A SHORT PAPER

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I think LaTex is pretty cool! Instead of using some silly equation editor, I just type little codes and LaTex makes it look pretty and professional. For example, $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Big and important equations deserve their own line. For this, I use double-dollar signs around the math. And here it goes!

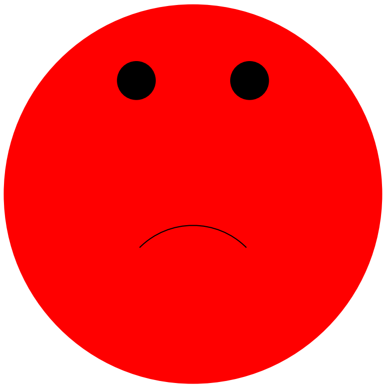
$$\sqrt{1+\sqrt{1+\sqrt{1}}} = \sqrt{1+\sqrt{2}}.$$

Sometimes, I like to put equations line after line after line.

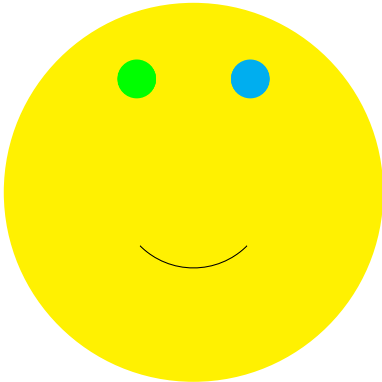
$$\begin{aligned} x &= x+1-1 \\ &= x+1-1+1-1 \\ &= x+1-1+1-1+1-1. \end{aligned}$$

We use special symbols for the sets of natural numbers, integers, rational numbers, real numbers complex numbers, quaternions, and octonians. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$. You can write down any equations. There are many manuals of LaTeX you can consult with. Some of these are listed in Module tab of Math 100 Canvas page.

I can even draw pictures in LaTex using the TikZ/PGF package!



Before LaTex



After LaTex

You can see more examples of TikZ and PGF at `http://www.texample.net/tikz/` Just search for TikZ, and you will get more introductory examples. Read more about LaTeX by looking at Gratzer's book[1].

1. Some cool Theorems

A theorem I quite like is known as the Separating Axis Theorem (SAT), which is a special case of the Separating Hyperplane Theorem. It is commonly used in Physics simulation since it simplifies the logic.

Here is a proof of the Separating Hyperplane Theorem from Wikipedia.

Theorem 1. *Let A and B be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exist a nonzero vector v and a real number c such that:*

$$\langle x, v\rangle \geq c \text{ and } \langle y, v\rangle \leq c \tag{1}$$

For all $x \in A$ and $y \in B$.

If both sets are closed and at least one is compact, then the separation can be strict, that is:

$$\langle x, v\rangle > c \text{ and } \langle y, v\rangle < c \tag{2}$$

Remark 2. It should be noted that $\langle a, b\rangle$ denotes the inner product. In common applications to linear algebra and geometry this would be the dot product, but it had the property of taking in $a, b \in \mathbb{R}^n$ and mapping them to some value $c \in \mathbb{R}$.

Assume A and B to be disjoint, nonempty, and convex subsets of $\mathbb{R}^n$. The summary is as follows:

A	B	$\langle x, v\rangle$	$\langle y, v\rangle$
		$\geq c$	$\leq c$
closed compact	*closed*	$> c_1$	$< c_2$ *with* $c_2 < c_1$
closed	*closed compact*	$> c_1$	$< c_2$ *with* $c_2 < c_1$
open		$> c$	$\leq c$
open	*open*	$> c$	$< c$

The number of dimensions must also be finite. If the sets are compact then it is possible to generalize to infinite dimensions, which is known as the Hahn-Banach separation theorem.

Lemma 3. *Let A and B be two disjoint closed subsets of $\mathbb{R}^n$, and assume A is compact. Then there exists points $a_0 \in A$ and $b_0 \in B$ minimizing the distance $||a - b||$ over $a \in A$ and $b \in B$.*

Proof of Lemma 2:

Let $a \in A$ and $b \in B$ be any pair of points, and let $r_1 = ||b - a||$. Since A is compact, it is contained in some ball centered on a; let the radius of this ball be r_2.

