Recursive Metagraphs (RMG): DPOI Semantics, Confluence, Hypergraph Embedding, and Rulial Distance

James Ross • RMG Core • Project "Echo"

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Abstract

We formalize the execution model of Recursive Metagraphs (RMG) using Double-Pushout with Interfaces (DPOI) in the adhesive category of typed open graphs. We prove *tick-level confluence* (deterministic batches) and *two-plane commutation* (attachments-first is correct), and give conditions for *global confluence* via critical pairs. We provide a faithful incidence encoding of typed open hypergraphs into our setting, preserving DPO steps and lifting multiway derivations, and define a task- and resource-aware *rulial distance* as an MDL-based pseudometric over observers.

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1 Notation and Setting

Typed open graphs $\mathbf{OGraph}_T$ (cospans of monos) form an adhesive category. Rules are linear spans $(L \leftarrow K \rightarrow R)$; matches are boundary-preserving monos satisfying gluing. RMG states $(G; \alpha, \beta)$ carry attachments in the fibers over nodes and edges; publishing is two-plane: attachments then skeleton.

2 Confluence and Two-Plane Commutation (DPOI)

Setting. Fix a type set T. $\mathbf{Graph}_T$ is the category of T-typed directed graphs; $\mathbf{OGraph}_T$ is the adhesive category of typed open graphs (cospans $I \rightarrow G \leftarrow O$ with monos) [LS06]. Rules and DPO rewriting follow the standard treatment [EL97, EEPT06]. A *DPOI rule* is a span of monos $p = (L \xleftarrow{\ell} K \xrightarrow{r} R)$; a *match* is a boundary-preserving mono $m : L \hookrightarrow G$ satisfying gluing (dangling/identification). The step $G \Rightarrow_p H$ is given by the standard double square:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & L \arrow[d, hook] \\ D \arrow[r, hook] & G \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & R \arrow[d, hook] \\ D \arrow[r, hook] & H \end{tikzcd}$$

Typed ports are enforced by boundary typing; if violated, the pushout complement does not exist.

RMG two-plane state. An RMG state is $(G; \alpha, \beta)$ with skeleton $G \in \mathbf{OGraph}_T$ and attachments $\alpha(v)$, $\beta(e)$ in fibers over nodes/edges. A *tick* applies attachment steps (fibers) *then* skeleton steps (base), under the invariant **no-delete-under-descent** (and a "no-clone" policy on preserved items).

Scheduler independence. For $m : L \hookrightarrow G$ of $p = (L \leftarrow K \to R)$, let $\mathrm{Del}(m) = m(L \setminus K)$ and $\mathrm{Use}(m) = m(L)$. Two matches m_1, m_2 are *parallel independent* iff $\mathrm{Del}(m_1) \cap \mathrm{Use}(m_2) = \varnothing$ and $\mathrm{Del}(m_2) \cap \mathrm{Use}(m_1) = \varnothing$, and both satisfy gluing. Operationally we use a safe over-approximation, the *touch set* $\mathrm{Use}(m) \cup \mathrm{Halo}_r(\mathrm{Use}(m))$ (kernel radius r), and select a maximal independent set (MIS).

Theorem 1 (Tick-level confluence (Theorem A))**.** *Given a scheduler-admissible batch (pairwise parallel independent in the base; attachments under no-delete-under-descent), applying the batch in any serial order consistent with attachments-first yields a unique result up to typed open-graph isomorphism.*

Sketch. By the Concurrency/Parallel Independence Theorem for DPO in adhesive categories, independent base steps commute. Attachment steps commute in the product of fibers; applied first, they are unaffected by base updates. □

Theorem 2 (Two-plane commutation (Theorem B))**.** *Under no-delete-under-descent, performing all attachment updates then base updates equals (up to iso) performing base updates then transporting and applying the attachment updates in the new fibers.*

Sketch. Base updates are pushouts along monos in $\mathbf{OGraph}_T$. Reindexing along base monos preserves pushouts in fibers (Van Kampen). Hence the square "attachments vs base" commutes up to isomorphism. □

Theorem 3 (Conditional global confluence)**.** *Let R be a finite DPOI rule set. If all DPOI critical pairs are joinable (modulo boundary iso) and rewriting terminates (or admits a decreasing-diagrams labelling), then $\Rightarrow_R$ is confluent.*

Idea. Critical Pair Lemma $\Rightarrow$ local confluence; combine with Newman's Lemma (or Decreasing Diagrams) for global confluence. □

Math-to-code contract. *Independence check*: require $\mathrm{Del}(m_1) \cap \mathrm{Use}(m_2) = \varnothing$ and symmetric, plus gluing. *Two-plane discipline*: forbid delete/clone of any position touched in fibers; publish attachments before skeleton.

