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# ECE259: Electromagnetism

## Term Test 1 - February 15th, 2022

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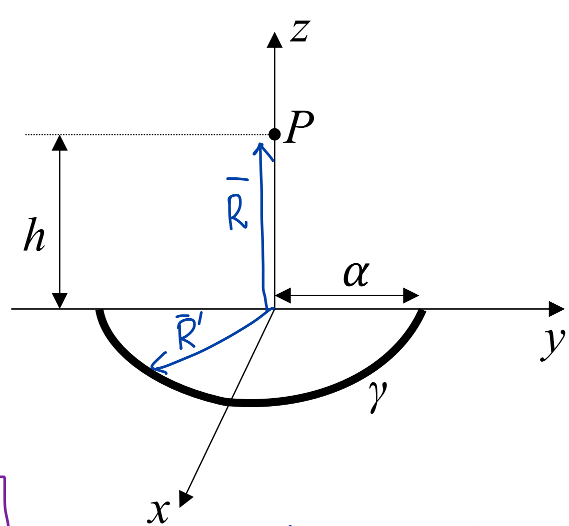
SOLUTION

### Instructions

- Duration: 1 hour 30 minutes (18:10 to 19:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- **Only answers that are fully justified will be given full marks!**

## Question 1

The path $\gamma$ shown in the figure consists of *half* of a circle of radius $\alpha$ centered at the origin and laying in the $x$-$y$ plane. The arc is in free space. A total amount of charge $Q$ is uniformly distributed along $\gamma$. A point $P$ is defined along the positive $z$ axis, at a distance $h > 0$ from the origin. Find the electric field vector $\mathbf{E}$ at $P$. [20 points]




$\rho_\ell = \frac{Q}{\pi\alpha}$ [2pt]

Superposition $\bar{E} = \int_\gamma \frac{\rho_\ell \, d\ell'}{4\pi\varepsilon_0} \frac{\bar{R}-\bar{R}'}{|\bar{R}-\bar{R}'|^3}$

Cylindrical coordinates

$\varphi' \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $r' = \alpha$ $z' = 0$

$\bar{R}' = \alpha\, \bar{a}_r'$ [2pt] $\bar{R} = h\, \bar{a}_z$ [2pt]

$d\ell' = \alpha\, d\varphi'$ [1pt]

[2pt] $\bar{R}-\bar{R}' = h\,\bar{a}_z - \alpha\,\bar{a}_r' = h\,\bar{a}_z - \alpha\cos\varphi'\,\bar{a}_x - \alpha\sin\varphi'\,\bar{a}_y$

$|\bar{R}-\bar{R}'| = \sqrt{h^2+\alpha^2}$ [1pt]

integration limits [2pt]

$$\bar{E} = \int_{\varphi'=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{Q\,\cancel{\alpha}\,d\varphi'}{\pi\,\cancel{\alpha}\,4\pi\varepsilon_0} \cdot \frac{1}{\left[h^2+\alpha^2\right]^{3/2}} \left[-\alpha\cos\varphi'\,\bar{a}_x - \underbrace{\alpha\sin\varphi'\,\bar{a}_y}_{\text{integrates to zero}} + h\,\bar{a}_z\right] =$$

$$= \frac{Q}{4\pi^2\varepsilon_0\left[h^2+\alpha^2\right]^{3/2}} \left[-\alpha\,\sin\varphi'\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{a}_x + \pi h\,\bar{a}_z\right] =$$

integral calculation [6pt]

$$= \frac{Q}{4\pi^2\varepsilon_0\left[h^2+\alpha^2\right]^{3/2}} \left[-\alpha\,2\,\bar{a}_x + \pi h\,\bar{a}_z\right]$$

Final answer: (*)

correct → [2pt]

partially → [1pt]

wrong/missing [0pt]

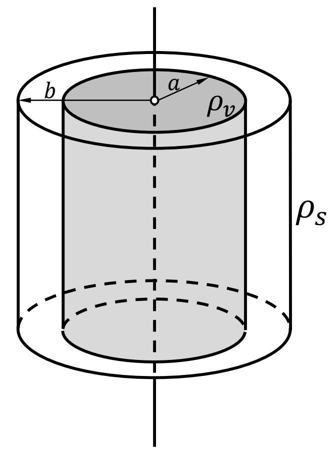
*: irrespectively of carry-over mistakes

