ECE259H1: Electromagnetism

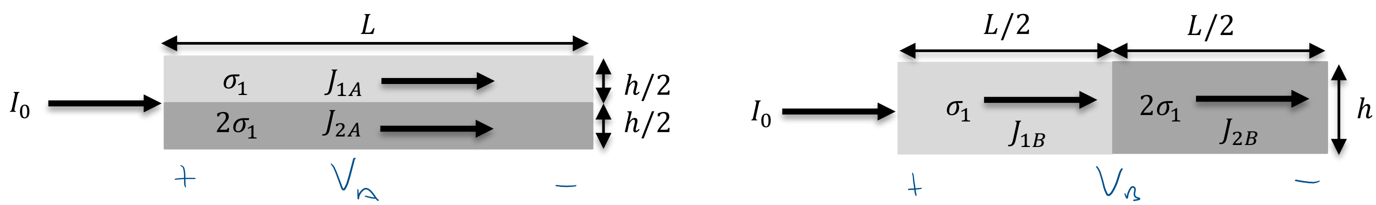
Homework Review Quiz 3 – Friday March 24, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The quiz is worth 20 marks and has two questions. Question 1 is worth 6 marks, and Question 2 is work 14 marks.
- Show all of your work.
- The final page has some reference material that you might find helpful.
- Take a deep breath and relax 😊.

Question #1 (6 marks)

Two resistors are shown below as *Resistor A* and *Resistor B*. Both have a current of I_0 flowing through them from left to right. You can assume that each current density is uniformly distributed throughout the cross-sectional areas of each part of these resistors. The resistors have a square cross-sectional area (h x h) and the same total length, L.



Resistor A

~ Voltages are different due to different R's

Resistor B

For the statements below, identify if they are *True* or *False* and justify your answer.

True *False* The total resistance of *Resistor A* is larger than that of *Resistor B*.

* Recognizing that $E_{1A} = E_{2A} = \frac{V_A}{L}$, then $J_{1A} = \sigma_1 \frac{V_A}{L}$ and $J_{2A} = 2\sigma_1 \frac{V_A}{L}$

$\therefore I_0 = J_{1A}(h)(h/2) + J_{2A}(h)(h/2) = \left(\frac{h^2}{2}\right)\left(\sigma_1 \frac{V_A}{L} + 2\sigma_1 \frac{V_A}{L}\right) = V_A\left(\frac{h^2}{2L}\right)(3\sigma_1)$

$\therefore R_A = \frac{V_A}{I_0} = \frac{2L}{3h^2\sigma_1}$

For R_B: $J_{1B} = J_{2B} = \frac{I_0}{h^2}$ (due to the continuity of current)

But $E_{1B} = \frac{J_{1B}}{\sigma_1} = \frac{I_0}{\sigma_1 h^2}$ and $E_{2B} = \frac{J_{2B}}{2\sigma_1} = \frac{I_0}{2\sigma_1 h^2}$

And $V_B = E_{1B}\left(\frac{L}{2}\right) + E_{2B}\left(\frac{L}{2}\right) = \frac{I_0}{\sigma_1 h^2}\left(\frac{L}{2}\right) + \frac{I_0}{2\sigma_1 h^2}\left(\frac{L}{2}\right) \rightarrow R_B = \frac{V_B}{I_0} = \frac{3L}{4\sigma_1 h^2}$

$\therefore R_B > R_A$

OR: For R_A: This is like two resistors in parallel:

$$R_A = R_{1A} \,//\, R_{2A} = \frac{L}{\sigma_1 (h)(h/2)} \,//\, \frac{L}{2\sigma_1 h (h/2)} = \left(\frac{\sigma_1 h^2}{2L} + \frac{2\sigma_1 h^2}{2L}\right)^{-1} = \frac{2L}{3\sigma_1 h^2}$$

For R_B: This is like two resistors in series:

$$R_B = R_{1B} + R_{2B} = \frac{L/2}{\sigma_1 (h^2)} + \frac{L/2}{2\sigma_1 h^2} = \frac{3L}{4\sigma_1 h^2}$$

$\therefore R_B > R_A$

True *False* The electric field intensity in Part 1 of *Resistor B* is smaller than the electric field intensity of Part 2 of *Resistor B*. Meaning $E_{1B} < E_{2B}$.

