AER210F VECTOR CALCULUS AND FLUID MECHANICS

Quiz 2

19 October 2017 8:50 am - 9:50 am

Closed Book, No aid sheets, No calculators

Instructor: J. W. Davis

Last Name: J W Davis

Given Name: Solutions

Student #:

Tutorial/TA:

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	8	
3	10	
4	15	
5	10	
TOTAL	53	/ 50

Note: The following integrals may be useful.

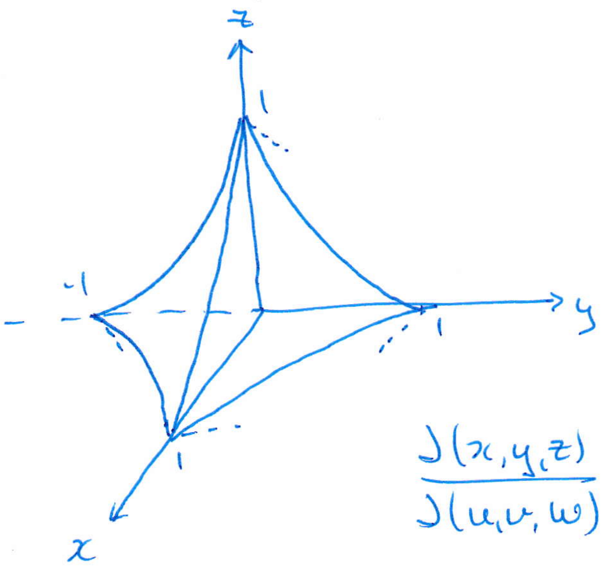
$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C; \qquad \int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$$

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR; \qquad \oiint_S \bar{F}\cdot\hat{n}\, dS = \iiint_V \nabla\cdot\bar{F}\, dV; \qquad \oint_C \bar{F}\cdot d\bar{r} = \iint_S \nabla\times\bar{F}\cdot d\bar{S}$$

1) Use a triple integral to find the volume defined by: $x^{2/3} + y^{2/3} + z^{2/3} \le 1$
Provide a sketch of the volume.

Hint: Use a coordinate transformation such that the shape defined by the new variables is a sphere.

(10 marks)



let $u^3 = x$, $v^3 = y$, $w^3 = z$ $\Big\}$ $x^{2/3} = u^2$, $y^{2/3} = v^2$, $z^{2/3} = w^2$

$\Rightarrow u^2 + v^2 + w^2 \le 1$

$\Rightarrow$ The volume is a sphere in uvw space.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 3u^2 & 0 & 0 \\ 0 & 3v^2 & 0 \\ 0 & 0 & 3w^2 \end{vmatrix} = 27u^2v^2w^2$$

$$\therefore V = \int_{u^2+v^2+w^2 \le 1} 27u^2v^2w^2 \, du\, dv\, dw \cdot 1$$

$\Rightarrow$ spherical coordinates: $u = \rho\sin\phi\cos\theta$, $v = \rho\sin\phi\sin\theta$, $w = \rho\cos\phi$

$$\therefore V = \int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^1 d\rho \cdot \rho^2 \sin\phi \cdot 27\rho^2\sin^2\phi\cos^2\theta \cdot \rho^2\sin^2\phi\sin^2\theta \cdot \rho^2\cos^2\phi$$

$$= 27 \int_0^{2\pi} \cos^2\theta\sin^2\theta\, d\theta \int_0^{\pi} \sin^5\phi\cos^2\phi\, d\phi \int_0^1 \rho^8\, d\rho$$

$$= 27 \int_0^{2\pi} \frac{1}{4}\sin^2 2\theta\, d\theta \int_0^{\pi} (1-\cos^2\phi)(1-\cos^2\phi)\cos^2\phi\sin\phi \left[\frac{\rho^9}{9}\right]_0^1$$

$$= \frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{8}\sin 4\theta\right]_0^{2\pi} \left[-\frac{\cos^3\phi}{3} + \frac{2\cos^5\phi}{5} - \frac{\cos^7\phi}{7}\right]_0^{\pi}$$

$$= \frac{3}{4}\cdot\pi\cdot\left(\frac{1}{3}+\frac{1}{3}-\frac{2}{5}-\frac{2}{5}+\frac{1}{7}+\frac{1}{7}\right) = \frac{3\pi}{4}\left(\frac{2}{3}+\frac{2}{7}-\frac{4}{5}\right) = \frac{4\pi}{35}$$

