The Edward S. Rogers Sr. Department
of Electrical & Computer Engineering
UNIVERSITY OF TORONTO

ECE259: Electromagnetism

Term Test 2 - Monday March 9, 2020

Instructors: Profs. Micah Stickel and Piero Triverio

Last name: Solutions / Marking

First name:

Student number:

Tutorial section number:

Section	Day	Time	Room	TA
TUT0101	Monday	14:00-15:00	BA1230	Shashwat
TUT0102	Monday	14:00-15:00	BA2175	Paul
TUT0103	Monday	14:00-15:00	BA2135	Sameer
TUT0104	Monday	14:00-15:00	BA2159	Damian
TUT0105	Wednesday	13:00-14:00	BA2165	Shashwat
TUT0106	Wednesday	13:00-14:00	BA2195	Paul
TUT0107	Wednesday	13:00-14:00	BA1230	Sameer
TUT0108	Wednesday	13:00-14:00	BAB024	Damian

Instructions

- Duration: 1 hour 30 minutes (9:10 to 10:40)
- Exam Paper Type: A. Closed book. Only the aid sheet provided at the end of this booklet is permitted.
- Calculator Type: 2. All non-programmable electronic calculators are allowed.
- Answers written in pen are typically eligible for remarking. Answers written in pencil may *not* be eligible for remarking.
- **Only answers that are fully justified will be given full credit!**

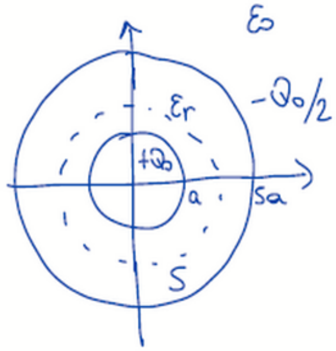
Marks:

Q1: /20	Q2: /20	Q3: /20	TOTAL: /60

Question 1 General: Don't carry mistakes through. −0.5 simple math or copy error

Two metallic spheres are centred about the origin and have radii given by a and $5a$. Both spheres are uniformly charged, with total charge of Q_0 on the sphere of radius a, and total charge of $-Q_0/2$ on the sphere of radius $5a$. The space between the spheres ($a < R < 5a$) is filled with a dielectric with relative permittivity of ε_r. All other space can be considered to be free space, and you may assume the thickness of the spheres is negligible.

(a) Calculate the potential $V(R)$ for $R > 5a$, for $a < R < 5a$ and for $R < a$. You may assume that $V = 0$ Volts for $R \to \infty$. [9 points]



Find $\bar{E}$ by Gauss' Law. Because of symmetry $\bar{D} = D_R(R)\,\bar{a}_R$

For $R<a \Rightarrow \bar{E}=0$ (1)

For $a<R<5a \Rightarrow \oint_S \bar{D}\cdot d\bar{S} = Q_0$ $\quad D_R(R)\,4\pi R^2 = Q_0$

$\bar{D} = \dfrac{Q_0}{4\pi R^2}\bar{a}_R$ $\quad \bar{E} = \dfrac{Q_0}{4\pi\varepsilon_0\varepsilon_r R^2}\bar{a}_R$ (1)

For $R>5a$ $\quad Q_{enc} = \dfrac{Q_0}{2} \Rightarrow \bar{D} = \dfrac{Q_0}{8\pi R^2}\bar{a}_R$

$\bar{E} = \dfrac{Q_0}{8\pi\varepsilon_0 R^2}\bar{a}_R$ (1)

Correct fields 3 marks

(1) limits

For $R>5a$ $\quad V(R) = \underbrace{V(\infty)}_{=0} + \displaystyle\int_R^\infty \frac{Q_0}{8\pi\varepsilon_0 R^2}\bar{a}_R\cdot\bar{a}_R\,dR =$

$= \dfrac{Q_0}{8\pi\varepsilon_0}\left[-\dfrac{1}{R}\right]_R^\infty = \dfrac{Q_0}{8\pi\varepsilon_0 R}$ (1) evaluation ~~V(5a) = 89.9 kV~~

V(R>5a) 2 marks

For $a<R<5a$ $\quad V(R) = V(5a) + \displaystyle\int_R^{5a} \frac{Q_0}{4\pi\varepsilon_0\varepsilon_r R^2}\bar{a}_R\cdot\bar{a}_R\,dR =$

(0.5) V(5a) (0.5) limits

$= \dfrac{Q_0}{8\pi\varepsilon_0 5a} + \dfrac{Q_0}{4\pi\varepsilon_0\varepsilon_r}\left[-\dfrac{1}{R}\right]_R^{5a} = \dfrac{Q_0}{8\pi\varepsilon_0 5a} + \dfrac{Q_0}{4\pi\varepsilon_0\varepsilon_r}\left(\dfrac{1}{R} - \dfrac{1}{5a}\right) =$

