

Fourier Transforms

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Abstract—This manual provides a quick introduction to the Fourier transform.

1 SINUSOIDAL RESPONSE

1. Fig. 1 shows an RC circuit with input $x(t)$ and output $y(t)$. Show that

$$RC \frac{dy}{dt} + y(t) = x(t) \quad (1.1)$$

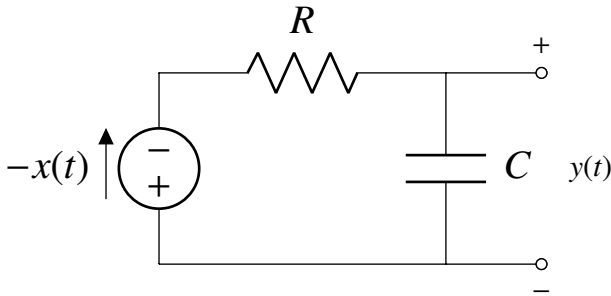


Fig. 1: RC Circuit

2. Let $x(t) = \cos 2\pi f_0 t$. Show that

$$y(t) = \frac{1}{\sqrt{1 + (2\pi f_0 RC)^2}} \cos \left[2\pi f_0 \left(t - \tan^{-1} RC \right) \right] \quad (1.2)$$

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by solving the differential equation in (1.1) using the integrating factor.

(1.3)

2 FOURIER TRANSFORM

The Fourier transform of a signal $g(t)$ is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \quad (2.1)$$

1. Define the Dirac delta function as

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.2)$$

$$\delta(t) = 0, \quad t \neq 0.$$

and show that

$$\delta(t) \stackrel{\mathcal{F}}{\rightleftharpoons} 1 \quad (2.3)$$

2. Show that

$$\delta(t - t_0) \stackrel{\mathcal{F}}{\rightleftharpoons} e^{-j2\pi f t_0} \quad (2.4)$$

3. Show that

$$g(t) \stackrel{\mathcal{F}}{\rightleftharpoons} G(f) \quad (2.5)$$

$$\Rightarrow G(t) \stackrel{\mathcal{F}}{\rightleftharpoons} g(-f)$$

assuming

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df. \quad (2.6)$$

(2.6) is known as the inverse Fourier transform.

4. Show that

$$\cos 2\pi f_0 t \stackrel{\mathcal{F}}{\rightleftharpoons} \frac{1}{2} [\delta(f - f_0) + \delta(f + f_0)] \quad (2.7)$$

5. Show that

$$\frac{dy}{dt} \stackrel{\mathcal{F}}{\rightleftharpoons} j2\pi f Y(f) \quad (2.8)$$

6. Define

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (2.9)$$

Show that

$$e^{-at}u(t) \xrightarrow{\mathcal{F}} \frac{1}{a + j2\pi f} \quad (2.10)$$

7. If $x(t) = \delta(t)$ in (1.1), find $y(t)$ using Fourier transforms. This is known as the *impulse response* and is denoted by $h(t)$.
8. Solve (1.1) for $x(t) = \cos 2\pi f_0 t$ using Fourier transforms.
9. Verify that

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau) d\tau \quad (2.11)$$

This is known as the *convolution integral*.

10. Show that

$$x(t) * h(t) \xrightarrow{\mathcal{F}} X(f)H(f) \quad (2.12)$$

11. In (1.1), show that

$$H(f) = \frac{1}{1 + j2\pi fRC} \quad (2.13)$$

12. Show that

$$y(t) = |H(f_0)| \cos [2\pi f_0 t + \angle H(f_0)] \quad (2.14)$$

3 FILTERING

1. Plot $|H(f)|$ for different values of RC . The following code plots Fig. 1.

```
import numpy as np
import matplotlib.pyplot as plt
import subprocess
import shlex

#Filter Characteristics

R = 5*1e2; #500 ohm resistance
C = 10e-6; #10 uF capacitance

#Plotting the filter amplitude response
f_0 = 50.0
f = np.linspace(-3*f_0, 3*f_0, 1e2)
s = 1j*2*np.pi*f

den = np.polyval([1,1],s*C*R);
H = 1/den;

plt.plot(f,abs(H))
plt.grid()# minor
plt.xlabel('$f$ (Hz)')
plt.ylabel('$|H(f)|$')
```

```
#Save figure
#plt.savefig('..figs/lpf.eps')
plt.savefig('..figs/lpf.pdf')
subprocess.run(shlex.split("termux-open ../figs/
lpf.pdf"))
plt.show()
```

