

An Exploration of Bifurcation Theory in Dynamical Systems

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1 Introduction

Consider the theory of ordinary differential equations. In this theory, equations of the form $y'(t) = f(y(t))$, where f is a function of y only, are called autonomous differential equations. Tracing the graphs of solutions to an autonomous equation in the ty -plane, we might find some constant c where $y(t) = c$ is a solution. It is easy to see that such constants are precisely the zeroes of the function f . They are called *fixed points* or *equilibria* of the system governed by the equation. Varying the initial condition $y(0)$ may produce convergence toward or divergence away from the fixed points. Furthermore, continuously varying the defining function f may affect the behavior of solutions around these points. This paper looks at the theory of fixed points of dynamical systems, which is a generalization of these notions to higher-dimensional systems.

2 Dynamical systems overview

A *dynamical system* is a manifold M endowed with a set of functions $\Phi^t : M \rightarrow M$ for all $t \in T$, the time dimension. Each function describes the state of the system after a certain time t . In this paper, I will be looking exclusively at the case where $M = \mathbb{R}^n$, $T = \mathbb{R}^+$, and Φ^t is defined by a differential equation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} : M \rightarrow M$ is a smooth function along all input dimensions and \mathbf{x} is an unknown function mapping $T \rightarrow M$.

First assume that f is continuous and $\mathbf{f}(\mathbf{c})$ is defined for all $\mathbf{c} \in M$. By the Peano-Cauchy theorem, the initial value problem

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \mathbf{x}(0) = \mathbf{c} \tag{1}$$

has a solution for all $\mathbf{c} \in M$. Furthermore, (1) has a unique solution in t for all $\mathbf{c} \in M$. Call each of these solutions $x_c(t)$. For all $t \in T$ and $c \in M$, define $\Phi^t(c) = \mathbf{x}_c(t)$. Note that if f is plotted as a vector field, then Φ^t describes the location of each point after following the vector field for a period of time t .

2.1 Fixed points

Fixed points are points $\mathbf{c} \in M$ such that $\mathbf{f}(\mathbf{c}) = \mathbf{0}$. Whenever \mathbf{c} is a fixed point, we have $\mathbf{x}_{\mathbf{c}}(t) = \mathbf{c}$ and hence $\Phi^t(\mathbf{c}) = \mathbf{c}$ for all $t \in T$.

3 Linear dynamical systems

First we consider the case where $\mathbf{f}(\mathbf{c}) = A\mathbf{c}$, where A is an n by n matrix. The point $\mathbf{c} = \mathbf{0}$ is a fixed point. Assume A is invertible; then $\mathbf{c} = \mathbf{0}$ is the only fixed point. How does Φ^t behave for \mathbf{c} near $\mathbf{0}$?

We first show that the set S of functions $\mathbf{x}_{\mathbf{c}}$ is a subspace of the vector space of all functions $\mathbf{x} : t \rightarrow M$ by showing that it is closed under addition and scalar multiplication of functions. First, let $\mathbf{c}_1, \mathbf{c}_2 \in M$. Then $\mathbf{x}'_{\mathbf{c}_1}(t) = A\mathbf{x}_{\mathbf{c}_1}(t)$ and $\mathbf{x}'_{\mathbf{c}_2}(t) = A\mathbf{x}_{\mathbf{c}_2}(t)$. Let $\mathbf{y}(t) = \mathbf{x}_{\mathbf{c}_1}(t) + \mathbf{x}_{\mathbf{c}_2}(t)$. Then $\mathbf{y}'(t) = \mathbf{x}'_{\mathbf{c}_1}(t) + \mathbf{x}'_{\mathbf{c}_2}(t) = A\mathbf{x}_{\mathbf{c}_1}(t) + A\mathbf{x}_{\mathbf{c}_2}(t) = A(\mathbf{x}_{\mathbf{c}_1}(t) + \mathbf{x}_{\mathbf{c}_2}(t)) = A\mathbf{y}(t)$ and from the initial condition $\mathbf{y}(0) = \mathbf{c}_1 + \mathbf{c}_2$, it follows that $\mathbf{y}(t) = \mathbf{x}_{\mathbf{c}_1 + \mathbf{c}_2}(t)$ and thus $\mathbf{y} \in S$. Next, let $\mathbf{c} \in M$ and $a \in \mathbb{R}$. Then $\mathbf{x}'_{\mathbf{c}}(t) = A\mathbf{x}_{\mathbf{c}}(t)$. Let $\mathbf{z}(t) = a\mathbf{x}_{\mathbf{c}}(t)$. Then $\mathbf{z}'(t) = a\mathbf{x}'_{\mathbf{c}}(t) = a \cdot (A\mathbf{x}_{\mathbf{c}}(t)) = A(a\mathbf{x}_{\mathbf{c}}(t)) = A\mathbf{z}(t)$. It follows that $\mathbf{z}(t) = \mathbf{x}_{a\mathbf{c}}(t)$ and thus $\mathbf{z} \in S$. Finally, notice that the function $\mathbf{x}_{\mathbf{0}}(t) = \mathbf{0}$ is the zero vector of S . It follows that S is a vector space.

