Fourier Transforms

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Contents

***Abstract*—This manual provides a quick introduction to the Fourier transform.**

1 Sinusoidal Response

1. Fig. 1 shows an RC circuit with input $x(t)$ and output $y(t)$. Show that

$$RC\frac{dy}{dt} + y(t) = x(t) \tag{1.1}$$

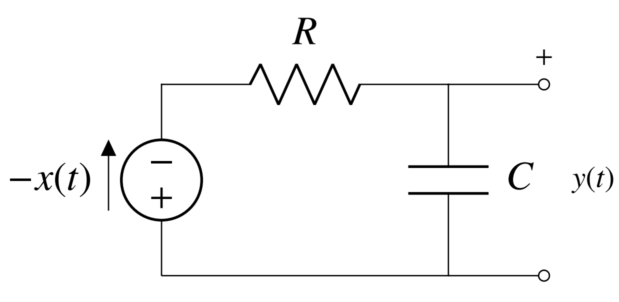

Fig. 1: RC Circuit

2. Let $x(t) = \cos 2\pi f_0 t$. Show that

$$y(t) = \frac{1}{\sqrt{1 + (2\pi f_0 RC)^2}} \cos\left[2\pi f_0 \left(t - RC\right)\right] \tag{1.2}$$

by solving the differential equation in (1.1) using the integrating factor.

(1.3)

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2 Fourier Transform

The Fourier transform of a signal $g(t)$ is defined as

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi f t}\, dt \tag{2.1}$$

1. Define the Dirac delta function as

$$\begin{aligned} \int_{\infty}^{\infty} \delta(t)\, dt &= 1 \\ \delta(t) &= 0, \quad t \neq 0. \end{aligned} \tag{2.2}$$

and show that

$$\delta(t) \overset{\mathcal{F}}{\rightleftharpoons} 1 \tag{2.3}$$

2. Show that

$$\delta(t - t_0) \overset{\mathcal{F}}{\rightleftharpoons} e^{-\jmath 2\pi f t_0} \tag{2.4}$$

3. Assuming

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{2\pi f t}\, dt, \tag{2.5}$$

show that

$$\begin{aligned} g(t) &\overset{\mathcal{F}}{\rightleftharpoons} G(f) \\ \implies G(t) &\overset{\mathcal{F}}{\rightleftharpoons} g(-f) \end{aligned} \tag{2.6}$$

4. Show that

$$\cos 2\pi f_0 t \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{2}\left[\delta\left(f - f_0\right) + \delta\left(f + f_0\right)\right] \tag{2.7}$$

5. Show that

$$\frac{dy}{dt} \overset{\mathcal{F}}{\rightleftharpoons} \jmath 2\pi f Y(f) \tag{2.8}$$

6. Define

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{2.9}$$

Show that

$$e^{-at}u(t) \overset{\mathcal{F}}{\rightleftharpoons} \frac{1}{a + \jmath 2\pi f} \tag{2.10}$$

7. If $x(t) = \delta(t)$ in (1.1), find $y(t)$ using Fourier transforms. This is known as the *impulse response* and is denoted by $h(t)$.
8. Solve (1.1) for $x(t) = \cos 2\pi f_0 t$ using Fourier transforms.
9. Verify that

$$y(t) = h(t) * x(t) = \int_{\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \tag{2.11}$$

This is known as the *convolution integral*.

10. Show that

$$x(t) * h(t) \overset{\mathcal{F}}{\rightleftharpoons} X(f)H(f) \tag{2.12}$$

11. In (1.1), show that

$$H(f) = \frac{1}{1 + \jmath 2\pi f RC} \tag{2.13}$$

12. Show that

$$y(t) = |H(f_0)| \cos\left[2\pi f_0 t + \angle H(f_0)\right] \tag{2.14}$$

3 Filtering

1. Plot $|H(f)|$ for different values of RC. The following code plots Fig. 1.

```
import numpy as np
import matplotlib.pyplot as plt
import subprocess
import shlex

#Filter Characteristics

R = 5*1e2; #1K ohm resistance
C = 10e−6;#10 uF capacitance

#Plotting the filter amplitude response
f_0 = 50.0
f = np.linspace(−3*f_0,3*f_0,1e2)
s = 1j*2*np.pi*f

den = np.polyval([1,1],s*C*R);
H = 1/den;

plt.plot(f,abs(H))
plt.grid()# minor
plt.xlabel('$f$ (Hz)')
plt.ylabel('$H(f)$')
#Save figure
#plt.savefig('../figs/lpf.eps')
plt.savefig('lpf.pdf')
subprocess.run(shlex.split("termux−open lpf.pdf
    "))
#plt.show()
```

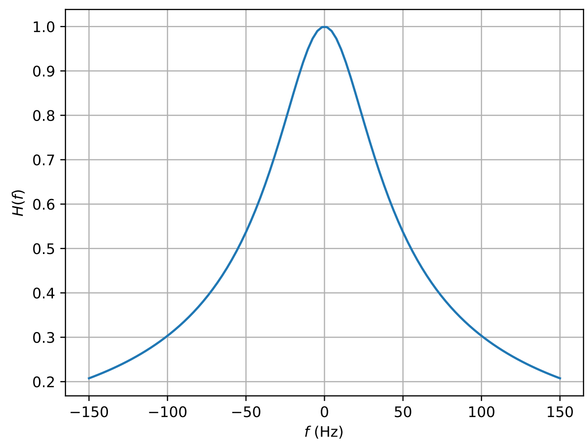

Fig. 1: Low Pass Filter

2. Sketch the output when $x(t) = \cos 100\pi t + \cos 300\pi t, R = 500\Omega$ and $C = 10\mu$F. Comment.