Stats 130
Day 15 Notes

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Markov identity

$$\pi = \lim_{n\to\infty} v'P^n \tag{1}$$

Where P is the probability matrix, and v' is the initial distribution, and π is the equilibrium distribution.

Obtaining the Equilibrium Distribution

- Markov chain
- What will be the stable distribution over time.
  ie: what is the probability of the system being in a given state as time goes to infinity.

The equilibrium satisfies $\pi' = \pi' P$. This implies that $\pi'(P-I) = 0$. Unfortunately there isn't a unique solution since $\pi* = c\pi$ also satisfies $(\pi*)(P-I) = 0$ for all c.

idk why the slides use π its a terrible choice for a variable name.

To make the solution unique we constrain π to be positive and sum to 1. This is saying that $\sum_{j=1}^{k} \pi_j = 1$

$$(\pi_1, ..., \pi_k)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix} = \pi' 1. = 1$$

Let h_j be the j-th column of $P-I$. Define $G = [h_1, ..., h_{k-1}, 1]$.

$$\pi' G = (0, ..., 0, 1) \implies \pi' = (0, ...0, 1)G^{-1}$$

How to get the Equilibrium

- Start with transition matrix
- We subtract from the transition matrix the identity matrix.
- Take last column and replace it with all ones
- Calculate the inverse of that matrix
- Multiply by a vector with all zeros and only a single 1 in the last entry.

Example

$$P = \begin{bmatrix}\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{1}{3}\end{bmatrix} \tag{2}$$

$$P - I = \begin{bmatrix}-\frac{2}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3}\end{bmatrix} \tag{3}$$

$$G = \begin{bmatrix}-\frac{2}{3} & 1\\ \frac{2}{3} & 1\end{bmatrix} \tag{4}$$

$$G^{-1} = \begin{bmatrix}-\frac{3}{4} & \frac{3}{4}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} \tag{5}$$

$$\pi' = (0, 1)\begin{bmatrix}-\frac{3}{4} & \frac{3}{4}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\end{bmatrix} \tag{6}$$

This only works with homogeneous Markov Chains, where the probabilities are constant.

Expected Value / Expectation

Let X be a random variable with probability function $f_X(x)$.

$$E(X) = \sum_x x f_X(x)$$

This is only useful if the sum is convergent. We require the sum is absolute convergence.

$$\sum_x |x| f_X(x) < +\infty$$

Many well defined random variables fail this requirement. For those variables we say that $E(X)$ does not exist.

The Expectation of a variable only depends on its distribution, and two random variables with the same distribution will always have the same Expectation.

Continuous Random Variables

For a continuous random variable X with p.d.f. $f_X(x)$ the expectation is defined.

$$E(x) = \int_{-\infty}^{+\infty} x f_X(x) dx$$

This exists if:

$$E(x) = \int_{-\infty}^{+\infty} |x| f_X(x) dx$$

Otherwise the expectation is undefined.

Functions of Random Varibles

For any function $r(X)$ we have that

$$E(r(X)) = \int_{-\infty}^{+\infty} r(x) f_X(x) dx$$

Therefore we do not need the density of the random variables that result from transforming X in order to obtain the expectation.

Even more generally:
For two random variables X and Y, consider a function $r(X, Y)$, then

$$E(r(X,Y)) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} r(x,y) f(x,y) dx dy$$

Example

Consider a random variable with density

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$E(X) = \int_0^1 x(2x)dx = \int_0^1 2x^2 dx = 23$$

Consider the exponential density defined as:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\begin{aligned} E(X) &= \int_0^\infty x\lambda e^{-\lambda x} dx = -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x} dx \\ &= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \frac{1}{\lambda} \end{aligned}$$

Cauchy Density is defined as:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, x \in \mathbb{R}$$

To check if the expectation exists

$$\int_{-\infty}^{+\infty} \frac{|x|}{\pi(1+x^2)} dx = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2} dx = \frac{1}{\pi} log(1+x^2)\Big|_0^\infty = \infty$$

Therefore the expectation of the Cauchy distribution does not exist.

Expectation Properties

1. $Y = aX + b \implies EY = aEX + b, \forall a, b \in \mathbb{R}$

$$EY = \int_{\mathbb{R}} (ax+b) f_X(x) dx = a\int_{\mathbb{R}} x f_X(x) dx + b\int_{\mathbb{R}} f_X(x) dx$$

2. $Pr(X \geq [\leq] a) = 1 \implies EX \geq [\leq] a$

$$EY = \int_{x \geq a} x f_X(x) dx \geq a \int_{\mathbb{R}} f_X(x) dx = a, f_X(x) = 0, x > a$$

3. $E(X+Y) = EX + EY$ The expectation distributes linearly over sums.

$$E(X+Y) = \int\int_{\mathbb{R}^{\npreceq}} (x+y) f(x,y) dx dy = \int_{\mathbb{R}} x f_X(x) dx + \int_{\mathbb{R}} y f_Y(y) dy$$

4. In general for a collection of random variables $X_1, ..., X_k$, EX_i exists $\forall i$, $E\sum_{i=1}^k X_i = \sum_{i=1}^k EX_i$
5. For a collection of independent random variables $X_1, ..., X_k$ then EX_i exists $\forall i$, $E\prod_{i=1}^k X_i = \prod_{i=1}^k EX_i$

Bernoulli Trials

Consider a random variable $X \sim Ber(p)$. Then, $Pr(X=1)=p$, and $Pr(X=0)=1-p=q$.

$$EX = 1 \times p + 0 \times q = p$$

Repeat the Bernoulli trials n times, and define Y as the random variable that counts the number of successes. Then

$$Y = X_1 + ... + X_n$$

and

$$EY = EX_1 + ... + EX_n = np$$

And $Y \sim Bin(n,p)$

Hypergeometric Distribution

Select n balls from a box containing A red balls and B blue balls. Define the binary variable $X_i = 1$ if the i-th ball is red, and $X_i = 0$ if the i-th ball is blue.

$$Pr(X=1) = \frac{A}{A+B} \implies EX_i = \frac{A}{A+B}$$

That implies that

$$Y = \sum_{i=1}^{n} X_i \implies EY = \sum i = 1^n EX_i = n\frac{A}{A+B}$$

Poisson Distribution

A random variable with a Poisson distribution has function

$$f_X(x) = \begin{cases} \frac{e^{-\lambda}\lambda^x}{x!} & x = 0,1,2,... \\ 0 & \text{otherwise} \end{cases}$$

$$EX = \sum_{i=0}^{\infty} i\frac{e^{-\lambda}\lambda^i}{i!} = \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^i}{(i-1)!} = \lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} = \lambda$$

The last step works because the sum is the taylor expansion of e^{λ} which cancels with $e^{-\lambda}$.

Geometric Distribution

A random variable with a geometric distrubution has probability function.

$$f_X(x) = \begin{cases} pq^x & x = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases}$$

$$EX = \sum_{i=0}^{\infty} ipq^i = p \sum_{i=1}^{\infty} iq^i = pq \sum_{i=1}^{\infty} iq^{i-1}$$

$$= pq \sum_{i=0} \frac{d}{dq} q^i = pq \frac{d}{dq} \sum_{i=0}^{\infty} q^i = pq \frac{d}{dq} \frac{1}{1-q}$$

$$= pq \frac{1}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Negative Binomial

A random variable with a negative binomial corrospond to the sum of r independent random variables X_i where $X_i \sim Geo(p)$

$$EY = \frac{rq}{p}$$