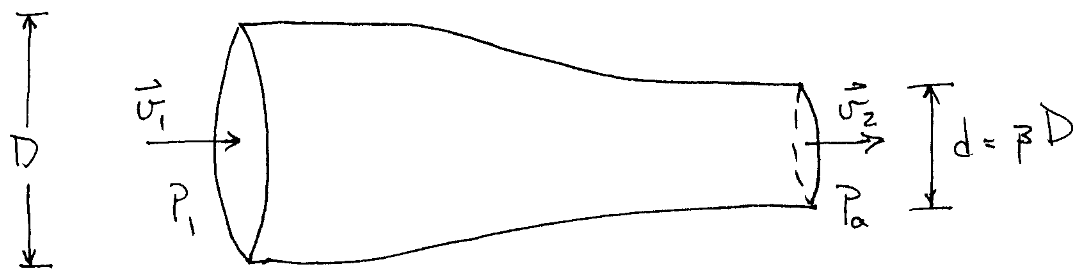
5.1 Force Acting on a Nozzle



$$A_1 = \frac{\pi D^2}{4} \qquad\qquad A_2 = \frac{\pi d^2}{4} = \beta^2 A_1$$

1) Continuity : $A_1 q_1 = A_2 q_2 = Q \quad \Rightarrow \quad q_1 = Q/A_1$

$$q_2 = Q/A_2 = Q/\beta^2 A_1$$

2) Bernoulli : $\dfrac{P_1}{\rho} + \dfrac{1}{2} q_1^2 + g z_1 = \dfrac{P_a}{\rho} + \dfrac{1}{2} q_2^2 + g z_2$

$\Rightarrow$ for horizontal configuration : $z_1 = z_2$

$$\therefore \ (P_1 - P_a) = \frac{\rho}{2}\left(q_2^2 - q_1^2\right) = \frac{\rho Q^2}{2 A_1^2}\left(\frac{1}{\beta^4} - 1\right)$$

3) Momentum : $\vec{F}_E = \dot{m}_f\left(\vec{v}_2 - \vec{v}_1\right) = -\vec{F}_N + \vec{F}_P$

pressure force

force on nozzle

$\Rightarrow$ assume atmospheric pressure acts equally on exterior of nozzle, as well as the exit :

$$\therefore \quad \vec{F}_P = \left(P_1 A_1 \; - \; P_a A_2 \; - \; P_a (A_1 - A_2)\right) \hat{i}_x$$

pressure force at nozzle entrance

pressure force at nozzle exit

atmospheric pressure acting on outside of nozzle

$$\vec{F}_P = (P_1 - P_a) A_1 \hat{i}_x$$

$$\Rightarrow \dot{m}_f (\vec{v}_2 - \vec{v}_1) = \rho Q (q_2 - q_1) \hat{i}_x = \rho \frac{Q^2}{A_1} \left(\frac{1}{\beta^2} - 1 \right) \hat{i}_x$$

$$\therefore \vec{F}_N = \vec{F}_P - \dot{m}_f (\vec{v}_2 - \vec{v}_1)$$

$$= (P_1 - P_a) A_1 \hat{i}_x - \rho \frac{Q^2}{A_1} \left(\frac{1}{\beta^2} - 1 \right) \hat{i}_x$$

$$\| \vec{F}_N \| = \frac{\rho Q^2}{2 A_1} \left(\frac{1}{\beta^4} - 1 \right) - \rho \frac{Q^2}{A_1} \left(\frac{1}{\beta^2} - 1 \right)$$

$$= \frac{\rho Q^2}{A_1} \left(\frac{1 - \beta^4}{2\beta^4} - \frac{2\beta^2 - 2\beta^4}{2\beta^4} \right)$$

$$= \frac{\rho Q^2}{A_1} \left(\frac{1 - 2\beta^2 + \beta^4}{2\beta^4} \right)$$

4) Numerical Example: $D = 0.06$, $d = 0.03$, $q_2 = 30$ m/s

$\Rightarrow \beta = 0.5$

$A_1 = 0.00283 \text{ m}^2$

$Q = q_2 \beta^2 A_1 = 30 \cdot 0.5^2 \cdot 0.00283 = 0.0212 \text{ m}^3/\text{s}$

$$\Rightarrow \| \vec{F}_N \| = \frac{1000 \cdot (0.0212)^2}{0.00283} \left(\frac{1 - 2(0.5)^2 + (0.5)^4}{2 (0.5)^4} \right) = 7.4 \text{ N}$$

$\boxed{5.2 \ \& \ 5.3}$ Pelton Wheel

1) We start by forming a vector diagram of velocity components:

