Print legibly: **First name:**________________

Last name:________________

Student #:________________

Q1:_____Q2:_____Q3:_____Q4:_____Q5:_____Q6:_____

UNIVERSITY OF TORONTO

FACULTY OF APPLIED SCIENCE AND ENGINEERING

ESC103F – Engineering Mathematics and Computation

Final Exam

December 19, 2017

Instructor – W.R. Cluett

Closed book.

Allowable calculators:

- Sharp EL-520X
- Sharp EL-520W
- Casio FX-991
- Casio FX-991EX
- Casio FX-991ES Plus
- Casio FX-991MS

All questions of equal value.

All work to be marked <u>must</u> appear on front of page. Use back of page for rough work only.

Given information:

$$cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||}$$

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}$$

$$proj_{\vec{d}}\vec{u} = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d}$$

The inverse of a 2x2 matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is equal to $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The normal system of equations corresponding to $A\vec{x} = \vec{b}$ is given by:

$A^T A\vec{x} = A^T\vec{b}$

Trapezoidal formula: $T_n = \sum_{i=1}^{n} (\frac{f(x_{i-1}) + f(x_i)}{2})\Delta x$

Q1

Consider matrix A

$$A = \begin{bmatrix} 1 & 0 & a & 1 \\ -1 & -1 & b & -2 \\ 3 & 1 & c & 0 \end{bmatrix}$$

And its corresponding reduced normal form (R)

$$R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a) Using R, determine the solution to $A\vec{x} = \vec{0}$ and give a geometric interpretation of this solution.

b) Given matrices A and R, determine the numerical values for a, b and c found in matrix A.

Q2

All 3 parts of this question deal with eigenvalues and eigenvectors but all 3 parts are separate questions.

a) Let $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$. Show that $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A and determine the corresponding eigenvalue.

b) Suppose that A is an nxn matrix such that the sum of the entries in each row is the same and equal to scalar c. Show that $\vec{v} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector of matrix A and determine the corresponding eigenvalue.

c) Suppose that matrix A has an eigenvector $\vec{v}$ with a corresponding eigenvalue $\lambda = 5$. Show that $\vec{v}$ is also an eigenvector of A^4 and determine its corresponding eigenvalue.

Q3

Let $\vec{v}$ be an $nx1$ column vector of unit length, i.e. $\|\vec{v}\| = 1$. Since $\vec{v}$ is a column vector, then $\vec{v} \cdot \vec{v} = \vec{v}^T\vec{v} = \|\vec{v}\|^2$. Let matrix $A = I - 2\vec{v}\vec{v}^T$ where A is an nxn matrix, I is the nxn identity matrix, and $\vec{v}\vec{v}^T$ is an nxn matrix.

Each of the following four statements found on this page and the next three pages is true. Clearly show why each statement is true.

i. $(\vec{v}\vec{v}^T)\vec{v} = \vec{v}$.

ii. $\vec{v}$ is an eigenvector of A with corresponding eigenvalue $\lambda = -1$.

iii. A is symmetric ($A = A^T$).

iv. A is the inverse of A.

Q4

Given $A = \begin{bmatrix} 2 & -2 & -2 \\ 0 & -2 & 2 \\ -1 & 5 & 2 \end{bmatrix}$, find an LU decomposition of matrix A.

Q5

Experimental data from a PHY180F lab has been collected and five experiments have been performed to determine if there is a relationship between a particular independent variable and a dependent response variable. The table below shows the different values of the independent variable (v_1) used for each experiment and the corresponding dependent response variable (v_2).

Experiment #	v_1	v_2
1	2	5
2	3	7
3	4	8
4	5	11
5	6	12

a) Let's say that the objective is to fit a straight line $v_2 = c_1 + c_2 v_1$ exactly through all of the five data points. Set up the corresponding system of linear equations that would need to be solved ($A\vec{x} = \vec{b}$). Without attempting to solve this system of equations, how do you know the system is inconsistent?

b) Find the least squares fit of a straight line to this data.

c) Determine the value for the error vector that corresponds to the least squares fit, i.e. $\vec{E} = \vec{b} - A\vec{x}_{LS}$. In the v_1-v_2 plane, sketch the five data points, the least squares straight line fit obtained in part (b), and the 5 elements of the error vector $\vec{E}$.