Since I_o is continuous throughout the Resistor B,

$$I_o = J_{1B}(h^2) = J_{2B}(h^2) \Rightarrow J_{1B} = J_{2B} = \frac{I_o}{h^2}$$

Since $\bar{J} = \sigma \bar{E} \rightarrow E_{1B} = \frac{J_{1B}}{\sigma_1} = \frac{I_o}{\sigma_1 h^2}$ and $E_{2B} = \frac{J_{2B}}{2\sigma_1} = \frac{I_o}{2\sigma_1 h^2}$

$$\therefore E_{2B} = \frac{1}{2} E_{1B} \rightarrow E_{2B} < E_{1B}$$

OR: Since $E_{1B} = \frac{V_{1B}}{L/2}$ and $V_{1B} = I_o R_{1B} = I_o \left(\frac{L/2}{\sigma_1 h^2}\right)$

$$= \frac{I_o}{\sigma_1 h^2}$$

and $E_{2B} = \frac{V_{2B}}{L/2}$ and $V_{2B} = I_o R_{2B} = I_o \left(\frac{L/2}{2\sigma_1 h^2}\right)$

$$= \frac{I_o}{2\sigma_1 h^2}$$

$$\therefore E_{2B} = \frac{1}{2} E_{1B} \rightarrow E_{2B} < E_{1B}$$

Question #2 (14 marks)

A spherical capacitor consists of two concentric perfectly conducting spheres of radii $R_i = a$ and $R_o = 2a$. Between the conducting spheres exists a volume charge density given by $\rho_v = -\rho_o \varepsilon_0$ [C/m³] in air (i.e., $\varepsilon_r = 1$), where ρ_o is a positive constant. It is known that that the outer sphere is grounded, and the inner sphere is held at a potential of V_0.

(*10 marks*) i) Determine the expression for the electric scalar potential $V(R)$ between these spheres.

* Using Poisson's Equation: $\bar{\nabla}\cdot(\varepsilon_r\varepsilon_0\bar{\nabla}V) = \bar{\nabla}^2 V = -\frac{\rho_v}{\varepsilon_0} = +\rho_o$

↳ In spherical coordinates: $\boxed{\frac{1}{R^2}\frac{d}{dR}}\left(R^2\frac{dV}{dR}\right) = +\rho_o$

Integration #1: $\int d\left(R^2\frac{dV}{dR}\right) = \boxed{R^2}\frac{dV}{dR} = \int \rho_o R^2\, dR = \frac{\rho_o}{3}R^3 + C_1$

Integration #2: $\int dV = V(R) = \int\left(\frac{\rho_o}{3}R + \frac{C_1}{R^2}\right)dR = \frac{\rho_o}{6}R^2 - \frac{C_1}{R} + C_2$

Using Boundary Values: $V(a) = V_0 = \frac{\rho_o}{6}a^2 - \frac{C_1}{a} + C_2$ ①

$V(2a) = 0 = \frac{\rho_o}{6}(4a^2) - \frac{C_1}{2a} + C_2$ ②

① − ②: $V_0 = -\frac{\cancel{3}\rho_o a^2}{\cancel{6}\,2} - \frac{C_1}{2a} \rightarrow C_1 = -\rho_o a^3 - 2aV_0$ ③

③ → ②: $C_2 = -\frac{2\rho_o a^2}{3} + \frac{1}{2a}(-\rho_o a^3 - 2aV_0) = -\frac{2\rho_o a^2}{3} - \frac{\rho_o a^2}{2} - V_0 = -\frac{7\rho_o a^2}{6} - V_0$

