Math 100

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Set Review Cont.

- Cartesian Product

$$A \times B = \text{Set of all ordered pairs} \tag{1}$$
$$= \{(a, b) | a \in A, b \in B\} \tag{2}$$

Lemma:

$$\boxed{|A \times B| = |A| \times |B|} \tag{3}$$

- Could we make set arithmetic analogous to real arithmetic?

$$A^B \text{ or } \frac{A}{B} \tag{4}$$

The property of expontentials we want to preserve is:

$$A^{B_1 + B_2} = A^{B_1} \times A^{B_2} \tag{5}$$
$$A^{\emptyset} = A \tag{6}$$

A/B only makes sense for some versions of A and B.

- How do we define Set Exponentiation for Arbitrary sets A and B?

One possibility:

$$A^B := \{f : B \to A\} \tag{7}$$

$$A = \{1, 2\} \tag{8}$$
$$A^B = \mathcal{P}(B) \tag{9}$$
$$2^B = \mathcal{P}(B) \tag{10}$$

This means that the powerset is a special case of set exponentiation.

$$A^{B_1 + B_2} = A^{B_1} \times A_{B_2} \tag{11}$$
$$A^{B_1 + B_2} = \{f ; B_1 + B_2 \to A\} \tag{12}$$
$$= \{f_1 : B_1 \to A\} \times \{f_2 : B_2 \to A\} \tag{13}$$
$$f \leftrightarrow (f_1, f_2) \tag{14}$$

Chapter 2 - Logic

A statement is a declarative sentance which can be objectively determined to be True or False.

Example

1. An integer 11 is divisible by 4
   False
2. An integer 11 is big.
   Subjective, not falsifiable. Not a logical statement.
3. An integer 11 is odd.
   True

Logical Negation

Given a statement P.

$\neg P$ is a statement which is true if P is false, and false if P is true.

P	$\neg P$
T	F
F	T

Example

P: an integer 11 is odd.
$\neg P$: an integer 11 is not odd.

Disjunction

P, Q are statements.

The disjunction of P and Q, $P \vee Q$ is a statement which is true if P is true, or if Q is true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Example

P is a statement.

$$P \vee \neg P$$

That statement is always true, meaning it is a tautology.

Conjuction

P, Q are statements.

The conjunction of P and Q, $P \wedge Q$ is a statement which is true only if both P is true, and Q is true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Implication

P, Q statements.

For P to imply Q, $P \implies Q$ means that if P is true, Q is true. If P is true and Q is false, the statement is false. Otherwise the statement is true.

Example

P: It is raining
Q: I will stay home

$P \implies Q$: If it is raining, I will stay home.

If P is false the statement is trivially true.

Note: $P \implies Q$ is equivalent to $\neg(P \land \neg Q)$ or $\neg P \lor Q$

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical Identities

- $P \implies Q$ if P then Q
  P implies Q
  Q, if P.
  P only if Q.
  P sufficient for Q
  Q is necessary for P

What is the negation of an implication?

$$\neg(P \implies Q) \tag{15}$$
$$= \neg(\neg P \lor Q) \tag{16}$$
$$= P \land \neg Q \tag{17}$$
$$\tag{18}$$

P	Q	$P \implies Q$	$\neg(P \implies Q)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

Theorem

$$\neg(P \implies Q) \equiv P \land (\neg Q) \tag{19}$$

Following:

$$P \implies Q \equiv \neg P \lor Q \tag{20}$$

Definition

An open sentence is a declarative sentence which cantains variables.

Each variable can assume any value in a given set, called the domain of the variables, and an open sentence becomes a statement when variables are replaced with values.

Example

$$P(x) : |x| = 3, x \in \mathbb{R} \tag{21}$$

The open sentence above only becomes objectively true or false when we assign a value to x.

$P(x)$ is not a statement but $P(2)$ is a statement which is false, and $P(3)$ is a statement which is true.