University of Toronto

Faculty of Applied Science and Engineering

ESC194F Calculus
Midterm Test
9:10 – 10:55, 25 October 2021
105 minutes
No calculators or aids
There are 10 questions, each question is worth 10 marks

Examiners: P.C. Stangeby and J.W. Davis

1. Evaluate the following limits if they exist. Indicate the limit laws used in your solution.

(a) $\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12}$ (b) $\lim_{x\to 4} \frac{x^2+3x}{x^2-x-12}$ (c) $\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2}$

(d) $\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h}$ (e) $\lim_{h\to 0} \frac{(x+h)^{-2}-x^{-2}}{h}$

a) $\lim_{x\to -3} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to -3} \frac{x(x+3)}{(x+3)(x-4)} = \lim_{x\to -3} \frac{x}{x-4} = \frac{3}{7}$

$\Rightarrow$ quotient law, sum/difference law, cancel common factor, direct substitution

b) $\lim_{x\to 4} \frac{x^2+3x}{x^2-x-12} = \lim_{x\to 4} \frac{x(x+3)}{(x+3)(x-4)} = \lim_{x\to 4} \frac{x}{x-4} \longrightarrow \frac{4}{0}$ DNE

$\Rightarrow$ cancel common factor, definition of infinite limit

c) $\lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} = \lim_{x\to 2} \frac{\sqrt{4x+1}-3}{x-2} \cdot \frac{\sqrt{4x+1}+3}{\sqrt{4x+1}+3} = \lim_{x\to 2} \frac{4x-8}{(x-2)(\sqrt{4x+1}+3)}$

$= \lim_{x\to 2} \frac{4}{\sqrt{4x+1}+3} = \frac{2}{3}$

$\Rightarrow$ sum/difference, quotient, root laws

$\Rightarrow$ cancel common factor, direct substitution

d) $\lim_{h\to 0} \frac{(3+h)^{-1}-3^{-1}}{h} = \lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-3-h}{3(3+h)}}{h} = \lim_{h\to 0} \frac{-h}{3h(3+h)}$

$= \lim_{h\to 0} \frac{-1}{3(3+h)} = -\frac{1}{9}$

$\Rightarrow$ sum, difference, quotient laws,

$\Rightarrow$ cancel common factor, direct substitution

e) $\lim_{h\to 0} \frac{(x+h)^{-2}-x^{-2}}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{h x^2 (x+h)^2}$

$= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)} = \frac{-2}{x^3}$

$\Rightarrow$ sum, difference, quotient laws

$\Rightarrow$ cancel common factor

$\Rightarrow$ direct substitution

2. Calculate the derivative of the following functions, citing all theorems used:

(a) $f(x) = 3x^2$ (b) $f(x) = 2/x^3$ (c) $f(x) = 3\cos(2x)$

(d) $f(x) = 2\sin^2(3x^3)$ (f) $f(x) = 3/(2 - x)^2$.

a) $f(x) = 3x^2 \Rightarrow f'(x) = 6x$

- constant multiple rule
- power rule

b) $f(x) = \frac{2}{x^3} \Rightarrow f'(x) = \frac{-6}{x^4}$

- general power rule

c) $f(x) = 3\cos(2x) \Rightarrow f'(x) = -6\sin(2x)$

- basic trig derivative
- chain rule with constant multiplier rule

d) $f(x) = 2\sin^2(3x^3) \Rightarrow f'(x) = 2 \cdot 2\sin(3x^2)\cos(3x^2) \cdot 9x^2$

$= 36x^2 \sin(3x^2)\cos(3x^2)$

- power rule
- chain rule
- basic trig derivative
- chain rule
- power rule & constant multiplier rule

e) $f(x) = \frac{3}{(2-x)^2} \Rightarrow f'(x) = 3(-2)(2-x)^{-3}(-1)$

- general power rule
- chain rule

3. Use the definition of the derivative to find: $\frac{d}{dx}\frac{1}{x+2}$

Note: no marks will be given for simply evaluating the derivative.

