MAT185 Linear Algebra
Term Test 1

Full Name: ______________ **Solutions** ______________

Student number: ______________________________

Email : ______________________________@mail.utoronto.ca

Signature: ______________________________

Instructions:

1. This test contains a total of 9 pages.
2. DO NOT DETACH ANY PAGES FROM THIS TEST.
3. There are no aids permitted for this test, including calculators.
4. Cellphones, smartwatches, or any other electronic devices are not permitted. They must be turned off and in your bag under your desk or chair. These devices may **not** be left in your pockets.
5. Write clearly and concisely in a linear fashion. Organize your work in a reasonably neat and coherent way.
6. Show your work and justify your steps on every question unless otherwise indicated. A correct answer without explanation will receive no credit unless otherwise noted; an incorrect answer supported by substantially correct calculations and explanations may receive partial credit.
7. For questions with a boxed area, ensure your answer is completely inside the box.
8. **The back side of pages will not be scanned nor graded**. Use the back side of pages for rough work only.
9. You must use the methods learned in this course to solve all of the problems.
10. DO NOT START the test until instructed to do so.

GOOD LUCK!

Multiple Choice: No justification is required. Only your final answer will be graded.

1. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. Define vector addition and scalar multiplication in V by

$$\begin{aligned}(x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2 - 2) \\ c(x_1, x_2) &= (cx_1, cx_2 - 2c + 2), \quad c \in \mathbb{R}\end{aligned}$$

Then V is a vector space. The additive inverse of $(2, 1)$ is ______. [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ $(-2, -3)$
- ○ $(-2, -1)$
- ○ $(0, 0)$
- ○ $(2, -3)$
- ● $(-2, 3)$

2. Let $V = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$. Define vector addition in V as the usual entry-wise addition, and define scalar multiplication by

$$c(x_1, x_2) = \begin{cases} (0, 0), & \text{if } c = 0 \\ (cx_1, \frac{x_2}{c}), & \text{if } c \neq 0 \end{cases}$$

Then V is *not* a vector space. Which of the following axioms fails to hold? [1 mark]

Indicate your final answer by **filling in exactly one circle** below (unfilled ○ filled ●).

- ○ V is closed under scalar multiplication.
- ○ For all $\mathbf{x} \in V$, and $c, d \in \mathbb{R}$, $(cd)\mathbf{x} = c(d\mathbf{x})$.
- ○ For all $\mathbf{x}, \mathbf{y} \in V$, and $c \in \mathbb{R}$, $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$.
- ● For all $\mathbf{x} \in V$, and $c, d \in \mathbb{R}$, $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$.
- ○ For all $\mathbf{x} \in V$, $1\mathbf{x} = \mathbf{x}$.

Multiple Choice: No justification is required. Only your final answer will be graded.

3. Which of the following subsets of $\mathbb{R}^2$ are subspaces of $\mathbb{R}^2$ with respect to the usual entry-wise vector addition and scalar multiplication in $\mathbb{R}^2$? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ○ $\{(x_1, x_2) \mid x_1 x_2 \leq 0\}$
- ● $\{(x_1, x_2) \mid x_1^2 + x_2^2 = 0\}$
- ● $\{(x_1, x_2) \mid (x_1 + x_2)^2 = 0\}$
- ○ $\{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1\}$
- ● $\{(x_1, x_2) \mid \frac{x_1}{2} = \frac{x_2}{3}\}$

4. Let S and T be subsets of a vector space V. Which of the following statements are true? [2 marks]

You can fill in more than one option for this question (unfilled ○ filled ●).

- ● $S \subseteq \operatorname{span} S$.
- ● $S = \operatorname{span} S$ if and only if S is a subspace.
- ● $\operatorname{span} S = \operatorname{span}(\operatorname{span} S)$.
- ● If $S \subseteq T$ then $\operatorname{span} S \subseteq \operatorname{span} T$.
- ○ If $\operatorname{span} S = \operatorname{span} T$ then there is a vector $\mathbf{x}$ such that $\mathbf{x} \in S$ and $\mathbf{x} \in T$.

True or False: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

5. Let $V = \{(x_1, x_2) \mid x_1, x_2 > 0, x_1 + x_2 = 1\}$. Define vector addition and scalar multiplication in V by

$$(x_1, x_2) + (y_1, y_2) = \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right)$$
$$c(x_1, x_2) = (x_1, x_2), \quad c \in \mathbb{R}$$

Indicate your final answers by **filling in exactly one circle** for each part below (unfilled ○ filled ●). Each part is worth 2 marks: 1 mark for a correct final answer; 1 mark for a correct explanantion.

