

# Semidefinite Optimization and Relaxation

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# Preface

This is the textbook for Harvard ENG-SCI 257: Semidefinite Optimization and Relaxation.

## Feedback

I would like to invite you to provide comments to the textbook via the following two ways:

- Inline comments with Hypothesis:
  - Go to Hypothesis and create an account
  - Install the Chrome extension of Hypothesis
  - Provide public comments to textbook contents and I will try to address them
- Blog-style comments with Disqus:
  - At the end of each Chapter, there is a Disqus module where you can leave feedback

I would recommend using Disqus for high-level and general feedback regarding the entire Chapter, but using Hypothesis for feedback and questions about the technical details.

## Offerings

Information about the offerings of the class is listed below.

**2024 Spring**

**Time:** Mon/Wed 2:15 - 3:30pm

**Location:** Science and Engineering Complex, 1.413

**Instructor:** Heng Yang

**Teaching Fellow:** Safwan Hossain

**Syllabus**

# Notation

We will use the following standard notation throughout this book.

## Basics

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$\mathbb{R}$	real numbers
$\mathbb{R}_+$	nonnegative real
$\mathbb{R}_{++}$	positive real
$\mathbb{Z}$	integers
$\mathbb{N}$	nonnegative integers
$\mathbb{N}_+$	positive integers
$\mathbb{R}^n$	$n$ -D column vector
$\mathbb{R}_+^n$	nonnegative orthant
$\mathbb{R}_{++}^n$	positive orthant
$e_i$	standard basic vector
$\Delta_n := \{x \in \mathbb{R}_+^n \mid \sum x_i = 1\}$	standard simplex

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## Matrices

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$\mathbb{R}^{m \times n}$	$m \times n$ real matrices
$\mathbb{S}^n$	$n \times n$ symmetric matrices
$\mathbb{S}_+^n$	$n \times n$ positive semidefinite matrices
$\mathbb{S}_{++}^n$	$n \times n$ positive definite matrices
$\langle A, B \rangle$ or $\bullet$	inner product in $\mathbb{R}^{m \times n}$
$\text{tr}(A)$	trace of $A \in \mathbb{R}^{n \times n}$
$A^\top$	matrix transpose
$\det(A)$	matrix determinant
$\text{rank}(A)$	rank of a matrix
$\text{diag}(A)$	diagonal of a matrix
	$A$ as a vector

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$\text{Diag}(a)$	turning a vector into a diagonal matrix
$\text{BlkDiag}(A, B, \dots)$	block diagonal matrix with blocks $A, B, \dots$
$\succeq 0$ and $\preceq 0$	positive / negative semidefinite
$\succ 0$ and $\prec 0$	positive / negative definite
$\lambda_{\max}$ and $\lambda_{\min}$	maximum / minimum eigenvalue
$\sigma_{\max}$ and $\sigma_{\min}$	maximum / minimum singular value
$\text{vec}(A)$	vectorization of $A \in \mathbb{R}^{m \times n}$
$\text{svec}(A)$	symmetric vectorization of $A \in \mathbb{S}^n$
$\ A\ _{\text{F}}$	Frobenius norm

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### Geometry

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$\ a\ _p$	$p$ -norm
$\ a\ $	2-norm
$B(o, r)$	ball with center $o$ and radius $r$
$\text{aff}(S)$	affine hull of set $S$
$\text{conv}(S)$	convex hull of set $S$
$\text{cone}(S)$	conical hull of set $S$
$\text{int}(S)$	interior of set $S$
$\text{ri}(S)$	relative interior of set $S$
$\partial S$	boundary of set $S$
$P^\circ$	polar of convex body
$P^*$	dual of set $P$
$\text{SO}(d)$	special orthogonal group of dimension $d$
$\mathcal{S}^{d-1}$	unit sphere in $\mathbb{R}^d$

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### Optimization

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KKT	Karush–Kuhn–Tucker
LP	linear program
QP	quadratic program
SOCP	second-order cone program
SDP	semidefinite program

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**Algebra**


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$\mathbb{R}[x]$	polynomial ring in $x$ with real coefficients
$\deg$	degree of a monomial / polynomial
$\mathbb{R}[x]_d$	polynomials in $x$ of degree up to $d$
$[x]_d$	vector of monomials of degree up to $d$
$\llbracket x \rrbracket_d$	vector of monomials of degree $d$

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# Chapter 1

## Mathematical Background

### 1.1 Convexity

A very important notion in modern optimization is that of *convexity*. To a large extent, an optimization problem is “easy” if it is convex, and “difficult” when convexity is lost, i.e., *nonconvex*. We give a basic review of convexity here and refer the reader to (Rockafellar, 1970), (Boyd and Vandenberghe, 2004), and (Bertsekas et al., 2003) for comprehensive treatments.

