ECE259H1: Electromagnetism

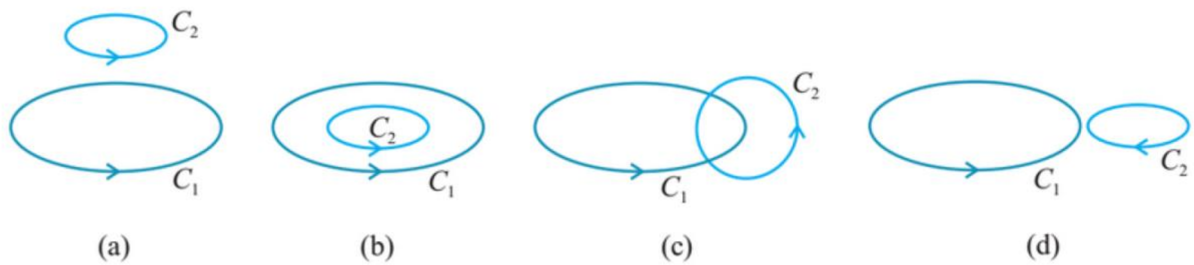
Homework Review Quiz 4 – Friday April 14, 2023

$\boxed{\text{Solutions}}$

- Make sure to ***accurately*** enter your first name, last name, and student number above.
- The quiz is worth 20 marks and has two questions. Question 1 is worth 6 marks, and Question 2 is work 14 marks.
- Show all of your work.
- The final page has some reference material that you might find helpful.
- Take a deep breath and relax 😊.

Question #1 (6 marks total, 2 marks for each part)

1. Consider the four configurations of two metallic loops shown below.



(i) Which configuration, if any, would maximize the mutual inductance between the two loops? Briefly justify your response.

(b) This will maximize the mutual flux between the two loops.

(ii) Which configuration, if any, would maximize the self-inductance of loop C_1? Briefly justify your response.

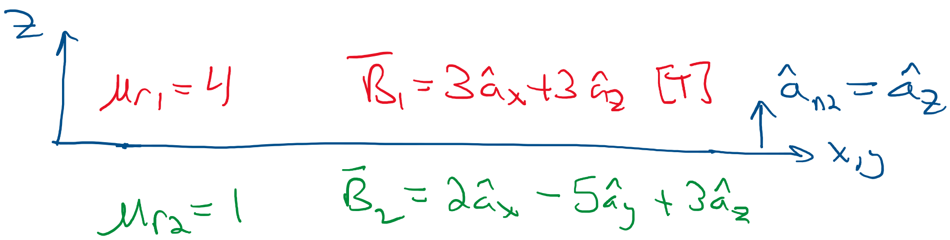
None. The self-inductance of one loop does not depend on the relationship with the other loop.

(iii) For configuration (b), if you doubled the radius of loop C_1, how would that change the mutual inductance between the two loops? Would this change increase the mutual inductance, decrease it, or have no effect?

It would have no effect. Even if the size of C_1 increases the "mutual area" will stay the same, since C_2 remains the same.

Question #2 (14 marks – 7 marks for each part)

(a) An infinite sheet of current exists in the xy-plane and has a surface current density $\mathbf{J}_s$. Below the sheet ($z < 0$) the material is air, i.e., $\mu_{r2} = 1$. Above the sheet ($z > 0$) there is a magnetic material with $\mu_{r1} = 4$. If a constant magnetic flux density given by $\mathbf{B}_2 = 2\hat{\mathbf{a}}_x - 5\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z$ [T] exists in the air, then what would the current density, $\mathbf{J}_s$, need to be such that the magnetic flux density in the magnetic material is $\mathbf{B}_1 = 3\hat{\mathbf{a}}_x + 3\hat{\mathbf{a}}_z$ [T]?



$$\hat{a}_{n2} \times (\overline{H}_1 - \overline{H}_2) = \overline{J}_s$$

$$\therefore \overline{J}_s = \hat{a}_z \times \left(\frac{\overline{B}_1}{\mu_{r1}\mu_0} - \frac{\overline{B}_2}{\mu_{r2}\mu_0} \right)$$

$$\overline{J}_s = \hat{a}_z \times \left(\frac{3\hat{a}_x + 3\hat{a}_z}{4\mu_0} - \frac{2\hat{a}_x - 5\hat{a}_y + 3\hat{a}_z}{\mu_0} \right)$$

$$= \frac{1}{\mu_0}\left[\frac{3}{4}\hat{a}_y - 2\hat{a}_y + (-5\hat{a}_x) \right]$$

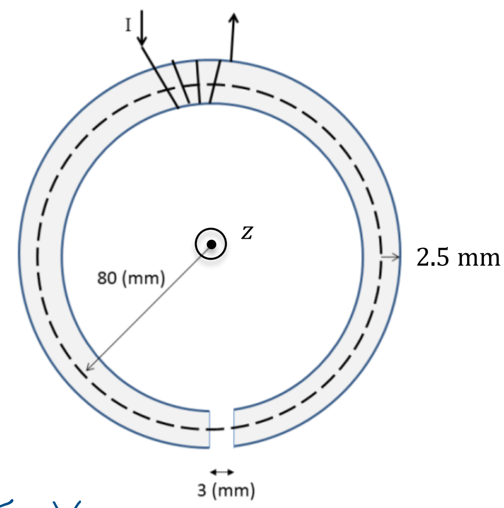
$$\therefore \overline{J}_s = \frac{1}{\mu_0}\left(-5\hat{a}_x - \frac{5}{4}\hat{a}_y \right) \text{ [A/m]}$$

Question #2 (continued)

(b) A magnetic circuit with a toroidal core ($\mu_r = 1000$) has a small 3 mm air gap and a circular cross-section area with radius $r_{\text{core}} = 2.5$ mm, as shown to the right. It lies in the xy-plane and is centered about the z-axis (i.e., the z-axis points out of the page). It is fed as shown by a current of $I = 2$ A, through a coil with $N = 500$ turns, such that the magnetic flux in the circuit is given by $\Phi = 7.05$ µWb.