V(a<R<5a) 2 marks

① evaluation

$$= \frac{Q_0}{40\pi\varepsilon_0 a} - \frac{Q_0}{100\pi\varepsilon_0 a} + \frac{Q_0}{20\pi\varepsilon_0 R} = \frac{Q_0}{\pi\varepsilon_0 a}\left(\frac{1}{40} - \frac{1}{100}\right) + \frac{Q_0}{20\pi\varepsilon_0 R}$$

For $R<a$ ① $\underbrace{V = \text{const} = V(a)}$ $= \frac{Q_0}{\pi\varepsilon_0 a}\left(\frac{1}{40} - \frac{1}{100} + \frac{1}{20}\right) =$ $V(R<a)$ 2 marks

$$= \frac{Q_0}{\pi\varepsilon_0 a}\,\frac{5-2+10}{200} = \underbrace{\frac{Q_0}{\pi\varepsilon_0 a}\,\frac{13}{200}}_{\text{① evaluation}}$$

no numeric results are expected

(b) Assume $a = 1$ cm, $Q_0 = 1\mu$C, and $\varepsilon_r = 5$. Determine the electrostatic energy W_e stored in this system of charges. [9 points]

① start ① limits ① integrand

$$W_e = \frac{1}{2}\iiint_{Vol} \varepsilon_r\varepsilon_0 |\bar{E}|^2 = \frac{1}{2}\int_a^{5a}\int_0^{2\pi}\int_0^{\pi} \varepsilon_r\varepsilon_0\left(\frac{Q_0}{4\pi\varepsilon_r\varepsilon_0 R^2}\right)^2 R^2 \sin\theta\, d\theta\, d\phi\, dR$$

① integrand

$$+ \frac{1}{2}\int_{5a}^{\infty}\int_0^{2\pi}\int_0^{\pi} \varepsilon_0\left(\frac{Q_0}{8\pi\varepsilon_0 R^2}\right)^2 R^2 \sin\theta\, d\theta\, d\phi\, dR$$

① limits

$$= \frac{1}{2}\cancel{(4\pi)}\cancel{(\varepsilon_r\varepsilon_0)}\left(\frac{Q_0}{\cancel{4\pi\varepsilon_r\varepsilon_0}}\right)^2\int_a^{5a}\frac{dR}{R^2} + \frac{1}{2}\cancel{(4\pi)}\varepsilon_0\left(\frac{Q_0}{\cancel{8\pi\varepsilon_0}}\right)^{\cancel{2}}\int_{5a}^{\infty}\frac{dR}{R^2}$$

$4\pi\varepsilon_r\varepsilon_0$ $16\pi\varepsilon_0$

$$= \frac{Q_0^2}{8\pi\varepsilon_r\varepsilon_0}\left(\frac{1}{a} - \frac{1}{5a}\right) ② + \frac{Q_0^2}{32\pi\varepsilon_0}\left(\frac{1}{5a}\right) ②$$

$$= \frac{Q_0^2}{40\pi\varepsilon_0}\left(\frac{4}{5a}\right) + \frac{Q_0^2}{160\pi\varepsilon_0 a} = 71.9\text{ mJ} + 22.5\text{ mJ} = \underline{\underline{94.4\text{ mJ}}}$$

5 marks set-up: ① correct start
② limits / two regions
② Correct integrands

4 marks evaluation: ② for each integral
↳ −1 for wrong or no units

1 limits

1 integrand

Alt. Soln to (b):

$$W_e = \frac{1}{2}\iint \rho_s V \, ds = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{Q_0}{4\pi a^2}\right)V(a)a^2\sin\theta \, d\theta \, d\phi$$

1 start

1 integrand

$$+\frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\left[\frac{-Q_0/2}{4\pi(5a)^2}\right]V(5a)(5a)^2\sin\theta \, d\theta \, d\phi$$

1 limits

$$= \frac{1}{2}(\cancel{4\pi})\left(\frac{Q_0}{\cancel{4\pi}}\right)\left[\frac{Q_0}{40\pi\varepsilon_0 a} + \frac{Q_0}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{4}{5a}\right)\right] + \frac{1}{2}(\cancel{4\pi})\left(\frac{-Q_0}{\cancel{8\pi}_{\,2}}\right)\left(\frac{Q_0}{40\pi\varepsilon_0 a}\right)$$

2 2

$$= \frac{1}{2}\left[\frac{Q_0^2}{80\pi\varepsilon_0 a} + \frac{Q_0^2}{20\pi\varepsilon_0}\left(\frac{4}{5a}\right)\right] = \frac{Q_0^2}{40\pi\varepsilon_0}\left(\frac{4}{5a}\right) + \frac{Q_0^2}{160\pi\varepsilon_0 a} = \underline{\underline{94.4\text{ mJ}}}$$