## Question 2

Two very long cylinders have equal but opposite charge. Inner cylinder of radius $a$ has a volume charge distribution of

$$\rho_v = \begin{cases} \rho_0 \frac{r}{a} & 0 \le r \le a, \quad \rho_0 > 0 \\ 0 & 0 < r < b \end{cases}$$

Outer cylinder has radius $b$, is very thin, and has a uniform surface charge distribution. Permittivity is $\epsilon_0$ everywhere.




Find:

(1) the surface charge density $\rho_s$ on the outer cylindrical surface in terms of $\rho_0$. [4 points]

(2) the electric field $\mathbf{E}$ everywhere. [8 points]

(3) the potential $V$ everywhere, taking the potential of the outer cylinder as reference (that is $V(r = b) = 0$). [8 points]

a) consider a section of length L

Charge in inner cylinder

$$Q_{in} = \int_{z=0}^{L}\int_{r=0}^{a}\int_{\varphi=0}^{2\pi} \rho_0 \frac{r}{a}\, r\, d\varphi\, dz\, dr = \frac{\rho_0}{a} 2\pi L \int_0^a r^2 dr = \frac{\rho_0 2\pi L}{a}\frac{a^3}{3}$$

$$= \frac{2\pi \rho_0 L a^2}{3}$$

Writing correctly the integral for Qin 1 mark

getting the correct Qin 2 marks

Charge on outer cylinder $Q_{out} = -Q_{in}$

$Q_{out} = 2\pi b L \rho_s$ ;

$$\rho_s = \frac{-Q_{in}}{2\pi b L} = -\frac{\cancel{2\pi} \rho_0 \cancel{L} a^2}{3\, \cancel{2\pi} b \cancel{L}} = -\frac{\rho_0 a^2}{3b}$$

Getting the correct \rho_s 1 mark. (If the sign is wrong, then subtract 1/2 mark)

b) Use Gauss' law

Cylindrical symmetry $\Rightarrow \bar{E} = E(r)\,\bar{a}_r$

Gaussian surface: cylindrical surface, radius $r$, height $L$

4 marks in total for Region r<a:
- 1 mark for using Gauss's Law correctly
- 1 mark for writing the correct integration expression
- 2 marks for carrying out correct calculations and having the correct direction

-subtract 1/2 mark for each small inconsequential mistake
- subtract 1/2 mark for missing the direction
-subtract /2 mark for the wrong sign.

Region $r<a$

$$\int_S \bar{E}\cdot d\bar{S} = \frac{Q}{\varepsilon_0} \qquad E(r)\,2\pi r L = \frac{Q}{\varepsilon_0}$$

$$Q = \iiint_{r'=0}^{r} \frac{\rho_0 r'}{a}\, r'\,d\varphi\,dr'\,dz = \frac{\rho_0}{a} 2\pi L \int_{r'=0}^{r'=r} (r')^2\,dr' = \frac{\rho_0 2\pi L}{a}\frac{r^3}{3}$$

$$E(r) = \frac{\rho_0 \cancel{2\pi}\cancel{L}\, r^{\cancel{3}\,2}}{\cancel{2\pi}\cancel{r}\,\varepsilon_0\, 3a\,\cancel{L}} = \frac{\rho_0 r^2}{3a\,\varepsilon_0}$$

$$\boxed{\bar{E} = \frac{\rho_0 r^2}{3a\,\varepsilon_0}\,\bar{a}_r}$$

## Region $r \in [a,b]$

$$\int_S \bar{E} \cdot d\bar{S} = \frac{Q}{\varepsilon_0} \qquad Q = \frac{2\pi \rho_0 L a^2}{3}$$

2 marks for Region a<r<b:
- 1 mark for applying Gauss's Law correctly
- 1 mark for getting the correct expression and direction

$$\bar{E} = \frac{\cancel{2\pi} \rho_0 \cancel{L} a^2}{3 \cdot \cancel{2\pi} r \cancel{L} \varepsilon_0} \bar{a}_r = \frac{\rho_0 a^2}{3 \varepsilon_0 r} \bar{a}_r$$

## Region $r > b$

$$\bar{E} = 0$$

2 marks

## c) Potential

For $r > b$, $\bar{E} = 0$, $\Rightarrow V(r) = V(\infty) = 0$

2 marks for r>b region.

