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Abstract—This manual provides a simple introduction to digital signal processing.

1 DIFFERENCE EQUATION

1.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (1.1)$$

Sketch $x(n)$.

1.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (1.2)$$

Sketch $y(n)$.

Solution: The following code yields Fig. 1.2.

```
wget https://github.com/gadepall/EE1310/raw/master/filter/codes/xnyn.py
```

2 Z-TRANSFORM

2.1 The Z-transform of $x(n]$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (2.1)$$

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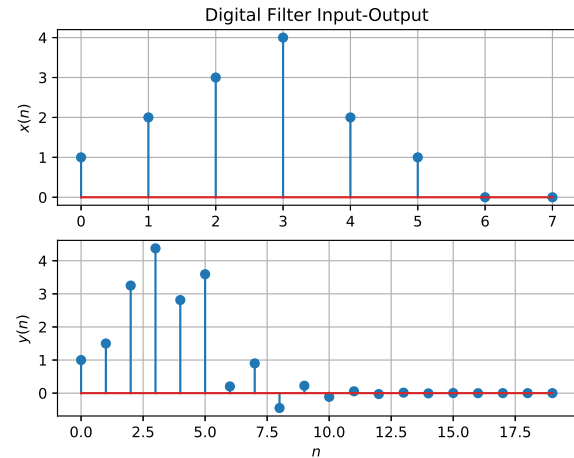


Fig. 1.2

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (2.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (2.3)$$

Solution: From (2.1),

$$\begin{aligned} \mathcal{Z}\{x(n-k)\} &= \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \end{aligned} \quad (2.4)$$

$$= z^{-1} X(z) \quad (2.5)$$

resulting in (2.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \quad (2.6)$$

2.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (2.7)$$

from (1.2) assuming that the Z-transform is a linear operation.

Solution: Applying (2.6) in (1.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (2.8)$$

$$\Rightarrow \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (2.9)$$

2.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.10)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.11)$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (2.12)$$

Solution: It is easy to show that

$$\delta(n) \stackrel{Z}{\rightleftharpoons} 1 \quad (2.13)$$

and from (2.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (2.14)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (2.15)$$

using the formula for the sum of an infinite geometric progression.

2.4 Show that

$$a^n u(n) \stackrel{Z}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (2.16)$$

2.5 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (2.17)$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution: The following code plots Fig. 2.5.

```
wget https://raw.githubusercontent.com/
gadepall/EE1310/master/filter/codes/dtft.
py
```

3 IMPULSE RESPONSE

3.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \stackrel{Z}{\rightleftharpoons} H(z) \quad (3.1)$$

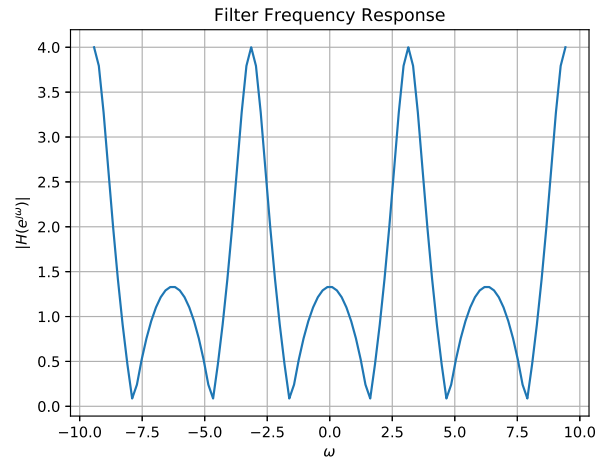


Fig. 2.5: $|H(e^{j\omega})|$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (1.2).

Solution: From (2.9),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (3.2)$$

$$\Rightarrow h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (3.3)$$

using (2.16) and (2.6).

3.2 Sketch $h(n)$. Is it bounded? Convergent?

Solution: The following code plots Fig. 3.2.

```
wget https://raw.githubusercontent.com/
gadepall/EE1310/master/filter/codes/hn.py
```

3.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (3.4)$$

Is the system defined by (1.2) stable for the impulse response in (3.1)?

3.4 Compute and sketch $h(n)$ using

$$\sum_{m=0}^M a(m) h(n-m) = \sum_{k=0}^N b(k) \delta(n-k) \quad (3.5)$$

This is the definition of $h(n)$.

Problem 1. Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (3.6)$$

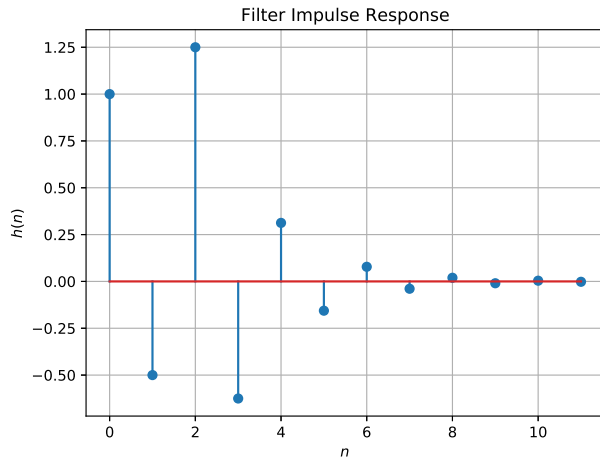


Fig. 3.2: $|H(e^{j\omega})|$

where $x(k)$ is the **input_signal** in Problem ??. You will need to suitably truncate $h(n)$ calculated in Problem 1. Use $y(n)$ as **output_signal** in Problem ??. Comment. The operation in (3.6) is known as *convolution*.

Problem 2. Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (3.7)$$

4 DFT AND FFT

Problem 3. Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (4.1)$$

and $H(k)$ using $h(n)$.

Problem 4. Compute

$$Y(k) = X(k)H(k) \quad (4.2)$$

Problem 5. Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (4.3)$$

Use $y(n)$ as **output_signal** in Problem ??.

Problem 6. Repeat the previous exercise by computing $X(k)$, $H(k)$ and $y(n)$ through FFT and IFFT.

Problem 7. Wherever possible, express all the above equations as matrix equations.