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x+2 - (x+h+2)}{(x+h+2)(x+2)}}{h} = \lim_{h \to 0} \frac{\frac{-h}{(x+h+2)(x+2)}}{h}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h+2)(x+2)} = \frac{-1}{(x+2)^2} \qquad x \neq -2$$

4. a) Use a $\varepsilon - \delta$ type of proof to prove that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$.

b) Use $\varepsilon - \delta$ arguments to prove the Pinching Theorem:
Let $p > 0$. Suppose that, for all x such that $0 < |x - c| < p$, $h(x) \le f(x) \le g(x)$.
If $\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$ then $\lim_{x \to c} f(x) = L$.

a) given $N < 0$, find $\delta > 0$ st. for $-\delta < x < 0$ $f(x) = \frac{1}{x} < N$

$\Rightarrow \frac{1}{-\delta} < N \Rightarrow \delta < \frac{-1}{N} \Rightarrow$ choose $\delta = -\frac{1}{N}$

proof: given $N < 0$, choose $\delta = \frac{-1}{N}$

$\therefore$ for $-\delta < x < 0$ $f(x) = \frac{1}{x} < \frac{-1}{\delta} = N$

$\therefore$ by the definition of an infinite limit, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

b) Let $\epsilon > 0$; let $p > 0$ st. if $0 < |x - c| < p$ then $h(x) \le f(x) \le g(x)$

- Choose δ_1 such that:
  if $0 < |x - c| < \delta_1$ then $L - \epsilon < h(x) < L + \epsilon$
- Choose δ_2 such that
  if $0 < |x - c| < \delta_2$ then $L - \epsilon < g(x) < L + \epsilon$

Let $\delta = \min\{p, \delta_1, \delta_2\}$

$\therefore$ for $0 < |x - c| < \delta$

$L - \epsilon \le h(x) \le f(x) \le g(x) \le L + \epsilon$

$\therefore |f(x) - L| < \epsilon$

5. Sketch the curve, marking all important features on the plots of:

a) $y = x^5 - 5x$

b) $y = \frac{x^3}{x-2}$

a) Domain : $(-\infty, \infty)$ Range $(-\infty, \infty)$

Intercepts: $x = 0 \rightarrow y = 0$: $(0,0)$

$y = 0 \rightarrow x = 0, x = \pm\sqrt[4]{5}$: $(0,0), (\pm\sqrt[4]{5}, 0)$

Symmetry: $y(x) = -y(-x)$ $\therefore$ odd f'n; symmetric abt. origin

No asymptotes; $y \sim x^5$ as $x \rightarrow \pm\infty$

$y = x^5 - 5x \rightarrow y' = 5x^4 - 5 \Rightarrow y' = 0 \rightarrow x^4 = 1 \rightarrow x = \pm 1$

$\therefore$ local extremes $(1, -4)$ & $(-1, +4)$

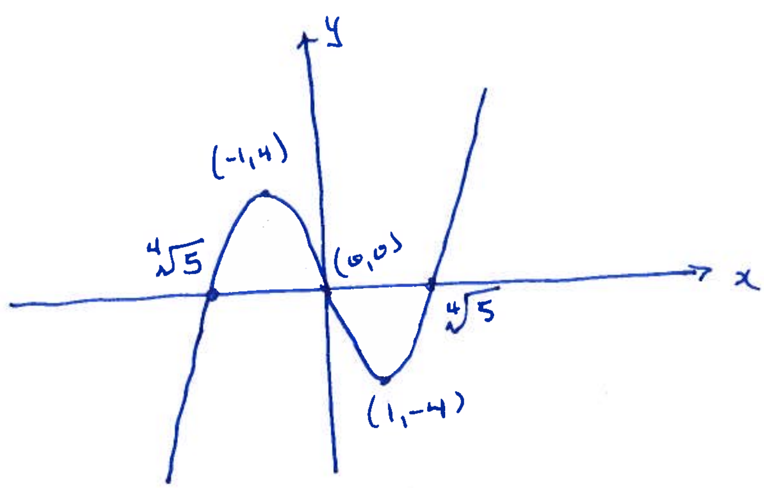
$y' < 0$ for $-1 < x < 1$ $\therefore$ decreasing

$y' > 0$ for $x < -1$ & $x > 1$ $\therefore$ increasing

$y'' = 20x^3 \Rightarrow y''(1) = 20 > 0$ $\therefore$ local min

$y''(-1) = -20 < 0$ $\therefore$ local max

$$\left.\begin{array}{l} y'' < 0 \text{ for } x < 0 \ \therefore \text{ concave down} \\ y'' > 0 \text{ for } x > 0 \ \therefore \text{ concave up} \end{array}\right\} \begin{array}{l} x = 0 \\ \text{is a pt of} \\ \text{inflection} \end{array}$$



b) $y = \frac{x^3}{x-2}$

Domain: $x \neq 2$ Range $(-\infty, \infty)$

Intercepts: $\left.\begin{array}{l} x=0 \Rightarrow y=0 \\ y=0 \Rightarrow x=0 \end{array}\right\} (0,0)$

