General Parametric Splines in carEx

2019-03-30

Introduction

The parametric polynomial splines implemented in the 'carEx' package are piecewise polynomial functions on $k+1$ intervals formed by k knots partitioning the real line:

$$(-\infty, t_1], (t_1, t_2], ..., (t_{i-1}, t_i], ..., (t_k, \infty)$$

with degree d_i on the ith interval $(t_{i-1}, t_i]$, $i = 1, ..., k+1$, and order of continuity c_i at the ith knot, $i = 1, ..., k$.

The order of continuity refers to the highest order for which the derivatives of the polynomial on the interval to the left and to the right of a knot, t_i, have the same limits at t_i. For all orders above c_i, derivatives, if any, are not constained to have the same limit.

Such a spline is parametrized by three vectors: a vector of knots, $t_1 < t_2 < ... < t_k$, of length $k > 0$, a vector of polynomial degrees, $d_1, d_2, ..., d_{k+1}$, of length $k+1$, and a vector of orders of continuity or 'smoothness', $c_1, c_2, ..., c_k$, of length k.

Theory

We first describe the general principles that underly the implemention of splines in this package.

Let X_f be a $n \times q$ matrix for a model whose coefficients are subject to c linearly independent constraints given by a $c \times q$ matrix C. That is, the linear space for the model is:

$$\mathcal{M} = \{\eta = X_f \phi : \phi \in \mathbb{R}^q, C\phi = 0\}$$

We wish to construct a $n \times p$ design matrix X with $p = q - c$ so that

$$\mathcal{M} = \{\eta = X\beta : \beta \in \mathbb{R}^p\}$$

Supppose further that we want the parameters β to provide p specified linearly independent function of ϕ represented by the rows of the $p \times q$ matrix E whose rows are linearly independent of the rows of C to ensure that they are not equal to 0 on $\mathcal{M}$.

Consider the $q \times q$ partitioned matrix $\left[\begin{array}{c} C \\ E \end{array} \right]$. Since its rows are linearly independent, it is invertible and has a conformably partitioned inverse:

$$\left[\begin{array}{cc} F & G \end{array} \right] = \left[\begin{array}{c} C \\ E \end{array} \right]^{-1}$$

Thus $FC + GE = I$, $CF = I$, etc.

Consider the model matrix $X = X_f G$. We show that $\mathcal{M} = \{X\beta : \beta \in \mathbb{R}^p\}$ and that for any $\phi \in \mathbb{R}^q$, such that $C\phi = 0$, $\beta = E\phi$.

Suppose $C\phi = 0$. Then

$$\phi = \left[\begin{array}{cc} F & G \end{array} \right] \left[\begin{array}{c} C \\ E \end{array} \right] \phi = \left[\begin{array}{cc} F & G \end{array} \right] \left[\begin{array}{c} 0 \\ E\phi \end{array} \right] = GE\phi$$

Thus, with $\beta = E\phi$, we have

$$X_f \phi = X_f GE\phi = X\beta$$

We therefore have a 1-1 correspondence between $\beta \in \mathbb{R}^p$ and $\{\phi \in \mathbb{R}^q : C\phi = 0\}$ given by $\beta = E\phi$ and $\phi = G\beta$.

If X is of full rank, we can obtain the least-squares estimator $\hat{\beta} = (X'X)^{-1}X'Y$. We can then estimate any linear function $\psi = L\phi$ of ϕ under the constraint $C\phi = 0$ with the estimator $\hat{\psi} = A\hat{\beta}$ with

$$A = LG$$

Thus, the matrix G serves as a post-multiplier to transform X_f into a model matrix $X = X_f G$ that can be used in a linear model. The matrix G also serves as a post-multiplier to transform any general linear hypothesis matrix expressed in terms of ϕ into a general linear hypothesis matrix in terms of of β.

Application to Splines

Our goal is to generate model matrices for splines in a way that produces interpretable coefficients and lends itself to easily estimating and testing properties of the spline that are linear functions of parameters: slope, curvature, discontinuities, etc.

Given k knots, $-\infty = t_0 < t_1 < ... < t_k < t_{k+1} = \infty$, the spline in the ith interval, $(t_{i-1}, t_i]$, is a polynomial of degree d_i, a non-negative integer with the value 0 signifying a constant over the corresponding interval.

The order of smoothness c_i at t_i is either a non-negative integer or -1 to allow a discontinuity. (TODO: control direction of discontinuity)

Generating a model matrix for some piecewise polynomial functions is simple. For example, if the degrees, d_i, are non-decreasing and the order of continuity is a constant c less than $\min(d_i)$, one can add terms using 'plus' functions at each knot. For example, a quadratic spline (degree 2, continuity 1) with one knot at 1 can be generated with a model matrix with three columns, in addition to the intercept term:

$$x, x^2, (x-1)_+^2$$

where

$$(y)_+ = \begin{cases} 0 & \text{if } y < 0 \\ y & \text{otherwise} \end{cases}$$

A spline that is quadratic on the interval $(-\infty, 1]$ and cubic on $(1, \infty)$ with continuity of order 1, $c_1 = 1$, at $t_1 = 1$, can be generated by the columns:

$$x, x^2, (x-1)_+^2, (x-1)_+^3$$

However, if one allows the degree of the polynomial or the order of smoothness to vary in diffenret parts of the spline, the approach above works only in special cases.

Generating model matrices in more general situations, for example with degrees that are not monotone, nor monotone increasing as the index radiates from a central value, is more challenging. The approach described here works for any pattern of degrees, d_i and smoothness constraints, c_i.

We start by constucting a matrix, X_f, for a spline in which the polynomial degree in each interval is the maximal value, $\max(d_i)$. We then construct constraints for the coefficients of this model to produce the desired spline.

As an example, consider a spline, $\mathcal{S}$, with knots at 3 and 7, polynomial degrees, $(2, 3, 2)$, and smoothness, $(1, 2)$, meaning that $\mathcal{S}$ is smooth of order 1 at $x = 3$, and smooth of order 2 at $x = 7$. Columns of the full matrix X_f contain the intercept, linear and quadratic and cubic terms in each interval of the spline.

To create an instance of X_f we need to specify the values over which the matrix is evaluated. Evaluating X_f at $x = 0, 1, ...9$, we obtain the following matrix, which happens here to be block diagonal because of the ordering of the x values:

```r
Xf(0:9, knots = c(3,7), degree = 3)
```

```
     X0 X1 X2 X3 X0 X1 X2  X3 X0 X1 X2  X3
f(0)  1  0  0  0  0  0  0   0  0  0  0   0
f(1)  1  1  1  1  0  0  0   0  0  0  0   0
f(2)  1  2  4  8  0  0  0   0  0  0  0   0
f(3)  1  3  9 27  0  0  0   0  0  0  0   0
f(4)  0  0  0  0  1  4 16  64  0  0  0   0
f(5)  0  0  0  0  1  5 25 125  0  0  0   0
f(6)  0  0  0  0  1  6 36 216  0  0  0   0
f(7)  0  0  0  0  1  7 49 343  0  0  0   0
f(8)  0  0  0  0  0  0  0   0  1  8 64 512
f(9)  0  0  0  0  0  0  0   0  1  9 81 729
attr(,"class")
[1] "gspline_matrix" "matrix"
```

The model for the unconstrained maximal polynomial is $X_f\phi : \phi \in \mathbb{R}^{12}$.

We impose three types of constraints on ϕ.

1. $X_f\phi$ should evaluate to 0 at $x = 0$ so an intercept term in the model will have the correct interpretation,
2. the limits of the value and of the first derivative of the spline must be the same when approaching the first knot from the right or from the left, and the limits of the value, the first and second derivatives should be the same when approaching the second knot from the right or from the left, and
3. the degree of the polynomial in the first and third intervals must be reduced to 2.

The constraint marix, C is created by the 'Cmat' function:

```r
Cmat(knots = c(3, 7), degree = c(2, 3, 2), smooth = c(1, 2))
```

```
       X0 X1 X2  X3 X0 X1  X2   X3 X0 X1 X2  X3
f(0)    1  0  0   0  0  0   0    0  0  0  0   0
C(3).0 -1 -3 -9 -27  1  3   9   27  0  0  0   0
C(3).1  0 -1 -6 -27  0  1   6   27  0  0  0   0
C(7).0  0  0  0   0 -1 -7 -49 -343  1  7 49 343
C(7).1  0  0  0   0  0 -1 -14 -147  0  1 14 147
C(7).2  0  0  0   0  0  0  -2  -42  0  0  2  42
I.1.3   0  0  0   1  0  0   0    0  0  0  0   0
I.3.3   0  0  0   0  0  0   0    0  0  0  0   1
attr(,"ranks")
  npar.full         C.n      C.rank spline.rank
         12           8           8           4
attr(,"d")
[1] 536.66701452  48.80391245  10.85308819   3.18591258   0.97504352
[6]   0.81688866   0.35905212   0.08458296
```

The row labels of the constraint matrix show the role of each row. For example, "f(0)" is the value of the spline when $x = 0$ which is constrained to 0 so that an intercept term in a linear model can have its usual interpretation, "C(3).0" ensures continuity at $x = 3$, "C(7).2" forces continuity of the second derivative at $x = 7$, "I.1.3" constrains the cubic term to be 0 in the first interval, etc.

Attributes give the length of the ϕ vector as 'npar.full', the number of constraints as 'C.n', the rank of the constraint matrix as 'C.rank' and the rank of the spline, omitting the intercept term, as 'spline.rank'.

The 'd' attribute contains the vector of singular values of the constraint matrix.

