AER210 VECTOR CALCULUS and FLUID MECHANICS

Quiz 2

Duration: 75 minutes

28 October 2019

Closed Book, no aid sheets, no calculators

Instructor: Prof. Alis Ekmekci

Family Name: Alis Ekmekci

Given Name: Solutions

Student #: ______________________

TA Name/Tutorial #: ______________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	8	
2	10	
3	12	
4	5	
5	8	
6	8	
7	10	
TOTAL	61	/60

Note the following integrals may be useful:

$\int cos^2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}sin2\theta + C$; $\int sin^2\theta d\theta = \frac{1}{2}\theta - \frac{1}{4}sin2\theta + C$

$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$; $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$

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1) a) Let a parameterization to the shape of a wire be $\vec{r}(t) = (3t-2)\vec{\imath} + (t+1)\vec{\jmath}$ where $1 \le t \le 2$. If the density (mass per unit length) of the wire at any given point is given by $\rho(x,y) = x + y$. Compute the mass of the wire.
(4 marks)

$$m = \int_C \rho(x,y)\,ds = \int_{t=1}^{2} \rho(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

$$\vec{r}(t) = (3t-2)\vec{\imath} + (t+1)\vec{\jmath}$$

$$\vec{r}\,'(t) = 3\vec{\imath} + \vec{\jmath} \quad \Rightarrow \quad \|\vec{r}\,'(t)\| = \sqrt{3^2+1^2} = \sqrt{10}$$

$$m = \int_1^2 \Big[\underbrace{(3t-2)+(t+1)}_{\rho(\vec{r}(t))}\Big]\underbrace{\sqrt{10}}_{\|\vec{r}\,'(t)\|}\,dt = \sqrt{10}\int_1^2 (4t-1)\,dt = \sqrt{10}\,(2t^2-t)\Big|_{t=1}^{2}$$

$$= \sqrt{10}\big[(8-2)-(2-1)\big] = 5\sqrt{10}$$

b) An object moves along the parabola $y = 3x^2$ from point (0,0) to the point (1,3). One of the forces acting on the object is $\vec{F} = x^3\,\vec{\imath} + y\,\vec{\jmath}$. Calculate the work done by $\vec{F}$.
(4 marks)

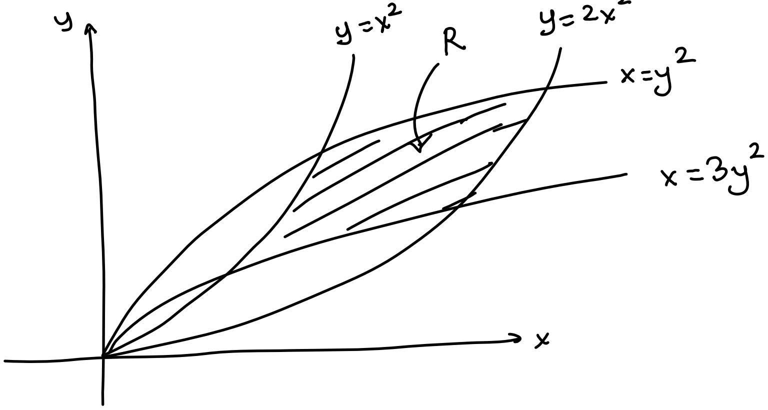
The parabola $y = 3x^2$ can be parametrized by letting

$$\left.\begin{aligned} x &= t \\ y &= 3t^2 \end{aligned}\right\} \Rightarrow \vec{r}(t) = t\vec{\imath} + 3t^2\vec{\jmath} \quad \text{where} \quad 0 \le t \le 1$$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_{t=0}^{1} \left(t^3\,\vec{\imath} + 3t^2\vec{\jmath}\right)\cdot\left(\vec{\imath} + 6t\vec{\jmath}\right)dt$$

