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Sources:

- Qiskit Textbook: https://qiskit.org/textbook/
- Microsoft Quantum Documentation – Concepts: https://docs.microsoft.com/en-us/quantum/concepts/

Linear Algebra

Vectors

Adjoint of a vector is its complex conjugate transpose.

$$v = \begin{bmatrix} v_1 \\ \cdots \\ v_n \end{bmatrix}, \quad v^\dagger = [v_1^* \ldots v_n^*]$$

Inner product in a complex vector space

$$\langle u, v \rangle = u^\dagger v = u_1^* v_1 + \cdots + u_n^* v_n$$

Note that inner product on complex vector space (Hermitian inner product) has a different set of axioms (more general) than those of real vector space

- $\langle u, v \rangle = \overline{\langle v, u \rangle}$
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle, \ \langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
- $\langle cu, v \rangle = \bar{c} \langle u, v \rangle, \ \langle u, cv \rangle = c \langle u, v \rangle$
- $\langle u, u \rangle \geq 0, \ \langle u, u \rangle = 0 \Leftrightarrow u = 0$

Matrix

Adjoint of a matrix is its complex conjugate transpose.

A matrix U is **unitary** when

$$UU^\dagger = U^\dagger U = 1$$

Or

$$U^{-1} = U^\dagger$$

Unitary matrices preserve the norm of a vector

$$\langle v, v \rangle = v^\dagger v = v^\dagger U^{-1} U v = v^\dagger U^\dagger U v = \langle Uv, Uv \rangle$$

A matrix M is **Hermitian** (self-adjoint) when

$$M = M^\dagger$$

Tensor product (or Kronecker product) of M $(m \times n)$ and N $(p \times q)$ is P $(mp \times nq)$

$$P = M \otimes N = \begin{bmatrix} M_{11}N & \ldots & M_{1n}N \\ \vdots & \ddots & \vdots \\ M_{m1}N & \ldots & M_{mn}N \end{bmatrix}$$

For any vector v or matrix M, $v^{\otimes n}$ or $M^{\otimes n}$ is n-fold repeated tensor product

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\otimes 2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\otimes 2} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Spectral theorem For any Hermitian or unitary matrix M

$$M = U^\dagger D U$$

Where D is a diagonal matrix, whose entries are eigenvalues of M, and
U is a unitary matrix. Note that this is similar to diagonalization in real matrices, since unitary U has the property $U^\dagger = U^{-1}$

D and M are related. If D has an eigenvector-value pair $v, \ c$, then $U^\dagger v, \ c$ will be an eigenvector-value pair of M

$$MU^\dagger v = U^\dagger D U U^\dagger v = U^\dagger D v = c U^\dagger v$$

Matrix exponential is defined in exact analogy to exponential function

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

We can compute matrix exponential using its decomposition form
Since

$$A^n = U^\dagger D^n U = U^\dagger \begin{bmatrix} D_{11}^n & & \\ & \ddots & \\ & & D_{NN}^n \end{bmatrix} U$$

$$e^A = U^\dagger \begin{bmatrix} \exp D_{11} & & \\ & \ddots & \\ & & \exp D_{NN} \end{bmatrix} U$$

If matrix B is both unitary and Hermitian

$$e^{iBx} = I\cos x + iB\sin x\,, \quad x \in \mathbb{R}$$

Quantum Computing Theories

Qubit and its representation

Statevector Describe the state of the system

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|q_0\rangle = a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}, \quad |a|^2 + |b|^2 = 1, \quad a, b \in \mathbb{C}$$

Dirac notation
$|x\rangle$ is a column vector, called **ket**
$\langle x|$ is a row vector, called **bra**

Bra and ket of the same vector are their complex conjugate transpose (or adjoint)

$$|x\rangle = \begin{bmatrix} x_0 \\ \dots \\ x_n \end{bmatrix}, \quad \langle x| = \begin{bmatrix} x_0^* & \dots & x_n^* \end{bmatrix}$$

