MAT292 - Calculus III - Fall 2015

Term Test 2 - November 12, 2015

Time allotted: 90 minutes. Aids permitted: None.

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Last First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page). Make sure you have all of them.
- You can use page 14 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGE 14.

GOOD LUCK!

PART I No explanation is necessary.

For questions **1.** to **3.** , consider the system

$$\vec{x}\,' = A\vec{x}.$$

1. **(2 marks)** If A has eigenvalues $r = \alpha \pm \beta i$ and the equilibrium $\vec{0}$ is unstable, then

$$\alpha \in \Big(\underline{\hspace{4cm}}\ ,\ \underline{\hspace{4cm}}\Big)$$

$$\beta \in \Big(\underline{\hspace{4cm}}\ ,\ \underline{\hspace{4cm}}\Big)$$

2. **(2 marks)** If A has eigenvalues $r_1 < 0 < r_2$ with eigenvectors $\vec{\xi}_1, \vec{\xi}_2$, then the equilibrium $\vec{0}$ is unstable. However, some solutions don't diverge to infinity.

Find all the possible initial conditions $\vec{x}(0) = \vec{x}_0$ such that $\lim\limits_{t\to\infty} |\vec{x}(t)| = 0$:

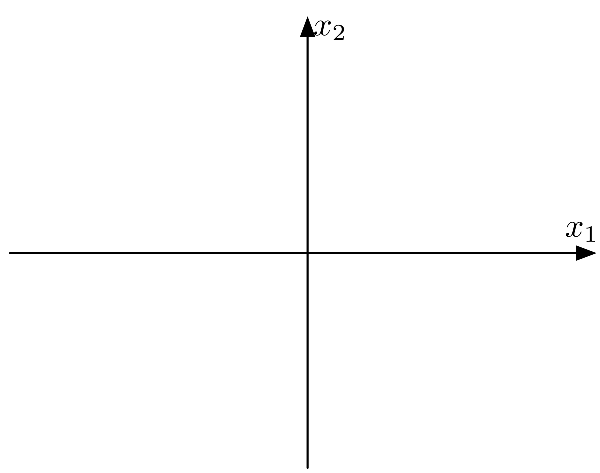
$$\vec{x}_0 = \underline{\hspace{12cm}}.$$

3. **(3 marks)** If A has the eigenvalues and eigenvectors

$$r_1 = 1 \quad , \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_2 = 5 \quad , \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

sketch the phase portrait.



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For questions **4.** to **5.** , consider the system

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

4. **(2 marks)** The equilibrium solution is

$$\vec{x}_{\text{eq}} = \begin{pmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{pmatrix}.$$

5. **(2 marks)** The deviation from equilibrium $\vec{x}_h = \vec{x} - \vec{x}_{\text{eq}}$ satisfies the system

$$\vec{x}_h{}' = \begin{pmatrix} \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} & \underline{\hspace{3cm}} \end{pmatrix} \vec{x}_h + \begin{pmatrix} \underline{\hspace{3cm}} \\ \underline{\hspace{3cm}} \end{pmatrix}.$$

6. **(3 marks)** Write a second-order linear differential equation with solutions

$$y_1 = e^{2t} \qquad \text{and} \qquad y_2 = -e^{2t} + e^{3t}.$$

$$y'' \quad + \quad \underline{\hspace{3cm}} \quad y' \quad + \quad \underline{\hspace{3cm}} \quad y \quad = \quad \underline{\hspace{3cm}}$$

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PART II **Justify** your answers.

7. Consider the system of differential equations **(10 marks)**

$$\vec{x}\,' = \begin{pmatrix} 2a & -a \\ 5a & 0 \end{pmatrix} \vec{x},$$

where $a \neq 0$.

(a) **(8 marks)** Sketch all the possible phase portraits for this system. Justify your answer.

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(b) **(2 marks)** Give an example of a system of differential equations where the phase portrait is a centre.

(1 extra mark) if it is clockwise.

