Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Difference Equation

1.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{1.1}$$

Sketch $x(n)$.

1.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{1.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 1.2.

```
wget https://github.com/gadepall/EE1310/raw/
    master/filter/codes/xnyn.py
```

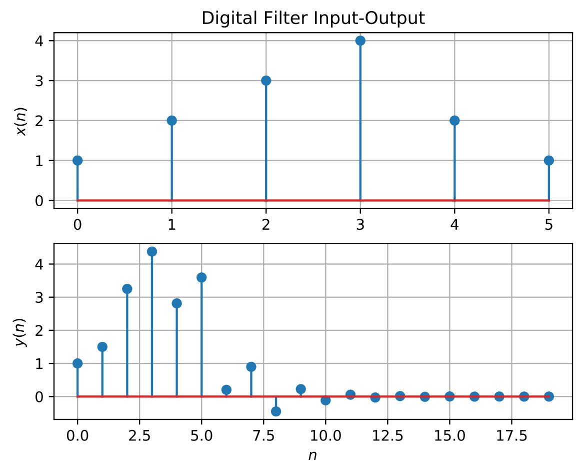

Fig. 1.2

2 Z-transform

2.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{2.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{2.3}$$

Solution: From (2.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{2.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{2.5}$$

resulting in (2.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{2.6}$$

2.2 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{2.7}$$

from (1.2) assuming that the Z-transform is a linear operation.

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Solution: Applying (2.6) in (1.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{2.8}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{2.9}$$

2.3 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{2.12}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{2.13}$$

and from (2.11),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{2.14}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{2.15}$$

using the fomula for the sum of an infinite geometric progression.

2.4 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{2.16}$$

2.5 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{2.17}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: The following code plots Fig. 2.5.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/dtft.
    py
```

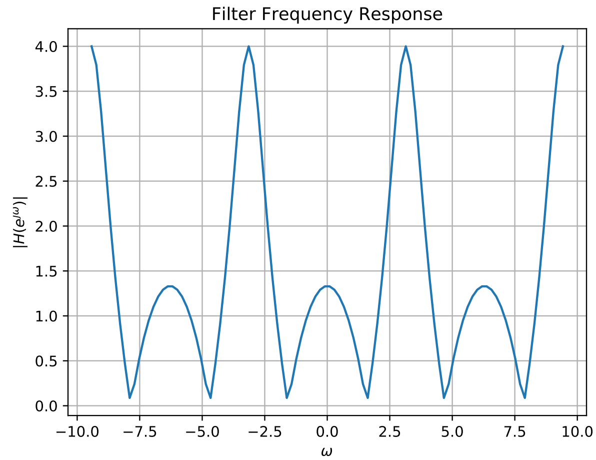

Fig. 2.5: $|H\left(e^{\jmath\omega}\right)|$

3 Impulse Response

3.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{3.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (1.2).
Solution: From (2.9),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{3.2}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{3.3}$$

using (2.16) and (2.6).

3.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hn.py
```

3.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{3.4}$$

Is the system defined by (1.2) stable for the impulse response in (3.1)?

3.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{3.5}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 3.4. Note that this is the same as Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/hndef.
    py
```

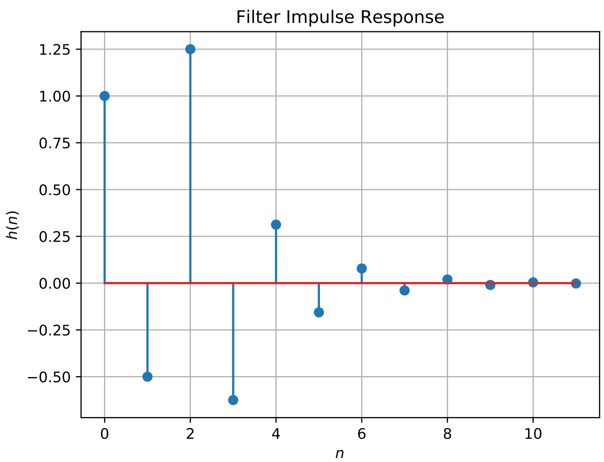

Fig. 3.2: $h(n)$ as the inverse of $H(z)$

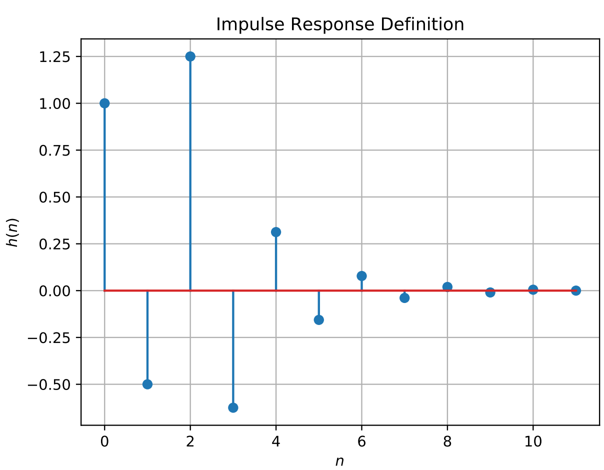

Fig. 3.4: $h(n)$ from the definition

3.5 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (3.6)$$

Comment. The operation in (3.6) is known as *convolution*.

Solution: The following code plots Fig. 3.5. Note that this is the same as $y(n)$ in Fig. 1.2.

```
wget https://raw.githubusercontent.com/
    gadepall/EE1310/master/filter/codes/
    ynconv.py
```

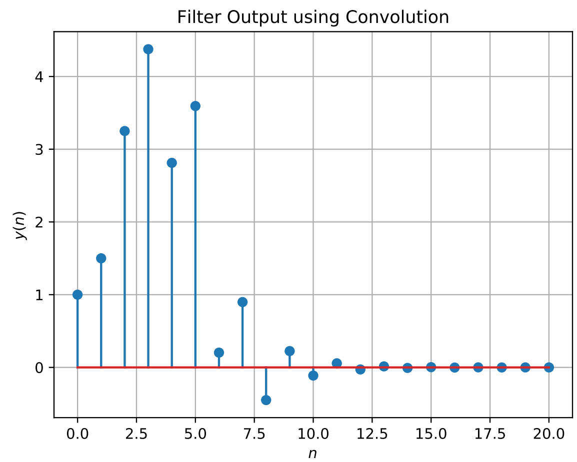

Fig. 3.5: $y(n)$ from the definition of convolution

3.6 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \quad (3.7)$$

4 DFT and FFT

4.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (4.1)$$

and $H(k)$ using $h(n)$.

4.2 Compute

$$Y(k) = X(k)H(k) \quad (4.2)$$

4.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (4.3)$$

Use $y(n)$ as **output_signal** in Problem **??**.

4.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

4.5 Wherever possible, express all the above equations as matrix equations.