This paper outlines the architecture of the ''WARP Core deterministic graph rewrite engine, a high-performance, real-time simulation engine with bit-level perfect deterministism, embarassingly-high coucurrency, and first-class time travel, by construction.

1 Introduction

To conclude the **AIΩN Foundations Series**, we describe the construction of the WARP Core, a real-time deterministic graph rewrite simulation engine. This technology is already running this project's homepage, https://flyingrobots.dev, known herein as the "WARPSite"—a "website" powered by the WARP Paper I introduces WARP graphs as a minimal recursively nested state object [Ros25c]. Paper II defines a deterministic multiway semantics (via a two-plane DPO discipline) so that executions become replayable *worldlines* [Ros25b]. Paper III shows that deterministic worldlines admit a boundary representation: a *provenance payload* is sufficient to reconstruct the full interior derivation volume (*computational holography*) [Ros25a].

This paper addresses the remaining mathematical question: *how should a computation be compared across observers?* In practice, consumers rarely require a raw microstep-by-microstep derivation. Engineering use-cases demand derived views: summaries for interpretation, invariants for compilation, provenance for audit, and counterfactual branches for adversarial analysis. These viewpoints correspond to different *observers* of the same underlying history.

The correct comparison problem is not "which observer is right?" but:

> *Given two observers that emit different trace languages, what is the cost of translating between them under explicit resource constraints, and how much distortion is unavoidable?*

We operationalise this as a geometry on observer space: a distance defined by translator description length (MDL) plus trace distortion. This distance is *budgeted*: two observers may be equivalent under unbounded resources but far apart at finite time/memory budgets.

Context within the Series. Paper V is about ethics: what perfect provenance implies for accountability, privacy, and power. Paper VI is about architecture: how to implement the semantics as a system. Paper IV is therefore the final mathematics-oriented paper in the series. Accordingly, we take the opportunity to give the Ruliad connection and observer geometry the full formal weight they require, rather than deferring mathematical structure to later papers.

Contributions. The contributions of this paper are:

1. A stand-alone account of *history categories* for WARP rewriting, relating deterministic worldlines to multiway systems (§2, §6).

2. A formal definition of *observers* as resource-bounded functors out of $\mathrm{Hist}(\mathcal{U}, R)$ into an observation space, including boundary-versus-bulk observer pairs induced by holography (§3).

3. A translation framework between observers, equipped with MDL description length [Ris78] and trace distortion, together with explicit assumptions needed for compositionality (§4).

4. The definition of the *rulial distance* $D_{\tau,m}$ and its core properties: non-negativity, symmetry, monotonicity under budget relaxation, and a triangle inequality up to a constant overhead inherited from prefix coding, together with a Lawvere-metric/enriched-category interpretation of directed cost (§5).

5. A formalisation of the Chronos–Kairos–Aion triad as a three-layer time model embedded in the multiway space, and an interpretation of rulial distance as "frame separation" in the Ruliad (§6).

6. A minimal temporal logic aligned with Chronos–Kairos–Aion, with concrete liveness/reconciliation examples and a transport lemma relating temporal satisfaction to observer translation cost (§6).

Scope management. We deliberately do *not* attempt to axiomatise a single canonical trace metric, nor do we claim that MDL gives an optimal notion of semantic similarity for all domains. Our aim is narrower and foundational: to provide a mathematically explicit, computable mechanism that turns *translation cost* into geometry, so that later work can specialise it to concrete trace languages and security goals.

Roadmap. We begin by restating the minimal background from Papers I–III needed for a stand-alone reading (§2). We then formalise observers as resource-bounded functors out of history categories and motivate canonical observer families induced by holography (§3). Next we introduce translators, MDL description length, and lifted distortion as the ingredients for a quantitative comparison of observers (§4). Rulial distance is defined and analysed in §5, including the Lawvere-metric/enriched-category interpretation of directed cost (§5.5). We then connect deterministic worldlines to multiway systems and the Ruliad, formalise the Chronos–Kairos–Aion time model, and develop a minimal temporal logic whose semantics range over worldlines and branching histories (§6, §6.4). Finally, we summarise related work (§7), discuss implications and open directions (§8), and provide a notation summary (§9).

2 Preliminaries and Standing Assumptions

We briefly restate the fragments of Papers I–III needed for a self-contained treatment. Throughout, we adopt the deterministic replay discipline of Paper II (fixed boundary data determines a unique committed tick worldline) and the boundary encoding of Paper III.

2.1 WARP states and deterministic worldlines

A WARP graph is a finite directed multigraph whose vertices and edges carry recursively attached WARP graphs [Ros25c]. A WARP *state* $U \in$ WState is a typed open graph skeleton together with recursively attached WARP states on each vertex and edge. We write $\mathrm{skel}(U)$ for the skeleton component.

Deterministic evolution is expressed in ticks. Let Labels denote the space of tick patches: finite records sufficient to advance the state by one tick under the deterministic semantics of Paper II. Write

$$\mathsf{Apply} : \mathsf{WState} \times \mathsf{Labels} \rightharpoonup \mathsf{WState}$$

for the deterministic tick-application function. A *tick patch* is an element $\mu \in$ Labels (intended to be applied via Apply). Intuitively, a tick patch is the serialised record of the within-tick batch committed at that tick. A *tick* is the unit of concurrent evolution: it groups attachment-plane rewrites together with a scheduler-selected batch of independent skeleton rewrites, committed atomically (Paper II, Def. 4.2). A deterministic worldline is a sequence

$$U_0 \Rightarrow U_1 \Rightarrow \cdots \Rightarrow U_n \qquad \text{with} \qquad U_{i+1} = \mathsf{Apply}(U_i, \mu_i)$$

whenever defined [Ros25b]. Paper II shows how to construct such an Apply from DPO rewriting in adhesive categories under a two-plane discipline, and how to package scheduling decisions so that replay is bit-level deterministic.

2.2 Boundary encoding and wormholes

Paper III introduces a *provenance payload*

$$P = (\mu_0, \ldots, \mu_{n-1})$$

and the boundary encoding (U_0, P) [Ros25a]. Under patch sufficiency,[1] (U_0, P) reconstructs the interior worldline uniquely. A *wormhole* is a provenance-preserving compression of a multi-tick segment into a single edge labelled by a sub-payload. For the present paper, holography induces two natural classes of observers:

- *bulk observers* that inspect some or all of the interior worldline; and

- *boundary observers* that operate only on the compact boundary artefact (U_0, P).

Rulial distance will quantify the cost of translating between these viewpoints.

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[1] Patch sufficiency is the condition that (U_0, P) uniquely determines the interior worldline under the deterministic semantics.

2.3 Multiway graphs and history categories

Fix a universe $\mathcal{U} \subseteq \mathsf{WState}$ and a rule pack R (a finite set of rewrite rules plus the fixed typing/open-graph discipline). The associated *multiway graph* is the directed graph

$$\mathrm{MW}(\mathcal{U}, R) = (V, E)$$

whose vertices are states $V = \mathcal{U}$ and whose directed edges are individual rewrite steps generated by R (including alternative matches and orderings where applicable). In general $\mathrm{MW}(\mathcal{U}, R)$ branches and merges.

Definition 2.1 (History category). Let $\mathrm{MW}(\mathcal{U}, R)$ be a multiway graph. Its *history category* $\mathrm{Hist}(\mathcal{U}, R)$ is the path category of $\mathrm{MW}(\mathcal{U}, R)$:

- objects are states $U \in \mathcal{U}$;

- morphisms $h : U \to V$ are finite directed paths in $\mathrm{MW}(\mathcal{U}, R)$ from U to V;

- composition is path concatenation.