The following is the matrix E of estimable functions created by the 'Emat' function:

```r
Emat(knots = c(3, 7), degree = c(2, 3, 2), smooth = c(1, 2))
```

```
       X0 X1 X2  X3 X0 X1 X2 X3 X0 X1 X2 X3
D1(0)   0  1  0   0  0  0  0  0  0  0  0  0
D2(0)   0  0  2   0  0  0  0  0  0  0  0  0
C(3).2  0  0 -2 -18  0  0  2 18  0  0  0  0
C(3).3  0  0  0  -6  0  0  0  6  0  0  0  0
```

The row labels signify the first derivative at $x = 0$, 'D1(0)', the second derivative at $x = 0$, 'D2(0)', the saltus in the second derivative at $x = 3$ and the saltus in the third derivative at $x = 3$.

The full rank model for the spline is generated by a matrix $X = X_f G$ as described in the previous section.

The spline modelling function is a closure generated by the `gspline` function.

```r
sp <- gspline(knots = c(3, 7), degree = c(2, 3, 2), smoothness = c(1, 2))
sp(0:9)
```

```
     D1(0) D2(0)       C(3).2        C(3).3
f(0)     0   0.0 0.000000e+00  0.000000e+00
f(1)     1   0.5 9.621933e-16  7.031412e-16
f(2)     2   2.0 1.813364e-15 -2.238950e-15
f(3)     3   4.5 2.664535e-15 -1.243450e-14
f(4)     4   8.0 5.000000e-01  1.666667e-01
f(5)     5  12.5 2.000000e+00  1.333333e+00
f(6)     6  18.0 4.500000e+00  4.500000e+00
f(7)     7  24.5 8.000000e+00  1.066667e+01
f(8)     8  32.0 1.250000e+01  2.066667e+01
f(9)     9  40.5 1.800000e+01  3.466667e+01
```

produce a matrix $X = X_f G$ that will generate the desired spline parametrized by linear estimable coefficients.

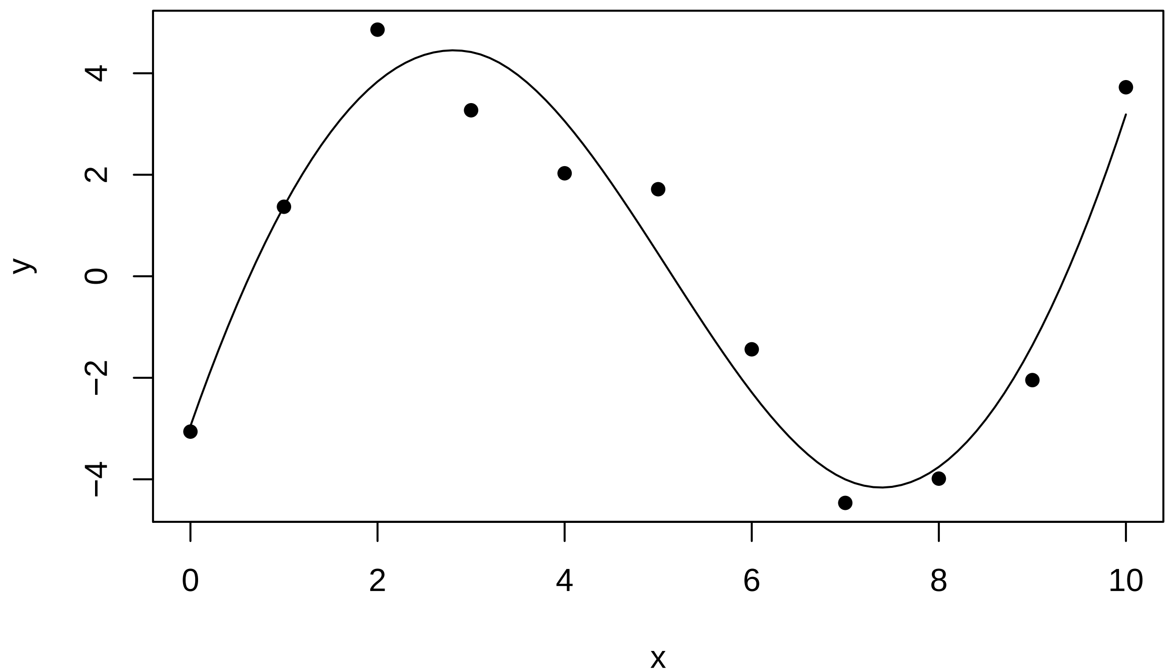
The closure created by the `gspline` function can be used in a linear model formulas. We illustrate its use with a small example. Note that the spline function can be used in any linear model formula. It can, for example, be modelled as interacting with other predictors.

```r
df <- data.frame(x = 0:10)
set.seed(123)
df <- within(df, y <- -2* (x-5) + .1 * (x-5)^3 + rnorm(x))
df <- rbind(df, data.frame(x = seq(0,10,.1), y = NA))
df <- sortdf(df, ~ x)
plot(y~x, df, pch = 16)
fit <- lm(y ~ sp(x), data = df)
summary(fit)
```

```

Call:
lm(formula = y ~ sp(x), data = df)

Residuals:
    Min      1Q  Median      3Q     Max 
-1.1476 -0.5748 -0.1091  0.6914  1.2704 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)   
(Intercept)  -2.9513     1.0165  -2.903  0.02721 * 
sp(x)D1(0)    5.2685     1.3117   4.017  0.00699 **
sp(x)D2(0)   -1.8747     0.6726  -2.787  0.03169 * 
```

```
sp(x)C(3).2  -0.5129     1.3846  -0.370  0.72381
sp(x)C(3).3   1.1346     0.2749   4.127  0.00616 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 1.064 on 6 degrees of freedom
  (101 observations deleted due to missingness)
Multiple R-squared:  0.9372,    Adjusted R-squared:  0.8954
F-statistic:  22.4 on 4 and 6 DF,  p-value: 0.0009419
```

```
lines(df$x , predict(fit, df))
```



Linear hypotheses

Linear hypotheses about a spline may be easy to formulate in terms of its 'full' parameter vector ϕ but challenging in terms of the 'working' parameters, β. For example, the derivative or curvature of the spline over a range of values is easily expressed in terms of ϕ. To do this We use the relationship between linear hypotheses in terms of ϕ with those in terms of β to generate linear hypotheses based on $\hat{\beta}$. Namely the least-squares estimator of $\psi = L\phi$ under the contraint $C\phi = 0$ is $\hat{\psi} = A\hat{\beta}$ where $A = LG$.

Given a spline function `sp` created by the `gspline` function:

```
sp <- gspline(knots = c(3,7), degree = c(2,3,2), smoothness = c(1,2))
sp(0:9)
```

```
     D1(0) D2(0)       C(3).2       C(3).3
f(0)     0   0.0 0.000000e+00 0.000000e+00
```

```
f(1)      1   0.5 9.621933e-16  7.031412e-16
f(2)      2   2.0 1.813364e-15 -2.238950e-15
f(3)      3   4.5 2.664535e-15 -1.243450e-14
f(4)      4   8.0 5.000000e-01  1.666667e-01
f(5)      5  12.5 2.000000e+00  1.333333e+00
f(6)      6  18.0 4.500000e+00  4.500000e+00
f(7)      7  24.5 8.000000e+00  1.066667e+01
f(8)      8  32.0 1.250000e+01  2.066667e+01
f(9)      9  40.5 1.800000e+01  3.466667e+01
```

The `sp` function will generate a hypothesis matrix to query values and derivatives of the spline.

```
sp(c(2, 3, 7), D = 1)
```

Denoting the matrix above by A, $A\hat{\beta}$ will estimate the first derivative of the spline at $x = 2$ and its limit from the right at the knots $x = 3, 7$. The `limit` parameter to the `sc` function is used to select whether the value estimated is a limit from the right, from the left, or the saltus (jump) in value if discontinuous. For example, at $x = 3$ where the spline has a discontinuous second derivatives:

```
sp(c(3, 3, 3), D = 2, limit = c(-1,0,1))
```

Using the 'wald' function it is possible to graph these estimates as a function of of x.

```
# xpred <- seq(0,10, .05)
# A.1 <- cbind(0, sp(xpred, D = 1))
# A.2 <- cbind(0, sp(xpred, D = 2))
# ww.1 <- as.data.frame(wald(fit, A.1))
# ww.2 <- as.data.frame(wald(fit, A.2))
#
# plot(xpred, ww.1$coef, type = 'l', lwd = 2)
# plot(xpred, ww.2$coef, type = 'l', lwd = 2)
# library(latticeExtra)
# ww.1$x <- xpred
# xyplot(coef ~ x, ww.1, type = 'l',
#      lower = ww.1$L2, upper = ww.1$U2,
#      subscripts = TRUE) +
#   layer(gpanel.fit(...))
# head(ww.1)
```

```
knitr::knit_exit()
```

Finer control

The approach detailed above generates splines with arbitrary degrees in each interval and arbitrary orders of smoothness, i.e. continuous derivatives of all orders up to a specified order, and unconstrained for higher orders up to the degree of the polynomials.

One can have finer control to generate more complex splines, for example periodic splines and splines with a possible disconuity at a lower order but with continuity at some higher order(s). This can be done by explicitly specifying the constraint matrix or by specifying the smoothness parameter as a list with each element of the list consisting of a vector indicating which orders of derivatives are constrained to be continuous. It is thus possible to fit a spline with a discontinuity at a knot but first and second derivatives that have the same right and left limits at the knot.

For example, consider a spline that is cubic above and below a knot at $x = 1$, with a possible jump in value at $x = 5$, but the same slope and curvature to the right and left of $x = 1$ and a possibly different third derivative on either side of the knot.