Q6

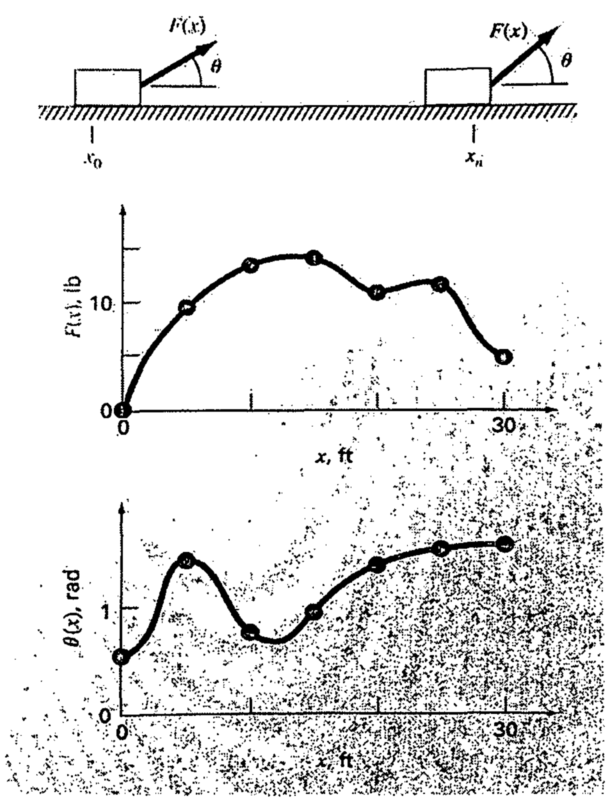
Many engineering problems involve the calculation of work. The general formula is $work = force\ x\ distance$. In realistic problems, the force varies as a function of position as does the angle between the force and the direction of movement. Consider the figure below showing a force acting on a block where both the force and the angle vary with position. In this case, the work must be calculated as follows:

$$Work(W) = \int_a^b F(x)\cos[\theta(x)]\,dx$$

The following data has been collected from an experiment at 5 foot intervals:

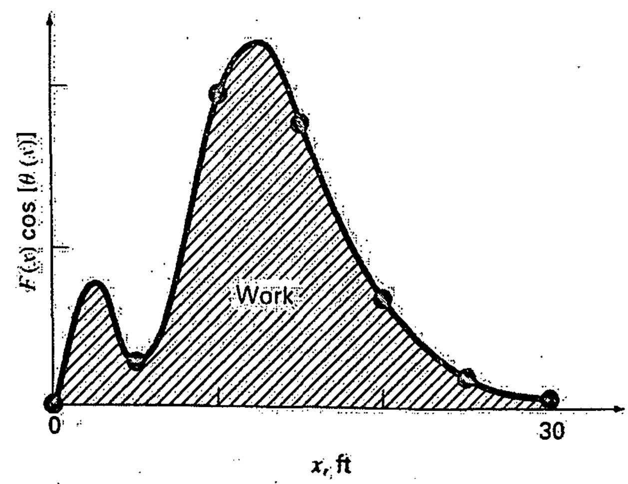
x, ft	$F(x), lb$	$\theta, radians$	$F(x)cos\theta$
0	0.0	0.50	0.0000
5	9.0	1.40	1.5297
10	13.0	0.75	9.5120
15	14.0	0.90	8.7025
20	10.5	1.30	2.8087
25	12.0	1.48	1.0881
30	5.0	1.50	0.3537

Units of work using this data is in $foot - pounds\ (ft \cdot lb)$.



a). Using the data given in the table, calculate an estimate of the work using a trapezoidal approximation with 2 subintervals. Also, calculate an estimate of the work using a trapezoidal approximation with 6 subintervals.

b) A continuous plot of $F(x)\cos[\theta(x)]$ versus position along with the seven discrete points given in the table are shown in the figure below. A "true" value of the work has been estimated as $129.52\ ft \cdot lb$ based on values collected at 1 foot intervals. Directly on the figure below, show your trapezoidal approximation with 2 subintervals and use this to explain why the trapezoidal approximation is reasonably accurate with only 2 subintervals.



c) Again, the same continuous plot of $F(x)\cos[\theta(x)]$ versus position along with the seven discrete points given in the table are shown in the figure below. As stated above, a "true" value of the work has been estimated as $129.52\ ft \cdot lb$ based on values collected at 1 foot intervals. Directly on the figure below, show your trapezoidal approximation with 6 subintervals and use this to explain why the trapezoidal approximation is less accurate with 6 subintervals as compared to the trapezoidal approximation obtained using only 2 subintervals.