No symmetry

Asymptotes: $\lim_{x\to 2^-} \frac{x^3}{x-2} = -\infty$ & $\lim_{x\to 2^+} \frac{x^3}{x-2} = \infty$

$\therefore x=2$ is a vertical asymptote

$f'(x) = x^3(-1)(x-2)^{-2} + 3x^2(x-2)^{-1} = \frac{-x^3 + 3x^2(x-2)}{(x-2)^2}$

$= \frac{2x^3 - 6x^2}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2}$

$\Rightarrow f'(x) > 0$ for $x > 3$ $\therefore$ incr.

$f'(x) < 0$ for $2 < x < 3$ $\therefore$ decr.

$f'(x) < 0$ for $x < 2$ $\therefore$ decr.

$f'(x) = 0 \Rightarrow x = 0$ or $x = 3 \Rightarrow$ $f(3) = 27$ is local min

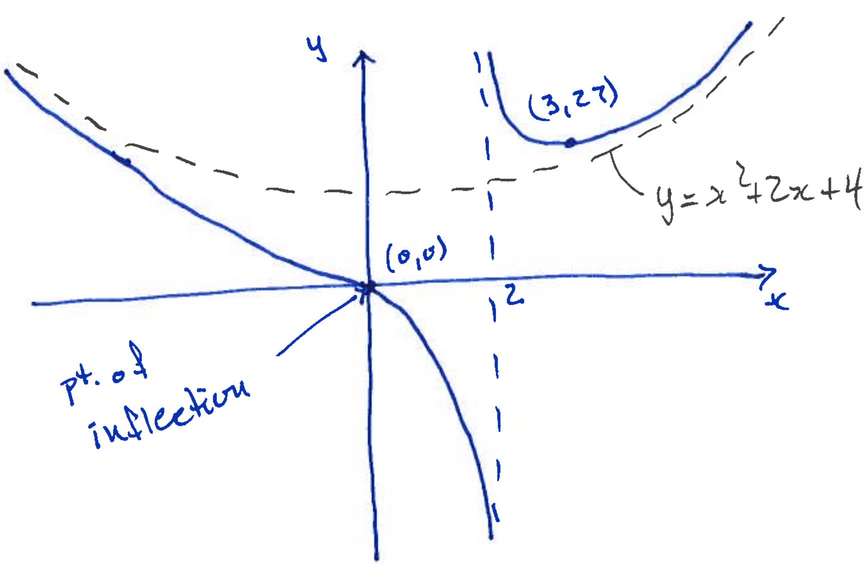
$f(0) = 0$ not a local extreme

$f''(x) = \frac{2x(x^2 - 6x + 12)}{(x-2)^3}$ $\Rightarrow$ $x^2 - 6x + 12$ has no Real roots

$\Rightarrow f''(x) > 0$ for $x < 0$: concave up

$f''(x) < 0$ for $0 < x < 2$: concave down

$f''(x) > 0$ for $x > 2$: concave up



Bonus:

$y = \frac{x^3}{x-2} = x^2 + 2x + 4 + \frac{8}{x-2}$

(by long division)

$\therefore$ as $x \to \pm\infty$, $\frac{8}{x-2} \to 0$

$\therefore y$ approaches parabola $y = x^2 + 2x + 4$

6. Use differentials to estimate $\sqrt{100.3}$.

Let $f(x) = \sqrt{x}$

We approximate $f(x)$ with $L(x) = f(a) + f'(a)(x-a)$

where we choose $a = 100$

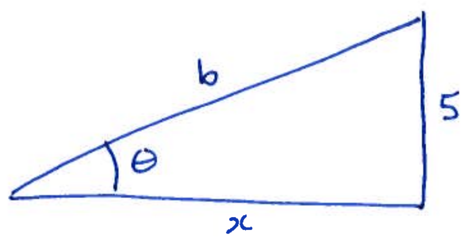
$$f'(x) = \frac{1}{2\sqrt{x}} \longrightarrow f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

$$\therefore f(100.3) \simeq L(100.3) = 10 + \frac{1}{20}(100.3 - 100)$$

$$= 10 + \frac{0.3}{20} = 10.015$$

cf : $\sqrt{100.3} = 10.0149888$

7. A man whose eye level is 2 m above the ground walks straight toward a billboard at a rate of 1 m/s. The bottom of the billboard is 3 m above the ground, and it is 5 m high. The man's viewing angle is the angle formed by the lines between the man's eyes and the top and bottom of the billboard. At what rate is the viewing angle changing when the man is 10 m from the billboard?