−1 no or wrong units

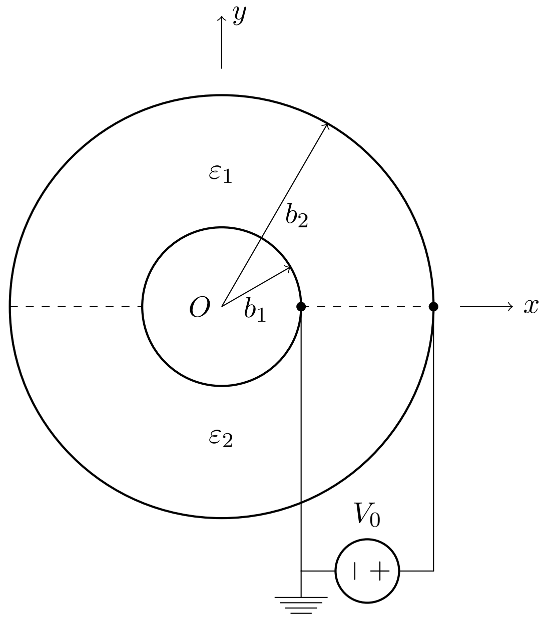
(c) The stored energy for the same charge distribution as described above, with the dielectric region replaced with free space, is given by $W_e' = 382\,\text{mJ}$. Is W_e' less, equal or greater than the stored energy W_e calculated in point (b) above? Provide a physical justification of your answer. [2 points]

1 larger

* W_e' (freespace) is larger than W_e (dielectric)

This make sense as it requires some energy to polarize the dielectric. 1 polarization

Question 2



The structure shown in the figure above consists of two thin cylindrical conductors, concentric, with axis along the z axis. The inner conductor has radius b_1, while the outer conductor has radius b_2. Both cylindrical conductors extend from $z = 0$ to $z = L$. The volume between the conductors is filled by two perfect dielectrics. The dielectric in the region $\varphi \in [0, \pi]$ has permittivity ε_1. The dielectric in the region $\varphi \in [\pi, 2\pi]$ has permittivity ε_2. The inner conductor is held at a potential $V = 0$, while the outer conductor is held at a potential $V = V_0$.

a) Neglecting fringing (edge) effects, use the Poisson or Laplace equation to calculate the potential V between the two conductors. You can assume that V is a function only of the radial coordinate, that is $V = V(r)$, and that there is no free charge anywhere in between the two cylindrical conductors. [10 points]

$V = V(r)$ (1) Assumption $V(r)$

(2) Correct form of Laplace's Eq^n

Poisson equation $-\nabla \cdot (\varepsilon \nabla V) = 0$ since between the conductors there is no free charge (both materials are perfect) (1)

Region $\varphi \in [0, \pi]$: $-\nabla \cdot (\varepsilon_1 \nabla V) = 0$

$-\cancel{\varepsilon_1} \nabla^2 V = 0$ (1)

$\nabla^2 V = 0$ → this is valid also in region $\varphi \in [\pi, 2\pi]$

(1) form in cylindrical coords

$$\frac{1}{\cancel{r}}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right]=0 \qquad r\frac{\partial V}{\partial r}=C_1 \;;\; \frac{\partial V}{\partial r}=\frac{C_1}{r} \;;\; V(r)=C_1\ln r+C_2$$

(1) int #1 (1) int #2 (3) General Sol^{n}.

Impose boundary conditions

$$\begin{cases} V(b_1)=0 \\ V(b_2)=V_0 \end{cases} \quad \begin{cases} C_1\ln b_1+C_2=0 \;; \\ C_1\ln b_2+C_2=V_0 \end{cases} \quad \begin{matrix} C_2=-C_1\ln b_1 \\ C_1\ln b_2-C_1\ln b_1=V_0 \end{matrix}$$

(1)

$$C_1\ln\frac{b_2}{b_1}=V_0 \qquad C_1=\frac{V_0}{\ln b_2/b_1} \qquad C_2=-V_0\frac{\ln b_1}{\ln\frac{b_2}{b_1}}$$

(1) (1) (3) Boundary Conditions

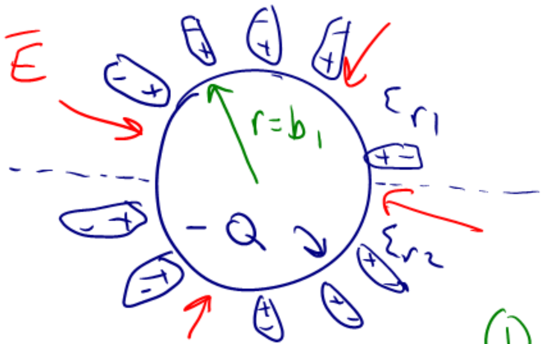
$$V(r)=\frac{V_0}{\ln b_2/b_1}\ln r-V_0\frac{\ln b_1}{\ln b_2/b_1}$$

$$\boxed{V(r)=V_0\frac{\ln r/b_1}{\ln b_2/b_1}}$$

(1) final simplified form

b) At the inner surface of the dielectric regions, i.e., at $r = b_1$, is there any surface density of bound charge? If so, is it positive or negative, and if not, why not? [2 points]

1 justification



→ Given the polarity of the source, a negative free charge would exist on the inner metallic surface.

