AER210 VECTOR CALCULUS and FLUID MECHANICS

Quiz 4

Duration: 60 minutes

2 December 2019

Closed Book, no aid sheets

Non-programmable calculators allowed

Instructor: Prof. Alis Ekmekci

Family Name: (Solutions)

Given Name: ______________________________

Student #: ______________________________

TA Name/Tutorial #: ______________________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	8	
2	10	
3	10	
4	10	
5	12	
TOTAL	/50	/50

The following formula may be useful:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial z}, \qquad -\nabla p + \rho\vec{g} = \rho\vec{a}, \qquad \frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

$$\Sigma\overrightarrow{F_{CV}} = \frac{d}{dt}\iiint_{CV} \vec{V}(\rho d\forall) + \oiint_{S} \vec{V}(\rho\vec{V}\cdot\overrightarrow{dA})$$

1) (a) Consider the following velocity field: $\vec{V} = (2x)\,\vec{i} + (t^2)\,\vec{j}$

- Is this a steady or unsteady flow? [1 mark]

Unsteady flow. $\vec{V} = \vec{V}(x,t)$

- Also, determine if this flow is compressible or incompressible [1 marks].

$\vec{\nabla}\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2 \neq 0$ (Flow is compressible)

(b) Indicate true (T) or false (F): [3 marks]

F The amount of mass entering a control volume have to be equal to the amount of mass leaving during an *unsteady-flow process*.

F The Eulerian method of fluid flow analysis is similar to the study of a system.

T The variation of pressure with elevation in steady incompressible flow with straight streamlines is the same as that in the stationary fluid.

(c) Fill the following: [3 marks]

A streakline is a line that connects all fluid particles that have passed through the same point in space at a previous time.

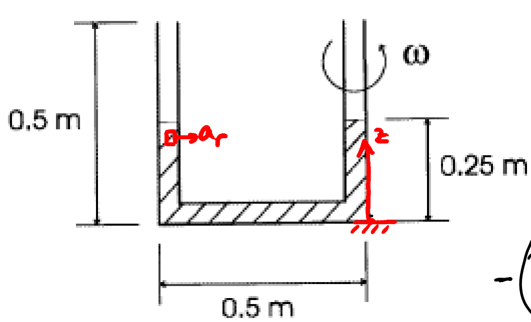
A streamline is a line that is tangent to the local velocity vector at every point along the line at that instant.

A pathline is a line traced out by a particle as it moves from one point to another in the flow.

2) A U-tube is rotated with a constant angular velocity ω about one leg as shown in the figure. Before being rotated, the liquid in the tube fills 0.25 m of each leg. The length of the base of the U-tube is 0.5 m and each leg is 0.5 m long. What would be the maximum rotation rate to ensure that no liquid is lost from the outer leg? [10 marks]

Hint: The magnitude of the centripetal acceleration is $r\omega^2$ (where r is the distance from the axis of rotation), and the gradient of pressure in cylindrical coordinates is as follows:

$$\vec{\nabla} p = \frac{\partial p}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\vec{e_\theta} + \frac{\partial p}{\partial z}\vec{e_z}$$



$a_r = r\omega^2$, $a_\theta = 0$, $a_z = 0$ } $\vec{a} = -r\omega^2\vec{e_r}$ (−) as it acts toward the center of rotation

$-\vec{\nabla}p - \rho g\vec{k} = \rho\vec{a} \implies -\vec{\nabla}p - \rho g\vec{e_z} = \rho\vec{a}$ (in cyl. coord. system)

$-\left(\frac{\partial p}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial p}{\partial \theta}\vec{e_\theta} + \frac{\partial p}{\partial z}\vec{e_z}\right) - \rho g\vec{e_z} = \rho(-r\omega^2)\vec{e_r}$

$-\frac{\partial p}{\partial r} = -\rho r\omega^2, \quad -\frac{1}{r}\frac{\partial p}{\partial \theta} = 0, \quad -\frac{\partial p}{\partial z} = \rho g$

