UNIVERSITY OF TORONTO
Faculty of Applied Science and Engineering

Winter 2024 Term Test 1

ECE 286 H1S

Duration: 90 minutes

Aids Allowed: The exam is Type D. One two-sided aid sheet is provided. One non-programmable calculator.

*Do **not** turn this page until you have received the signal to start.*
In the meantime, write your name, student number, UTORid and email address below (please do this now!) and *carefully* read *all* the information on the rest of this page.

Marking Guide

Nº 1:	______/ 9
Nº 2:	______/ 5
Nº 3:	______/10
Nº 4:	______/ 7
Nº 5:	______/ 6
Nº 6:	______/ 7
Nº 7:	______/ 6
TOTAL:	______/50

- This test consists of 7 question on 12 page (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete.*
- Answer each question directly on the test paper, in the space provided.

Question 1. Counting [9 MARKS]

A bag contains 8 balls: 3 red, 3 blue, and 2 green. (Note: Every assumption made in each question is only valid for such a question.)

Part (a) [1 MARK] If all the balls are taken out of the bag one at a time, how many ways can the balls be drawn? (note that the balls of the same color are distinguishable for this question.)

Solution: The different ways the balls can be drawn is:

$$8! = 40320$$

Part (b) [2 MARKS] If the balls of the same color are indistinguishable, how many different ways can the balls be arranged in a straight line?

Solution: The different ways the balls can be arranged is:

$$\frac{8!}{3!3!2!} = 560$$

Part (c) [3 MARKS] If the balls of the same color are indistinguishable, in how many ways can the 8 balls be partitioned (with no order) into two non-labeled groups of 4 balls each, such that at least one group contains 3 balls of the same color and the second group contains the remaining balls?

Solution: There are only three scenarios in this case as the order does not matter and the balls of the same color are indistinguishable:

1. RRRB BBGG
2. RRRG BBBG
3. BBBR RRGG

Part (d) [3 MARKS] If 4 balls are to be selected at random simultaneously, what is the probability that exactly 2 red balls and 2 blue balls are selected?

Solution: The probability is:

$$\frac{\binom{3}{2}\binom{3}{2}}{\binom{8}{4}} = \frac{9}{70} \approx 0.1286$$

Question 2. Basic Probability Calculations [5 marks]

In a software development company, two teams A and B are working on different modules of the same project. Let A denote the event that Team A meets the project deadline, and B denote the event that Team B meets the project deadline. It is known that $P(B) = 0.8$, the probability that Team A meets the deadline given that Team B meets the deadline is 0.9, and the probability that Team B meets the deadline given that Team A meets the deadline is 0.85. Calculate the probability that at least one of the teams meets the project deadline.

Calculate $P(A \text{ or } B)$.

Solution: First, find $P(A)$ using Bayes' Theorem:

$$\begin{aligned} P(A) &= P(A|B)\frac{P(B)}{P(B|A)} \\ &= 0.9 \times \frac{0.8}{0.85} \\ &= 0.8471 \end{aligned}$$

Next, calculate $P(A \text{ and } B)$:

$$\begin{aligned} P(A \text{ and } B) &= P(B|A)P(A) \\ &= 0.85 \times 0.8471 \\ &= 0.7190 \end{aligned}$$

Finally, calculate $P(A \text{ or } B)$:

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.8471 + 0.8 - 0.7190 \\ &= 0.9281 \end{aligned}$$

Question 3. Discrete and Continuous Distributions [10 marks]

A physicist is observing a particle that can be in one of four energy states: 0 eV, 1 eV, 2 eV, or 3 eV. The probability of finding the particle in each state is 0.1, 0.2, 0.3, and 0.4, respectively. Let X be a random variable representing the energy state of a particle.

Part (a) [2 marks] Write the Probability Mass Function and calculate the probability of finding the particle in any state between 2 eV and 4 eV, both inclusive.

Solution: The PMF is:

$$\begin{aligned} P(X = x) = f(x) = 0.1 &\quad \text{for } x = 0, \\ 0.2 &\quad \text{for } x = 1, \\ 0.3 &\quad \text{for } x = 2, \\ 0.4 &\quad \text{for } x = 3, \\ 0 &\quad \text{elsewhere} \end{aligned}$$

$$P(2 \leq X \leq 4) = 1 - P(X < 2) = 1 - (P(X = 0) + P(X = 1)) = 1 - 0.1 - 0.2 = 0.7$$

Part (b) [2 marks] Calculate the mean energy and the variance of the particle.

Solution:

$$\sum(x \cdot P(x)) = 0 \cdot 0.1 + 1 \cdot 0.2 + 2 \cdot 0.3 + 3 \cdot 0.4$$
$$E[X] = 2.0 \text{ eV}$$

$$\sum((x - \mu)^2 \cdot P(x)) = (0 - 2)^2 \cdot 0.1 + (1 - 2)^2 \cdot 0.2 + (2 - 2)^2 \cdot 0.3 + (3 - 2)^2 \cdot 0.4$$
$$Var[X] = 1.0 \text{ eV}^2$$

The physicist now models the energy of the particle as a continuous random variable with PDF:

$$f(x) = \begin{cases} x/3 & \text{if } 0 \le x \le 3, \\ 0 & \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} \frac{2}{9}x & \text{if } 0 \le x \le 3, \\ 0 & \text{otherwise} \end{cases}$$

Part (c) [2 marks] Calculate the probability that the particle's energy is between 2 eV and 4 eV.

