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Lecture 26ns18

Games and traffic networks

Course: **Complex Networks Analysis and Visualization**
Sub-Module: **NetSci**

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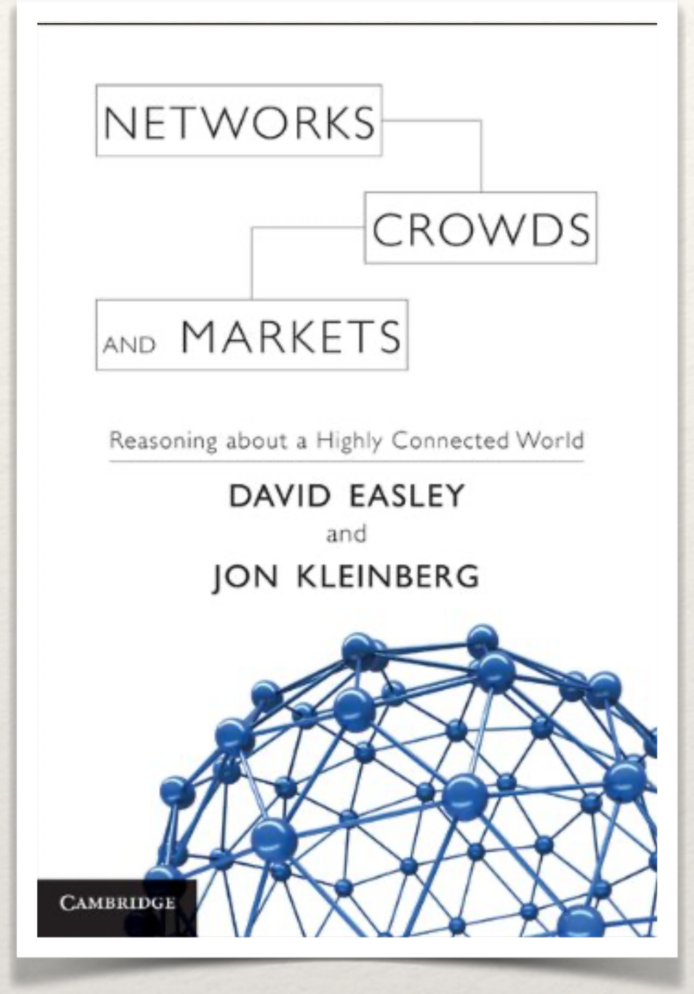
References



NETWORKS
CROWDS
AND MARKETS

Reasoning about a Highly Connected World

DAVID EASLEY
and
JON KLEINBERG

CAMBRIDGE

[ns2] Chapter 6 (6.1 - 6.9)

"Games" —->

https://www.cs.cornell.edu/home/kleinber/networks-book/networks-book-ch06.pdf

[ns2] Chapter 8 (8.1 - 8.3)

"Modeling Network Traffic using Game Theory" —->

https://www.cs.cornell.edu/home/kleinber/networks-book/networks-book-ch08.pdf

What is a Game?

- Complex networks describe the interactions between a set of items. For this reason they are characterized by intrinsic "interdependence"
- In a game each "decision maker" has an **individual satisfaction** to maximize (e.g. a profit) but the its strategy depends also on other people's choices
- **Individual satisfaction** does not depend only on individual's choices
- **Game Theory**: decision makers need to interact with each other

A beautiful mind

Example

- Two students want to pass an exam
  - **assumptions**:
    - they cannot study AND prepare the presentation;
    - they cannot communicate with each other
  - **Exam**:
    - if one studies ---> gets **92** points
    - if one does not study -> **80**;
  - **Presentation**:
    - if one or (xor) the other prepare it: **92** for both;
    - if neither of them prepare it: **84**;
    - if both of them prepare it: **100**;
  - **Final vote**: average on Exam and Presentation

Payoff matrix: it describes the set up

		Player 2	
		P	**E**
Player 1	**P**	90, 90	86, 92
	E	92, 86	88, 88

P_1 plays $P \wedge P_2$ plays $P \Rightarrow$

P_1 gets 100 for P and 80 for E $\Rightarrow P_1$ gets $\dfrac{80+100}{2} = 90$

P_2 gets $= 90$, too.

Basic Ingredients

1. **Players**
2. **Strategies**: set of options for each player
3. **Payoff**: the outcome for each selected strategy

