Q1: (parts (a) and (b) are separate)

a) Assume matrix A is nxn. By working with the eigenvalue/eigenvector equation $A\vec{x} = \lambda\vec{x}$, where λ is a scalar, prove the following statements if they are true or provide a counterexample if they are not true:

   i. λ^2 is an eigenvalue of A^2.

   ii. λ^{-1} is an eigenvalue of A^{-1} (assuming $\lambda \neq 0$ and A is invertible).

   iii. $\lambda + 1$ is an eigenvalue of $A + I$.

b) Let $A = \begin{bmatrix} 0 & 10^4 \\ 0 & 0 \end{bmatrix}$. Find the eigenvalues of A. For one of these eigenvalues, find the corresponding eigenvector.

Solutions:

a) All statements are true.

i. $A\vec{x} = \lambda\vec{x}$

$\therefore A(A\vec{x}) = A(\lambda\vec{x})$

$\therefore A^2\vec{x} = \lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$

Therefore, λ^2 is an eigenvalue of A^2.

ii. $A\vec{x} = \lambda\vec{x}$

$\therefore A^{-1}(A\vec{x}) = A^{-1}(\lambda\vec{x})$

$\therefore I\vec{x} = \vec{x} = \lambda(A^{-1}\vec{x})$

$$\therefore A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$$

Therefore, λ^{-1} is an eigenvalue of A^{-1}.

iii. $A\vec{x} = \lambda\vec{x}$

$\therefore A\vec{x} + I\vec{x} = \lambda\vec{x} + \vec{x}$

$\therefore (A + I)\vec{x} = (\lambda + 1)\vec{x}$

Therefore, $\lambda + 1$ is an eigenvalue of $A + I$.

b) $\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 10^4 \\ 0 & -\lambda \end{bmatrix} = \lambda^2 = 0$

Therefore, the eigenvalues of A are 0,0.

Solving $A\vec{x} = \lambda\vec{x}$ with $\lambda = 0$:

$$\begin{bmatrix} 0 & 10^4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = 0$ and x_1 is a free variable.

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the corresponding eigenvector.