1 positive

∴ A positive bound charge would exist on this inner surface.

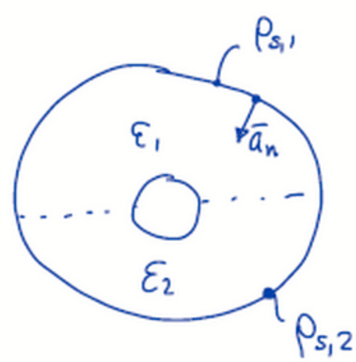
c) Calculate the capacitance between the two conductors. [8 points]

Capacitance $C = \frac{Q}{V_o}$

must find total charge Q on outer conductor

Use boundary cond's

Piero



$\bar{a}_n \cdot \bar{D} = \rho_s$ 1 $\qquad \bar{a}_n = -\bar{a}_r$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\bar{a}_r = \frac{-V_o}{\ln b_2/b_1}\frac{\cancel{b_1}}{r}\frac{1}{\cancel{b_1}}\bar{a}_r$$

$$\bar{E} = -\frac{V_o}{\ln b_2/b_1}\frac{1}{r}\bar{a}_r \quad (1) \qquad \bar{D} = -\varepsilon\frac{V_o}{\ln b_2/b_1}\frac{1}{r}\bar{a}_r \quad (1)$$

$$\rho_{s,1} = \varepsilon_1\frac{V_o}{\ln b_2/b_1}\frac{1}{b_2} \quad (1) \qquad \rho_{s,2} = \varepsilon_2\frac{V_o}{\ln b_2/b_1}\frac{1}{b_2} \quad (1)$$

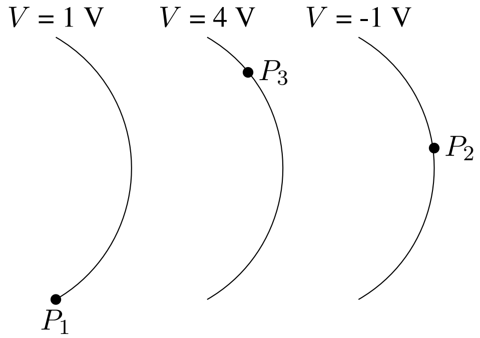
5 marks ρ_{s1}, ρ_{s2}

$$Q = \rho_{s,1} \cdot \pi b_2 L + \rho_{s,2} \pi b_2 L = (\rho_{s,1} + \rho_{s,2}) \pi b_2 L$$ 1 1 2 marks Correct Q

$$Q = (\varepsilon_1 + \varepsilon_2) \frac{V_0}{\ln b_2/b_1} \frac{1}{\cancel{b_2}} \pi \cancel{b_2} L$$

$$\boxed{C = \frac{Q}{V_0} = (\varepsilon_1 + \varepsilon_2) \frac{\pi L}{\ln b_2/b_1}}$$ ✓ 1 mark final C

Question 3.1



Consider the equipotential surfaces shown in the figure above. What is the total work W needed to move a point charge $q = 1\,\text{C}$ from P_1 to P_3 passing through P_2?

Briefly justify your answer. [2 points]

$$W = qV = q[V(P_3) - V(P_1)]$$

1

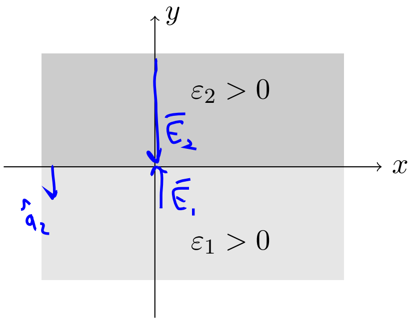
This is path independant so P_2 doesn't matter.

$$= (1)(4-1) = \underline{\underline{3\text{ J}}}$$

1

→ −0.5 no or wrong units

Question 3.2



Consider the interface between the two materials shown in the above figure. The field in medium 2 right above the interface is $\mathbf{E}_2 = -7\mathbf{a}_y$ V/m. The field in medium 1 right below the interface is $\mathbf{E}_1 = 4\mathbf{a}_y$ V/m. Which statement is correct?

a) a surface density of free charge $\rho_s > 0$ exists at the interface;

b) a surface density of free charge $\rho_s < 0$ exists at the interface;

c) fields $\mathbf{E}_1$ and $\mathbf{E}_2$ do not satisfy boundary conditions;

d) the values of ε_1 and ε_2 are needed to answer this question.