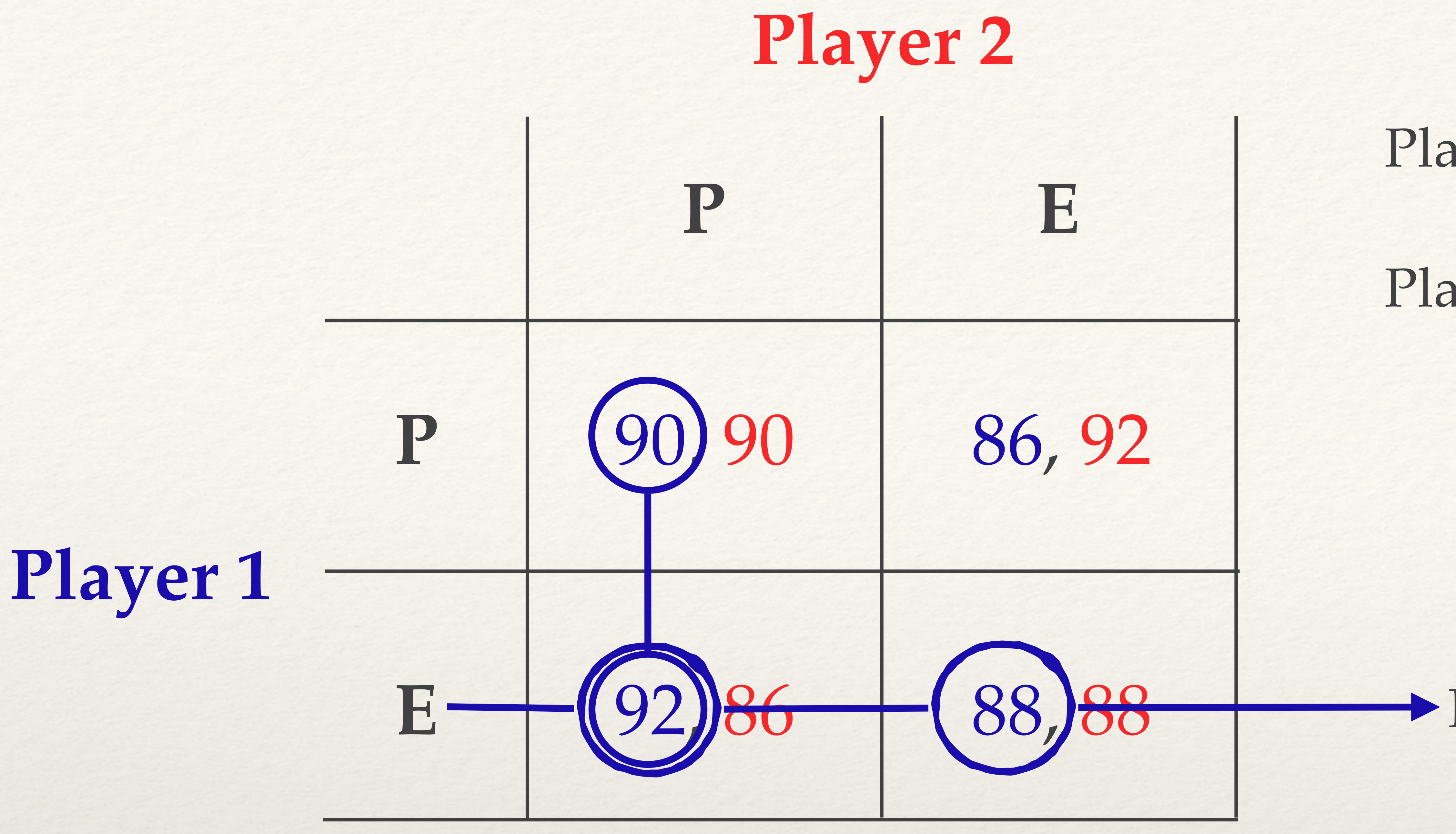
} ⇒ Payoff matrix

We want to reason about how two players will behave in a given game

Reasoning about Behavior in a Game

We need a tractable problem ⇒ assumptions:

a. everything that a player cares of is in the payoff matrix
b. everything about the structure of the game is known
c. players are rational



Player 2

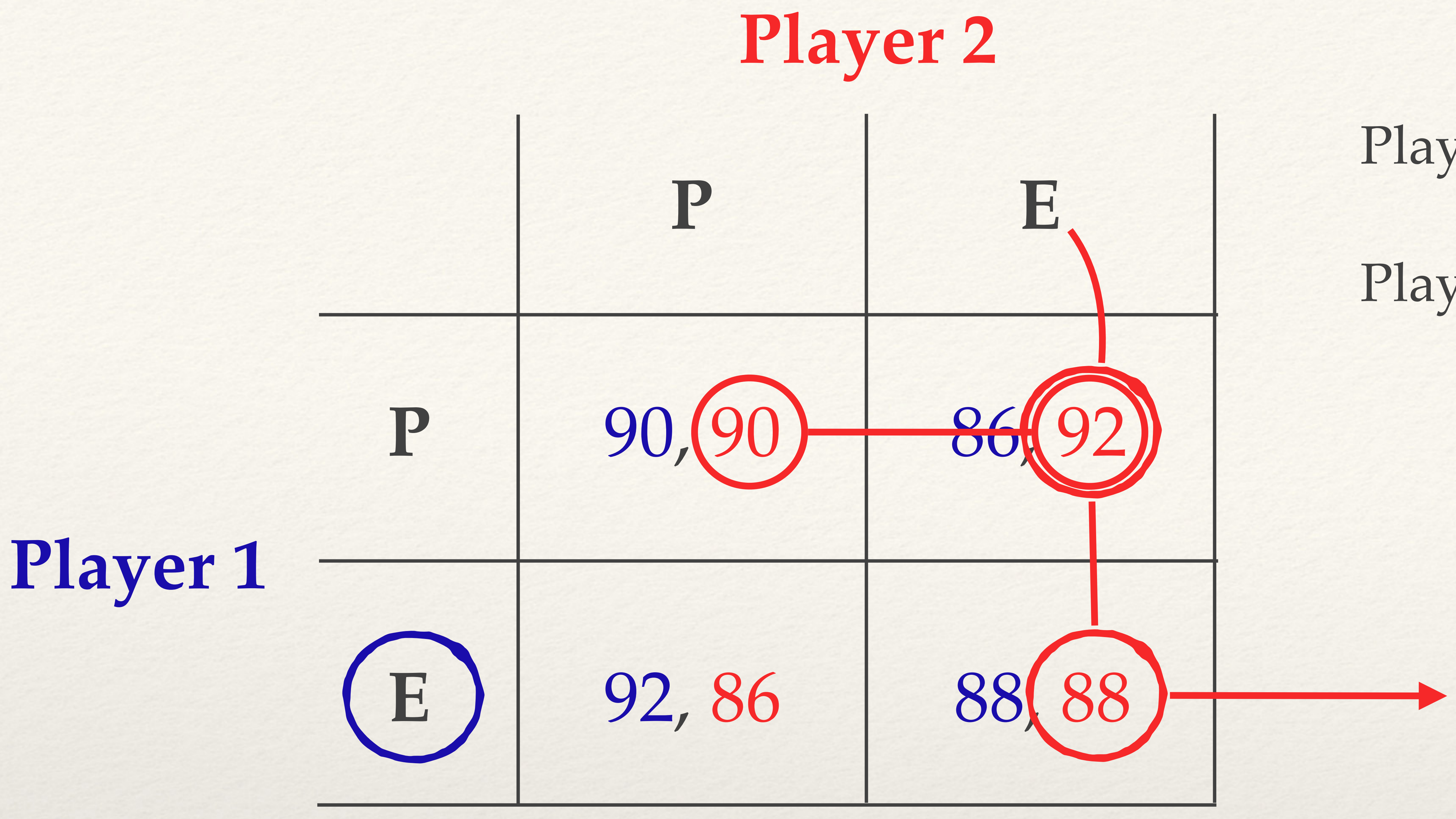
Player 1

	P	E
P	90, 90	86, 92
E	92, 86	88, 88

Player 1's (strict) dominant strategy!

