NAME:______________________________

STUDENT NUMBER:______________________________

TUTORIAL GROUP:______________________________

16 December 2022

18:30 – 21:00

Time: 150 minutes

This is a closed-book exam worth a total of 120 points. Please answer all questions.

ONLY THE FOLLOWING ITEMS ARE ALLOWED ON YOUR DESK DURING THE EXAM:

1. **THIS EXAM BOOK –** It contains this cover page, eight question pages and two formula sheet pages (which may be torn off). Make sure you start by putting your **NAME, ID NUMBER,** and **TUTORIAL GROUP** on the front (cover) page of the exam. The entire exam book (minus the formula sheet) **will be handed in** at the end of the exam and marked.
    a. **FORMULA SHEET**, for the "Waves and Oscillations" part of the course – printed from Quercus on letterpaper and annotated on the printed page.
2. **A CALCULATOR**, which can be anything that does NOT connect to wifi and does NOT communicate with other devices. ACCEPTABLE calculators include programmable and graphing calculators, scientific calculators, etc. UNACCEPTABLE calculators include: cell phones, tablets, laptops, etc.
3. **A PEN OR/AND A PENCIL.**
4. **YOUR STUDENT ID CARD**, used to check your identity.

If you are missing anything from the above items, raise your hand and ask an exam supervisor to supply what is missing. If you are missing an item that should have been brought by you (e.g., calculator, pen/pencil) there is a limited supply of extras and are on a first come, first served basis.

The exam is printed on one side of the paper. You can use blank sides for additional calculations, but keep in mind that only the printed pages will be scanned and marked.

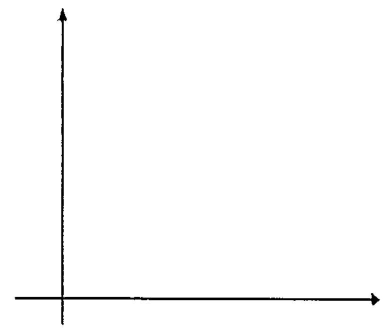
COMPLETE SOLUTION INCLUDES:

a) A sketch with a coordinate system, where appropriate. Where a diagram is provided, additional markings can be added to it.
b) Clear and logical workings, including quoting the equations that were used, proper substitutions of numbers (when necessary) with all steps clearly indicated.
c) The final answer with units and three significant figures, unless specificed in the question.

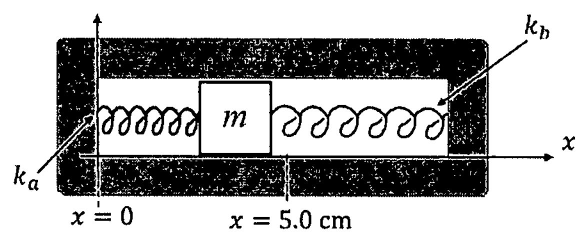
FOR OFFICE USE ONLY							
QUESTION	I	II	III	IV	V	VI	TOTAL
MARK							
MAXIMUM	20	20	20	20	20	20	120

Question I

1. A physical pendulum is shaped like a uniform bar of mass M and length L. Use the graph on the right to sketch the dependence of the period of the pendulum on the distance between the pivot point and the center of mass. [4 points]



2. Mass $m = 0.0780$ kg is attached to two horizontal springs, with spring constants $k_a = 9.00$ N/m and $k_b = 4.00$ N/m and placed in an empty container in a way that fastens the springs to the sides of the container, as shown in the picture.
The motion of the mass is described by the equation $x(t) = x_0 + A\cos(\omega t + \phi_0)$.
When both springs are in equilibrium the mass is at rest at a position $x = 5.0$ cm from the left-hand side of the container. At time $t = 0$ s the mass is at the position $x = 3.40$ cm and is moving to the right with the velocity $v = 19.5$ cm/s.



a. Determine the angular frequency of of the motion. [3 points]

b. Determine the initial phase constant of the motion. [6 points]

c. A viscous liquid is added to the container, causing additional force $F = -bv$ acting on the mass. What is the value of the constant b that would cause critical damping? [3 points]

d. If at time $t = 0$ s the mass is displaced by the distance $\Delta x(0) = \alpha$ and is moving with speed $v(0) = \beta$ towards the equilibrium, what are the values of the coefficients A and B in the solution to the equation of motion for this critically damped oscillator? Express your answers in terms of α, β, and γ. [4 points]

Question II

1. On a cool day ($v_{sound} = 330\ \mathrm{m/s}$) two **consecutive** standing waves observed in the $1.75\ \mathrm{m}$ long pipe have wavelengths of $1.4\ \mathrm{m}$ and $1.0\ \mathrm{m}$. The end of the pipe located at $x = 0\ \mathrm{m}$ is opened.