Let $S = B \cap \overline{B_{r_1+r_2}(a)}$ be the intersection of B with a closed ball of radius $r_1 + r_2$ centered around a.

Then S is compact and nonempty because it contains b. Since the distance function is continuous, there exist points a_0 and b_0 whose distance $||a_0 - b_0||$ is the minimum over all pairs of points in $A \times S$.

To prove that a_0 and b_0 have the minimum distance over all pairs of points $A \times B$. Suppose that there exists points a' and b' such that $||a' - b'|| < ||a_0 - b_0||$. Then in particular $||a' - b'|| < r_1$, and by the triangle inequality, $||a - b'|| \leq ||a' - b'|| + ||a - a'|| < r_1 + r_2$.

Therefore b' is contained in S, which contradicts the fact that a_0 and b_0 had minimum distance over $A \times S$. $\square$

Proof of Theorem:

We first prove the second case.

Without loss of generality, A is compact. By the lemma, there exist points $a_0 \in A$ and $b_0 \in B$ of minimum distance to each other. Since A and B are disjoint, we have $a_0 \neq b_0$. Now, construct two hyperplanes L_A,L_B perpendicular to the line segment $[a_0, b_0]$, with L_A across a_0 and L_B across b_0. We claim, neither A nor B enters the space between L_A, L_B, and thus the perpendicular hyperplanes to (a_0, b_0) satisfy the requirement of the theorem.

Algebraically, the hyperpanes L_A and L_B are defined by the vector $v := b_0 - a_0$, and the two constants $c_A := \langle v, a_0 \rangle < c_B := \langle v, b_0 \rangle$, such that $L_A = \{x : \langle v, x \rangle = c_A\}$, $L_B = \{x \langle v, x \rangle = c_B\}$. Our claim is that $\forall a \in A, \langle v, a \rangle \leq c_A$ and $\forall b \in B, \langle v, b \rangle \geq c_B$.

Suppose that $a \in A$ such that $\langle v, a \rangle > c_a$, then let a' be the foot of perpendicular from b_0 to the line segment $[a_0, a]$. Since A is convex, a' is inside A, and by planar geometry, a' is closer to b_0 than a_0, contradiction.

Similar arguements apply to B.

For the first case,

Approach both A,B, from the inside by $A_1 \subseteq A_2 \subseteq ... \subseteq A$ and $B_1 \subseteq B_2 \subseteq ... \subseteq B$. such that A_k,B_k is closed and compact, and the unions are the relative interiors $relint(A)$, $relint(B)$.

Remark 4. I am somewhat unfamiliar with the relative interior. Apparently it is a generalization of the interior which allows low dimensional sets in high dimensional spaces to have meaningful interiors.

Now by the second case, for each pair A_k, B_k there exists some unit vector v_k and real number c_k such that $\langle v_k, A_k \rangle < c_k < \langle v_k, B_k \rangle$.

Since the unit sphere is compact, we can take a convergent subsequence, so that $v_k \to v$. Let $c_A := sup_{a \in A} \langle v, a \rangle$, $c_B := inf_{b \in B} \langle v, b \rangle$. thus separating A and B.

Assume not, then there exists some $a \in A$, $b \in B$ such that $\langle v, a \rangle > \langle v, b \rangle$, then since $v_k \to v$ for large enough k, we have $\langle v_k, a \rangle > \langle v_k, b \rangle$, contradiction.

There is more proof for each case as well as specific properties of each case however I think this illustrates the core theorem.

Generally in simulations the property that if two convex, compact sets are disjoint there must exist some projection in which they are disjoint is used. However in recent years this method has been superseeded by the Gilbert-Johnson-Keeli algorithm which has a better algorithmic performance for 3 dimensions and high complexity sets.

References

[1] G. Grätzer, "More Math Into LaTeX, 4th Edition," Springer 2007.
[2] "Hyperplane Separation Theorem," Wikipedia