Player 1's perspective: ○

Player 2's perspective: ○



		Player 2	
		P	**E**
Player 1	**P**	90, 90	86, 92
	E	92, 86	88, 88

Player 2's (strict) dominant strategy!

Player 1's perspective: —○—

Player 2's perspective: —○—

		Player 2	
		P	**E**
Player 1	**P**	90, 90	86, 92
	E	92, 86	88, 88

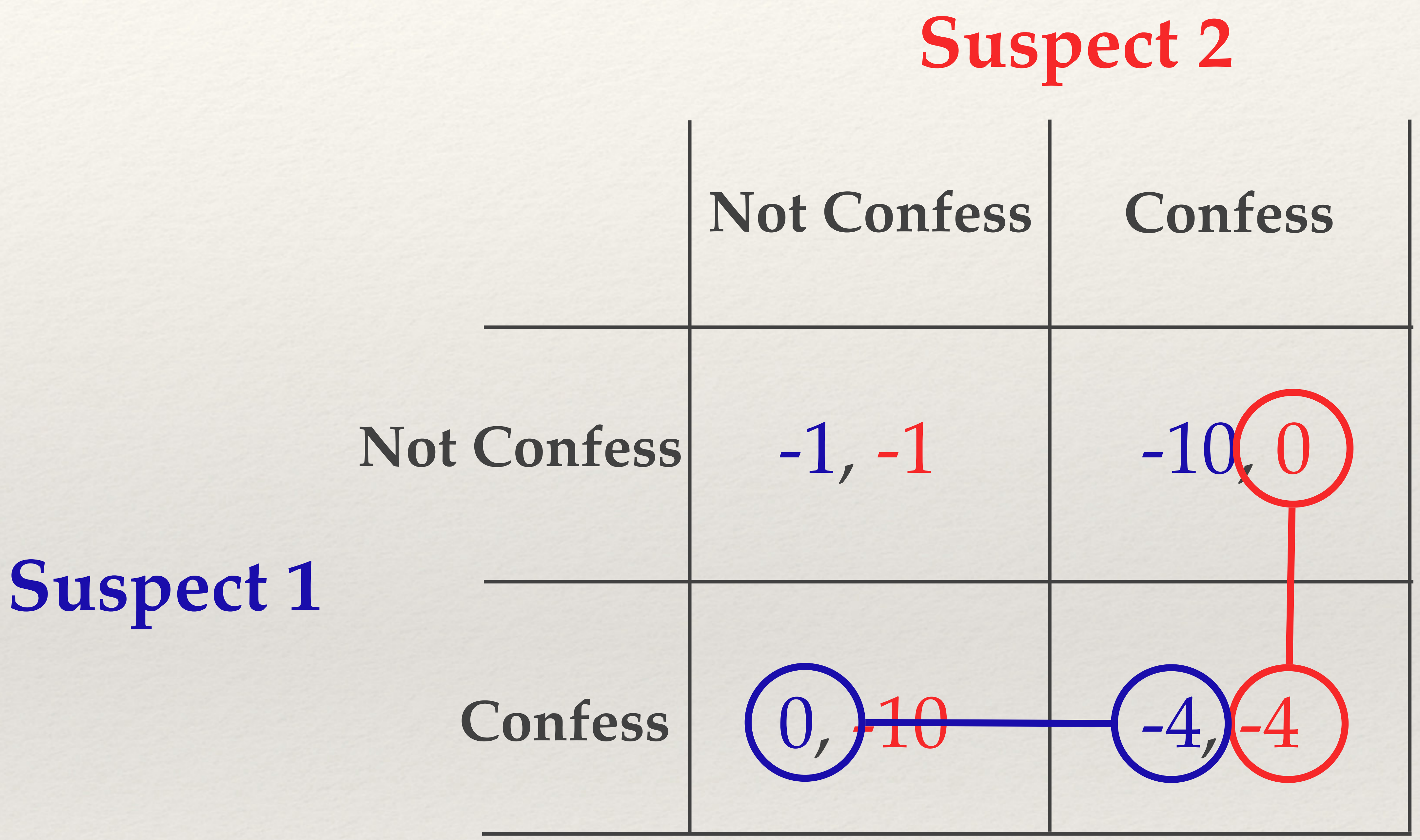
Player 1's perspective:

Player 2's perspective:

- **Strict dominant strategy** for both players: E
  - counterintuitive: (P,P) would be better for both!
  - explanation: if P_1 decide to prepare P, P_2 would be tempted anyhow to try her/his dominant strategy (s/he would get 92!)