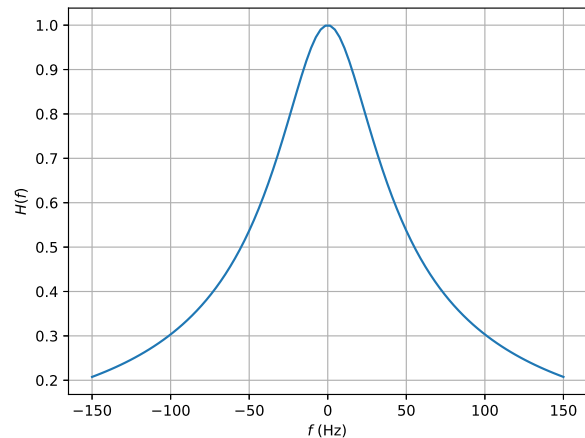


Fig. 1: Low Pass Filter

2. Sketch $x(t) = \cos 100\pi t$, $t \in (-50ms, 50ms)$.
3. Sketch $y(t)$ in (1.2).

Solution: The following code plots Fig. 3

```
import numpy as np
import matplotlib.pyplot as plt
#comment the following 2 lines
import subprocess
import shlex

def H(f):
    s = 1j*2*np.pi*f
    den = np.polyval([1,1],s*C*R)
    H = 1/den
    return H

#Filter Characteristics
R = 5*1e2; #500 ohm resistance
C = 10e-6; #10 uF capacitance

#Input and Output
f_0 = 50.0
t = np.linspace(-50e-3, 50e-3, 5e2)
xt = np.cos(2*np.pi*f_0*t)
yt = abs(H(f_0))*np.cos(2*np.pi*f_0*t + np.
    angle(H(f_0)))
```

```

#subplots
plt.subplot(2, 1, 1)
plt.plot(t, xt)
plt.title('Sinusoidal Response')
plt.ylabel('$x(t)$')
plt.grid()# minor

plt.subplot(2, 1, 2)
plt.plot(t, yt)
plt.xlabel(' t (ms)')
plt.ylabel('$y(t)$')
plt.grid()# minor

#Save figure. Comment the following two lines
plt.savefig('../figs/xyt.pdf')
subprocess.run(shlex.split("termux-open ../figs/
xyt.pdf"))
#uncomment this line
#plt.show()

```

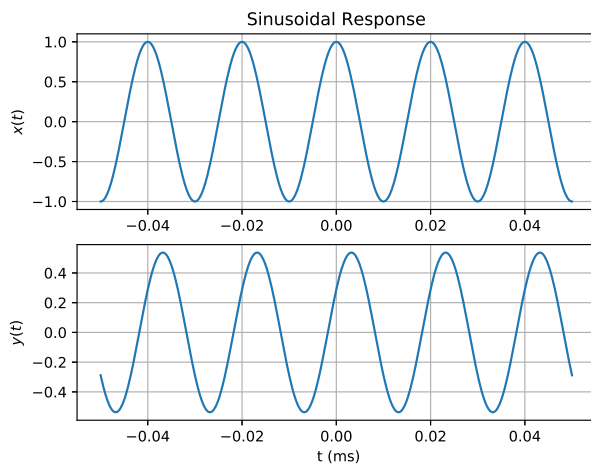


Fig. 3: Sinusoidal Response

4. Sketch the output $y(t)$ in Fig. 1 when $x(t) = \cos 100\pi t + \cos 300\pi t$, $R = 500\Omega$ and $C = 10\mu\text{F}$ for $t \in (-50\text{ms}, 50\text{ms})$. Comment.
5. Sketch the output when $x(t) = \cos 100\pi t + \cos 2\pi kt$, $R = 500\Omega$ and $C = 10\mu\text{F}$ for different values of k . Find the value of k for which the output of $H(f)$ has frequency 50 Hz.