ESC103F Engineering Mathematics and Computation: Tutorial #5

Question 1: Test the "truth" of the associative law (AB)C=A(BC):

i) $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$

ii) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

Solution:

i) $\left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$

ii) $\left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 0 & 1 \end{bmatrix}$

Question 2:

Let $A = \begin{bmatrix} 2 & -2 & 1 & 6 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 3 & -3 & 0 & 6 & 1 \end{bmatrix}$. We want to factor this matrix, $A = CR$.

i) Construct matrix C from matrix A by going from left to right and putting each column of A into C if that column is not a combination of earlier columns.

ii) Construct matrix R. Note: if C has r columns, then R must have r rows.

Solution:

i) $C = \begin{bmatrix} 2 & 1 & ? & ? \\ 1 & 0 & ? & ? \\ 3 & 0 & ? & ? \end{bmatrix}$

Since col 2 of A is a multiple of col 1, it does not go into C.

Is col 4 of A a combination of the first two columns of C? Let's see:

$c \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix} \rightarrow c = 2 \,.\, d = 2$

Yes, it is, so it does not get included in C.

Is col 5 of A a combination of the first two columns of C? Let's see:

$c\begin{bmatrix}2\\1\\3\end{bmatrix}+d\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ → there are no values for c and d that will satisfy this equation.

Therefore, col 5 of A does get included in C.

$$\therefore C=\begin{bmatrix}2 & 1 & 0\\1 & 0 & 0\\3 & 0 & 1\end{bmatrix}$$

ii) Given $A=CR$, A is 3x5 and C is 3x3, R must be 3x5 and the 5 columns in R must tell us how to produce the 5 columns of A from the columns of C.

$$R=\begin{bmatrix}1 & -1 & 0 & 2 & 0\\0 & 0 & 1 & 2 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Question 3: If all columns of $A=[\vec{a}\quad\vec{a}\quad\vec{a}]$ are vectors in R^n and where $\vec{a}\neq\vec{0}$, what are C and R, where $A=CR$?

Solution:

$A=[\vec{a}][1\quad 1\quad 1]=CR$ where C is nx1 and R is 1x3.

Question 4: Why is it not possible for a matrix with 4 rows and 7 columns to have 5 independent columns?

Solution:

These 7 column vectors are in 4-D (R^4). You cannot have 5 independent column vectors in a 4-D space, the maximum is 4.

Question 5: Complete the 2x2 matrices to meet the requirements specified:

i) $\begin{bmatrix}3 & 6\\5 & \end{bmatrix}$ (rank 1)

ii) $\begin{bmatrix}6 & \\7 & \end{bmatrix}$ (orthogonal columns)

iii) $\begin{bmatrix} 2 & \\ 3 & 6 \end{bmatrix}$ (rank 2)

iv) $\begin{bmatrix} 3 & 4 \\ & -3 \end{bmatrix}$ $(A^2 = I)$

Solution:

i) $\begin{bmatrix} 3 & 6 \\ 5 & 10 \end{bmatrix}$ (rank 1)

ii) $\begin{bmatrix} 6 & 7 \\ 7 & -6 \end{bmatrix}$ or $\begin{bmatrix} 6 & -7 \\ 7 & 6 \end{bmatrix}$ (orthogonal columns)

iii) $\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$ or any value other than 4 (rank 2)

iv) $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ $(A^2 = I)$