For $r \in [a,b]$:

$$V = -\int_{r'=b}^{r'=r} \bar{E} \cdot d\bar{\ell} = \int_{r'=r}^{r'=b} \frac{\rho_0 a^2}{3\varepsilon_0 r'} \bar{a}_r \cdot \bar{a}_r \, dr' =$$

$$= \frac{\rho_0 a^2}{3\varepsilon_0} \int_r^b \frac{1}{r'} dr' = \frac{\rho_0 a^2}{3\varepsilon_0} \ln\left(\frac{b}{r}\right)$$

3 marks for region a<r<b:
- 1 mark for correct integral limits
- 1 mark for correct integral expression
- 1 mark for carrying out the integration correctly.

For $r < a$

$$V(r=a) = \frac{\rho_0 a^2}{3\varepsilon_0} \ln\left(\frac{b}{a}\right)$$

$$V(r) = V(a) + \int_{r'=r}^{r'=a} \frac{\rho_0 (r')^2}{3a\varepsilon_0} \bar{a}_r \cdot \bar{a}_r \, dr' =$$

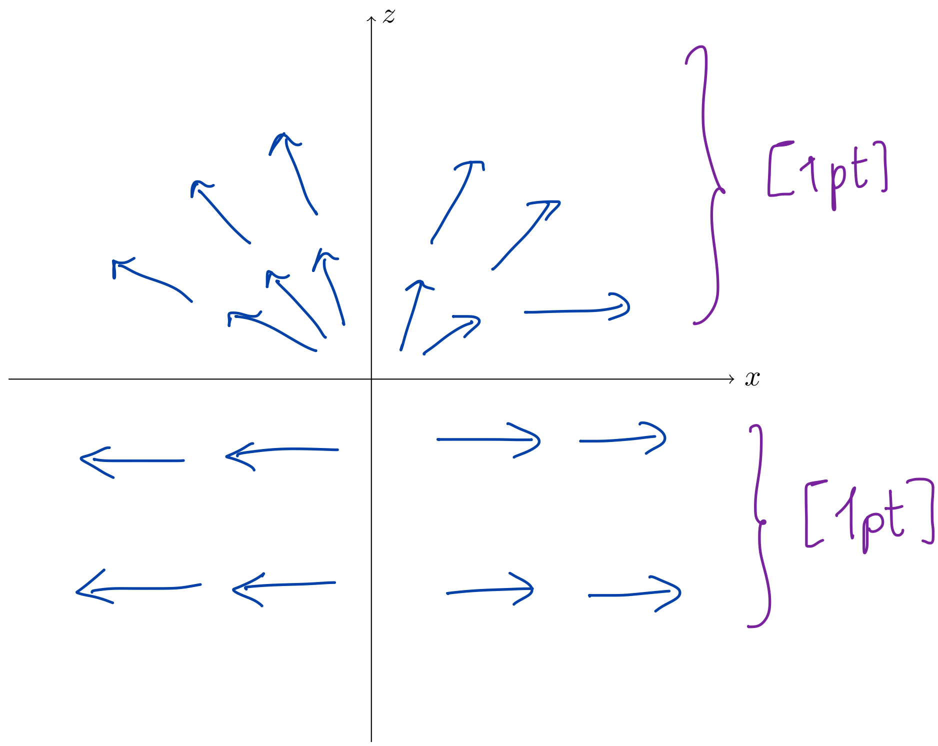
3 marks for r<a region:
- 1 mark for using V(r=a) as a starting point
- 1 mark for correct integral expression and limits
- 1 mark for correct final expression

$$= \frac{\rho_0 a^2}{3\varepsilon_0} \ln\left(\frac{b}{a}\right) + \frac{\rho_0}{3a\varepsilon_0}\left[\frac{a^3}{3} - \frac{r^3}{3}\right]$$