The bra-ket notation $\langle a|b\rangle$ simply means the inner product between the two vector $|a\rangle$ and $|b\rangle$

$$\langle a|b\rangle = \begin{bmatrix} a_0^* & \dots & a_n^* \end{bmatrix} \begin{bmatrix} b_0 \\ \dots \\ b_n \end{bmatrix} = a_0^* b_0 + \cdots + a_n^* b_n$$

The ket-bra notation $|b\rangle\langle a|$ in turns means the outer product

$$|b\rangle\langle a| = \begin{bmatrix} b_0 \\ \dots \\ b_n \end{bmatrix} \begin{bmatrix} a_0^* & \dots & a_n^* \end{bmatrix} = \begin{bmatrix} a_0^* b_0 & \dots & a_n^* b_0 \\ \vdots & \ddots & \vdots \\ a_0^* b_n & \dots & a_n^* b_n \end{bmatrix}$$

Measurement The probability of measuring a state $|\psi\rangle$ and find the state $|x\rangle$

$$p(|x\rangle) = |\langle\psi|x\rangle|^2$$

Note that $\langle x|x\rangle = 1$ due to normalization, and $\langle 1|0\rangle = \langle 0|1\rangle = 0$ due to orthogonaility

This follows that: given a state $|q_0\rangle = a|0\rangle + b|1\rangle$

$$\langle q_0|0\rangle = a\langle 0|0\rangle + b\langle 1|0\rangle = a$$
$$\Rightarrow p(|0\rangle) = a^2$$

Implications

- Normalization: $\langle\psi|\psi\rangle = 1$
  - If $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then $\sqrt{|\alpha|^2 + |\beta|^2} = 1$
- Global phase: does not matter, only relative phase (between $|0\rangle$ and $|1\rangle$) matters. We also cannot determine the global phase through measurements
  - $|\langle x|a\rangle|^2 = \left|\langle x|e^{i\theta}|a\rangle\right|^2 \;\; \forall\theta \in \mathbb{R}$
- Qubit collapse: once a qubit is measured, its superposition is destroyed → permanently change the state to either $|0\rangle$ or $|1\rangle$

Restricted Qubit state

Since we are only concerned with relative phase, we can make our qubit more restricted

$$|q\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle, \;\; \alpha, \; \beta, \phi \in \mathbb{R}$$

Normalization condition: $\sqrt{\alpha^2 + \beta^2} = 1$
We will transform α, β to θ to remove the normalization constraint

$$\alpha = \cos\frac{\theta}{2}, \;\; \beta = \sin\frac{\theta}{2}$$

Our qubit now can be described using two free variables ϕ, θ

$$|q\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \;\; \theta, \; \phi \in \mathbb{R}$$

We will interpret θ and ϕ as spherical co-ordinates, with unit radius → Bloch sphere
Thus the Block sphere represents all posible Qubit states

Single Qubit gates

Pauli gates

The X-gate Switch amplitudes of the states $|0\rangle$ and $|1\rangle$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$X|0\rangle = |1\rangle, \;\; X|1\rangle = |0\rangle$$

The Y and Z-gates Rotations by π along the y and z-axis of the Bloch sphere

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i|0\rangle\langle 1| - i|1\rangle\langle 0|$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$Y|0\rangle = -i|1\rangle, \;\; Y|1\rangle = i|0\rangle$$
$$Z|0\rangle = |0\rangle, \;\; Z|1\rangle = -|1\rangle$$

Basis

Z-gate has eigenvectors $|0\rangle$ and $|1\rangle$ (as shown above)

- Z-gate has no effect on either $|0\rangle$ or $|1\rangle$ (note that phase change does not matter)
- Z-basis → the **computational basis** $\{\,|0\rangle, \;\; |1\rangle\,\}$

X-basis $|+\rangle$ and $|-\rangle$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \;\; |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$X|+\rangle = |+\rangle, \;\; X|-\rangle = -|-\rangle$$