1 - correct conclusion

Briefly justify your answer. [4 points]

$$\hat{a}_2 \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s \quad 1$$

$$\hat{a}_2 = -\hat{a}_y \quad 1$$

$$-\hat{a}_y \cdot (4\varepsilon_1 \hat{a}_y + 7\varepsilon_2 \hat{a}_y) = \rho_s$$

$$-(4\varepsilon_{r1}\varepsilon_0 + 7\varepsilon_{r2}\varepsilon_0) = \rho_s$$

$$\therefore \underline{\rho_s < 0} \quad 1$$

3 - justification

Question 3.3

Two ideal parallel plate capacitors are described below. You can assume the electric fields inside these capacitors are uniform (that is edge effects are neglected), and that the entire space between the plates is filled with the specific dielectric material given. For this situation, determine whether each statement below is True or False (circle your answer for each). Briefly justify your answers. [8 points]

Capacitor A:
Plate dimensions = 1 cm x 1 cm
Plate spacing = 0.1 mm
Dielectric constant (relative permittivity) = 4
Total charge on the positive plate = 1×10^{-6} C

Capacitor B:
Plate dimensions = 2 cm x 2 cm
Plate spacing = 0.5 mm
Dielectric constant (relative permittivity) = 3
Total charge on the positive plate = 5×10^{-6} C

↳ 2 marks each
(1) - conclusion
(1) - justification
-0.5 marks simple/math mistakes

True / False 1) The capacitance of Capacitor A is larger than that of Capacitor B.

(1)

$$\frac{C_A}{C_B} = \frac{\dfrac{4\cancel{\varepsilon_0}(\cancel{1cm})^2}{\cancel{0.1mm}}}{\dfrac{3\cancel{\varepsilon_0}(\cancel{2cm})^2\ 4}{\cancel{0.5mm}\ 5}} = \frac{\cancel{4}(5)}{3(\cancel{4})} = \frac{5}{3} \quad (1) \quad C_A > C_B$$

(1)

True / False 2) The electric field intensity in Capacitor A has a larger magnitude than that of Capacitor B.

$$\text{Sol}^n \#1: \; E_A = \frac{V_A}{d_A} = \frac{Q_A}{C_A d_A}, \; E_B = \frac{Q_B}{C_B d_B}, \quad (1)$$

$$\therefore \frac{E_A}{E_B} = \left(\frac{Q_A}{Q_B}\right)\left(\frac{C_B}{C_A}\right)\left(\frac{d_B}{d_A}\right) = \left(\frac{1}{5}\right)\left(\frac{3}{5}\right)\left(\frac{5}{1}\right) < 1$$

$$\text{Sol}^n \#2: \; D_A = \rho_{sA} = \frac{Q_A}{S_A} = \varepsilon_{rA}\varepsilon_0 E_A$$

$$\therefore \frac{E_A}{E_B} = \left(\frac{Q_A}{Q_B}\right)\left(\frac{S_B}{S_A}\right)\left(\frac{\varepsilon_{rB}}{\varepsilon_{rA}}\right) = \left(\frac{1}{5}\right)\left(\frac{4}{1}\right)\left(\frac{3}{4}\right) < 1$$

True / False 3) The polarization vector in Capacitor A is less than that of Capacitor B.

1

$$P_A = \varepsilon_0(\varepsilon_{rA}-1)E_A\,,\quad P_B = \varepsilon_0(\varepsilon_{rB}-1)E_B$$

$$\therefore\ \frac{P_A}{P_B} = \frac{\cancel{\varepsilon_0}(\varepsilon_{rA}-1)}{\cancel{\varepsilon_0}(\varepsilon_{rB}-1)}\frac{E_A}{E_B} = \left(\frac{3}{2}\right)\left(\frac{3}{5}\right) = \frac{9}{10} < 1$$

1

True / False 4) The magnitude of the electric flux density in Capacitor A is 10μ C/m^2.

1

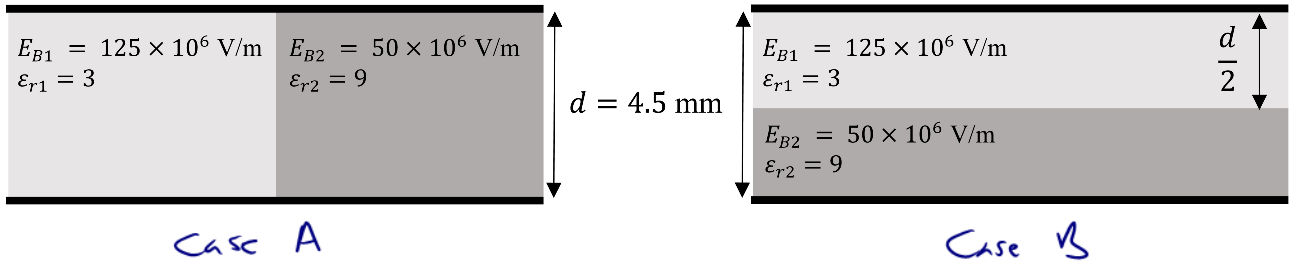
$$D_A = \rho_{sA} = \frac{Q_A}{S_A} = \frac{1\times10^{-6}}{1\times10^{-4}} = 10\ \mathrm{mC/m^2}$$

1

In general, give the mark for the conclusion if it is correctly based on a simple mistaken calculation.