Prisoner's dilemma

- police and two suspects
- no evidence
- suspects are asked to confess
- penalties are in the payoff matrix (the larger the better)

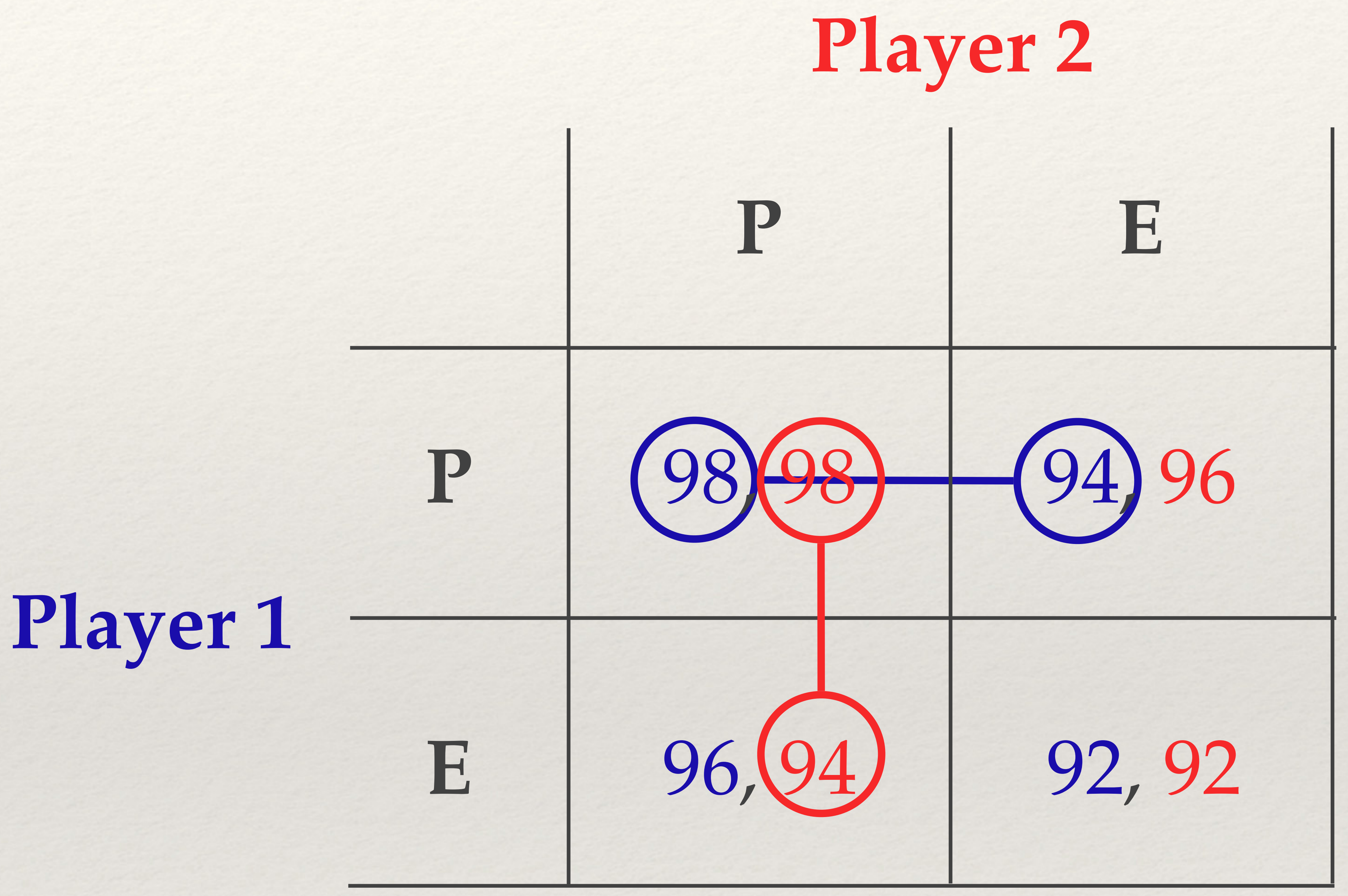


		Suspect 2	
		Not Confess	**Confess**
Suspect 1	**Not Confess**	-1, -1	-10, 0
	Confess	0, -10	-4, -4

Strict dominant strategy for both: Confess

Wrapping up the Prisoner's dilemma

- It arises only when payoffs are designed in a certain way
  - simple changes ⇒ more benign outcomes
- Example
  - An easier exam: you will get 96 if you don't study



Player 2

Player 1

	P	E
P	98, 98	94, 96
E	96, 94	92, 92

Strict dominant strategy for both: Presentation!

Formalization: Best Responses

- **Players**: 1, 2 (it can be generalized for more players)
- **Strategies**: S, T (we can have more strategies)
- $P_1(S, T)$: Payoff of playing S for P_1, and T (fixed) for P_2
- **Best Response** S for P_1: $\forall S' : P_1(S, T) \geq P_1(S', T)$
- **Strict Best Response** S for P_1: $\forall S' : P_1(S, T) > P_1(S', T)$
- For P_2 we have symmetrical definitions

Formalization: Dominant Strategies

- **Dominant Strategy**: a P_1's strategy that is best response to every strategy of P_2