Y-basis $|↺\rangle$ and $|↻\rangle$

$$|↺\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}i|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix},\ |↻\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\i\end{bmatrix}$$

$$Y|↺\rangle = -|↺\rangle,\ Y|↻\rangle = |↻\rangle$$

The Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$H|0\rangle = |+\rangle,\ H|1\rangle = |-\rangle$$

The Hadamard gate is also known as the superposition gate: it turns a single state into superposition of two states

- It can be thought as a rotation around the Bloch vector $[1,\ 0,\ 1]$, or transforming the state of qubit between the X and Z bases

The H-gate can be used to transform from Z-basis to X-basis while preserving the amplitude

$$|q\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$H|q\rangle = \alpha|+\rangle + \beta|-\rangle$$

A mathematical check show that H-gate is its own inverse $H = H^{-1}$

$$HH = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$$

Thus H-gate can be used to transform X-basis back to Z-basis

$$|q\rangle = \alpha|+\rangle + \beta|-\rangle$$
$$H|q\rangle = \alpha|0\rangle + \beta|1\rangle$$

Measurement in different bases

We don't have to measure in the computational basis (the Z-basis), we can measure in any basis. Measurement in another basis can be represented as (e.g. X-basis)

$$p(|+\rangle) = |\langle +|q\rangle|^2,\ p(|-\rangle) = |\langle -|q\rangle|^2$$

To measure in X-basis, we transform from Z-basis to X-basis using a single Hadamard gate, then measure it.

$$|q\rangle \rightarrow H|q\rangle \rightarrow measure$$

Measurement error mitigation

Two notable sources of error

- Energy relaxation of qubit during measurement. The relaxation takes the $|1\rangle$ state to the $|0\rangle$ state for each qubit
- Descrimination after measurement. Signal obtained from measurement is wrongly identified

Obtain the **calibration matrix**

- We prepare and measure each of the 2^n basis states, given n qubits.
- Outcome statistics (probabilities) are captured by the **calibration matrix** M
  - Row → output (measured) state
  - Column → prepared state
- Model the noisy output

$$|\psi_{noisy}\rangle = M|\psi\rangle$$

Goal: apply classical correction that undoes the errors

$$|\psi\rangle = M^{-1}|\psi_{noisy}\rangle$$

Note:

- M is usually non-invertible
- Noise is not deterministic → reduce noise, but cannot eliminate noise entirely

Using IBM Qiskit

Installation

pip install qiskit

Setup

from qiskit import QuantumCircuit, execute, Aer
from qiskit.visualization import plot_histogram

%config InlineBackend.figure_format = 'svg' # Makes the images look nice

qc = QuantumCircuit(n_q, n_b)	Create a new quantum circuit • First argument: number of qubits • Second argument: number of output (classical) bits • Each qubit is initialized to $\|0\rangle$
qc.initialize([0,1], 0)	Initialize the qubit
qc.measure(0, 0) qc.measure_all()	Measure the qubit • Store the output to the classical bit (2nd argument)
qc.draw()	Draw qubit and operators
backend = Aer.get_backend('**qasm_simulator**') result = execute(qc, backend, shots=1000).result() counts = result.get_counts() plot_histogram(counts)	Run a simulator to make measurements • qasm_simulator: interact with real quantum computer
backend = Aer.get_backend('**statevector_simulator**') result = execute(qc,backend).result() out_state = result.get_statevector()	Get statevector • Return type is a Python list of complex number

minicomposer(1, dirac=True, qsphere=True)	Create a widget to manipulate qubits with GUI

Operators (gate)

.x(0)	X-gate → Bit flip
.y(0)	Bit and phase flip
.z(0)	Phase flip
.h(0)	H-gate → Superposition
.s(0)	Quantum phase rotation by $\pi/2$
.sdg(0)	Quantum phase rotation by $-\pi/2$