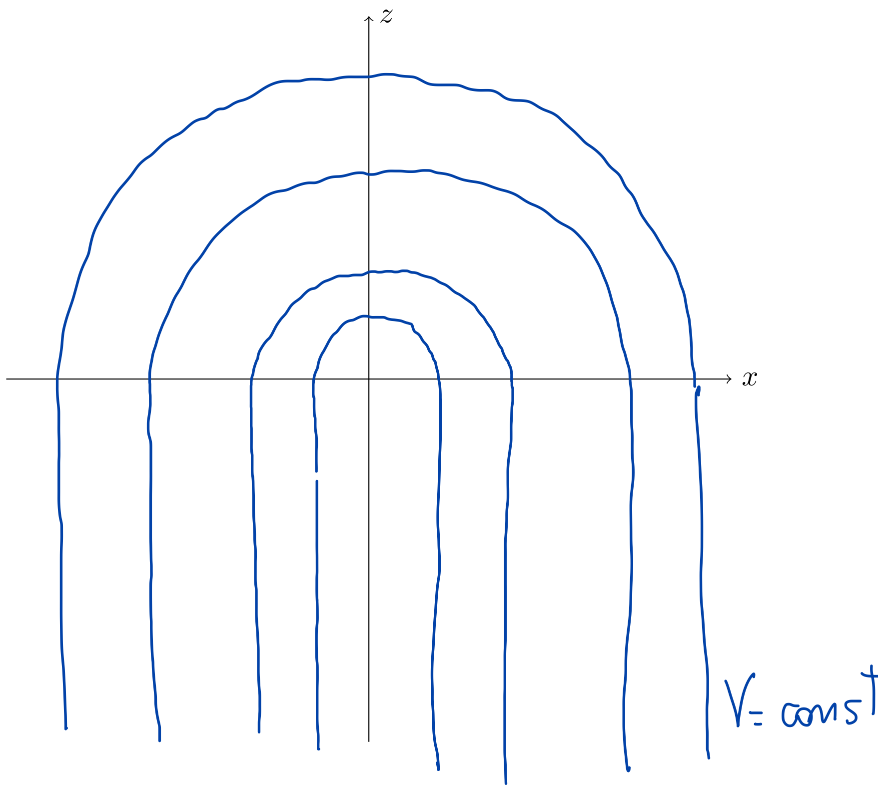
## Question 3.1

Consider the following electric field distribution $\mathbf{E}(x,y,z) = \begin{cases} 5\mathbf{a}_R & z \geq 0 \\ 5\mathbf{a}_r & z < 0 \end{cases}$.

a) In the graph below, sketch the direction of $\mathbf{E}$ in all four quadrants. Draw *at least* four arrows per quadrant [2 points].



b) In the graph below, sketch four different equipotential surfaces. Briefly justify your answer [2 points].



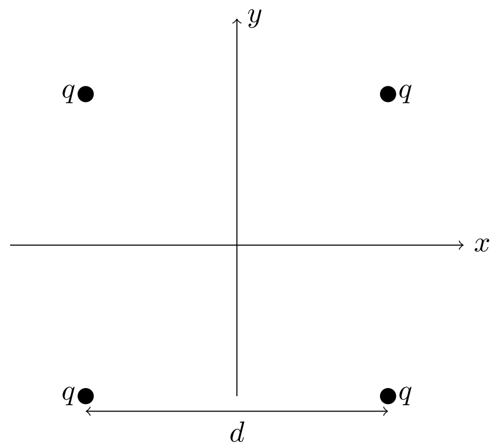

graph: [1pt]

The equipotential surfaces are normal to $\bar{E}$ everywhere

(since $\bar{E} = -\nabla V$)

justific. [1pt]

## Question 3.2




Four equal charges of value $q > 0$ are located at the four vertices of a square in vacuum. The zero reference for potential is at infinity.

a) Calculate the electric field at the origin. [2 points]

Zero.

2 marks awarded to correct answer with or without calculation.

