Contents

1 Construction

2 Asymptotics

We fix a large d, and set $q := \lfloor (2/\sqrt{3})^d \rfloor$ and $N := q^d$. Then,

$$\#A_r \geq N^2 (dq^2)^{-1} \left(\frac{3}{4} + O\left(\frac{1}{q}\right)\right)^d$$

Writing $o(1)$ for a quantity tending to 0 as $N \to \infty$, we note that $q = \left(\frac{2}{\sqrt{3}} + o(1)\right)^d$ ...

Theorem 2.1: ✓ (March 6)

We have $q = (2/\sqrt{3})^d + O(1) = \ldots$, see 3

and that $d = (1+o(1))\sqrt{\frac{\log N}{\log(2/\sqrt{3})}}$.

Theorem 2.2: ✓ (March 14)

From $N = q^d$ we know $d = \frac{\log N}{\log q} = \ldots$, see 4

A short calculation then confirms that $\#A_r \geq N^2 2^{-(c+o(1))\sqrt{\log_2 N}}$, where $c = 2\sqrt{2\log_2\left(\frac{4}{3}\right)} \approx 1.822\ldots$.

Theorem 2.3

See 5

3 Asymptotics 1

Fix $c \geq 1$. We want to say that $c^d + 1 = (c + o(1))^d$. In Lean, we may write this as:

Theorem 3.1: Lean Formulation 1

There exists $f : \mathbb{N} \to \mathbb{R}$ such that $f = o(1)$, and $c^d + 1 = (c + f(d))^d$ for all $d \in \mathbb{N}$.

However, this is already wrong, since when $d = 0$ the relation cannot be satisfied. Hence, we have to modify the statement slightly.

Theorem 3.2: Lean Formulation 2

There exists $f : \mathbb{N} \to \mathbb{R}$ such that $f = o(1)$, and $c^d + 1 = (c + f(d))^d$ for all $d \in \mathbb{N}_{>0}$.

Let us try to prove the theorem *without* using the explicit form of the solution. We can make the observation that the "correct solution" satisfies $f(d) \leq \frac{1}{d}$, as $c^d + 1 \leq \left(c + \frac{1}{d}\right)^d$, from which $f = o(1)$ follows easily. Hence, in some sense $f = o(1)$ is "easy". This inspires our first attempt:

Theorem 3.3: Attempt 1

The following hold:

(1) There exists a function $f : \mathbb{N} \to \mathbb{R}$ such that $c^d + 1 = (c + f(d))^d$ for all d.

(2) For *all* functions $f : \mathbb{N} \to \mathbb{R}$ such that $c^d + 1 = (c + f(d))^d$ for all d, $f = o(1)$.

However, (2) turns out to be false, as when d is even, we can take $c + f(d)$ to be the negative of that for the "correct solution", meaning $f(d)$ does not tend to 0. Here is the fixed version:

Theorem 3.4: ✓ Attempt 2

The following hold:

(1) There exists a function $f : \mathbb{N} \to \mathbb{R}$ where $f(d) \geq 0$ and $c^d + 1 = (c + f(d))^d$.

(2) For *all* functions $f : \mathbb{N} \to \mathbb{R}$ such that $f(d) \geq 0$ and $c^d + 1 = (c + f(d))^d$, $f = o(1)$.

For (1), the approach we take here is to use the continuity of $x \mapsto x^d$ and the fact that $c^d \leq c^d + 1 \leq (c+1)^d$, the latter of which can be proven by noting $c^d + 1 \leq c^d + d \leq (c+1)^d$. For (2), note that $(c + f(d))^d = c^d + 1 \leq \left(c + \frac{1}{d}\right)^d$, so $f(d) \leq \frac{1}{d}$. This combined with $f(d) \geq 0$ shows $f = o(1)$.

The final formalisation can be found here.

4 Asymptotics 2

Let $c > 1$ be a constant. We want to show that $\sqrt{\frac{\log N}{\log(c+o(1))}} = (1+o(1))\sqrt{\frac{\log N}{\log c}}$ as $N \to \infty$.

Theorem 4.1: Attempt 1

$\log(c+o(1)) = \log(c) + \log(1+o(1))$.

Simply write $\log(c+o(1)) = \log(c) + \log(1+o(1)/c) = \log(c) + \log(1+o(1))$.

At this point, I realised that my formulation was slightly wrong again. Recalled from the last section that there was a problem with 3, as the statement was not satisfiable at $d = 0$. Here, for the equality $\log(xy) = \log(x) + \log(y)$ hold (or to be nicely defined), we need $0 < x$ and $0 < y$ too.

Hence, if we phrase the theorem in the typical Lean way of $\forall f \in o(1), \exists g, g \in o(1) \wedge \log(c+f(N)) = \log(c) + \log(1+g(N))$, then this might not be satisfable, since f can be negative at the beginning. The correct formulation is

Theorem 4.2: ✓ Step 1 (Mar 11)

$\forall f \in o(1), \exists g, g \in o(1) \wedge \log(c+f(N)) =^{f} \log(c) + \log(1+g(N))$.

From which we can argue that since $f \in o(1)$, we get $f \to 0$, so $1 + f/c \to 1$ i.e. it is eventually positive.

Theorem 4.3: ✓ Step 2 (Mar 14)

$\log(1+o(1)) = o(1)$.

From the elementary inequality $1 + x \leq \exp(x)$ we get $\log(1+x) \leq x = O(x)$ when x is large enough (non-negative). Hence, $\log(1+o(1)) = O(o(1)) = o(1)$.

Theorem 4.4: ✓ Step 3 (Mar 14)

$\sqrt{\frac{1}{1+o(1)}} = 1 + o(1)$.

To show this, we simply recall that $f(x) \in o(1) \iff \lim_{x\to\infty} f(x) = 0$. Let $f(n)$ be a function such that $\lim_{n\to\infty} f(n) = 0$. Then, we can immediate get $\sqrt{\frac{1}{1+f(x)}} \to 1$ by substitution.

5 Asymptotics 3