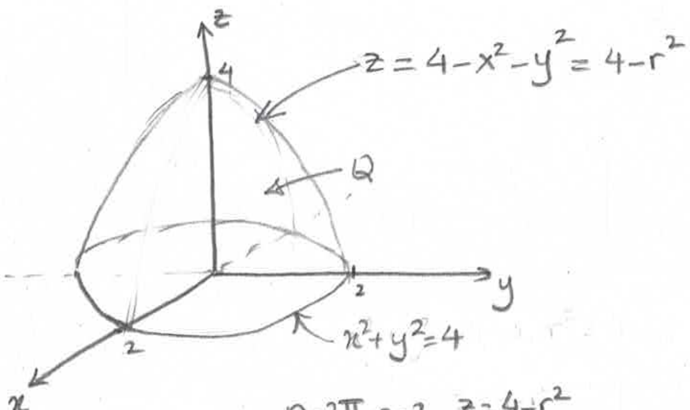
$$= \frac{1}{2}\left(6\theta + 8\cos\theta - \sin 2\theta\right)\Big|_{\theta=0}^{2\pi}$$

$$= \frac{1}{2}\left(12\pi + 8\underbrace{\cos 2\pi}_{1} - \underbrace{\cancel{\sin 4\pi}}_{0} - 8\underbrace{\cos 0}_{1} + \underbrace{\cancel{\sin 0}}_{0}\right)$$

$$= \frac{1}{2}(12\pi) = 6\pi$$

Name: ____________________

2) (8 marks) Use **cylindrical coordinates** to form the appropriate **triple integral** to find the volume of the solid given by the following surfaces: $z = 4 - x^2 - y^2$ and the xy-plane. Make sure to sketch the solid.



$$V = \iiint_Q dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=0}^{z=4-r^2} r \, dz \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} r z \Big|_{z=0}^{z=4-r^2} dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4r - r^3) \, dr \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(2r^2 - \frac{r^4}{4}\right) \Big|_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(8 - \frac{16}{4}\right) d\theta$$

$$= 4 \cdot 2\pi$$

$$= 8\pi$$

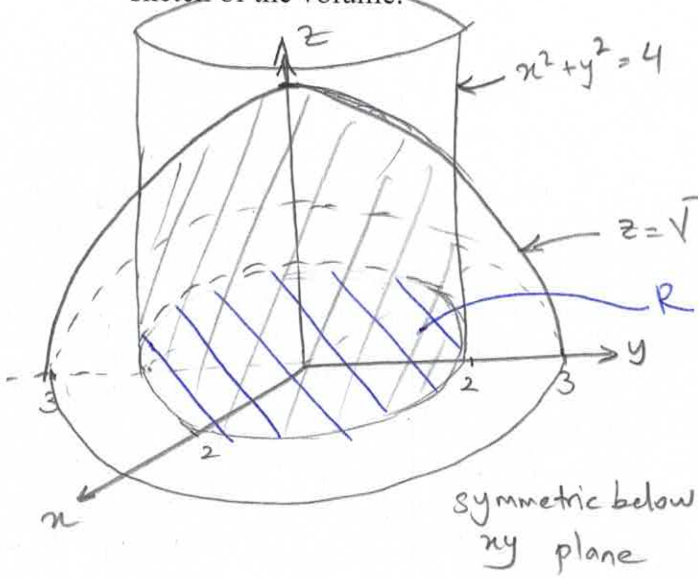
Name: ____________________

3) (10 marks) Use **a double integral in polar coordinates** to find the volume of the solid that lies inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $2x^2 + 2y^2 + z^2 = 18$. Provide a sketch of the volume.

$$\Downarrow$$

$$z^2 = 18 - 2x^2 - 2y^2$$

$$z = \mp\sqrt{18 - 2x^2 - 2y^2}$$



$$V = 2\cdot\iint_R z\,dA = 2\iint_R \sqrt{18-2x^2-2y^2}\,dA = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\sqrt{18-2r^2}\,r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left(-\frac{1}{4}\right)\sqrt{\underbrace{18-2r^2}_{u}}\;\underbrace{(-4)r\,dr}_{du}\,d\theta$$

