ESC195 - Midterm Test #1
February 11, 2021
9:10 - 10:40 am, EST

The following materials are considered to be acceptable aids during the writing of this test:

- The Stewart textbook and the student solution manuals
- Any course notes or problem solutions prepared by the student
- Any handouts or other materials posed on the ESC195 course website
- Any non-programmable, non-graphing calculator

All questions are worth 10 marks

1. Use l'hospital's rule to evaluate the following limits:

$$\text{a) } \lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)} \qquad \text{b) } \lim_{x\to\infty} x^{e^{-x}} \qquad \text{c) } \lim_{x\to o}\left(\frac{1}{x} - \cot x\right)$$

a) $\lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)} \quad \left(\to \frac{0}{0}\right) \overset{*}{=} \lim_{x\to 0} \frac{e^x}{\frac{1}{1+x}} = \frac{1}{1} = 1$

b) $\lim_{x\to\infty} x^{e^{-x}} \quad (\to \infty^0) = \lim_{x\to\infty} e^{\ln x^{e^{-x}}}$

consider $\lim_{x\to\infty} \ln x^{e^{-x}} = \lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} \to \frac{\infty}{\infty}$

$\overset{*}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} \longrightarrow 0$

$\therefore \lim_{x\to\infty} x^{e^{-x}} = e^0 = 1$

c) $\lim_{x\to 0}\left(\frac{1}{x} - \cot x\right) = \lim_{x\to 0}\left(\frac{\sin x - x\cos x}{x\sin x}\right) \quad \left(\to \frac{0}{0}\right)$

$\overset{*}{=} \lim_{x\to 0}\left(\frac{\cos x + x\sin x - \cos x}{x\cos x + \sin x}\right) = \lim_{x\to 0}\left(\frac{x\sin x}{x\cos x + \sin x}\right) \quad \left(\to \frac{0}{0}\right)$

$\overset{*}{=} \lim_{x\to 0}\left(\frac{x\cos x + \sin x}{-x\sin x + \cos x + \cos x}\right) = \frac{0}{2} = 0$

2. Evaluate the integrals:

a) $\displaystyle\int \cos 4x \cos 3x \, dx$ b) $\displaystyle\int \frac{x^5}{\sqrt{x^2+2}} \, dx$ c) $\displaystyle\int \frac{x(3-5x)}{(3x-1)(x-1)^2} \, dx$

a) $\displaystyle\int \cos 4x \cos 3x \, dx = \int \frac{1}{2}\left(\cos x + \cos 7x \right) dx$

$= \frac{1}{2} \sin x + \frac{1}{14} \sin 7x + C$

b) $\displaystyle\int \frac{x^5}{\sqrt{x^2+2}} \, dx$ let $x = \sqrt{2} \tan\theta$

$dx = \sqrt{2} \sec^2\theta \, d\theta$

$\sqrt{x^2+2} = \sqrt{2} \sec\theta$

$\displaystyle= \int \frac{4\sqrt{2} \tan^5\theta \cdot \sqrt{2} \sec^2\theta \, d\theta}{\sqrt{2} \sec\theta} = 4\sqrt{2} \int \tan^5\theta \sec\theta \, d\theta$

$\displaystyle= 4\sqrt{2} \int (\sec^2\theta - 1)(\sec^2\theta - 1) \sec\theta \tan\theta \, d\theta$

$\displaystyle= 4\sqrt{2} \left[\frac{\sec^5\theta}{5} - 2 \frac{\sec^3\theta}{3} + \sec\theta \right] + C$

$\displaystyle= \frac{(x^2+2)^{5/2}}{5} - 4\frac{(x^2+2)^{3/2}}{3} + 4\sqrt{x^2+2} + C$

c) $\displaystyle\frac{x(3-5x)}{(3x-1)(x-1)^2} = \frac{A}{3x-1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow 3x - 5x^2 = A(x^2 - 2x + 1) + B(3x^2 - 4x + 1) + C(3x-1)$

x^2: $-5 = A + 3B$ $\Rightarrow -5 = B + 3 + 3B$ $\Rightarrow B = -2$

x: $3 = -2A - 4B + 3C$ $\Rightarrow 3 = A - B$ or $A = B + 3$

1: $0 = A + B - C$ $\Rightarrow C = A + B$

$\therefore A = 1, \; C = -1$

$\displaystyle\int \frac{x(3-5x)}{(3x-1)(x-1)^2} dx = \int \frac{dx}{3x-1} - \int \frac{2dx}{x-1} - \int \frac{dx}{(x-1)^2}$