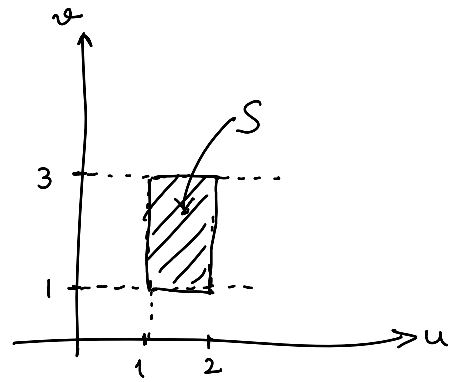
$$= \int_0^1 (t^3 + 18t^3)\,dt = \int_0^1 19t^3\,dt = 19\,\frac{t^4}{4}\Big|_0^1 = \frac{19}{4}$$

2) Find the area of the planar region bounded by the four parabolas: $y = x^2, y = 2x^2, x = y^2$ and $x = 3y^2$ using an appropriate coordinate transformation. Also provide a sketch of the region in the new coordinate systems.
(10 marks)



$$\left.\begin{aligned} y = x^2 &\Rightarrow \frac{y}{x^2} = 1 \\ y = 2x^2 &\Rightarrow \frac{y}{x^2} = 2 \end{aligned}\right\} \quad \frac{y}{x^2} = u \Rightarrow \begin{cases} u = 1 \\ u = 2 \end{cases}$$

$$\left.\begin{aligned} x = y^2 &\Rightarrow \frac{x}{y^2} = 1 \\ x = 3y^2 &\Rightarrow \frac{x}{y^2} = 3 \end{aligned}\right\} \quad \frac{x}{y^2} = v \Rightarrow \begin{cases} v = 1 \\ v = 3 \end{cases}$$



The new region is the rectangular region S in the uv-plane given by $1 \le u \le 2$, $1 \le v \le 3$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -2\frac{y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -2\frac{x}{y^3} \end{vmatrix} = \underbrace{\left(-2\frac{y}{x^3}\right)\left(-2\frac{x}{y^3}\right)}_{\frac{4}{x^2y^2}} - \underbrace{\left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right)}_{\frac{1}{x^2y^2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{3}{x^2y^2} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{x^2y^2}{3} = \frac{1}{3u^2v^2}$$

$$\left.\begin{aligned} x^2 &= \frac{y}{u} \\ y^2 &= \frac{x}{v} \end{aligned}\right\} x^2y^2 = \frac{y}{u}\cdot\frac{x}{v} \Rightarrow xy = \frac{1}{uv} \Rightarrow x^2y^2 = \frac{1}{u^2v^2}$$

$$\iint_R 1 \cdot dx\,dy = \iint_S \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv = \int_{v=1}^{3}\int_{u=1}^{2} \frac{1}{3u^2v^2}\,du\,dv = \frac{1}{3}\int -\frac{1}{u}\cdot\frac{1}{v^2}\Big|_{u=1}^{2} dv = \frac{1}{6}\int_1^3 \frac{1}{v^2}\,dv = \underline{\underline{\frac{1}{9}}}$$

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3) (a) For what value of the constant A is the following vector field conservative?

$$\vec{F} = \vec{F}(x, y) = Ax\, sin(\pi y)\vec{\imath} + x^2 \cos(\pi y)\vec{\jmath}$$

(b) For this value of A, find the potential function.

(c) For this value of A, evaluate the line integral $\int_C \vec{F} \cdot \overrightarrow{dr}$ where C is the curve given by $\vec{r}(t) = \cos t\,\vec{\imath} + \sin 2t\,\vec{\jmath} \qquad (0 \le t \le 2\pi)$.

(d) For this value of A, evaluate the line integral $\int_C \vec{F} \cdot \overrightarrow{dr}$ where C is the curve formed by the intersection of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 3x + 2y$ from point (0, 0) to point (0, 1/2).