2) Give $\vec{F} = (3z+2y)\hat{i} + (2x+z)\hat{j} + (3x+y)\hat{k}$, evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ in two ways: a) by evaluating the line integral where C is the straight line path between (3, 1, 2) and (3, -1, 3), and b) by finding a function f such that $\nabla f = \vec{F}$, and using the fundamental theorem for line integrals.

(8 marks)

a) let $\vec{r}(t) = 3\hat{i} + (1-2t)\hat{j} + (2+t)\hat{k} \quad 0 \le t \le 1$

$d\vec{r} = \vec{r}\,'(t)\,dt = (0, -2, 1)$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left((3z+2y)\cdot 0 + (2x+z)(-2) + (3x+y)\cdot 1\right) dt$$

$$= \int_0^1 \left((6+2+t)(-2) + (9+1-2t)\cdot 1\right) dt$$

$$= \int_0^1 (-6-4t)\,dt = \left[-6t - \frac{4t^2}{2}\right]_0^1 = -8$$

b) let $f = 3xz + 2xy + yz$

$$\therefore \nabla f = (3z+2y)\hat{i} + (2x+z)\hat{j} + (3x+y)\hat{k} = \vec{F}$$

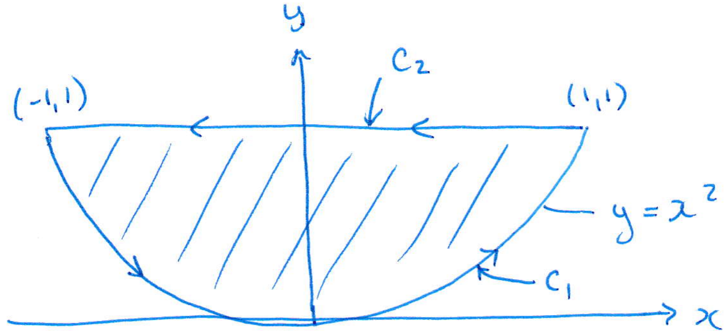
$$\therefore \int_C \vec{F} \cdot d\vec{r} = \left[3xz + 2xy + yz\right]_{(3,1,2)}^{(3,-1,3)}$$

$$= 27 - 6 - 3 - 18 - 6 - 2$$

$$= -8$$

3) Verify Green's Theorem for the line Integral $\int_C xy^2\,dx + xy\,dy$ where C consists of the parabola $y = x^2$ from (-1, 1) to (1, 1) and the line segment from (1, 1) to (-1, 1).

(10 marks)



Green's Theorem

$$\oint_C P\,dx + Q\,dy = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR$$

a) C_1 : let $x = t$, $y = t^2$ $\quad -1 \le t \le 1$: $dx = dt \quad dy = 2t\,dt$

C_2 : let $x = t$, $y = 1$ $\quad 1 \ge t \ge 1$: $dx = dt \quad dy = 0$

$$\oint_C xy^2\,dx + xy\,dy = \int_{C_1} xy^2\,dx + xy\,dy + \int_{C_2} xy^2\,dx + xy\,dy$$

$$= \int_{-1}^{1} t^5\,dt + t^3(2t\,dt) + \int_{1}^{-1} t\,dt$$

$$= \int_{-1}^{1} (t^5 + 2t^4)\,dt - \int_{-1}^{1} t\,dt = \left[\frac{t^6}{6} + \frac{2t^5}{5}\right]_{-1}^{1} - \left[\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{6} - \frac{1}{6} + \frac{2}{5} + \frac{2}{5} - \frac{1}{2} + \frac{1}{2} = \frac{4}{5}$$

b) $$\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dR = \int_R (y - 2xy)\,dR$$

