Semidefinite Optimization and Relaxation

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Contents

Preface

This is the textbook for Harvard ENG-SCI 257: Semidefinite Optimization and Relaxation. Information about the offerings of the class is listed below.

2024 Spring

Time: Mon/Wed 2:15 - 3:30pm

Location: Science and Engineering Complex, 1.413

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Teaching Fellow: Safwan Hossain

Syllabus

Acknowledgment

Notation

We will use the following standard notation throughout this book.

Basics

$\mathbb{R}$	real numbers
$\mathbb{R}_+$	nonnegative real
$\mathbb{R}_{++}$	positive real
$\mathbb{Z}$	integers
$\mathbb{N}$	nonnegative integers
$\mathbb{R}^n$	n-D column vector
$\mathbb{R}^n_+$	nonnegative orthant
$\mathbb{R}^n_{++}$	positive orthant
e_i	standard basic vector
$\Delta_n := \{x \in \mathbb{R}^n_+ \mid \sum x_i = 1\}$	standard simplex

Matrices

$\mathbb{R}^{m\times n}$	$m \times n$ real matrices
$\mathbb{S}^n$	$n \times n$ symmetric matrices
$\mathbb{S}^n_+$	$n \times n$ positive semidefinite matrices
$\mathbb{S}^n_{++}$	$n \times n$ positive definite matrices
$\langle A, B\rangle$ or $\bullet$	inner product in $\mathbb{R}^{m\times n}$
$\mathrm{tr}(A)$	trace of $A \in \mathbb{R}^{n\times n}$
$A^\top$	matrix transpose
$\det(A)$	matrix determinant
$\mathrm{rank}(A)$	rank of a matrix
$\mathrm{diag}(A)$	diagonal of a matrix A as a vector

$\mathrm{Diag}(a)$	turning a vector into a diagonal matrix
$\mathrm{BlkDiag}(A, B, \dots)$	block diagonal matrix with blocks $A, B, \dots$
$\succeq 0$ and $\preceq 0$	positive / negative semidefinite
$\succ 0$ and $\prec 0$	positive / negative definite
$\lambda_{\max}$ and $\lambda_{\min}$	maximum / minimum eigenvalue
$\sigma_{\max}$ and $\sigma_{\min}$	maximum / minimum singular value
$\mathrm{vec}(A)$	vectorization of $A \in \mathbb{R}^{m \times n}$
$\mathrm{svec}(A)$	symmetric vectorization of $A \in \mathbb{S}^n$
$\|A\|_{\mathrm{F}}$	Frobenius norm

Geometry

$\|a\|_p$	p-norm
$\|a\|$	2-norm
$B(o, r)$	ball with center o and radius r
$\mathrm{conv}(S)$	convex hull of set S
$\mathrm{cone}(S)$	conical hull of set S
$\mathrm{int}(S)$	interior of set S
∂S	boundary of set S
P°	polar dual of convex body
$\mathrm{SO}(d)$	special orthogonal group of dimension d
$\mathcal{S}^{d-1}$	unit sphere in $\mathbb{R}^d$

Optimization

KKT	Karush–Kuhn–Tucker
LP	linear program
QP	quadratic program
SOCP	second-order cone program
SDP	semidefinite program

Algebra

$\mathbb{R}[x]$	polynomial ring in x with real coefficients
deg	degree of a monomial / polynomial
$\mathbb{R}[x]_d$	polynomials in x of degree up to d
$[x]_d$	vector of monomials of degree up to d
$[\![x]\!]_d$	vector of monomials of degree d

Chapter 1

Mathematical Background

- Convexity
- Convex Optimization
- Convex Geometry
- Linear Programming

1.1 Convexity

A very important notion in modern optimization is that of *convexity*. To a large extent, an optimization problem is "easy" if it is convex, and "difficult" when convexity is lost, i.e., *nonconvex*. We give a basic review of convexity here and refer the reader to (Rockafellar, 1970), (Boyd and Vandenberghe, 2004), and (Bertsekas et al., 2003) for comprehensive treatments.

We will work on a finite-dimensional real vector space, which we will identify with $\mathbb{R}^n$.

Definition 1.1 (Convex Set)**.** A set S is convex if $x_1, x_2 \in S$ implies $\lambda x_1 + (1-\lambda)x_2 \in S$ for any $\lambda \in [0,1]$. In other words, if $x_1, x_2 \in S$, then the line segment connecting x_1 and x_2 lies inside S.

Conversely, a set S is nonconvex if Definition 1.1 does not hold.

A *hyperplane* is a common convex set defined as

$$P = \{x \in \mathbb{R}^n \mid \langle c, x \rangle = d\}$$

for some $c \in \mathbb{R}^n$ and scalar d. A *halfspace* is a convex set defined as

$$H = \{x \in \mathbb{R}^n \mid \langle c, x \rangle \geq d\}.$$

An important property of a convex set is that we can *certify* when a point is not in the set. This is usually done via a separation theorem.

Theorem 1.1 (Separation Theorem). *Let S_1, S_2 be two convex sets in $\mathbb{R}^n$ and $S_1 \cap S_2 = \emptyset$, then there exists a hyperplane that separates S_1 and S_2, i.e., there exists c and d such that*

$$\begin{aligned} \langle c, x \rangle &\geq d, \forall x \in S_1, \\ \langle c, x \rangle &\leq d, \forall x \in S_2. \end{aligned} \tag{1.1}$$

Further, if S_1 is compact (i.e., closed and bounded) and S_2 is closed, then the separation is strict, i.e., the inequalities in (1.1) *are strict.*

The strict separation theorem is used typically when S_1 is a single point (hence compact).

The intersection of convex sets is always convex (try to prove this).

1.2 Convex Geometry

Definition 1.2 (Extreme Point). Given a convex set, a point $x \in S$ is *extreme* if

$$\forall x_1, x_2 \in S, \exists \lambda \in (0,1) \text{ such that } x = \lambda x_1 + (1-\lambda) x_2 \Longrightarrow x = x_1 = x_2.$$

In other words, any line segment that contains x and lies inside S must be x itself.

Chapter 2

Semidefinite Optimization

- Positive Semidefinite Matrices
- Spectrahedra

Bibliography

Bertsekas, D., Nedic, A., and Ozdaglar, A. (2003). *Convex analysis and optimization*, volume 1. Athena Scientific. 11

Boyd, S. P. and Vandenberghe, L. (2004). *Convex optimization*. Cambridge university press. 11

Rockafellar, R. T. (1970). Convex analysis. 11