We will work on a finite-dimensional real vector space, which we will identify with  $\mathbb{R}^n$ .

**Definition 1.1** (Convex Set). A set  $S$  is convex if  $x_1, x_2 \in S$  implies  $\lambda x_1 + (1 - \lambda)x_2 \in S$  for any  $\lambda \in [0, 1]$ . In other words, if  $x_1, x_2 \in S$ , then the line segment connecting  $x_1$  and  $x_2$  lies inside  $S$ .

Conversely, a set  $S$  is nonconvex if Definition 1.1 does not hold.

Given  $x_1, x_2 \in S$ ,  $\lambda x_1 + (1 - \lambda)x_2$  is called a *convex combination* when  $\lambda \in [0, 1]$ . For convenience, we will use the following notation

$$\begin{aligned} (x_1, x_2) &= \{\lambda x_1 + (1 - \lambda)x_2 \mid \lambda \in (0, 1)\}, \\ [x_1, x_2] &= \{\lambda x_1 + (1 - \lambda)x_2 \mid \lambda \in [0, 1]\}. \end{aligned} \tag{1.1}$$

A **hyperplane** is a common convex set defined as

$$H = \{x \in \mathbb{R}^n \mid \langle c, x \rangle = d\} \tag{1.2}$$

for some  $c \in \mathbb{R}^n$  and scalar  $d$ . A **halfspace** is a convex set defined as

$$H^+ = \{x \in \mathbb{R}^n \mid \langle c, x \rangle \geq d\}. \tag{1.3}$$

Given two nonempty convex sets  $C_1$  and  $C_2$ , the **distance** between  $C_1$  and  $C_2$  is defined as

$$\text{dist}(C_1, C_2) = \inf\{\|c_1 - c_2\| \mid c_1 \in C_1, c_2 \in C_2\}. \quad (1.4)$$

For a convex set  $C$ , the hyperplane  $H$  in (1.2) is called a **supporting hyperplane** for  $C$  if  $C$  is contained in the half space  $H^+$  and the distance between  $H$  and  $C$  is zero. For example, the hyperplane  $x_1 = 0$  is supporting for the hyperboloid  $\{(x_1, x_2) \mid x_1 x_2 \geq 1, x_1 \geq 0, x_2 \geq 0\}$  in  $\mathbb{R}^2$ .

An important property of a convex set is that we can *certify* when a point is not in the set. This is usually done via a separation theorem.

**Theorem 1.1** (Separation Theorem). *Let  $S_1, S_2$  be two convex sets in  $\mathbb{R}^n$  and  $S_1 \cap S_2 = \emptyset$ , then there exists a hyperplane that separates  $S_1$  and  $S_2$ , i.e., there exists  $c$  and  $d$  such that*

$$\begin{aligned} \langle c, x \rangle &\geq d, \forall x \in S_1, \\ \langle c, x \rangle &\leq d, \forall x \in S_2. \end{aligned} \quad (1.5)$$

*Further, if  $S_1$  is compact (i.e., closed and bounded) and  $S_2$  is closed, then the separation is strict, i.e., the inequalities in (1.5) are strict.*

The strict separation theorem is used typically when  $S_1$  is a single point (hence compact).

We will see a generation of the separation theorem for nonconvex sets later after we introduce the idea of sums of squares.

**Exercise 1.1.** Provide examples of two disjoint convex sets such that the separation in (1.5) is not strict in one way and both ways.

**Exercise 1.2.** Provide a constructive proof that the separation hyperplane exists in Theorem 1.1 when (1) both  $S_1$  and  $S_2$  are closed, and (2) at least one of them is bounded.

The intersection of convex sets is always convex (try to prove this).

