DO NOT WRITE ABOVE THIS LINE

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Term Test

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This is a closed book test. No calculators are permitted.

All six questions are of equal value.

This test contains 20 pages including the cover page 1 and this information page 2.

The test is printed two-sided.

Do <u>not</u> tear any pages from this test.

Present complete solutions in the space provided for the associated question.

Given information:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{v}}\, \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

Q1:

Consider the two vectors in R^3:

$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{i} + \vec{j} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

a) Are either of these two vectors a unit vector? Explain your answer.

Solution:

$\left\| \vec{i} \right\| = 1$, $\therefore \vec{\imath}$ is a unit vector

$\left\| \vec{i} + \vec{j} \right\| = \sqrt{2}$, $\therefore$ $\vec{i} + \vec{j}$ is not a unit vector

b) Give a vector equation of the plane that is described by all linear combinations of these two vectors and goes through the origin.

Solution:

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ where c, d are scalars

c) Give a scalar equation of the plane described by all linear combinations of these two vectors that goes through the origin. Give an explanation for how you arrived at your answer.

Solution:

A normal to the plane may be found by finding the cross product of $\vec{\imath}$ and $\vec{\imath} + \vec{\jmath}$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore 0x + 0y + z = d$$

Since the plane contains the origin, $d = 0$. Therefore, the scalar equation of the plane is given by:

$$z = 0$$

d) Give a scalar equation of the plane that is orthogonal to the plane in parts b) and c) and contains the points (1,0,0) and (1,1,0). Give an explanation for how you arrived at your answer.

Solution:

This plane is parallel to the normal of the plane in parts b) and c) and is also parallel to the vector given by the two points known to be in the plane:

$$\begin{bmatrix} 1-1 \\ 1-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A normal to the plane may be found by finding the cross product between these two vectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore -1x + 0y + 0z = d$$

Since the plane contains (1,0,0):

$$-1 = d$$

Therefore, the scalar equation of the plane is given by:

$$x = 1$$

Q2: Note that the vectors used in each of the four parts (a,b,c,d) of this question are not related to one another.

a) In your responses below, you must show your intermediate steps. For two unit vectors $\vec{v}$ and $\vec{w}$ in any dimension, find numerical (scalar) values for the dot products of:

i. $\vec{v}$ and $-0.5\vec{v}$

Solution:

$$\vec{v} \cdot (-0.5\vec{v}) = -0.5(\vec{v} \cdot \vec{v}) = -0.5\|\vec{v}\|^2 = -0.5(1) = -0.5$$

ii. $\vec{v} - \vec{w}$ and $\vec{v} + \vec{w}$

Solution:

$$\begin{aligned}(\vec{v} - \vec{w}) \cdot (\vec{v} + \vec{w}) &= \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{v} + \vec{v} \cdot \vec{w} - \vec{w} \cdot \vec{w} \\ &= \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{v} + \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{w} \\ &= \|\vec{v}\|^2 - \|\vec{w}\|^2 = 1 - 1 = 0\end{aligned}$$

b) Give a vector equation of the line in R^2 that goes through (2,-3) and is orthogonal to $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Solution:

A vector orthogonal to $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ because $\begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0$. Therefore, $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ can serve as a direction vector for the line. Using (2,-3) as a known point on the line, a vector equation for the line is given by:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ where } t \text{ is a scalar}$$

c) Is the angle between $\vec{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ obtuse, acute, or right? Explain your answer.

Solution:

$$\vec{v} \cdot \vec{w} = 2(2) + 2(-2) - 1(1) = -1$$

From the cosine formula:

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}$$

Since $\vec{v} \cdot \vec{w}$ is negative, $\cos\theta$ is negative because the vector magnitudes are positive.

Since $\cos\theta$ is negative, θ is an obtuse angle.

d) True or False: Consider 3 nonzero vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$, in any dimension. If $\vec{u}$ is orthogonal to $\vec{v}$ and to $\vec{w}$, then $\vec{u}$ is orthogonal to all linear combinations of $\vec{v}$ and $\vec{w}$. If this statement is true, give a proof that does not assume specific values for the vectors. If this statement is false, provide a counterexample.