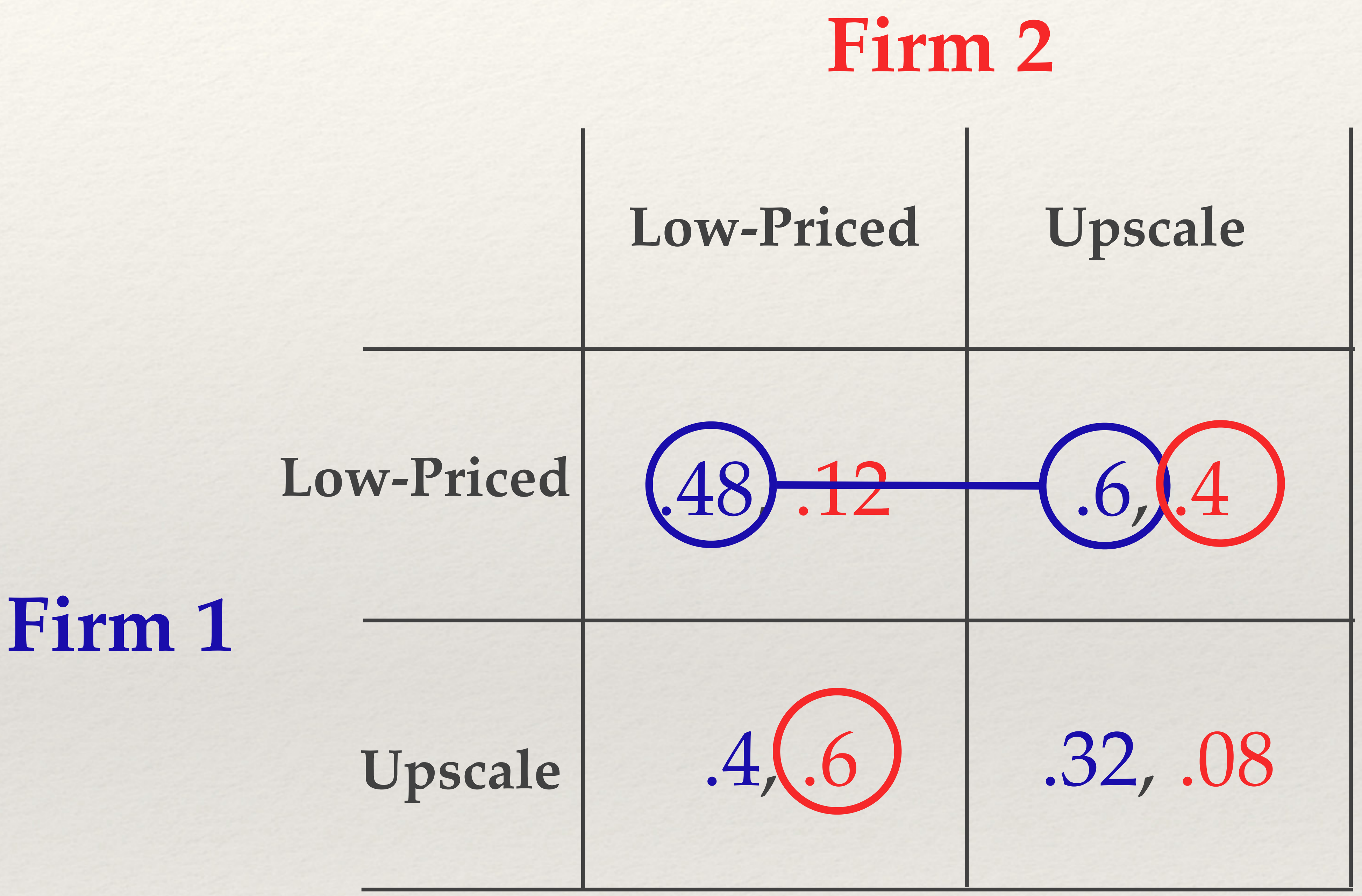
$$\forall S', T : P_1(S, T) \geq P_1(S', T)$$

- **Strict Dominant Strategy**: a P_1's strategy that is strict best response to every strategy of P_2

$$\forall S', T : P_1(S, T) > P_1(S', T)$$

What if only one player has a strictly dominant strategy?

- New example:
  - two firms planning to produce and market a new product
  - two market segments:
    - people who would buy a low-priced version of the product (60%)
    - people who would buy a upscale version (40%)
  - Firm1: more powerful, reaches 80% of the sales
  - Firm2: will get 20% of the sales



		Firm 2	
		Low-Priced	Upscale
Firm 1	Low-Priced	.48, .12	.6, .4
	Upscale	.4, .6	.32, .08

Strict dominant strategy for P_1: Low-Priced

No dominant strategy for P_2!

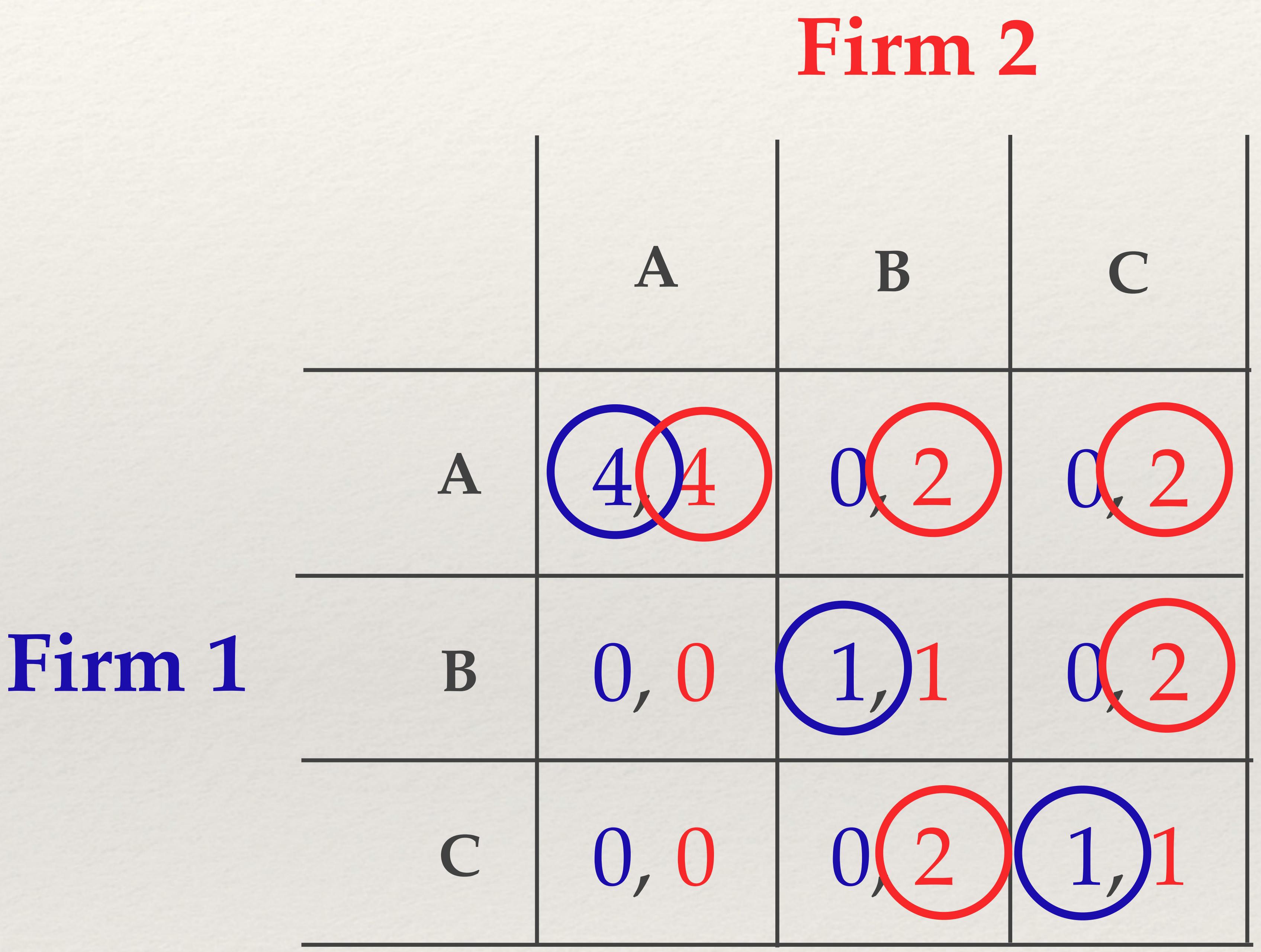
- Players must move simultaneously
- Firm1 can decide its strategy with no regards of Firm2's move
- secrecy: Firm2 must move without knowing Firm1's move
- but payoff matrix $\Longleftrightarrow$ full knowledge
- Firm2 is subordinate to Firm1:
    - its best strategy is to stay away from Firm1 market segment