Remark 2.2 (Deterministic worldlines as functors). A deterministic worldline $U_0 \Rightarrow U_1 \Rightarrow \cdots$ defines a functor $W : \mathbb{N} \to \mathrm{Hist}(\mathcal{U}, R)$ sending $i \mapsto U_i$ and $(i \to i+1) \mapsto (U_i \to U_{i+1})$. The determinism discipline of Paper II can be understood as selecting a unique such functor for fixed boundary data. We later formalise its finite restriction as the Chronos functor (Definition 6.4 in §6.3).

3 Observers

Observers are the interface between a WARP history and a consumer. We treat observers as functors out of the history category into a structured space of traces.

3.1 Observation spaces

An *observation space* is an object that supports: (i) a notion of trace value; and (ii) a distortion measure between traces. The minimal structure we require is a set Tr equipped with a metric (or pseudometric)

$$\mathrm{dist}_{\mathrm{tr}} : \mathsf{Tr} \times \mathsf{Tr} \to \mathbb{R}_{\geq 0}.$$

In applications, Tr may be: symbol streams, labelled paths, graphs of causal dependencies, certificates, or slices of provenance payloads.

When it is convenient to keep categorical structure explicit, we may regard Tr as the object set of a category $\mathcal{Y}$ and work objectwise. Nothing in the core definitions requires nontrivial morphisms in $\mathcal{Y}$; the geometry is carried by $\mathrm{dist}_{\mathrm{tr}}$.

3.2 Observers as budgeted functors

Definition 3.1 (Observer). Fix $\mathrm{Hist}(\mathcal{U}, R)$ and an observation space $(\mathsf{Tr}, \mathrm{dist}_{\mathrm{tr}})$. An *observer* is a functor

$$O : \mathrm{Hist}(\mathcal{U}, R) \to \mathsf{Tr},$$

where we regard Tr as a discrete category. Operationally, O is realised by an algorithm that maps any derivation path h to a trace value $O(h)$.

Definition 3.2 (Resource-bounded observer). Let (τ, m) be time and memory budgets (in any fixed machine model). An observer O is (τ, m)-*bounded* if it admits an implementation that, on any history input h in its domain, runs within time τ and memory m.

Remark 3.3 (Why we bound observers). Without explicit budgets, all observers collapse into an uninformative equivalence: "compute the full worldline and output it". Budgets ensure the geometry respects real computational constraints: replaying a wormhole is algorithmically simple (low description length) but may be infeasible at small τ.

3.3 Canonical observer families induced by holography

Holographic boundary encoding induces a practical taxonomy of observers:

- *boundary observers* that inspect only (U_0, P) (or its authenticated packaging such as a BTR);

- *bulk observers* that inspect interior states, matches, receipts, or causal cones; and

- *semantic observers* that collapse syntactic evolution into invariant properties (types, safety checks, query semantics).

Example 3.4 (Boundary vs bulk). Let O_∂ map a history h to the boundary artefact (U_0, P) that generates it, and let O_{bulk} map h to the full state sequence $(U_0, \ldots, U_n)$. There is a natural translator T_{replay} from O_∂ to O_{bulk} given by deterministic replay. Its *description length* is small (it is essentially the interpreter Apply), but its *time cost* grows with the length of P. This example will be revisited in §5.6.
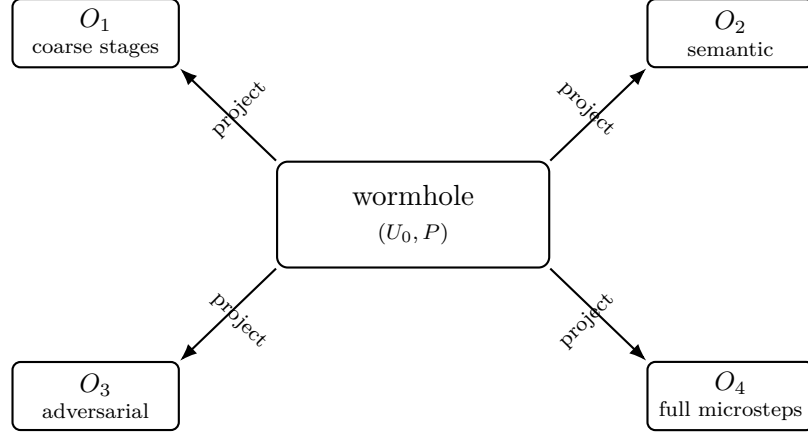
Figure 3.1: Multiple observers projecting the same wormhole boundary (U_0, P) into different trace formats. The rulial distance measures the complexity of translating between such views, balancing translator description length against residual distortion.

3.4 Observer projections of wormholes

Given a wormhole boundary encoding (U_0, P), different observers may:

- expose only coarse-grained stages of P (e.g. AST→IR→plan);

- restrict to semantic effects (e.g. schema and invariants);

- highlight only adversarial or counterfactual branches;

- or inspect every microstep.

4 Translators, MDL Complexity, and Distortion

To compare observers we require a compositional notion of translation, a complexity measure for translators, and a distortion measure between outputs.

4.1 Translators

Let $O_1, O_2 : \mathrm{Hist}(\mathcal{U}, R) \to \mathsf{Tr}$ be observers into a common trace space. A translator should map traces produced by O_1 into traces in the format of O_2.

Definition 4.1 (Translator). A *translator* from O_1 to O_2 is an algorithmic operator

$$T_{12} : \mathsf{Tr} \to \mathsf{Tr}$$

such that $T_{12} \circ O_1$ is a well-defined observer and is intended to approximate O_2. We write $T_{12} \in \mathrm{Trans}(O_1, O_2)$.

Remark 4.2 (Why we translate by post-composition). This definition makes typing explicit: $T_{12} \circ O_1$ is an observer with the same domain as O_2. If one prefers to keep the functor category $\mathsf{Tr}^{\mathrm{Hist}(\mathcal{U},R)}$ explicit, a translator can be regarded as an endofunctor on Tr together with the induced action on observers by post-composition.

Definition 4.3 (Budgeted translators). For budgets (τ, m), let $\mathrm{Trans}_{\tau,m}(O_1, O_2) \subseteq \mathrm{Trans}(O_1, O_2)$ denote the translators realisable within those budgets.

Assumption 4.4 (Budgeted translator axioms). For each budget pair (τ, m):

1. *Identity.* For every observer O, the identity translator I belongs to $\mathrm{Trans}_{\tau,m}(O, O)$, and we normalise codes so that $\mathrm{DL}(I) = 0$.

2. *Composition.* If $T_{12} \in \mathrm{Trans}_{\tau,m}(O_1, O_2)$ and $T_{23} \in \mathrm{Trans}_{\tau,m}(O_2, O_3)$, then

$$T_{23} \circ T_{12} \in \mathrm{Trans}_{\tau,m}(O_1, O_3).$$

Example 4.5 (SQL $\leftrightarrow$ AST). Consider a WARP universe modelling a database query planner. Observer O_1 outputs a trace of AST transformations, while observer O_2 outputs only the initial SQL string and a final execution summary. A translator T_{12} must compile an AST evolution into a SQL-like summary, while T_{21} must infer a plausible AST evolution consistent with SQL and execution effects. The description lengths $\mathrm{DL}(T_{12}), \mathrm{DL}(T_{21})$ and their residual distortions quantify the separation of these two views.