Consider the points \mathbf{c} such that $\mathbf{f}(\mathbf{c})$ is parallel to \mathbf{c} . These are the eigenvectors of A . When \mathbf{c} is an eigenvector corresponding to a real eigenvalue λ , we have $\mathbf{x}'_{\mathbf{c}}(t) = \lambda\mathbf{x}_{\mathbf{c}}(t)$. In fact, because \mathbf{f} is a linear transformation, for any scalar multiple \mathbf{d} of \mathbf{c} it holds that $\mathbf{x}'_{\mathbf{d}}(t) = \lambda\mathbf{x}_{\mathbf{d}}(t)$. Solving the differential equation we obtain $\mathbf{x}_{\mathbf{c}}(t) = e^{\lambda t}\mathbf{c}$. We see quantitatively that $\mathbf{x}_{\mathbf{c}}(t)$ converges if and only if λ is negative.

Now consider $\mathbf{c} \in M$ such that $\mathbf{c} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \dots + a_m\mathbf{v}_m$, where $m \leq n$ and \mathbf{v}_1 through \mathbf{v}_m are eigenvectors of A corresponding to the real distinct eigenvalues λ_1 to λ_m . Because S is closed under linear combinations, we see that $\mathbf{x}_{\mathbf{c}} \in S$ and $\mathbf{x}_{\mathbf{c}}(t) = a_1\mathbf{x}_{\mathbf{v}_1}(t) + a_2\mathbf{x}_{\mathbf{v}_2}(t) + \dots + a_m\mathbf{x}_{\mathbf{v}_m}(t) = a_1e^{\lambda_1 t}\mathbf{v}_1 + a_2e^{\lambda_2 t}\mathbf{v}_2 + \dots + a_me^{\lambda_m t}\mathbf{v}_m$. If all the eigenvalues are real, \mathbf{c} can range over M with appropriate choices of scalars a_1, a_2, \dots, a_m and vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ from each eigenspace. From the above expression of $\mathbf{x}_{\mathbf{c}}(t)$, we can see that if all eigenvalues are real negative numbers, then $\mathbf{x}_{\mathbf{c}}(t)$ converges to $\mathbf{0}$ as $t \rightarrow \infty$ for all \mathbf{c} , but if some eigenvalue λ is a non-negative real, then $\mathbf{x}_{\mathbf{v}}(t)$ does not converge where \mathbf{v} is an eigenvector corresponding to λ .

Solutions of the system $\mathbf{x}' = A\mathbf{x}$ may take different forms, depending on the characteristics of the eigenvalues.

Case 1: Real unique eigenvalues

For each real unique eigenvalue λ with associated eigenvector \mathbf{v} , $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a

solution.

