Math 100

Elijah Hantman

Definition:

Given $P \implies Q$, the converse is $Q \implies P$.

Definition:

The biconditional of P and Q is the statment

$$(P \implies Q) \wedge (Q \implies P)$$

It is also written as

$$P \iff Q$$

And uses the word iff or the phrase if and only if.
Also in English, necessary and sufficient also means the same thing.
This works because necessary means $Q \implies P$ and sufficient means $P \implies Q$.
Also means that P and Q are equivalent because the statement $P = Q$ will be tautalogical.

Implication Truth Table

P	Q	$P \implies Q$	$Q \implies P$	$P \iff Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

0.0.1 Tautologies and Contradictions

Definition:

A compound statement is a statement consisting of at least one statement involving at least one logical connective (and, or, not, implies, bicond).

Definition:

A tautology is a compound statement which is always true regardless of the truth values of component statements.

Definition:

A contradiction is a compound statement which is always false regardless of the truth values of component statements.

Example:

$$P \wedge \neg P = F \tag{1}$$

A simple contradiction.

$$P \vee \neg P = T \tag{2}$$

A simple tautology.

Other Useful tautologies:

- $$P \wedge (P \implies Q) \implies Q \tag{3}$$

  This is a simple tautology from the definitions. If P implies Q and P is true, then Q must be true.

- $$(P \implies Q) \wedge (Q \implies R) \implies (P \implies R) \tag{4}$$

0.0.2 Logical Equivalence

Definition:

Given statements R, and S. R and S are logically equivlanet if they have the same truth value for all combinations of truth values of each of their component statements.

Denoted: $R \equiv S$

$$R \equiv S$$

When

$$R \iff S$$

Is a tautology.

Theorem 0.0.1.

$$(P \implies Q) \equiv (\neg P) \vee Q \tag{5}$$

0.0.3 Fundamental Properties of Logical Equivalences

1. $\neg(\neg P) \equiv P$
2. Commutivity:
$$P \vee Q \equiv Q \vee P \tag{6}$$
$$Q \wedge P \equiv P \wedge Q \tag{7}$$
3. Associativity:
$$(P \vee Q) \vee R \equiv P \vee (Q \vee R) \tag{8}$$
$$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R) \tag{9}$$
4. Distributive Laws:
$$(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R) \tag{10}$$
$$(P \wedge Q) \vee R \equiv (P \vee R) \wedge (Q \vee R) \tag{11}$$
5. DeMorgan's Law:
$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q) \tag{12}$$
$$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q) \tag{13}$$

These are the same as the set theory version. It works since sets can also be defined as some logical statement where truth means it is in the set and false means it is excluded. Then the laws are literally the same.

How do we use Equivalences?

1. $\neg(P \implies Q) \equiv P \wedge (\neg Q)$ We can show this is the same using De Morgan's law. And the tautology $P \implies Q \equiv \neg P \vee Q$

$$\neg(P \implies Q) \equiv \neg(\neg P \vee Q) \tag{14}$$
$$P \wedge (\neg Q) \equiv P \wedge (\neg Q) \tag{15}$$

Example:

$$((\neg Q) \implies (P \land \neg P)) \equiv Q \tag{16}$$

Example:

Proof by Contradiction

Goal:

$$P \implies Q \tag{17}$$

$$P \implies Q \equiv P \land (\neg Q) \implies (R \land \neg R) \tag{18}$$

We get benefits by getting to assume more things. Rather than just assuming P we assume P and $\neg Q$. If we can reach a contradiction then we have shown $P \implies Q$.

$$P \land (\neg Q) \implies (R \land \neg R) \tag{19}$$
$$\equiv \neg(P \land (\neg Q)) \lor (R \land \neg R) \tag{20}$$
$$\equiv \neg(P \land (\neg Q)) \lor F \tag{21}$$
$$\equiv \neg(P \land (\neg Q)) \tag{22}$$
$$\equiv \neg P \lor Q \tag{23}$$
$$\equiv P \implies Q \tag{24}$$

0.0.4 Quantified Statements

$P(x)$ is an open sentence.

Consider:

$$\forall x \in S | P(x) \tag{25}$$
$$\exists x \in S | P(x) \tag{26}$$

These mean, for all x and there exists an x respectively.

Negation

$$\neg(\forall x \in S, P(x)) \equiv \exists x \in S, \neg P(x) \tag{27}$$
$$\neg(\exists x \in S, P(x)) \equiv \forall x \in S, \neg P(x) \tag{28}$$