4.2 MDL and description length

We measure translator complexity using MDL: a translator is "simple" if it admits a short prefix-free description.

Definition 4.6 (Description length). Fix a prefix-free code over translator programmes. For a translator T, let $\mathrm{DL}(T) \in \mathbb{R}_{\geq 0}$ denote the length of its code word.

The constant-overhead behaviour of prefix codes gives the subadditivity we require.

Assumption 4.7 (Subadditivity up to a constant). There exists a constant $c \geq 0$ such that for any composable translators T_{12}, T_{23} we have

$$\mathrm{DL}(T_{23} \circ T_{12}) \leq \mathrm{DL}(T_{12}) + \mathrm{DL}(T_{23}) + c.$$

Remark 4.8 (On the constant c). The constant c is the code overhead required to describe "run T_{12} then T_{23}" under the chosen universal coding scheme. MDL theory [Ris78] (and related invariance results for prefix complexity [LV19]) justify treating such overhead as $O(1)$: it does not scale with the size of the translators being composed.

Remark 4.9 (Relation to information distance and rate–distortion). At $\lambda \to \infty$ with the constraint $\mathrm{Dist}(O_2, T \circ O_1) = 0$, the directed cost reduces to the description length of the shortest exact translator from O_1 to O_2. The resulting symmetrised distance is closely related in spirit to *algorithmic information distance*: the Kolmogorov-style cost of converting one description into another [BGL$^+$98, LV19]. At finite λ, the objective is an MDL-flavoured instance of a rate–distortion trade-off: we pay *rate* (translator description length) to purchase lower residual distortion.

4.3 Trace distortion and lifted observer distortion

Fix a metric (or pseudometric) $\mathrm{dist}_{\mathrm{tr}}$ on trace space Tr. We lift it to a distortion between observers by taking a supremum over histories.

Definition 4.10 (Lifted distortion). For observers $O, O' : \mathrm{Hist}(\mathcal{U}, R) \to \mathsf{Tr}$, define

$$\mathrm{Dist}(O, O') := \sup_{h \in \mathrm{Mor}(\mathrm{Hist}(\mathcal{U},R))} \mathrm{dist}_{\mathrm{tr}}\big(O(h), O'(h)\big).$$

Assumption 4.11 (Bounded diameter). All observers under comparison take values in a common trace space Tr of uniformly bounded diameter, so that $\mathrm{Dist}(O, O')$ is finite.

Assumption 4.12 (Non-expansive translators). Post-composition by any translator is 1-Lipschitz:

$$\mathrm{Dist}(T \circ O, \, T \circ O') \leq \mathrm{Dist}(O, O')$$

for all translators T and observers O, O'.

Remark 4.13 (Alternative liftings). The supremum lifting is conservative: it protects against worst-case histories and adversarial inputs. In statistical settings we may instead use an expected distortion over a distribution on histories, or restrict to histories within a time cone. Our results adapt to any lifting that preserves the triangle inequality and non-expansiveness properties used in §5.

5 Rulial Distance

We now define the rulial distance and prove its core properties. Throughout we fix a weighting parameter $\lambda > 0$ trading off translator complexity against residual distortion.

5.1 Directed and symmetrised distance

It is useful to separate the directed translation problem from its symmetrisation.

Definition 5.1 (Directed rulial cost). For observers O_1, O_2 define the directed cost

$$\vec{D}_{\tau,m}(O_1 \to O_2) := \inf_{T_{12} \in \mathrm{Trans}_{\tau,m}(O_1,O_2)} \Big(\mathrm{DL}(T_{12}) + \lambda \, \mathrm{Dist}(O_2, T_{12} \circ O_1) \Big),$$

with the convention that the infimum over an empty set is $+\infty$.

Definition 5.2 (Rulial distance). The (symmetrised) *rulial distance* is

$$D_{\tau,m}(O_1, O_2) := \vec{D}_{\tau,m}(O_1 \to O_2) \ + \ \vec{D}_{\tau,m}(O_2 \to O_1).$$

Equivalently, expanding the two infima yields the joint infimum formulation used in earlier drafts:

$$D_{\tau,m}(O_1, O_2) = \inf_{\substack{T_{12} \in \mathrm{Trans}_{\tau,m}(O_1,O_2) \\ T_{21} \in \mathrm{Trans}_{\tau,m}(O_2,O_1)}} \Big(\mathrm{DL}(T_{12}) + \mathrm{DL}(T_{21}) + \lambda \big(\mathrm{Dist}(O_2, T_{12} \circ O_1) + \mathrm{Dist}(O_1, T_{21} \circ O_2) \big) \Big).$$

5.2 Basic properties

Theorem 5.3 (Basic properties). *For all observers O_1, O_2 and budgets (τ, m):*

1. *$D_{\tau,m}(O_1, O_2) \geq 0$;*

2. *$D_{\tau,m}(O_1, O_2) = D_{\tau,m}(O_2, O_1)$;*

3. *$D_{\tau,m}(O, O) = 0$ for every observer O.*

Proof. Non-negativity follows because $\mathrm{DL} \geq 0$ and $\mathrm{Dist} \geq 0$. Symmetry holds by definition of $D_{\tau,m}$ as the sum of two directed terms. For reflexivity, the identity translator I is admissible by Assumption 4.4 and satisfies $\mathrm{DL}(I) = 0$ and $\mathrm{Dist}(O, I \circ O) = 0$, so both directed costs vanish. $\square$

Corollary 5.4 (Observer equivalence). *Let O_1, O_2 be observers. Then $D_{\tau,m}(O_1, O_2) = 0$ if and only if there exist translators $T_{12} \in \mathrm{Trans}_{\tau,m}(O_1, O_2)$ and $T_{21} \in \mathrm{Trans}_{\tau,m}(O_2, O_1)$ such that:*

1. *$\mathrm{Dist}(O_2, T_{12} \circ O_1) = 0$ and $\mathrm{Dist}(O_1, T_{21} \circ O_2) = 0$; and*

2. *$\mathrm{DL}(T_{12})$ and $\mathrm{DL}(T_{21})$ are bounded by a constant independent of the histories under consideration.*

In this case the observers are equivalent under the rulial geometry: they differ only by constant-overhead, distortion-free translation.

Proof sketch. If such translators exist then both directed costs are bounded by constants independent of the histories (distortion is 0 and description length is constant), so $D_{\tau,m}(O_1, O_2)$ is bounded by a constant. Under the constant-overhead convention of the remark below, we identify such constant separation with 0, yielding $D_{\tau,m}(O_1, O_2) = 0$. Conversely, if $D_{\tau,m}(O_1, O_2) = 0$ then (by definition of $D_{\tau,m}$ as the sum of two directed costs) both directed costs vanish modulo constant overhead, hence there exist translators in both directions with zero residual distortion and constant description length, as claimed. $\square$

Remark 5.5. Observer equivalence is defined modulo constant description overhead; exact zero-length translators are not required and depend on the choice of coding scheme.

5.3 Monotonicity under budget relaxation

The budgeted nature of rulial distance is essential: it distinguishes translations that are short in description length but exceed available time/memory resources from those that are admissible under the deployment constraints.