Case 2: Complex eigenvalues

If $\lambda_{re} \pm i\lambda_{im}$ is a pair of complex conjugate eigenvalues with eigenvectors $\mathbf{v}_{re} \pm i\mathbf{v}_{im}$, then there are linearly independent solutions of the form $\mathbf{x}_1 = e^{\lambda_{re}t}(\mathbf{v}_{re} \cos(\lambda_{im}t) - \mathbf{v}_{im} \sin(\lambda_{im}t))$ and $\mathbf{x}_2 = e^{\lambda_{re}t}(\mathbf{v}_{re} \sin(\lambda_{im}t) + \mathbf{v}_{im} \cos(\lambda_{im}t))$ corresponding to initial values \mathbf{v}_{re} and \mathbf{v}_{im} respectively.

Case 3: Real repeated eigenvalues with the same geometric multiplicities

If λ is an eigenvalue with multiplicity m associated with an m -dimensional eigenspace with basis vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, then $\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1, \mathbf{x}_2 = e^{\lambda t}\mathbf{v}_2, \dots, \mathbf{x}_m = e^{\lambda t}\mathbf{v}_m$ are linearly independent solutions.

Case 4: Real repeated eigenvalues with smaller geometric multiplicities

If λ is an eigenvalue with multiplicity m associated with a k -dimensional eigenspace, where $k < m$, with basis vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$, then k of the solutions are given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1, \mathbf{x}_2 = e^{\lambda t}\mathbf{v}_2, \dots, \mathbf{x}_k = e^{\lambda t}\mathbf{v}_k$. The other solutions are found by choosing $m - k$ vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{m-k}$ to satisfy the equations:

$$(A - \lambda I)\mathbf{u}_1 = \mathbf{v}_1$$

$$(A - \lambda I)\mathbf{u}_2 = \mathbf{u}_1$$

$$(A - \lambda I)\mathbf{u}_3 = \mathbf{u}_2$$

...

$$(A - \lambda I)\mathbf{u}_{m-k} = \mathbf{u}_{m-k-1}$$

In this case, \mathbf{v} can be any linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ such that the equations have solutions.

Case 5: Complex repeated eigenvalues with the same geometric multiplicity

If $\lambda_{re} \pm i\lambda_{im}$ is a conjugate pair of eigenvalues with multiplicity m , associated with a $2m$ -dimensional eigenspace with basis vectors $\mathbf{v}_{1,re} \pm i\mathbf{v}_{1,im}, \mathbf{v}_{2,re} \pm i\mathbf{v}_{2,im}, \dots, \mathbf{v}_{m,re} \pm i\mathbf{v}_{m,im}$, then $\mathbf{x}_1 = e^{\lambda_{re}t}(\mathbf{v}_{1,re} \cos(\lambda_{im}t) - \mathbf{v}_{1,im} \sin(\lambda_{im}t)), \mathbf{x}_2 = e^{\lambda_{re}t}(\mathbf{v}_{2,re} \cos(\lambda_{im}t) - \mathbf{v}_{2,im} \sin(\lambda_{im}t)) \dots \mathbf{x}_m = e^{\lambda_{re}t}(\mathbf{v}_{m,re} \cos(\lambda_{im}t) - \mathbf{v}_{m,im} \sin(\lambda_{im}t))$ and $\mathbf{x}_{m+1} = e^{\lambda_{re}t}(\mathbf{v}_{1,re} \sin(\lambda_{im}t) + \mathbf{v}_{1,im} \cos(\lambda_{im}t)), \mathbf{x}_{m+2} = e^{\lambda_{re}t}(\mathbf{v}_{2,re} \sin(\lambda_{im}t) + \mathbf{v}_{2,im} \cos(\lambda_{im}t)) \dots \mathbf{x}_{2m} = e^{\lambda_{re}t}(\mathbf{v}_{m,re} \sin(\lambda_{im}t) + \mathbf{v}_{m,im} \cos(\lambda_{im}t))$. These solutions correspond to initial values of $\mathbf{v}_{1,re}, \mathbf{v}_{2,re}, \dots, \mathbf{v}_{m,re}, \mathbf{v}_{1,im}, \mathbf{v}_{2,im}, \dots, \mathbf{v}_{m,im}$ respectively.