$= \frac{1}{3} \ln|3x-1| - 2\ln|x-1| + (x-1)^{-1} + C$

3. a) Use a comparison test to show that $\int_0^\infty x^r e^{-x}\,dx$ is convergent for all r.

b) Use mathematical induction to show that $\int_0^\infty x^n e^{-x}\,dx = n!$, $n = 1, 2, 3, \ldots$

a) given that $\lim_{x\to\infty} \frac{x^r}{e^{x/2}} \to 0$ for all r, there must be some number k st. $x^r < e^{x/2}$ for $x > k$

$$\therefore \int_0^\infty x^r e^{-x}\,dx = \int_0^k x^r e^{-x}\,dx + \int_k^\infty x^r e^{-x}$$

$$< \int_0^k x^r e^{-x}\,dx + \int_k^\infty e^{x/2} \cdot e^{-x}\,dx$$

$$= \underbrace{\int_0^k x^r e^{-x}\,dx}_{\text{not improper}} + \underbrace{\int_k^\infty e^{-x/2}\,dx}_{= -2e^{-x/2}\Big]_k^\infty} = +2e^{-k/2}$$

$\therefore \int_0^\infty x^r e^{-x}\,dx$ converges for all values of r

b) $n=1$: $\int_0^\infty x e^{-x}\,dx$ let $u = x$, $du = dx$; $dv = e^{-x}dx$, $v = -e^{-x}$

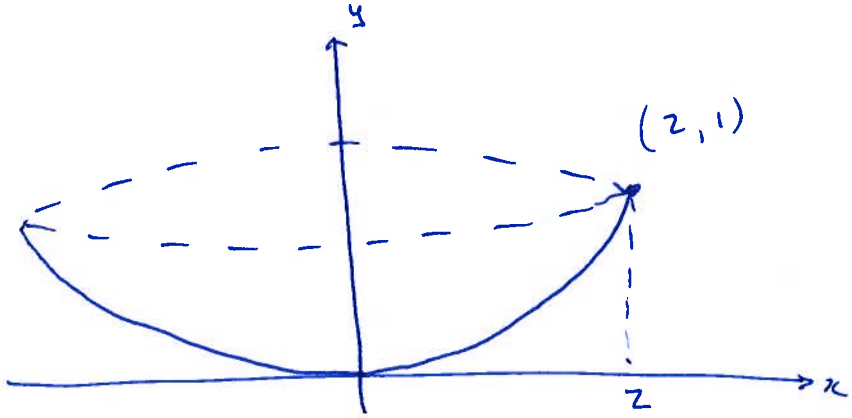
$$= \left[-x e^{-x}\right]_0^\infty + \int_0^\infty e^{-x}\,dx = 0 + \left[-e^{-x}\right]_0^\infty = 1$$

assume $\int_0^\infty x^n e^{-x}\,dx = n!$ $\Rightarrow$ show $\int_0^\infty x^{n+1} e^{-x}\,dx = (n+1)!$

$\int_0^\infty x^{n+1} e^{-x}\,dx$ let $u = x^{n+1}$, $du = (n+1)x^n\,dx$; $dv = e^{-x}dx$, $v = -e^{-x}$

$$= \left[-x^{n+1} e^{-x}\right]_0^\infty + (n+1)\int_0^\infty x^n e^{-x}\,dx = 0 + (n+1)\cdot n! = (n+1)!$$

4. Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = \dfrac{x^2}{4}$ for $0 \le x \le 2$ about the y-axis.