given $\frac{dx}{dt} = -1$ m/s, find $\frac{d\theta}{dt}$

$$b = \sqrt{x^2+5^2} \Rightarrow \sin\theta = \frac{5}{b} = \frac{5}{\sqrt{x^2+5^2}}$$

$$\cos\theta = \frac{x}{b} = \frac{x}{\sqrt{x^2+5^2}}$$

$$\Rightarrow 5 = \sqrt{x^2+25} \cdot \sin\theta$$

$$\therefore 0 = \frac{1}{2}(x^2+25)^{-1/2} \cdot 2x \cdot \frac{dx}{dt} \sin\theta + (x^2+25)^{1/2} \cos\theta \cdot \frac{d\theta}{dt}$$

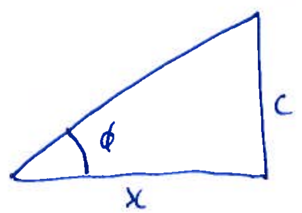
$$\Rightarrow \frac{d\theta}{dt} = \frac{-x(x^2+25)^{-1/2} \frac{dx}{dt} \sin\theta}{(x^2+25)^{1/2} \cos\theta} = \frac{-x \frac{dx}{dt}}{x^2+25} \cdot \frac{5}{x}$$

$$= \frac{-5 \frac{dx}{dt}}{x^2+25}$$

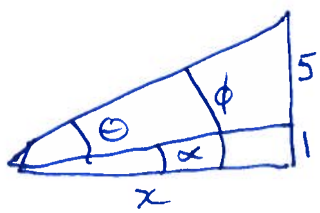
at $x = 10$, $\frac{dx}{dt} = -1 \Rightarrow \frac{d\theta}{dt} = \frac{5}{125} = 0.04$ rad/s

Q7 Solutions with bottom of bill board 3m above the ground

$\Rightarrow$ Following the solution on the previous page:



$$\frac{d\phi}{dt} = \frac{-c\,\frac{dx}{dt}}{x^2+c^2}$$

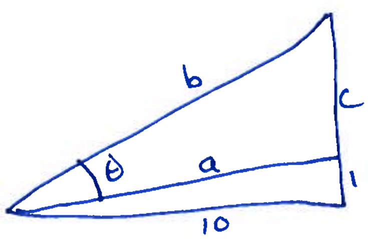


$$\frac{d\phi}{dt} = \frac{-6(-1)}{10^2+6^2} = \frac{6}{136} \text{ rad/s}$$

$$\frac{d\alpha}{dt} = \frac{-1(-1)}{10^2+1^2} = \frac{1}{101} \text{ rad/s}$$

$$\therefore \frac{d\theta}{dt} = \frac{d\phi}{dt} - \frac{d\alpha}{dt} = \frac{6}{136} - \frac{1}{101} = 0.0342 \text{ rad/s}$$

$\Rightarrow$ Alternate solution using cosine law (not recommended)



$$a^2 = 1^2 + x^2$$

$$b^2 = 6^2 + x^2$$

cosine law: $c^2 = a^2 + b^2 - 2ab\cos\theta$

$$\therefore 25 = 1 + x^2 + 36 + x^2 - 2\sqrt{1+x^2}\sqrt{36+x^2}\cos\theta$$

$$-6 = x^2 - \sqrt{1+x^2}\sqrt{36+x^2}\cos\theta$$

$$\Rightarrow 0 = 2x\frac{dx}{dt} - \frac{1}{2}(1+x^2)^{-1/2}\left(2x\frac{dx}{dt}\right)\sqrt{36+x^2}\cos\theta - \frac{1}{2}(36+x^2)^{-1/2}\left(2x\frac{dx}{dt}\right)\sqrt{1+x^2}\cos\theta$$

$$+ \sin\theta\frac{d\theta}{dt}\sqrt{1+x^2}\sqrt{36+x^2}$$

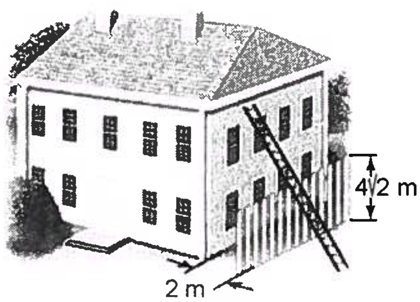
$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} = \frac{101+136-25}{2\sqrt{101}\sqrt{136}} = \frac{106}{\sqrt{101}\sqrt{136}} = 0.9044$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{106^2}{101\cdot 136}} = 0.4266$$