## 1.2 Convex Geometry

### 1.2.1 Basic Facts

Given a set  $S$ , its **affine hull** is the set

$$\text{aff}(S) = \left\{ \sum_{i=1}^k \lambda_i u_i \mid \lambda_1 + \cdots + \lambda_k = 1, u_i \in S, k \in \mathbb{N}_+ \right\},$$

where  $\sum_{i=1}^k \lambda_i u_i$  is called an *affine combination* of  $u_1, \dots, u_k$  when  $\sum_i \lambda_i = 1$ . The affine hull of the empty set is the empty set, of a singleton is the singleton itself. The affine hull of a set of two different points is the line going through them. The affine hull of a set of three points not on one line is the plane going through them. The affine hull of a set of four points not in a plane in  $\mathbb{R}^3$  is the entire space  $\mathbb{R}^3$ .

For a convex set  $C \subseteq \mathbb{R}^n$ , the **interior** of  $C$  is defined as

$$\text{int}(C) := \{u \in C \mid \exists \epsilon > 0, B(u, \epsilon) \subseteq C\},$$

where  $B(u, \epsilon)$  denotes a ball centered at  $u$  with radius  $\epsilon$  (using the usual 2-norm). Each point in  $\text{int}(C)$  is called an *interior point* of  $C$ . If  $\text{int}(C) = C$ , then  $C$  is said to be an **open set**. A convex set with nonempty interior is called a **convex domain**, while a compact (i.e., closed and bounded) convex domain is called a **convex body**.

The **boundary** of  $C$  is the subset of points that are not in the interior of  $C$  and we denote it as  $\partial C$ .

It is possible that a convex set has empty interior. For example, a hyperplane has no interior, and neither does a singleton. In such cases, the **relative interior** can be defined as

$$\text{ri}(C) := \{u \in C \mid \exists \epsilon > 0, B(u, \epsilon) \cap \text{aff}(C) \subseteq C\}.$$

For a nonempty convex set, the relative interior always exists. If  $\text{ri}(C) = C$ , then  $C$  is said to be **relatively open**. For example, the relative interior of a singleton is the singleton itself, and hence a singleton is relatively open.

For a convex set  $C$ , a point  $u \in C$  is called an **extreme point** if

$$u \in (x, y), x \in C, y \in C \quad \Rightarrow \quad u = x = y.$$

For example, consider  $C = \{(x, y) \mid x^2 + y^2 \leq 1\}$ , then all the points on the boundary  $\partial C = \{(x, y) \mid x^2 + y^2 = 1\}$  are extreme points.

A subset  $F \subseteq C$  is called a **face** if  $F$  itself is convex and

$$u \in (x, y), u \in F, x, y \in C \quad \Rightarrow \quad x, y \in F.$$

Clearly, the empty set  $\emptyset$  and the entire set  $C$  are faces of  $C$ , which are called *trivial faces*. The face  $F$  is said to be *proper* if  $F \neq C$ . The set of any single extreme point is also a face. A face  $F$  of  $C$  is called **exposed** if there exists a supporting hyperplane  $H$  for  $C$  such that

$$F = H \cap C.$$

### 1.2.2 Cones, Duality, Polarity

**Definition 1.2** (Polar). For a nonempty set  $T \subseteq \mathbb{R}^n$ , its polar is the set

$$T^\circ := \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 1, \forall x \in T\}. \quad (1.6)$$

The polar  $T^\circ$  is a closed convex set and contains the origin. Note that  $T$  is always contained in the polar of  $T^\circ$ , i.e.,  $T \subseteq (T^\circ)^\circ$ . Indeed, they are equal under some assumptions.

**Theorem 1.2** (Bipolar). *If  $T \subseteq \mathbb{R}^n$  is a closed convex set containing the origin, then  $(T^\circ)^\circ = T$ .*

An important class of convex sets are those that are invariant under nonnegative scalings. A set  $K \subseteq \mathbb{R}^n$  is a **cone** if  $tx \in K$  for all  $x \in K$  and for all  $t > 0$ . For example, the positive real line  $\{x \in \mathbb{R} \mid x > 0\}$  is a cone. The cone  $K$  is **pointed** if  $K \cap -K = \{0\}$ . It is said to be **solid** if its interior  $\text{int}(K) \neq \emptyset$ . Any nonzero point of a cone cannot be extreme. If a cone is pointed, the only extreme point is the origin.

