3.6 Exercises for Chapter 3

3.1) Show that August Ritter's formula for the period of a Cepheid variable star, equation 1.4.12, is dimensionally homogeneous.

3.2) In thinking about stellar structure, astrophysicists often begin by estimating the time T_C that a stellar gas sphere of radius R and mass M would take to collapse as a result of the mutual gravitational attraction of the gas particles comprising the sphere. Following this they make inferences about the mechanism that prevent such collapse. With $G = 6.67 \times 10^{-11}$ in SI units being the constant in Newton's Law of Gravitation, an order of magnitude estimate of T_C can be obtained by dimensional analysis. To perform such an analysis, assume that $T_C = CR^x M^y G^z$ with the constant C being dimensionless, and use the principle of dimensional homogeneity to find the exponents x, y and z. By assuming that $C = 1$, estimate T_C for the Sun, which has $R = 6.96\times10^{8}$ m, and $M = 2\times10^{30}$ kg.

3.3) Use the method of indices to show that, if the law of propagation of a solitary water wave had the form

$$V_W = \sqrt{CT_1 + C_2T_2} \quad \text{with} \quad T_j = \rho^{a_j} g^{b_j} H^{c_j} h^{d_j} \mu^{e_j} \quad \text{for } j = 1 \text{ and } 2$$

then the nondimensional equivalent is consistent with equation 3.3.5 as inferred from the generalized method of indices.

3.4) A large variety of tests in both liquids and gases show that in incompressible flow the drag force F_D exerted on a sphere of diameter D immersed in uniform stream is a function of the speed U of the oncoming stream, and of its density ρ and viscosity μ; that is, $F_D = f(\rho, \mu, D, U)$. Starting from the assumption that

$$F_D = \sum_{j=1}^{\infty} C_j \rho^{x_j} D^{y_j} U^{z_j} \mu^{w_j}$$

convert this to dimensionless form, allowing one to infer that

$$\frac{F_D}{\rho D^2 U^2} = f_1(Re) \quad \text{or that} \quad \frac{F_D}{\mu D U} = f_2(Re) \quad \text{where} \quad Re \equiv \frac{\rho D U}{\mu}$$

is the Reynolds Number. For $Re \ll 1$, that is, for small D or U, or for large μ, it can be shown that the flow is inertialess, that is $F_D = f(\mu, D, U)$ only. Show that in this case, with K being a dimensionless constant,

$$F_D = K\mu DU \quad \Rightarrow \quad \frac{F_D}{\rho D^2 U^2} = \frac{K}{Re}$$

3.5) Repeat the dimensional analysis of Problem 3.4, this time using the dimensional matrix method of Example 3.1. Do this two ways: (1), by using ρ, U and D as reference variables: and (2), by using μ, D and U as references. What do you infer?

3.6) The speed c of surface waves on deep water which is otherwise at rest is known to be a function of wavelength λ but not the height H if H/λ is small enough. Also the surface tension γ (Ð [β] = force/length) can affect c. Hence one might expect $c = f(\lambda, g, \rho, \beta)$. Obtain nondimensional equivalents of this expression which enable examination of the functional dependence of c on g, ρ and γ in the limits of large and small λ. In particular, show that they suggest that c is independent of β in the limit $\lambda \to \infty$, and of g in the limit $\lambda \to 0$. In the latter case the waves are called *ripples*. Given that, for sea water, ρ is about 1023 kg/m^2 and β is about 0.073 N/m, what would you conclude for λ = 1 m and 5 mm respectively? If experiments in a laboratory wave show that c = 1.25 m/s at λ = 1 m, what would be the speed of a 100 m wave generated by an oceanic storm?

3.7) As discussed in Example 1.2, Newton was the first to propose a theory for the speed of sound in a fluid. In modern terms his theory predicts that, if during a pressure and density fluctuation pressure p is related to the density ρ by an equation of state of the form $p = f(\rho)$, then the speed c is given by equation (1.1.16b): $c = \sqrt{(dp/d\rho)}$, where the derivative is evaluated at the undisturbed conditions. For all fluids, compressibility can be described in terms of the bulk modulus K given by equations (1.4.10). First, show that $c = \sqrt{(K/\rho)}$, and then use this result to guide a dimensional analysis of propagation of a pure tone of frequency ω in a viscous fluid. That is, assume that $c = f(\rho, K, \mu, \omega)$ where, as usual, μ is the viscosity. Choose ρ, K and ω as reference dimensions to obtain a suitable Π_μ. Given that, for air at 20°C and 101.3 kPa, for which ρ = 1.20 kg/m^3 and $\mu = 1.81\times10^{-5}$ Ns/m^2, calculate Π_μ for frequencies $f = \omega/(2\pi) = 10^3$ and 10^5 Hz. The first of these frequencies corresponds to audible sound, and the second is called *ultrasound.* What would you conclude about the importance of viscosity at these two frequencies?