What if none has a dominant strategy?

- What if none has (strict) dominant strategy?
    - We need another way to predict what is likely to happen
- Example: a three clients game
    - Players: two firms
    - Three large clients: A, B, C
    - Three strategies: A, B, C

		Firm 2		
		A	B	C
Firm 1	A			
	B			
	C			

- If the firms approach the same client: 50% of the general business
- Firm1 too small: if it approaches a client on its own ⇒ payoff = 0
- A is larger than B and C: it wants to do business with both of the firms (or nothing)
- A worths 8
- B and C worth 2



		Firm 2		
		A	B	C
	A	4, 4	0, 2	0, 2
Firm 1	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

No dominant strategy!

Nash Equilibrium

- What about (A,A)?
- Observe that no players have incentive to change their strategy!
- The system got an equilibrium state, with no force pushing towards a different configuration
- (S, T) is a **Nash Equilibrium** if S is a best response to T and T is best response to S

		Firm 2		
		A	B	C
Firm 1	A	4, 4	0, 2	0, 2
	B	0, 0	1, 1	0, 2
	C	0, 0	0, 2	1, 1

Multiple Equilibria: Coordination Games

- What if we have more than one NE?
- Example
    - two players
    - joint presentation: which software (Keynote or PowerPoint)?
    - Players need to coordinate with no communication

Player 2

		PowerPoint	Keynote
Player 1	**PowerPoint**	1, 1	0, 0
	Keynote	0, 0	1, 1

Two different Nash Equilibria!

- What to do?
- Thomas Schelling's idea of **Focal Points**:
  - look for *natural reasons* to focus on one of the NE
  - (social) *conventions* outside the payoff matrix can help
  - Then, try to embed in the payoff matrix the *intrinsic features* that help you to select an equilibrium

Player 2

		PowerPoint	Keynote
Player 1	PowerPoint	2, 2	0, 0
	Keynote	0, 0	1, 1

No Nash Equilibria

- **Matching Pennies** game:
  - each player has a penny
  - they can show head or tail
  - match: player 1 loses
  - no match: player 1 wins
- Also called: "Attack-Defense" or "zero-sum" games
- With no N.E. at all: let's introduce **randomization** and **probabilities**

Player 2

		Head	**Tail**
Player 1	**Head**	-1, 1	1, -1
	Tail	1, -1	-1, 1

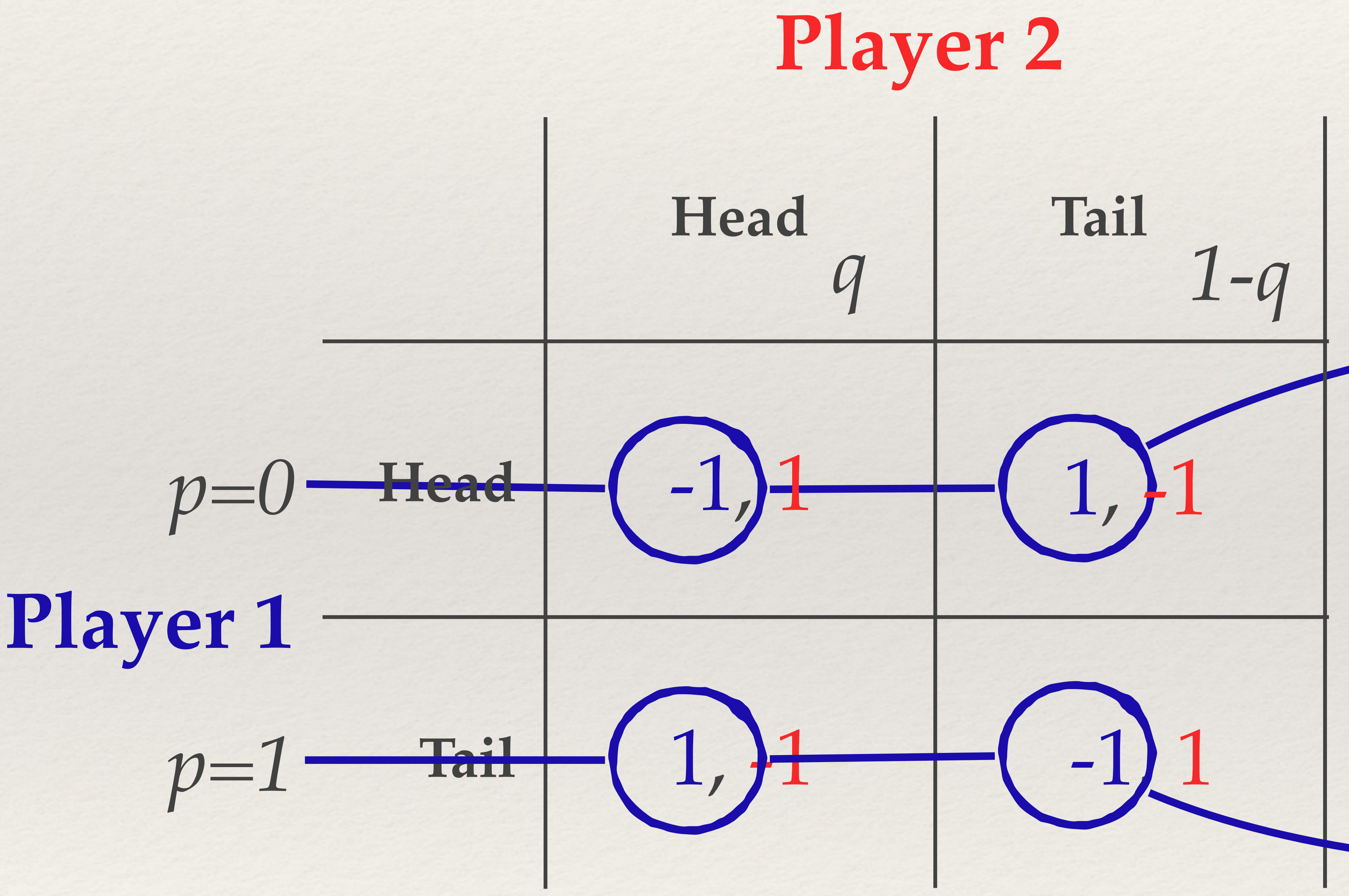
No Nash Equilibria!