The analogue of extreme point for convex cones is the **extreme ray**. For a convex cone  $K$  and  $0 \neq u \in K$ , the line segment

$$u \cdot [0, \infty) := \{tu \mid t \geq 0\}$$

is called an extreme ray of  $K$  if

$$u \in (x, y), x, y \in K \quad \Rightarrow \quad u, x, y \text{ are parallel to each other.}$$

If  $u \cdot [0, \infty)$  is an extreme ray, then we say  $u$  generates the extreme ray.

**Definition 1.3** (Proper Cone). A cone  $K$  is proper if it is closed, convex, pointed, and solid.

A proper cone  $K$  induces a **partial order** on the vector space, via  $x \succeq y$  if  $x - y \in K$ . We also use  $x \succ y$  if  $x - y$  is in  $\text{int}(K)$ . Important examples of proper cones are the nonnegative orthant, the second-order cone, the set of symmetric positive semidefinite matrices, and the set of nonnegative polynomials, which we will describe later in the book.

**Definition 1.4** (Dual). The dual of a nonempty set  $S$  is

$$S^* := \{y \in \mathbb{R}^n \mid \langle y, x \rangle \geq 0, \forall x \in S\}.$$

Given any set  $S$ , its dual  $S^*$  is always a closed convex cone. Duality reverses inclusion, that is,

$$S_1 \subseteq S_2 \quad \Rightarrow \quad S_1^* \supseteq S_2^*.$$

If  $S$  is a closed convex cone, then  $S^{**} = S$ . Otherwise,  $S^{**}$  is the closure of the smallest convex cone that contains  $S$ .

For a cone  $K \subseteq \mathbb{R}^n$ , one can show that

$$K^\circ = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leq 0, \forall x \in K\}.$$

The set  $K^\circ$  is called the **polar cone** of  $K$ . The negative of  $K^\circ$  is just the **dual cone**

$$K^* = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \geq 0, \forall x \in K\}.$$

**Definition 1.5** (Self-dual). A cone  $K$  is self-dual if  $K^* = K$ .

As an easy example, the nonnegative orthant  $\mathbb{R}_+^n$  is self-dual.

**Example 1.1** (Second-order Cone). The second-order cone, or the Lorentz cone, or the ice cream cone

$$\mathcal{Q}_n := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sqrt{x_1^2 + \dots + x_n^2} \leq x_0\}$$

is a proper cone of  $\mathbb{R}^{n+1}$ . We will show that it is also self-dual.

**Proof.** Consider  $(y_0, y_1, \dots, y_n) \in \mathcal{Q}_n$ , we want to show that

$$x_0 y_0 + x_1 y_1 + \dots + x_n y_n \geq 0, \forall (x_0, x_1, \dots, x_n) \in \mathcal{Q}_n. \quad (1.7)$$

This is easy to verify because

$$x_1 y_1 + \dots + x_n y_n \geq -\sqrt{x_1^2 + \dots + x_n^2} \sqrt{y_1^2 + \dots + y_n^2} \geq -x_0 y_0.$$

Hence we have  $\mathcal{Q}_n \subseteq \mathcal{Q}_n^*$ .

Conversely, if (1.7) holds, then take

$$x_1 = -y_1, \dots, x_n = -y_n, \quad x_0 = \sqrt{x_1^2 + \dots + x_n^2},$$

we have

$$y_0 \geq \sqrt{y_1^2 + \dots + y_n^2},$$

hence  $\mathcal{Q}_n^* \subseteq \mathcal{Q}_n$ . ■

Not every proper cone is self-dual.

**Exercise 1.3.** Consider the following proper cone in  $\mathbb{R}^2$

$$K = \{(x_1, x_2) \mid 2x_1 - x_2 \geq 0, 2x_2 - x_1 \geq 0\}.$$

Show that it is not self-dual.

## 1.3 Convex Optimization

## 1.4 Linear Optimization





## Chapter 2

# Semidefinite Optimization

- Positive Semidefinite Matrices
- Spectrahedra



## Chapter 3

# Problem Sets

**Exercise 3.1** (Test). Test

**Exercise 3.2** (Test). Test



# Bibliography

- Bertsekas, D., Nedic, A., and Ozdaglar, A. (2003). *Convex analysis and optimization*, volume 1. Athena Scientific. 11
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- Rockafellar, R. T. (1970). Convex analysis. 11