2014 EXAM SOLUTIONS

Q1 START BY FINDING A^{-1} AS A PRODUCT OF ELEMENTARY MATRICES.

A

$$\left[\begin{array}{cc|cc} -1 & -2 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right]$$

SAME ROW OPERATIONS APPLIED TO I

$R1 \times (-1)$ $$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 3 & 8 & 0 & 1 \end{array}\right] \qquad E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$R2 - 3R1$ $$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 1 \end{array}\right] \qquad E_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$R2 \div 2$ $$\left[\begin{array}{cc|cc} 1 & 2 & -1 & 0 \\ 0 & 1 & 1.5 & 0.5 \end{array}\right] \qquad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$R1 - 2R2$ $$\left[\begin{array}{cc|cc} 1 & 0 & -4 & -1 \\ 0 & 1 & 1.5 & 0.5 \end{array}\right] \qquad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

A^{-1}

$$\therefore A^{-1} = E_4 E_3 E_2 E_1$$

$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$$

Q1 CONT'D

INVERSE ROW OPERATIONS APPLIED TO I

$E_1^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $R1 \times (-1)$

$E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $R2 + 3R1$

$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $R2 \times 2$

$E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $R1 + 2R2$

Q2

a) $$M-\lambda I = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix}$$

b) $(M-\lambda I)\vec{u} = \vec{0}$ IS A HOMOGENEOUS SYSTEM AND $M-\lambda I$ IS A SQUARE MATRIX. IF $M-\lambda I$ WAS INVERTIBLE, THEN THE RNF OF $M-\lambda I$ WOULD BE I AND THE SOLUTION WOULD BE $\vec{u} = \vec{0}$ (THE TRIVIAL SOLUTION). HOWEVER $M-\lambda I$ IS NOT INVERTIBLE AND THEREFORE THE RNF OF $M-\lambda I$ IS NOT I AND THERE WOULD BE INFINITE SOLUTIONS.

Q2 CONT'D

c) $\lambda=1$ THEREFORE NEED TO SOLVE
$(M-I)\vec{u}=\vec{0}$ FOR $\vec{u}\neq\vec{0}$.

$$\left[\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

$R1 \rightleftharpoons R2$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right]$$

$R3+R1$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R2\div 2$ $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R1-R2$ $$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

LET $$\vec{u}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} -2u_3 \\ u_3 \\ u_3 \end{bmatrix}=u_3\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Q2 CONT'D

d) LET $u_3 = 1$

$$\vec{u} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$M\vec{u} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \lambda \vec{u}$$

$\lambda = 1$

THEREFORE WHEN $\vec{u}$ IS CHOSEN TO BE AN EIGENVECTOR, $M\vec{u}$ PRODUCES A VECTOR PARALLEL TO $\vec{u}$ AND SCALED BY λ.

Q3

a)
$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 20 \\ -20 \\ 20 \\ -10 \\ -10 \end{bmatrix}$$

b)
$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

$$R4-R1 \quad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & -1 & 0 & 0 & -1 & -30 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

$$R2 \leftrightarrow R3 \quad \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & -1 & 0 & 0 & -1 & -30 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

Q3 CONT'D

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

R4+R2

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & -1 & 1 & -10 \end{array}\right]$$

R4-R3

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

R5+R4

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

R3+R4

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

R2-R3

Q3 CONT'D

$$R1-R2 \quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

c)
$$\begin{aligned} x_1 - x_5 &= -10 \\ x_2 + x_5 &= 30 \\ x_3 - x_5 &= -10 \\ x_4 - x_5 &= 10 \end{aligned}$$

d) x_1, x_2, x_3, x_4 LEADING
x_5 FREE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \\ -10 \\ 10 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

LINE IN 5-SPACE

e) $x_1 = x_5 - 10$

$x_3 = x_5 - 10$

$\therefore$ SET $x_5 = 10$ TO REDUCE x_1 AND x_3 TO ZERO.

Q4

a) LET $A = \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 3 & 1 \end{bmatrix}$ $\quad \vec{B} = \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 19 & -1 \\ -1 & 9 \end{bmatrix}$$

$$A^T \vec{B} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -7 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \end{bmatrix}$$

SOLVE NORMAL SYSTEM

$$\left[\begin{array}{cc|c} 19 & -1 & -7 \\ -1 & 9 & -3 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -66/170 \\ 0 & 1 & -64/170 \end{array}\right]$$

OR

$$(A^T A)^{-1} = \frac{1}{(19)(9)-1} \begin{bmatrix} 9 & 1 \\ 1 & 19 \end{bmatrix}$$

$$= \frac{1}{170} \begin{bmatrix} 9 & 1 \\ 1 & 19 \end{bmatrix}$$

$$(A^T A)^{-1} A^T \vec{B} = \frac{1}{170} \begin{bmatrix} 9 & 1 \\ 1 & 19 \end{bmatrix} \begin{bmatrix} -7 \\ -3 \end{bmatrix} = \frac{1}{170} \begin{bmatrix} -66 \\ -64 \end{bmatrix}$$

Q4 CONT'D

$$\therefore \; X_{LS} = \begin{bmatrix} -66/170 \\ -64/170 \end{bmatrix}$$

ORTHOGONAL PROJECTION ONTO THE PLANE

$$AX_{LS} = \frac{-66}{170}\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} - \frac{64}{170}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