(No calculation is needed)

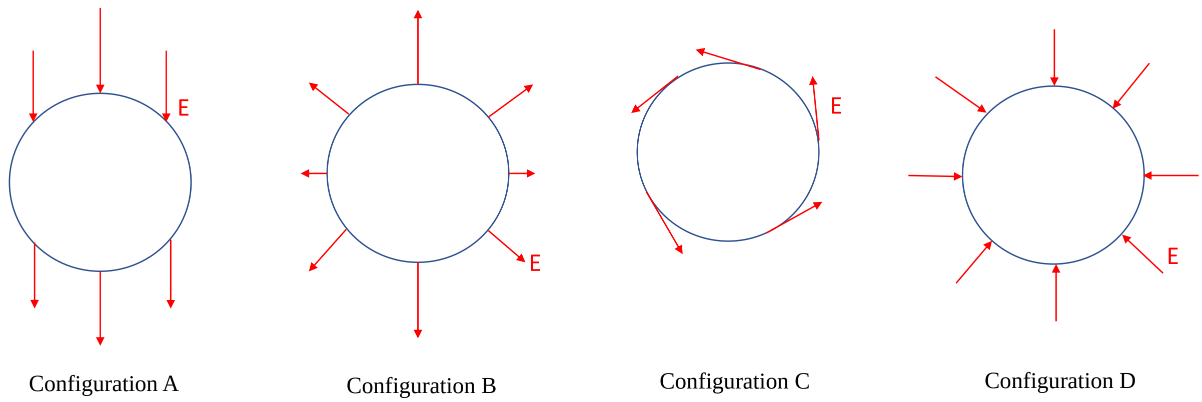
b) Calculate the external work needed to bring a positive test charge ($q_{test} > 0$) from infinity to the origin. [4 points]

3 marks for writing the correct potential V(0) at the origin
1 mark for knowing external work = q_test * V(0)

$$V(0) = \sum_{k=1}^{4} \frac{q_k}{4\pi\epsilon_o(\frac{d}{\sqrt{2}})} = \frac{\sqrt{2}q}{\pi\epsilon_o d} \qquad W_{ext} = q_{test}V(0) = \frac{\sqrt{2}qq_{test}}{\pi\epsilon_o d}$$

### Question 3.3

The cylindrical object in the figures is made by a perfectly conducting material. The arrows indicate the direction of a static electric field on the boundary of the object. There are four configurations labelled as A, B, C and D.




A B C D

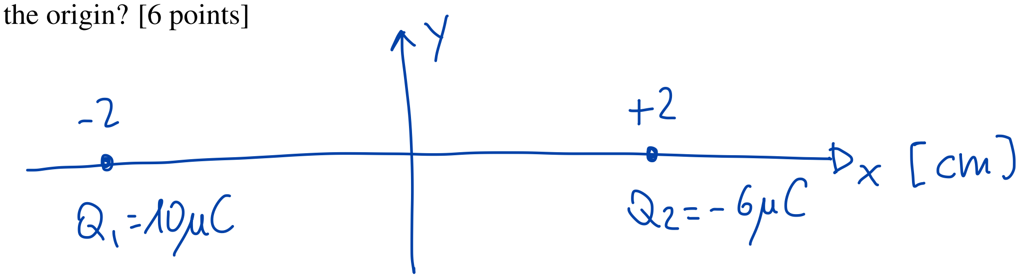
Indicate which configurations are physically possible, and which ones are not. Justify your answer. [4 points]

$\bar{E}$ must always be normal to the surface of a perfect conductor ($E_t = 0$) ] 2pt justif.

