Math 100

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Review/Intro

- Midterm is take home, start when convenient.
- Practice writing papers
- Using LaTeX
- One problem per lecture, due Sunday
- 2 Papers, normal math paper.

Goal is to practice writing, so the problems are not particularly advanced. Nor is it systematic, it is whatever problem is best to practice the given writing skill.

Set Theory Notation

Capital letters denote sets.

Lower case letters denote elements.

Example:

$$a \in A$$

a is an element in the set A.

$$a \notin A$$

a is not contained in A.

How to Describe a Set?

1. List all elements

$$A = \{-4, -2, 0, 2, 4\}$$

2. State properties of elements

$$A = \{x \in \mathbb{Z} | IsEven(x), |x| \leq 4\}$$

Empty Set

Denoted $\emptyset$, which is the greek letter 'phi'. Also denoted $= \{\}$.

Note: $\{\emptyset\}$

A set whose single element is an empty set. This set has a size of 1. This is distinct from the empty set.

This also implies $\emptyset \in A$ is reasonable.

Cardinality

$$|A| = \text{ the cardinality of the set A} \tag{1}$$

There are multiple infinite hierarchies of cardinality, regarding sets whose definitions contain infinite but distinct quanities.

The Cardinality is the number of elements. The empty set has a cardinality of 0, while the set containing the empty set has a cardinality of 1.

Subset

Let B, be a set.

a set A is a subset of B if every element of A is an element of B.

$$A \subseteq B \iff \forall a \in A | a \in B \tag{2}$$

If $A \subseteq B$, then it is possible that $A = B$.

Proper Subset

A proper subset excludes the possibility that the subset is equal to the set.

$$A \subset B \implies A \neq B \tag{3}$$

Example

$$B = \{\emptyset, \{\emptyset\}, 1, 2, \{1, 2\}\} \tag{4}$$

We can write:

$$\emptyset \in B, \quad \{\emptyset\} \subseteq B \tag{5}$$

Both mean the same thing, however the first directly states that the element is in B, and the second says that the set containing the element is a subset, which is equivalent.

$$1, 2 \in B \quad \{1, 2\} \subseteq B \tag{6}$$
$$\{1, 2\} \in B \tag{7}$$

All the above are true, however the first line refer to the same elements, but the second line refers to specifically the set which is an element of B.

Powerset

A: a set

$$\mathcal{P}(A) = \text{ The power set} \tag{8}$$
$$= \text{ the set of all subsets of A} \tag{9}$$

For Example:

$$A = \{1, 2, 3\} \tag{10}$$
$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, `\{3, 1\}, \{1, 2, 3\}\} \tag{11}$$

Lemma

A is a set with a finite number of elements.

Then:

$$|\mathcal{P}(A)| = 2^{|A|} \tag{12}$$

This follows since each element of the set can either be included or not included, meaning there are 2 variants per element, meaning for each element we double the number of subsets.

This also follows from:

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{13}$$

This means we are counting the number of ways to choose k elements from n elements. We sum all of these choices for each size of subset.

This can be directly shown via algebra since we have a closed form solution for $\binom{n}{k}$

Example

$$A = \{\emptyset, \{\emptyset\}\} \tag{14}$$
$$\text{Then} \tag{15}$$
$$\mathcal{P}(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} \tag{16}$$

Set Operations

A, B, C are subsets of U

1. $A \cup B$ are the elements of U which are in A or B.
   $\{x \in U | x \in A \vee x \in B\}$
2. $A \cap B$ are elements in U which are in both A and B.
   $\{x \in U | x \in A \wedge x \in B\}$
3. $A - B$ The elements of U which are in A but not in B.
   $\{x \in U | x \in A \wedge x \notin B\}$
   Like in arithmetic, subtraction is not commutative.
4. $A^c = \bar{A}$ The complement of A. It means all the elements of U which are not in A.
   $\{x \in U | x \notin A\}$
   Two common notations, the bar or a small c.

Some identities

- $A - B = A \cap \bar{B}$
- $\overline{A \cup B} = \bar{A} \cap \bar{B}$
- $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Note: The previous two identities are corrollaries of de Morgan's Laws of logic. They are also called de Morgan's Laws here.