3 Typed Open Hypergraphs Embed Faithfully into Typed Open-Graph DPOI

Let T_V be vertex types and $\Sigma = \{(s, \mathrm{ar}(s))\}$ hyperedge signatures. $\mathbf{Hyp}_T$ is the category of typed directed hypergraphs; $\mathbf{OHyp}_T$ is open hypergraphs (cospans of monos). $\mathbf{OGraph}_T$ is typed open graphs (adhesive).

Incidence encoding. Define $T^\star := T_V \sqcup \{E_s\}_{s\in\Sigma} \sqcup \{P_{s,i}\}_{s\in\Sigma, 1\le i\le \mathrm{ar}(s)}$. For $H \in \mathbf{OHyp}_T$, build $J(H) \in \mathbf{OGraph}_T$ with a node for each $v \in V$ (typed in T_V), an *edge-node* v_e of type $E_{s(e)}$ for each hyperedge e, and a *port-edge* for each incidence $(e,i) \mapsto v$. Boundaries map identically. This extends to a functor $J : \mathbf{OHyp}_T \to \mathbf{OGraph}_T$. (Open cospans and decorated wiring are standard [Fon15].)

Proposition 1 (Full & faithful on monos)**.** *J is full/faithful on monomorphisms: a mono of hypergraphs corresponds uniquely to a mono of incidence-respecting images, and vice versa.*

Proposition 2 (Creates pushouts along monos)**.** *For a span of monos $H_1 \leftarrow K \to H_2$ in $\mathbf{OHyp}_T$, the pushout exists and $J(H_1 +_K H_2) \cong J(H_1) +_{J(K)} J(H_2)$ in $\mathbf{OGraph}_T$.*

Theorem 4 (DPO preservation/reflection)**.** *For any DPOI rule p and match m in $\mathbf{OHyp}_T$, the DPO step $H \Rightarrow_p H'$ exists iff the DPOI step $J(H) \Rightarrow_{J(p)} J(H')$ exists in $\mathbf{OGraph}_T$, and the results correspond up to iso.*

Derivations and multiway. The functor J lifts to a homomorphism of derivation bicategories $J_\star : \mathrm{Der}(\mathbf{OHyp}_T) \to \mathrm{Der}(\mathbf{OGraph}_T)$ that is locally full/faithful. Thus causal/branchial constructions transport functorially into RMG.

4 Rulial Distance as a Pseudometric via MDL Translators

Fix an RMG universe (U,R) and its history category $\mathrm{Hist}(U,R)$. An *observer* is a boundary-preserving functor $O : \mathrm{Hist}(U,R) \to \mathcal{Y}$ (symbol streams or causal-annotated traces) under budgets (τ, m). A *translator* $T : O_1 \Rightarrow O_2$ is an open-graph transducer (small DPOI rule pack) with $O_2 \approx T \circ O_1$.

Let $\mathrm{DL}(T)$ be a prefix-code description length (MDL) and let Dist be a task-appropriate distortion on outputs. Define the symmetric distance

$$D^{(\tau,m)}(O_1,O_2) = \inf_{T_{12},T_{21}} \mathrm{DL}(T_{12}) + \mathrm{DL}(T_{21}) + \lambda\big(\mathrm{Dist}(O_2, T_{12}\circ O_1) + \mathrm{Dist}(O_1, T_{21}\circ O_2)\big).$$

Assume DL is subadditive up to a constant c and Dist is a metric/pseudometric.

Proposition 3 (Pseudometric)**.** *$D^{(\tau,m)}$ is a pseudometric (nonnegative, symmetric, $D(O,O) = 0$).*

Theorem 5 (Triangle inequality)**.** *$D^{(\tau,m)}(O_1,O_3) \le D^{(\tau,m)}(O_1,O_2) + D^{(\tau,m)}(O_2,O_3) + 2c.$*

Sketch. Choose near-minimizers for the two terms; compose translators: $T_{13} = T_{23} \circ T_{12}$ and $T_{31} = T_{21} \circ T_{32}$. Subadditivity of DL and the metric triangle for Dist bound the composed cost; take infima. □

Operational estimator. Compile translators as DPOI rule packs; measure DL by compressed bundle size and Dist on a fixed test suite under resource budgets. This yields an empirical (approximate) $D^{(\tau,m)}$.

A Scheduler Contract: Math $\leftrightarrow$ Engine

For a compiled rule $p = (L \leftarrow K \to R)$ and match m:

- $\mathrm{Del}(m) = \mathrm{im}(L \setminus K)$; $\mathrm{Use}(m) = \mathrm{im}(L)$.
- Independence requires $\mathrm{Del}(m_1) \cap \mathrm{Use}(m_2) = \varnothing$ and symmetrically, plus gluing.
- The scheduler computes an MIS over $\mathrm{Touch}(m) = \mathrm{Use}(m) \cup \mathrm{Halo}_r(\mathrm{Use}(m))$.
- Two-plane: if a fiber update touches $\alpha(v)$ or $\beta(e)$, no concurrent base step may delete/clone v or e; publish attachments, then base.

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