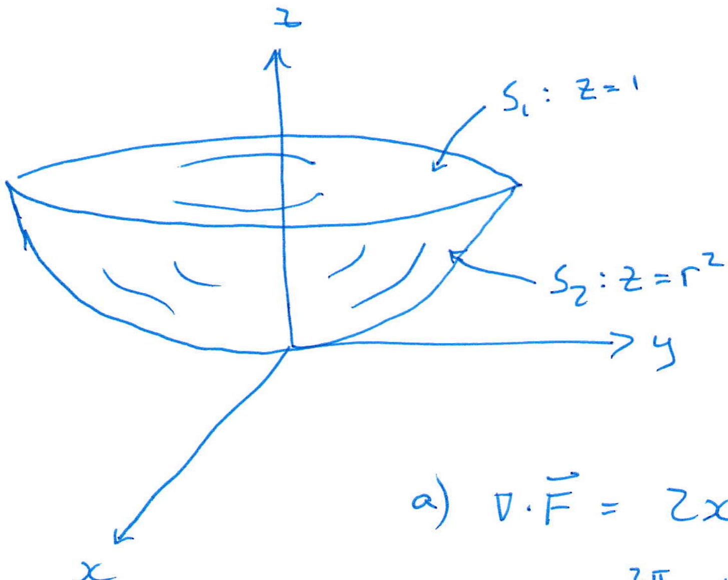
$$= \int_{-1}^{1} dx \int_{x^2}^{1} dy\,(1-2x)y = \int_{-1}^{1} (1-2x)\,dx \left[\frac{y^2}{2}\right]_{x^2}^{1}$$

$$= \int_{-1}^{1} (1-2x)\left(\frac{1}{2} - \frac{x^4}{2}\right) dx = \int_{-1}^{1} \left(\frac{1}{2} - x - \frac{x^4}{2} + x^5\right) dx$$

$$= \left[\frac{1}{2}x - \frac{x^2}{2} - \frac{x^5}{10} + \frac{x^6}{6}\right]_{-1}^{1} = \frac{1}{2} + \frac{1}{2} - \frac{1}{10} - \frac{1}{10} = \frac{4}{5}$$

4) Given the vector field $\bar{F}(x,y,z) = x^2\,\hat{i} + (x-y)\,\hat{j} + 2z\,\hat{k}$, confirm the divergence theorem over the surface which consists of the paraboloid $z = x^2 + y^2,\ 0 \le z \le 1$, and the disk $x^2 + y^2 \le 1,\ z = 1$.

(15 marks)



Divergence Theorem

$$\int_S \vec{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{F}\,dV$$

a) $\nabla\cdot\vec{F} = 2x - 1 + 2 = 2x + 1$

$$\Rightarrow \int_V \nabla\cdot\vec{F}\,dV = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{r^2}^1 dz\,(2\cdot r\cos\theta + 1)$$

$$= \int_0^{2\pi} d\theta \int_0^1 r(1-r^2)(2r\cancel{\cos\theta}+1)\,dr$$

0

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$

b) $S_1 : z = 1,\ \hat{n} = \hat{k},\ x^2 + y^2 \le 1$

$$\therefore \int_{S_1} \vec{F}\cdot\hat{n}\,dS = \int_{x^2+y^2\le 1} 2z\,dx\,dy = 2\int_{x^2+y^2\le 1} dx\,dy = 2\pi$$

$S_2 : z = r^2 \Rightarrow \vec{r}(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r^2)$ $\quad 0 \le \theta \le 2\pi$, $0 \le r \le 1$

4) Continued ...

$$\vec{N} = \vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\Rightarrow$ we want outward normal, $\therefore$ take $-ve$

$$\therefore \int_{S_2} \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 dr \left((r\cos\theta)^2, r\cos\theta - r\sin\theta, 2r^2\right) \cdot \left(2r^2\cos\theta, 2r^2\sin\theta, -r\right)$$

$$= \int_0^{2\pi} d\theta \int_0^1 dr \left(\underbrace{2r^4\cos^3\theta}_{0} + \underbrace{2r^3\cos\theta\sin\theta}_{0} - 2r^3\sin^2\theta - 2r^3 \right)$$

$$= \int_0^{2\pi} d\theta \left[-\frac{2r^4}{4}\sin^2\theta - \frac{2r^4}{4} \right]_0^1$$

$$= \int_0^{2\pi} \left(-\frac{1}{2}\sin^2\theta - \frac{1}{2} \right) d\theta$$

$$= -\frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta + \theta \right]_0^{2\pi}$$

$$= -\frac{1}{2}(\pi + 2\pi) = -\frac{3\pi}{2}$$

$$\therefore \oint_S \vec{F}\cdot d\vec{S} = 2\pi - \frac{3\pi}{2} = \frac{\pi}{2} = \int_V \nabla\cdot\vec{F}\,dV$$