$\frac{\partial p}{\partial r} = \rho r\omega^2$, $\frac{\partial p}{\partial \theta} = 0$, $\frac{\partial p}{\partial z} = -\rho g$ } $p = p(r,z) \Rightarrow dp = \underbrace{\frac{\partial p}{\partial r}}_{\rho r\omega^2}dr + \underbrace{\frac{\partial p}{\partial z}}_{-\rho g}dz$

$dp = \rho r\omega^2 dr - \rho g\,dz$

integrating

$$\boxed{p = \rho\omega^2\frac{r^2}{2} - \rho g z + C}$$

At the free surface $z = z_s$, $p = p_{atm}$ (from this z_s function (a functional relationship for the free surface) can be found).

$p_{atm} = \rho\omega^2\frac{r^2}{2} - \rho g z_s + C \Rightarrow z_s = \frac{\omega^2 r^2}{2g} + \underbrace{\frac{C - p_{atm}}{\rho g}}_{C_1 \text{ (a new constant)}} \implies \boxed{z_s = \frac{\omega^2 r^2}{2g} + C_1}$

A constant volume of liquid requires: $z_R + z_L = 0.5$

At ω_{max}: $\underbrace{z_L}_{z_s(r=0.5)} = 0.5$, $\underbrace{z_R}_{z_s(r=0)} = 0 \implies z_R = \frac{\omega_{max}^2 \cdot 0^2}{2g} + C_1 \Rightarrow \boxed{C_1 = z_R = 0}$

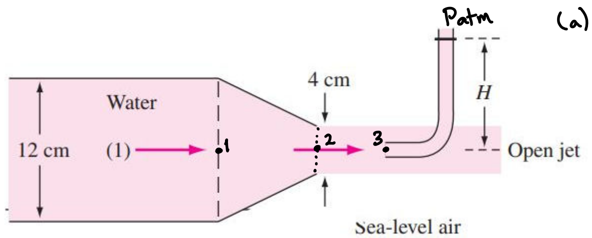
$z_L = 0.5 = \frac{\omega_{max}^2 (0.5)^2}{2g} + 0 \Rightarrow \omega_{max} = \sqrt{4g} = 6.32 \text{ rad/s}$

3) As shown in the figure below, the open jet of water at 20°C exits a nozzle into sea-level air and strikes a pitot tube, tip of which is placed at the centre of the jet as shown. If the pressure at the centerline at section 1 is 110 kPa, and losses are neglected, estimate:

(a) the mass flow rate [5 marks],

(b) the height H of the fluid in the pitot tube [5 marks].

Take density of water as $\rho = 1000\ kg/m^3$, atmospheric pressure at sea level as $p_{atm} = 101.35$ kPa, gravitational acceleration as $g = 10\ m/s^2$, and assume one dimensional flow.



(a) Apply Bernoulli between 1 and 2:

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + \cancel{g z_1} = \frac{P_2}{\rho} + \frac{V_2^2}{2} + \cancel{g z_2} \quad \Leftarrow \text{Eqn. 1.}$$

$(z_1 = z_2)$

From continuity eqn:

$$V_1 \frac{\cancel{\pi}(0.12)^2}{\cancel{4}} = V_2 \frac{\cancel{\pi}(0.04)^2}{\cancel{4}} \Rightarrow V_2 = 9\, V_1 \quad \Leftarrow \text{Eqn. 2}$$

Combining eqn. 1 and eqn. 2: $\frac{P_1}{\rho} + \frac{V_1^2}{2} = \frac{P_2}{\rho} + \frac{(9V_1)^2}{2} \Rightarrow V_1 = \sqrt{\frac{P_1 - P_2}{40\rho}} = \sqrt{\frac{110000 - 101350}{(40)(1000)}} = 0.465\ m/s$

$$V_2 = 9V_1 = (9)(0.465) = 4.185\ m/s$$

mass flow rate: $\dot{m} = \rho A_1 V_1 = (1000)\,\pi \frac{(0.12)^2}{4} (0.465) = 5.26\ kg/s$