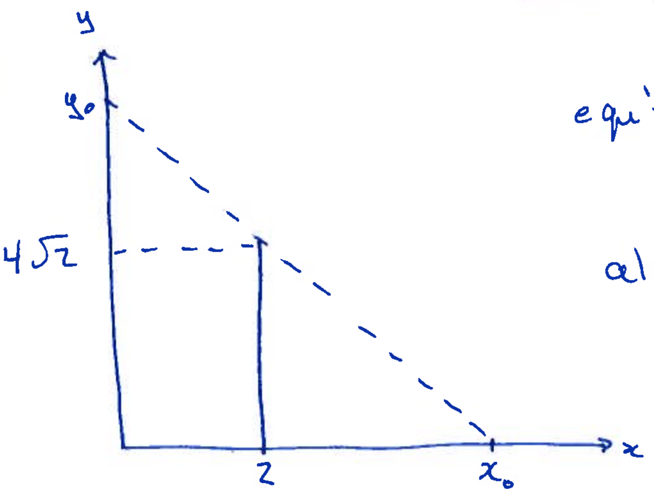
$$0 = -20 + \frac{1}{2}\left(\sqrt{\frac{136}{101}} + \sqrt{\frac{101}{136}}\right)(20)\cos\theta + \sin\theta\sqrt{101}\sqrt{136}\,\frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = 0.0342 \text{ rad/s}$$

8. A $4\sqrt{2}$ m tall fence runs parallel to the wall of a building at a distance of 2 m. Find the length of the shortest ladder that extends from the ground (outside the fence) to the building without touching the fence.
   a) For the first case, assume the vertical wall of the building and the horizontal ground have infinite extent.
   b) For a second case, assume the vertical wall of the building is 15 m high, and the horizontal ground extends 2 m from the fence.



a)



$b = y_0$

equ'n of line: $y = mx + b \Rightarrow m = -\frac{y_0}{x_0}$

$\therefore y = y_0 - \frac{y_0}{x_0}x$

also: $4\sqrt{2} = y_0\left(1 - \frac{2}{x_0}\right) \Rightarrow y_0 = \frac{4\sqrt{2}}{1 - \frac{2}{x_0}}$

minimize $l^2 = x_0^2 + y_0^2 = x_0^2 + \frac{32}{\left(1-\frac{2}{x_0}\right)^2} = x_0^2 + 32\left(1 - \frac{2}{x_0}\right)^{-2}$

$$\frac{d(l^2)}{dx_0} = 2x_0 + 32(-2)\left(1 - \frac{2}{x_0}\right)^{-3}(-2)(-1)x_0^{-2}$$

$$\Rightarrow 0 = x_0 - \frac{64}{x_0^2\left(1-\frac{2}{x_0}\right)^3} \Rightarrow x_0^3\left(1 - \frac{2}{x_0}\right)^3 = 64$$

$$\Rightarrow x_0\left(1 - \frac{2}{x_0}\right) = x_0 - 2 = (64)^{1/3} = 4 \Rightarrow x_0 = 6,\ y_0 = 6\sqrt{2}$$

$\therefore l = \sqrt{x_0^2 + y_0^2} = 6\sqrt{3}$

test: for $x_0 = 6\sqrt{3}$, $y_0 > 0 \therefore l > 6\sqrt{3}$

$y_0 = 6\sqrt{3}$, $x_0 > 0 \therefore l > 6\sqrt{3}$

$\therefore l = 6\sqrt{3}$ must be a minimum

b) In (a), $x_0 = 6 > 4$

$\therefore$ If a solution with $x \le 4$ exists it must be at the boundary point, $x_0 = 4$

$$\Rightarrow y_0 = \frac{4\sqrt{2}}{1-\frac{2}{4}} = 8\sqrt{2} < 15$$

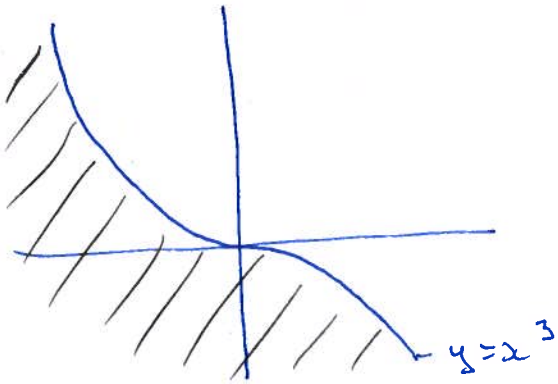
$\therefore$ $x_0 = 4$, $y_0 = 8\sqrt{2}$ is a valid solution

$$\therefore l = \sqrt{16+128} = 12$$

9. a) Sketch the graph of the inequality $x^3 + y \le 0$.