Question 3.4

Of the two parallel plate capacitors shown below, which would be able to store the most electric energy before dielectric breakdown occurred? Both capacitors have the same plate area. The dielectric strength of each material is denoted with E_B. You can ignore the effect of fringing fields. Briefly justify your answer. [6 points]



(1)

In the capacitor on the left, $E_{t1} = E_{t2}$ requires that $|\bar{E}_1| = |\bar{E}_2|$ → so to avoid breakdown $E_{max} = 50\times10^6$ V/m (1)

3 - Case A

$$\therefore W_{eA} = \frac{1}{2}\iiint \bar{D}\cdot\bar{E}\,dv = \frac{1}{2}(3\varepsilon_0)(50\times10^6)^2\left(\frac{S}{2}\right)d + \frac{1}{2}(9\varepsilon_0)(50\times10^6)^2\left(\frac{S}{2}\right)d$$

$$= 3\varepsilon_0(50\times10^6)^2 Sd \quad (1)$$

For case B: $D_{n1} = D_{n2}$ (1) requires $\varepsilon_{r1}\varepsilon_0 E_1 = \varepsilon_{r2}\varepsilon_0 E_2$

$$3E_1 = 9E_2 \rightarrow E_1 = 3E_2$$

$\therefore$ If $E_2 = 50\times10^6$ V/m, $E_1 = 150\times10^6$ V/m, thus E_{B1} is (1) the limiting value → $E_1 = 125\times10^6$ and $E_2 = \frac{125}{3}\times10^6$ V/m.

3 - Case B

$$\therefore W_{eB} = \frac{1}{2}\iiint \varepsilon_r\varepsilon_0|\bar{E}|^2\,dv = \frac{1}{2}(3\varepsilon_0)(125\times10^6)^2(S)\left(\frac{d}{2}\right) + \frac{1}{2}(\cancel{9}\varepsilon_0)\left(\frac{125}{\cancel{3}}\times10^6\right)^2(S)\left(\frac{d}{2}\right)$$

$$= \frac{3\varepsilon_0}{4}(Sd)(125\times10^6)^2 + \frac{\varepsilon_0}{4}(Sd)(125\times10^6)^2 = \varepsilon_0(Sd)(125\times10^6)^2$$

$\therefore W_{eB} > W_{eA} \rightarrow \frac{W_{eA}}{W_{eB}} = 0.48$ (1)

(blank page) Alternate Sol^n

Left capacitor. The E field in both materials is $\frac{V}{d}$. The second material will break down first for $V = E_{B2} \cdot d = 225$ kV (1)

3-marks Case A

Capacitance: $\varepsilon_0 3 \cdot \frac{A}{2d} + \varepsilon_0 9 \cdot \frac{A}{2d} = \varepsilon_0 6 \frac{A}{d}$ (1)

Max energy: $W_1 = \frac{1}{2} C V^2 = \varepsilon_0 \frac{1}{2} 6 \frac{A}{d} \cdot (E_{B2} d)^2 = 3\varepsilon_0 A E_{B2}^2 d = 299$ A (1)

Right capacitor

Assume $+Q/-Q$ on plates $D = \rho_S = \frac{Q}{A} \Rightarrow$ $E_1 = \frac{Q}{A 3\varepsilon_0}$, $E_2 = \frac{Q}{A 9 \varepsilon_0}$

dielectric 1 will break down first for $E_1 = E_{B1}$ (1)

3-marks case B

$Q_{max} = 3 \varepsilon_0 A E_{B1}$

$$C = \left(\frac{d}{2 \cdot 3\varepsilon_0 A} + \frac{d}{2 \cdot 9 \varepsilon_0 A} \right)^{-1} = \left(\frac{d}{6 \varepsilon_0 A} + \frac{d}{18 \varepsilon_0 A} \right)^{-1} = \frac{\varepsilon_0 A}{d} \left(\frac{1}{6} + \frac{1}{18} \right)^{-1} =$$

$$= \frac{\varepsilon_0 A}{d} \left(\frac{3+1}{18} \right)^{-1} = \frac{\varepsilon_0 A}{d} \cdot \frac{\cancel{18}^{\,9}}{\cancel{4}_{\,2}} = \frac{\varepsilon_0 A}{d} \cdot 4.5$$ (1)