Mixed Strategies

- We model with **randomization**
- Strategies are **probabilities** between [0,1]
    - P_1 chooses S with prob. p (and T with prob. $1\text{-}p$)
    - P_2 chooses S with prob. q (and T with prob. $1\text{-}q$)
- That means that each player chooses a **mixing** between the given **strategies**

$$\left.\begin{array}{l} p = 0 \Rightarrow P_1 \textit{ is playing } T \\ p = 1 \Rightarrow P_1 \textit{ is playing } S \end{array}\right\} \Rightarrow \textbf{pure strategies}$$

Payoffs for Mixed Strategies



		Player 2	
		Head q	Tail $1-q$
Player 1	$p=0$ Head	-1, 1	1, -1
	$p=1$ Tail	1, -1	-1, 1

- Payoffs are random. How to compare them?
- P_1 point of view:
  - expected payoff of pure strategy *Head*

$$-1 \cdot q + 1 \cdot (1 - q)$$

$$= 1 - 2q$$

  - expected payoff of pure strategy *Tail*

$$1 \cdot q + (-1) \cdot (1 - q)$$

$$= 2q - 1$$

Equilibrium with mixed strategies

Which is the relationship between $1-2q$ and $2q-1$?

$$1-2q=2q-1 \qquad \Rightarrow q=\frac{1}{2}$$

Symmetrically, from P_2 perspective:

$$\Rightarrow p=\frac{1}{2}$$

$$\Rightarrow \left(p=\frac{1}{2}, q=\frac{1}{2}\right)$$ **is a Nash Equilibrium**

Interpretation of the "indifference principle"

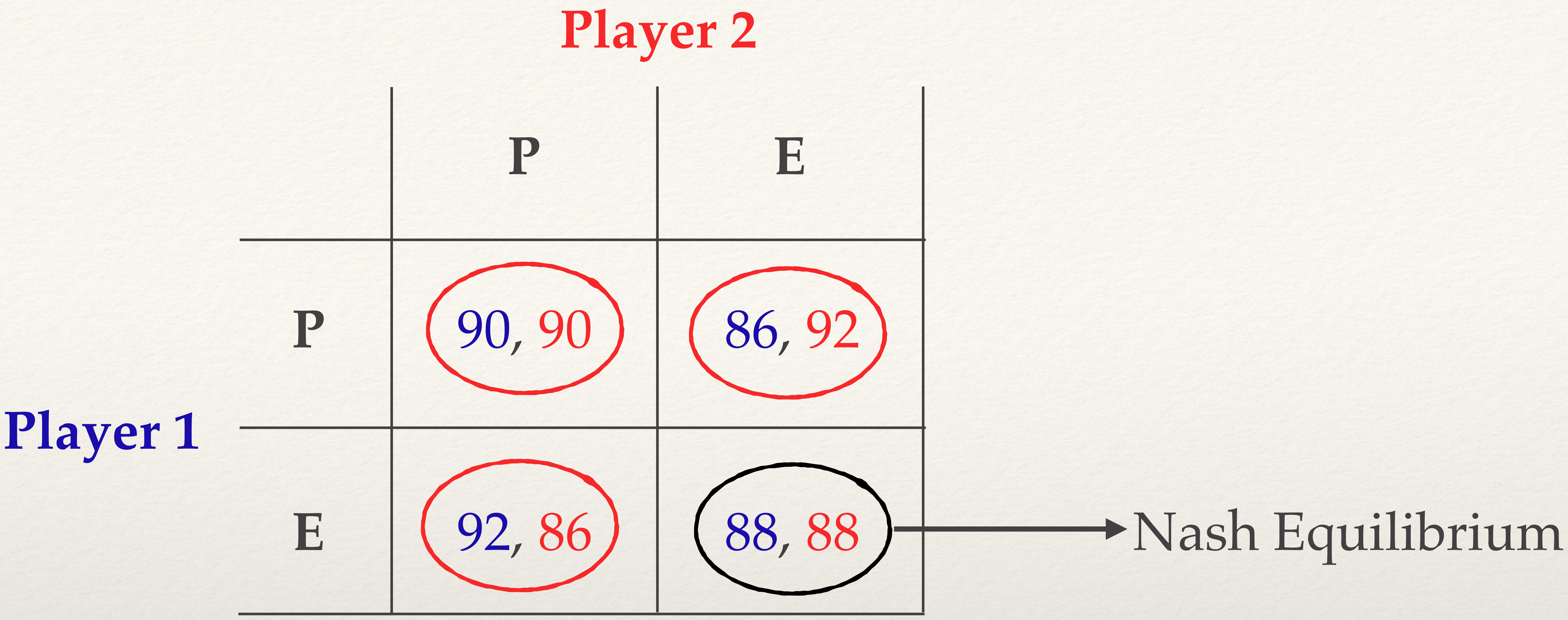
- If P_1 believes that P_2 will choose Head more than half of the times, then s/he will win more than half of the times simply choosing Tail
  - Symmetric reasoning applies for P_2 as well.
- The choice of $q = \frac{1}{2}$ is un-exploitable for P_1
- **indifference principle:** the choice of p and q are un-exploitable for the other player to decide their strategies
- Nash main results (deserving a Nobel prize): he proved that **every game has at least on Nash Equilibrium**