Proposition 5.6 (Budget monotonicity). *If $(\tau', m') \succeq (\tau, m)$ (i.e. $\tau' \geq \tau$ and $m' \geq m$) then*

$$D_{\tau',m'}(O_1, O_2) \leq D_{\tau,m}(O_1, O_2).$$

Proof. By definition, $\mathrm{Trans}_{\tau,m}(O_i, O_j) \subseteq \mathrm{Trans}_{\tau',m'}(O_i, O_j)$ under budget relaxation, so the infimum is taken over a larger set and cannot increase. □

5.4 Triangle inequality up to a constant

Theorem 5.7 (Triangle inequality up to additive slack). *Assume:*

1. *Assumption 4.7 (subadditivity of DL up to constant c);*

2. Dist *is a metric on observers (triangle inequality) and translators are non-expansive (Assumption 4.12);*

3. *budget classes are closed under composition (Assumption 4.4).*

Then for all observers O_1, O_2, O_3 we have

$$D_{\tau,m}(O_1, O_3) \leq D_{\tau,m}(O_1, O_2) + D_{\tau,m}(O_2, O_3) + 2c.$$

Proof. Fix $\varepsilon > 0$ and choose near-optimal translators for the two distances: pick T_{12}, T_{21} such that the objective for $D_{\tau,m}(O_1, O_2)$ is within $\varepsilon/2$ of the infimum, and T_{23}, T_{32} similarly for $D_{\tau,m}(O_2, O_3)$. By closure under composition, $T_{13} = T_{23} \circ T_{12}$ and $T_{31} = T_{21} \circ T_{32}$ are admissible budgeted translators.

Subadditivity gives $\mathrm{DL}(T_{13}) \leq \mathrm{DL}(T_{12}) + \mathrm{DL}(T_{23}) + c$ and $\mathrm{DL}(T_{31}) \leq \mathrm{DL}(T_{21}) + \mathrm{DL}(T_{32}) + c$. For distortion, the triangle inequality and non-expansiveness yield

$$\begin{aligned}
\mathrm{Dist}(O_3, T_{13} \circ O_1) &= \mathrm{Dist}(O_3, T_{23} \circ T_{12} \circ O_1) \\
&\leq \mathrm{Dist}(O_3, T_{23} \circ O_2) + \mathrm{Dist}(T_{23} \circ O_2, T_{23} \circ T_{12} \circ O_1) \\
&\leq \mathrm{Dist}(O_3, T_{23} \circ O_2) + \mathrm{Dist}(O_2, T_{12} \circ O_1),
\end{aligned}$$

and similarly for $\mathrm{Dist}(O_1, T_{31} \circ O_3)$. Summing the bounds and using near-optimality yields the stated inequality up to ε. Letting $\varepsilon \to 0$ completes the proof. □

Remark 5.8 (Quasi-pseudometric). Together with Theorem 5.3, Theorem 5.7 makes $D_{\tau,m}$ a quasi-pseudometric: it satisfies all pseudometric axioms except that the triangle inequality holds only up to an additive constant $2c$. In practice c is a small, fixed prefix-coding overhead; it may also be absorbed into λ if desired. The geometry is most informative when translation costs scale nontrivially with history size, or when comparing asymptotically distinct observer classes (e.g. $O(1)$ vs $O(N)$), in which regime the constant c becomes negligible.

5.5 Lawvere-metric (enriched category) viewpoint

The symmetrised distance $D_{\tau,m}$ is convenient for neighbourhoods and "frame separation", but the underlying translation problem is inherently *directed*: decompressing a boundary view into a bulk view can be infeasible under strict budgets, whereas projection from bulk into boundary is typically admissible under the same budgets. This asymmetry is captured by Lawvere's observation that metric spaces are categories enriched in the monoidal poset $([0,\infty], \geq, +, 0)$ [Law73, Kel82].

Definition 5.9 (Lawvere metric space)**.** A *Lawvere metric space* is a category enriched over the monoidal poset $([0,\infty], \geq, +, 0)$. Concretely, it is a collection of objects together with a function $d(x,y) \in [0,\infty]$ such that: (i) $d(x,x) = 0$ for all x; and (ii) $d(x,z) \leq d(x,y) + d(y,z)$ for all x,y,z. No symmetry condition is imposed; $d(x,y)$ and $d(y,x)$ may differ. The value $+\infty$ is permitted and represents "no morphism" (infeasible translation).

Definition 5.10 (Directed rulial hom)**.** Fix budgets (τ, m). For observers O_1, O_2 define the *directed hom-value*

$$d_{\tau,m}(O_1, O_2) := \vec{D}_{\tau,m}(O_1 \to O_2) \in [0,\infty],$$

with the convention $d_{\tau,m}(O_1, O_2) = +\infty$ when $\mathrm{Trans}_{\tau,m}(O_1, O_2) = \varnothing$. The symmetrised rulial distance is the induced symmetrisation $D_{\tau,m}(O_1, O_2) = d_{\tau,m}(O_1, O_2) + d_{\tau,m}(O_2, O_1)$.

For notational convenience, we treat $\vec{D}_{\tau,m}(O_1 \to O_2)$ and $d_{\tau,m}(O_1, O_2)$ as interchangeable; we use $d_{\tau,m}$ when emphasising the Lawvere-enriched interpretation.

Proposition 5.11 (Composition as triangle inequality)**.** *Assume: (i) Assumption 4.7; (ii)* Dist *satisfies the triangle inequality and translators are non-expansive (Assumption 4.12); and (iii) budget classes are closed under composition (Assumption 4.4). Then for all observers O_1, O_2, O_3,*

$$d_{\tau,m}(O_1, O_3) \leq d_{\tau,m}(O_1, O_2) + d_{\tau,m}(O_2, O_3) + c.$$

Proof sketch. The argument is the directed half of the proof of Theorem 5.7. Choose near-optimal translators $T_{12} \in \mathrm{Trans}_{\tau,m}(O_1, O_2)$ and $T_{23} \in \mathrm{Trans}_{\tau,m}(O_2, O_3)$. Closure under composition gives an admissible translator $T_{13} = T_{23} \circ T_{12}$. Subadditivity bounds $\mathrm{DL}(T_{13}) \leq \mathrm{DL}(T_{12}) + \mathrm{DL}(T_{23}) + c$. The distortion term satisfies $\mathrm{Dist}(O_3, T_{13} \circ O_1) \leq \mathrm{Dist}(O_3, T_{23} \circ O_2) + \mathrm{Dist}(O_2, T_{12} \circ O_1)$ by the triangle inequality and non-expansiveness. Taking infima yields the stated inequality. $\square$

Remark 5.12 (Strict enrichment vs $O(1)$ slack)*.* If we treat description lengths modulo constant additive overhead (as is standard in Kolmogorov/MDL-style arguments), or adopt a translator description language with a primitive sequencing combinator whose size is absorbed into the base machine model, then the constant c may be taken as 0. In that regime, $d_{\tau,m}$ satisfies the Lawvere triangle inequality exactly and the "space of observers" is a $[0,\infty]$-enriched category. When $c > 0$, the enrichment is accurate up to fixed $O(1)$ slack, matching the quasi-pseudometric remark above.

The enriched viewpoint encodes several familiar facts: directed costs compose by addition (triangle inequality); budgets produce $+\infty$ hom-values (no admissible translator); and asymmetry is the generic case rather than an exception. It also exposes standard categorical tools: the enriched Yoneda embedding associates to each observer O its distance profile $d_{\tau,m}(O, -)$, and Cauchy completion corresponds to freely adjoining "ideal observers" realising limits of Cauchy weights (useful when taking refinement limits) [Kel82].

