Signals & Circuits

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Contents

1 Fourier Series

Problem 1.1. Type the following program in octave to obtain $g(t)$. $g(t)$ is a periodic signal called a square wave with amplitude $A = 5V$ and time period $T = 20ms$.

Solution:

```
clear;
close;
clc;


%Square wave
A = 5;
T = 0.02;


t = linspace(-2.5*T,2.5*T,1e4);
s = A/2*(1+square(2*pi*t/T));
plot(t,s,"Linewidth",4)
grid
axis([-0.06,0.06,-1,6])
xlabel('t')
ylabel('g(t)')

print -deps -color ../figs/1.1.eps
```

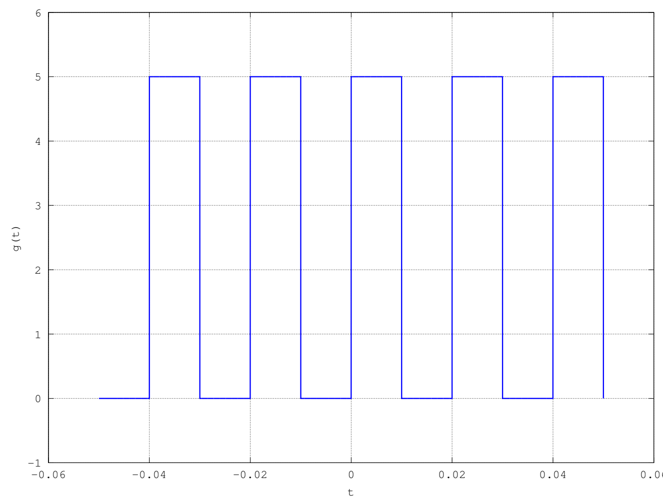

Fig. 1.1: Generating square wave.

Problem 1.2. The following expression

$$g(t) = \sum_{n=0}^{\infty} a_n \cos 2\pi n f t + b_n \sin 2\pi n f t \tag{1.1}$$

is known as the Fourier series expansion of $g(t)$, where $f = \frac{1}{T}$. Find

$$a_n = \frac{2}{T}\int_0^T g(t)\cos 2\pi n f t \, dt \tag{1.2}$$

$$b_n = \frac{2}{T}\int_0^T g(t)\sin 2\pi n f t \, dt \tag{1.3}$$

Solution:

$$a_0 = \frac{A}{T}\int_0^{T_0} dt \tag{1.4}$$

$$= \frac{AT_0}{T} \tag{1.5}$$

and

$$a_n = \frac{2A}{T}\int_0^{T_0} \cos 2\pi n f t \, dt \tag{1.6}$$

$$= \frac{A}{\pi n f T}\sin 2\pi n f T_0 \tag{1.7}$$

Similarly,

$$b_n = \frac{2A}{T}\int_0^{T_0} \sin 2\pi n f t \, dt \tag{1.8}$$

$$= \frac{A}{\pi n f T}\left[1 - \cos 2\pi n f T_0\right] \tag{1.9}$$

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Problem 1.3. Using Octave, compute the series

$$\sum_{n=0}^{15} a_n \cos 2\pi n f t + b_n \sin 2\pi n f t \tag{1.10}$$

for $A = 5, T = 20ms$ and a_n, b_n obtained in the previous problem. Comment.

Solution: Type the following program

```
%Filter input: Square Wave and
    Fourier series

clear;
close;

T_0 = 0.01;
T = 0.02;
f = 1/T;
A = 5;
simlen = 1e3;

t = linspace(0,0.1,simlen); %
    generating points in t-axis
n = 1:15; %series range

%g = zeros(1,1e2);%initialising
    sum

for n = 0:20,
        if n == 0,
                g = A*T_0/T;
        else
                cost = cos(2*pi*n*
                    f*t);%Computing
                    cosine term
                sint = sin(2*pi*n*
                    f*t);%Computing
                    sine term

                an = 2*A*sin(2*pi*
                    n*f*T_0)./(2*pi*
                    n*f*T); %
                    Computing
                    coefficients
                bn = 2*A*(1 - cos
                    (2*pi*n*f*T_0))
                    ./(2*pi*n*f*T);
                    %Computing
                    coefficients
                g = g + an*cost +
                    bn*sint; %
                    evaluating the
                    summation
        end
end

plot(t,g,"Linewidth",4)
grid
xlabel('t')
ylabel('g(t)')

print -deps -color ../figs/1.4.eps
```

to obtain the following figure.