(b) Bernoulli between (2) & (3):

$$\frac{P_2}{\rho} + \frac{V_2^2}{2} + \cancel{g z_2} = \frac{P_3}{\rho} + \cancel{\frac{V_3^2}{2}} + \cancel{g z_3} \qquad (z_2 = z_3)$$

$V_3 = 0$ (stagnation point)

$$P_3 = \overset{P_{atm}}{P_2} + \rho \frac{V_2^2}{2} \Rightarrow P_3 = P_{atm} + \rho \frac{V_2^2}{2}$$ (the pressure at the stagnation point (point 3))

From hydrostatic relations we can also write: $P_3 - \rho g H = P_{atm} \Rightarrow H = \frac{P_3 - P_{atm}}{\rho g}$

combining these two:

$$H = \frac{P_3 - P_{atm}}{\rho g} = \frac{\cancel{P_{atm}} + \rho \frac{V_2^2}{2} - \cancel{P_{atm}}}{\rho g} = \frac{V_2^2}{2g} = \frac{(4.185)^2}{2 \times 10} = 0.875\ m = 87.5\ cm$$

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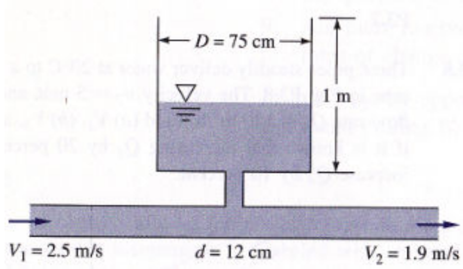
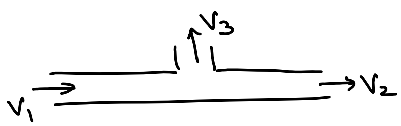
4) The pipe flow, shown in the figure below, fills a cylindrical tank. The dimensions of this tank and the pipe are given in the figure. At time t = 0, the water depth in the tank is 30 cm. Estimate the time required to fill the remainder of the tank. [10 marks]



Hint: *Select an appropriate control volume, and then apply the integral form of the continuity equation to this control volume, which is:*

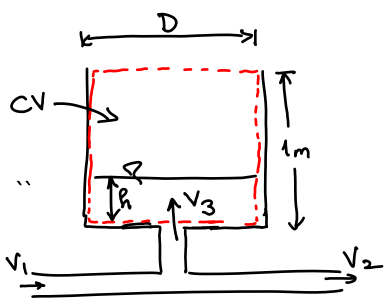
$$\frac{\partial}{\partial t}\iiint_{V\!\!\!/} \rho\, dV\!\!\!/ + \oint\!\!\!\oint_S \rho \vec{V}\cdot\vec{dS} = 0$$



$\dot{V\!\!\!/}_1 = \dot{V\!\!\!/}_2 + \dot{V\!\!\!/}_3$ (from continuity)

$V_1 \pi \frac{d^2}{4} = V_2 \pi \frac{d^2}{4} + \dot{V\!\!\!/}_3$

$\boxed{\dot{V\!\!\!/}_3 = \pi \frac{d^2}{4}(V_1 - V_2)}$



Integral form of the continuity eqn.:

$$\underbrace{\frac{d}{dt}\iiint_{V\!\!\!/} \rho\, dV\!\!\!/}_{\text{Term 1}} + \underbrace{\oint\!\!\!\oint_S \rho \vec{V}\cdot d\vec{S}}_{\text{Term 2}} = 0$$

$$\text{Term 1} = \frac{d}{dt}\underbrace{\iiint_{V\!\!\!/} \rho\, dV\!\!\!/}_{\text{mass inside the CV.}} = \frac{d}{dt}\left[\rho \frac{\pi D^2}{4} h\right] = \rho \frac{\pi D^2}{4}\frac{dh}{dt}$$

$$\text{Term 2} = \oint\!\!\!\oint_S \rho \vec{V}\cdot d\vec{S} = \rho \dot{V\!\!\!/}_3$$