(A) $y = \dfrac{x^2}{4} \Rightarrow x = 2\sqrt{y} \quad \therefore x' = 2 \cdot \dfrac{1}{2\sqrt{y}} = \dfrac{1}{\sqrt{y}}$

$$A = \int 2\pi x \, ds = \int_0^1 2\pi x \sqrt{1 + (x')^2} \, dy = \int_0^1 2\pi \cdot 2\sqrt{y} \sqrt{1 + \frac{1}{y}} \, dy$$

$$= 4\pi \int_0^1 \sqrt{y+1} \, dy = 4\pi \left[\frac{2}{3} (y+1)^{3/2} \right]_0^1 = \frac{8\pi}{3} \left[2^{3/2} - 1 \right]$$

$$= \frac{8\pi}{3} (2\sqrt{2} - 1)$$

(B) $y = \dfrac{x^2}{4} \quad \therefore y' = \dfrac{x}{2}$

$$A = \int 2\pi x \sqrt{1 + (y')^2} \, dx = \int_0^2 2\pi x \sqrt{1 + \frac{x^2}{4}} \, dx$$

$$= \int_0^2 \pi x \sqrt{x^2 + 4} \, dx = \pi \left[\frac{2}{3} (x^2+4)^{3/2} \cdot \frac{1}{2} \right]_0^2$$

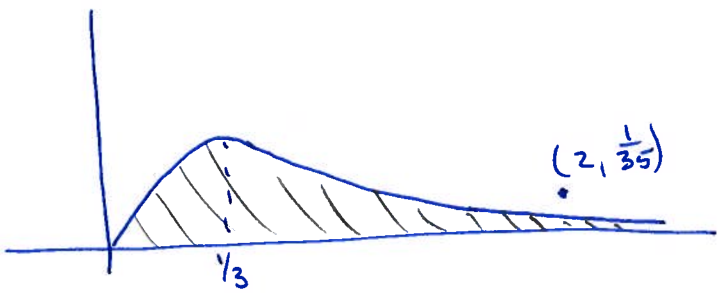
$$= \frac{\pi}{3} \left(8^{3/2} - 4^{3/2} \right) = \frac{\pi}{3} \cdot 4^{3/2} \left(2^{3/2} - 1 \right) = \frac{8\pi}{3} (2\sqrt{2} - 1)$$

5. Find the centroid of the infinitely long region lying between the x-axis and the curve $y = \dfrac{x}{(x+1)^4}$, and to the right of the y-axis. Provide a sketch of the region showing the location of the centroid.

$f(x) = x(x+1)^{-4}$

$f'(x) = (x+1)^{-4} + x(-4)(x+1)^{-5} = \dfrac{1-3x}{(x+1)^5}$

$f'(x) = 0 \Rightarrow x = \frac{1}{3}$; $f' > 0$ for $x < \frac{1}{3}$, $f' < 0$ for $x > \frac{1}{3}$



$\text{Area} = \displaystyle\int_0^\infty \frac{x}{(x+1)^4}\,dx$ let $u = x+1$, $du = dx$, $x = u-1$

$= \displaystyle\int_1^\infty \frac{(u-1)\,du}{u^4} = \int_1^\infty \frac{du}{u^3} - \int_1^\infty \frac{du}{u^4} = \left[-\frac{1}{2}\frac{1}{u^2}\right]_1^\infty - \left[-\frac{1}{3}\frac{1}{u^3}\right]_1^\infty = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\bar{x}A = \displaystyle\int_0^\infty \frac{x^2}{(x+1)^4}\,dx$ let $u = x+1$, $du = dx$, $x^2 = u^2 - 2u + 1$

$= \displaystyle\int_1^\infty \frac{(u^2-2u+1)}{u^4}\,du = \int_1^\infty \left(\frac{1}{u^2} - \frac{2}{u^3} + \frac{1}{u^4}\right)du = \left[-\frac{1}{u} + \frac{1}{u^2} - \frac{1}{3u^3}\right]_1^\infty$

$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$ $\longrightarrow$ $\boxed{\bar{x} = \dfrac{1/3}{1/6} = 2}$