Example 5.13 (Boundary vs bulk as an asymmetric hom)*.* Let O_∂ be the boundary observer of Example 3.4. Let O_{bulk}^+ be a bulk observer whose trace format includes the boundary payload as a visible
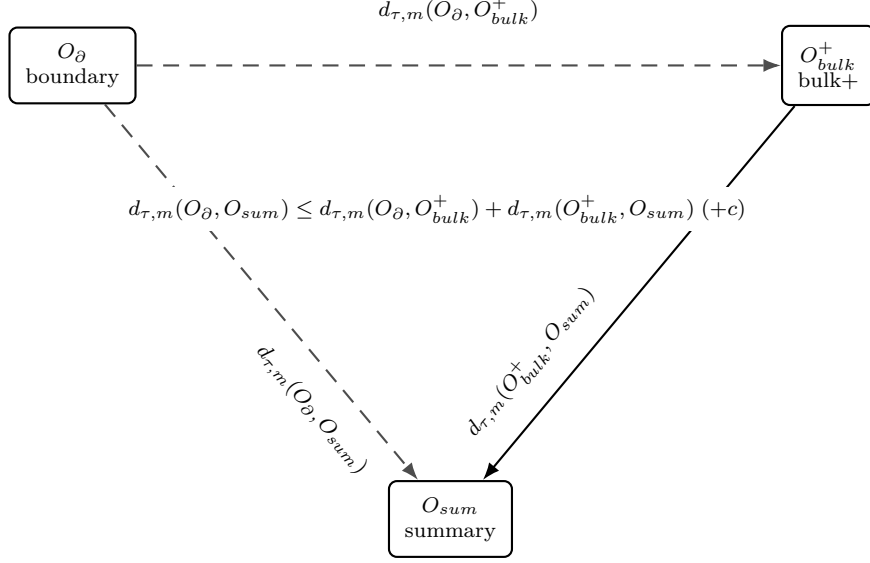
Figure 5.1: Directed translation costs form a Lawvere-style geometry: costs compose additively (triangle inequality), asymmetry is expected, and strict budgets can force $+\infty$ distances. The dashed arrows emphasise that some translations (e.g. boundary→bulk+) may be infeasible at fixed (τ, m).

component (e.g. it outputs (U_0, P) together with additional interior witnesses such as $(U_1, \ldots, U_n)$, match receipts, or causal cones). Then the forgetful projection translator T_{forget} extracting (U_0, P) is admissible with $\text{DL}(T_{\text{forget}}) = O(1)$ and zero residual distortion, so $d_{\tau,m}(O_{\text{bulk}}^+, O_\partial) = O(1)$. In the opposite direction, Proposition 5.14 shows that $d_{\tau,m}(O_\partial, O_{\text{bulk}}^+)$ can be $+\infty$ under strict budgets (replay is infeasible under the time bound), but reduces to $O(1)$ when (τ, m) are unbounded. This is typical of Lawvere metric spaces: translation is compositional, but symmetry is not assumed.

5.6 Budget effects: replay is short but not fast

The boundary/bulk example illustrates why we insist on explicit budgets.

Proposition 5.14 (Boundary-to-bulk translation). *Let O_∂ be a boundary observer and O_{bulk} a bulk observer as in Example 3.4. Assume deterministic replay is available as a translator T_{replay}. Then:*

1. $\text{DL}(T_{\text{replay}})$ *is $O(1)$ relative to the fixed semantics* Apply *(it is essentially the interpreter);*

2. *for fixed finite budgets (τ, m), $T_{\text{replay}} \notin \text{Trans}_{\tau,m}(O_\partial, O_{\text{bulk}})$ once the payload length exceeds τ, so $\vec{D}_{\tau,m}(O_\partial \to O_{\text{bulk}}) = +\infty$ beyond that regime;*

3. *for unbounded budgets, the directed distortion term can be 0 (exact replay), so $\vec{D}_{\infty,\infty}(O_\partial \to O_{\text{bulk}}) = O(1)$.*

Proof. (1) follows from the fact that the replay algorithm is fixed once the operational semantics is fixed. (2) is immediate: replay must apply the tick patches sequentially and therefore requires time proportional to payload length. (3) follows because replay is exact, so distortion vanishes, and only the constant description length remains. $\square$

Remark 5.15 (Interpretation). At unbounded resources, boundary and bulk descriptions may be close in rulial distance. At bounded resources, they can be infinitely far. This captures an engineering

reality: a short programme can still be computationally infeasible under strict budgets. In particular, under the two-plane WARP semantics (Paper II), a state decomposes into a skeleton together with recursively attached sub-states; translating a boundary observer into a bulk observer amounts to *expanding* these attachment fibres across the committed tick sequence, work that can be infeasible under strict (τ, m) budgets. Wormholes (Paper III) are precisely provenance-preserving compressions of multi-tick segments into single labelled edges; a bulk observer "sees inside" only by replaying (and hence expanding) the corresponding sub-payload.
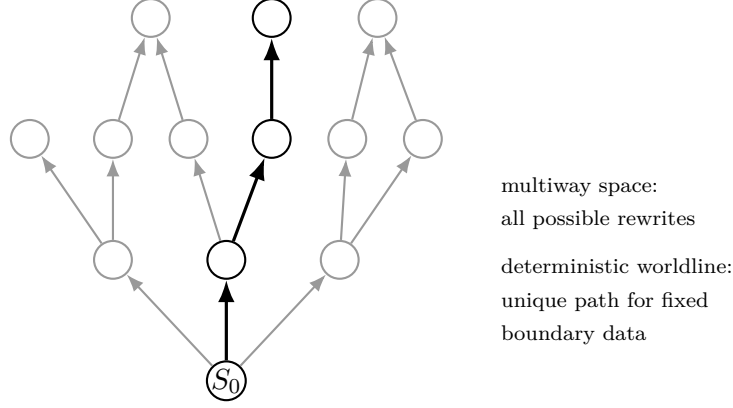
Figure 6.1: A deterministic worldline (thick) through the multiway space of all possible WARP rewrites. Fixing the rule pack, initial state, and scheduling/tie-break data selects a unique path; alternative branches represent different matches, schedules, or rule-pack choices.

6 Multiway Systems, the Ruliad, and Observer Geometry

We treat the Ruliad connection with full mathematical detail, since later papers in the series move from mathematics to ethics and architecture.

6.1 Multiway space induced by WARP rewriting

A rule pack R induces a multiway graph $\mathrm{MW}(\mathcal{U}, R)$. The determinism discipline of Paper II does *not* remove branching from the underlying multiway space; rather, it ensures that once a boundary encoding is fixed (initial state, rule pack, scheduler policy, and tie-breaks), the realised evolution is a unique path.

Remark 6.1 (Confluence vs determinism). Confluence is a property of a rewrite system: different rewrite orders lead to a common result. Determinism in Paper II is stronger and more operational: given fixed boundary data, there is a unique committed tick outcome. The multiway graph still exists as the ambient possibility space in which observers may reason about counterfactuals.

6.2 The Ruliad as a large history space

Wolfram's Ruliad is informally the limit of all possible computations; in our setting it is natural to model it as a large history space built from multiway systems [Wol20].