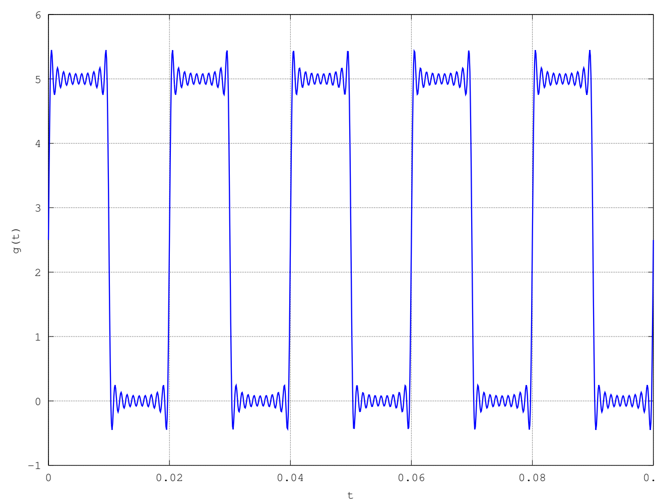

Fig. 1.3: Gibbs phenomenon.

Problem 1.4. Generate $g(t)$ using an arduino for $A = 5\,V$ and $T = 20\,ms$ using the blink.ino program.

2 Filter

2.1 RC Circuit

Problem 2.1. Refer to the circuit in Fig. 2.1. Suppose you are told that C has a resistance given by $\frac{1}{sC}$. Find the ratio $H(s)$ of the output voltage and input voltage using node analysis. The above circuit is known as a low pass filter and $H(s)$ is known as the transfer function.

Solution: The equations at the nodes are given by

$$\frac{V_1 - V_i}{R} + sCV_1 + \frac{V_1 - V_2}{R} = 0 \tag{2.1}$$

$$\frac{V_2 - V_1}{R} + sCV_2 + \frac{V_2 - V_o}{R} = 0 \tag{2.2}$$

$$\frac{V_o - V_2}{R} + sCV_o = 0 \tag{2.3}$$

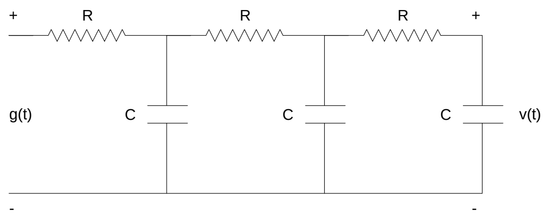

Fig. 2.1: Three stage $R-C$ low pass filter circuit

which can be expressed as

$$\begin{pmatrix} sC+\frac{2}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & sC+\frac{2}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & sC+\frac{1}{R} \end{pmatrix}\begin{pmatrix} \frac{V_1}{V_i} \\ \frac{V_2}{V_i} \\ \frac{V_o}{V_i} \end{pmatrix} = \begin{pmatrix} \frac{1}{R} \\ 0 \\ 0 \end{pmatrix} \tag{2.4}$$

Thus,

$$H(s) = \frac{V_o}{V_i} = \frac{\begin{vmatrix} sC+\frac{2}{R} & -\frac{1}{R} & \frac{1}{R} \\ -\frac{1}{R} & sC+\frac{2}{R} & 0 \\ 0 & -\frac{1}{R} & 0 \end{vmatrix}}{\begin{vmatrix} sC+\frac{2}{R} & -\frac{1}{R} & 0 \\ -\frac{1}{R} & sC+\frac{2}{R} & -\frac{1}{R} \\ 0 & -\frac{1}{R} & sC+\frac{1}{R} \end{vmatrix}} \tag{2.5}$$

$$= \frac{1/R^3}{\left(sC+\frac{1}{R}\right)\left\{\left(sC+\frac{2}{R}\right)^2-\frac{1}{R^2}\right\}-\frac{1}{R^2}\left(sC+\frac{2}{R}\right)} \tag{2.6}$$

which can be expressed as

$$H(s) = \frac{1}{(sCR+1)\left\{(sCR+2)^2-1\right\}-(sCR+2)} \tag{2.7}$$

$$= \frac{1}{(sCR+2)^3-(sCR+2)^2-2\,(sCR+2)+1} \tag{2.8}$$

$$= \frac{1}{(sCR)^3-5\,(sCR)^2+6sCR+1} \tag{2.9}$$

Problem 2.2. Substitute $s = \jmath 2\pi f, \jmath = \sqrt{-1}$ in (2.9) to obtain $H(f)$. $H(f)$ is known as the frequency response. Plot $|H(f)|$ in octave for $-20 < f < 20$, given that $R = 1\,k\Omega$ and $C = 10\,\mu F$.

