Print name legibly (first name, last name):__________,__________

Student #:________________

Q1:_____Q2:____Q3:____Q4:____Q5:____Q6:____

Total:_________

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

December 22, 2015

Instructor – W.R. Cluett

Closed book.

Allowable calculators:

Casio FX-991EX or FX-991ES PLUS/PLUS C or FX-991MS

or

Sharp EL-520X or EL-520W

All questions of equal value.

All work to be marked <u>must</u> appear on front of page. Use back of page for rough work only.

Given information:

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\,\vec{u} = \frac{\vec{u}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d}$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The normal system of equations corresponding to $AX = B$ is given by:

$A^T AX = A^T B$

Euler's method:

$t_{n+1} = t_n + \Delta t$; $y_{n+1} = y_n + \Delta t f(t_n, y_n)$; $\frac{dy}{dt} = f(t, y(t))$

Different finite difference approximations to the first derivative $f'(x)$:

Forward difference: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$

Backward difference: $\frac{f(x) - f(x-\Delta x)}{\Delta x}$

Central difference: $\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

Q1

a) Find all linear combinations of the three vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ that produce $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e. $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{b}$.

b) Using your solution in part a), find two different linear combinations of the three vectors $\vec{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ that produce $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c) If you take any three vectors in R^2, will there always be two different linear combinations that produce $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? Explain your answer.

Q2

a) Start from the four equations:

$$-x_{i+1} + 2x_i - x_{i-1} = i \text{ for } i = 1,2,3,4$$

with $x_0 = x_5 = 0$, and write these equations in their matrix form $AX = B$, being sure to define A, X and B.

b) Use Gaussian elimination to find the inverse of matrix A, i.e. A^{-1}.

c) Use your answer from part b) to solve this system of linear equations for x_1, x_2, x_3, x_4.

d) Give a geometric interpretation of your solution found in part c).

Q3

a) Project the vector $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$ onto the line through the origin and the point $(2,2,1)$.

These next parts b), c) and d) require you to project the vector $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$ onto the plane R containing the origin, the point $(2,2,1)$ and the point $(1,0,0)$.

b) State the matrix A that has R as its column space.

c) Derive the matrix P that transforms vector $\vec{u}$ to the projection of $\vec{u}$ onto the plane R (your answer should be expressed in terms of A).

d) Using your answers to parts b) and c), find this projection matrix P and use this matrix to find the vector in R that is closest to $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$.

Q4 (treat parts a) and b) as separate questions)

a) Suppose that matrix A is square and invertible and you exchange its first two rows to reach matrix B. Is the new matrix B invertible? How would you find B^{-1} from A^{-1}? Use elementary matrices to make your argument.

b) Assume matrix B is square and is the inverse of the matrix A^2. Prove that $AB = BA$.

Q5 (treat parts a) and b) as separate questions)

a) Consider the following initial value problem consisting of two first order differential equations:

$$\frac{dy}{dt} = [(-y+z)e^{(1-t)} + 0.5y] \text{ with } y(0) = 3$$

$$\frac{dz}{dt} = [y - z^2] \text{ with } z(0) = 0.2$$

Using $\Delta t = 0.25$, solve this system for the first two time steps ($t = 0.25$ and 0.5) using Euler's method. Note that this system of differential equations is nonlinear and therefore cannot be formulated using a state matrix.

b) The midpoint method is an improvement on the Euler's method that differs from the improved Euler's method studied in this course. With the midpoint method, the slope used for calculating y_{n+1} is an estimate of the slope at the midpoint of the interval (Δt). In this algorithm, Euler's method is used to calculate an approximate value of y (let's call it y_m) at the midpoint of the interval $t_m = t_n + \frac{\Delta t}{2}$. Then, the estimated slope at the midpoint is calculated by substituting (t_m, y_m) into the differential equation $\frac{dy}{dt} = f(t, y(t))$. Finally, this slope is used for calculating the update for y_{n+1} based on y_n. From this word description of the algorithm, write out the update equations for t_{n+1} and y_{n+1} corresponding to this midpoint method. Feel free to use more than 2 equations in your answer.

Q6

Consider the following boundary value problem described by the differential equation:

$$-2\frac{d^2y}{dx^2} + y = e^{-0.2x} \text{ for } 0 \le x \le 1$$

with the following boundary conditions:

$$y = 1 \text{ at } x = 0 \text{ and } \frac{dy}{dx} = -y \text{ at } x = 1$$

a) Using the following approximation to the second derivative:

$$\frac{d^2y}{dx^2} \approx \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{(\Delta x)^2}$$

convert the second order differential equation given above into an algebraic equation and simplify this algebraic equation by grouping like terms.

b) By dividing the interval [0,1] into 4 evenly spaced subintervals ($\Delta x = 0.25$), write down the 3 algebraic equations corresponding to the 3 interior grid points using your result from part a).

c) The boundary condition given for y at $x = 0$ may be substituted into the appropriate equation in part b). However, the boundary condition at $x = 1$ is given in terms of the derivative. Of the three finite differences discussed in this course for approximating the first derivative (forward, backward, central), which one would you recommend be used here (note, these approximations are found in the Given Information on page 2 of the exam)? Explain your answer.

d) Using the backward difference, approximate the boundary condition for $x = 1$ in order to provide a 4th algebraic equation and then write down the system of equations that need to be solved in the form $AX = B$ in order to find a numerical solution for $X = \begin{bmatrix} y(0.25) \\ y(0.5) \\ y(0.75) \\ y(1) \end{bmatrix}$. Do not attempt to solve for X.