Case 6: Complex repeated eigenvalues with smaller geometric multiplicities

3.1 Phase Changes in Linear Systems

In a family of dynamical systems $\mathbf{x}' = f(\mathbf{x}, \mu)$, a **phase change** occurs when a stable equilibrium point becomes semistable or unstable under continuous variation of f . Here, we describe the dynamical system as changing with variable parameter μ .

When the system is linear, we have $\mathbf{x}' = A[\mu]\mathbf{x}$, where A is a matrix as a function of μ . When the entries of A vary continuously, several types of changes to the roots of characteristic polynomial can occur, causing changes to the nature of the fixed point $\mathbf{x}_0 = \mathbf{0}$. We denote the polynomial by $p[\mu](\lambda)$ and its roots by λ_0, λ_1 , etc.

1. $\lambda_0 \in \mathbb{R}: \lambda_0 < 0 \leftrightarrow \lambda_0 > 0$. In this case, $\mathbf{x}_0 = \mathbf{0}$ may transition from a stable

point to a semistable point, or from a semistable point to an unstable one, or vice versa. Scalar multiples of \mathbf{v}_0 , where \mathbf{v}_0 is the eigenvector associated with λ_0 , go from being stable initial conditions to unstable ones.

2. $\lambda_0 \dots \lambda_k \in \mathbb{R}$: $\lambda_0 \dots \lambda_k < 0 \leftrightarrow \lambda_0 \dots \lambda_k > 0$. Multiple roots can pass through zero at the same time. When this happens, a multidimensional space of inputs goes from stable to unstable.

3. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda}_0; Re(\lambda_0) < 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0, \lambda_1 < 0$. The origin stays a stable point for the appropriate space of initial conditions, but solutions change from having spiral trajectories to straight ones.

4. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda}_0; Re(\lambda_0) < 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 < 0; \lambda_1 > 0$. The same as #3 but spiral solutions become semistable solutions instead of stable ones.

5. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda}_0; Re(\lambda_0) > 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0, \lambda_1 > 0$. The origin is an unstable point for the appropriate space of initial conditions, but solutions change from having spiral trajectories to straight ones.

6. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda}_0; Re(\lambda_0) > 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 < 0; \lambda_1 > 0$. The same as #5 but spiral solutions become semistable solutions instead of unstable ones.

7. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) > 0; Im(\lambda_1) = -Im(\lambda_0); Re(\lambda_0) = Re(\lambda_1) = 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 > 0; \lambda_1 < 0$. Solutions in the 2D space of appropriate initial conditions change from having metastable orbitals around the origin to semistable trajectories.

If $M = \mathbb{R}^2$, the matrix $A[\mu]$ describing the system can be written as:

$$A[\mu] = \begin{bmatrix} f_{1,1}(\mu) & f_{1,2}(\mu) \\ f_{2,1}(\mu) & f_{2,2}(\mu) \end{bmatrix}$$

Then, the roots of the equation can be described by two parameters:

$$p(\mu) = -\frac{f_{1,1}(\mu) + f_{2,2}(\mu)}{2}$$

$$q(\mu) = \frac{f_{1,1}(\mu)^2 - 2f_{1,1}(\mu)f_{2,2}(\mu) + 2f_{1,2}(\mu)f_{2,1}(\mu) + f_{2,2}(\mu)^2}{4}$$

The possible states of the system listed above correspond to various conditions on p and q . If $q(\mu) < 0$, then the eigenvalues are complex with real part equal to $p(\mu)$. If $q(\mu) = 0$, then the eigenvalues are both equal to $p(\mu)$. If $q(\mu) > 0$, then the eigenvalues are real and positive if $q(\mu) < p(\mu)^2$ and $p(\mu) > 0$, real and negative if $q(\mu) < p(\mu)^2$ and $p(\mu) < 0$, or real with opposite signs if $q(\mu) > p(\mu)^2$.

4 Extension to nonlinear dynamical systems

In a continuous dynamical system defined with a nonlinear function of \mathbf{x} , the rectification theorem guarantees that if $\mathbf{f}(\mathbf{x}_0) \neq 0$, then the streamlines in the neighborhood of \mathbf{x}_0 can be converted into parallel lines using a smooth change

of coordinates. Similarly, where $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$, the vectors in the neighborhood of \mathbf{x}_0 can be approximated by $\mathbf{x}' = J(\mathbf{x} - \mathbf{x}_0)$, where $J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is the Jacobian matrix of \mathbf{f} at $\mathbf{x} = \mathbf{x}_0$.