B, D possible
A, C not possible } 0.5pt for each correct answer (up to 2pt]

## Question 3.4

A charge $Q_1 = 10\mu C$ is located at $(x, y) = (-2\,cm, 0\,cm)$. Another charge $Q_2 = -6\,\mu C$ is located at $(x, y) = (2\,cm, 0\,cm)$. Where should another point charge with $Q_3 = -2\,\mu C$ be located to make the electric field zero at the origin? [6 points]




$$\bar{E} = \frac{Q_1}{4\pi\varepsilon_0}\frac{0+2\bar{a}_x}{2^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{-2\bar{a}_x}{2^3} + \frac{Q_3}{4\pi\varepsilon_0}\frac{-\bar{R}_3}{|\bar{R}_3|^3} =$$

$$= \frac{Q_1}{\cancel{4\pi\varepsilon_0}}\frac{\bar{a}_x}{4} + \frac{Q_2}{\cancel{4\pi\varepsilon_0}}\frac{-\bar{a}_x}{4} + \frac{Q_3}{\cancel{4\pi\varepsilon_0}}\frac{-\bar{R}_3}{|\bar{R}_3|^3} = 0$$

$$\frac{Q_1-Q_2}{4}\bar{a}_x + Q_3\frac{-\bar{R}_3}{|\bar{R}_3|^3} = 0 \qquad \bar{R}_3 = x_3\bar{a}_x$$

$$\frac{Q_1-Q_2}{4}\cancel{\bar{a}_x} + Q_3\frac{-x_3\cancel{\bar{a}_x}}{|x_3|^3} = 0$$

$$\frac{Q_1-Q_2}{4} - Q_3\frac{x_3}{|x_3|^3} = 0 \quad ; \quad \frac{x_3}{|x_3|^3} = \left.\frac{\overbrace{Q_1-Q_2}^{16\mu C}}{\underbrace{4Q_3}_{<0}}\right\} < 0$$

$$\Rightarrow x_3 < 0$$

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$$-\frac{1}{x_3^2} = \frac{16\cdot\cancel{10^{-6}}}{-8\cdot\cancel{10^{-6}}} \qquad \frac{1}{x_3^2} = 2 \;;\; x_3 = \frac{-1}{\sqrt{2}}$$

2 marks for knowing the
correct expression for E field
due to a single charge
(both magnitude and direction)

1 mark for knowing vector
superposition

2 marks for carrying through
the calculation correctly

1 mark for stating the final location
correctly (-1/sqrt(2) cm , 0cm)

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**1. Coordinate Systems**

*1.1 Cartesian coordinates*

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

*1.2 Cylindrical coordinates*

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

*1.3 Spherical coordinates*

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

**2. Relationship between coordinate variables**

| - | Cartesian | Cylindrical | Spherical |
|---|---|---|---|
| $x$ | $x$ | $r\cos\phi$ | $R\sin\theta\cos\phi$ |
| $y$ | $y$ | $r\sin\phi$ | $R\sin\theta\sin\phi$ |
| $z$ | $z$ | $z$ | $R\cos\theta$ |
| $r$ | $\sqrt{x^2+y^2}$ | $r$ | $R\sin\theta$ |
| $\phi$ | $\tan^{-1}\frac{y}{x}$ | $\phi$ | $\phi$ |
| $z$ | $z$ | $z$ | $R\cos\theta$ |
| $R$ | $\sqrt{x^2+y^2+z^2}$ | $\sqrt{r^2+z^2}$ | $R$ |
| $\theta$ | $\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$ | $\tan^{-1}\frac{r}{z}$ | $\theta$ |
| $\phi$ | $\tan^{-1}\frac{y}{x}$ | $\phi$ | $\phi$ |