11

$$W_2 = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{\cancel{2}} \frac{d}{\varepsilon_0 \cancel{A}\, 4.5} \cdot \cancel{9}\, \varepsilon_0^{\cancel{2}} A^{\cancel{2}} E_{B1}^2 =$$

$C = \frac{Q}{V}$

$V = \frac{Q}{C}$

$= d\, \varepsilon_0\, A\, E_{B1}^2 = 622.5$ A (1)

1. Coordinate Systems

1.1 Cartesian coordinates

Position vector: $\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_x = \mathbf{a}_x dx$, $\mathbf{dl}_y = \mathbf{a}_y dy$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_x = \mathbf{a}_x dydz$, $\mathbf{dS}_y = \mathbf{a}_y dxdz$, $\mathbf{dS}_z = \mathbf{a}_z dxdy$

Differential volume element: $dV = dxdydz$

1.2 Cylindrical coordinates

Position vector: $\mathbf{R} = r\mathbf{a}_r + z\mathbf{a}_z$

Differential length elements: $\mathbf{dl}_r = \mathbf{a}_r dr$, $\mathbf{dl}_\phi = \mathbf{a}_\phi r d\phi$, $\mathbf{dl}_z = \mathbf{a}_z dz$

Differential surface elements: $\mathbf{dS}_r = \mathbf{a}_r r d\phi dz$, $\mathbf{dS}_\phi = \mathbf{a}_\phi dr dz$, $\mathbf{dS}_z = \mathbf{a}_z r d\phi dr$

Differential volume element: $dV = r dr d\phi dz$

1.3 Spherical coordinates

Position vector: $\mathbf{R} = R\mathbf{a}_R$

Differential length elements: $\mathbf{dl}_R = \mathbf{a}_R dR$, $\mathbf{dl}_\theta = \mathbf{a}_\theta R d\theta$, $\mathbf{dl}_\phi = \mathbf{a}_\phi R \sin\theta d\phi$

Differential surface elements: $\mathbf{dS}_R = \mathbf{a}_R R^2 \sin\theta d\theta d\phi$, $\mathbf{dS}_\theta = \mathbf{a}_\theta R \sin\theta dR d\phi$, $\mathbf{dS}_\phi = \mathbf{a}_\phi R dR d\theta$

Differential volume element: $dV = R^2 \sin\theta dR d\theta d\phi$

2. Relationship between coordinate variables

-	Cartesian	Cylindrical	Spherical
x	x	$r\cos\phi$	$R\sin\theta\cos\phi$
y	y	$r\sin\phi$	$R\sin\theta\sin\phi$
z	z	z	$R\cos\theta$
r	$\sqrt{x^2+y^2}$	r	$R\sin\theta$
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ
z	z	z	$R\cos\theta$
R	$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
θ	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	$\tan^{-1}\frac{r}{z}$	θ
ϕ	$\tan^{-1}\frac{y}{x}$	ϕ	ϕ

3. Dot products of unit vectors

$\cdot$	$\mathbf{a}_x$	$\mathbf{a}_y$	$\mathbf{a}_z$	$\mathbf{a}_r$	$\mathbf{a}_\phi$	$\mathbf{a}_z$	$\mathbf{a}_R$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x$	1	0	0	$\cos\phi$	$-\sin\phi$	0	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y$	0	1	0	$\sin\phi$	$\cos\phi$	0	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_r$	$\cos\phi$	$\sin\phi$	0	1	0	0	$\sin\theta$	$\cos\theta$	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1
$\mathbf{a}_z$	0	0	1	0	0	1	$\cos\theta$	$-\sin\theta$	0
$\mathbf{a}_R$	$\sin\theta\cos\phi$	$\sin\theta\sin\phi$	$\cos\theta$	$\sin\theta$	0	$\cos\theta$	1	0	0
$\mathbf{a}_\theta$	$\cos\theta\cos\phi$	$\cos\theta\sin\phi$	$-\sin\theta$	$\cos\theta$	0	$-\sin\theta$	0	1	0
$\mathbf{a}_\phi$	- $\sin\phi$	$\cos\phi$	0	0	1	0	0	0	1

4. Relationship between vector components

=	Cartesian	Cylindrical	Spherical
A_x	A_x	$A_r\cos\phi - A_\phi\sin\phi$	$A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$
A_y	A_y	$A_r\sin\phi + A_\phi\cos\phi$	$A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_r	$A_x\cos\phi + A_y\sin\phi$	A_r	$A_R\sin\theta + A_\theta\cos\theta$
A_ϕ	-$A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ
A_z	A_z	A_z	$A_R\cos\theta - A_\theta\sin\theta$
A_R	$A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$	$A_r\sin\theta + A_z\cos\theta$	A_R
A_θ	$A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$	$A_r\cos\theta - A_z\sin\theta$	A_θ
A_ϕ	$-A_x\sin\phi + A_y\cos\phi$	A_ϕ	A_ϕ

