ESC195S - Calculus II

Midterm Test #1

February 13, 2024

9:10 - 10:50 am

Instructor: J. W. Davis

Closed book, no aid sheets, no calculators

There are 8 questions worth 10 marks.

Plus a bonus question worth 5 marks.

1. Use l'Hospital's rule to evaluate the following limits:

a) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$ b) $\lim_{x\to 0} \frac{\tan 3x}{\sin 2x}$ c) $\lim_{x\to 1^+} [\ln(x^7-1) - \ln(x^5-1)]$ d) $\lim_{x\to 0^+} x^{\sqrt{x}}$

a) $\lim_{x\to 0} \frac{x^2}{1-\cos x} \overset{*}{=} \lim_{x\to 0} \frac{2x}{\sin x} \overset{*}{=} \lim_{x\to 0} \frac{2}{\cos x} = \frac{2}{1} = 2$

$(0/0)$ $(0/0)$

b) $\lim_{x\to 0} \frac{\tan 3x}{\sin 2x} \overset{*}{=} \lim_{x\to 0} \frac{3\sec^2 3x}{2\cos 2x} = \frac{3}{2}$

$(0/0)$

c) $\lim_{x\to 1^+} \ln\left(\frac{x^7-1}{x^5-1}\right) \Rightarrow \lim_{x\to 1^+} \frac{x^7-1}{x^5-1} \overset{*}{=} \lim_{x\to 1^+} \frac{7x^6}{5x^4} = \frac{7}{5}$

$(0/0)$

$\therefore \lim_{x\to 1^+} [\ln(x^7-1) - \ln(x^5-1)] = \lim_{x\to 1^+} \ln\left(\frac{x^7-1}{x^5-1}\right) = \ln\frac{7}{5}$

d) $\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln x^{\sqrt{x}}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln x}$

(0^0)

$\Rightarrow \lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \overset{*}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}}$

$(0\cdot\infty)$ (∞/∞)

$= \lim_{x\to 0^+} -2x^{1/2} = 0$

$\therefore \lim_{x\to 0^+} e^{\ln x^{\sqrt{x}}} = e^0 = 1 = \lim_{x\to 0^+} x^{\sqrt{x}}$

2. Evaluate the integrals:

a) $\displaystyle\int_1^2 x^5 \ln x\, dx$ b) $\displaystyle\int_{-1}^3 \frac{x}{1+|x|}\, dx$ c) $\displaystyle\int \frac{x^2+8x-3}{x^3+3x^2}\, dx$ d) $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$

a) $\displaystyle\int_1^2 x^5 \ln x\, dx$ let $u = \ln x$ $du = dx/x$ $dv = x^5 dx$ $v = x^6/6$

$$= \left[\frac{x^6 \ln x}{6}\right]_1^2 - \int_1^2 \frac{x^5}{6}\, dx = \frac{64\ln 2}{6} - \left[\frac{x^6}{36}\right]_1^2 = \frac{32\ln 2}{3} - \frac{64-1}{36} = \frac{32\ln 2}{3} - \frac{7}{4}$$

b) $$\int_{-1}^3 \frac{x}{1+|x|}\, dx = \int_{-1}^0 \frac{x}{1-x} dx + \int_0^3 \frac{x}{1+x}\, dx = \int_{-1}^0 \left(-1+\frac{1}{1-x}\right) dx + \int_0^3 \left(1 - \frac{1}{1+x}\right) dx$$

$$= \left[-x - \ln|1-x|\right]_{-1}^0 + \left[x - \ln|1+x|\right]_0^3 = -1+\ln 2 + 3 - \ln 4 = 2 - \ln 2$$