The constraint matrix to impose continuity of all orders is:

```
Cmat(knots = 5, degree = c(3,3), smooth = 3)
```

```
       X0 X1  X2   X3 X0 X1 X2  X3
f(0)    1  0   0    0  0  0  0   0
C(5).0 -1 -5 -25 -125  1  5 25 125
C(5).1  0 -1 -10  -75  0  1 10  75
C(5).2  0  0  -2  -30  0  0  2  30
C(5).3  0  0   0   -6  0  0  0   6
attr(,"ranks")
  npar.full         C.n      C.rank spline.rank
          8           5           5           3
attr(,"d")
[1] 214.0457593   8.2534602   1.0202829   0.5009444   0.0375901
```

To relax the constraint for the value of the function and for the third derivative, we select only the rows of the constraint matrix for which smoothness is desired:

```
C <- Cmat(5, c(3,3), 3)[c(1,3,4),]
C
```

```
       X0 X1  X2  X3 X0 X1 X2 X3
f(0)    1  0   0   0  0  0  0  0
C(5).1  0 -1 -10 -75  0  1 10 75
C(5).2  0  0  -2 -30  0  0  2 30
```

The function to fit this spline in a linear model can then be generated with the 'lin' paramter of 'gsp':

```
sp2 <- function(x) gsp(x, knots = 5, smooth = -1, degree = c(3,3), lin = C)
sp2(seq(0,2,.5))
```

```
       D1(0) D2(0)      D3(0)        C(5).0        C(5).3
f(0)     0.0 0.000 0.00000000  0.000000e+00  0.000000e+00
f(0.5)   0.5 0.125 0.02083333 -5.273559e-18 -1.509903e-16
f(1)     1.0 0.500 0.16666667 -8.881784e-18 -1.421085e-16
f(1.5)   1.5 1.125 0.56250000 -9.159340e-18 -7.993606e-17
f(2)     2.0 2.000 1.33333333 -4.440892e-18 -7.105427e-17
attr(,"spline.attr")
attr(,"spline.attr")$knots
[1] 5

attr(,"spline.attr")$degree
[1] 3 3

attr(,"spline.attr")$smooth
[1] -1

attr(,"spline.attr")$lin
       X0 X1  X2  X3 X0 X1 X2 X3
f(0)    1  0   0   0  0  0  0  0
C(5).1  0 -1 -10 -75  0  1 10 75
C(5).2  0  0  -2 -30  0  0  2 30

attr(,"spline.attr")$intercept
[1] 0
```

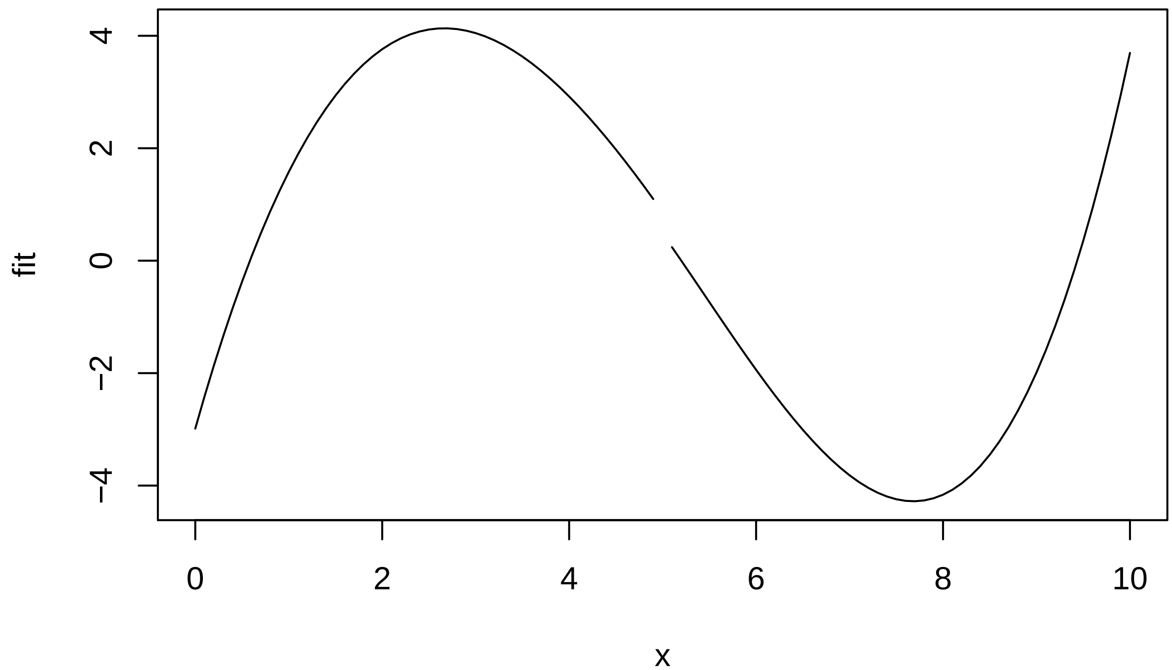
```
attr(,"spline.attr")$signif
[1] 3

attr(,"class")
[1] "gsp"
```

Fitting this spline to our previous data:

```r
fit <- lm(y ~ sp2(x), df)
summary(fit)
```

```

Call:
lm(formula = y ~ sp2(x), data = df)

Residuals:
       1        2        3        4        5        6        7        8 
-0.07363 -0.20805  1.09727 -0.77709 -0.89083  0.85233  0.50106 -0.64811 
       9       10       11 
 0.17563 -0.05761  0.02903 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)   
(Intercept)   -2.9868     0.8612  -3.468  0.01788 * 
sp2(x)D1(0)    5.9166     1.1881   4.980  0.00418 **
sp2(x)D2(0)   -2.8649     0.8342  -3.434  0.01855 * 
sp2(x)D3(0)    0.4838     0.2386   2.027  0.09844 . 
sp2(x)C(5).0  -0.3835     1.5854  -0.242  0.81847   
sp2(x)C(5).3   0.5049     0.4842   1.043  0.34483   
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.9036 on 5 degrees of freedom
  (101 observations deleted due to missingness)
Multiple R-squared:  0.9623,	Adjusted R-squared:  0.9245 
F-statistic:  25.5 on 5 and 5 DF,  p-value: 0.001442
```

```r
df$fit <- predict(fit, df)
df$fit[df$x == 5] <- NA
plot(fit ~ x, df, type = 'l')
```

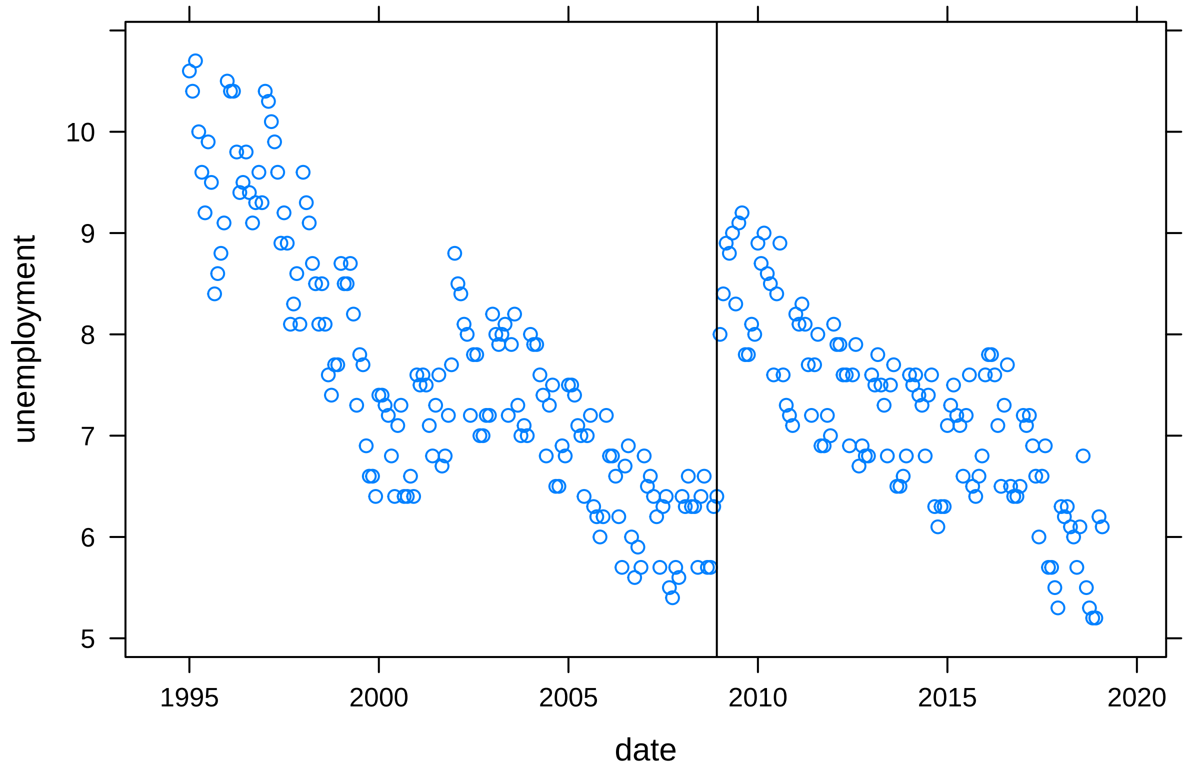


Periodic splines

```r
unemp <- as.data.frame(spida2::Unemp)
head(unemp)
```

```
        date unemployment x
1 1995-01-01         10.6 1
2 1995-02-01         10.4 2
3 1995-03-01         10.7 3
4 1995-04-01         10.0 4
5 1995-05-01          9.6 5
6 1995-06-01          9.2 6
```

```r
library(latticeExtra)
xyplot(unemployment ~ date, unemp) + layer(panel.abline(v = as.Date('2008-12-01')))
```