$\bar{y}A = \displaystyle\int_0^\infty \frac{1}{2}\left(\frac{x}{(x+1)^4}\right)^2 dx = \int_1^\infty \frac{1}{2}\,\frac{u^2-2u+1}{u^8}\,du = \frac{1}{2}\left[-\frac{1}{5u^5} + \frac{2}{6u^6} - \frac{1}{7u^7}\right]_1^\infty$

$= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right) = \frac{1}{2}\left(\frac{21-35+15}{105}\right) = \frac{1}{210}$ $\Rightarrow$ $\boxed{\bar{y} = \dfrac{1/210}{1/6} = \dfrac{1}{35}}$

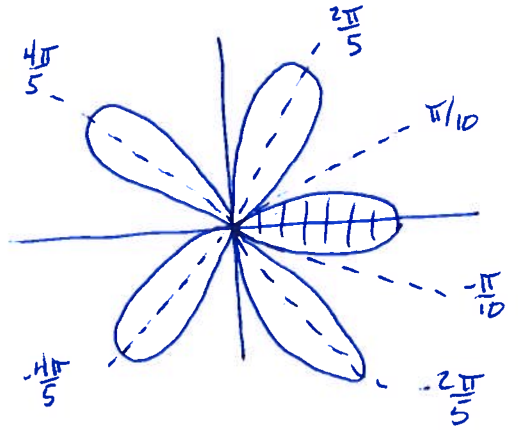
6. Sketch the curves and regions indicated, and find an integral representing the area of the region. Do not evaluate the integrals.

a) The region inside one petal of the curve $r = \cos 5\theta$.

b) The region that lies inside the cardioid $r = 1 - \cos\theta$ but outside the circle $r = 3\cos\theta$.

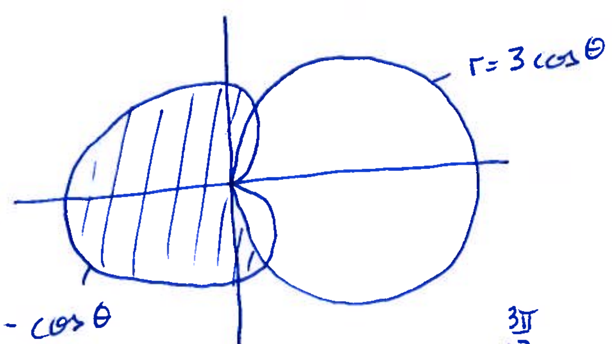
c) The region that lies inside both the circle $r = \sqrt{2}\sin\theta$ and inside the lemniscate $r^2 = \sin 2\theta$.

a)



$$A = \int_{-\pi/10}^{\pi/10} \frac{1}{2}(\cos 5\theta)^2\, d\theta$$

b)



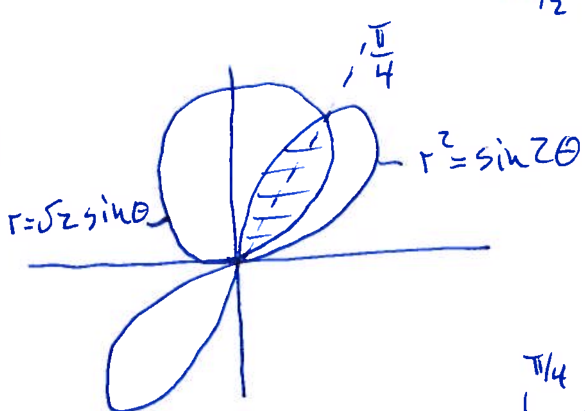
Intersection: $1 - \cos\theta = 3\cos\theta$

$\cos\theta = \frac{1}{4}$

$\Rightarrow \theta = \cos^{-1}\frac{1}{4} \quad (\pm 75.5°)$

$$A = \int_{\pi/2}^{3\pi/2} \frac{1}{2}(1-\cos\theta)^2\, d\theta + 2\cdot\frac{1}{2}\int_{\cos^{-1}\frac{1}{4}}^{\pi/2} \left[(1-\cos\theta)^2 - 9\cos^2\theta\right] d\theta$$

c)



Intersection:

$\sqrt{2}\sin\theta = \sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$$A = \frac{1}{2}\int_0^{\pi/4} (\sqrt{2}\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \sin 2\theta\, d\theta$$

7. Sketch a graph of the parametric curve: $x = t^2 - 2$

$$y = t^3 - t$$

Show all vertical and horizontal tangents, the tangents at $(-1, 0)$, and identify the asymptotic behaviour.

