Math 181

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Problem 1

Attached on Back.

Problem 2

George A. Miller was born Lynnville Pensylvania, on the 31st of July, 1863. He got his undergraduate in Muhlenberg College, and his PhD at Cumberland University. In 1895 he got his passport, and according to newspapers studied at the Universities of Leipzig and Paris. He taught at multiple American universities throughout the 1890s, before settling into the University of Illinois in 1906. In 1900 he won the International Mathematics Prize and wrote more than 800 artiles. There he became very wealthy via investments. One of this mathematical contributions was a history of abstract groups in 1927. Group theory would go on to be a massively important field in abstract algebra and have connections to other disciplines like physics. He retired in 1931, and passed away in 1951 leaving almost a million dollars to the University of Illinois.

As a side note: the census is impossible to read, the resolution is both too low, and the cursive strokes too light to make out properly. In addition, initials are used making it difficult to tell who each row refers to.

Problem 3

I would be interested to know several things about Miller. One is what he won the International Mathematics Prize for. I presume there are original documentation of all the winners of the prize, as well as newspapers, published announcements, and so on that can be found.

Another interesting thing I am curious about has to do with the paper on Abstract groups. I am curious if he had any other contributions to Group theory, abstract algebra, and so on. His published work should still be avalible via libraries and archives of mathematical papers. There are also some citations of other work of his in his own paper which could lead to answers.

Problem 4

The problem presented is as such:

Solve and Discuss:

$$\begin{cases} x^2 + y^2 = a^2 \\ log(x) + log(y) = n \end{cases}$$

This is very vague, so I set out a couple of goals to orient my work.

1. Combine both equations into one.
2. Write a in terms of n and vice versa.
3. Write x and y in terms of only a and n.

First Lets begin by rewriting the second equation to remove the logarithms.

$$log(x) + log(y) = n \tag{1}$$
$$log(xy) = n \tag{2}$$
$$e^(log(xy)) = e^n \tag{3}$$
$$xy = e^n \tag{4}$$

We already get a very nice simplification which is begging for further manipulation.

Next lets solve for y and plug it into the first equation.

$$xy = e^n \tag{5}$$
$$y = \frac{e^n}{x} \tag{6}$$
$$y^2 = \frac{e^{2n}}{x^2} \tag{7}$$

Now to substitute.

$$x^2 + y^2 = a^2 \tag{8}$$
$$x^2 + \frac{e^{2n}}{x^2} = a^2 \tag{9}$$
$$x^4 + e^{2n} = a^2x^2 \tag{10}$$
$$x^4 - a^2x^2 = -e^{2n} \tag{11}$$

We have already acomplished the first goal, equation 11 combines both constraints into a single statement.

To write a in terms of n we need to do a substitution. We set $u = x^2$ and substitute.

$$u^2 - a^2u = -e^{2n} \tag{12}$$

This is clearly a quadratic, so we first put the quadratic into standard form, then solve using the quadratic formula.

$$u^2 - a^2u + e^{2n} = 0 \tag{13}$$

The roots of this quadratic therefor are:

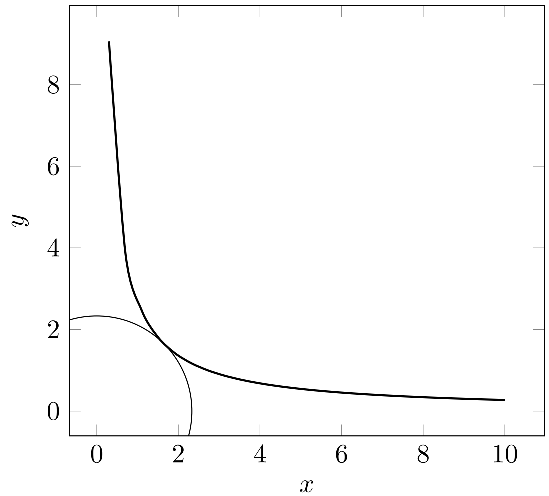
$$u = \frac{a^2 \pm \sqrt{a^4 - 4e^{2n}}}{2} \tag{14}$$

Undoing our substitution of u we can obtain the possible values of x.

$$x = \sqrt{\frac{a^2 \pm \sqrt{a^4 - 4e^{2n}}}{2}} \tag{15}$$

We can eliminate all negative values of x by noticing that the second equation is only valid when x and y are positive, because logarithms are only defined on $\mathbb{R}^+$. This means only positive solutions are valid.

The number of solutions will depend on both a and n. This can be seen clearly by graphing both equations.