Definition 6.2 (Aion/Ruliad history space). Fix a class $\mathfrak{R}$ of admissible rule packs and a class $\mathfrak{U}$ of admissible initial states. Define the *Aion history space* (the *Ruliad* in our setting) as the disjoint union of history categories

$$\mathcal{R} := \bigsqcup_{(U_0, R) \in \mathfrak{U} \times \mathfrak{R}} \mathrm{Hist}(\mathcal{U}_{U_0, R}, R),$$

where $\mathcal{U}_{U_0, R}$ is the forward closure of U_0 under R (the reachable states).

Remark 6.3 (Large-category caveat). $\mathcal{R}$ is a large category (indeed a proper class in many settings). The disjoint union is deliberate: we treat histories as provenance-bearing artefacts, so components are not quotiented by extensional state equality. In particular, even if two reachable states are *identical* as graph-shaped data, we keep them as distinct objects of $\mathcal{R}$ when they arise from different

causal origins (different initial states and/or rule packs). This contrasts with a "merging" view of the Ruliad that identifies states across components and thereby erases origin information. We use it as a conceptual container: the purpose is to make explicit that a single deterministic worldline is a small, selected path within a vastly larger possibility space. None of the metric arguments in §5 require manipulating $\mathcal{R}$ as a set-theoretic object.

6.3 Chronos, Kairos, Aion

We formalise the three-layer time model alluded to in earlier drafts.

Definition 6.4 (Chronos). *Chronos time* is the linear time of a fixed worldline: given a replayable payload $P = (\mu_0, \ldots, \mu_{n-1})$ of *tick patches*, Chronos is the finite linear order $0 < 1 < \cdots < n$ on committed ticks, or the functor $\mathsf{Chronos} : [n] \to \mathrm{Hist}(\mathcal{U}, R)$ selecting the unique replay path. If a tick patch μ_i is internally realised by a within-tick microstep trace, we treat that trace as structure *inside* the tick edge label rather than as additional Chronos instants.

Definition 6.5 (Kairos). *Kairos time* is the structure of *branch events*: points at which alternative continuations exist in the multiway graph (alternative matches, schedules, rule packs, or inputs), or, internally, points within a tick at which candidate matches conflict (witnessed by the within-tick trace posets of Paper II). Formally, Kairos may be modelled as a subgraph (or event structure) of $\mathrm{MW}(\mathcal{U}, R)$ capturing divergence and merge loci.

Definition 6.6 (Aion). *Aion time* is the history category $\mathrm{Hist}(\mathcal{U}, R)$: the possibility space of all finite derivations in the multiway graph. At the largest scale, the Ruliad history space $\mathcal{R}$ (Definition 6.2) is the disjoint union of such Aion categories across initial states and rule packs.

6.4 Temporal logic on Chronos–Kairos–Aion

The three-layer time model is most useful when it supports clear *statements* about computation: safety properties ("nothing bad ever happens"), liveness properties ("something good eventually happens"), and branch-sensitive claims ("every possible continuation eventually reconciles"). Temporal logics provide a compact language for such statements [Pnu77, CES86].

Atomic propositions are observer-relative. In our setting a "state" may carry arbitrarily rich provenance and interior structure, but an observer sees only a trace value. Accordingly, we interpret atomic propositions as predicates on trace space. Fix a set AP of atomic propositions and, for each $p \in \mathsf{AP}$, a predicate (by abuse of notation)

$$p \; : \; \mathsf{Tr} \to \{0, 1\}.$$

Given an observer $O : \mathrm{Hist}(\mathcal{U}, R) \to \mathsf{Tr}$, we write $O \models_h p$ to mean $p(O(h)) = 1$ for a history morphism $h \in \mathrm{Mor}(\mathrm{Hist}(\mathcal{U}, R))$.

A minimal CTL*-style language on histories. To engineer reliable systems on top of the Aion possibility space, we require a formal language to state invariants such as "all branches eventually merge" or "provenance is always preserved", and to reason about how such claims behave under observer translation. We evaluate formulas on *history prefixes* rather than bare states, because observers can be path-dependent. Let φ range over *state formulas* and ψ over *path formulas*:

$$\varphi \quad ::= \quad p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathbf{A}\psi \mid \mathbf{E}\psi,$$
$$\psi \quad ::= \quad \varphi \mid \neg\psi \mid (\psi \wedge \psi) \mid \mathbf{X}\psi \mid \mathbf{F}\psi \mid \mathbf{G}\psi \mid (\psi \, \mathbf{U} \, \psi),$$

where $p \in \mathsf{AP}$, and $\mathbf{A}$ ("for all paths") / $\mathbf{E}$ ("there exists a path") quantify over continuations in the Aion/multiway space. The operators $\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U}$ are read as "next", "eventually", "always", and "until", respectively.

Semantics. Write $\mathrm{MW}(\mathcal{U}, R)$ for the multiway graph, and interpret a *one-step* extension of a history prefix $h : U_0 \to U$ as composition with a single outgoing rewrite edge $e : U \to U'$ in MW. A (maximal) *continuation path* from h is then an infinite sequence of such one-step extensions (or, for finite computations, a finite path padded by stuttering at its terminal object). We write $h[i]$ for the length-i extension along a continuation, with $h[0] = h$.

State satisfaction $O \models_h \varphi$ and path satisfaction $O \models_{h,\pi} \psi$ (where π is a continuation of h) are defined by the standard CTL* clauses:

- $O \models_h p$ iff $p(O(h)) = 1$;

- $O \models_h \mathbf{A}\psi$ iff for every continuation π of h, $O \models_{h,\pi} \psi$; and $O \models_h \mathbf{E}\psi$ iff for some continuation π of h, $O \models_{h,\pi} \psi$;

- $O \models_{h,\pi} \mathbf{X}\psi$ iff $O \models_{h[1],\pi} \psi$;

- $O \models_{h,\pi} \mathbf{F}\psi$ iff there exists $i \geq 0$ such that $O \models_{h[i],\pi} \psi$;

- $O \models_{h,\pi} \mathbf{G}\psi$ iff for all $i \geq 0$, $O \models_{h[i],\pi} \psi$;

- $O \models_{h,\pi} (\psi_1 \mathbf{U} \psi_2)$ iff there exists $k \geq 0$ with $O \models_{h[k],\pi} \psi_2$ and for all $0 \leq i < k$, $O \models_{h[i],\pi} \psi_1$.

Chronos is recovered by evaluating path formulas along the unique continuation selected by deterministic replay (Definition 6.4); Aion semantics uses the full continuation set; and Kairos points are the prefixes h whose endpoint admits multiple distinct one-step extensions (Definition 6.5).

Example 6.7 (Liveness of provenance exposure on Chronos). Let $p_{\text{expose}} \in \mathsf{AP}$ be the atomic proposition "a specified provenance receipt has been exposed", defined as a predicate on trace space (e.g. the trace contains a receipt hash, or a certificate has been emitted). On a fixed deterministic worldline, the liveness claim

$$\mathbf{G}\,\mathbf{F}\,p_{\text{expose}}$$

asserts that *at every time*, exposure eventually occurs again. This is an instance of an "always eventually" property: exposure may be delayed by compression or batching, but it cannot be postponed indefinitely along Chronos.

Example 6.8 (Eventual reconciliation across branches). Let p_{merge} be the proposition "the current prefix lies at (or after) a merge locus", for example detected by the multiway graph structure or by a trace witness of branch reconciliation. The Aion-level claim

$$\mathbf{A}\,\mathbf{F}\,p_{\text{merge}}$$

says that *every* possible continuation from the current prefix eventually reaches a merged regime. In contrast, $\mathbf{E}\,\mathbf{F}\,p_{\text{merge}}$ asserts only that *some* continuation eventually reconciles, which is strictly weaker when divergence is possible.