Alternate: $\displaystyle\int_{-1}^1 \frac{x}{1+|x|} dx = 0$ (odd f'n) $\therefore \displaystyle\int_{-1}^3 \frac{x}{1+|x|} dx = \int_1^3 \frac{x}{1+x}\, dx$

c) $$\frac{x^2+8x-3}{x^3+3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3} \Rightarrow Ax(x+3) + B(x+3) + Cx^2 = x^2+8x-3$$

let $x=0$: $3B = -3$ $\therefore B = -1$

$x = -3$: $9C = 9-24-3 = -18$ $\therefore C = -2$

$x = 1$: $4A - 4 - 2 = 6$ $\therefore A = 3$

$$\int \frac{x^2+8x-3}{x^3+3x^2}\, dx = \int \frac{3}{x} dx - \int \frac{dx}{x^2} - \int \frac{2}{x+3} dx$$

$$= 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$$

d) $\displaystyle\int \frac{dx}{x\sqrt{x^2+1}}$ let $x = \tan\theta$ $dx = \sec^2\theta\, d\theta$ $\sqrt{x^2+1} = \sec\theta$

$$= \int \frac{\sec^2\theta\, d\theta}{\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta} d\theta = \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + C$$

3. Determine whether the integral is convergent or divergent. Evaluate the integrals that are convergent.

a) $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ b) $\displaystyle\int_0^{\pi} \tan^2 x\,dx$

a) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$ let $u = e^x$ $du = e^x\,dx$

$$= \int \frac{du}{1+u^2} = \tan^{-1} u + C = \tan^{-1}(e^x) + C$$

$$\Rightarrow \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx = \lim_{a\to-\infty}\left[\tan^{-1} e^x\right]_a^0 = \tan^{-1} 1 - \lim_{a\to-\infty} \tan^{-1} e^a = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \lim_{b\to\infty}\left[\tan^{-1} e^x\right]_0^b = \lim_{b\to\infty} \tan^{-1} e^b - \tan^{-1} 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore \int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

b) $\displaystyle\int_0^{\pi} \tan^2 x\,dx = \int_0^{\pi/2} \tan^2 x\,dx + \int_{\pi/2}^{\pi} \tan^2 x\,dx$ discontinuity at $x = \pi/2$

consider $\displaystyle\int_0^{\pi/2} \tan^2 x\,dx = \int_0^{\pi/2} (\sec^2 x - 1)\,dx = \left[\tan x - x\right]_0^{\pi/2}$

$\displaystyle\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$ $\therefore \displaystyle\int_0^{\pi/2} \tan^2 x\,dx$ diverges

$\therefore \displaystyle\int_0^{\pi} \tan^2 x\,dx$ diverges.

Note: $\displaystyle\int_{\pi/2}^{\pi} \tan^2 x\,dx$ also diverges

4. Find the area of the surface generated by rotating the curve:

$$x = e^t \cos t, \qquad y = e^t \sin t, \qquad 0 \le t \le \frac{\pi}{2}$$

about the y-axis.

$$S = \int 2\pi \cdot \text{radius} \cdot ds = \int 2\pi x \sqrt{(x')^2 + (y')^2}\, dt$$

$$x = e^t \cos t \Rightarrow x' = -e^t \sin t + e^t \cos t = e^t(\cos t - \sin t)$$

$$y = e^t \sin t \Rightarrow y' = e^t(\cos t + \sin t)$$

$$(x')^2 + (y')^2 = e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t) + e^{2t}(\cos^2 t + 2\cos t \sin t + \sin^2 t)$$

$$= e^{2t}(2\cos^2 t + 2\sin^2 t) = 2e^{2t}$$

$$\therefore S = \int_0^{\pi/2} 2\pi e^t \cos t \cdot \sqrt{2e^{2t}}\, dt = 2\sqrt{2}\pi \int_0^{\pi/2} e^{2t} \cos t\, dt$$