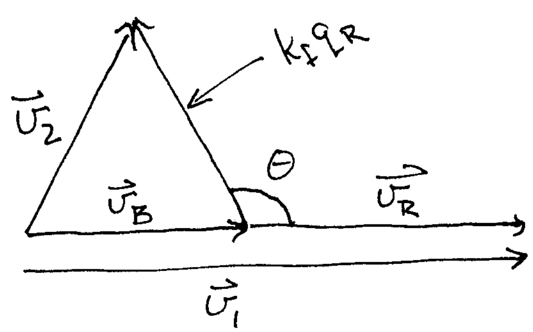
$\vec{v}_1 = q_J \hat{i}_x$ = exit velocity of water jet from nozzle

$\vec{v}_B = q_B \hat{i}_x$ = forward velocity of bucket

$\vec{v}_R = q_R \hat{i}_x = \vec{v}_1 - \vec{v}_B$ = relative velocity of water jet to bucket

$k_f q_R$ = speed of water relative to bucket, slowed by friction factor k_f.

$\vec{v}_2$ = final velocity of water in stationary frame



$$\Rightarrow \vec{v}_2 = k_f q_R (\cos\theta \, \hat{i}_x + \sin\theta \, \hat{i}_y) + \vec{v}_B$$

$$= k_f (q_J - q_B)(\cos\theta \, \hat{i}_x + \sin\theta \, \hat{i}_y) + q_B \hat{i}_x$$

2) Force on the bucket

⇒ From momentum balance we find:

$$\vec{F}_B = -\vec{F}_{Ext} = -\dot{m}\,(\vec{v}_2 - \vec{v}_1)$$

⇒ From symmetry, there is no net force in the y-direction:

$$F_x = -\dot{m}\,(\vec{v}_2 - \vec{v}_1)\cdot\hat{i}_x$$

$$= \dot{m}\,(q_J - k_f(q_J - q_B)\cos\theta - q_B)$$

$$= \dot{m}\,(q_J - q_B)(1 - k_f\cos\theta)$$

⇒ In determining $\dot{m}$, we note that the ~~total~~ mass flux from the nozzle interacts with the wheel; as one bucket moves away, another intercepts the water jet ⇒ $\dot{m} = q_J\,\rho\,A$

⇒ There is no correction for the moving bucket.

Another way of looking at it, is to realize that the water jet will sometimes be pushing on two buckets at the same time; as a new bucket intercepts the flow, the will still be water moving towards the previous one.

3) Power is given by force x velocity

$$\therefore P = F_x \times q_B = q_J \rho A (q_J - q_B)(1 - k_f \cos\theta) \cdot q_B$$

The maximum power is found when $\frac{dP}{dq_B} = 0$

$$\Rightarrow \frac{d}{dq_B} q_B(q_J - q_B) = 0 \quad \text{or} \quad q_B = \frac{1}{2} q_J$$

Note that if $k_f = 1$ and $\theta = \pi$, corresponding to complete frictionless turning, then:

$$\text{if } q_B = \frac{1}{2} q_J \rightarrow \vec{v}_2 = 1\left(q_J - \frac{1}{2}q_J\right)\left(-1\,\hat{i}_x + 0\,\hat{i}_y\right) + \frac{1}{2}q_J\,\hat{i}_x$$

$$= \left(-q_J + \frac{1}{2}q_J + \frac{1}{2}q_J\right)\hat{i}_x$$

$$= 0$$

$\Rightarrow$ All of the kinetic energy in the jet is transferred to the wheel. $\Rightarrow P_w = \frac{\rho A q_J^3}{2}$

$$\text{At } q_B = \frac{1}{2} q_J \Rightarrow P_w = \frac{\rho A q_J^3}{4}(1 - k_f \cos\theta)$$

$$\eta_H = \frac{1}{2}(1 - k_f \cos\theta)$$

4) Numerical Example

$R = 1.0\ m$

$$\left.\begin{aligned} \Theta &= \tfrac{3}{4}\pi \quad (135^\circ) \\ k_f &= 0.95 \end{aligned}\right\} \quad \eta_H = \tfrac{1}{2}\left(1 - 0.95 \cos 135^\circ\right) = 0.836$$

$$d_j = 0.2m \implies A = \frac{\pi d_j^2}{4} = 0.0314\ m^2$$

$$H = 800\ m \implies q_J = \sqrt{2gH} = \sqrt{2 \cdot 9.8 \cdot 800} = 125.2\ m/s$$