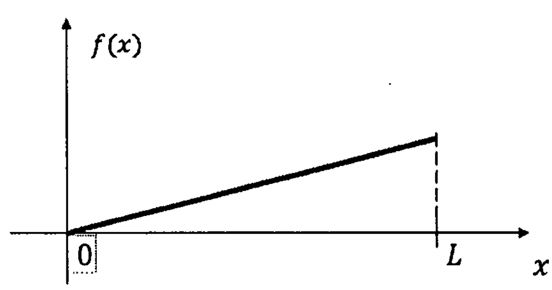
   a. Is the other end of the pipe opened or closed? Justify your answer with calculations. [4 points]

   b. What is the fundamendal frequency that can resonate in this pipe? [2 points]

   c. Write the expression for the n^{th} standing wave in the pipe as a function of x and t if at time $t = 0\ \mathrm{s}$ all particles are in the equilibrium positions. [4 points]

2. A function shown in the figure on the right is defined as $f(x) = ax$ where a is a positive constant for $0 \leq x \leq L$. The function can be represented by a Fourier series

$$f(x) = \sum_{n=1} A_n \sin\left(\frac{n\pi x}{L}\right).$$



   a. Write the expression for the coefficients of the Fourier series in an integral form, without evaluating anything. [2 points]

   b. Show that the coefficients can be expressed as $A_n = -(-1)^2 \frac{2aL}{n\pi}$ and determine the values of the first two coefficients. [8 points]

Question III

1. A dispersion relation for a wave can take the form of power law with $\omega = Ak^{\alpha}$. Show that the group velocity v_g is directly proportional to the phase velocity v_p with $v_g = \alpha v_p$. [4 points]

2. A wave of amplitude A coming from water ($v_{water} = 1480\ \mathrm{m/s}$, $\rho = 1000\ \mathrm{kg/m^3}$) is incident on a boundary with another liquid. You are looking for a liquid that allows for **at most** 1% of the incident intensity to be reflected back at the boundary.
   a. Determine the reflection and transmission coefficients for intensity and amplitude of the wave when 1% of the incident power is reflected [4 points]

   b. What is (are) the range(s) of the acoustic impedance values for the unknown material? [8 points]

   c. The traveling wave is attenuated in the unknown liquid. After traveling the distance of $100\ \mathrm{m}$ the intensity is equal to 85% of the initial transmitted intensity. What is the attenuation coefficient? [4 points]

Question IV

1. If an electron's kinetic energy is 100 keV, what is its speed (in units of m/s)? Explain why you need (or do not need) the relativistic expression. [5 points]

2. Electron and proton beams, each with a kinetic energy of 100 keV, are incident on a rectangular potential barrier of height 500 keV and thickness 1.0 Å. Which particle has the greater probability of tunneling through the barrier? You do not need to calculate the transmission coefficient explicitly, if you can explain your reasoning. [5 points]

3. An electron beam with a kinetic energy of 100 keV is often used to excite an atom from its ground state into an excited state. However, these excited states are short-lived and the excited electrons transition back to the ground state, emitting photons in the X-ray range. For example, an excited Cesium atom will emit an X-ray photon of 31000 eV.

   (a) When you measure the X-ray photon energy experimentally, the spectral line width (or energy uncertainty) is 12 eV. What is the lifetime of the excited state of the Cesium atom? [5 points]

   (b) What would be the wavelength of the X-ray (in units of m) if seen by an observer in a spaceship moving away from the Earth at a constant speed of 0.5c? [5 points]

Question V

1. For the following questions, consider a particle of mass m in the ground state of a one-dimensional infinite square well with walls at $x = 0$ and $x = a$.

