Question 1

Part a)

This part can be done by separately writing down equations for conservation of energy and momentum and solving these for the requested quantities, or by the use of four-vector invariants. (*Method 1* and *Method 2* below).

Labeling the initial state particles as 1 and 2 respective, for both methods we will need the Lorentz factors relevant to the two velocities:

$$\gamma_1 = (1-(0.8)^2)^{-1/2} = 1.667 \qquad \gamma_2 = (1-(0.7)^2)^{-1/2} = 1.400$$

Method 1

For the first method write down the expression for conservation of momentum:

$$\begin{aligned}\gamma_f M_f \vec{v}_f &= \gamma_1 m_1 \vec{v}_1 + \gamma_2 m_2 \vec{v}_2 \\ &= 1.667(3m)(0.8c\,\hat{x}) + 1.400(8m)(-0.7c\,\hat{x}) \\ &= (4.00mc - 7.84mc)\,\hat{x} \\ &= -3.84mc\,\hat{x}\end{aligned}$$

Now use conservation of energy:

$$\begin{aligned}\gamma_f M_f c^2 &= \gamma_1 m_1 c^2 + \gamma_2 m_2 c^2 \\ \gamma_f M_f &= \gamma_1 m_1 + \gamma_2 m_2 \\ &= 1.667(3m) + 1.400(8m) \\ &= 5.00m + 11.2m \\ &= 16.2m\end{aligned}$$

Above we have expressions for both $\gamma_f M_f \vec{v}_f$ and $\gamma_f M_f$. Taking the ratio gives us $\vec{v}_f$

$$\vec{v}_f = \frac{-3.84mc\,\hat{x}}{16.2m} = -0.237c\,\hat{x}.$$

The γ factor associated with this speed is $\gamma_f = (1-(0.237)^2)^{-1/2} = 1.029$

Putting this back into the equation $\gamma_f M_f = 16.2m$ yields $M_f = 15.74m$.

Method 2

Doing the problem with four-vector invariant goes as follows: the energy-momentum four-vectors in the initial state are:

$$P_1 = \left(\gamma_1(3m)(0.8c), 0, 0, \gamma_1(3m)c\right) \text{ and } P_2 = \left(\gamma_2(8m)(-0.7c), 0, 0, \gamma_2(8m)c\right)$$

So we have

$P_1 = \left(4.00mc, 0, 0, 5.00mc\right)$

$P_2 = (-7.84mc, 0, 0, 11.20mc)$

$P_{TOT} = P_1 + P_2 = (-3.84mc, 0, 0, 16.20mc)$

$P_{TOT}^2 = [(16.20)^2 - (3.84)^2]m^2c^2 = (15.74)^2 m^2c^2$

For the final state with four-momentum P_f we know that $P_f^2 = M_f^2c^2 = P_{TOT}^2$ so the mass of the final state object is $M_f = 15.74m$

The total energy is conserved. The value can be read from the P_{TOT} four-vector above ($16.2mc^2$) so the kinetic energy in the final state is $16.20mc^2 - 15.74mc^2 = 0.46mc^2$. So the velocity of the final-state object can be obtained from:

$(\gamma_f - 1)M_f c^2 = 0.46mc^2 \Rightarrow (\gamma_f - 1)15.74mc^2 = 0.46mc^2$ so we have

$(\gamma_f - 1) = 0.46 / 15.74 = 0.029 \Rightarrow \gamma_f = 1.029 \Rightarrow v_f = -0.236c\hat{x}$, where the sign comes from sign of the momentum, which must be conserved. (The difference of .001c with respect to the other method is just rounding error.)

Part b)

From above the total energy of $16.2mc^2$. The kinetic energy in the initial and final states can be obtained from the masses and velocities, but also just by subtracting the mass energy from the total (since mass energy and kinetic energy must sum to the total). So the kinetic energy in the initial state is

$$(16.2 - 3.0 - 8.0)mc^2 = 5.2mc^2$$

In the final state it is

$$(16.2 - 15.74)mc^2 = 0.46mc^2$$

so the difference is $-4.74mc^2$.