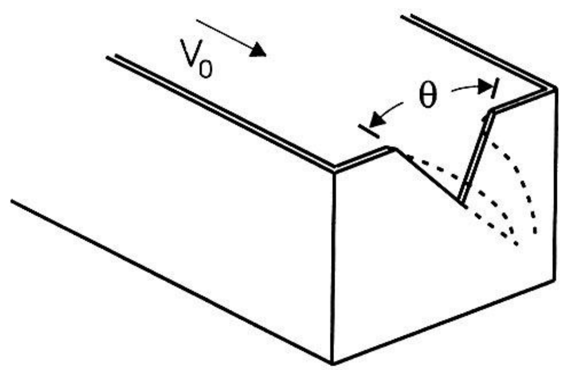
3.8) Perform the dimensional analysis of Example 3.5.2 in Section 3.5 using the matrix method.

3.9) Perform the dimensional analysis of the right-angled pipe bend flow meter described in Example 3.5.3

3.10) Perform the dimensional analysis for pipe flow meters described in Section 4.3, leading to equation 4.3.19.

3.11) A viscous liquid flows down a plane inclined at an angle α to the horizontal. This forms a uniform sheet of thickness H, so that it may be assumed that all the liquid particles move parallel to the plane. Use dimensional analysis to find an expression for the volume flux Q^* per unit plane width down the plane, given an assumed functional relationship $Q^* = f(\rho, \mu, H, g\sin\alpha)$. In this expression $g\sin\alpha$ is the component of g acting down the plane.

3.12) The volume flux Q of water being conveyed in an open channel is usually measured by a *weir*. A weir is a barrier having a specified geometry placed across the channel, and Q is related to the height H above the lowest point on the barrier at which water can flow across it. A geometry used for measurements in a small channels is known as a *V-notch weir*. As depicted in the diagram, the aperture through which the water passes is an inverted isosceles triangle of included angle θ, with the upstream edge being made sharp in order to fix the position at which the water separates from the barrier; H is measured from the apex. Like all weirs, to allow accurate measurements, this device has to be calibrated. With V_0 being the assumed uniform speed of approach of the water in the channel, and assuming that the flow processes through the notch is predominantly frictionless, starting from $Q = f(\rho, \mu, g, H, \theta, V_0)$, perform a dimensional analysis which would show the effects of viscosity and approach speed. What would you conclude about the functional dependence of Q on H when these two



effects are small? Test this conclusion by plotting in appropriate nondimensional form the measurements for a 90° weir given in Table 3.6.1.

H	0.5340	0.6133	0.6215	0.7965	0.8348	0.8949	0.9660	0.9906	1.034
Q	0.5311	0.7484	0.7727	1.425	1.597	1.901	2.303	2.446	2.713
H	1.119	1.136	1.234	1.239	1.379	1.469	1.609	1.693	
Q	3.321	3.431	4.235	4.257	5.542	6.511	8.168	9.276	

Table 3.6.1 Calibration measurements obtained by Woodburn (1932) for a 90° V-notch weir. The data is in US engineering units; H is in feet, and Q is in ft^3/sec. In such units g = 32.16 ft/sec^2; also take the laboratory temperature to be 60°F.

3.13) Another weir geometry suitable for small and moderate flow rates is a circle. Advantages of this geometry are: ease of machining to obtain an accurately known geometry with the requisite sharp edge, and simplicity of alignment. Stevens (1957) presented the data in Table 3.6.2 for such a device; in this table D is the circle diameter and the flow height is measured from the bottom of the circle.