Based on this graph we can see that there are either, one, two, or no solutions. Lets see if we can find the values of a and n that yield a unique solution. These are the values for which our formula above only has a single value, which means that:

$$a^4 - 4e^{2n} = 0$$

This would remove the term under the radical thus simplifying the formula to yield only a single value. All that's left is solving for a.

$$a^4 - 4e^{2n} = 0 \tag{16}$$
$$a^4 = 4e^{2n} \tag{17}$$
$$a^2 = 2e^n \tag{18}$$
$$a = \sqrt{2e^n} \tag{19}$$

This also means we can write n as:

$$n = ln(\frac{a^2}{2}) \tag{20}$$

If we are interested in any solutions at all, we simply replace the equality with an inequality:

$$a^4 - 4e^{2n} \geq 0 \tag{21}$$
$$a^4 \geq 4e^{2n} \tag{22}$$
$$a^2 \geq 2e^n \tag{23}$$
$$a \geq \sqrt{2e^n} \tag{24}$$

To solve the last goal all we need to do is plug our solution back into one of the equations and solve for y.

$$x^2 + y^2 = a^2 \tag{25}$$

$$y = \sqrt{a^2 - x^2} \tag{26}$$

$$y = \sqrt{a^2 - \frac{a^2 \pm \sqrt{a^4 - 4e^{2n}}}{2}} \tag{27}$$

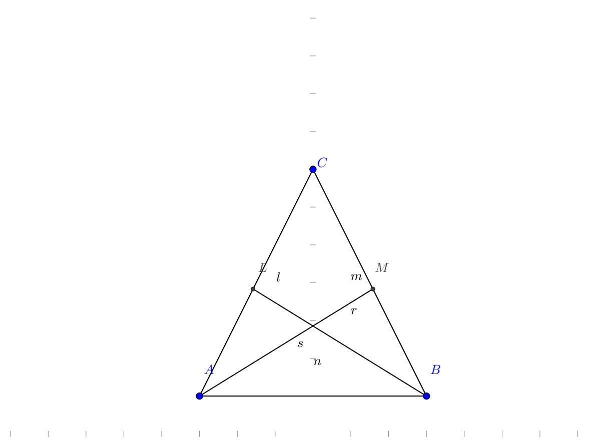
$$y = \sqrt{a^2 - \frac{a^2}{2} \mp \frac{\sqrt{a^4 - 4e^{2n}}}{2}} \tag{28}$$

$$y = \sqrt{\frac{a^2 \mp \sqrt{a^4 - 4e^{2n}}}{2}} \tag{29}$$

This gives something which is equivalent to the formula we obtained for x. This makes sense since both equations are symmetric about the $y = x$ axis.

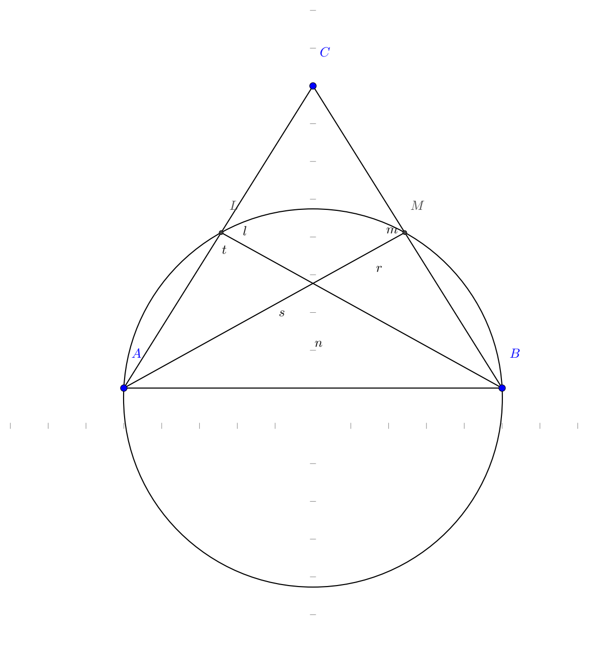
Problem 5

Given a triangle ABC, we can bisect two of its angles with lines. Prove that if the segments of each line inside the triangle are of equivalent length the entire triangle is isosceles.

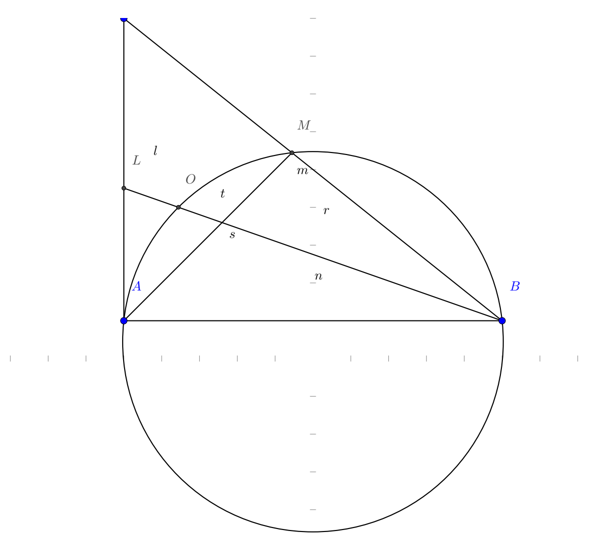


In the figure above we are asked to prove that if the lines AM and BL are the same length, then the triangle is isosceles.

To begin the proof we circumscribe a circle with points A, B, and M defining the circle.



Lets imagine our circle intersects the line BL at a point between B and L.