$\dot{V\!\!\!/}_3 = \pi \frac{d^2}{4}(V_1 - V_2)$

$$\cancel{\rho}\frac{\pi D^2}{4}\frac{dh}{dt} = \cancel{\rho}\dot{V\!\!\!/}_3 \Rightarrow \frac{dh}{dt} = \dot{V\!\!\!/}_3 \frac{4}{\pi D^2} = \cancel{\pi}\frac{d^2}{\cancel{4}}(V_1 - V_2)\cdot\frac{\cancel{4}}{\cancel{\pi} D^2} = \frac{d^2}{D^2}(V_1 - V_2) = \frac{12^2}{75^2}(2.5 - 1.9) = 0.0153\ \text{m/s}$$

$$\Delta t = \frac{(1 - 0.3)}{0.0153} = 45.7\text{s}$$

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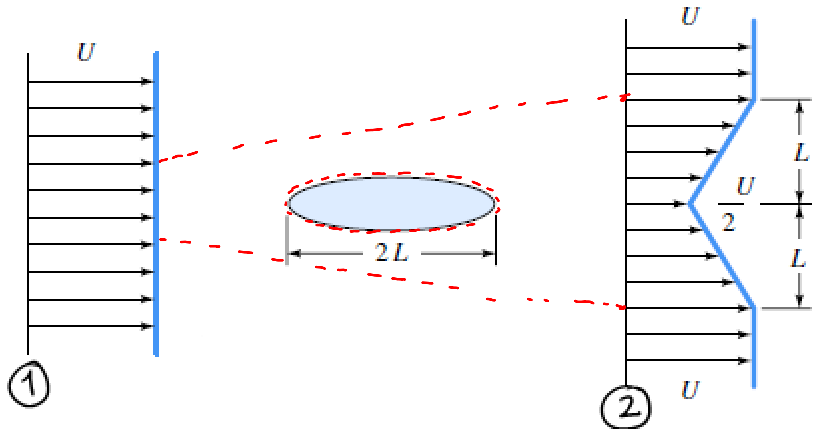
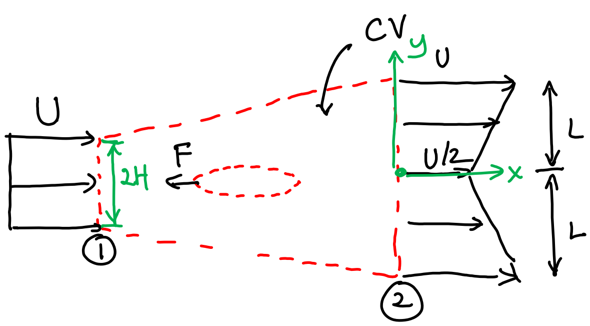
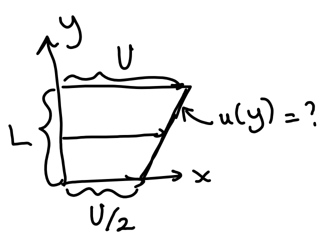
5) When a uniform stream flows past an immersed thick cylinder, a low-velocity wake is created downstream, idealized as a V shape as shown in the figure below. Pressures p_1 and p_2 are approximately equal. If the flow is two-dimensional and incompressible, with width b into the paper, derive a formula for the drag force F on the cylinder. [12 marks]







$u(y) = Ay + B$

BCs: @ $y=0$ $u(y=0) = U/2$

@ $y=L$ $u(y=L) = U$

@ $y=0 \Rightarrow \frac{U}{2} = A(0) + B \Rightarrow \boxed{B = \frac{U}{2}}$

@ $y=L \Rightarrow U = AL + \underbrace{B}_{U/2} \Rightarrow A = \left(U - \frac{U}{2}\right)\frac{1}{L}$