Solution:

Given:

$$\vec{u} \cdot \vec{v} = 0 \text{ and } \vec{u} \cdot \vec{w} = 0$$

Then, for scalars c, d:

$$\begin{aligned} &\vec{u} \cdot (c\vec{v} + d\vec{w}) = \vec{u} \cdot (c\vec{v}) + \vec{u} \cdot (d\vec{w}) \\ &= c(\vec{u} \cdot \vec{v}) + d(\vec{u} \cdot \vec{w}) \\ &= c(0) + d(0) \\ &= 0 \end{aligned}$$

Therefore, $\vec{u}$ is orthogonal to all linear combinations of $\vec{v}$ and $\vec{w}$.

Q3: Note that the matrices used in each of the three parts (a,b,c) of this question are not related to one another.

a) For each matrix A given below, which number(s) q would leave matrix A with two independent columns?

i. $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 9 \\ 5 & 0 & q \end{bmatrix}$

Solution:

Columns 1 and 2 are independent. For column 3 to be dependent on columns 1 and 2, there need to be scalars c, d such that:

$$c \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ q \end{bmatrix}$$

$$\begin{aligned} &c(1) + d(0) = c = 2 \\ &c(3) + d(1) = 6 + d = 9 \rightarrow d = 3 \\ &c(5) + d(0) = 10 = q \end{aligned}$$

Therefore, $q = 10$ leaves matrix A with two independent columns.

ii. $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 0 & q \end{bmatrix}$

Columns 1 and 2 are dependent. For column 3 to be independent of columns 1 and 2, column 3 cannot be a scalar multiple of columns 1 and 2. Therefore, any value of $q \neq 0$ leaves matrix A with two independent columns.

b) Suppose $A\vec{x} = \vec{b}$. If you add column vector $\vec{b}$ as an extra column in matrix A, explain why the rank of this new matrix stays the same as that of matrix A.

Solution:

Given $A\vec{x} = \vec{b}$, then $\vec{b}$ is a combination of the columns of matrix A. Therefore, if $\vec{b}$ is added as an extra column in matrix A, that column is dependent on the other columns and it does not change the rank.

c) The column space of a matrix contains all combinations of the columns. For each matrix A given below, describe the column space (in words) in R^3 of matrix A and give a vector equation for this column space:

i. $A = \begin{bmatrix} 2 & -2 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$

Solution:

Since column 1 and column 2 are parallel, the column space is a line in R^3 through the origin:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \text{ where } c \text{ is a scalar}$$

ii. $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 2 & 4 \end{bmatrix}$

Solution:

There are only two independent columns, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ because column 3 is the sum of columns 1 and 2, and column 4 is parallel to column 3. Therefore, the column space is a plane in R^3 through the origin:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \text{ where } c, d \text{ are scalars}$$

Q4:

a) For each matrix A given below, factor matrix $A = CR$ and state the rank of matrix A.

i. $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 5 \end{bmatrix}$

Solution:

$$C = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

Column 4 does not go into matrix C because it is a combination of columns 2 and 3 in matrix A:

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 5 \end{bmatrix} = CR = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Rank of matrix A is equal to 2, the number of columns in matrix C.

ii. $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix}$

Solution:

$$C = \begin{bmatrix} 2 & 2 \\ 0 & 4 \\ 0 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} = CR = \begin{bmatrix} 2 & 2 \\ 0 & 4 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of matrix A is equal to 2, the number of columns in matrix C.

b) Students often ask why matrix multiplication AB is defined the way it is. The main reason is that we want AB times $\vec{x}$ to be equal to A times $B\vec{x}$, which leads to the all-important associative law for matrix multiplication: $(AB)C = A(BC)$.

Given:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} ; B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} ; \ \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i. Calculate AB.

Solution:

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \\ 9 & 9 \end{bmatrix}$$

ii. Calculate $B\vec{x}$.

Solution:

$$B\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

iii. Using your results from i. and ii., write $(AB)\vec{x}$ as a product of AB and $\vec{x}$ and $A(B\vec{x})$ as a product of A and $B\vec{x}$ and then calculate these two products to show that they produce the same result.