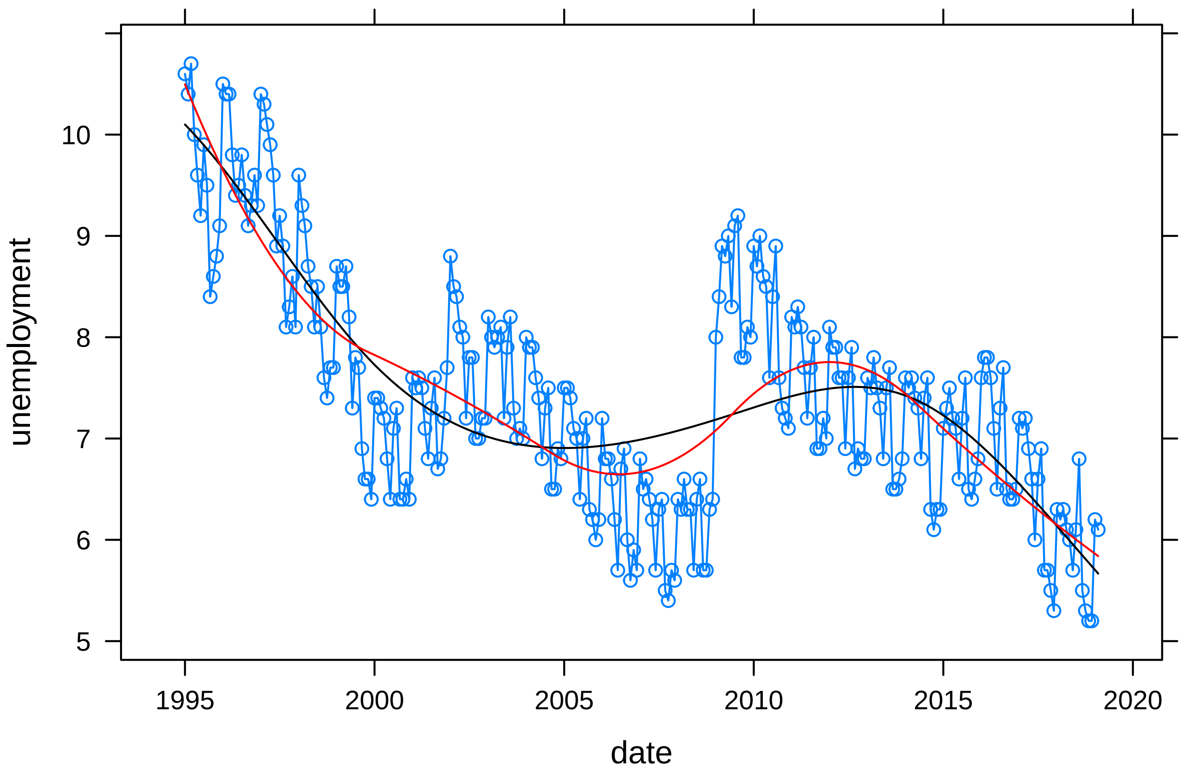



```
toyear <- function(x) {
  (as.numeric(x) - as.numeric(as.Date('2000-01-01')))/365.25
}

unemp <- within(
  unemp,
  {
    year <- toyear(date)
    month <- as.numeric(format(date, '%m'))

  })
summary(unemp)
```

```
      date            unemployment         x              month
 Min.   :1995-01-01   Min.   : 5.200   Min.   :  1.00   Min.   : 1.000
 1st Qu.:2001-01-08   1st Qu.: 6.600   1st Qu.: 73.25   1st Qu.: 3.000
 Median :2007-01-16   Median : 7.300   Median :145.50   Median : 6.000
 Mean   :2007-01-15   Mean   : 7.448   Mean   :145.50   Mean   : 6.466
 3rd Qu.:2013-01-24   3rd Qu.: 8.100   3rd Qu.:217.75   3rd Qu.: 9.000
 Max.   :2019-02-01   Max.   :10.700   Max.   :290.00   Max.   :12.000
      year
 Min.   :-4.999
 1st Qu.: 1.023
 Median : 7.043
 Mean   : 7.041
 3rd Qu.:13.066
```

```
 Max.   :19.086
```

```
quintiles <- quantile(unemp$year, 1:4/5)
sp3 <- gspline(quintiles, 3, 2)
sp2 <- gspline(quintiles, 2, 1)
fit2 <- lm(unemployment ~ sp2(year), unemp)
unemp$fit2 <- predict(fit2)
fit3 <- lm(unemployment ~ sp3(year), unemp)
unemp$fit3 <- predict(fit3)



pp <- xyplot(unemployment ~ date, unemp, type = 'b') +
  layer(panel.lines(x, unemp$fit3, col = 'black')) +
  layer(panel.lines(x, unemp$fit2, col = 'red'))
pp
```

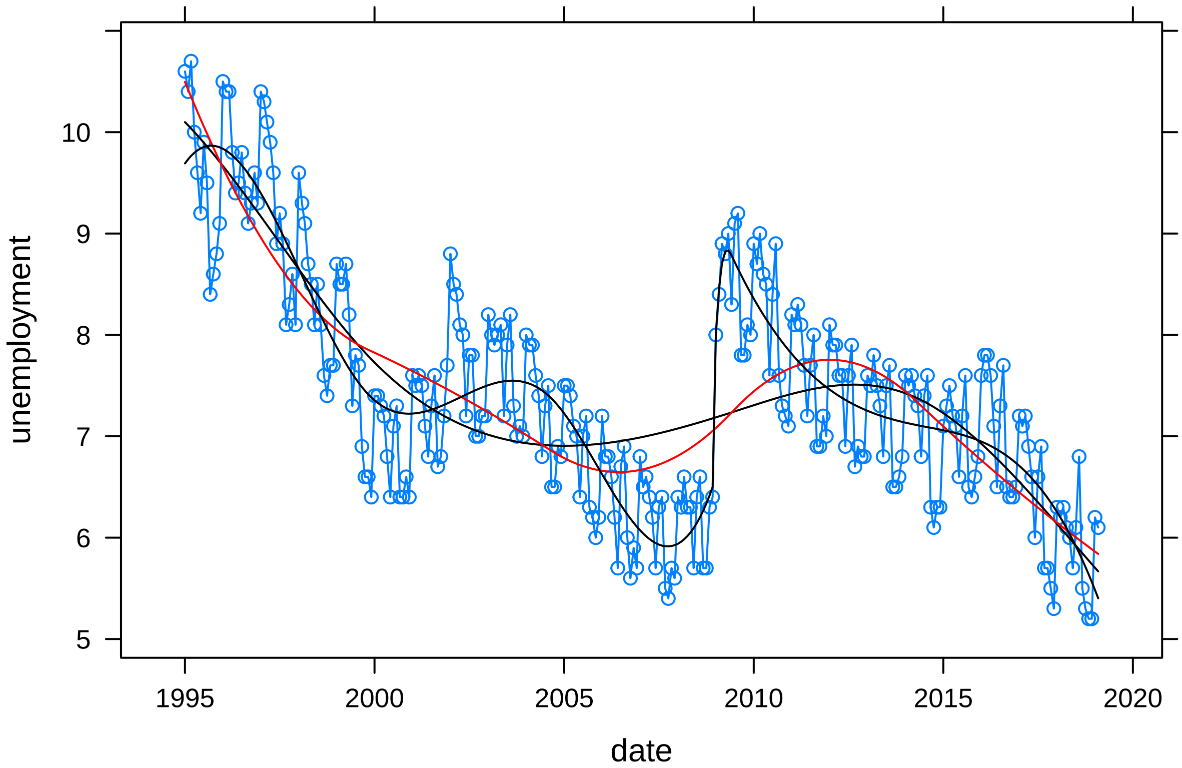



```
crash2008 <- toyear(as.Date('2008-12-15'))
knots08 <- sort(c(quintiles, crash2008))
knots08
```

```
       20%        40%                    60%        80%
-0.1839836  4.6340862  8.9555099  9.4483231 14.2642026
```

```
sp08 <- gspline(knots08, 3, c(2,2,-1,2,2))
fit08 <- lm(unemployment ~ sp08(year), unemp)
unemp$fit08 <- predict(fit08)
pp <- pp + layer(panel.lines(x, unemp$fit08, col = 'black'))
```

```r
pp
```