## 3. Dot products of unit vectors

| $\cdot$ | $\mathbf{a}_x$ | $\mathbf{a}_y$ | $\mathbf{a}_z$ | $\mathbf{a}_r$ | $\mathbf{a}_\phi$ | $\mathbf{a}_z$ | $\mathbf{a}_R$ | $\mathbf{a}_\theta$ | $\mathbf{a}_\phi$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{a}_x$ | 1 | 0 | 0 | $\cos\phi$ | $-\sin\phi$ | 0 | $\sin\theta\cos\phi$ | $\cos\theta\cos\phi$ | $-\sin\phi$ |
| $\mathbf{a}_y$ | 0 | 1 | 0 | $\sin\phi$ | $\cos\phi$ | 0 | $\sin\theta\sin\phi$ | $\cos\theta\sin\phi$ | $\cos\phi$ |
| $\mathbf{a}_z$ | 0 | 0 | 1 | 0 | 0 | 1 | $\cos\theta$ | $-\sin\theta$ | 0 |
| $\mathbf{a}_r$ | $\cos\phi$ | $\sin\phi$ | 0 | 1 | 0 | 0 | $\sin\theta$ | $\cos\theta$ | 0 |
| $\mathbf{a}_\phi$ | - $\sin\phi$ | $\cos\phi$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 |
| $\mathbf{a}_z$ | 0 | 0 | 1 | 0 | 0 | 1 | $\cos\theta$ | $-\sin\theta$ | 0 |
| $\mathbf{a}_R$ | $\sin\theta\cos\phi$ | $\sin\theta\sin\phi$ | $\cos\theta$ | $\sin\theta$ | 0 | $\cos\theta$ | 1 | 0 | 0 |
| $\mathbf{a}_\theta$ | $\cos\theta\cos\phi$ | $\cos\theta\sin\phi$ | $-\sin\theta$ | $\cos\theta$ | 0 | $-\sin\theta$ | 0 | 1 | 0 |
| $\mathbf{a}_\phi$ | - $\sin\phi$ | $\cos\phi$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 |

## 4. Differential operators

*4.1 Gradient*

$$\nabla V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z = \frac{\partial V}{\partial R}\vec{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

*4.2 Divergence*

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial (R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial (\sin\theta A_\theta)}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

*4.3 Laplacian*

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

*4.4 Curl*

$$\begin{aligned}
\nabla \times \vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\vec{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial (\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\vec{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial R}\right)\vec{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right)\vec{a}_\phi
\end{aligned}$$

**5. Electromagnetic formulas**

**Table 1** Electrostatics

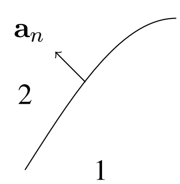
| | |
|---|---|
| $\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$ | $\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3} dQ'$ |
| $\nabla \times \mathbf{E} = 0$ | $\nabla \cdot \mathbf{D} = \rho_v$ |
| $\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$ | $\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$ |
| $\mathbf{E} = -\nabla V$ | $V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$ |
| $V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}$ | |
| $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$ | $\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$ |
| $\rho_{p,v} = -\nabla \cdot \mathbf{P}$ |  |
| $\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$ | |
| $\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$ | |
| $E_{1,t} = E_{2,t}$ | |
| $Q = CV$ | $W_e = \frac{1}{2} QV$ |
| $W_e = \frac{1}{2}\int_v \rho_v V dv = \frac{1}{2}\int_{\mathbb{R}^3} \mathbf{D} \cdot \mathbf{E} dv$ | |
| $\nabla \cdot (\varepsilon \nabla V) = -\rho_v$ | $\nabla \cdot (\varepsilon \nabla V) = 0$ |

**Table 2** Magnetostatics

| | |
|---|---|
| $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ | $\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$ |
| $\nabla \times \mathbf{B} = \mu \mathbf{J}$ | $\nabla \cdot \mathbf{B} = 0$ |
| $\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$ | $\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$ |
| $\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$ | $\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$ |
| $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$ | $\mathbf{M} = \chi_m \mathbf{H}$ |
| $\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$ | $\mathbf{J}_m = \nabla \times \mathbf{M}$ |
| $B_{1,n} = B_{2,n}$ | $\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$ |
| $\nabla \times \mathbf{H} = \mathbf{J}$ | $\oint \mathbf{H} \cdot \mathbf{dl} = I$ |
| | $W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$ |
| $L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$ | $L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$ |
| $\mathcal{R} = \frac{l}{\mu S}$ | $V_{mmf} = NI$ |

**Table 3** Faraday's law, Ampere-Maxwell law

| | |
|---|---|
| $V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$ | $V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$ |
| $\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$ | $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ |

**Table 4** Currents

| | |
|---|---|
| $I = \int_S \mathbf{J} \cdot \mathbf{dS}$ | $\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$ |
| $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$ | $\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$ |
| $J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$ | $\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$ |
| $R = \frac{l}{\sigma S}$ | $\sigma = -\rho_e \mu_e = \frac{1}{\rho}$ |