Indexed Sets

$$\{A_1, A_2, ..., A_n\} = \{A_i\}_1^n = \{A_i\}_{i \in I} \tag{17}$$
$$I = \{1, 2, ..., n\} \tag{18}$$
$$A_1 \cap A_2 \cap ... \cap A_n = \bigcap_{i \in I} A_i \tag{19}$$
$$A_1 \cup A_2 \cup ... \cup A_n = \bigcup_{i \in I} A_i \tag{20}$$
$$\tag{21}$$

Convenient for expressing infinite combinations of sets or infinite sets of sets. This is easy to denote if I is an infinite set.

$$\{S_\alpha, S_\beta, S_\gamma, ...\} = \{S_\lambda\}_{\lambda \in I} \tag{22}$$
$$I = \{\alpha, \beta, \gamma, ...\} \tag{23}$$

Example

$$I = [1, \infty) \tag{24}$$
$$\text{For each } r \in I\text{, consider a set} \tag{25}$$
$$S_r = (-\frac{1}{r}, \frac{1}{r}) \tag{26}$$
$$\text{What is the intersection and union?} \tag{27}$$
$$\bigcap_{r \in I} S_r, \quad \bigcup_{r \in I} S_r \tag{28}$$

$$\bigcap_{r \in I} S_r = \{x \in \mathbb{R} | x \in S_r, \forall r \in I\} \tag{29}$$
$$= \{0\} \tag{30}$$
$$\text{For all } x \neq 0 \text{ we can choose } r \geq 1 \text{ such that } |x| > \frac{1}{r} \tag{31}$$

This means that for all $x \neq 0$ it cannot be in the intersection.

$$\bigcup_{r \in I} S_r = (-1, 1) = S_1 \tag{32}$$

In general, infinite union of open intervals is a collection of open intervals. However, infinite intersection of infinite open intervals may not be open.

One of the core concepts of topology apparently. Leads directly to topological spaces.

Partition of Sets

A is a set

$$\mathcal{S} = \{X_a\}_{a \in I} \tag{33}$$

A collection of nonempty subsets of A, indexed by I.

Definition:

A collection S of subsets of A is a partition of A if

1. $\bigcap_{a \in I} X_a = \emptyset$ All sets have no elements in common, they are disjoint.
2. $\bigcup_{a \in I} X_a = A$ The union of all sets is equal to A.

An equivalence relation is related to partitions, each relation implies a partition and each partition implies an equivalence relation.

Example

$$A = \mathbb{Z} \tag{34}$$
$$\text{Def: For } n \in \mathbb{Z}^+\text{, and } a, b \in \mathbb{Z} \tag{35}$$
$$a \equiv b \mod n \iff n|a - b \tag{36}$$

$$n > 0 \text{ For } 0 \leq r \leq n - 1\text{, let} \tag{37}$$
$$[r]_n = \{\text{all integers with remainder r after dividing by n}\} \tag{38}$$
$$= \{k \in \mathbb{Z}^+ | k \equiv r \mod n\} \tag{39}$$
$$= \{..., r - 2n, r - n, r, r + n, r + 2n, ...\} \tag{40}$$

$$[2]_3 = \{..., -11, -8, -5, -2, 2, 5, 8, 11, ...\} \tag{41}$$

Lemma

$$\{[0]_n, [1]_n, [2]_n, ...[n - 1]_n\} \tag{42}$$

Is a partition of $\mathbb{Z}$.

This is called a conguence class.

Example

For $n = 3$ then $0 \leq r \leq 2$

That implies three subsets.

$$[0]_n = \{..., -6, -3, 0, 3, 6, ...\} \tag{43}$$
$$[1]_n = \{..., -5, -2, 1, 4, 7, ...\} \tag{44}$$
$$[2]_n = \{..., -4, -1, 2, 5, 8, ...\} \tag{45}$$

Together, no element is shared, and they cover all elements in $\mathbb{Z}$.

We can treat these sets as algebraic objects we can do arithmetic on.

We can define division on these conguence classes which is contained to the sets. This is a finite field.