Fourier Transforms

G V V Sharma*

Contents

***Abstract*—This manual provides a quick introduction to the Fourier transform.**

1 Sinusoidal Response

1. Fig. 1 shows an RC circuit with input $x(t)$ and output $y(t)$. Show that

$$RC\frac{dy}{dt} + y(t) = x(t) \tag{1.1}$$

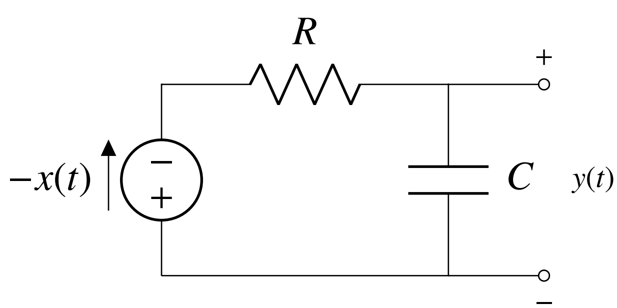

Fig. 1: RC Circuit

2. Let $x(t) = \cos 2\pi f_0 t$. Show that

$$y(t) = \frac{1}{\sqrt{1 + (2\pi f_0 RC)^2}} \cos\left[2\pi f_0\left(t - \tan^{-1} RC\right)\right] \tag{1.2}$$

by solving the differential equation in (1.1) using the integrating factor.

$$\tag{1.3}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

2 Fourier Transform

The Fourier transform of a signal $g(t)$ is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-\jmath 2\pi f t}\, dt \tag{2.1}$$

1. Define the Dirac delta function as

$$\int_{\infty}^{\infty} \delta(t)\, dt = 1$$
$$\delta(t) = 0, \quad t \neq 0. \tag{2.2}$$

and show that

$$\delta(t) \overset{\mathcal{F}}{\rightleftharpoons} 1 \tag{2.3}$$

2. Show that

$$\delta(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-\jmath 2\pi f t_0} \tag{2.4}$$

3. Show that

$$g(t) \overset{\mathcal{F}}{\rightleftharpoons} G(f)$$
$$\implies G(t) \overset{\mathcal{F}}{\rightleftharpoons} g(-f) \tag{2.5}$$

assuming

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{\jmath 2\pi f t}\, df. \tag{2.6}$$

(2.6) is known as the inverse Fourier transform.

4. Show that

$$\cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - f_0\right) + \delta\left(f + f_0\right)\right] \tag{2.7}$$

5. Show that

$$\frac{dy}{dt} \overset{\mathcal{F}}{\rightleftharpoons} \jmath 2\pi f Y(f) \tag{2.8}$$

6. Define

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{2.9}$$

Show that

$$e^{-at}u(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{a+\jmath 2\pi f} \tag{2.10}$$

7. If $x(t) = \delta(t)$ in (1.1), find $y(t)$ using Fourier transforms. This is known as the *impulse response* and is denoted by $h(t)$.
8. Solve (1.1) for $x(t) = \cos 2\pi f_0 t$ using Fourier transforms.
9. Verify that

$$y(t) = h(t) * x(t) = \int_{\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \tag{2.11}$$

This is known as the *convolution integral*.

10. Show that

$$x(t) * h(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f)H(f) \tag{2.12}$$

11. In (1.1), show that

$$H(f) = \frac{1}{1+\jmath 2\pi fRC} \tag{2.13}$$

12. Show that

$$y(t) = |H(f_0)| \cos\left[2\pi f_0 t + \angle H(f_0)\right] \tag{2.14}$$

3 Filtering

1. Plot $|H(f)|$ for different values of RC. The following code plots Fig. 1.

```
import numpy as np
import matplotlib.pyplot as plt
import subprocess
import shlex

#RC filter frequency response
def H(f):
        s = 1j*2*np.pi*f
        den = np.polyval([1,1],s*C*R)
        H = 1/den
        return H


#Filter Characteristics

R = 5*1e2; #500 ohm resistance
C = 10e-6;#10 uF capacitance

#Plotting the filter amplitude response
f_0 = 50.0
f = np.linspace(-3*f_0,3*f_0,1e2)

plt.plot(f,abs(H(f)))
plt.grid()# minor
plt.xlabel('$f$ (Hz)')
plt.ylabel('$H(f)$')
#Save figure
#plt.savefig('../figs/lpf.eps')
plt.savefig('../figs/lpf.pdf')
subprocess.run(shlex.split("termux-open ../figs/
    lpf.pdf"))
#plt.show()
```

