UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

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Closed book.

All questions are of equal value.

Permitted calculators (any suffix is acceptable):

- Sharp EL-W516
- Casio FX-991

This test contains XX pages including this page and the cover page, printed two-sided. Do <u>not</u> tear any pages from this test. Present complete solutions in the space provided.

Given information:

$$cos\,\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\,\vec{u} = \frac{\vec{u}\cdot\vec{d}}{||\,\vec{d}\,||^2}\vec{d}$$

$$det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The normal system of equations corresponding to $A\vec{x} = \vec{b}$ is given by:

$$A^T A\vec{x}_{LS} = A^T\vec{b}$$

Euler's method for solving a first order differential equation $y'(t) = f(t, y(t))$ is given by:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

Q1: Consider the system of linear equations represented by the following augmented matrix:

$$[A|\vec{b}] = \left[\begin{matrix} 0 & 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{matrix}\middle| \begin{matrix} 7 \\ 8 \\ 9 \end{matrix}\right]$$

1.5 a) What is the rank of the augmented matrix?

2 b) What $3x3$ elementary matrix when multiplied on the left of the augmented matrix will bring the augmented matrix to its reduced normal form?

1.5 c) Declare the leading variables and the free variables associated with this system. Please use $x_1, \cdots, x_6$ to denote the variables.

3 d) Write the vector form of the solution to this system of linear equations.

2 e) How does the solution to this system change if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$? Write the solution to this revised system $[A|\vec{0}]$ in vector form.

Solution:

a) Multiplying row 3 by $\frac{1}{3}$ brings the augmented matrix to its reduced normal form. The number of leading one's is then 3, so the rank is 3.

b) The $3x3$ elementary matrix required to bring the augmented matrix to its reduced normal form is given by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

c) Leading variables: x_2, x_4, x_5

Free variables: x_1, x_3, x_6

d) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 8 \\ 3 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -4 \\ 0 \\ -5 \\ -2 \\ 1 \end{bmatrix}$$

e) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ -4 \\ 0 \\ -5 \\ -2 \\ 1 \end{bmatrix}$$

Q2: Consider the two matrices given below:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

3 a) Find the eigenvalues (λ) and eigenvectors of matrix A.

3 b) Find the eigenvalues (β) and eigenvectors of matrix B.

2 c) Consider the following argument:

$$(A + B)\vec{x} = A\vec{x} + B\vec{x} = \lambda\vec{x} + \beta\vec{x} = (\lambda + \beta)\vec{x}$$

It would appear from this argument that an eigenvalue λ of matrix A plus an eigenvalue β of matrix B gives an eigenvalue of matrix $A + B$. Show that this is in fact not true by finding the eigenvalues (α) of the matrix $C = A + B$ using matrices A and B given at the beginning of this question.

2 d) Identify the flaw in the argument presented in part (c) and the step in the argument that is incorrect.

Solution:

a) $A - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 - (1)(0) = \lambda^2 = 0$

$\therefore \lambda = 0{,}0$

$A\vec{x} = \lambda\vec{x}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ x_2 is leading and x_1 is free.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore$ eigenvectors are all non-zero vectors parallel to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) $B - \beta I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} -\beta & 0 \\ 1 & -\beta \end{bmatrix}$

$\det(B - \beta I) = \beta^2 - (1)(0) = \beta^2 = 0$

$\therefore \beta = 0{,}0$

$B\vec{x} = \beta\vec{x}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ x_1 is leading and x_2 is free.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ eigenvectors are all non-zero vectors parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

c) $C = A + B = \begin{bmatrix} 0+0 & 1+0 \\ 0+1 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$C - \alpha I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} -\alpha & 1 \\ 1 & -\alpha \end{bmatrix}$

$\det(C - \alpha I) = \alpha^2 - (1)(1) = \alpha^2 - 1 = 0$

$\therefore \alpha = +1, -1$

$B\vec{x} = \beta\vec{x}$

$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ x_1 is leading and x_2 is free.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ eigenvectors are all non-zero vectors parallel to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ this example shows that an eigenvalue λ of matrix A (0,0) plus an eigenvalue β of matrix B (0,0) does not give an eigenvalue of matrix $C = A + B$ $(+1, -1)$.

d) The flaw in the argument is that matrix A and matrix B do not necessarily have the same eigenvectors. Therefore, the flaw occurs in this step $A\vec{x} + B\vec{x} = \lambda\vec{x} + \beta\vec{x}$.