(12 marks = (a) 3 marks + (b) 3 marks + (c) 3 marks + (d) 3 marks)

a) $\vec{F} = \underbrace{Ax\sin(\pi y)}_{P(x,y)}\vec{\imath} + \underbrace{x^2\cos(\pi y)}_{Q(x,y)}\vec{\jmath}$

$\vec{F}$ cannot be conservative unless $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial P}{\partial y} = \pi A x\cos(\pi y)$

$\frac{\partial Q}{\partial x} = 2x\cos(\pi y)$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \pi A = 2 \Rightarrow \boxed{A = \frac{2}{\pi}}$

b) $\vec{F} = P\vec{\imath} + Q\vec{\jmath} = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} \Rightarrow P = \frac{\partial f}{\partial x}$ & $Q = \frac{\partial f}{\partial y}$

$P = \frac{\partial f}{\partial x} \Rightarrow \frac{2}{\pi}x\sin(\pi y) = \frac{\partial f}{\partial x} \Rightarrow f(x,y) = \int \frac{2}{\pi}x\sin(\pi y)\,dx$

$f(x,y) = \frac{x^2}{\pi}\sin(\pi y) + g(y)$

$Q = \frac{\partial f}{\partial y} \Rightarrow x^2\cos(\pi y) = \frac{\partial}{\partial y}\left(\frac{x^2}{\pi}\sin(\pi y) + g(y)\right) \Rightarrow \cancel{x^2\cos(\pi y)} = \cancel{x^2\cos(\pi y)} + g'(y)$

$g'(y) = 0 \Rightarrow g(y) = C_1$

$\therefore$ The potential function is: $\boxed{f(x,y) = \frac{x^2}{\pi}\sin(\pi y) + C_1}$

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c) curve $C : \vec{r}(t)\ \cos t\,\vec{i} + \sin(2t)\,\vec{j} \quad (0 \le t \le 2\pi)$

$\vec{r}(0) = \vec{r}(2\pi) = \vec{i} \Rightarrow$ the start and finishing points of this curve is the same.

So, it is a closed curve.

As $\vec{F}$ is conservative, $\oint_C \vec{F} \cdot d\vec{r} = 0$

d) Because $\vec{F}$ is conservative, path between the two points does not matter. We simply use the potential function and evaluate the line integral from the values of the potential funct. at the endpoints:

$$\left.\begin{array}{l} (0,0) : \text{starting point of } C \\ (1, 1/2) : \text{terminal point of } C \end{array}\right\}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f\left(1, \tfrac{1}{2}\right) - f(0,0) \quad \text{where } f(x,y) = \frac{x^2}{\pi} \sin(\pi y)$$

$$= \frac{1}{\pi} \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} - 0$$

$$= \frac{1}{\pi}$$

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4) Find the surface area of the surface given parametrically by $x = u^2, y = uv, z = \frac{1}{2}v^2$ where $0 \le u \le 1, 0 \le v \le 2$.

(5 marks)

$$\vec{r}(u,v) = u^2\vec{i} + uv\vec{j} + \frac{1}{2}v^2\vec{k}$$

$$\vec{r}_u = 2u\vec{i} + v\vec{j} + 0\vec{k}$$

$$\vec{r}_v = 0\vec{i} + u\vec{j} + v\vec{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} = v^2\vec{i} - 2uv\vec{j} + 2u^2\vec{k}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = v^2 + 2u^2$$

$$S = \iint_S dS = \int_{v=0}^{2}\int_{u=0}^{1} \|\vec{r}_u \times \vec{r}_v\|\, du\, dv = \int_0^2\int_0^1 (v^2 + 2u^2)\, du\, dv$$

$$S = \int_0^2 \left[v^2 u + \frac{2u^3}{3}\right]_{u=0}^{1} dv = \int_0^2 \left[v^2 + \frac{2}{3}\right] dv = \left.\frac{v^3}{3} + \frac{2}{3}v\right]_0^2$$

$$= \frac{8}{3} + \frac{4}{3} = 4$$

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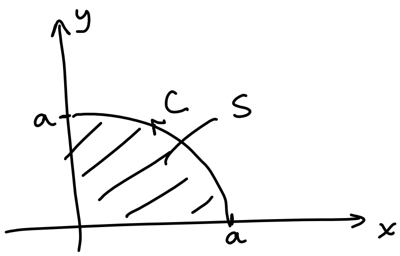
5) Using Green's theorem, evaluate $\oint_C (x - y^3)dx + (y^3 + x^3)dy$, where C is the positively oriented boundary of the quarter disk S, given by $0 \le x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$.
(8 marks)



$$I = \oint_C \underbrace{(x-y^3)}_{P}\,dx + \underbrace{(y^3+x^3)}_{Q}\,dy$$

Green's thrm: $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$$I = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_R (3x^2 + 3y^2)\, dA = 3\iint_R (x^2+y^2)\, dA$$