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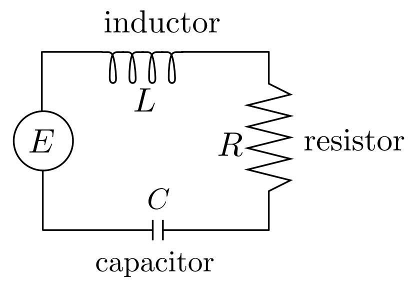
8. Consider an oscillator. We can model the charge $q(t)$ of the capacitor by using Kirchhoff's Second Law: **(6 marks)**



Sum of the voltage drops over the components of the circuit is equal to the impressed voltage.

We obtain the following DE:

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

and $q' = i =$ current.

(a) **(2 marks)** Assuming $E(t) = 0$, find a condition on the constants R, L, C that makes sure that there will be oscillations in $q(t)$, and if the condition is not satisfied, there won't be any oscillations.

(b) **(2 marks)** Still with $E(t) = 0$, if there are no oscillations, then what is the limiting behaviour of the solution? Justify using the Differential Equation.

$\lim_{t \to \infty} q(t) =$

Continued...

(c) **(2 marks)** Give an example of constants R, L, C and $E(t)$ such that $E(t)$ is bounded and

$$\lim_{t \to \infty} |q(t)| = \infty.$$

Continued...

9. Consider the initial-value problem **(10 marks)**

$$\begin{cases} ty'' + (2+t)y' + y = 0 \\ y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = v_0 \end{cases}$$

(a) **(2 marks)** For which values of t_0, y_0, v_0 can we guarantee that there is a unique solution? What is the domain of the solution?

(b) **(2 marks)** Show that $y_1 = \frac{1}{t}$ is a solution of the differential equation.

Continued...

(c) **(3 marks)** Find a second solution y_2 of the differential equation.

(d) **(1 mark)** Compute the Wronskian of the solutions found in **(b)** and **(c)** and show that it is never 0, as long as the solutions are defined.

Hint. $W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

Continued...

(e) **(2 marks)** Find the general solution to the DE

$$ty'' + (2 + t)y' + y = t^2$$

Continued...

10. When a baseball is flying through the air, spin will affect its motion. **(10 marks)**

Consider a baseball with $m = \frac{1}{7}\ kg$ and assume that $g = 9.8 = \frac{49}{5}\ m/s^2$.

Consider also the following functions:

- $x(t)$ = x-position of the baseball at time t
- $y(t)$ = y-position of the baseball at time t
- $v_\theta(t)$ = counter-clockwise velocity of the baseball in radians per second (if the ball is rotating clockwise, v_θ is negative) at time t

Then there are three forces acting on the baseball:

- Gravity
- Air drag: for simplicity, assume it is proportional to each component's velocity (including the spinning velocity) with proportionality constant $\gamma = 1$
- Magnus Effect (due to spin): Counterclockwise spin creates vertical lift proportional to spinning velocity (with proportionality constant $k = 1$)

(a) **(3 marks)** Define

- $v_x(t)$ = horizontal velocity of the baseball at time t
- $v_y(t)$ = vertical velocity of the baseball at time t

Find a system of DEs that describes the motion of the ball.

$$\begin{cases} m\,v_x' = \\ m\,v_y' = \\ m\,v_\theta' = \end{cases}$$

Continued...

(b) **(3 marks)** Assuming that the baseball was thrown horizontally with speed 40 m/s and spin 10 rad$/s$, find the solution of the system found in **(a)** .

$$\begin{cases} v_x(t) = \\ v_y(t) = \\ v_\theta(t) = \end{cases}$$

(c) **(2 marks)** Assume that the ball was thrown from a height of 2 m. Find the position of the ball at time t.

$$\begin{cases} x(t) = \\ y(t) = \end{cases}$$

Continued...

(d) **(2 marks)** Terms of the form e^{-rt} become very small very quickly. Ignore those terms in your solution and estimate when the ball lands on the ground.

(bonus) **(1 mark)** Once the ball lands, will it roll on the ground?

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