Solution: Type the following code to get Fig. 2.2. You will find that $H(f)$ is a low pass filter.

```
%Filter Characteristics
clear;
close;


R = 1e3; %10K ohm resistance
C = 10e-6;%10 uF capacitance


%Plotting the filter amplitude
   response

f = linspace(-20,20,1e2);
s = j*2*pi*f;
den = polyval([1 -5 6 1],s*C*R);
H = 1./den;

plot(f,abs(H),"Linewidth",4)
grid minor
xlabel('f')
ylabel('H(f)')

print -deps -color ../figs/2.2.eps
```

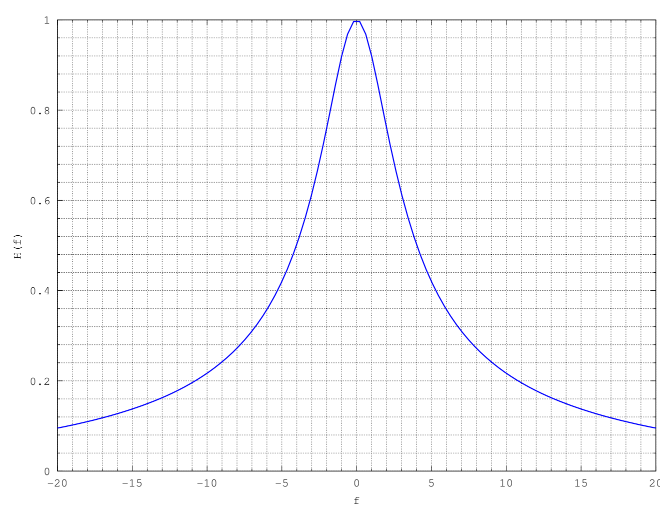

Fig. 2.2: Frequency response of the $R-C$ filter

Problem 2.3. Find the frequency at which $|H(f)|^2 = \frac{1}{2}$. This frequency is known as the 3-dB bandwidth of $H(f)$.

Solution: Substituting sCR = ȷx in (2.9),

$$\left|H(\jmath x)\right| = \frac{1}{\sqrt{2}} \tag{2.10}$$

$$\Rightarrow -jx^3 + 5x^2 + j6x + 1 = \sqrt{2} \tag{2.11}$$

$$\Rightarrow x^2(6 - x^2)^2 + (1 + 5x^2)^2 = 2 \tag{2.12}$$

$$\Rightarrow x^6 + 13x^4 + 46x^2 - 1 = 0 \tag{2.13}$$

$$\tag{2.14}$$

Letting $y = x^2$, we obtain the cubic equation

$$y^3 + 13y^2 + 46y - 1 = 0 \tag{2.15}$$

The following script gives the 3 dB bandwidth for the filter H by choosing the real root.

```
%Filter Characteristics
clear;
close;


R = 1e3; %1K ohm resistance
C = 10e-6;%10 uF capacitance

%finding 3 dB bandwidth
    numerically
sqrt(roots([1  13  46  -1]))/(2*pi
    *R*C)
```

This yields the value $f_{3dB} = 2.3395$ Hz.

Problem 2.4. Obtain the 3 dB bandwidth by solving the cubic equation in the previous problem

Solution: In the above, let $y = z - \frac{13}{3}$. Then the equation becomes

$$\Rightarrow z^3 - (31/3)z - 1015/27 = 0 \tag{2.16}$$

This equation has the theoretical solution evaluated by the following script

```
%Filter Characteristics
clear;
close;

R = 1e3; %1K ohm resistance
C = 10e-6;%10 uF capacitance



%finding 3 dB bandwidth
    theoretically

q = -31/3;
r = -1015/27;

sqrt((-r/2 + sqrt(r^2/4 +q^3/27))
    ^(1/3) + (-r/2 - sqrt(r^2/4 +q
    ^3/27))^(1/3) - 13/3)/(2*pi*R*C)
```

Note that this script gives the same result as the one in the previous problem.

Problem 2.5. Suppose the square wave in Fig. 1.1 is given as input to the filter in Fig. 2.2. Find and plot the filter output.