$18 - 2r^2 = u$

$-4r\,dr = du$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi}\frac{2}{3}(18-2r^2)^{3/2}\Big|_{r=0}^{2}\,d\theta$$

$$= -\frac{1}{2}\cdot\frac{2}{3}\int_{\theta=0}^{2\pi}\left(10^{3/2} - 18^{3/2}\right)d\theta$$

$$= -\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot 2\pi\cdot\left(10^{3/2} - 18^{3/2}\right)$$

$$= \frac{2\pi}{3}\left(18^{3/2} - 10^{3/2}\right)$$

4) (a) (6 marks) Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$

where $\vec{F}(x,y,z) = x\vec{\imath} - z\vec{\jmath} + y\vec{k}$ and C is given by $\vec{r}(t) = 2t\vec{\imath} + 3t\vec{\jmath} - t^2\vec{k}, \quad -1 \le t \le 1$

$$\vec{r}\,'(t) = 2\vec{i} + 3\vec{j} - 2t\vec{k}, \quad -1 \le t \le 1$$

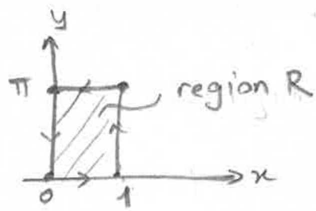
$$F(\vec{r}(t)) = 2t\vec{i} + t^2\vec{j} + 3t\vec{k}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=-1}^{1} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_{t=-1}^{1} \left(2t\vec{i} + t^2\vec{j} + 3t\vec{k}\right)\cdot\left(2\vec{i} + 3\vec{j} - 2t\vec{k}\right)dt$$

$$= \int_{t=-1}^{1} (4t - 3t^2)\,dt = (2t^2 - t^3)\Big|_{t=-1}^{1} = (2-1-2-1) = -2$$

(b) (7 marks) Evaluate the following line integral by Green's theorem where C is the rectangle with vertices $(0,0)$, $(1,0)$, $(1,\pi)$, $(0,\pi)$.

$$\oint_C \underbrace{e^x \cos y}_{P}\, dx + \underbrace{e^x \sin y}_{Q}\, dy$$



$$\oint_C \underbrace{e^x \cos y}_{P}\, dx + \underbrace{e^x \sin y}_{Q}\, dy = \oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_R \left[\frac{\partial}{\partial x}(e^x \sin y) - \frac{\partial}{\partial y}(e^x \cos y)\right] dA$$

$$= \iint_R (e^x \sin y + e^x \sin y)\,dA = 2\int_{y=0}^{\pi}\int_{x=0}^{1} e^x \sin y\,dx\,dy$$

$$= 2\int_0^{\pi} e^x \sin y\Big|_{x=0}^{1} dy = 2\int_0^{\pi} (e \sin y - \sin y)\,dy$$

$$= 2\Big[-e\cdot\cos y + \cos y\Big]_{y=0}^{\pi} = 2\Big[-e\cos\pi + \cos\pi - e\cos 0 - \cos 0$$

$$= 2\big[e - 1 + e - 1\big] = \boxed{4(e-1)}$$

Name: ____________________

5) (10 marks) If R is the region bounded by the lines

$$y = 2x - 1, \quad y = 2x + 5, \quad y = 1 - 3x, \quad y = -1 - 3x$$

using an appropriate coordinate transformation evaluate the following integral:

$$\iint_R (y + 3x)\, dA$$

$y = 2x - 1 \rightarrow y - 2x = -1$
$y = 2x + 5 \rightarrow y - 2x = 5$ } letting $\boxed{u = y - 2x}$

$y = 1 - 3x \rightarrow y + 3x = 1$
$y = -1 - 3x \rightarrow y + 3x = -1$ } letting $\boxed{v = y + 3x}$