Let's switch to polar coordinates:

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \text{ where } 0 \le r \le a,\ 0 \le \theta \le \pi/2$$

$$I = 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \underbrace{(r^2\cos^2\theta + r^2\sin^2\theta)}_{r^2}\, r\,dr\,d\theta = 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3\,dr\,d\theta$$

$$= 3\int_0^{\pi/2} \frac{r^4}{4}\Big|_{r=0}^{a} d\theta = 3\int_0^{\pi/2} \frac{a^4}{4}\, d\theta = \frac{3a^4}{4}\int_0^{\pi/2} d\theta = \frac{3a^4}{4}\,\theta\Big|_0^{\pi/2} = \frac{3}{8}\pi a^4$$

6) Determine the flux of the vector field $\vec{F} = -xy^2\,\vec{\imath} + z\,\vec{\jmath}$ through the surface S given by $z = xy$ where $0 \le x \le 1$, $0 \le y \le 2$, using a parametric representation of the surface and taking the upward oriented unit normal vector side of the surface as the positive side.
(8 marks)

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = ?$$

Parametric form of the surface $z = xy$ ($z = f(x,y)$ form)

$$\left.\begin{aligned} x(u,v) &= u \\ y(u,v) &= v \\ z(u,v) &= uv \end{aligned}\right\} \Rightarrow \vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$$\boxed{\vec{r}(u,v) = u\vec{i} + v\vec{j} + uv\vec{k}}$$

$$\boxed{\text{where } 0 \le u \le 1 \,,\ 0 \le v \le 2}$$

$$\vec{r}_u = \vec{i} + v\vec{k}$$

$$\vec{r}_v = \vec{j} + u\vec{k}$$

Normal vector $\Rightarrow \vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & v \\ 0 & 1 & u \end{vmatrix} = -v\vec{i} - u\vec{j} + \vec{k}$

Normal unit vector $\Rightarrow \vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

$\vec{k}$ component is positive. So, the $\vec{n}$ vector direction found by $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ is in the upper direction.

$$\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,dS = \int_{v=0}^{2}\int_{u=0}^{1} \vec{F}(\vec{r}(u,v))\cdot \underbrace{\frac{\vec{r}_u \times \vec{r}_v}{\cancel{\|\vec{r}_u \times \vec{r}_v\|}}}_{\vec{n}} \underbrace{\cancel{\|\vec{r}_u \times \vec{r}_v\|}\,du\,dv}_{dS}$$

$$= \int_{v=0}^{2}\int_{u=0}^{1} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

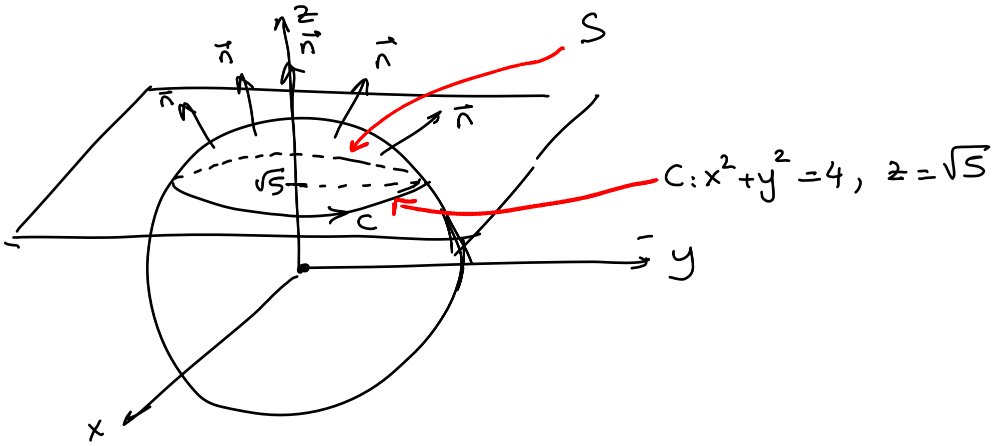
$$\text{Flux} = \int_{v=0}^{2}\int_{u=0}^{1} \Big(\underbrace{-uv^2\,\vec{i} + uv\,\vec{j}}_{\vec{F}(\vec{r}(u,v))}\Big) \cdot \Big(\underbrace{-v\,\vec{i} - u\,\vec{j} + \vec{k}}_{\vec{r}_u \times \vec{r}_v}\Big)\, du\, dv$$