(a) V is closed under addition.

● True.

○ False.

Given the definition for addition, we note that

$$\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2} > 0, \qquad \frac{x_1 + y_1}{2} + \frac{x_2 + y_2}{2} = \frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} = \frac{1}{2} + \frac{1}{2} = 1$$

As such, $(x_1, x_2) + (y_1, y_2) \in V$ and hence V is closed under addition.

(b) V is closed under scalar multiplication.

● True.

○ False.

The result of scalar multiplication is (x_1, x_2), the original element in V, which of course remains in V. Thus V is closed under scalar multiplication.

(c) V is a vector space.

○ True.

● False.

Associativity for addition fails: Consider $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in V$. Now

$$[(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) = \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right) + (z_1, z_2) = \left(\frac{x_1 + y_1 + 2z_1}{4}, \frac{x_2 + y_2 + 2z_2}{4}\right)$$

However,

$$(x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] = (x_1, x_2) + \left(\frac{y_1 + z_1}{2}, \frac{y_2 + z_2}{2}\right) = \left(\frac{2x_1 + y_1 + z_1}{4}, \frac{2x_2 + y_2 + z_2}{4}\right)$$

These are not the same in general as the specific example of $(\frac{3}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4})$ reveals. (By the former calculation, the result is $(\frac{7}{16}, \frac{9}{16})$ and, by the latter, $(\frac{9}{16}, \frac{7}{16})$.)

The zero does not exist either: Let (o_1, o_2) be the zero, if it exists. Then, in general, we must have

$$(x_1, x_2) + (o_1, o_2) = \left(\frac{x_1 + o_1}{2}, \frac{x_2 + o_2}{2}\right) = (x_1, x_2)$$

This entails $x_1 + o_1 = 2x_1$ implying $o_1 = x_1$; similarly $o_2 = x_2$. The zero would depend on the element (x_1, x_2). Hence a zero does not exist.

As a result, the additive inverse cannot exist either.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

6. Consider the subset $W = \{p(x) \mid p(0) = 0\}$ of $P_3(\mathbb{R})$.

(a) Show that W is a subspace of $P_3(\mathbb{R})$ with respect to the usual vector addition and scalar multiplication in $P_3(\mathbb{R})$. [3 marks]

By the Subspace Test,

SI. The zero polynomial $z : \mathbb{R} \to \{0\}$ satisfied $z(0) = 0$ and thus is included in W.

SII. W is closed under vector addition as, for any $p, q \in W$,

$$(p + q)(0) = p(0) + q(0) = 0 + 0 = 0$$

SIII. W is also closed under scalar multiplication because, for any $p \in W$ and any $\lambda \in \mathbb{R}$,

$$(\lambda p)(0) = \lambda p(0) = \lambda \cdot 0 = 0$$

Alternatively, we can argue that W is a subspace by noting

SI′. W is nonempty because $z : \mathbb{R} \to \{0\}$ is included.

SII′. W is closed under the operation $\lambda p + q$ for any $p, q \in W$ and any $\lambda \in \mathbb{R}$. Consider that

$$(\lambda p + q)(0) = (\lambda p)(0) + q(0) = \lambda p(0) + q(0) = \lambda \cdot 0 + 0 = 0$$

By either procedure, W is shown to be a subspace of $\mathbb{P}_3$.

(b). Show that $W = \operatorname{span}\{x, x^2, x^3\}$. [3 marks]

In general, a vector in $\mathbb{P}_3$ can be written as

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

But for p to be admitted into W, we must have $p(0) = 0$. This requires $a_0 = 0$. So a general vector in W can be expressed as

$$p(x) = a_1 x + a_2 x^2 + a_3 x^3 \in \operatorname{span}\{x, x^2, x^3\}$$

This shows that $W \subseteq \operatorname{span}\{x, x^2, x^3\}$.

Conversely, every vector in the spanning set is in W. Therefore any linear combination of these vectors is in W, i.e., $\operatorname{span}\{x, x^2, x^3\} \subseteq W$.

Thus $W = \operatorname{span}\{x, x^2, x^3\}$ as was required to show.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way.

7.