Solution:

$$(AB)\vec{x} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \\ 9 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$A(B\vec{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

Q5:

a) Reduce the following system of equations to upper triangular form using elimination. Then find values for the unknowns x, y, z and describe the row picture of the system and the system's solution.

$$2x + 3y + z = 8$$
$$4x + 7y + 5z = 20$$
$$-2y + 2z = 0$$

Solution:

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 4 & 7 & 5 & 20 \\ 0 & -2 & 2 & 0 \end{array}\right] R2 - 2R1$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & -2 & 2 & 0 \end{array}\right] R3 + 2R2$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 8 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 8 & 8 \end{array}\right] = [U|\vec{c}]$$

Solving:

$$8z = 8 \rightarrow z = 1$$
$$y = 4 - 3(1) = 1$$
$$x = \frac{1}{2}(8 - 3(1) - 1) = 2$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The row picture is 3 planes in R^3 that intersect at a single point (2,1,1).

b) Which number b leads to a row exchange during the process of reducing the following system to upper triangular form?

$$\begin{aligned} x + by &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0 \end{aligned}$$

Solution:

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] R2 - R1$$

$$\to \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

If $b = -2$, then:

$$\to \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

And then in order to proceed, we would make a row exchange $R2 \leftrightarrow R3$.

c) For the system of equations in part b),

$$\begin{aligned} x + by &= 0 \\ x - 2y - z &= 0 \\ y + z &= 0 \end{aligned}$$

which number b leads to a missing (zero) pivot when reducing the system to upper triangular form?

Solution:

$$\left[A\middle|\vec{b}\right] = \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] R2 - R1$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & b & 0 & 0 \\ 0 & -2-b & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

If $b = -1$:

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] R3 + R2$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This leads to a missing pivot.

Q6:

Consider the standard unit vectors in R^2 given by $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These two vectors form a basis for R^2. Now we are going to define a new basis for R^2 by rotating $\vec{i}$ and $\vec{j}$ both clockwise by 45 degrees without changing their length. We will refer to these new basis vectors as $\vec{i'}$ and $\vec{j}'$. These new basis vectors are given by:

$$\vec{i'} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \text{ and } \vec{j}' = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

a) The point (3,2) in the xy space can be expressed as a vector in standard position using a linear combination of the standard unit vectors as:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\vec{i} + 2\vec{j}$$

Express $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as a linear combination of the new basis vectors $\vec{i'}$ and $\vec{j}'$.

Solution:

Solve for scalars c, d:

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} + d\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$3 = \frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}}$$

$$3\sqrt{2} = c + d \quad (1)$$

$$2 = \frac{-c}{\sqrt{2}} + \frac{d}{\sqrt{2}}$$

$$2\sqrt{2} = -c + d \quad (2)$$

Adding (1) and (2): $5\sqrt{2} = 2d \rightarrow d = \frac{5\sqrt{2}}{2}$

Substituting into (1): $c = 3\sqrt{2} - \frac{5\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$

Giving: $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} + \frac{5\sqrt{2}}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

b) Project $\begin{bmatrix}3\\2\end{bmatrix}$ onto $\vec{i'}$ and then project $\begin{bmatrix}3\\2\end{bmatrix}$ onto $\vec{j'}$.

Solution:

$$proj_{\vec{i'}}\begin{bmatrix}3\\2\end{bmatrix} = \frac{\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}}{\left\|\vec{i'}\right\|^2}\vec{i'} = \frac{\frac{3}{\sqrt{2}}-\frac{2}{\sqrt{2}}}{1^2}\vec{i'} = \frac{1}{\sqrt{2}}\vec{i'} = \frac{\sqrt{2}}{2}\vec{i'}$$

$$proj_{\vec{j'}}\begin{bmatrix}3\\2\end{bmatrix} = \frac{\begin{bmatrix}3\\2\end{bmatrix}\cdot\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}}{\left\|\vec{j'}\right\|^2}\vec{j'} = \frac{\frac{3}{\sqrt{2}}+\frac{2}{\sqrt{2}}}{1^2}\vec{j'} = \frac{5}{\sqrt{2}}\vec{j'} = \frac{5\sqrt{2}}{2}\vec{j'}$$

c) Your answer to part a) and your answer to part b) are closely related. What is the connection between these two answers?

Solution:

In part a), the scalars c, d are the coordinates of (3,2) expressed in the new coordinate system defined by the basis vectors $\overrightarrow{i'}$ and $\overrightarrow{j}'$. In part (b), those same coordinates are found by projecting the vector $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ on the new basis vectors $\overrightarrow{i'}$ and $\overrightarrow{j}'$.