Periodic spline and Fourier analysis

```r
per3 <- gspline(12 * 1:5/5, 3, 2, periodic = TRUE)
per3
```

```
Spline function created by gspline
$A
               X0   X1    X2      X3 X0   X1     X2       X3 X0   X1     X2
C(2.4).0       -1 -2.4 -5.76 -13.824  1  2.4   5.76   13.824  0  0.0   0.00
C(2.4).1        0 -1.0 -4.80 -17.280  0  1.0   4.80   17.280  0  0.0   0.00
C(2.4).2        0  0.0 -2.00 -14.400  0  0.0   2.00   14.400  0  0.0   0.00
C(4.8).0        0  0.0  0.00   0.000 -1 -4.8 -23.04 -110.592  1  4.8  23.04
C(4.8).1        0  0.0  0.00   0.000  0 -1.0  -9.60  -69.120  0  1.0   9.60
C(4.8).2        0  0.0  0.00   0.000  0  0.0  -2.00  -28.800  0  0.0   2.00
C(7.2).0        0  0.0  0.00   0.000  0  0.0   0.00    0.000 -1 -7.2 -51.84
C(7.2).1        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0 -1.0 -14.40
C(7.2).2        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0  -2.00
C(9.6).0        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).1        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).2        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).0  1  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).2  0  0.0  2.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
D1(12 mod 12)  0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
```

```
D2(12 mod 12)  0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
D3(12 mod 12)  0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(2.4).3       0  0.0  0.00  -6.000  0  0.0   0.00    6.000  0  0.0   0.00
                    X3 X0   X1      X2        X3 X0    X1      X2         X3
C(2.4).0          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(2.4).1          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(2.4).2          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).0        110.592  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).1         69.120  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).2         28.800  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(7.2).0       -373.248  1  7.2  51.84   373.248  0   0.0    0.00      0.000
C(7.2).1       -155.520  0  1.0  14.40   155.520  0   0.0    0.00      0.000
C(7.2).2        -43.200  0  0.0   2.00    43.200  0   0.0    0.00      0.000
C(9.6).0          0.000 -1 -9.6 -92.16 -884.736  1   9.6   92.16    884.736
C(9.6).1          0.000  0 -1.0 -19.20 -276.480  0   1.0   19.20    276.480
C(9.6).2          0.000  0  0.0  -2.00  -57.600  0   0.0    2.00     57.600
C(0 mod 12).0     0.000  0  0.0   0.00     0.000 -1 -12.0 -144.00 -1728.000
C(0 mod 12).1     0.000  0  0.0   0.00     0.000  0  -1.0  -24.00  -432.000
C(0 mod 12).2     0.000  0  0.0   0.00     0.000  0   0.0   -2.00   -72.000
f(12 mod 12)      0.000  0  0.0   0.00     0.000  1  12.0  144.00  1728.000
D1(12 mod 12)     0.000  0  0.0   0.00     0.000  0   1.0   24.00   432.000
D2(12 mod 12)     0.000  0  0.0   0.00     0.000  0   0.0    2.00    72.000
D3(12 mod 12)     0.000  0  0.0   0.00     0.000  0   0.0    0.00     6.000
C(2.4).3          0.000  0  0.0   0.00     0.000  0   0.0    0.00     0.000

$Cmat
               X0   X1    X2       X3 X0   X1      X2        X3 X0   X1     X2
C(2.4).0       -1 -2.4 -5.76 -13.824  1  2.4   5.76   13.824  0  0.0   0.00
C(2.4).1        0 -1.0 -4.80 -17.280  0  1.0   4.80   17.280  0  0.0   0.00
C(2.4).2        0  0.0 -2.00 -14.400  0  0.0   2.00   14.400  0  0.0   0.00
C(4.8).0        0  0.0  0.00   0.000 -1 -4.8 -23.04 -110.592  1  4.8  23.04
C(4.8).1        0  0.0  0.00   0.000  0 -1.0  -9.60  -69.120  0  1.0   9.60
C(4.8).2        0  0.0  0.00   0.000  0  0.0  -2.00  -28.800  0  0.0   2.00
C(7.2).0        0  0.0  0.00   0.000  0  0.0   0.00    0.000 -1 -7.2 -51.84
C(7.2).1        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0 -1.0 -14.40
C(7.2).2        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0  -2.00
C(9.6).0        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).1        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).2        0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).0  1  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).2  0  0.0  2.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
                    X3 X0   X1      X2        X3 X0    X1      X2         X3
C(2.4).0          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(2.4).1          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(2.4).2          0.000  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).0        110.592  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).1         69.120  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(4.8).2         28.800  0  0.0   0.00     0.000  0   0.0    0.00      0.000
C(7.2).0       -373.248  1  7.2  51.84   373.248  0   0.0    0.00      0.000
C(7.2).1       -155.520  0  1.0  14.40   155.520  0   0.0    0.00      0.000
C(7.2).2        -43.200  0  0.0   2.00    43.200  0   0.0    0.00      0.000
C(9.6).0          0.000 -1 -9.6 -92.16 -884.736  1   9.6   92.16    884.736
```

```
C(9.6).1         0.000  0 -1.0 -19.20 -276.480  0   1.0   19.20   276.480
C(9.6).2         0.000  0  0.0  -2.00  -57.600  0   0.0    2.00    57.600
C(0 mod 12).0    0.000  0  0.0   0.00    0.000 -1 -12.0 -144.00 -1728.000
C(0 mod 12).1    0.000  0  0.0   0.00    0.000  0  -1.0  -24.00  -432.000
C(0 mod 12).2    0.000  0  0.0   0.00    0.000  0   0.0   -2.00   -72.000
f(12 mod 12)     0.000  0  0.0   0.00    0.000  1  12.0  144.00  1728.000

$constraint_mat
              X0   X1    X2      X3 X0   X1     X2       X3 X0   X1     X2
C(2.4).0      -1 -2.4 -5.76 -13.824  1  2.4   5.76   13.824  0  0.0   0.00
C(2.4).1       0 -1.0 -4.80 -17.280  0  1.0   4.80   17.280  0  0.0   0.00
C(2.4).2       0  0.0 -2.00 -14.400  0  0.0   2.00   14.400  0  0.0   0.00
C(4.8).0       0  0.0  0.00   0.000 -1 -4.8 -23.04 -110.592  1  4.8  23.04
C(4.8).1       0  0.0  0.00   0.000  0 -1.0  -9.60  -69.120  0  1.0   9.60
C(4.8).2       0  0.0  0.00   0.000  0  0.0  -2.00  -28.800  0  0.0   2.00
C(7.2).0       0  0.0  0.00   0.000  0  0.0   0.00    0.000 -1 -7.2 -51.84
C(7.2).1       0  0.0  0.00   0.000  0  0.0   0.00    0.000  0 -1.0 -14.40
C(7.2).2       0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0  -2.00
C(9.6).0       0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).1       0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(9.6).2       0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).0  1  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
C(0 mod 12).2  0  0.0  2.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00   0.000  0  0.0   0.00    0.000  0  0.0   0.00
                    X3 X0   X1     X2       X3 X0    X1      X2        X3
C(2.4).0         0.000  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(2.4).1         0.000  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(2.4).2         0.000  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(4.8).0       110.592  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(4.8).1        69.120  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(4.8).2        28.800  0  0.0   0.00    0.000  0   0.0    0.00     0.000
C(7.2).0      -373.248  1  7.2  51.84  373.248  0   0.0    0.00     0.000
C(7.2).1      -155.520  0  1.0  14.40  155.520  0   0.0    0.00     0.000
C(7.2).2       -43.200  0  0.0   2.00   43.200  0   0.0    0.00     0.000
C(9.6).0         0.000 -1 -9.6 -92.16 -884.736  1   9.6   92.16   884.736
C(9.6).1         0.000  0 -1.0 -19.20 -276.480  0   1.0   19.20   276.480
C(9.6).2         0.000  0  0.0  -2.00  -57.600  0   0.0    2.00    57.600
C(0 mod 12).0    0.000  0  0.0   0.00    0.000 -1 -12.0 -144.00 -1728.000
C(0 mod 12).1    0.000  0  0.0   0.00    0.000  0  -1.0  -24.00  -432.000
C(0 mod 12).2    0.000  0  0.0   0.00    0.000  0   0.0   -2.00   -72.000
f(12 mod 12)     0.000  0  0.0   0.00    0.000  1  12.0  144.00  1728.000

$constraints
NULL

$debug
[1] FALSE

$degree
[1] 3 3 3 3 3 3

$Dmat_smoothness_indices
 [1]  1  5  6  7  9 10 11 13 14 15 17 18 19 21 22 23
```