$\boxed{A = \frac{U}{2L}}$

Velocity profile at section 2 : $\boxed{u(y) = \frac{U}{2L}y + \frac{U}{2}}$

$\dot{m}_1 = \rho U (2H) b$

$\dot{m}_2 = 2\int_{y=0}^{L} \rho u \, dA = 2\int_{y=0}^{L} \rho \left(\frac{U}{2L}y + \frac{U}{2}\right) b \, dy = 2\rho b \frac{U}{2} \int_{y=0}^{L} \left(\frac{y}{L} + 1\right) dy = 2\rho b \frac{U}{2}\left(\frac{y^2}{2L} + y\right)\Big|_0^L$

$$\Rightarrow \dot{m}_2 = \cancel{2}\rho b \frac{U}{\cancel{2}}\left(\frac{L^2}{2L}+L\right) = \rho b U \frac{3L}{2}$$

From continuity : $\dot{m}_1 = \dot{m}_2$

$$\cancel{\rho}\cancel{U}\,2H\cancel{b} = \cancel{\rho}\cancel{U}\cancel{b}\,\frac{3}{2}L \Rightarrow \boxed{H = \frac{3}{4}L}$$

From momentum equation: $\sum \vec{F}_{cv} = \underbrace{\frac{d}{dt}\iiint_{CV} \vec{V}(\rho\, d\forall)}_{=0 \text{ (steady flow}} + \oint\!\!\oint_{CS} \vec{V}(\rho \vec{V}\cdot d\vec{A})$

$$\sum \vec{F}_{cv} = \iint_{A_1} \vec{V}_1(\rho \vec{V}_1 \cdot d\vec{A}) + \iint_{A_2} \vec{V}_2(\rho \vec{V}_2 \cdot d\vec{A})$$

$$\sum \vec{F}_{cv} = \iint_{A_1} U\vec{i}\left(\rho\, U\vec{i}\cdot(-dA\vec{i})\right) + \iint_{A_2} u\vec{i}(\rho u\vec{i}\cdot dA\vec{i})$$

$u = \frac{U}{2L}y + \frac{U}{2}$

$$= \underbrace{-\rho U^2 A_1 \vec{i}}_{\text{Term 1}} + \underbrace{\iint_{A_2} \rho u^2 dA\vec{i}}_{\text{Term 2}}$$

Term 1 $= -\rho U^2 A_1\vec{i} = -2\rho U^2 bH\vec{i} = -2\rho U^2 b\frac{3L}{4}\vec{i} = -\rho U^2 b\frac{3L}{2}\vec{i}$

$A_1 = b(2H)$ $\quad H = \frac{3L}{4}$

Term 2 $= 2\int_{y=0}^{L} \rho\left(\frac{U}{2L}y + \frac{U}{2}\right)^2 b\,dy\,\vec{i} = 2\rho\frac{U^2}{4}b\int_{y=0}^{L}\left(\frac{y}{L}+1\right)^2 dy\,\vec{i}$

$$= \rho\frac{U^2}{2}b\int_{y=0}^{L}\left(\frac{y^2}{L^2}+2\frac{y}{L}+1\right)dy\,\vec{i} = \rho\frac{U^2}{2}b\left[\frac{y^3}{3L^2}+\frac{y^2}{L}+y\right]_{y=0}^{L}\vec{i}$$

$$= \rho\frac{U^2}{2}b\left[\frac{L^3}{3L^2}+\frac{L^2}{L}+L\right]\vec{i} = \rho\frac{U^2}{2}b\left[\frac{L}{3}+L+L\right]\vec{i} = \rho\frac{U^2}{2}b\left(\frac{7}{3}L\right)$$

Putting back these terms to the momentum eqn, the x direction momentum eqn is:

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in x direction: $\Sigma F_x = -\rho U^2 b \frac{3L}{2} + \rho \frac{U^2}{2} b \frac{7}{3} L$

$$-F = \rho U^2 b L \left(\underbrace{\frac{7}{6} - \frac{3}{2}}_{-1/3} \right)$$

$$F = \frac{1}{3} \rho U^2 b L \Rightarrow \vec{F} = -\frac{1}{3} \rho U^2 b L \vec{i}$$

this force acts on the control volume to the left (-x direction)

$$\vec{F}_{drag} = -\vec{F} = \frac{1}{3} \rho U^2 b L \vec{i}$$ (acts to the right (in +x direction))