To ensure temporal claims are stable under small representation-level discrepancies between traces, we will assume a δ-robustness condition on atomic propositions.

Proposition 6.9 (Temporal properties transport along low-distortion translation). *Fix $\delta \geq 0$ and assume each atomic proposition $p \in \mathsf{AP}$ is δ-robust in trace space: whenever $\mathrm{dist}_{\mathrm{tr}}(t, t') \leq \delta$ we have $p(t) = p(t')$. Let O_1, O_2 be observers and let $T \in \mathrm{Trans}_{\tau,m}(O_1, O_2)$ satisfy*

$$\mathrm{Dist}(O_2, T \circ O_1) \leq \delta.$$

Then for every history prefix h and every state formula φ in the above language,

$$O_2 \models_h \varphi \quad \Longleftrightarrow \quad (T \circ O_1) \models_h \varphi.$$

Likewise, for every continuation π of h and every path formula ψ,

$$O_2 \models_{h,\pi} \psi \quad \Longleftrightarrow \quad (T \circ O_1) \models_{h,\pi} \psi.$$

In particular, temporal truth in O_2 can be decided from O_1 by applying an admissible translator whose cost is controlled by the directed hom-value $d_{\tau,m}(O_1, O_2)$ (Definition 5.10) and hence by $D_{\tau,m}$ (Definition 5.2).

Remark 6.10 (Robustness in practice). The δ-robustness hypothesis is mathematically clean, but it can be strict for discrete trace structures (hashes, exact logs, serialisations): a tiny change in representation can flip an atomic predicate. In practice, robustness is enforced by canonicalisation or coarse-graining of traces before predicate evaluation. We then define propositions on such a canonicalised or coarse-grained view of the trace (normalising order, discarding nonce-like fields, bucketing timestamps), or we use approximate matching, so that minor log variation does not alter the high-level truth values being checked. Conversely, if an observer suppresses events relevant to a predicate (for example by strong summarisation), then either the predicate is not δ-robust under the chosen trace metric or the translation distortion exceeds δ; Proposition 6.9 is intentionally restricted to the robust regime.

Proof. The hypothesis $\mathrm{Dist}(O_2, T \circ O_1) \leq \delta$ means that for every history prefix h,

$$\mathrm{dist}_{\mathrm{tr}}(O_2(h), T(O_1(h))) \leq \delta.$$

By δ-robustness, all atomic propositions have identical truth values under O_2 and $T \circ O_1$ at every prefix. Since the temporal semantics above depends only on these pointwise truth values along continuations, the equivalence follows by induction on the structure of formulas. More explicitly, the equivalence follows by structural induction on formulas: boolean connectives preserve equivalence immediately, and the temporal operators $(\mathbf{X}, \mathbf{F}, \mathbf{G}, \mathbf{U})$ and path quantifiers $(\mathbf{A}, \mathbf{E})$ depend only on truth values at successive prefixes, which by hypothesis agree pointwise between O_2 and $T \circ O_1$. $\square$

6.5 Observer geometry as frame separation in the Ruliad

Within $\mathcal{R}$, an observer is a functor that assigns traces to histories. Two observers may differ substantially on causal structure, yet still be near each other in rulial distance when translation cost is low under the given budgets. Conversely, two observers may agree semantically but be distant if the only translation requires large algorithmic overhead under the given budgets.

The temporal-logic layer of §6.4 makes this observer-dependence explicit: atomic propositions live on trace space, so temporal truth is phrased in what a given observer can *see*. Proposition 6.9 then formalises when such temporal claims transport across observers: low-distortion translation preserves satisfaction for robust predicates. Meanwhile, the Lawvere-enriched viewpoint of §5.5

reinterprets directed translation costs as enriched hom-values, so that "translation geometry" is precisely the compositional structure encoded by the triangle inequality.

A central point is technical: *rulial distance is not a semantic truth predicate*; it is a geometry induced by translation constraints. In this sense, it is analogous to choosing a coordinate chart in physics: an ill-conditioned chart can make simple structure appear complex.

Remark 6.11 (Rulial balls and neighbourhoods). Given an observer O and radius r, the *rulial ball* $B_r(O) = \{O' : D_{\tau,m}(O, O') \leq r\}$ collects observers mutually reachable within a fixed translation cost. This provides a mathematically explicit way to talk about "nearby frames" without privileging a single canonical observer.

6.6 Engineering implication

The observer geometry is not merely a matter of exposition. It provides a quantitative tool for system design: if a compliance observer is far from a diagnostic observer under deployment budgets, then emitting only compliance traces is not a neutral optimisation — it is a choice that can make subsequent diagnosis prohibitively expensive or impossible. Conversely, if a summary observer is close to a full-bulk observer under expected budgets, we may deploy summary traces without forfeiting the ability to reconstruct details when required. Equivalently, we choose a deployed observer O so that the required secondary views lie within a small rulial ball $B_r(O)$ under the relevant budgets.

Computability.   In full generality, $D_{\tau,m}$ is defined by an infimum over all admissible translator programmes; like other Kolmogorov/MDL-flavoured optima, it is a useful *specification* but not something we claim to compute exactly in a deployed system. The engineering task is to construct explicit translators. We then use these concrete translators to bound the mathematical distance from above, rather than attempting to evaluate the infimum itself. Concretely, we build a specific translator T_{12} (and, if needed, T_{21}), measure or estimate its resource usage under (τ, m), and use

$$\mathrm{DL}(T_{12}) + \lambda \, \mathrm{Dist}(O_2, T_{12} \circ O_1)$$

as a computable upper bound on the directed cost $\vec{D}_{\tau,m}(O_1 \to O_2)$ (and similarly in the reverse direction, yielding an upper bound on $D_{\tau,m}$). This turns rulial distance from an abstract infimum into an actionable design parameter: constructing better translators tightens these bounds.

7 Related Work

Multiway systems and the Ruliad.   Wolfram introduces multiway systems and the Ruliad as a framework for fundamental physics and metamathematics [Wol20]. Our WARP rewriting induces multiway graphs directly; our determinism discipline selects unique worldlines within this possibility space. The rulial distance is our contribution to the specific problem of quantifying observer differences.

Minimum Description Length.   The MDL principle, pioneered by Rissanen [Ris78], provides an information-theoretic basis for balancing model complexity and fit. We apply MDL to measure translator complexity. Connections to prefix complexity and invariance are treated classically in the literature on Kolmogorov complexity [LV19]; we require only the constant-overhead subadditivity property.

Categorical computation and diagrammatic reasoning.   String diagrams and categorical algebra have been applied successfully to concurrency and quantum computing [CD11]. Our two-plane view of WARP state (skeleton plus attachment fibres) sits in this tradition: attachments live in fibres, and reindexing along skeleton morphisms transports attachment updates coherently. The novelty is not any single component, but the synthesis across recursion, rewriting, determinism, and provenance.

8 Discussion and Outlook

This paper completes the mathematics-oriented foundations of AIΩN by making observer-dependence explicit and computable. Taken together, Papers I–IV trace a single line: *structure* (recursive WARP state), *dynamics* (deterministic worldlines), *provenance* (holographic boundary payloads), and finally *geometry* (rulial distance on observers). Deterministic replay and holographic provenance mean the same computation can be represented compactly in many forms. Rulial distance turns the question "how different are these forms?" into a quantitative, resource-aware statement.