   (a) Using the uncertainty principle, explain why the ground state cannot have the quantum number $n = 0$. [5 points]

   (b) At $t = 0$, the well suddenly expands to twice its original size (the right-hand side wall is moved instantaneously from $x = a$ to $x = 2a$), leaving the wave function (momentarily) undisturbed. What is the new ground state energy, E_n, and the wave function, $\psi_n(x)$, (of the new bigger well)? Express your answer in terms of a, m, and other constants. [5 points]

   (c) Calculate the probability of finding the particle, which was originally ($t < 0$) in the small well, in the ground state of the new (bigger) well, $\psi_1(x)$, for $t > 0$. [10 points]

More space for Question V on the next page.

Question VI

1. In some proton accelerators, proton beams are directed toward each other for head-on collisions. Suppose that in such an accelerator, protons move with a speed of $0.99c$ in the laboratory reference frame.

   (a) In the rest frame of one proton, calculate the speed of approach of the other proton with which it is about to collide head on. Express your answer as a multiple of c, using six significant digits. [5 points]

   (b) What is the kinetic energy of each proton beam (in units of MeV) in the laboratory reference frame? [5 points]

Question VI continued on the next page.

Question VI continued.

2. Astronaut Nick is space-travelling from planet X to planet Y at a speed of $0.6c$ relative to the planets. Nick's spaceship, Livian, is 1000 m long in its rest frame.

   (a) When Nick is precisely halfway between the planets, a distance of 1 light-hour from each, nuclear devices are detonated on both planets. The explosions are simultaneous in the frame of the planets. In Nick's reference frame, which explosion occurs first, and what is the difference in time between the explosions (in unites of light-hours)? [5 points]

   (b) As Nick continues his journey, another astronaut Nancy, onboard spaceship Rucid, overtakes Livian. Rucid is travelling at $0.8c$ relative to the planets, and is also 1000 m long in its rest frame. According to Nick, how long does Rucid take to completely pass? That is, how long is it from the instant the nose of Rucid is at the tail of Livian until the tail of Livian is at the nose of Rucid? [5 points]

Oscillations

$\omega = 2\pi f = \frac{2\pi}{T}$	$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{mgd}{I}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
$x(t) = A\cos(\omega t + \phi_i)$	$x(t) = A_0 \exp\left(-\frac{\gamma t}{2}\right)\cos(\omega t + \phi_i))$	$x(t) = A(\omega)\cos(\omega t - \delta)$	
$x(t) = A\exp\left(-\frac{\gamma t}{2}\right) + Bt\exp\left(-\frac{\gamma t}{2}\right)$	$x(t) = A\exp\left(\left(-\frac{\gamma}{2} + \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right) + B\exp\left(\left(-\frac{\gamma}{2} - \left(\left\lvert\omega_0^2 - \frac{\gamma^2}{4}\right\rvert\right)^{\frac{1}{2}}\right)t\right)$		
$q_0(\omega) = \frac{\varepsilon_0}{\omega Z}$	$q(t) = q_0(\omega)\cos(\omega t - \delta)$	$Z = \sqrt{\left(\frac{1}{\omega C} - \omega L\right)^2 + R^2}$	$i = \frac{dq}{dt}$
$V_R = i(t)R$	$V_C = \frac{q}{C}$	$V_L = L\frac{di}{dt}$	
$K = \frac{1}{2}mv^2$	$U = \frac{1}{2}kx^2$	$E(t) = E_0\exp(-\gamma t)$	$P = \frac{dE}{dt} = Fv$
$Q = \frac{\omega_0}{\gamma}$	$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$		
$A(\omega) = \frac{a\omega_0^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2}}$		$\tan\delta = \frac{\omega\gamma}{\omega_0^2 - \omega^2}$	
$\bar{P}(\omega) = \frac{b[v_0(\omega)]^2}{2}$	$\bar{P}_{max} = \frac{F_0^2}{2m\gamma}$	$\bar{P}(\omega) = \frac{F_0^2}{2m\gamma\left[\frac{4(\Delta\omega)^2}{\gamma^2} + 1\right]}$	