for $\int e^{2t} \cos t\, dt$: let $u = e^{2t}$, $dv = \cos t\, dt$

$du = 2e^{2t}$, $v = \sin t$

$$= e^{2t} \sin t - \int 2e^{2t} \sin t\, dt$$

let $u = 2e^{2t}$, $dv = \sin t\, dt$

$du = 4e^{2t}$, $v = -\cos t$

$$= e^{2t} \sin t + 2e^{2t} \cos t - \int 4e^{2t} \cos t\, dt$$

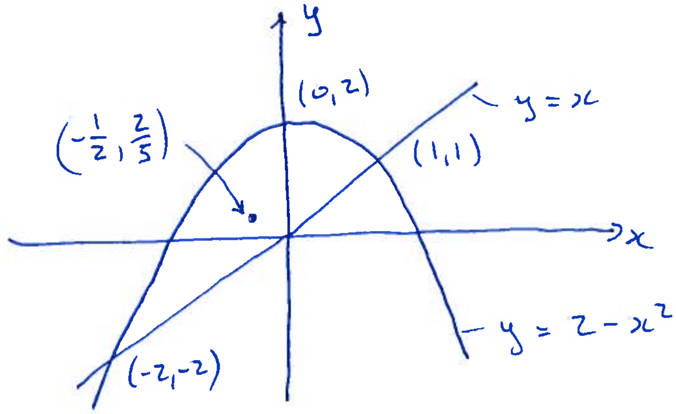
$$\Rightarrow 5\int e^{2t} \cos t\, dt = e^{2t}(\sin t + 2\cos t) + C'$$

$$\int e^{2t} \cos t\, dt = \frac{1}{5} e^{2t}(\sin t + 2\cos t) + C$$

$$\therefore S = 2\sqrt{2}\pi \left[\frac{1}{5} e^{2t}(\sin t + 2\cos t)\right]_0^{\pi/2} = 2\sqrt{2}\pi \left(\frac{1}{5} e^{\pi}(1) - \frac{1}{5}(2)\right)$$

$$= \frac{2\sqrt{2}\pi}{5}\left(e^{\pi} - 2\right) \text{ square units}$$

5. Find the centroid of the region bounded by the curves: $y = 2 - x^2$, $y = x$
Provide a sketch of the region indicating the location of the centroid.



Intersections: $2 - x^2 = x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$\therefore x = -2, 1$

$$A = \int_{-2}^{1} (2 - x^2 - x)\,dx = \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} = \left(2 - \frac{1}{3} - \frac{1}{2} + 4 - \frac{8}{3} + \frac{4}{2}\right) = \frac{9}{2}$$

$$\bar{x}A = \int_{-2}^{1} x(2 - x^2 - x)\,dx = \left[2\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^3}{3}\right]_{-2}^{1} = \left(1 - \frac{1}{4} - \frac{1}{3} - 4 + \frac{16}{4} - \frac{8}{3}\right) = -\frac{9}{4}$$

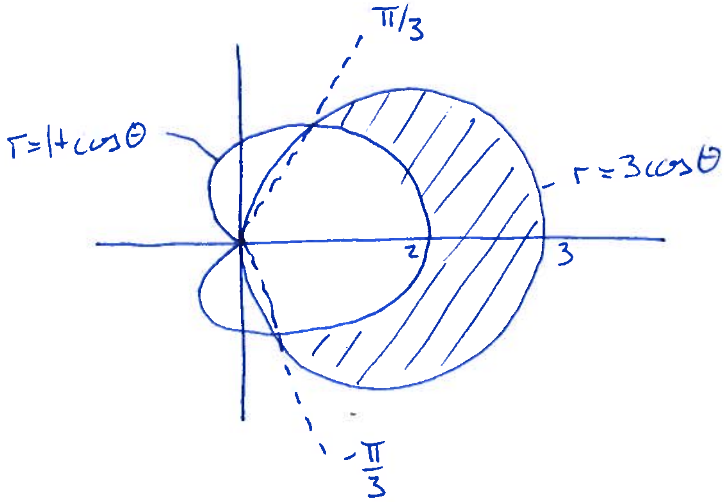
$$\therefore \bar{x} = -\frac{9}{4} \cdot \frac{2}{9} = -\frac{1}{2}$$

