PGSS: Math Finance Notes

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1 Asset Classes

The class will focus on assets which are indistinguishable from each other in terms of uniqueness (i.e. one share of a stock is identical to another). indicates a topic we will focus on and? indicates one that may be applicable.

- Real Estate

! Cash/Currencies

! Stocks → Equities

- Artwork → Collectibles
- Vehicles → Durable goods

! Certificates of Deposit (CDs) & Bonds → Fixed Income

? Cryptocurrencies

- NFTs

? Precious metals → Commodities

2 Trading

- The lowest selling price on a share is the **Asking Price**.
- The highest buying price on a share is the **Bid Price**.
- The **Bid-Ask spread** is the difference between bid price and ask price.
- The bid-ask spread tends to be smaller in actively traded, larger stocks.

Derivative Securities

Securities that derive value from the values of other assets.

Forward Contract

An agreement on the parameters of an exchange at a specific date before that date.

Call/Stock Options

Gives a person the right (not an obligation) to sell an assets for a particular price on a particular day.

Put Options

Gives the right (but not the obligation) to sell an asset.

3 Mathematical Models of Financial Markets

Assumptions

1. Single price to buy or sell (no bid-ask spread)
2. Can buy or sell any amount without moving the price
3. No trading fees
4. No taxes

4 Interest

- Borrow some amount (**Principal**)
- Pay back at some later datePay an extra amount (**Interest**)
- Borrow P, repay $P + I$ at time $T > 0$ (**maturity**) at $t = 0$

Why do banks charge interest on loans?

1. To compensate for default risk
2. To compensate for inflation risk
3. 'Time value of money'
4. Compensate for 'opportunity cost'

Why do banks pay interest on deposits?

Investors demand it

Why do banks accept deposits?

To borrow at a low interest rate and loan at a higher rate using your money

4.1 Interest Payments

4.1.1 Simple Interest

- Proportional to the size of the principal
- Proportional to the loaning period length

Given a loan with principal P, maturity T, and interest rate r where maturity is in years: principal is in dollars (or some other currency), and the interest rate is a rate in units $\frac{1}{\text{year}}$ The formula for the total interest paid would be as follows:

$$I = r[\frac{1}{\text{year}}] \cdot P[\$] \cdot T[\text{years}]$$
$$I = rPT[\$]$$

The total payment would be

$$P + I = P + rPT$$
$$P + I = P(1 + rT)$$

Note These principles assume simple interest

Interest is earned uniformly throughout the loan. i.e. A_0 borrowed at $t = 0$ grows to $A_t = (1 + rT)A_0$

4.1.2 Compound Interest

Interest is *compounded* to the interest during the loan.

Example
\$100 borrowed for 1 year at interest rate r compounded quarterly (3 months)

$$A_0 = \$100$$
$$t = \frac{1}{4}$$
$$A_{\frac{1}{4}} = 100 + (\frac{100}{4})r$$
$$A_{\frac{1}{4}} = 100(1 + \frac{r}{4})$$

The above becomes the new principal

$$t = \frac{1}{2}$$
$$A_{\frac{1}{2}} = [100(1 + \frac{r}{4})] + \frac{r}{4}[100(1 + \frac{r}{4})]$$
$$A_{\frac{1}{2}} = (1 + \frac{r}{4})[100(1 + \frac{r}{4})]$$
$$A_{\frac{1}{2}} = 100(1 + \frac{r}{4})^2$$

The above becomes the new principal again

$$t = \frac{3}{4}$$
$$A_{\frac{3}{4}} = (1 + \frac{r}{4})[100(1 + \frac{r}{4})^2]$$
$$A_{\frac{3}{4}} = 100(1 + \frac{r}{4})^3$$

The above becomes the new principal again

$$A_1 = 100(1 + \frac{r}{4})^4$$

More generally

$$A_T = A_0(1 + \frac{r}{m})^{mT}$$

Where m is the number of compounding periods per year

Compound interest allows for exponential growth of interest.

The *quoted* interest rate depends on the compounding convention even if payments are the same.

4.1.3 Negative Rates

- Interests rates can be negative
- But $(1 + \frac{r}{12})^{12T}$ must be positive
- Thus mathematically $r > -12$

4.2 The time value of money

If you were to receive \$1000 one year from now, what is the value of that future payment today?

Note A deposit of $\frac{1000}{(1+\frac{r}{12})^{12}}$ it will grow to $(\frac{1000}{(1+\frac{r}{12})^{12}})(1 + \frac{r}{12})^{12}$ or \$1000 at $t = 1$

We say $\frac{1000}{(1+\frac{r}{12})^{12}}$ is the **present value** of 1000 to be paid one year from now. Thus you can take a loan for that amount and pay off the loan with interest using the credit from the \$1000 you are to receive in a year (it all cancels).

For the present value of several payments, add up the values of each individual payment

5 Fixed Income Securities

5.1 Zero Coupon Bonds

Zero Coupon Bonds make a single payment at a single time. They pay a face value f at the maturity $T > 0$.

If we have a bank with interest rate r the discounted present value of the ZCB is $P_0 = \frac{F}{(1+\frac{r}{12})^12T}$.

If we deposit $A_0 = \frac{F}{(1+\frac{r}{12})^12T}$ at $t = 0$ until T_1 then $A_T = F$.

5.2 Annuities

An **annuity** is a series of same-sized payments made at regular intervals. It will make payments of $\$A$, m times per year, for T years

Example An annuity makes payments of \$200 for 2 years. To rewrite it, you could make it a sum of ZCBs, each being a month apart from each other in 24 fixed payments (emulating the annuity).

This makes the present value of an annuity be

$$P_0^A = \sum_{i=1}^{24} \frac{F}{(1+\frac{r}{12})^i}$$

Example An annuity makes payments of \$500 quarterly for 1 year. The net present value is $P_0^{A_2} = \frac{500}{(1+\frac{r}{12})^3} + \frac{500}{(1+\frac{r}{12})^6} + \frac{500}{(1+\frac{r}{12})^9} + \frac{500}{(1+\frac{r}{12})^{12}}$ or $\sum_{i=1}^4 \frac{500}{(1+\frac{r}{12})^{3i}}$

5.3 Coupon Bonds

To be completed

6 Portfolios

A **portfolio** is a collection of assets and and rules for trading among them. In mathematical finance it is generally thought of as a situation fo assets bought and sold at specific times.

If we were to use X as a label for a portfolio, then X_t is the value of the portfolio at time t. The value x_0 is called the **initial capital** of the portfolio.

6.1 Arbitrage

A potential for risk free profit. An **arbitrage portfolio** is defines as a portfolio X such that $X_0 = 0$ and $X_T \geq 0$ with $X_T > 0$ is possible.