b) Sketch a graph of the region in the x-y plane defined by: $|x+y| - |x-y| \le 2$.
Hint: use symmetry to avoid repetitive calculations.

a) $x^3 + y \le 0 \Rightarrow$ find boundary $x^3 + y = 0 \Rightarrow y = -x^3$



test $(1,0) \Rightarrow 1 \le 0$ ✗

$(-1,0) \Rightarrow -1 \le 0$ OK

∴ the inequality defines the region to the left of the curve.

b) $|x+y| - |x-y| \le 2$

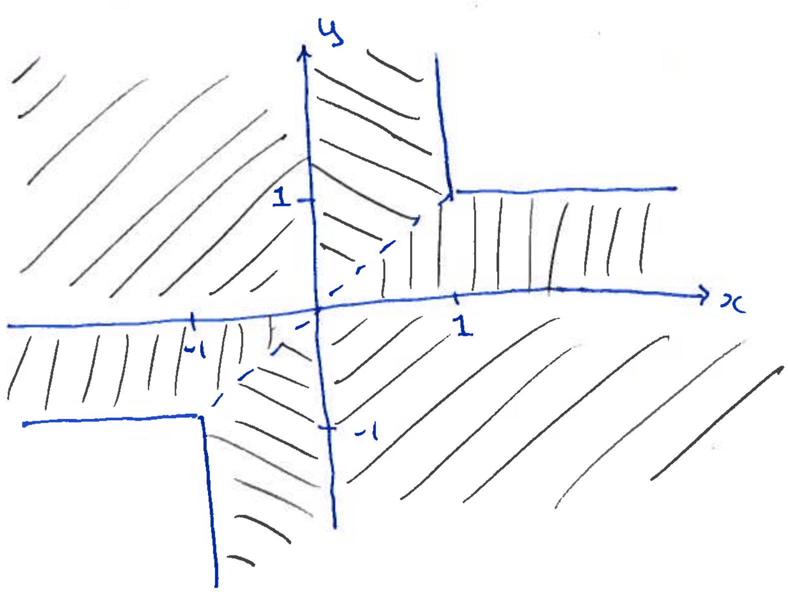
$x, y > 0,\ x > y \Rightarrow x+y-x+y \le 2 \Rightarrow y \le 1$

$x, y > 0,\ x < y \Rightarrow x+y+x-y \le 2 \Rightarrow x \le 1$

$\Rightarrow 0 \le 2 \Rightarrow$ always true

$x < 0,\ y > 0 \Rightarrow -x+y-(-x+y) \le 2 \Rightarrow 0 \le 2 \Rightarrow$ always true

$x > 0,\ y < 0 \Rightarrow x-y-(x-y) \le 2 \Rightarrow 0 \le 2$



⇒ Replace x with $-x$ and y with $-y$ gives the same equation ∴ symmetry about the origin.

10. a) Show that the absolute value function, $F(x) = |x|$, is continuous everywhere.

b) Prove that if f is a continuous function on an interval, then so is $|f|$.

c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counter example.

a) - For $a > 0$; $\lim_{x\to a} F(x) = \lim_{x\to a} x = a = F(a)$

- For $a < 0$; $\lim_{x\to a} F(x) = \lim_{x\to a} -x = -a = F(a)$

- For $a = 0$; $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} x = 0 = F(a)$

$\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} -x = 0 = F(a)$

$\therefore F(x) = |x|$ is continuous everywhere

b) Arguments similar to part (a) can be made;

alternately: let $g(x) = |f(x)|$

$\Rightarrow \lim_{x\to a} g(x) = \lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)| = |f(a)|$

By composite function theorem; given $|x|$ is continuous.

c) Counter example:

$$\text{let } f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$\Rightarrow |f(x)| = 1$ is continuous everywhere

but $f(x)$ is discontinuous at $x = 0$