$\Rightarrow$ $u = -1$, $u = 5$, $v = 1$, $v = -1$

The new region is the rectangular region in the uv-plane given by:

$$-1 \le u \le 5, \qquad -1 \le v \le 1$$

$$\iint_R f(x,y)\, dx\, dy = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 3 & 1 \end{vmatrix} = -2 - 3 = -5$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{5}$$

$$\iint_R (y+3x)\, dA = \int_{v=-1}^{1} \int_{u=-1}^{5} v \left| -\frac{1}{5} \right| du\, dv = \frac{1}{5} \int_{v=-1}^{1} \int_{u=-1}^{5} v\, du\, dv = \frac{1}{5} \int_{v=-1}^{1} v u \Big|_{u=-1}^{5} dv$$

$$= \frac{1}{5} \int_{v=-1}^{1} v (5+1)\, dv = \frac{6}{5} \frac{v^2}{2} \Big|_{-1}^{1} = \frac{6}{5}\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

Name: ____________________

6) The field $\vec{F} = \overbrace{(axy+z)}^{P}\vec{\imath} + \overbrace{x^2}^{Q}\vec{\jmath} + \overbrace{(bx+2z)}^{R}\vec{k}$ is a conservative vector field.

a) (4 marks) Find a and b.

b) (6 marks) Find a potential function for $\vec{F}$.

c) (2 marks) Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve from $(1,1,0)$ to $(0,0,3)$ that lies on the intersection of the surfaces $2x+y+z=3$ and $9x^2+9y^2+2z^2=18$ in the octant $x\geq 0, y\geq 0, z\geq 0$.

a) Conservative means $\vec{F} = \vec{\nabla} f \Rightarrow P\vec{i} + Q\vec{j} + R\vec{k} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\therefore \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow ax = 2x \Rightarrow \boxed{a=2}$

$\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \Rightarrow 1 = b \Rightarrow \boxed{b=1}$

b) Potential function $f=?$

$\frac{\partial f}{\partial x} = \underbrace{2xy+z}_{P} \Rightarrow \int df(x,y,z) = \int (2xy+z)\,dx$

$\boxed{f(x,y,z) = x^2y + zx + h(y,z)}$

$\frac{\partial f}{\partial y} = \underbrace{x^2}_{Q} \Rightarrow \frac{\partial}{\partial y}\left(x^2y + zx + h(y,z)\right) = x^2 \Rightarrow x^2 + \frac{\partial h(y,z)}{\partial y} = x^2$

$\frac{\partial h(y,z)}{\partial y} = 0 \Rightarrow h(y,z) = g(z)$

$\frac{\partial f}{\partial z} = \underbrace{x+2z}_{R} \Rightarrow \frac{\partial}{\partial z}\left(x^2y + zx + g(z)\right) = x + 2z$

$x + \frac{d}{dz}g(z) = x + 2z$

$\frac{dg(z)}{dz} = 2z \Rightarrow g(z) = \int 2z\,dz = z^2 + C$

$\boxed{f(x,y,z) = x^2y + zx + z^2 + C}$ → (Note to TAs: here, in potential function, they can also omit C)

c) C: from $(1,1,0)$ to $(0,0,3)$

$\int_C \vec{F}\cdot d\vec{r} = f(0,0,3) - f(1,1,0) = 9 - 1 = 8$

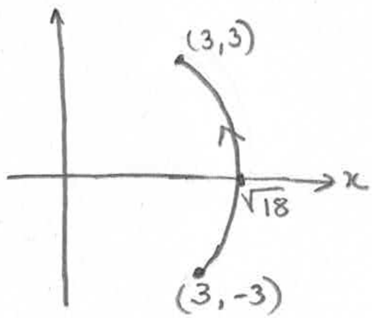
Name: ______________

7) a) (10 marks) Evaluate the following line integral

$$\int_C (3x - y)ds$$

where curve C is the portion of the circle $x^2 + y^2 = 18$ traversed from $(3, -3)$ to $(3, 3)$.

b) (2 marks) Write the value of the integral if the curve C was traversed in the opposite direction (that is, clockwise from $(3, 3)$ to $(3, -3)$). Here, you are expected to write the value without re-evaluating the integral.