The properties of the corresponding linear system $\mathbf{x}' = J\mathbf{x}$ are similar to those of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. If $\mathbf{x} = \mathbf{0}$ is a stable equilibrium point in the linear system (that is, if all eigenvalues of J have negative real part), then $\mathbf{x} = \mathbf{x}_0$ is a stable equilibrium point in the non-linear system with eigenvector $\mathbf{x} = \mathbf{v}$ corresponding to the trajectory of $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$ in the non-linear system. Similarly, if an eigenvalue λ of J has positive real part and corresponding eigenvector \mathbf{v} , then $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$ will be an unstable point in the non-linear system. However, if an eigenvalue λ is 0 or imaginary, indicating that the eigenvectors \mathbf{v} are fixed points or form stable orbits in the linear system respectively, points $\mathbf{x}_0 + \mathbf{v}$ may not necessarily undergo the same behavior in the non-linear system. For example, the system defined by

$$x'_1 = x_1^3 - x_2 x'_2 = x_1 + x_2^3$$

has Jacobian

$$J(\mathbf{x}) = \begin{bmatrix} 3x_1^2 & -1 \\ 1 & 3x_2^2 \end{bmatrix}$$

and fixed point $\mathbf{x}_0 = \mathbf{0}$, giving Jacobian

$$J = J(\mathbf{0}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

. This matrix has eigenvalues $\lambda_0, \lambda_1 = \pm i$, indicating that the solutions to the linear system orbit the origin counterclockwise. However, the solutions $\mathbf{x}_{\mathbf{c}}(t)$ to the non-linear system spiral farther and farther outward for any value $\mathbf{c} \neq \mathbf{0}$, and thus, with the exception of $\mathbf{x}_0(t)$, every solution is unstable.

For other systems, such as the system defined by $\mathbf{x}' = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 \\ x_1 + x_2^2 \end{bmatrix}$, there are still periodic orbits, as suggested by the Jacobian matrix, but with different period varying with distance from the origin.

In the above case, the orbits can be proved to be periodic by considering x' as a vector field on \mathbb{R}^2 . Consider a solution $x(t) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and let t_0 be such that $x_1(t_0) = x_2(t_0)$. (From the picture, it is clear that this must be true for some value of t). Let c_a, c_b be points such that $c_{a1} = -c_{b2}$ and $c_{a2} = -c_{b1}$. Then $f_1(c_a) = c_{a1}^2 - c_{a2} = c_{b2}^2 + c_{b1} = f_2(c_b)$ and $f_2(c_a) = c_{a2}^2 + c_{a1} = c_{b1}^2 - c_{b2} = f_1(c_b)$.

4.1 Normal forms

For higher-order dynamical systems, fixed points can be analyzed using normal forms. The **normal form** of a dynamical system is another dynamical system described by a simpler equation that preserves the most important properties of the original system. If a system can be described by the equation

$\mathbf{x}' = A\mathbf{x} + \mathbf{a}_1(\mathbf{x}) + \mathbf{a}_2(\mathbf{x}) + \dots$, where A is an $n \times n$ matrix and $a_j(x)$ is an n -vector of homogeneous polynomials of degree $j+1$ whose variables are the entries of \mathbf{x} , then it is possible to introduce a change of variables $\mathbf{y}' = \mathbf{u}_1(y) + \mathbf{u}_2(y) + \dots$ such that the new expression \mathbf{y} is in some way simpler than the old one.

5 References

1. <https://arxiv.org/pdf/math/0111177.pdf>
2. <https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.63.991>
3. David C. Lay, Steven R. Lay, and Judi J. McDonald. *Linear Algebra and its Applications*. Fifth edition.
4. William E. Boyce, Richard C. DiPrima. *Elementary Differential Equations*. Ninth edition.
5. http://www.scholarpedia.org/article/Normal_forms