LINEAR COMBINATION OF THE TWO DIRECTION VECTORS

$$= \frac{1}{170}\begin{bmatrix} -194 \\ -70 \\ -262 \end{bmatrix}$$

SINGLE VECTOR

b) $X_{LS} = (A^TA)^{-1}A^TB$

$$AX_{LS} = A\,(A^TA)^{-1}A^TB.$$

MATRIX THAT MAPS AN ARBITRARY VECTOR ($\vec{B}$) TO ITS ORTHOGONAL PROJECTION ONTO THE PLANE IN PART a) IS GIVEN BY

$$A\,(A^TA)^{-1}A^T$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & -2 \\ 3 & 1 \end{bmatrix} \frac{1}{170}\begin{bmatrix} 9 & 1 \\ 1 & 19 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

Q4 CONT'D

$$= \frac{1}{170}\begin{bmatrix} 11 & 39 \\ 25 & -35 \\ 28 & 22 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{170}\begin{bmatrix} 89 & -45 & 72 \\ -45 & 145 & 40 \\ 72 & 40 & 106 \end{bmatrix}$$

Q5

a) GIVEN B IS INVERTIBLE
AND $AB = BA$

$$\therefore ABB^{-1} = BAB^{-1}$$
$$AI = BAB^{-1}$$
$$A = BAB^{-1}$$
$$B^{-1}A = B^{-1}BAB^{-1}$$
$$B^{-1}A = IAB^{-1}$$
$$B^{-1}A = AB^{-1}$$ AS DESIRED.

b) GIVEN A IS INVERTIBLE
AND $AB = 0$

$$\therefore A^{-1}AB = A^{-1}0$$
$$IB = 0$$
$$B = 0$$ AS DESIRED.

c) GIVEN A IS NOT INVERTIBLE

$\therefore$ RNF OF $[A|0]$ IS NOT $[I|0]$

$\therefore$ $AX = \vec{0}$ HAS INFINITE SOLUTIONS

LET X_1 DENOTE A NONTRIVIAL SOLUTION
X_2 " " "
$\vdots$
X_n " " "

Ø5 CONT'D

LET $B = [X_1\ X_2 \cdots X_n]$

$\therefore\ AB = [AX_1\ AX_2 \cdots AX_n]$

$= [\vec{0}\ \vec{0} \cdots \vec{0}]$

$= 0$ AS DESIRED.

Q6

a) GIVEN $T(x)$

$$\frac{dT}{dx} = 20.4671\sqrt{.05}\, e^{\sqrt{.05}x} + 79.5329(-\sqrt{.05})\, e^{-\sqrt{.05}x}$$

$$\frac{d^2T}{dx^2} = (20.4671)(.05)\, e^{\sqrt{.05}x} + (79.5329)(.05)\, e^{-\sqrt{.05}x}$$

SUBSTITUTING INTO THE D.E.

$$(20.4671)(.05)\, e^{\sqrt{.05}x} + (79.5329)(.05)\, e^{-\sqrt{.05}x}$$

$$+ .05\left(200 - 200 - 20.4671 e^{\sqrt{.05}x} - 79.5329 e^{-\sqrt{.05}x}\right)$$

$= 0$ AS DESIRED

$T(0) = 300 \quad T(10) = 400$

$$T(0) = 200 + 20.4671\, e^{0} + 79.5329\, e^{0}$$

$= 300$ AS DESIRED

$$T(10) = 200 + 20.4671\, e^{\sqrt{5}} + 79.5329\, e^{-\sqrt{5}}$$

$= 200 + 191.5 + 8.5$

$= 400$ AS DESIRED

Q6 CONT'D

b) $$\frac{T(x+\Delta x)-2T(x)+T(x-\Delta x)}{(\Delta x)^2} + h'\left(T_\infty - T(x)\right) = 0$$

$$T(x+\Delta x)-2T(x)+T(x-\Delta x)+h'(\Delta x)^2\left(T_\infty - T(x)\right) = 0$$

$$T(x+\Delta x)-\left(2+h'(\Delta x)^2\right)T(x) + T(x-\Delta x) = -h'(\Delta x)^2\, T_\infty$$

c) WITH $\Delta x = 2$ AND $h' = .05$ AND $T_\infty = 200$

$$T(x+\Delta x)-\left(2+(.05)(2^2)\right)T(x) + T(x-\Delta x) = -(.05)(2^2)\,T_\infty$$

$$T(x+\Delta x) - 2.2\,T(x) + T(x-\Delta x) = -(0.2)(200) = -40$$

$x=2$

$$T(4) - 2.2\,T(2) + \underbrace{T(0)}_{300\ (B.C.)} = -40$$

$x=4$

$$T(6) - 2.2\,T(4) + T(2) = -40$$

$x=6$

$$T(8) - 2.2\,T(6) + T(4) = -40$$

$x=8$

$$\underbrace{T(10)}_{400\ (B.C.)} - 2.2\,T(8) + T(6) = -40$$

Q6 CONT'D

$$AX = B$$

$$\begin{bmatrix} -2.2 & 1 & 0 & 0 \\ 1 & -2.2 & 1 & 0 \\ 0 & 1 & -2.2 & 1 \\ 0 & 0 & 1 & -2.2 \end{bmatrix} \begin{bmatrix} T(2) \\ T(4) \\ T(6) \\ T(8) \end{bmatrix} = \begin{bmatrix} -340 \\ -40 \\ -40 \\ -440 \end{bmatrix}$$

d) DECREASE ΔX UNTIL THERE IS NEGLIGIBLE CHANGE IN THE SOLUTION.