Solution:

$$\begin{aligned} \int_2^3 \frac{x}{3}\,dx &= \frac{1}{3}\int_2^3 x\,dx \\ &= \frac{1}{3}\left[\frac{x^2}{2}\right]_2^3 = \frac{1}{3}\left(\frac{3^2}{2} - \frac{2^2}{2}\right) \\ &= \frac{1}{3}\left(\frac{9}{2} - \frac{4}{2}\right) = \frac{1}{3} \times \frac{5}{2} \\ &= \frac{5}{6} = 0.8333 \end{aligned}$$

$$\begin{aligned} \int_2^3 \frac{2x}{9}\,dx &= \frac{2}{9}\int_2^3 x\,dx \\ &= \frac{1}{9}\left[x^2\right]_2^3 \\ &= \frac{5}{9} = 0.5556 \end{aligned}$$

Part (d) [2 marks] Calculate the mean energy of the particle.

Solution:

$$\begin{aligned} \int_0^3 x \cdot \frac{x}{3}\,dx = \int_0^3 \frac{x^2}{3}\,dx &= \frac{1}{3}\int_0^3 x^2\,dx \\ &= \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\left(\frac{3^3}{3} - \frac{0^3}{3}\right) = \frac{1}{3} \times 9 = 3 \end{aligned}$$

$$\begin{aligned} \int_0^3 x \cdot \frac{2x}{9}\,dx = \int_0^3 \frac{2x^2}{9}\,dx &= \frac{2}{9}\int_0^3 x^2\,dx \\ &= \left[\frac{2x^3}{27}\right]_0^3 = 2 \end{aligned}$$

Part (e) [2 marks] Calculate the variance of the energy of the particle. **Solution:**

$$\int_0^3 (x-3)^2 \cdot \frac{x}{3}\,dx = \frac{1}{3}\int_0^3 x(x-3)^2\,dx = \frac{1}{3}\int_0^3 x(x^2-6x+9)\,dx = \frac{1}{3}\int_0^3 (x^3-6x^2+9x)\,dx$$
$$= \frac{1}{3}\left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2}\right]_0^3 = \frac{1}{3}\left(\frac{3^4}{4} - 2\times 3^3 + \frac{9\times 3^2}{2} - 0\right) = \frac{1}{3}\left(\frac{81}{4} - 54 + \frac{81}{2}\right)$$
$$= \frac{1}{3}\left(\frac{81}{4} - \frac{216}{4} + \frac{162}{4}\right) = \frac{1}{3}\times\frac{27}{4} = \frac{9}{4} = 2.25$$

$$\int_0^3 (x-2)^2 \cdot \frac{2x}{9}\,dx = \frac{2}{9}\int_0^3 x(x-2)^2\,dx = \frac{2}{9}\int_0^3 x(x^2-4x+4)\,dx = \frac{2}{9}\int_0^3 (x^3-4x^2+4x)\,dx$$
$$= \frac{2}{9}\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^3 = 0.5$$

Question 4. Joint and Marginal Distributions, Conditional Distribution, and Independence [7 MARKS]

Consider two continuous random variables, X and Y, representing the time (in hours) taken by two employees to complete two different, but related, tasks. The joint probability density function of X and Y is given by:

$$f(x, y) = \begin{cases} 2 & \text{if } 0 < x < 1, 0 < y < 1, x + y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Part (a) [5 MARKS] Find the expectation of X, as well as the expectation of Y.

Solution: The marginal PDF of X is given by integrating $f(x, y)$ over all values of Y, and similarly for Y:

$$f_X(x) = \int_0^{1-x} 2\, dy = 2 - 2x, \quad \text{for } 0 < x < 1$$

$$f_Y(y) = \int_0^{1-y} 2\, dx = 2 - 2y, \quad \text{for } 0 < y < 1$$

Calculate the Means: First, we find the means μ_X and μ_Y of X and Y, respectively.

$$\begin{aligned} \mu_X &= \int_0^1 x f_X(x)\, dx = \int_0^1 x(2 - 2x)\, dx \\ &= \left[x^2 - \frac{2}{3}x^3\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3} \end{aligned}$$

Similarly,

$$\mu_Y = \frac{1}{3}$$

Part (b) [2 MARKS] Determine if X and Y are independent.

Solution: X and Y are not independent given that $f_X(x) \cdot f_Y(y) \neq f(x, y)$ for all x and y.

Question 5. Discrete Probability Distributions - Part 1 [6 MARKS]

A political campaign team is recruiting volunteers for an upcoming election. The campaign's success with volunteers can be modeled with various probability distributions.