Parametrize the curve:

$$\left.\begin{array}{l} x = \sqrt{18}\cos\theta \\ y = \sqrt{18}\sin\theta \end{array}\right\} -\pi/4 \le \theta \le \pi/4$$

Vector form of the parametric eqn. of the curve:

$$\vec{r}(t) = \sqrt{18}\cos\theta\,\vec{i} + \sqrt{18}\sin\theta\,\vec{j}, \quad -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$$

$$\vec{r}\,'(t) = \sqrt{18}(-\sin\theta)\,\vec{i} + \sqrt{18}\cos\theta\,\vec{j}$$

$$\|\vec{r}\,'(t)\| = \sqrt{18\sin^2\theta + 18\cos^2\theta} = \sqrt{18}$$

$$\int_C (3x-y)\,ds = \int_{\theta=-\pi/4}^{\pi/4} (3\sqrt{18}\cos\theta - \sqrt{18}\sin\theta)\sqrt{18}\,d\theta$$

$ds = \|\vec{r}\,'(t)\|d\theta$

$$= 18\int_{-\pi/4}^{\pi/4}(3\cos\theta - \sin\theta)\,d\theta = 18\cdot\left[3\sin\theta\Big|_{-\pi/4}^{\pi/4} + \cos\theta\Big|_{\theta=-\pi/4}^{\pi/4}\right]$$

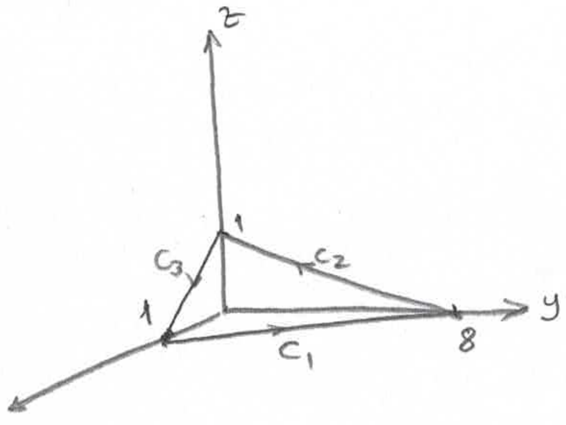
$$= 18\left[3.\left(\sin\frac{\pi}{4} - \sin\left(-\frac{\pi}{4}\right)\right) + \cos\frac{\pi}{4} - \cos\left(-\frac{\pi}{4}\right)\right]$$

$$= 54\sqrt{2}$$

b) The value would be the same! $54\sqrt{2}$. (Direction doesn't matter!)

Name: ____________________

8) (18 marks) Verify Stokes' theorem for the vector field $\vec{F}(x,y,z) = yz\vec{\imath} + 2xz\vec{\jmath} + y\vec{k}$ over the part of the plane $8x + y + 8z = 8$ in the first octant. Provide a sketch of the surface and the boundary curves.



Stoke's thrm:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\, dS$$

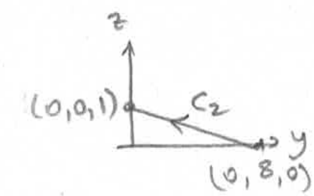
C_1: $\vec{r} = (1,0,0) + t\left[(0,8,0) - (1,0,0)\right]$

$\vec{r}(t) = (1-t,\ 8t,\ 0)$, $0 \le t \le 1$

$\vec{r}\,'(t) = (-1,\ 8,\ 0)$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (8t\vec{k})\cdot(-\vec{\imath} + 8\vec{\jmath} + 0\vec{k})\, dt = 0$$

$\vec{r}\,'(t)\,dt$



C_2: $\vec{r} = (0,8,0) + t\left[(0,0,1) - (0,8,0)\right]$

$\vec{r}(t) = (0,\ 8-8t,\ t)$, $0 \le t \le 1$

$\vec{r}\,'(t) = (0,\ -8,\ 1)$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \left[(8t-8t^2)\vec{\imath} + 0\vec{\jmath} + (8-8t)\vec{k}\right]\cdot(0\vec{\imath} - 8\vec{\jmath} + \vec{k})\, dt = \int_0^1 (8-8t)\,dt$$

