University of Toronto

Faculty of Applied Science and Engineering

Final Examination, 9:30 am 15 December 2017

First Year, Program 5

MAT194F Calculus I

Exam Type A

No aids of any kind are permitted.
No calculators of any kind are permitted.

Time allowed: 2 ½ hours.
There are 10 questions.
You can write on both sides of each page. There are also 2 extra pages at the end that you can use.

Examiners: P.C. Stangeby and F. Al-Faisal

Family Name: ______________________________

Given Name: ______________________________

Student #: ______________________________

FOR MARKER USE ONLY		
Question	Marks	Earned
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
TOTAL	100	/100

1. (a) Find the derivative of: $2x^2$, $\sin(\sqrt{x})$, $\ln(x^2)$, e^{-2x}, $3^{\sqrt{x}}$.

(b) Find the anti-derivative of: $3x^3$, $\sin(2x)$, $2x^2e^{x^3}$, $(4+x^2)^{-1}$, 3^x.

2. (a) Provide a $\delta - \varepsilon$ proof that $\lim_{x \to 2} x^2 = 4$.

   (b) Prove that $\lim_{x \to 2} x^2 \neq 6$ using a proof by contradiction: assume that $\lim_{x \to 2} x^2 = 6$ and use a $\delta - \varepsilon$ type of proof to show that this results in a contradiction.

3. Sketch the curve $y = x(x-4)^3$. Indicate on the sketch: intercepts with the 2 axes, if they exist; the regions where y is increasing, decreasing, concave up, concave down; local and absolute maxima and minima, if they exist; points of inflexion if they exist; vertical asymptotes, horizontal asymptotes and vertical tangents if they exist; symmetry or periodicity if they exist.

4. If 1200 cm^2 of thin sheet metal is available to make a box with a square base and an open top, find the largest possible volume of the box. What is the smallest possible volume and what are the dimensions of the box in that case?

5. Compute: (a) $\lim_{x \to 0} \frac{2x - \sin^{-1}(2x)}{x^3}$ (b) $\lim_{x \to \infty} \left(1 + \frac{\pi}{2} - \tan^{-1} x\right)^x$ (c) $\lim_{x \to \infty} x\left(e^{5/x} - 1\right)$.

6. Compute: (a) $\int x^3\sqrt{x^2+9}dx$ (b) $\int \frac{dx}{1-x^2+x}$ (c) $\int_0^1 \frac{dx}{1+e^{-x}}$ (d) $\int_0^1 \frac{dx}{x\sqrt{1-(\ln x)^2}}$

(e) $\int \sinh(2x)\cosh(x)dx$. For (e) first prove $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

7. (a) Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, for $x \in [0,1]$.

(b) Find the area of the region bounded by the curves $y = 2$ and $y = 2\ln x$ for $x \in [1, e]$. Assume that the antiderivative of $\ln x$ is not known.

(c) For $r > 0$ let $V(r)$ denote the volume of the solid obtained by rotating about the y-axis the curve $= \frac{1}{1+x^2}$ between $x = r$ and $x = r + 1$. Find the value of r that maximizes $V(r)$.

8. The minute and hour hands on a clock are 8 cm and 4 cm long, respectively. How fast is the distance between the tips of the hands changing at 3 o'clock?
Hint: the Law of Cosines $a^2 = b^2 + c^2 - 2bc\cos A$.

9. (a) Solve the differential equation $y'' - 4y' + 13y = 0$ with initial conditions
$y(0) = 0, y'(0) = 3$.

(b) Find the general solution to the differential equation $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$.

Hint: let $x = e^t$.

10. (a) Find all continuous functions $f(x)$ defined for all real x such that $\left(f(x)\right)^2 = \int_0^x \frac{tf(t)}{1+t^2}\,dt$.

(b) Suppose that for all real x, $g(x)$ satisfies $g^{(n)}(x) = g^{(n-1)}(x)$ and $g^{(n-1)}(0) = -1$, where $g^{(n)}(x)$ is the n^{th} derivative of $g(x)$.

Does $\lim\limits_{x\to\infty} \frac{g(x)}{e^x}$ exist? If so, find it; if not, explain.