$$= \int_0^2\int_0^1 \left(uv^3 - u^2 v\right) du\, dv = \int_0^2 \left(\frac{u^2v^3}{2} - \frac{u^3 v}{3}\right)\Bigg|_{u=0}^{u=1} dv$$

$$= \int_0^2 \left(\frac{v^3}{2} - \frac{v}{3}\right) dv = \left(\frac{v^4}{8} - \frac{v^2}{6}\right)\Bigg|_0^2 = \frac{2^4}{8} - \frac{2^2}{6} = 2 - \frac{2}{3} = \frac{4}{3}$$

7) Consider the surface S consisting of the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies above the plane $z = \sqrt{5}$. Let $\vec{n}$ denote the upward pointing unit normal vector on S. Making use of the Stokes' theorem, calculate $\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, dS$ for $\vec{F} = -y\,\vec{\imath} + xz\,\vec{\jmath} + y^2\,\vec{k}$.
(10 marks)

The intersection of the plane $z=\sqrt{5}$ and the sphere $x^2+y^2+z^2=9$ is the circle $x^2+y^2=4$, $z=\sqrt{5}$.



$$I = \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

$C: x^2+y^2=4,\ z=\sqrt{5}$ in parametric form: $\left.\begin{array}{l} x=2\cos t \\ y=2\sin t \\ z=\sqrt{5} \end{array}\right\}\ 0\le t\le 2\pi$

$\vec{r}(t) = 2\cos t\,\vec{i} + 2\sin t\,\vec{j} + \sqrt{5}\,\vec{k}$ (the curve C in vector form)

$\vec{r}\,'(t) = -2\sin t\,\vec{i} + 2\cos t\,\vec{j} + 0\,\vec{k}$

$\vec{F}(x,y,z) = -y\,\vec{i} + xz\,\vec{j} + y^2\,\vec{k} \Rightarrow \vec{F}(\vec{r}(t)) = -2\sin t\,\vec{i} + 2\sqrt{5}\cos t\,\vec{j} + 4\sin^2 t\,\vec{k}$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 4\sin^2 t + 4\sqrt{5}\cos^2 t$

$$I = \oint_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{2\pi} \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{2\pi} \left(4\sin^2 t + 4\sqrt{5}\cos^2 t\right)dt$$

$$= 4\int_0^{2\pi} \sin^2 t\,dt + 4\sqrt{5}\int_0^{2\pi}\cos^2 t\,dt = 4\cdot\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} + 4\sqrt{5}\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= 4\left[\frac{1}{2}\cdot 2\pi - \frac{1}{4}\underbrace{\cancel{\sin(4\pi)}}_{0} - \underbrace{\cancel{0}}_{0} - \frac{1}{4}\underbrace{\cancel{\sin 0}}_{0}\right] + 4\sqrt{5}\left[\frac{1}{2}\cdot 2\pi - \frac{1}{4}\underbrace{\cancel{\sin 4\pi}}_{0} - \underbrace{\cancel{0}}_{0} + \frac{1}{4}\underbrace{\cancel{\sin 0}}_{0}\right] = 4\pi + 4\sqrt{5}\,\pi$$