$d\vec{r} = \vec{r}\,'(t)\,dt$

$$= \left(8t - 4t^2\right)\Big|_0^1 = 8 - 4 = 4$$

C_3: $\vec{r} = (0,0,1) + t\left[(1,0,0) - (0,0,1)\right]$

$\vec{r}(t) = (t,\ 0,\ 1-t)$, $0 \le t \le 1$

$\vec{r}\,'(t) = (1,\ 0,\ -1)$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_0^1 \left[(0\vec{\imath} + 2t(1-t)\vec{\jmath} + 0\vec{k})\cdot(\vec{\imath} + 0\vec{\jmath} - \vec{k})\right] dt = \int_0^1 0\, dt = 0$$

Name: ____________________

EXTRA PAGE

$$\oint_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C_1} \vec{F}\cdot d\vec{r}}_{0} + \underbrace{\int_{C_2} \vec{F}\cdot d\vec{r}}_{4} + \underbrace{\int_{C_3} \vec{F}\cdot d\vec{r}}_{0} = \underline{\underline{4}}$$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & 2xz & y \end{vmatrix} = \underline{\underline{(1-2x)\vec{i} + y\vec{j} + z\vec{k}}}$$

Parametrize the surface: $8x + y + 8z = 8 \Rightarrow z = 1 - x - \frac{1}{8}y$

$$\left.\begin{array}{l} x = u \\ y = v \\ z = 1 - u - \frac{1}{8}v \end{array}\right\} \quad \vec{r}(u,v) = u\vec{i} + v\vec{j} + \left(1 - u - \frac{1}{8}v\right)\vec{k}$$

$$\left.\begin{array}{l} \vec{r}_u = \vec{i} + 0\vec{j} - \vec{k} \\ \vec{r}_v = 0\vec{i} + \vec{j} - \frac{1}{8}\vec{k} \end{array}\right\} \quad \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1/8 \end{vmatrix} = \vec{i} + \frac{1}{8}\vec{j} + \vec{k}$$

upward. Use $\vec{r}_u\times\vec{r}_v$ (not $\vec{r}_v\times\vec{r}_u$) to match the orientation of the surface with the orientation of the curves

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \iint_S \left[(1-2u)\vec{i} + v\vec{j} + \left(1-u-\frac{1}{8}v\right)\vec{k}\right]\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

$\vec{n} = \dfrac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}$ $\qquad dS = \|\vec{r}_u\times\vec{r}_v\|\,du\,dv$

ethod 1

Taking $0 \le u \le 1 - \frac{1}{8}v, \quad 0 \le v \le 8$

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \int_{v=0}^{8}\int_{u=0}^{1-\frac{1}{8}v} \left[(1-2u)\vec{i} + v\vec{j} + \left(1-u-\frac{1}{8}v\right)\vec{k}\right]\cdot\left(\vec{i} + \frac{1}{8}\vec{j} + \vec{k}\right)du\,dv$$

$$= \int_{v=0}^{8}\int_{u=0}^{1-\frac{1}{8}v} (2-3u)\,du\,dv = \int_{v=0}^{8} \left(2u - \frac{3u^2}{2}\right)\Bigg|_{u=0}^{u=1-\frac{1}{8}v} dv = \int_{v=0}^{8} \left(\frac{1}{2} + \frac{1}{8}v - \frac{3}{2\cdot 8\cdot 8}v^2\right)dv$$

$$= \left(\frac{1}{2}v + \frac{1}{8}\frac{v^2}{2} - \frac{3}{2\cdot 8\cdot 8}\cdot\frac{v^3}{3}\right)\Bigg|_{v=0}^{v=8} = \underline{\underline{4}}$$

Verified!

Extra Page

Method 2:

Taking : $0 \le v \le 8 - 8u \quad , \quad 0 \le u \le 1$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS = \int_{u=0}^{1} \int_{v=0}^{8-8u} (2 - 3u)\, dv\, du = \int_{u=0}^{1} \left(2v - 3uv\right)\Big|_{v=0}^{v=8-8u}$$

$$= \int_{u=0}^{1} \left(16 - 40u + 24u^2\right) du$$

$$= \left[16u - 20u^2 + 8u^3\right]_{u=0}^{u=1}$$

$$= 16 - 20 + 8$$

$$= 4$$

Verified!