Q3: Given matrix A below:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

3 a) Find three elementary matrices that bring matrix A to an upper triangular matrix U, i.e.

$$E_3 E_2 E_1 A = U$$

2 b) Multiply these three elementary matrices to produce matrix M such that:

$$MA = U$$

3 c) Now consider the system of linear equations represented by the following augmented matrix:

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 4 & 6 & 1 & 0 \\ -2 & 2 & 0 & 0 \end{array}\right]$$

Using matrix U and matrix M found in parts (a) and (b), solve this system of linear equations by backward substitution. Please use x_1, x_2, x_3 to denote the variables.

2 d) Give the column picture of the <u>solution</u> to this system of linear equations $[A|\vec{b}]$ found in part (c).

Solution:

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$; $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$; $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$; $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$= U$

b) $M = E_3E_2E_1 = (E_3E_2)E_1$

$$E_3E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$M = (E_3E_2)E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$$

Check $MA = U$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

c) Need to first find $M\vec{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$$

Now solve the following equivalent system by backward substitution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & -2 & 10 \end{array}\right]$$

$$-2x_3 = 10 \;\rightarrow\; x_3 = -5$$
$$2x_2 + x_3 = -4 \;\rightarrow\; x_2 = \frac{-4-(-5)}{2} = 0.5$$
$$x_1 + x_2 = 1 \;\rightarrow\; x_1 = 1 - 0.5 = 0.5$$

$$\vec{x} = \begin{bmatrix} 0.5 \\ 0.5 \\ -5 \end{bmatrix}$$

d) The solution is a unique linear combination of the column vectors of matrix A that equals $\vec{b}$, namely:

$$0.5\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + 0.5\begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} - 5\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Q4: Matrices A and B are both $3x3$ matrices. For each part of this question, choose the only matrix B that makes the given statement about A and B hold true for every matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

You are to treat each part of this question separately, so your answer for matrix B for each part will likely be different. Substitute your answer for matrix B in each part to demonstrate that the given statement regarding BA holds true.

2.5 a) $BA = 4A$
2.5 b) $BA = 4B$
2.5 c) BA has rows 1 and 3 of matrix A reversed and row 2 unchanged
2.5 d) All rows of BA are the same as row 1 of matrix A

Solutions:

a) Let $B = 4I = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$$\therefore BA = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 4a & 4b & 4c \\ 4d & 4e & 4f \\ 4g & 4h & 4i \end{bmatrix} = 4A$$

b) Let $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\therefore BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4B$$

c) Let $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore BA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix}$$

d) Let $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$\therefore BA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

Q5: In both parts of this question, you are required to find the least squares solution by solving the corresponding normal system of equations.

5 a) A person is conducting three experiments that involve selecting a value for the independent variable x and measuring the corresponding dependent variable y. The data collected from these three experiments is given in the table below:

x	y
1	2
2	2
3	2

Determine the least squares fit to this data, where the curve being fitted is given by:

$y = mx + b$ where m and b are unknown scalars and $-\infty < m < \infty$

If the least squares solution is unique, make an $x - y$ plot of the data along with the fitted curve. If the least squares solution is not unique or does not exist, make an $x - y$ plot of the data and explain what this means in terms of the fitted curve.

5 b) A person is conducting three experiments that involve selecting a value for the independent variable x and measuring the corresponding dependent variable y. The data collected from these three experiments is given in the table below:

x	y
2	1
2	2
2	3

Determine the least squares fit to this data, where the curve being fitted is given by:

$y = mx + b$ where m and b are unknown scalars and $-\infty < m < \infty$

If the least squares solution is unique, make an $x - y$ plot of the data along with the fitted curve. If the least squares solution is not unique or does not exist, make an $x - y$ plot of the data and explain what this means in terms of the fitted curve.