Several directions remain open, but they are now delineated:

- **Ethics (Paper V).** Once perfect provenance and deterministic replay are available, the principal questions shift from mathematics to governance. When provenance records cognitive processes—whether of autonomous digital agents or hybrid human–AI systems—replay becomes an ethically significant act, not a neutral debugging primitive. Paper V develops *provenance sovereignty*: an agent's right to control capture, inspection, replay, and branching of its worldline. We extract runtime-level requirements (consent, revocation, capability-scoped observers, cryptographic sealing, and fork rights) and show how they constrain the design of replay tooling and provenance payload formats.

- **Architecture (Paper VI).** The system design question is how to build a machine whose primitive step is a provenance-carrying WARP rewrite, with efficient storage, slicing, and multiverse debugging.

On the mathematical side, this paper adds two complementary lenses: a minimal temporal logic aligned with Chronos–Kairos–Aion (§6.4), and a Lawvere-metric/enriched-category view of directed translation costs (§5.5). Further work may refine these foundations in three directions: (i) develop fully directed rulial quasi-metrics without symmetrisation and with tighter control of $O(1)$ slack; (ii) study quantitative/robust temporal predicates and efficient model checking on WARP history spaces; and (iii) work out the enriched-categorical completion theory (Yoneda, Cauchy completion) and its computational meaning for observer limits.

9 Notation Summary

Symbol	Meaning
$\mathcal{U}$	a WARP universe (collection of states)
WState	class/category of WARP states
Labels	space of tick patches
Apply	deterministic tick-application function $\mathsf{WState} \times \mathsf{Labels} \to \mathsf{WState}$
R	rule pack / rewrite rule set
$\mathrm{MW}(\mathcal{U}, R)$	multiway graph induced by rewriting
$\mathrm{Hist}(\mathcal{U}, R)$	history category (path category of MW)
$(\mathsf{Tr}, \mathrm{dist}_{\mathrm{tr}})$	trace space with trace distortion metric
$O : \mathrm{Hist}(\mathcal{U}, R) \to \mathsf{Tr}$	observer (functor to traces)
$T : \mathsf{Tr} \to \mathsf{Tr}$	translator acting by post-composition
$\mathrm{Trans}_{\tau,m}(O_1, O_2)$	translators from O_1 to O_2 within budgets (τ, m)
$\mathrm{DL}(T)$	MDL description length of translator T
c	subadditivity constant (Assumption 4.7)
λ	weighting parameter trading off DL vs. distortion
$\mathrm{Dist}(O, O')$	lifted observer distortion (supremum over histories)
$\vec{D}_{\tau,m}(O_1 \to O_2)$	directed rulial cost
$d_{\tau,m}(O_1, O_2)$	the same quantity, viewed as a Lawvere hom-value
$D_{\tau,m}(O_1, O_2)$	symmetrised rulial distance
$B_r(O)$	rulial ball of radius r around observer O
BTR	Boundary Transition Record (Paper III)
$O \models_h \varphi$	temporal satisfaction at history prefix h
$\mathbf{A}, \mathbf{E}, \mathbf{X}, \mathbf{G}, \mathbf{F}, \mathbf{U}$	temporal operators (all/exists paths; next/always/eventually/until)
Chronos	linear time of a fixed worldline; functor $[n] \to \mathrm{Hist}(\mathcal{U}, R)$
Kairos	branch-event structure; subgraph of MW capturing divergence loci
Aion	encompassing possibility space; full history category or Ruliad
$\mathcal{R}$	Ruliad/Aion history space (large union of histories)

Appendix A. A worked example of rulial distance

This appendix provides a small concrete example illustrating how rulial distance is used in practice. The purpose is not to compute an exact infimum in Definition 5.2, but to demonstrate how explicit translators yield computable upper bounds on observer separation.

A.1 Setup

Consider a deterministic WARP worldline consisting of three committed ticks:

$$U_0 \overset{\mu_0}{\Longrightarrow} U_1 \overset{\mu_1}{\Longrightarrow} U_2 \overset{\mu_2}{\Longrightarrow} U_3.$$

Let the boundary encoding be (U_0, P), where the provenance payload is

$$P = (\mu_0, \mu_1, \mu_2).$$

We consider two observers of histories in $\mathrm{Hist}(\mathcal{U}, R)$:

- the *boundary observer* O_∂, which maps any history prefix to the compact boundary artefact (U_0, P); and

- the *bulk observer* O_{bulk}, which maps a history prefix to the explicit state sequence, bundled with the boundary artefact, (U_0, P, U_1, U_2, U_3).

Trace space clarification. In this example, the trace space Tr is taken large enough to include both boundary artefacts (U_0, P) and explicit state sequences. We equip Tr with a metric that yields zero distortion when traces agree up to representation or formatting. This makes the zero-distortion claims below immediate.

Both observers are deterministic and well-defined on all prefixes of the committed worldline.

A.2 Translators

There are two natural translators between these observers.

1. **Forgetful projection.**
$$T_{\mathrm{forget}} : O_{\mathrm{bulk}} \to O_\partial,$$
which extracts the boundary artefact (U_0, P) from the bulk trace. This translator has constant description length,
$$\mathrm{DL}(T_{\mathrm{forget}}) = O(1),$$
and zero residual distortion, since the boundary payload is already present in the bulk representation.

2. **Replay expansion.**
$$T_{\mathrm{replay}} : O_\partial \to O_{\mathrm{bulk}},$$
which deterministically replays the payload P using the fixed semantics Apply. Its description length is also constant relative to the fixed interpreter,
$$\mathrm{DL}(T_{\mathrm{replay}}) = O(1),$$
but its runtime cost grows linearly with $|P|$.

A.3 Directed rulial costs

Fix budgets (τ, m).

- For sufficiently small τ, replay is infeasible: $T_{\text{replay}} \notin \text{Trans}_{\tau,m}(O_\partial, O_{\text{bulk}})$. Hence

$$\vec{D}_{\tau,m}(O_\partial \to O_{\text{bulk}}) = +\infty.$$

- In the reverse direction, T_{forget} is admissible under any budget, with zero residual distortion, yielding (for any $\lambda > 0$)

$$\vec{D}_{\tau,m}(O_{\text{bulk}} \to O_\partial) = O(1).$$

For unbounded budgets $(\tau, m) = (\infty, \infty)$, replay is admissible and exact, so both directed costs are $O(1)$. Here the hidden constant is independent of $|P|$ (history length) once the fixed semantics Apply and the translator coding scheme are chosen.

A.4 Symmetrised distance

The symmetrised rulial distance is

$$D_{\tau,m}(O_\partial, O_{\text{bulk}}) = \vec{D}_{\tau,m}(O_\partial \to O_{\text{bulk}}) + \vec{D}_{\tau,m}(O_{\text{bulk}} \to O_\partial).$$

Thus:

- under strict budgets, $D_{\tau,m}(O_\partial, O_{\text{bulk}}) = +\infty$; and

- under relaxed budgets, $D_{\infty,\infty}(O_\partial, O_{\text{bulk}}) = O(1)$.

A.5 Interpretation

This example illustrates the operational use of rulial distance. In practice we do not compute the infimum in Definition 5.2 directly; rather, we construct explicit translators and thereby obtain concrete upper bounds. Improving translators reduces these bounds, whereas strong summarisation or information hiding can increase them—in the limiting case, to $+\infty$. The geometry therefore captures how observer separation depends on available resources rather than on semantic disagreement.

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