Part (a) [2 MARKS] Initially, the campaign team reaches out to 10 individuals, with each individual having a 60% chance of agreeing to volunteer. What is the probability that exactly 6 individuals will agree to volunteer? Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)

Solution: This can be modeled with a binomial distribution where the number of trials $n = 10$ and the probability of success $p = 0.6$. The probability of exactly 6 successes is given by:

$$P(X = 6) = \binom{10}{6}(0.6)^6(0.4)^4 = 0.2508$$

Part (b) [2 MARKS] At some point during the campaign, the team reaches out to 15 individuals. Each individual can either agree to volunteer (60%), decline (30%), or be undecided (10%). What is the probability that out of these 15 individuals, 9 agree to volunteer, 4 decline, and 2 are undecided? Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)

Solution: This can be modeled with a multinomial distribution with probabilities $p_1 = 0.6$, $p_2 = 0.3$, and $p_3 = 0.1$. The probability is given by:

$$P(X_1 = 9, X_2 = 4, X_3 = 2; p_1 = 0.6, p_2 = 0.3, p_3 = 0.1, n = 15) = \frac{15!}{9!4!2!}(0.6)^9(0.3)^4(0.1)^2 \approx .0613$$

Part (c) [2 MARKS] Towards the end of the campaign, there are 20 volunteers needed, but only 12 people are available, 8 of whom have previously expressed interest in volunteering. If 5 people are selected randomly from the 12 available, what is the probability that exactly 4 of those selected had previously expressed interest in volunteering? Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)

Solution: This can be modeled with a hypergeometric distribution where $N = 12$ (total number of people), $k = 8$ (number of people who have expressed interest), $n = 5$ (number of people chosen), and $x = 4$ (number of people chosen who have expressed interest). The probability is given by:

$$P(X = 4) = \frac{\binom{8}{4}\binom{4}{1}}{\binom{12}{5}} \approx 0.3535$$

Question 6. Discrete Probability Distributions - Part 2 [7 MARKS]

An ecologist is studying a rare species in a large forest. The sightings of these animals are rare and can be modeled using various probability distributions.

Part (a) [4 MARKS] Suppose the ecologist is interested in the number of non-sighting days before observing 5 sightings of the species. Assume the probability of sighting the species on any given day is 0.1.

(i) What is the probability that the ecologist observes the 5th sighting on the 10th day? Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)
Solution:

$$\begin{aligned} P(X = 10) &= \binom{10-1}{5-1}(0.1)^5(0.9)^{10-5} \\ &= \binom{9}{4}(0.1)^5(0.9)^5 \\ &= 126 \times 0.00001 \times 0.59049 \\ &= 0.000743 \end{aligned}$$

(ii) Calculate the mean and variance of the number of non-sighting days. Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)
Solution:

$$\begin{aligned} E(X-5) &= E(X) - 5 \\ Var(X-5) &= Var(X) \\ \text{Mean} &= \frac{k}{p} - 5 = \frac{5}{0.1} - 5 = 50 - 5 = 45 \\ \text{Variance} &= \frac{k(1-p)}{p^2} = \frac{5 \times 0.9}{0.1^2} = 450 \end{aligned}$$

Part (b) [3 MARKS] Due to changes in the environment, the average number of sightings has now increased to 2 per day. What is the probability of observing exactly 3 sightings on a given day? Mention what discrete probability distribution best models this random variable, along with the parameters (i.e., information that helps model the RV.)

Solution:

$$\begin{aligned} P(X = 3) &= \frac{e^{-2}2^3}{3!} \\ &= \frac{0.1353 \times 8}{6} \\ &= 0.1804 \end{aligned}$$

Question 7. Probability Calculations [6 MARKS]

In a manufacturing plant, an automated inspection system is deployed to detect defective items produced by a machine. Historically, it's known that 2% of the items produced are defective. The inspection system has a 95% detection accuracy for defective items and a 97% accuracy for identifying non-defective items as non-defective. Given that an item was flagged as defective by the inspection system, what is the probability that the item is indeed defective?

Let's define the events:

- D: The event that an item is defective.
- N: The event that an item is not defective.
- F: The event that an item is flagged as defective by the inspection system.

We are interested in finding $P(D|F)$, the probability that an item is defective given that it was flagged as defective.

According to Bayes' theorem, we have:

$$P(D|F) = \frac{P(F|D)P(D)}{P(F|D)P(D) + P(F|N)P(N)}$$

Where:

- $P(F|D) = 0.95$ (probability that a defective item is flagged as defective)
- $P(D) = 0.02$ (prior probability that an item is defective)
- $P(F|N) = 0.03$ (probability that a non-defective item is flagged as defective)
- $P(N) = 0.98$ (prior probability that an item is not defective)

Substituting these values into Bayes' formula gives:

$$P(D|F) = \frac{0.95 \times 0.02}{0.95 \times 0.02 + 0.03 \times 0.98} \approx 0.3926$$

Given that an item has been flagged as defective by the automated inspection system, the probability that the item is actually defective is approximately 39.26%.

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