University of Toronto

Faculty of Applied Science and Engineering

Final Examination, 6:30 pm 10 December 2018

First Year, Program 5

MAT194F Calculus I

Exam Type A

No aids of any kind are permitted.
No calculators of any kind are permitted.

Time allowed: 2 ½ hours.
There are 10 questions.
You can write on both sides of each page. There are also 2 extra pages at the end that you can use.

Examiners: P.C. Stangeby and F. Al-Faisal

Family Name: ______________________________

Given Name: ______________________________

Student #: ______________________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	/100

1. (a) [5 marks] Find the derivative of: $3x^4$, $\cos(\sqrt{x})$, $\ln(x^2)$, e^{-3x}, $7^{\sqrt{x}}$.

(b) [5 marks] Find the anti-derivative of: $2x^5$, $\cos(3x)$, $5x^3e^{x^4}$, $(9+x^2)^{-1}$, 7^x.

2. (a) [7 marks] Provide a $\delta - \varepsilon$ style of proof that $\lim_{x \to \infty} x^2 = \infty$.

(b) [3 marks] Prove that $\lim_{x \to \infty} x^2 \neq 10^{10}$ using a proof by contradiction: assume that $\lim_{x \to \infty} x^2 = 10^{10}$ and use a $\delta - \varepsilon$ style of proof to show that this results in a contradiction.

3. [10 marks] Sketch the curve $y = \frac{1}{x} + \ln x, x > 0$. Indicate on the sketch: intercepts with the 2 axes, if they exist; the regions where y is increasing, decreasing, concave up, concave down; local and absolute maxima and minima, if they exist; points of inflexion if they exist; vertical asymptotes, horizontal asymptotes and vertical tangents if they exist; symmetry or periodicity if they exist.

4. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^2$ and $y = 2x - x^2$.

Calculate the following quantities:

(a) [4 marks] The area of $\mathcal{R}$.

(b) [3 marks] The volume obtained by rotating $\mathcal{R}$ about the x-axis.

(c) [3 marks] The volume obtained by rotating $\mathcal{R}$ about the y-axis.

5. (a) [4 marks] Find two positive numbers whose product is 400 and whose sum is a minimum.

(b) [3 marks] Find two positive numbers whose product is 400 and whose difference is a minimum.

(c) [3 marks] Find the point on the parabola $y = 1 - x^2$ at which the tangent line intercepts the positive x-axis and positive y-axis to make the triangle of smallest area.

6. [10 marks] A curve has the property that the normal line through any point (x_0, y_0) on the curve passes through $(2, 0)$. If the curve contains the point $(2, 3)$, find its equation and sketch its graph. [Note: there are no marks for guessing the answer; you must prove your answer is correct.]

7. (a) [4 marks] Show that $\int \sqrt{a^2 - x^2}dx = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 sin^{-1}\left(\frac{x}{a}\right) + C \;\; (a > 0)$.
[Hint: do not integrate.]

(b) [2 marks] Hence find $\int \sqrt{1 - (\mathrm{x} - 1)^2}\mathrm{dx}$.

(c) [4 marks] Find the area of the region enclosed by the circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

8. (a) [5 marks] Solve the differential equation $y'' - 6y' + 10y = 0, \quad y(0) = 2, y'(0) = 3$.

(b) [3 marks] Consider the differential equation $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2y^2}$. Let $z = y^3$. Express $\frac{dz}{dx}$ in terms of y and $\frac{dy}{dx}$, and hence re-write the D.E. in terms of z and x.

(c) [2 marks] Solve the D.E. you found in (b) for z . Hence solve the original D.E.

9. [10 marks] Compute: (a) $\int e^x\sqrt{e^x+3}\,dx$ (b) $\int \frac{dx}{x(1+\ln x)^2}$

(c) $\int \frac{dx}{\sin 3x \cos 3x}$ (d) $\int \frac{x\,dx}{x^2+4x+6}$

10. Let $f(x) = \int_0^{x^2} e^{t^3}\,dt, x \in (0, \infty)$.

(a) [6 marks] Prove that $\frac{df}{dx}$ is increasing on $(0, \infty)$.

(b) [4 marks] Prove that every tangent line to $y = f(x)$ intersects it only once.

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