Waves

$v = \lambda f$	$y(x,t) = f(x \pm vt)$	$y(x,t) = A\cos(kx \pm \omega t + \phi_i)$	
$k = \frac{2\pi}{\lambda}$	$y(x,t) = (A\sin(kx) + B\cos(kx))\cos(\omega t)$		
$v = \sqrt{\frac{F_T}{\mu}}$	$v = \sqrt{\frac{B}{\rho}}$	$v = \sqrt{\frac{Y}{\rho}}$	$v = \sqrt{\frac{\gamma RT}{M}}$
$\frac{\partial y^2}{\partial t^2} = v^2\frac{\partial^2 y}{\partial x^2}$	$\omega = \frac{2\pi}{T}$	$f = \frac{1}{T}$	
$\omega_n = \frac{n\pi v}{L}$	$\omega_n = \frac{n\pi v}{2L}$	$f_n = nf_1$	$E = \frac{1}{4}\mu\omega_n^2 A_n^2 L$
$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$	$P_{ave} = \frac{1}{2}ZA^2\omega^2$	$I_{ave} = \frac{1}{2}Z_a A^2\omega^2$	$P = \mu vA^2\omega^2\sin^2(kx - \omega t + \phi_0)$
$Z = \sqrt{\mu\tau}$	$Z_a = \sqrt{Y\rho}$	$Z_a = \sqrt{B\rho}$	$R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$
$T = \frac{2Z_1}{Z_1 + Z_2}$			
$I(r) = I(r_0)\left[e^{-\alpha(r - r_0)}\right]\left(\frac{r_0}{r}\right)^{N-1}$	$v_g = \frac{d\omega}{dk}\big\rvert_{k=k_0}$	$v_g = v - \lambda\frac{dv}{d\lambda}$	$R^2 + \frac{Z_2}{Z_1}T^2 = 1$
$c = (\mu_0\varepsilon_0)^{-\frac{1}{2}}$	$v = \frac{c}{n}$		$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

Mathematical Formulae

$\cos\alpha + \cos\beta = 2\cos\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]$		$\cos\alpha - \cos\beta = -2\sin\left[\frac{\alpha+\beta}{2}\right]\sin\left[\frac{\alpha-\beta}{2}\right]$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$		$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$		$\tan^{-1}(x) = \{\theta, \theta + \pi\} + 2\pi n$ $\cos^{-1}(x) = \pm\theta + 2\pi n$
$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$		$\sin^{-1}(x) = \{\theta, \pi - \theta\} + 2\pi n$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$	$\tilde{A} = Ae^{j\theta} = A(\cos\theta + j\sin\theta)$
$\int\sin(ax)dx = -\frac{1}{a}\cos ax$		$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$

Constants

$c = 3.00 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$	$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T}\cdot\frac{\mathrm{m}}{\mathrm{A}}$	$\varepsilon_0 = 8.85 \times 10^{-12}\,\mathrm{N}\cdot\frac{\mathrm{m}}{\mathrm{C}^2}$	$g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}}$
$v_{sound\ at\ 20°C} = 343\,\frac{\mathrm{m}}{\mathrm{s}}$	$T_K = T_{°C} + 273.15°\mathrm{C}$		

Modern Physics Formulae

Useful constants:

$h = 6.626 \times 10^{-34}$ Js $= 4.14 \times 10^{-15}$eV $\cdot$ s (Planck constant)
$c = 3 \times 10^8$ m/s (speed of light)
$e = 1.602 \times 10^{-19}$ C (electron charge)
$m_e = 9.1 \times 10^{-31}$ kg $= 0.511$ MeV$/c^2$ (electron mass)
$m_p = 1.67 \times 10^{-27}$ kg $= 936.7$ MeV$/c^2$ (proton mass) $m_p/m_e = 1836$
$k = 8.99 \times 10^9$ Nm^2/C^2 (Coulomb constant)
$k_B = 1.38 \times 10^{-23}$ J/K $= 8.6 \times 10^{-5}$ eV/K (Boltzmann constant)
$N_A = 6.02 \times 10^{23}$ (Avogadro number)
$R_H = 1.096776 \times 10^7$ m^{-1} (Rydberg constant)

Quantum Mechanics:

$$\rho(\lambda, T) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \qquad E_{ph.el.} = hf - \phi \qquad E = hf = \hbar\omega$$