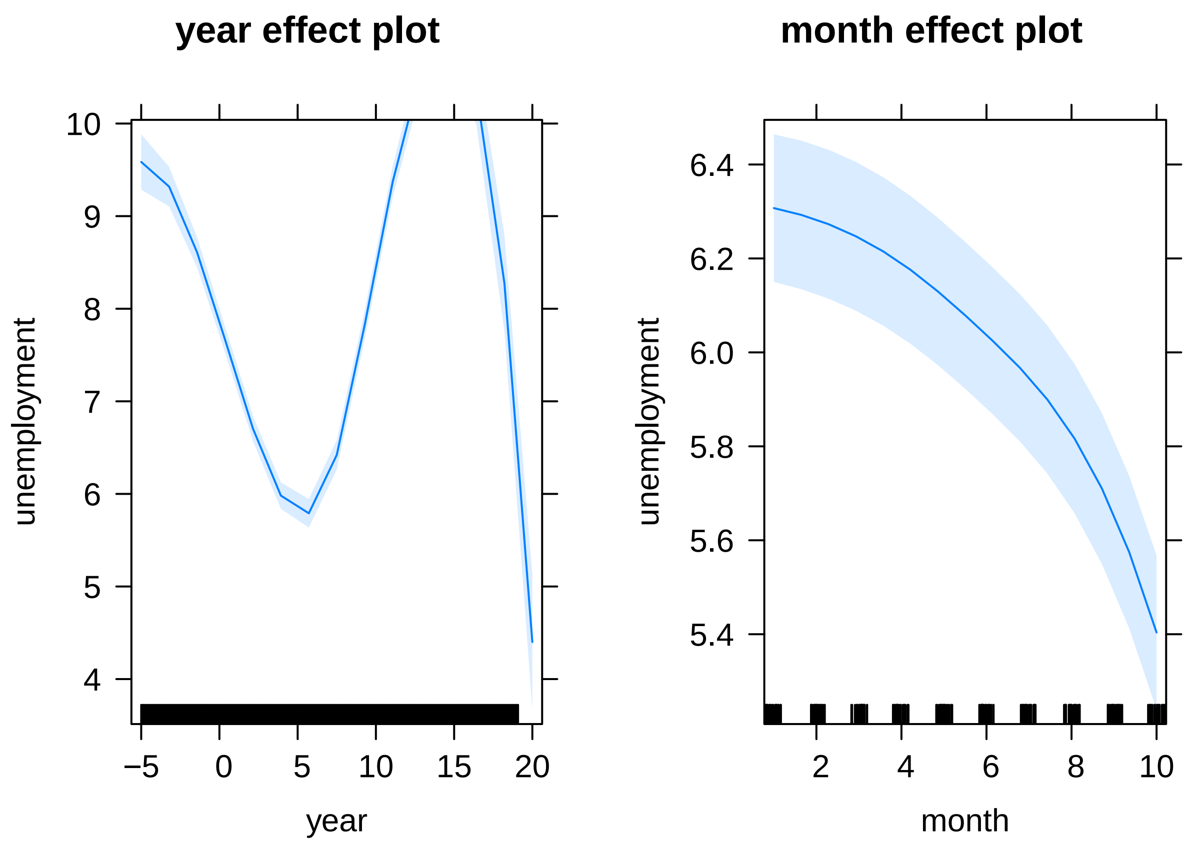
```
$Emat
                X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2  X3
D1(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1 24 432
D2(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  2  72
D3(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0   6
C(2.4).3       0  0  0 -6  0  0  0  6  0  0  0  0  0  0  0  0  0  0  0   0

$estimate_mat
                X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2 X3 X0 X1 X2  X3
D1(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  1 24 432
D2(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  2  72
D3(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0   6
C(2.4).3       0  0  0 -6  0  0  0  6  0  0  0  0  0  0  0  0  0  0  0   0

$estimates
NULL

$G
   D1(12 mod 12) D2(12 mod 12) D3(12 mod 12)      C(2.4).3
X0  0.000000e+00  0.000000e+00  0.000000e+00  3.944305e-31
X1  1.000000e+00 -7.956598e-15  1.063224e-13  1.350771e-15
X2 -4.625929e-17  5.000000e-01  7.123931e-15 -2.312965e-17
X3 -3.616898e-02 -4.340278e-02 -4.166667e-02 -4.166667e-02
X0  3.552714e-15 -4.263256e-16  9.521273e-15 -2.304000e+00
X1  1.000000e+00 -1.101341e-14  1.315392e-13  2.880000e+00
X2 -1.261617e-15  5.000000e-01 -1.887379e-14 -1.200000e+00
X3 -3.616898e-02 -4.340278e-02 -4.166667e-02  1.250000e-01
X0 -2.400000e+01  9.600000e+00 -4.608000e+01  2.534400e+01
X1  1.600000e+01 -6.000000e+00  2.880000e+01 -1.440000e+01
X2 -3.125000e+00  1.750000e+00 -6.000000e+00  2.400000e+00
X3  1.808449e-01 -1.302083e-01  3.750000e-01 -1.250000e-01
X0  8.400000e+01 -1.200000e+02  2.649600e+02 -3.686400e+01
X1 -2.900000e+01  4.800000e+01 -1.008000e+02  1.152000e+01
X2  3.125000e+00 -5.750000e+00  1.200000e+01 -1.200000e+00
X3 -1.085069e-01  2.170139e-01 -4.583333e-01  4.166667e-02
X0 -1.200000e+01  7.200000e+01 -2.880000e+02  0.000000e+00
X1  1.000000e+00 -1.200000e+01  7.200000e+01  0.000000e+00
X2  0.000000e+00  5.000000e-01 -6.000000e+00  0.000000e+00
X3  0.000000e+00  0.000000e+00  1.666667e-01  0.000000e+00

$intercept
[1] 0

$knots
[1]  2.4  4.8  7.2  9.6 12.0

$max_degree
[1] 3

$periodic
[1] TRUE

$smoothness
```

```
$smoothness[[1]]
[1] 0 1 2

$smoothness[[2]]
[1] 0 1 2

$smoothness[[3]]
[1] 0 1 2

$smoothness[[4]]
[1] 0 1 2

$smoothness[[5]]
[1] 0 1 2


$tolerance
[1] 1e-16
```

```
fitper3 <-  lm(unemployment ~ sp08(year) + per3(month), unemp)
summary(fitper3)
```

```

Call:
lm(formula = unemployment ~ sp08(year) + per3(month), data = unemp)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.01225 -0.23279 -0.00892  0.20521  1.18149 

Coefficients:
                           Estimate Std. Error t value Pr(>|t|)    
(Intercept)                 7.09603    0.07537  94.143  < 2e-16 ***
sp08(year)D1(0)            -0.32959    0.02339 -14.091  < 2e-16 ***
sp08(year)D2(0)             0.46948    0.03300  14.225  < 2e-16 ***
sp08(year)D3(0)            -0.20769    0.01484 -13.994  < 2e-16 ***
sp08(year)C(-0.184).3      -0.52094    0.04262 -12.223  < 2e-16 ***
sp08(year)C(4.63).3         0.62740    0.05253  11.942  < 2e-16 ***
sp08(year)C(8.96).0         0.11441    0.46534   0.246 0.805973    
sp08(year)C(8.96).1        11.37552    3.12632   3.639 0.000327 ***
sp08(year)C(8.96).2       -56.50467   13.07145  -4.323 2.16e-05 ***
sp08(year)C(8.96).3       112.04499   26.63734   4.206 3.52e-05 ***
sp08(year)C(9.45).3      -112.50572   26.65288  -4.221 3.31e-05 ***
sp08(year)C(14.3).3        -0.05644    0.06728  -0.839 0.402317    
per3(month)D1(12 mod 12)    0.45981    0.02659  17.290  < 2e-16 ***
per3(month)D2(12 mod 12)    0.24369    0.05616   4.339 2.02e-05 ***
per3(month)D3(12 mod 12)   -0.03609    0.04414  -0.818 0.414265    
per3(month)C(2.4).3         0.87800    0.08373  10.486  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3545 on 274 degrees of freedom
Multiple R-squared:  0.9119,	Adjusted R-squared:  0.907 
F-statistic:   189 on 15 and 274 DF,  p-value: < 2.2e-16
```

```
unemp$fitper3 <- predict(fitper3)
library(effects)
allEffects(fitper3) %>% plot
```




```
tab(unemp, ~ month)
```

```
month
    1     2     3     4     5     6     7     8     9    10    11    12
   25    25    24    24    24    24    24    24    24    24    24    24
Total
  290
```

```
Lper3 <- per3(rep(12*0:5/5, each = 9), D = rep(rep(1:3, each = 3),3), limit = rep(c(-1,0,1), 9))
wald(fitper3)
```

```
 numDF denDF  F.value p.value
    16   274 8179.134 <.00001
                          Estimate Std.Error  DF    t-value p-value
(Intercept)               7.096028  0.075375 274  94.143094 <.00001
sp08(year)D1(0)          -0.329592  0.023390 274 -14.091384 <.00001
sp08(year)D2(0)           0.469479  0.033004 274  14.224989 <.00001
sp08(year)D3(0)          -0.207694  0.014842 274 -13.993502 <.00001
sp08(year)C(-0.184).3    -0.520942  0.042618 274 -12.223490 <.00001
sp08(year)C(4.63).3       0.627396  0.052535 274  11.942477 <.00001
sp08(year)C(8.96).0       0.114410  0.465339 274   0.245863 0.80597
sp08(year)C(8.96).1      11.375517  3.126318 274   3.638631 0.00033
sp08(year)C(8.96).2     -56.504667 13.071453 274  -4.322753 0.00002
```

```
sp08(year)C(8.96).3       112.044989 26.637341 274   4.206313 0.00004
sp08(year)C(9.45).3      -112.505721 26.652881 274  -4.221147 0.00003
sp08(year)C(14.3).3        -0.056437  0.067283 274  -0.838793 0.40232
per3(month)D1(12 mod 12)    0.459807  0.026594 274  17.289671 <.00001
per3(month)D2(12 mod 12)    0.243691  0.056165 274   4.338873 0.00002
per3(month)D3(12 mod 12)   -0.036095  0.044144 274  -0.817655 0.41426
per3(month)C(2.4).3         0.877996  0.083731 274  10.485866 <.00001
                          Lower 0.95 Upper 0.95
(Intercept)                 6.947640   7.244415
sp08(year)D1(0)            -0.375638  -0.283546
sp08(year)D2(0)             0.404506   0.534452
sp08(year)D3(0)            -0.236913  -0.178475
sp08(year)C(-0.184).3      -0.604842  -0.437041
sp08(year)C(4.63).3         0.523972   0.730819
sp08(year)C(8.96).0        -0.801685   1.030504
sp08(year)C(8.96).1         5.220861  17.530174
sp08(year)C(8.96).2       -82.237909 -30.771425
sp08(year)C(8.96).3        59.605131 164.484847
sp08(year)C(9.45).3      -164.976172 -60.035270
sp08(year)C(14.3).3        -0.188894   0.076021
per3(month)D1(12 mod 12)    0.407452   0.512163
per3(month)D2(12 mod 12)    0.133122   0.354260
per3(month)D3(12 mod 12)   -0.123000   0.050810
per3(month)C(2.4).3         0.713158   1.042835
```

```
L <- cbind(0,0,0,0,0,0,0,0,0,0,0,0 ,
           per3(rep(12*0:5/5, each = 9), D = rep(rep(1:3, each = 3),3), limit = rep(c(-1,0,1), 9)))
rownames(L) <- rownames(Lper3)


per2 <- gspline(12 * 1:5/5, 2, 1, periodic = TRUE)
per2
```

```
Spline function created by gspline
$A
              X0   X1    X2 X0   X1     X2 X0   X1     X2 X0   X1     X2
C(2.4).0      -1 -2.4 -5.76  1  2.4   5.76  0  0.0   0.00  0  0.0   0.00
C(2.4).1       0 -1.0 -4.80  0  1.0   4.80  0  0.0   0.00  0  0.0   0.00
C(4.8).0       0  0.0  0.00 -1 -4.8 -23.04  1  4.8  23.04  0  0.0   0.00
C(4.8).1       0  0.0  0.00  0 -1.0  -9.60  0  1.0   9.60  0  0.0   0.00
C(7.2).0       0  0.0  0.00  0  0.0   0.00 -1 -7.2 -51.84  1  7.2  51.84
C(7.2).1       0  0.0  0.00  0  0.0   0.00  0 -1.0 -14.40  0  1.0  14.40
C(9.6).0       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00 -1 -9.6 -92.16
C(9.6).1       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0 -1.0 -19.20
C(0 mod 12).0  1  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
D1(12 mod 12)  0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
D2(12 mod 12)  0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
C(2.4).2       0  0.0 -2.00  0  0.0   2.00  0  0.0   0.00  0  0.0   0.00
C(4.8).2       0  0.0  0.00  0  0.0  -2.00  0  0.0   2.00  0  0.0   0.00
              X0   X1    X2
C(2.4).0       0   0.0   0.00
C(2.4).1       0   0.0   0.00
C(4.8).0       0   0.0   0.00
```

```
C(4.8).1        0   0.0    0.00
C(7.2).0        0   0.0    0.00
C(7.2).1        0   0.0    0.00
C(9.6).0        1   9.6   92.16
C(9.6).1        0   1.0   19.20
C(0 mod 12).0 -1 -12.0 -144.00
C(0 mod 12).1  0  -1.0  -24.00
f(12 mod 12)   1  12.0  144.00
D1(12 mod 12)  0   1.0   24.00
D2(12 mod 12)  0   0.0    2.00
C(2.4).2       0   0.0    0.00
C(4.8).2       0   0.0    0.00

$Cmat
              X0   X1    X2 X0   X1     X2 X0   X1     X2 X0   X1     X2
C(2.4).0      -1 -2.4 -5.76  1  2.4   5.76  0  0.0   0.00  0  0.0   0.00
C(2.4).1       0 -1.0 -4.80  0  1.0   4.80  0  0.0   0.00  0  0.0   0.00
C(4.8).0       0  0.0  0.00 -1 -4.8 -23.04  1  4.8  23.04  0  0.0   0.00
C(4.8).1       0  0.0  0.00  0 -1.0  -9.60  0  1.0   9.60  0  0.0   0.00
C(7.2).0       0  0.0  0.00  0  0.0   0.00 -1 -7.2 -51.84  1  7.2  51.84
C(7.2).1       0  0.0  0.00  0  0.0   0.00  0 -1.0 -14.40  0  1.0  14.40
C(9.6).0       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00 -1 -9.6 -92.16
C(9.6).1       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0 -1.0 -19.20
C(0 mod 12).0  1  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
              X0    X1      X2
C(2.4).0       0   0.0    0.00
C(2.4).1       0   0.0    0.00
C(4.8).0       0   0.0    0.00
C(4.8).1       0   0.0    0.00
C(7.2).0       0   0.0    0.00
C(7.2).1       0   0.0    0.00
C(9.6).0       1   9.6   92.16
C(9.6).1       0   1.0   19.20
C(0 mod 12).0 -1 -12.0 -144.00
C(0 mod 12).1  0  -1.0  -24.00
f(12 mod 12)   1  12.0  144.00

$constraint_mat
              X0   X1    X2 X0   X1     X2 X0   X1     X2 X0   X1     X2
C(2.4).0      -1 -2.4 -5.76  1  2.4   5.76  0  0.0   0.00  0  0.0   0.00
C(2.4).1       0 -1.0 -4.80  0  1.0   4.80  0  0.0   0.00  0  0.0   0.00
C(4.8).0       0  0.0  0.00 -1 -4.8 -23.04  1  4.8  23.04  0  0.0   0.00
C(4.8).1       0  0.0  0.00  0 -1.0  -9.60  0  1.0   9.60  0  0.0   0.00
C(7.2).0       0  0.0  0.00  0  0.0   0.00 -1 -7.2 -51.84  1  7.2  51.84
C(7.2).1       0  0.0  0.00  0  0.0   0.00  0 -1.0 -14.40  0  1.0  14.40
C(9.6).0       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00 -1 -9.6 -92.16
C(9.6).1       0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0 -1.0 -19.20
C(0 mod 12).0  1  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
C(0 mod 12).1  0  1.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
f(12 mod 12)   0  0.0  0.00  0  0.0   0.00  0  0.0   0.00  0  0.0   0.00
              X0    X1      X2
C(2.4).0       0   0.0    0.00
```

```
C(2.4).1       0   0.0    0.00
C(4.8).0       0   0.0    0.00
C(4.8).1       0   0.0    0.00
C(7.2).0       0   0.0    0.00
C(7.2).1       0   0.0    0.00
C(9.6).0       1   9.6   92.16
C(9.6).1       0   1.0   19.20
C(0 mod 12).0 -1 -12.0 -144.00
C(0 mod 12).1  0  -1.0  -24.00
f(12 mod 12)   1  12.0  144.00

$constraints
NULL

$debug
[1] FALSE

$degree
[1] 2 2 2 2 2 2

$Dmat_smoothness_indices
 [1]  1  4  5  7  8 10 11 13 14 16 17

$Emat
              X0 X1 X2 X0 X1 X2 X0 X1 X2 X0 X1 X2 X0 X1 X2
D1(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  1 24
D2(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  2
C(2.4).2       0  0 -2  0  0  2  0  0  0  0  0  0  0  0  0
C(4.8).2       0  0  0  0  0 -2  0  0  2  0  0  0  0  0  0

$estimate_mat
              X0 X1 X2 X0 X1 X2 X0 X1 X2 X0 X1 X2 X0 X1 X2
D1(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  1 24
D2(12 mod 12)  0  0  0  0  0  0  0  0  0  0  0  0  0  0  2
C(2.4).2       0  0 -2  0  0  2  0  0  0  0  0  0  0  0  0
C(4.8).2       0  0  0  0  0 -2  0  0  2  0  0  0  0  0  0

$estimates
NULL

$G
   D1(12 mod 12) D2(12 mod 12)      C(2.4).2      C(4.8).2
X0 -4.440892e-16  0.000000e+00  0.000000e+00  0.000000e+00
X1  1.000000e+00  1.156482e-16  1.850372e-16 -1.526557e-15
X2 -1.736111e-01  8.333333e-02 -2.500000e-01 -8.333333e-02
X0 -8.881784e-15 -2.264855e-15  2.880000e+00  2.087219e-15
X1  1.000000e+00  1.998401e-15 -2.400000e+00 -3.256654e-15
X2 -1.736111e-01  8.333333e-02  2.500000e-01 -8.333333e-02
X0  3.641532e-14  3.907985e-15  2.880000e+00  1.152000e+01
X1  1.000000e+00 -6.661338e-16 -2.400000e+00 -4.800000e+00
X2 -1.736111e-01  8.333333e-02  2.500000e-01  4.166667e-01
X0  3.600000e+01 -4.320000e+01 -2.304000e+01 -2.304000e+01
X1 -9.000000e+00  1.200000e+01  4.800000e+00  4.800000e+00
X2  5.208333e-01 -7.500000e-01 -2.500000e-01 -2.500000e-01
```