Solution: Using sinusoidal steady state analysis, if the input to the filter is $\cos 2\pi nft$, the output is given by

$$|H(nf)| \cos\{2\pi nft + \angle H(nf)\} \tag{2.17}$$

Using the principle of superposition, for the input

$$\sum_{n=0}^{\infty} a_n \cos 2\pi nft + b_n \sin 2\pi nft \tag{2.18}$$

the output will be

$$\sum_{n=0}^{\infty} a_n |H(nf)| \cos\{2\pi nft + \angle H(nf)\} + b_n |H(nf)| \sin\{2\pi nft + \angle H(nf)\} \tag{2.19}$$

Suitably modifying the program in Problem 1.3,

```
%Filter output

clear;
close;

R = 1e3; %10 k resistance
C = 10e-6;%10 uF capacitance
T_0 = 0.01;
T = 0.02;
f = 1/T;
A = 5; %input amplitude
simlen = 1e3; %time range

t = linspace(0,0.1,simlen); %
    generating points in t-axis
n = 1:15; %series range

for n = 0:20,
        if n == 0,
                g = A*T_0/T;
        else
```

```
                an = 2*A*sin(2*pi*n*f*T_0)./(2*pi*n*f*T); %Computing coefficients
                bn = 2*A*(1 - cos(2*pi*n*f*T_0))./(2*pi*n*f*T); %Computing coefficients
                s = j*2*pi*n*f;
                den = polyval([1 -5 6 1],s*C*R);
                Hn = 1./den; %Frequency response
                thetan = arg(Hn);


                cost = cos(2*pi*n*f*t + thetan);%Computing cosine term
                sint = sin(2*pi*n*f*t+ thetan);%Computing sine term


                g = g + abs(Hn)*an*cost + abs(Hn)*bn*sint; %evaluating the summation
        end
end

plot(t,g,"Linewidth",4)
grid

print -deps -color ../figs/2.5.eps
```

The output of the filter is shown in Fig. 2.5

Problem 2.6. Sketch $|H(nf)|$.

Solution:

```
%Filter output

clear;
```

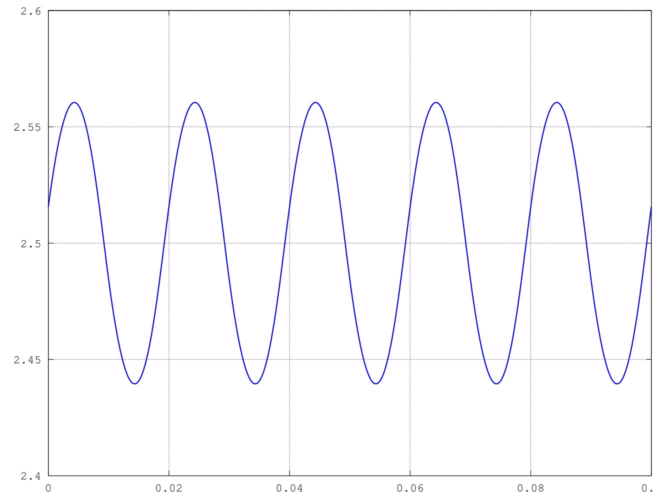

Fig. 2.5: Frequency response of the $R-C$ filter

```
close;

R = 1e3; %10 k resistance
C = 10e-6;%10 uF capacitance
T=0.02;
f = 1/T;


for n = 0:20,

                s = j*2*pi*n*f;
                den = polyval([1 -5 6 1],s*C*R);
                H(n+1) = 1./den; %Frequency response

end

stem(0:20,abs(H),"Linewidth",4)
xlabel('n')
ylabel('H(nf)')
grid

print -deps -color ../figs/2.6.eps
```

The output of the filter is shown in Fig. 2.6

2.2 Circuit Analysis

Problem 2.7. Obtain the expression for $H(s)$ using mesh analysis.

Problem 2.8. Repeat the above exercise using Thevenin's theorem.

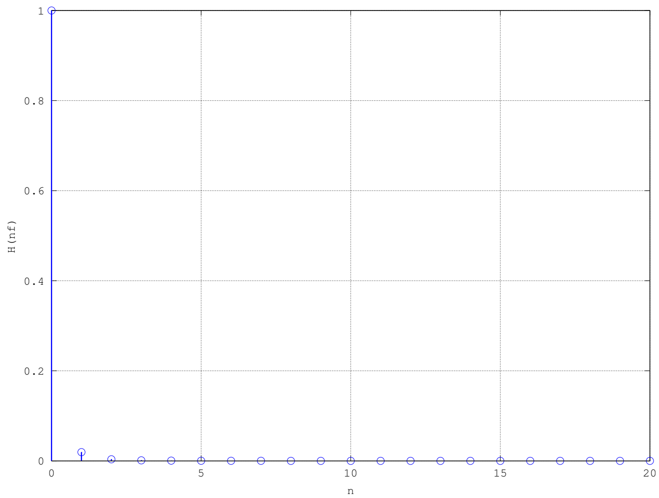

Fig. 2.6: Frequency response of the $R-C$ filter

Problem 2.9. Repeat the above exercise using Norton's theorem.

Problem 2.10. Repeat the above exercise using $Y-\Delta$ transformation.

Problem 2.11. Obtain all the two port network parameters for the circuit in Fig. 2.1.