$$\bar{y}A = \int_{-2}^{1} \frac{1}{2}\left[(2 - x^2)^2 - (x)^2\right] dx = \frac{1}{2}\int_{-2}^{1} (4 - 4x^2 + x^4 - x^2)\,dx$$

$$= \frac{1}{2}\left[4x - 5\frac{x^3}{3} + \frac{x^5}{5}\right]_{-2}^{1} = \frac{1}{2}\left(4 - \frac{5}{3} + \frac{1}{5} + 8 - \frac{40}{3} + \frac{32}{5}\right) = \frac{1}{2}\left(12 - \frac{45}{3} + \frac{33}{5}\right) = \frac{18}{10}$$

$$\therefore \bar{y} = \frac{18}{10} \cdot \frac{2}{9} = \frac{2}{5}$$

6. Find the area of the region that lies inside the polar curve: $r = 3\cos\theta$, but outside the curve $r = 1 + \cos\theta$. Provide a sketch of the region.



Intersection:

$3\cos\theta = 1 + \cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2}$

$\Rightarrow \theta = \pm \frac{\pi}{3}$

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}\left((3\cos\theta)^2 - (1+\cos\theta)^2 \right) d\theta$$

$$= \int_0^{\pi/3} \left(9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta \right) d\theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta \right) d\theta$$

$$= \int_0^{\pi/3} \left(8\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta\right) - 1 - 2\cos\theta \right] d\theta$$

$$= \left[4\theta + \frac{4}{2}\sin 2\theta - \theta - 2\sin\theta \right]_0^{\pi/3}$$

$$= \pi + 2\left(\frac{\sqrt{3}}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \pi$$

7. Sketch a graph of the parametric curve: $x = t^3 - 3t$. $y = t^2$

$x = t^3 - 3t$

$x' = 3t^2 - 3$

$x' = 0 \to t = \pm 1 \Rightarrow (-2, 1)$

$(2, 1)$

vertical tangent

$y = t^2$

$y' = 2t$

$y' = 0 \Rightarrow t = 0 \Rightarrow (0, 0)$

horizontal tangent

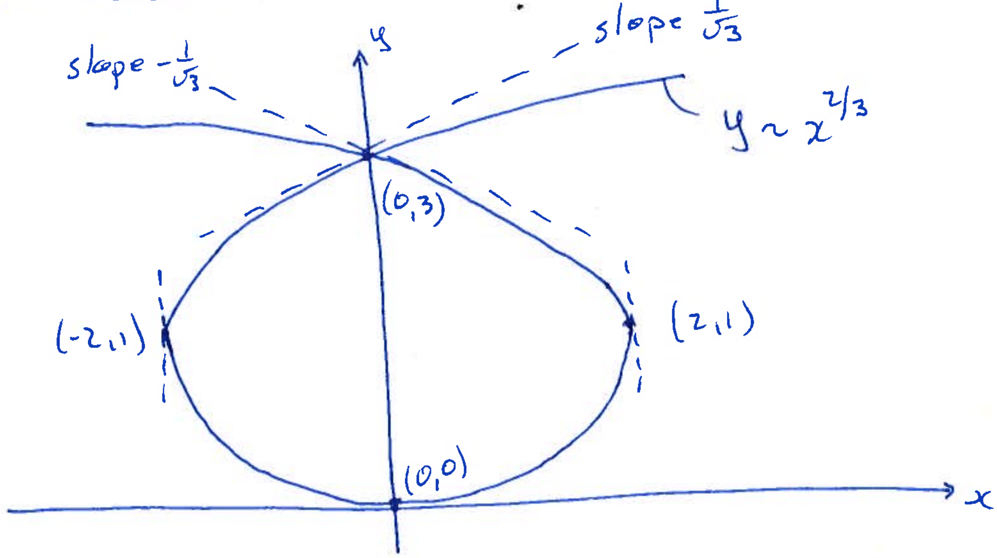
Intercepts: $y = 0 \Rightarrow t = 0 \to (0, 0)$