$\therefore V(R) = \frac{\rho_o}{6}R^2 + \frac{\rho_o a^3}{R} + \frac{2aV_0}{R} - \frac{7\rho_o a^2}{6} - V_0$

Question #2 (continued)

(*4 marks*) ii) Determine the expression for the electric field intensity $\mathbf{E}(R)$ between these spheres.

$$\bar{E}(R) = -\frac{\partial V}{\partial R}\hat{a}_R = -\frac{\partial}{\partial R}\left(\frac{\rho_0}{6}R^2 + \frac{\rho_0 a^3}{R} + \frac{2aV_0}{R} - \frac{7\rho_0 a^2}{6} - V_0\right)$$

$$= \underline{\underline{\left(-\frac{\rho_0 R}{3} - \frac{\rho_0 a^3}{R^2} - \frac{2aV_0}{R^2}\right)\hat{a}_R}}$$

<u>OR:</u> $\bar{E}(R) = -\frac{\partial V}{\partial R}\hat{a}_R = \left(-\frac{\rho_0}{3}R + \frac{C_1}{R^2}\right)\hat{a}_R$ But $C_1 = -\rho_0 a^3 - 2aV_0$

$$= \underline{\underline{\left(-\frac{\rho_0}{3}R - \frac{\rho_0 a^3}{R^2} - \frac{2aV_0}{R^2}\right)\hat{a}_R}}$$

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Reference Formulae

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

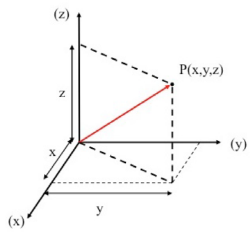
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$

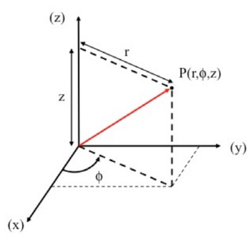
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

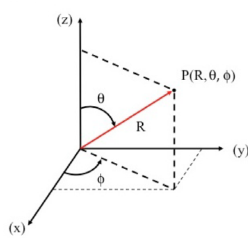
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ







3. Dot products of unit vectors

·	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

5. Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

5.4 Curl

$$\begin{aligned}
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right)\mathbf{a}_\phi
\end{aligned}$$

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

Table 1 Electrostatics

$$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{|\mathbf{R}_2-\mathbf{R}_1|^3}(\mathbf{R}_2-\mathbf{R}_1) \qquad \mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3}dQ'$$

$$\nabla\times\mathbf{E} = 0 \qquad \nabla\cdot\mathbf{D} = \rho_v$$

$$\oint_C \mathbf{E}\cdot\mathbf{dl} = 0 \qquad \oint_S \mathbf{D}\cdot\mathbf{dS} = Q$$

$$\mathbf{E} = -\nabla V \qquad V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{|\mathbf{R}-\mathbf{R}'|}$$

$$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2}\mathbf{E}\cdot\mathbf{dl}$$

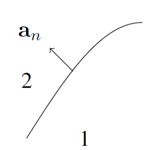
$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon\mathbf{E} \qquad \mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$$

$$\rho_{p,v} = -\nabla\cdot\mathbf{P}$$

$$\rho_{p,s} = -\mathbf{a}_n\cdot(\mathbf{P}_2-\mathbf{P}_1)$$

$$\mathbf{a}_n\cdot(\mathbf{D}_2-\mathbf{D}_1) = \rho_s$$

$$E_{1,t} = E_{2,t}$$



$$Q = CV \qquad W_e = \frac{1}{2}QV$$

$$W_e = \frac{1}{2}\int_{vol}\rho_v V dv = \frac{1}{2}\int_{vol}\mathbf{D}\cdot\mathbf{E}dv$$

$$\nabla\cdot(\varepsilon\nabla V) = -\rho_v \qquad \nabla\cdot(\varepsilon\nabla V) = 0$$