MAT194F Calculus
Midterm Test
9:00 – 10:45, 22 October 2015
105 minutes
No calculators or aids
Each question is worth 10 marks

1. Calculate $f'(x)$ for:

   (a) $f(x) = 5x + 2x^2$ (b) $f(x) = \sin(-2x)$ (c) $f(x) = \tan\sqrt{x}$

   (d) $f(x) = -2/x^2$ (e) $f(x) = \frac{2+3x-x^2/2}{7x^2-4}$

2. Evaluate the following limits if they exist. Do not use l'Hospital's Rule.

   (a) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$ (b) $\lim_{x\to 2}\frac{x^2-4x+4}{x^4-3x^2-4}$ (c) $\lim_{x\to 1}\frac{x^4-1}{x^3-1}$ (d) $\lim_{x\to 0}\frac{\sin 3x \sin 5x}{x^2}$ (e) $\lim_{x\to 0}\frac{\sin(x^2)}{x}$

3. Provide $\delta - \varepsilon$ proofs for:

   (a) $\lim_{x\to 5} x^2 = 25$ (b) $\lim_{x\to 1/5} x^2 = 1/25$

4. Sketch, indicating all important features:

   (a) $y = \sin x + \sin|x|$ (b) $y = \frac{\sqrt{x}}{1+\sqrt{x}}$

5. The radius of a sphere is increasing at a constant rate of 0.5 cm/s.

   (a) There will be a time at which the volume of the sphere and the area of a cross-section through the centre of the sphere are increasing at the same rate. At this time, what is the radius of the sphere?

   (b) At the time when the surface area of the sphere is increasing at a rate of 8π cm^2/s, how fast is the volume of the sphere increasing?

6. Consider the curve C defined implicitly by the equation $\cos(xy) = 1 + \sin y$.

   (a) Find dy/dx in terms of x and y.

   (b) Is there a line that is tangent to both the curve C and the parabola $y = x^2$? If so, give the equation of one such line.

7.

   (a) Find the absolute maximum and minimum values of $f(x) = \sin x + \cos^2(x)$ on $[0, \pi]$ and state where they occur

   (b) Let $f(x) = x^3 + ax + b$, where a, b are real, unknown constants. For what values of a, b does a local maximum and minimum of $f(x)$ exist? When they do exist, give their locations and values in terms of a, b.

8. Sketch, indicating all important features: $y = \sqrt{x^2 + x} - x$.

9. (a) f, a real-valued function defined for all real x, is differentiable and satisfies $\lim_{x\to\infty} f'(x) = 0$. Prove $\lim_{x\to\infty} [f(x+1) - f(x)] = 0$.

   (b) g, a real-valued function defined for all real x, satisfies
   (i) $g(x+y) = g(x) + g(y) + xy$ for all x, y and (ii) $\lim_{x\to 0} \frac{g(x)}{x} = 4$.
   Find $g(0)$ and $g'(0)$.

10. Evaluate the following limits if they exist. Do not use l'Hospital's Rule.

   (a) $\lim_{x\to 1} \frac{\sqrt{x^2+3}-2}{x^2-x}$ (b) $\lim_{x\to -\infty} 2x^5 - 6x^4 + 1$ (c) $\lim_{x\to 0} \frac{|x-1|-|2x+1|}{3x}$

   (d) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x^2, & \text{if } x \text{ is of the form } \frac{1}{n}, n \in \{1, 2, 3 \ldots\} \\ -x, & \text{otherwise} \end{cases}$

   (e) $\lim_{x\to\infty} \frac{2x^3-3}{3x^2-2} \sin\left(\frac{1}{x}\right)$