Case I: D = 0.25

H	0.131	0.133	0.137	0.146	0.176	0.179	0.240	0.244
Q	0.027	0.028	0.029	0.032	0.045	0.047	0.074	0.076

Case II: D = 0.50

H	0.131	0.160	0.212	0.273	0.305	0.344	0.391	0.440
Q	0.043	0.061	0.104	0.164	0.200	0.244	0.304	0.368

Case III: D = 1.00

H	0.205	0.254	0.307	0.364	0.520	0.634	0.732	0.873
Q	0.146	0.220	0.314	0.435	0.836	1.190	1.520	2.000

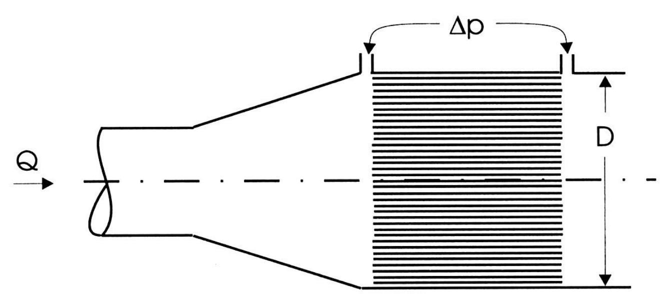
Case IV: D = 2.00

H	0.185	0.273	0.396	0.567	0.661	0.730	0.873	0.988
Q	0.186	0.392	0.792	1.560	2.070	2.504	3.485	4.430

Table 3.6.2 Measurements of flow obtained in the calibration of circular weirs and reported by Stevens (1957); D and H are in ft, and Q is in ft^3/s .

For this data check that viscous and surface tension effects are negligible by applying dimensional analysis to $Q = f(\rho, g, D, H)$ using ρ, g and H as reference quantities, and then by plotting the data as suggested by the analysis. As discussed in Section 3.5, nondimensional plots are not unique, and one can often use an understanding of the physics involved to improve the choice of nondimensional groups. In this case show that one can improve on the previous plot by nondimensionalizing Q with the product of a reference speed V_r based on H and a reference flow area A_r that includes the effect of D, and by replotting the data.

3.14) A *pneumotachometer* is a flow meter invented for medical researchers for the study of obstructive lung diseases; it is intended to measure human breathing air flow rates during exhalation. As depicted in the diagram it consists of a large number N of small tubes of internal diameter D_T and length L placed in parallel and forming a circular bank of diameter D = 5 - 6 cm; the volume flux Q to be measured is passed through this bank. The promoters of this had an incomplete understanding of this, so that they assumed that the pressure decrease Δp across the bank is proportional to the viscosity μ of the gas being measured (usually air) and not dependent on its density ρ. As the discussion of Section 3.2 indicates, provided that the flow is laminar, this can be the case. But, as is discussed in Section 4.3, this functional dependence only occurs for a section of the tube well downstream of the tube entrance, with the flow upstream of this entrance region involving fluid inertia, so that it is dependent on ρ. Furthermore, the human subject whose lung function is being tested breathes into a delivery tube of diameter D_D which is about 1 cm, and the flow is spread over the inlet face of the bank by a cone of semi-angle α. Owing to fluid inertia, this spreading process is not usually uniform; furthermore, turbulence may be generated, so that the flow entering the tube-bank may not be laminar. Assuming that the flow is incompressible, and starting from



$$\Delta p = f(D_B, D_T, L, N, D_D, \alpha, Q, \mu, \rho)$$

use dimensional analysis to suggest a method of plotting calibration data which will reveal the presence of any significant inertia effects.

3.15) For the atomic explosion of Example 1.4.1, undertake the dimensional analysis, leading to equation 1.4.1. That is, assuming, $R_S = f(\rho_a, E_0, t, \gamma)$, show that

$$R_s(t) = K(\gamma)\left[\frac{E_0 t^2}{\rho_a}\right]^{1/5} \equiv C\frac{t^{2/5}}{\rho_a^{1/5}}$$

3.16) An orifice plate flow meter, such as that depicted in Figure 4.3.4, is installed in a pipe of diameter D = 30cm. It is calibrated using water at T = 20° C, for which the density ρ and viscosity μ may be taken as 998 kg/m^3 and 1.00×10^{-3} kg/m/s respectively. The data is empirically fitted with a monomial of the form $Q = C\,\Delta p^{\,n}$, where Q is the volume flux in m^3/sec and Δp is the pressure difference in N/m^2 across the meter taps in Figure 4.3.4, and the coefficients are found to be C = 1.5 and n = 0.53. The meter is to be used to measure volume fluxes of water at pressures and temperatures which are not known. The density of the water may be assumed to be the same, but the viscosity differs by large amounts. Use dimensional analysis to find the error that would be incurred in using the value of C obtained at T=20°C, if at 100°C, $\mu = 0.28\times10^{-3}$ kg/m/s. Hint: assume a functional dependence of Q on all the parameters involved that is consistent with both the dictates of dimensional analysis and the empirical formula.

3.17) For the V-notch weir of Example 3.12, an experimenter is concerned that surface tension effects might become important at sufficiently small H. With V_0 assumed negligible, and with β being the surface tension of water, assume $Q = f(\rho, \mu, \beta, g, H, \theta)$. Perform a dimensional analysis showing the effects of μ and β. Evaluate the significance of these effects for H = 0.1 m and 0.01 m assuming that, for water at 20 °C, ρ = 998 kg/m^3, $\nu \approx 10^{-6}$ m^2/s and β = 0.073 N/m.