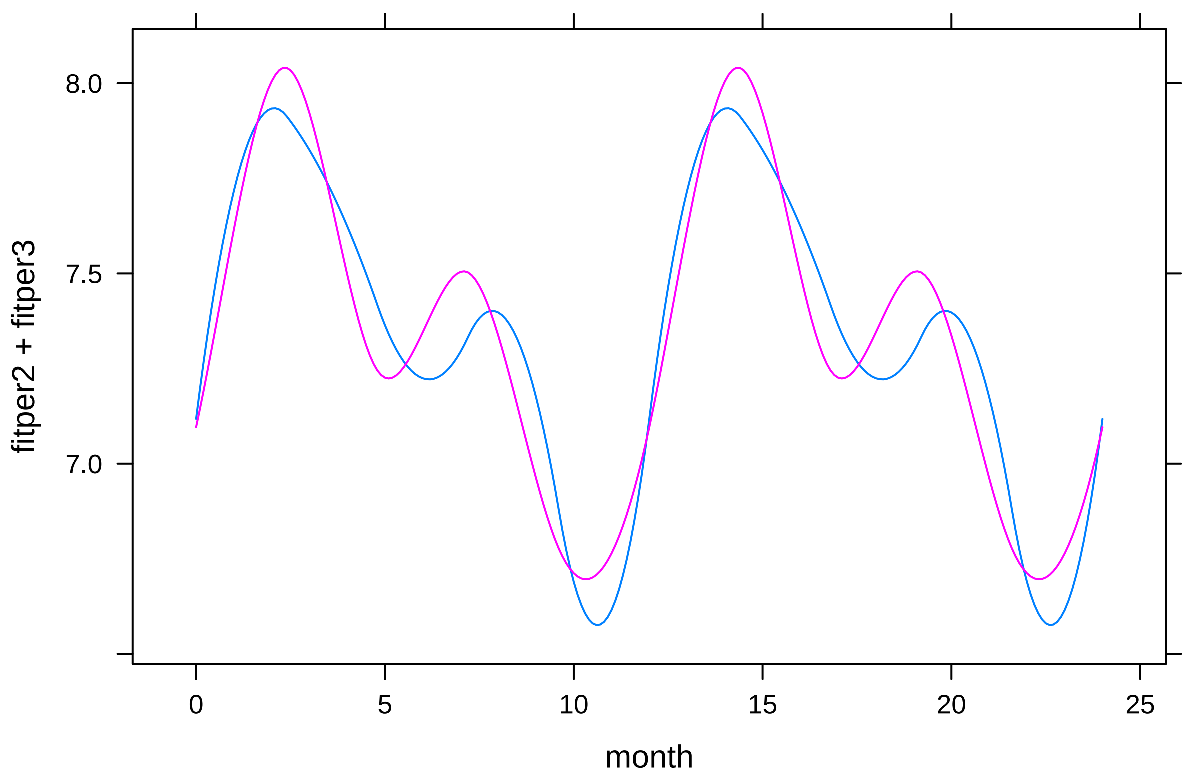
```
X0 -1.200000e+01  7.200000e+01  0.000000e+00  0.000000e+00
X1  1.000000e+00 -1.200000e+01  0.000000e+00  0.000000e+00
X2  0.000000e+00  5.000000e-01  0.000000e+00  0.000000e+00

$intercept
[1] 0

$knots
[1]  2.4  4.8  7.2  9.6 12.0

$max_degree
[1] 2

$periodic
[1] TRUE

$smoothness
$smoothness[[1]]
[1] 0 1

$smoothness[[2]]
[1] 0 1

$smoothness[[3]]
[1] 0 1

$smoothness[[4]]
[1] 0 1

$smoothness[[5]]
[1] 0 1


$tolerance
[1] 1e-16
```

```r
fitper2 <-  lm(unemployment ~ sp08(year) + per2(month), unemp)
summary(fitper2)
```

```
Call:
lm(formula = unemployment ~ sp08(year) + per2(month), data = unemp)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.22315 -0.24462 -0.00802  0.24895  1.07933 

Coefficients:
                           Estimate Std. Error t value Pr(>|t|)    
(Intercept)                 7.11767    0.06940 102.559  < 2e-16 ***
sp08(year)D1(0)            -0.33004    0.02339 -14.109  < 2e-16 ***
sp08(year)D2(0)             0.47060    0.03301  14.257  < 2e-16 ***
sp08(year)D3(0)            -0.20821    0.01484 -14.026  < 2e-16 ***
sp08(year)C(-0.184).3      -0.52275    0.04263 -12.264  < 2e-16 ***
sp08(year)C(4.63).3         0.62962    0.05254  11.983  < 2e-16 ***
```

```
sp08(year)C(8.96).0          0.04693    0.46654   0.101 0.919951
sp08(year)C(8.96).1         11.77222    3.13300   3.757 0.000210 ***
sp08(year)C(8.96).2        -58.18981   13.09978  -4.442 1.29e-05 ***
sp08(year)C(8.96).3        115.45875   26.69440   4.325 2.14e-05 ***
sp08(year)C(9.45).3       -115.92233   26.70997  -4.340 2.01e-05 ***
sp08(year)C(14.3).3         -0.05442    0.06729  -0.809 0.419397
per2(month)D1(12 mod 12)     0.79052    0.05484  14.416  < 2e-16 ***
per2(month)D2(12 mod 12)     0.57633    0.04202  13.715  < 2e-16 ***
per2(month)C(2.4).2          0.31674    0.07871   4.024 7.41e-05 ***
per2(month)C(4.8).2          0.27475    0.07860   3.496 0.000551 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3545 on 274 degrees of freedom
Multiple R-squared:  0.9118,    Adjusted R-squared:  0.907
F-statistic: 188.9 on 15 and 274 DF,  p-value: < 2.2e-16
```

```r
unemp$fitper2 <- predict(fitper2)
#library(effects)
#allEffects(fitper2) %>% plot

pred <- expand.grid(year = 0, month = seq(0,24,.1))
pred$fitper3 <- predict(fitper3, newdata = pred)
pred$fitper2 <- predict(fitper2, newdata = pred)
xyplot(fitper2 + fitper3 ~ month, pred, type = 'l')
```



```r
AIC(fitper2, fitper3)
```

```
        df      AIC
fitper2 17 239.0664
fitper3 17 239.0118
```

```r
circle <- function(x) cbind(sin=sin(2*pi*x), cos = cos(2*pi*x))

fitfourier1 <- lm(unemployment ~ sp08(year) + circle(month/12), unemp)
summary(fitfourier1)
```

```

Call:
lm(formula = unemployment ~ sp08(year) + circle(month/12), data = unemp)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.98951 -0.29773 -0.05293  0.26710  1.34342 

Coefficients:
                       Estimate Std. Error t value Pr(>|t|)    
(Intercept)             7.33870    0.07287 100.709  < 2e-16 ***
sp08(year)D1(0)        -0.32626    0.02870 -11.367  < 2e-16 ***
sp08(year)D2(0)         0.46050    0.04049  11.373  < 2e-16 ***
sp08(year)D3(0)        -0.20348    0.01821 -11.175  < 2e-16 ***
sp08(year)C(-0.184).3  -0.50643    0.05228  -9.687  < 2e-16 ***
sp08(year)C(4.63).3     0.60896    0.06444   9.450  < 2e-16 ***
sp08(year)C(8.96).0     0.57567    0.56902   1.012  0.31258    
sp08(year)C(8.96).1     8.87217    3.82674   2.318  0.02115 *  
sp08(year)C(8.96).2   -45.87442   15.99918  -2.867  0.00446 ** 
sp08(year)C(8.96).3    90.57417   32.60476   2.778  0.00585 ** 
sp08(year)C(9.45).3   -91.01930   32.62378  -2.790  0.00564 ** 
sp08(year)C(14.3).3    -0.05303    0.08257  -0.642  0.52123    
circle(month/12)sin     0.41625    0.03647  11.415  < 2e-16 ***
circle(month/12)cos    -0.10124    0.03641  -2.780  0.00581 ** 
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.435 on 276 degrees of freedom
Multiple R-squared:  0.8663,    Adjusted R-squared:   0.86 
F-statistic: 137.5 on 13 and 276 DF,  p-value: < 2.2e-16
```

```r
fitfourier2 <- lm(unemployment ~ sp08(year) + circle(month/12) +circle(month/6), unemp)
summary(fitfourier2)
```