$x = 0 \Rightarrow t = 0$

$t = \pm\sqrt{3} \Rightarrow (0, 3)$

slope at $t = \pm\sqrt{3}$: $\frac{y'}{x'} = \frac{2t}{3t^2 - 3} = \pm\frac{2\sqrt{3}}{6} = \pm\frac{1}{\sqrt{3}}$

Asymptotic behaviour: as $t \to \pm\infty$, $x \to t^3$ $\therefore y \to x^{2/3}$



8. Determine whether the sequence converges or diverges. If it converges, find the limit.

a) $a_n = \dfrac{3\sqrt{n}}{\sqrt{n}+2}$ b) $a_n = \dfrac{(\ln(n))^2}{n}$ c) $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)}{n!}$

a) $a_n = \dfrac{3\sqrt{n}}{\sqrt{n}+2} = \dfrac{3}{1+\frac{2}{\sqrt{n}}} \longrightarrow \dfrac{3}{1+0} = 3$

b) consider $f(x) = \dfrac{(\ln x)^2}{x}$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{*}{=} \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{1} \overset{*}{=} \lim_{x\to\infty} \frac{2\frac{1}{x}}{1} \longrightarrow 0$$

$$\therefore a_n = \frac{(\ln(n))^2}{n} \longrightarrow 0$$

c) $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots (2n-1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot n} = \left\{1, \frac{3}{2}, \frac{15}{6}, \frac{105}{24}, \cdots\right\}$

Now $a_2 = a_1 \cdot \frac{3}{2} = \frac{3}{2}$; $a_3 = a_2 \cdot \frac{5}{3} > a_2 \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^2$; $a_4 = a_3 \cdot \frac{7}{4} > a_3 \cdot \frac{3}{2} > \left(\frac{3}{2}\right)^3$

$\Rightarrow$ show $a_n \geq \left(\frac{3}{2}\right)^{n-1}$ for $n > 1$

given $a_n \geq \left(\frac{3}{2}\right)^{n-1}$ (true for $n = 2, 3, 4$ above) show $a_{n+1} > \left(\frac{3}{2}\right)^n$

Note: $\dfrac{2n+1}{n+1} > \dfrac{3}{2} \Rightarrow 4n+2 > 3n+3 \Rightarrow n > 1$

$a_{n+1} = \dfrac{2n+1}{n+1} a_n > \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1} > \left(\dfrac{3}{2}\right)^n$ as required

$\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$ is a diverging geometric sequence

$\therefore a_n$ diverges by comparison

9. Bonus Question:

$$\text{Given } \int_{-\infty}^{\infty} f(x)\,dx = L, \text{ show } \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right)\,dx = L. \ .$$

let $x = y - 1/y$ $\Rightarrow$ as y goes from $-\infty$ to 0,

$dx = \left(1 + \frac{1}{y^2}\right) dy$ $\quad$ x goes from $-\infty$ to $+\infty$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f\left(y - \frac{1}{y}\right)\left(1 + \frac{1}{y^2}\right) dy$$

$$= \int_{-\infty}^{0} f\left(y - \frac{1}{y}\right) dy + \int_{-\infty}^{0} f\left(y - \frac{1}{y}\right) \frac{1}{y^2}\,dy$$

In the second integral, we let $z = \frac{-1}{y}$ $\therefore dz = \frac{dy}{y^2}$

$$\therefore \int_{-\infty}^{0} f\left(y - \frac{1}{y}\right) \frac{dy}{y^2} = \int_{0}^{\infty} f\left(z - \frac{1}{z}\right) dz$$

Combining the two integrals we get:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f\left(y - \frac{1}{y}\right) dy + \int_{0}^{\infty} f\left(z - \frac{1}{z}\right) dz$$

$$= \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = L \quad \text{as required}$$