(a) Let V be a vector space. Define what it means for the list of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in V$ to be linearly independent. Be sure to give a precise definition. No partial credit will be given for statements that are "close" to the definition. [2 marks]

A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2 \ldots \mathbf{x}_n\} \subset V$ is *linearly independent* if and only if

$$\sum_{j=1}^{n} \lambda_j \mathbf{x}_j = \lambda_1 \mathbf{x}_1 + \cdots + \lambda_n \mathbf{x}_n = \mathbf{0}$$

implies that all $\lambda_j = 0$.

(b) Let $A \in^n \mathbb{R}^n$, and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in^n \mathbb{R}$ be such that $A\mathbf{x}_1 = \mathbf{0}$, $A^2\mathbf{x}_2 = \mathbf{0}$, and $A^3\mathbf{x}_3 = \mathbf{0}$ but $\mathbf{x}_1 \neq \mathbf{0}$, $A\mathbf{x}_2 \neq \mathbf{0}$, and $A^2\mathbf{x}_3 \neq \mathbf{0}$. Prove that the list $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is linearly independent. [4 marks]

We apply the above test to determine linear independence. Consider then

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \lambda_3 \mathbf{x}_3 = \mathbf{0}$$

Multiply through by A^2 and note that $A^2\mathbf{x}_1 = A^2\mathbf{x}_2 = \mathbf{0}$. This leaves $\lambda_3 A^2 \mathbf{x}_3 = \mathbf{0}$ but, as $\mathbf{A}^2\mathbf{x}_3 \neq \mathbf{0}$, $\lambda_3 = 0$.

What remains is

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 = \mathbf{0}$$

Multiply now by A giving $\lambda_2 A\mathbf{x}_2 = \mathbf{0}$ because $A\mathbf{x}_1 = \mathbf{0}$. This implies that $\lambda_2 = 0$ as $A\mathbf{x}_2 \neq \mathbf{0}$.

Finally, we are left with

$$\lambda_1 \mathbf{x}_1 = \mathbf{0}$$

But $\mathbf{x}_1 \neq \mathbf{0}$ necessitating $\lambda_1 = 0$.

Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and as a consequence $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is linearly independent.

Short Answer: Unsupported answers will not receive full credit. Organize your work in a reasonably neat and coherent way. Put your final answer in the box provided.

8.

(a) State the Fundamental Theorem of Linear Algebra. Be sure to give a precise statement. No partial credit will be given for statements that are "close" to the statement of the theorem. [2 marks]

Let V be a vector space spanned by n vectors. If a set of m vectors from V is linearly independent, then $m \leq n$.

(b) Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$ be linearly independent, let $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^3$ also be linearly independent, and suppose $W_1 = \operatorname{span}\{\mathbf{x}_1, \mathbf{x}_2\}$, and $W_2 = \operatorname{span}\{\mathbf{y}_1, \mathbf{y}_2\}$. Prove that $W_1 \cap W_2 \neq \{\mathbf{0}\}$. [4 marks]

We need to show that there is a vector $\mathbf{w} \neq \mathbf{0}$ in both W_1 and W_2. Let us seek such a vector, which we must be able to express as a linear combination of the spanning set of W_1, i.e.,

$$\mathbf{w} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 \in W_1, \qquad \alpha_1, \alpha_2 \in \mathbb{R}$$

For $\mathbf{w}$ to be in W_2, we must also be able to write

$$\mathbf{w} = \beta_1 \mathbf{y}_1 + \beta_2 \mathbf{y}_2 \in W_2, \qquad \beta_1, \beta_2 \in \mathbb{R}$$

Thus

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + (-\beta_1)\mathbf{y}_1 + (-\beta_2)\mathbf{y}_2 = \mathbf{0}$$

We claim that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2\}$ cannot be linearly independent because $\mathbb{R}^3$ can be spanned by 3 vectors; otherwise the Fundamental Theorem of Linear Algebra would be violated. So there must at least one coefficient among $\alpha_1, \alpha_2, \beta_1, \beta_2$ that is not zero.

Let's say $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. Then because $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent, $\mathbf{w}$ must be nonzero. (That is, the only way $\mathbf{w}$ can be zero, given its expression as a linear combination of $\mathbf{x}_1, \mathbf{x}_2$, is for both α_1 and α_2 to be zero.) Similarly, if $\beta_1 \neq 0$ or $\beta_2 \neq 0$, we conclude that $\mathbf{w}$ is nonzero. (In fact, one of α_1 or α_2 must be nonzero and one of β_1 or β_2 must be as well.)

Thus $W_1 \cap W_2 \neq \{\mathbf{0}\}$.