(i) What is the magnetic energy stored in the circuit?

(ii) What is the magnetization vector, $\mathbf{M}$, in the magnetic core? What is the magnetization vector in the air gap?



(i) $W_m = \frac{1}{2} L_{11} I^2$ with $L_{11} = \frac{N\Phi}{I} = \frac{(500)(7.05\times10^{-6})}{2} = \underline{1.76 \text{ mH}}$

$\therefore W_m = \frac{1}{2}(1.76\times10^{-3})(2)^2 = \underline{\underline{3.52 \text{ mJ}}}$

$\underline{\text{OR}}$ $B = \frac{\Phi}{\underline{\quad}} = \frac{7.05\times10^{-6}}{\pi(2.5\times10^{-3})^2} = \underline{0.359 \text{ T}} = \underbrace{B_{core} = B_{air}}_{\text{From } B_{n1} = B_{n2} \text{ boundary condition}}$

$W_M = \frac{1}{2}\int_V \vec{B}\cdot\vec{H}\, dv = \frac{1}{2}\iiint_{Vol} \frac{|\vec{B}|^2}{\mu_r\mu_0} dv$

$= \underbrace{\frac{1}{2}[2\pi(0.08) - 0.003](\pi)(2.5\times10^{-3})^2 \frac{(0.359)^2}{1000\mu_0}}_{\text{Energy stored in core}} + \underbrace{\frac{1}{2}(0.003)(\pi)(2.5\times10^{-3})^2 \frac{(0.359)^2}{\mu_0}}_{\text{Energy stored in air gap}}$

$= 0.503 \text{ mJ} + 3.02 \text{ mJ}$

$= \underline{\underline{3.52 \text{ mJ}}}$

(ii) $\vec{M} = \chi_m \vec{H} = (\mu_r - 1)\frac{\vec{B}}{\mu_r\mu_0} \rightarrow \vec{M}_{core} = \frac{(1000-1)}{1000\mu_0}(0.359\,\hat{a}_\phi)$

$= \underline{\underline{2.85\times10^5\,\hat{a}_\phi \text{ A/m}}}$

$\vec{M}_{air} = \frac{(1-1)(0.359\,\hat{a}_\phi)}{\mu_0} = \underline{\underline{0 \text{ A/m}}}$

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Reference Formulae

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

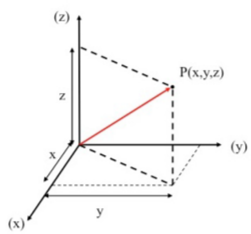
Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R\sin\theta d\phi$

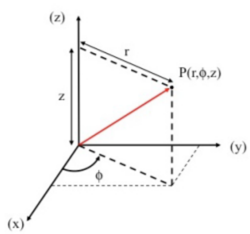
Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R\sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

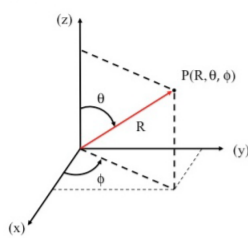
Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\frac{r}{\sin\theta}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ







3. Dot products of unit vectors

·	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	$-\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

5. Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial(R^2 A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$\begin{aligned}
\nabla\times\mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial\phi} - \frac{\partial(RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial(RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial\theta}\right)\mathbf{a}_\phi
\end{aligned}$$

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

Table 2 Magnetostatics

$$\mathbf{F}_m = q\mathbf{u}\times\mathbf{B} \qquad \mathbf{F}_m = I\mathbf{l}\times\mathbf{B}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{J} \qquad \nabla\cdot\mathbf{B} = 0$$

$$\oint_C \mathbf{B}\cdot\mathbf{dl} = \mu I \qquad \oint_S \mathbf{B}\cdot\mathbf{dS} = 0$$

$$\mathbf{B} = \frac{\mu I}{4\pi}\oint_C \frac{\mathbf{dl}'\times(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3} \qquad \Phi = \int_S \mathbf{B}\cdot\mathbf{dS}$$

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu\mathbf{H} \qquad \mathbf{M} = \chi_m\mathbf{H}$$

$$\mathbf{J}_{m,s} = \mathbf{M}\times\mathbf{a}_n \qquad \mathbf{J}_m = \nabla\times\mathbf{M}$$

$$B_{1,n} = B_{2,n} \qquad \mathbf{a}_{n2}\times(\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla\times\mathbf{H} = \mathbf{J} \qquad \oint \mathbf{H}\cdot\mathbf{dl} = I$$

$$W_m = \frac{1}{2}\int_{vol}\mathbf{B}\cdot\mathbf{H}dv$$

$$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2} \qquad L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{N_2}{I_1}\int_{S_2}\mathbf{B}_1\cdot\mathbf{dS}_2$$

$$\mathcal{R} = \frac{l}{\mu S} \qquad V_{mmf} = NI$$