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta) \text{ (Compton)} \qquad \lambda_C = \frac{h}{mc} = 2.4263 \times 10^{-12}\text{m}$$

$$\frac{1}{\lambda} = R_H\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \qquad n\lambda = 2d\sin\theta \text{ (Bragg's law)} \qquad F_{cent} = \frac{mv^2}{r}$$

$$r_n = \frac{(n\hbar)^2}{kme^2} \qquad r_1 = 0.053\text{nm (Bohr radius)} \qquad E_n = -\frac{m(ke^2)^2}{2\hbar^2}\frac{1}{n^2}$$

$$\lambda = \frac{h}{p} \qquad \Delta x\Delta p \geq \frac{\hbar}{2} \qquad \Delta E\Delta t \geq \frac{\hbar}{2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi = i\hbar\frac{\partial\Psi}{\partial t} \qquad \hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) \qquad \hat{H}\psi(x) = E\psi(x)$$

$$\langle\hat{O}\rangle = \int \Psi(x,t)^*\hat{O}\Psi(x,t)dx \qquad \langle x\rangle = \sum xP(x) \text{ (discrete)} \qquad \langle x\rangle = \int xP(x)dx \text{ (continuous)}$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad \psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a} \qquad E_n = \frac{\hbar^2\pi^2}{2ma^2}n^2$$

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n\psi_n(x)e^{-iE_nt/\hbar} \qquad c_n = \int \psi_n^*(x)\Psi(x,0)dx \qquad \langle\hat{H}\rangle = \sum_{n=1}^{\infty}|c_n|^2E_n$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int \phi(k)e^{ikx}dk \qquad \phi(k) = \frac{1}{\sqrt{2\pi}}\int \Psi(x,0)e^{-ikx}dx$$

$$T = \frac{16E(U_0 - E)}{U_0^2}e^{-2\alpha a} \quad (E \ll U_0,\ \ \alpha^2 = 2m(U_0 - E)/\hbar^2)$$

$$T = \exp\left[-\frac{4\sqrt{2m}}{3\hbar}\frac{W^{3/2}}{e\varepsilon}dx\right] \text{ (Field Emission)} \qquad \ln\tau_\alpha \propto 1/\sqrt{E}$$

Special relativity:

$$\Delta t' = \gamma\Delta t_0, \qquad l' = l_0/\gamma, \qquad \gamma = (1 - v^2/c^2)^{-1/2} \qquad \beta = v/c$$

$$\begin{aligned} x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ t' &= \gamma(t - vx/c^2) & t &= \gamma(t' + vx'/c^2) \end{aligned}$$

$$\begin{aligned} u_x' &= \frac{u_x - v}{1 - u_xv/c^2} & u_x &= \frac{u_x' + v}{1 + u_x'v/c^2} \\ u_y' &= \frac{u_y}{\gamma(1 - u_xv/c^2)} & u_y &= \frac{u_y'}{\gamma(1 + u_x'v/c^2)} \\ u_z' &= \frac{u_z}{\gamma(1 - u_xv/c^2)} & u_z &= \frac{u_z'}{\gamma(1 + u_x'v/c^2)} \end{aligned}$$

$$f_{obs} = f_{sce}\sqrt{\frac{1 \mp \beta}{1 \pm \beta}} \text{ (upper sign moving away)} \qquad E^2 = (pc)^2 + (mc^2)^2$$

$$\vec{p} = \gamma_p m\vec{u}, \qquad E = \gamma_p mc^2, \qquad K = (\gamma_p - 1)mc^2, \qquad \gamma_p = (1 - u^2/c^2)^{-1/2}$$

Math formulae:

$$c = a + ib \qquad c^* = a - ib \qquad |c|^2 = c^*c = a^2 + b^2 \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$\sin x = \frac{e^{+ix} - e^{-ix}}{2i} \qquad \cos x = \frac{e^{+ix} + e^{-ix}}{2} \qquad \int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx = \frac{\sqrt{\pi}}{\alpha} \qquad \int_{-\infty}^{\infty} x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

$\sigma_y^2 = \langle y^2\rangle - \langle y\rangle^2$

For $x \ll 1$, $e^x \approx 1 + x$, $\ln(1 + x) \approx x$, $\sin x \approx x$, $\cos x \approx 1 - x^2/2$.