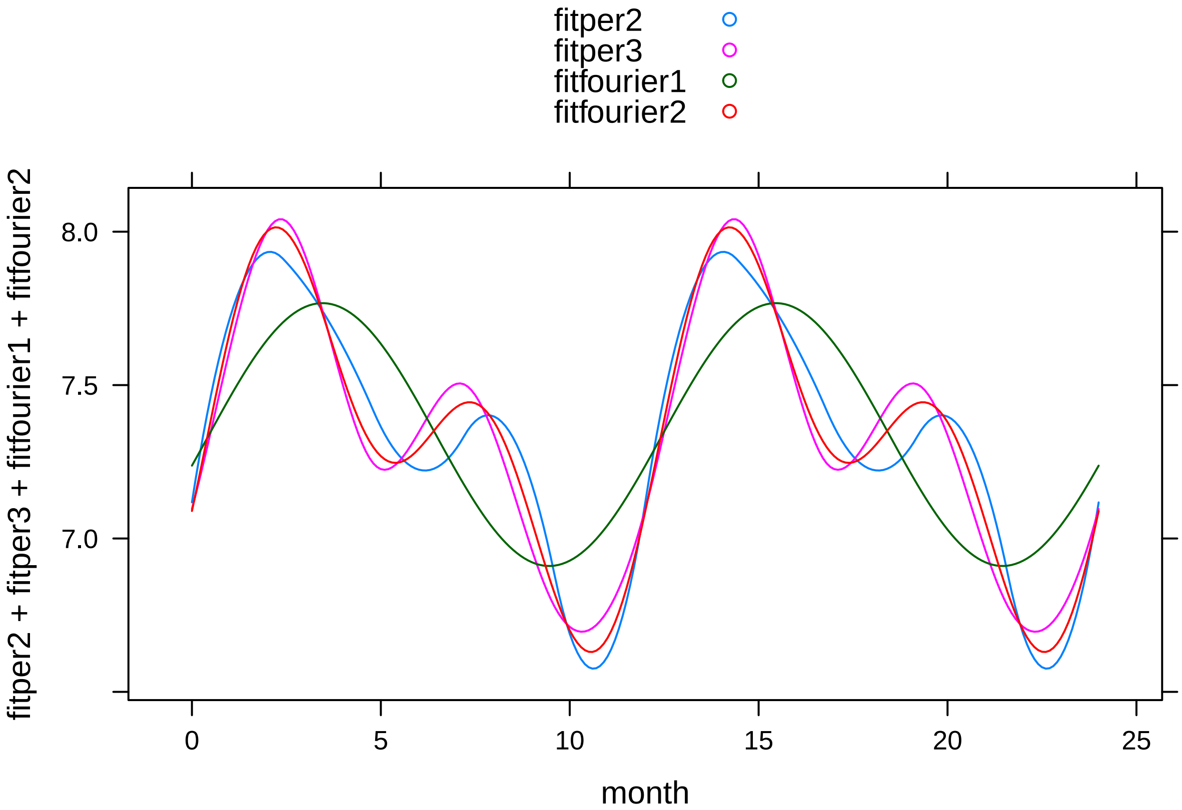
```

Call:
lm(formula = unemployment ~ sp08(year) + circle(month/12) + circle(month/6), 
    data = unemp)

Residuals:
     Min       1Q   Median       3Q      Max 
-1.10203 -0.23839 -0.00374  0.23491  1.12477 

Coefficients:
```

```
                        Estimate Std. Error t value Pr(>|t|)    
(Intercept)              7.33132    0.05846 125.416  < 2e-16 ***
sp08(year)D1(0)         -0.33004    0.02302 -14.334  < 2e-16 ***
sp08(year)D2(0)          0.47067    0.03249  14.487  < 2e-16 ***
sp08(year)D3(0)         -0.20825    0.01461 -14.253  < 2e-16 ***
sp08(year)C(-0.184).3   -0.52287    0.04195 -12.463  < 2e-16 ***
sp08(year)C(4.63).3      0.62984    0.05172  12.179  < 2e-16 ***
sp08(year)C(8.96).0      0.05038    0.45861   0.110 0.912607    
sp08(year)C(8.96).1     11.73214    3.08050   3.809 0.000173 ***
sp08(year)C(8.96).2    -58.02086   12.88004  -4.505 9.85e-06 ***
sp08(year)C(8.96).3    115.11074   26.24697   4.386 1.65e-05 ***
sp08(year)C(9.45).3   -115.57391   26.26228  -4.401 1.55e-05 ***
sp08(year)C(14.3).3     -0.05588    0.06623  -0.844 0.399568    
circle(month/12)sin      0.41790    0.02925  14.287  < 2e-16 ***
circle(month/12)cos     -0.10100    0.02921  -3.458 0.000632 ***
circle(month/6)sin       0.33368    0.02910  11.468  < 2e-16 ***
circle(month/6)cos      -0.14084    0.02905  -4.849 2.09e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.3489 on 274 degrees of freedom
Multiple R-squared:  0.9146,    Adjusted R-squared:  0.9099 
F-statistic: 195.6 on 15 and 274 DF,  p-value: < 2.2e-16
```

```
pred$fitfourier1 <- predict(fitfourier1, newdata = pred)
pred$fitfourier2 <- predict(fitfourier2, newdata = pred)

AIC(fitper2, fitper3, fitfourier1, fitfourier2)
```

```
            df      AIC
fitper2     17 239.0664
fitper3     17 239.0118
fitfourier1 15 355.8626
fitfourier2 17 229.8863
```

```
xyplot(fitper2 + fitper3  + fitfourier1 + fitfourier2 ~ month, pred, type = 'l',
       auto.key = T)
```



References to incorporate

- Spline derivatives