$x = t^2 - 2$

$x' = 2t$

$x' = 0 \Rightarrow t = 0 \Rightarrow (-2, 0)$

(vertical tangent)

$y = t^3 - t$

$y' = 3t^2 - 1$

$y' = 0 \Rightarrow t = \pm \frac{1}{\sqrt{3}} \Rightarrow \left(-\frac{5}{3}, \pm \frac{2}{3\sqrt{3}}\right)$

(horizontal tangents)

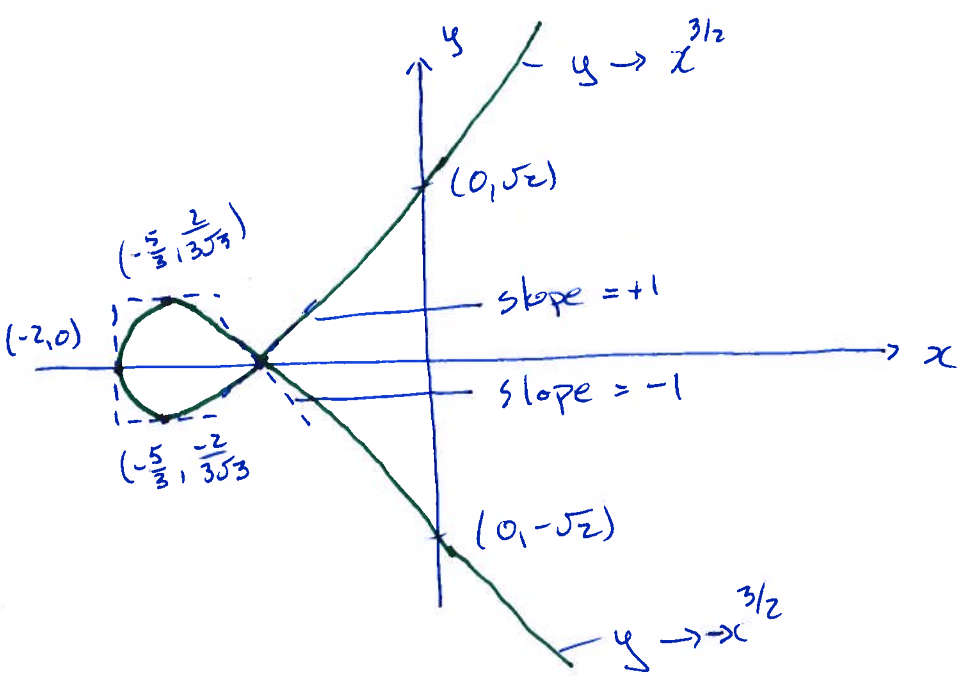
Intercepts: $x = 0 \Rightarrow t = \pm\sqrt{2} \Rightarrow (0, \pm\sqrt{2})$

$y = 0 \Rightarrow t = 0, t = \pm 1 \Rightarrow (-2, 0)$ & $(-1, 0)$

slope at $x = -1$: $t = \pm 1$: $t = 1 \Rightarrow \frac{y'}{x'} = \frac{2}{2} = 1$

$t = -1 \Rightarrow \frac{y'}{x'} = \frac{2}{-2} = -1$

Asymptotic behaviour: As $t \to \pm\infty$: $\left.\begin{matrix} x \to t^2 \\ y \to t^3 \end{matrix}\right\} y \sim \pm x^{3/2}$



8. Evaluate the integral $\int_0^1 e^x\,dx$ using the Reimann definition of the integral (not the Fundamental Theoram of Calculus).

