Stats 130
Day 5 Notes

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Example from last time

A fair coin is tossed until heads comes up. All tosses are independent. What is the probability that n tosses are required?

$$Pr(n) = Pr(\neg(n-1) \wedge 1) \tag{1}$$

$$Pr((\bigcap_{i=1}^{n-1} T_i) \cap H_n) \tag{2}$$

$$\frac{1}{2}^{n-1}\frac{1}{2} = \frac{1}{2}^{n} \tag{3}$$

If p is the probability of success, and q is failure. The probability of needing n tries is

$$q^{n-1}p \tag{4}$$

What is the probability that we eventually get a head?

$$Pr(H) = Pr(\bigcup_{i=1}^{\infty} H_i) \tag{5}$$

$$Pr(H) = \sum_{i=1}^{\infty} Pr(H_i) \tag{6}$$

$$Pr(H) = \sum_{i=1}^{\infty}(q^{i-1}p) \tag{7}$$

$$Pr(H) = \sum_{i=1}^{\infty} \frac{1}{2}^{i} \tag{8}$$

$$Pr(H) = -1 + \sum_{i=0}^{\infty} \frac{1}{2}^{i} \tag{9}$$

$$Pr(H) = 1 \tag{10}$$

Partitions

The notion of a partition is useful. The idea is to split sample space into mutually exclusive events.

Consider events $B_1, ..., B_n, B_i \subseteq S$, and suppose $B_i \cap B_j = \emptyset, \forall i, j$. Also, suppose that $\bigcup_{j=1}^{n} B_j = S$. Then the collection forms a partition.

All partitions are also equivalent to the equivalence classes of some equivalence relation.

The Law of Total Probabilities

$$Pr(A) = \sum_{i=1}^{n} Pr(A \cap B_i) \tag{11}$$

$$Pr(A) = \sum_{i=1}^{n} Pr(B_i)Pr(A|B_i) \tag{12}$$

The second formula only works given $A \cap B_i$ is non-zero. The first formula always works.

Can also be viewed via a weighted average of probability. Also, since the probabilties of the B_i sets add to 1, then the law is also a convex combination.

Example

Two boxes contain long bolts and short bolts. One contains 60 short and 40 long. The other contains 20 short and 10 long. We select a box at random, and then a bolt at random. What is the probability of selecting a long bolt?

$$L = (L \cap B_1) \cup (L \cap B_2) \tag{13}$$

$$Pr(L) = Pr(B_1)Pr(L|B_1) + Pr(B_2)Pr(L|B_2) \tag{14}$$

$$Pr(L) = \frac{1}{2} \times \frac{40}{100} + \frac{1}{2} \times \frac{10}{30} \tag{15}$$

$$Pr(L) = \frac{1}{5} + \frac{1}{6} = \frac{11}{30} \tag{16}$$

Bayes' Theorem

Bayes' theorem is a probability rule that allows for updating probabilities of the elements of a partition once information about a given event is avalible.

Theorem: Given a partition $B_1, ..., B_n$ of S, such that $Pr(B_j) > 0, \forall j$, and a given event A, $Pr(A) > 0$, then

$$Pr(B_i|A) = \frac{Pr(B_i)Pr(A|B_i}{\sum_{j=1}^{n} Pr(A|B_j)Pr(B_j)} \tag{17}$$

Proof:

$$Pr(B_i|A) = \frac{Pr(A \cap B_i)}{Pr(A)} = \frac{Pr(A|B_i)Pr(B_i)}{Pr(A)} \tag{18}$$

For the full formula, $Pr(A)$ is expressed using the Total Law of Probability.

Bayesian statistics uses Bayes' theorem to create models or deduce relations in data.

Example

Epidemiological data indicate a disease has a frequency of 1 in 10,000. A test is provided to determine the presense of the disease. The test has a 0.9 probability of returning a true positive, and a 0.05 false positive probability. If someone takes the test and gets a positive result, what is the probability that the person is sick?

$$Pr(S|P) = \frac{Pr(P|S)Pr(S)}{Pr(P)} \tag{19}$$

$$Pr(S|P) = \frac{Pr(P|S)Pr(S)}{Pr(P|S)Pr(S) + Pr(P|\neg S)Pr(\neg S)} \tag{20}$$

$$Pr(S|P) = \frac{0.00009}{0.00009 + 0.049995} \tag{21}$$

$$Pr(S|P) = 0.0018 \approx 0.18\% \tag{22}$$

Something to note is that the test doesn't give anything close to a garuntee, however it does mean you are 18x more likely to be sick.

Example

Three machines are used to make items. 20% is produced by Machine 1, 30% by machine 2, and the rest by Machine 3. Machines produce a defective items in 1%, 2%, and 3% of cases respectively. If an item is sampled at random what is the probability that it will be defective? If the item is defective what is the probability it was produced by Machine 2?

$$Pr(D|M_1) = 0.01 \tag{23}$$
$$Pr(D|M_2) = 0.02 \tag{24}$$
$$Pr(D|M_3) = 0.03 \tag{25}$$
$$Pr(D) = Pr(D|M_1)Pr(M_1) + Pr(D|M_2)Pr(M_2) + Pr(D|M_3)Pr(M_3) \tag{26}$$
$$Pr(D) = 0.01 \times 0.2 + 0.02 \times 0.3 + 0.03 \times 0.5 = 0.023 \tag{27}$$
$$Pr(M_s|D) = \frac{Pr(D|M_2)Pr(M_2)}{Pr(D)} = \frac{0.3 \times 0.02}{0.023} = 0.26 \tag{28}$$