Solution:

a) Setting up this problem in the $A\vec{x} = \vec{b}$ form:

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Forming the corresponding normal system of equations:

$$A^T A \vec{x}_{LS} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}$$

Note: $\det(A^T A) = 6, \therefore\ invertible, \therefore\ unique\ solution$

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -6 \\ -6 & 14 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

Solving for $\vec{x}_{LS}$:

$$\vec{x}_{LS} = \begin{bmatrix} m \\ b \end{bmatrix} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{6} \begin{bmatrix} 0 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

The least squares fit is given by:

$$y = 2 \quad (m = 0; b = 2)$$

b) Setting up this problem in the $A\vec{x} = \vec{b}$ form:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Forming the corresponding normal system of equations:

$$A^T A \vec{x}_{LS} = A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix}$$

Note: $\det(A^T A) = 0, \therefore \textit{not invertible}, \therefore \textit{no unique solution}$

$$A^T \vec{b} = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

Solving for $\vec{x}_{LS}$:

$$\left[\begin{array}{cc|c} 12 & 6 & 12 \\ 6 & 3 & 6 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 12 & 6 & 12 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0.5 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

m is leading and b is free:

$$\vec{x}_{LS} = \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}, \therefore \textit{infinite solutions}$$

The least squares fit is given by:

$$y = mx + b = (1 - 0.5b)x + b$$

With b being a free variable, this corresponds to an infinite set of straight lines that go through the point $(2,2)$.

Q6:

2 a) Given the following state-space representation of an initial value problem:

$$\frac{d}{dt}\begin{bmatrix} e(t) \\ f(t) \\ g(t) \end{bmatrix} = Z'(t) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} e(t) \\ f(t) \\ g(t) \end{bmatrix} = AZ(t)$$

with:

$$\begin{bmatrix} e(0) \\ f(0) \\ g(0) \end{bmatrix} = Z_0 = \begin{bmatrix} e_0 \\ f_0 \\ g_0 \end{bmatrix}$$

Write out the individual scalar differential equations and scalar initial conditions represented by the above state-space representation.

2 b) This system of equations can be solved analytically. Provide the analytical solution, i.e. find expressions for $e(t), f(t), g(t)$.

1 c) Although this system can be solved analytically, the decision has been made to solve it numerically using Euler's method. Using the above state-space representation, write the update equation for Z_{n+1} where n is an integer that denotes the time step.

2 d) Using your solution to part (c), write out the individual scalar difference equations that will be used in part (e) to solve this system numerically, assuming a step size of $\Delta t = 1$.

3 e) Provide the numerical solution to this system for three time steps, Z_0, Z_1 and Z_2 assuming $Z_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Give your solution in the table below:

n	e_n	f_n	g_n
0			
1			
2			

Solution:

a)

$$\frac{de(t)}{dt} = 0\,; e(0) = e_0$$

$$\frac{df(t)}{dt} = e(t)\,; f(0) = f_0$$

$$\frac{dg(t)}{dt} = 2f(t); g(0) = g_0$$

b)

$$\int_0^t \frac{de(u)}{du}\,du = \int_0^t 0\,du$$

$$\therefore e(t) = e_0$$

$$\int_0^t \frac{df(u)}{du}\,du = \int_0^t e(u)\,du = \int_0^t e_0\,du$$

$$\therefore f(t) = e_0 t + f_0$$

$$\int_0^t \frac{dg(u)}{du}\,du = \int_0^t 2f(u)\,du = \int_0^t (2e_0 u + 2f_0)\,du$$

$$\therefore g(t) = e_0 t^2 + 2f_0 t + g_0$$

c) $Z_{n+1} = Z_n + \Delta t A Z_n$

d) $\begin{bmatrix} e_{n+1} \\ f_{n+1} \\ g_{n+1} \end{bmatrix} = \begin{bmatrix} e_n \\ f_n \\ g_n \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} e_n \\ f_n \\ g_n \end{bmatrix}$

$e_{n+1} = e_n$
$f_{n+1} = f_n + e_n$
$g_{n+1} = g_n + 2f_n$

e)

n	e_n	f_n	g_n
0	1	1	1
1	1	2	3
2	1	3	7