Hint: to find the sum of a geometric sequence $S = \sum_{i=1}^{n} r^i$, take the difference between S and rS.

uniform partition : $\Delta x = \frac{1}{n}$

RH end point : $x_i^* = x_i = i \cdot \frac{1}{n}$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} e^{i/n}$$

$$S = \sum_{i=1}^{n} r^i = r + r^2 + r^3 + \dots + r^n$$

$$r.S = r^2 + r^3 + r^4 + \dots + r^n + r^{n+1}$$

$$\Rightarrow S - rS = r + (r^2 - r^2) + (r^3 - r^3) + \dots + (r^n - r^n) - r^{n+1}$$

$$S(1-r) = r - r^{n+1}$$

$$\therefore S = \frac{r - r^{n+1}}{1-r}$$

let $r = e^{1/n} \Rightarrow r^i = e^{i/n}$ $\therefore S = \dfrac{e^{1/n} - e^{\frac{n+1}{n}}}{1 - e^{1/n}}$

$$\therefore \int_0^1 e^x\,dx = \lim_{n\to\infty} \frac{1}{n}\left(\frac{e^{1/n} - e^{\frac{n+1}{n}}}{1-e^{1/n}}\right) = \lim_{n\to\infty} (1-e)\,\frac{e^{1/n}\cdot\frac{1}{n}}{1-e^{1/n}}$$

let $t = \frac{1}{n}$ $\therefore t \to 0^+$ as $n \to \infty$

$$= \lim_{t\to 0^+} (1-e)e^t \cdot \frac{t}{1-e^t} = \lim_{t\to 0^+}\left[(1-e)e^t\right] \cdot \underbrace{\lim_{t\to 0^+}\left[\frac{t}{1-e^t}\right]}_{\to \frac{0}{0}}$$

$$= (1-e) \lim_{t\to 0^+} \frac{t}{1-e^t} \overset{*}{=} (1-e) \lim_{t\to 0^+} \frac{1}{-e^t} = (1-e)(-1)$$

$$\boxed{= e - 1}$$

9. Evaluate the integral: $\int \frac{dx}{x^2+2x+1+\sqrt{x+1}}$

$$\int \frac{dx}{x^2+2x+1+\sqrt{x+1}} = \int \frac{dx}{(x+1)^2+\sqrt{x+1}}$$

let $u^2 = x+1$
$2u\,du = dx$

$$= \int \frac{2u\,du}{u^4+u} = 2\int \frac{du}{u^3+1} = 2\int \frac{du}{(u+1)(u^2-u+1)}$$

$$\Rightarrow \frac{1}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1} \Rightarrow 1 = A(u^2-u+1)+(Bu+C)(u+1)$$

u^2: $A+B=0$
u: $-A+B+C=0$
1: $A+C=0$

$\left.\begin{matrix} 2B+C=0 \\ -B+C=1 \end{matrix}\right\} -B-2B=1 \rightarrow$

$B = -1/3$
$A = 1/3$
$C = 2/3$

$$\therefore 2\int \frac{du}{(u+1)(u^2-u+1)} = \frac{2}{3}\int \frac{du}{u+1} - \frac{2}{3}\int \frac{u-2}{u^2-u+1}\,du$$

$$= \frac{2}{3}\ln|1+u| - \frac{1}{3}\int \frac{2u-1}{u^2-u+1}\,du + \frac{1}{3}\int \frac{3\,du}{u^2-u+1}$$

$$= \frac{2}{3}\ln|1+u| - \frac{1}{3}\ln|u^2-u+1| + \int \frac{du}{(u-1/2)^2+\frac{3}{4}}$$

$$= \frac{2}{3}\ln|1+u| - \frac{1}{3}\ln|u^2-u+1| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u-1/2}{\sqrt{3}/2}\right) + C$$

$$= \frac{2}{3}\ln\left(1+\sqrt{x+1}\right) - \frac{1}{3}\ln\left|x+2-\sqrt{x+1}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt{x+1}-1}{\sqrt{3}}\right) + C$$