Alternate parameterization for S_2:

$$S_2: \vec{r}(u,v) = (u, v, u^2+v^2) \qquad u^2+v^2 \leq 1$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

$\Rightarrow$ Inward facing $\therefore$ change sign $\Rightarrow \vec{N} = (2u, 2v, -1)$

$$\therefore \int_{S_2} \vec{F}\cdot\vec{N}\, du\,dv = \iint_{u^2+v^2\leq 1} (u^2, u-v, 2u^2+2v^2)\cdot(2u, 2v, -1)\, du\,dv$$

$$= \iint_{u^2+v^2\leq 1} (2u^3 + 2uv - 2v^2 - 2u^2 - 2v^2)\, du\,dv$$

let $u = r\cos\theta$, $v = r\sin\theta$, $du\,dv \to r\,dr\,d\theta$

$$= \int_0^{2\pi} d\theta \int_0^1 r\,dr \left(2r^3 \cancelto{0}{\cos^3\theta} + 2r^2\cancelto{0}{\cos\theta\sin\theta} - 2r^2\sin^2\theta - 2r^2\right)$$

$$= \int_0^{2\pi} d\theta \left[-\frac{2r^4}{4}\sin^2\theta - \frac{2r^4}{4}\right]_0^1 = \int_0^{2\pi} \left(-\frac{1}{2}\sin^2\theta - \frac{1}{2}\right) d\theta$$

$$= -\frac{1}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta + \theta\right]_0^{2\pi} = -\frac{1}{2}(\pi + 2\pi) = -\frac{3\pi}{2}$$

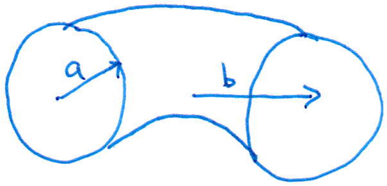
5) Use the formula for the surface area of a parametric surface to find the surface area of the torus given by the equations: $x = (b + a\cos\phi)\cos\theta$, $y = (b + a\cos\phi)\sin\theta$, $z = a\sin\phi$ where $0 \le \theta \le 2\pi$ and $0 \le \phi \le 2\pi$.

(10 marks)

$$x = (b + a\cos\phi)\cos\theta \qquad 0 \le \theta \le 2\pi$$

$$y = (b + a\cos\phi)\sin\theta \qquad 0 \le \phi \le 2\pi$$

$$z = a\sin\phi$$



$$\vec{N} = \begin{vmatrix} i & j & k \\ -(b+a\cos\phi)\sin\theta & (b+a\cos\phi)\cos\theta & 0 \\ -a\sin\phi\cos\theta & -a\sin\phi\sin\theta & a\cos\phi \end{vmatrix}$$

$$= \left(a(b+a\cos\phi)\cos\phi\cos\theta\right)\hat{i} + \left(a(b+a\cos\phi)\cos\phi\sin\theta\right)\hat{j}$$

$$+ \left(\underbrace{a(b+a\cos\phi)\sin\phi\sin^2\theta + a(b+a\cos\phi)\sin\phi\cos^2\theta}_{a(b+a\cos\phi)\sin\phi}\right)\hat{k}$$

$$\|\vec{N}\| = a(b+a\cos\phi)\sqrt{\cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi} = a(b+a\cos\phi)$$

$$\therefore S = \int_S \|\vec{N}\|\, d\phi\, d\theta = \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta\; a(b+a\cos\phi)$$

$$= 2\pi a\left[b\phi + a\sin\phi\right]_0^{2\pi} = 2\pi a \cdot 2\pi b$$