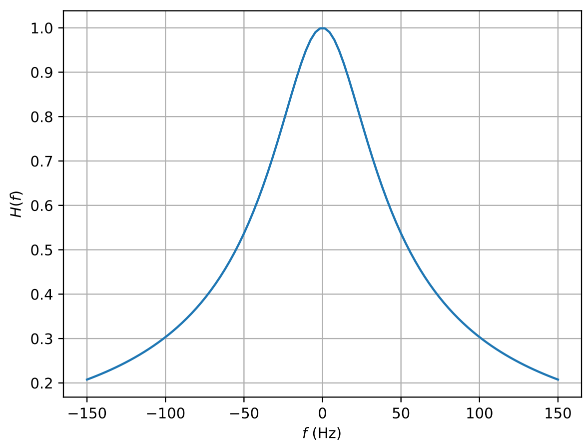

Fig. 1: Low Pass Filter

2. Sketch $x(t) = \cos 100\pi t, \quad t \in (-50ms, 50ms)$.
3. Sketch $y(t)$ in (1.2) using (2.14) for $f_0 = 50$ Hz.
Solution: The following code plots Fig. 3

```
import numpy as np
import matplotlib.pyplot as plt
#comment the following 2 lines
import subprocess
import shlex

def H(f):
        s = 1j*2*np.pi*f
        den = np.polyval([1,1],s*C*R)
        H = 1/den
        return H


#Filter Characteristics
R = 5*1e2; #500 ohm resistance
C = 10e-6;#10 uF capacitance

#Input and Output
f_0 = 50.0
```

```
t = np.linspace(−50e−3,50e−3,5e2)
xt = np.cos(2∗np.pi∗f_0∗t)
yt = abs(H(f_0))∗np.cos(2∗np.pi∗f_0∗t + np.
     angle(H(f_0)))

#subplots
plt.subplot(2, 1, 1)
plt.plot(t, xt)
plt.title('Sinusoidal Response')
plt.ylabel('$x(t)$')
plt.grid()# minor


plt.subplot(2, 1, 2)
plt.plot(t, yt)
plt.xlabel(' t (ms)')
plt.ylabel('$y(t)$')
plt.grid()# minor

#Save figure. Comment the following two lines
plt.savefig('../figs/xtyt.pdf')
subprocess.run(shlex.split("termux−open ../figs/
     xtyt.pdf"))
#uncomment this line
#plt.show()
```

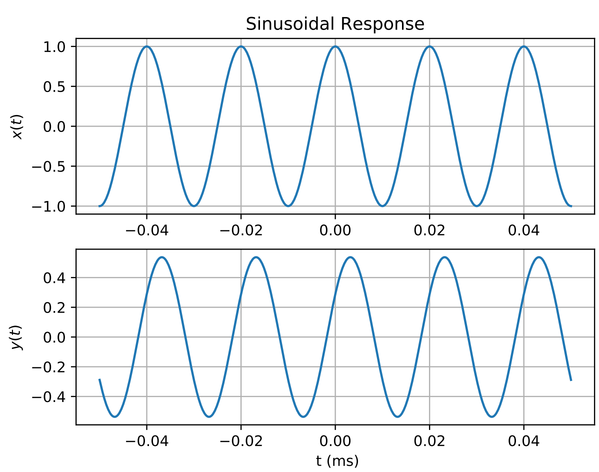

Fig. 3: Sinusoidal Response

4. Sketch the output $y(t)$ in Fig. 1 when $x(t) = \cos 100\pi t + \cos 300\pi t, R = 500\Omega$ and $C = 10\mu$F for $t \in (-50ms, 50ms)$. Comment.
5. Sketch the output when $x(t) = \cos 100\pi t + \cos 2\pi kt, R = 500\Omega$ and $C = 10\mu$F for different values of k. Find the value of k for which the output of $H(f)$ has frequency 50 Hz.