5. Differential operators

5.1 Gradient

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z = \frac{\partial V}{\partial R}\mathbf{a}_R + \frac{1}{R}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

5.2 Divergence

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = \frac{1}{R^2}\frac{\partial(R^2A_R)}{\partial R} + \frac{1}{R\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

5.3 Laplacian

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

5.4 Curl

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{a}_z \\
&= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\mathbf{a}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\mathbf{a}_\phi + \frac{1}{r}\left(\frac{\partial (rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi}\right)\mathbf{a}_z \\
&= \frac{1}{R\sin\theta}\left(\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right)\mathbf{a}_R + \frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial R}\right)\mathbf{a}_\theta \\
&+ \frac{1}{R}\left(\frac{\partial (RA_\theta)}{\partial R} - \frac{\partial A_R}{\partial \theta}\right)\mathbf{a}_\phi
\end{aligned}
$$

6. Electromagnetic formulas

Table 1 Electrostatics

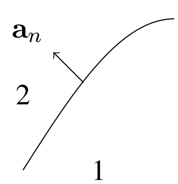
$\mathbf{F} = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{\lvert\mathbf{R}_2 - \mathbf{R}_1\rvert^3}(\mathbf{R}_2 - \mathbf{R}_1)$	$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_v \frac{(\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3} dQ'$
$\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{D} = \rho_v$
$\oint_C \mathbf{E} \cdot \mathbf{dl} = 0$	$\oint_S \mathbf{D} \cdot \mathbf{dS} = Q$
$\mathbf{E} = -\nabla V$	$V = \frac{1}{4\pi\varepsilon}\int_v \frac{dQ'}{\lvert\mathbf{R} - \mathbf{R}'\rvert}$
$V = V(P_2) - V(P_1) = \frac{W}{Q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot \mathbf{dl}$	
$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \mathbf{E}$	$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}$
$\rho_{p,v} = -\nabla \cdot \mathbf{P}$	
$\rho_{p,s} = -\mathbf{a}_n \cdot (\mathbf{P}_2 - \mathbf{P}_1)$	
$\mathbf{a}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$	
$E_{1,t} = E_{2,t}$	
$Q = CV$	$W_e = \frac{1}{2}QV$
$W_e = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_{vol} \mathbf{D} \cdot \mathbf{E} dv$	
$\nabla \cdot (\varepsilon \nabla V) = -\rho_v$	$\nabla \cdot (\varepsilon \nabla V) = 0$

Table 2 Magnetostatics

$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$	$\mathbf{F}_m = I\mathbf{l} \times \mathbf{B}$
$\nabla \times \mathbf{B} = \mu \mathbf{J}$	$\nabla \cdot \mathbf{B} = 0$
$\oint_C \mathbf{B} \cdot \mathbf{dl} = \mu I$	$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$
$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{\mathbf{dl}' \times (\mathbf{R} - \mathbf{R}')}{\lvert\mathbf{R} - \mathbf{R}'\rvert^3}$	$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS}$
$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu \mathbf{H}$	$\mathbf{M} = \chi_m \mathbf{H}$
$\mathbf{J}_{m,s} = \mathbf{M} \times \mathbf{a}_n$	$\mathbf{J}_m = \nabla \times \mathbf{M}$
$B_{1,n} = B_{2,n}$	$\mathbf{a}_n \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot \mathbf{dl} = I$
	$W_m = \frac{1}{2} \int_{vol} \mathbf{B} \cdot \mathbf{H} dv$
$L = \frac{N\Phi}{I} = \frac{2W_m}{I^2}$	$L_{12} = \frac{N_2 \Phi_{12}}{I_1} = \frac{N_2}{I_1} \int_{S_2} \mathbf{B}_1 \cdot \mathbf{dS}_2$
$\mathcal{R} = \frac{l}{\mu S}$	$V_{mmf} = NI$

Table 3 Faraday's law, Ampere-Maxwell law

$V_{emf} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{dl} - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{dS}$	$V_{emf} = \oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt}$
$\oint_C \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J} \cdot \mathbf{dS} + \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Table 4 Currents

$I = \int_S \mathbf{J} \cdot \mathbf{dS}$	$\mathbf{J} = \rho \mathbf{u} = \sigma \mathbf{E}$
$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$	$\mathbf{P} = \int_{vol} (\mathbf{E} \cdot \mathbf{J})\, dv$
$J_{2,n} - J_{1,n} = -\frac{\partial \rho_s}{\partial t}$	$\sigma_2 J_{1,t} = \sigma_1 J_{2,t}$
$R = \frac{l}{\sigma S}$	$\sigma = -\rho_e \mu_e = \frac{1}{\rho}$