For maximum operating efficiency, $q_B = \frac{1}{2} q_J = 62.6\ m/s$

$$\therefore \Omega = \frac{q_B}{R} = 62.6\ rad/s$$

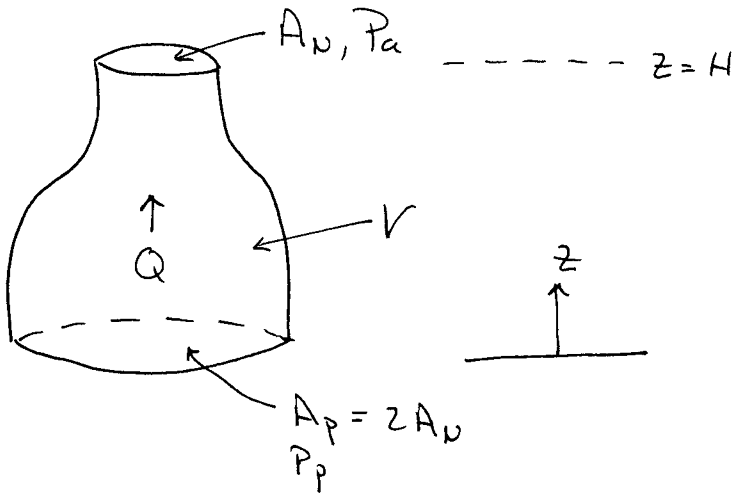
$$\text{or } N = \frac{\Omega \times 60}{2\pi} = 598\ rpm$$

$$P_w = \frac{\rho A q_J^3}{2} \eta_H = \frac{1000 \cdot 0.0314 \cdot (125.2)^3}{2} \cdot 0.836$$

$$= 25.8\ MW$$

Note: High head Pelton wheels produce power at these levels, and can get η_H as high as 90%.

5.4 Calculate the force on the nozzle:



$\vec{g} = -g\hat{k}$

⇒ Incompressible, inviscid flow:

Continuity: $Q = q_P A_P = q_N A_N \implies q_P = \frac{Q}{2A_N}$; $q_N = \frac{Q}{A_N}$

Momentum: $\vec{F}_{\substack{TOTAL \\ External}} = \dot{m}\left(\vec{v}_2 - \vec{v}_1\right) = \rho Q \left(q_N \hat{k} - q_P \hat{k}\right)$

$$= \rho Q \left(\frac{Q}{A_N} - \frac{Q}{2A_N}\right)\hat{k} = \frac{\rho Q^2}{2A_N}\hat{k}$$

The sum of the external forces acting on the fluid in the control volume has 4 components:

$$\vec{F}_{TOT} = \underset{\text{gravity}}{\vec{F}_g} + \underbrace{\vec{F}_{P_P} + \vec{F}_{P_N}}_{\text{pressure at inlet \& outlet}} + \vec{F}_N$$

force exerted on the water by the nozzle.

(Because atmospheric pressure acts evenly everywhere, it cannot lead to a net force acting on either the water or the nozzle. Thus we work with gage pressure.)

$$\Rightarrow \frac{\rho Q^2}{2A_N}\hat{k} = -\rho g V\hat{k} + P_P \cdot A_P \hat{k} - P_N \cdot A_N \hat{k} + F_N \hat{k}$$

$$\llcorner P_{N_{gage}} = 0$$

$$\therefore F_N = \frac{\rho Q^2}{2A_N} + \rho g V - P_P A_P$$

$$\llcorner A_P = 2A_N$$

Bernoulli: $\frac{P_P}{\rho} + g \cdot 0 + \frac{1}{2} q_P^2 = \frac{0}{\rho} + gH + \frac{1}{2} q_N^2$

$$P_P = \rho g H + \frac{\rho}{2}\left(q_N^2 - q_P^2\right) = \rho g H + \frac{\rho}{2}\left(\frac{Q^2}{A_N^2} - \frac{Q^2}{4A_N^2}\right)$$

$$= \rho g H + \frac{3}{8}\frac{\rho Q^2}{A_N^2}$$

$$\therefore F_N = \frac{\rho Q^2}{2A_N} + \rho g V - 2\rho g H A_N - \frac{3}{4}\frac{\rho Q^2}{A_N}$$

$$= \rho g V - 2\rho g H A_N - \frac{1}{4}\frac{\rho Q^2}{A_N}$$

The force exerted on the nozzle will be in the opposite direction.

<u>No Flow</u>: $P_P = \rho g H$; $F_{TOT} = 0 = F_g + F_P + F_N$

$$= -\rho g V + \rho g H \cdot A_P + F_N$$

$$\Rightarrow F_N = \rho g V - 2\rho g H A_N$$

<u>No Gravity</u>: Bernoulli: $P_P = \frac{3}{8}\frac{\rho Q^2}{A_N^2}$

Momentum: $F_{TOT} = \frac{\rho Q^2}{2A_N}$

$$\left.\right\} F_N = -\frac{1}{4}\frac{\rho Q^2}{A_N}$$