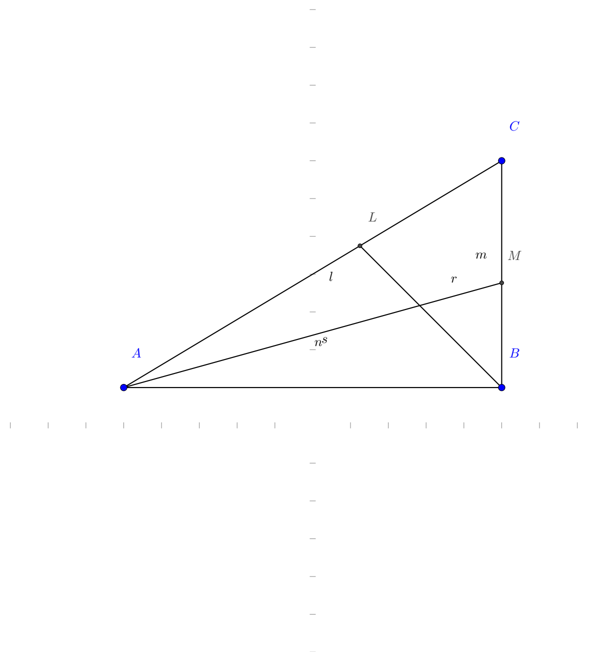


The arc from M to B, must be greater than the arc from O to M. We know this to be true because the angles LAM and MAB are equal, and the angle formed by OAM is contained within the angle LAM. This means MAB is larger than or equal to OAM.

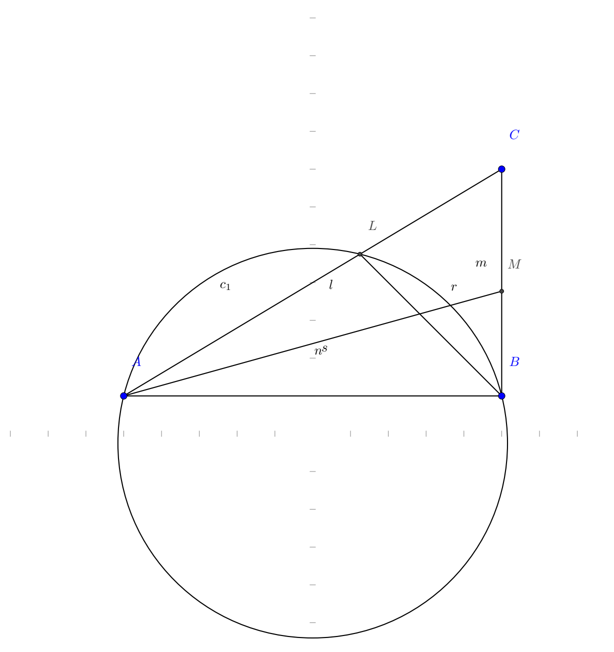
This line of reasoning works because for inscribed and tangent angles, the conversion between arc length and angle is linear. The arc O to M is equal to the arc O to A. This is true because the line BL which splits both bisects the angle ABM. This means both are equivalent arc length. Using this fact, we can conclude that the distance B to L is greater than A to M. However this contradicts the fact that B to L and A to M are equal by definition.

This means the point O cannot be between B and L.

If the point O is instead further from B than L.

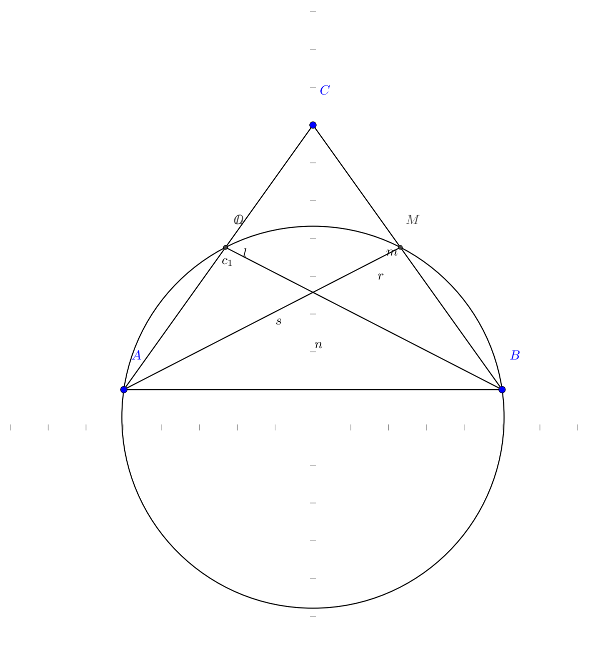


We can instead of using a circle which touches points A, B, and M, can instead use a circle that touchs A, B, and L.



Now the arguments from before apply symmetrically.

This means that the circle must pass through A, B, L, and M. This implies that the arcs A to L, and B to M are equal, which implies the angles CAB and CBA are equal, which is the definition of an isoceles triangle.



Q.E.D

I've added multiple pictures to this text, in addition I've added justification for some of the angle equivalences that I did not understand at first. I also added justification for the second case of O being outside of the triangle which the original proof glossed over. I believe the proof is correct however I am not particularly practiced when it comes to this style of proof so I likely would have missed any subtle errors.