Question 2

Part a)

(i) For this we need $t_1' = t_2'$. Using the expressions provided to transform from S to S' we have:

$$t_1' = \gamma(t_1 - \frac{\beta}{c}x_1) \quad t_2' = \gamma(t_2 - \frac{\beta}{c}x_2)$$

so

$$t_1' - t_2' = 0 \quad \Rightarrow \quad (t_1 - \frac{\beta}{c}x_1) - (t_2 - \frac{\beta}{c}x_2) = 0$$

which becomes

$$(\frac{a}{c} - \frac{\beta}{c}a) - (\frac{a}{3c} - \frac{\beta}{c}(3a)) = 0 \;\Rightarrow\; (\frac{a}{c} - \beta\frac{a}{c}) = (\frac{a}{3c} - \beta\frac{3a}{c}) \;\Rightarrow\; (\frac{a}{c} - \frac{a}{3c}) = \beta(\frac{a}{c} - \frac{3a}{c})$$

$\Rightarrow \dfrac{2a}{3c}) = \beta(-\dfrac{2a}{c}) \quad \Rightarrow \beta = -\dfrac{2a}{c}\dfrac{c}{2a} = -\dfrac{1}{3}$, so the required velocity is $-\dfrac{1}{3}c\hat{x}$.

(ii) For the time in S', just use this velocity expression on one of the expressions for t_1' or t_2' (or both, as a check that you get the same result).

$$t_1' = \gamma\left(\frac{a}{c} - \frac{1}{3c}a\right) = 1.06\left(\frac{a}{c} + \frac{a}{3c}\right) = 1.06\left(\frac{4a}{3c}\right)$$

$t_2' = \gamma\left(\dfrac{a}{3c} - \dfrac{1}{3c}3a\right) = 1.06\left(\dfrac{a}{3c} + \dfrac{a}{c}\right) = 1.06\left(\dfrac{4a}{3c}\right)$ which agrees.

Part b)

For this part we just need to obtain Bob's velocity as seen by Anna, and then use length contraction.

To move into Anna's reference frame we need to travel at her velocity, so at -0.5*c*. The expression for the addition of velocities is:

$$u_x' = \frac{u_x - v}{1 - (v/c^2)u_x}$$

and we need $u_x = 0.7c$ and $v = -0.5c$. This yields (for Bob's speed, as seen by Anna):

$$u'_x = \frac{0.7c - (-0.5)c}{1-(0.7)(-0.5)} = \frac{1.2c}{1.35} = 0.889c$$

For this velocity we have $\gamma = \dfrac{1}{\sqrt{1-(0.889)^2}} = 2.18$.

The length of the rod as observed by Anna is therefore 4*m* / 2.18 = 1.83*m*.

Question 3

Part a)

First we need to know the maximum kinetic energy of the electrons. It is sufficient to calculate this non-relativistically, since the velocity is rather low compared to the speed of light, but the relativistic calculation is also fine of course. Non-relativistically, we have

$$\text{K.E.} = \frac{1}{2}m_e v^2 = \frac{1}{2}(511\times 10^3\,\frac{\text{eV}}{c^2})(.002c)^2 = \frac{(511\times 10^3)(4\times 10^{-6})}{2} = 1.022\,\text{eV}$$

The fact that this value is very small relative to the electron mass supports the use of the non-relativistic calculation.

The expression for the relativistic Doppler shift was provided (in terms of frequency):

$$f_{obs} = f_{source}\cdot\frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}\cos\theta}$$

For a source moving towards the photo-cathode, $\cos\theta = -1$ and we have that

$$f_{obs} = f_{source}\cdot\frac{\sqrt{1-v/c}\cdot\sqrt{1+v/c}}{1-v/c} = f_{source}\cdot\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}$$

For the wavelength one therefore gets: $\lambda_{obs} = \lambda_{source}\cdot\frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}$ (since $c = f\lambda$).

For a kinetic energy of 1.022 eV and a work-function of 2.0 eV, the photon's total energy must be 3.022 eV, which corresponds to a wavelength of

$$\lambda = \frac{1240\,\text{eV}\cdot\text{nm}}{3.022\,\text{eV}} = 410.3\,\text{nm}$$

So we have that $\frac{410.3}{550} = 0.746 = \frac{\sqrt{1-v/c}}{\sqrt{1+v/c}}$. So we have $\frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} = 0.746$ which can be solved to yield $\beta = 0.285$, so the speed of the source is $0.285c$.

b) For the source moving away from the photocathode, repeating the above exercise starting with $\cos\theta = +1$ one gets that

$$\lambda_{obs} = \lambda_{source} \cdot \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} = 550\text{nm} \cdot \frac{\sqrt{1+0.285}}{\sqrt{1-0.285}} = 737.2\,\text{nm}.$$

This corresponds to an energy of $E = \dfrac{1240\,\text{eV}\cdot\text{nm}}{737.2\,\text{eV}} = 1.68\,\text{eV}.$

This is less than the work-function for this photocathode so no electrons will be ejected in this case.