Optimalities

- We have Nash Equilibria, s.t., each player's strategy is a best response to the other player's strategy
- This does not mean that the players will necessarily reach an outcome that is in any sense "good"
- It is possible to classify outcomes not just by their strategic or equilibrium properties, but also by whether they are "good" for ourselves and "the others"

Pareto Optimality

- Pareto optimality is a situation where no action or allocation is available that makes one individual better off without making another worse off
- A binding agreement to actually play the "superior" pairs of strategies is usually needed



		Player 2	
		P	E
Player 1	P	90, 90	86, 92
	E	92, 86	88, 88

Nash Equilibrium

- Three Pareto Optima
- Players have the incentive to change their strategy, unless they have a binding agreement

Social Optimality

- Stronger
- A choice of strategies, one by each players, is a **social welfare maximizer** (or **social optimum**) if it maximizes the sum of the players' payoffs
- (P,P) is a Social Optimum (and also a Pareto Optimum)
- Nash Equilibrium is not a social optimum

Player 2

		P	E
Player 1	**P**	90, 90	86, 92
	E	92, 86	88, 88

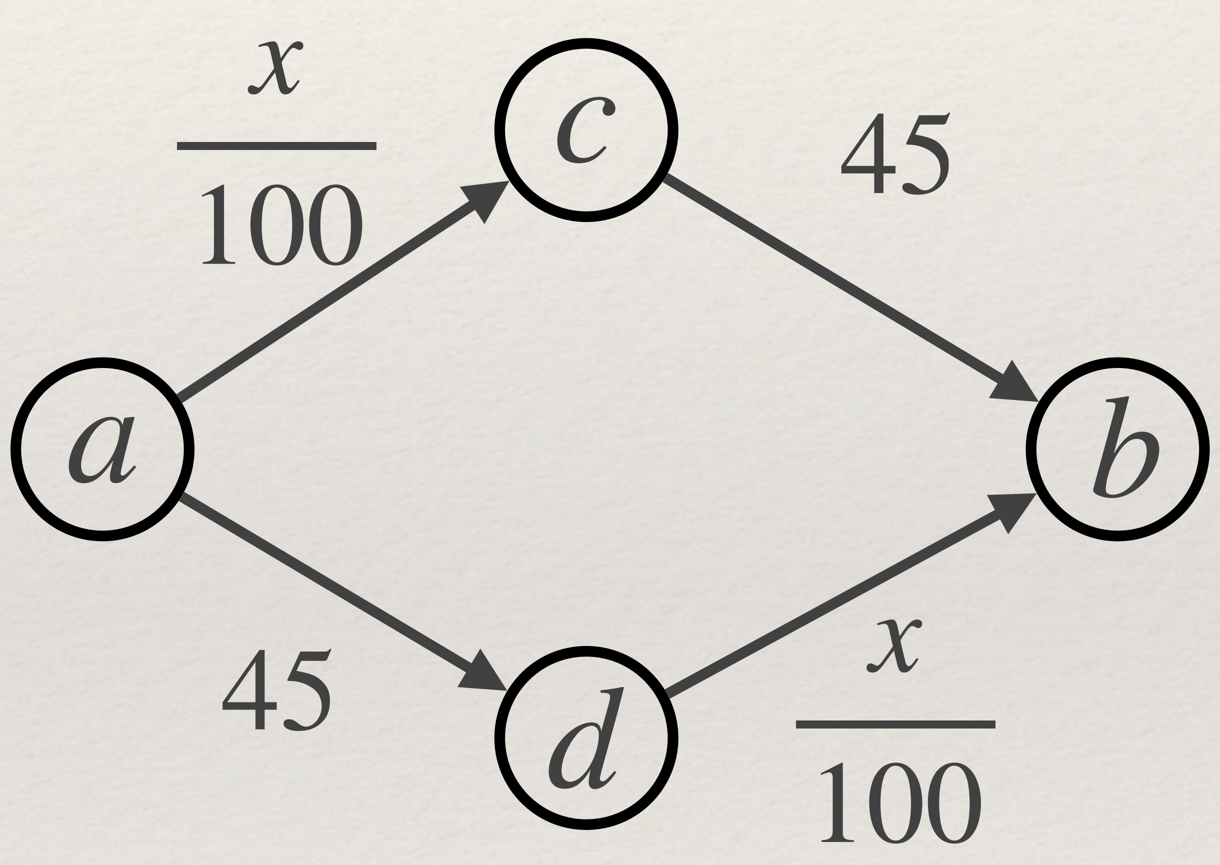
Networks and Game Theory

- Nodes connected with many other nodes: if one agent has to select, for some given purpose, one (or some) connection out of your choices, then you need a strategy
  - it is likely that agents will select the strategy that leads to the highest payoff
  - a multi agents system: each agent will evaluate payoffs according their and everyone's else strategies
- **Traffic network**: individuals need to evaluate routes in the presence of the congestion
  - congestion is the result of the decisions made by themselves and everyone else
- **Models for network traffic** may lead to unexpected results

Traffic at Equilibrium

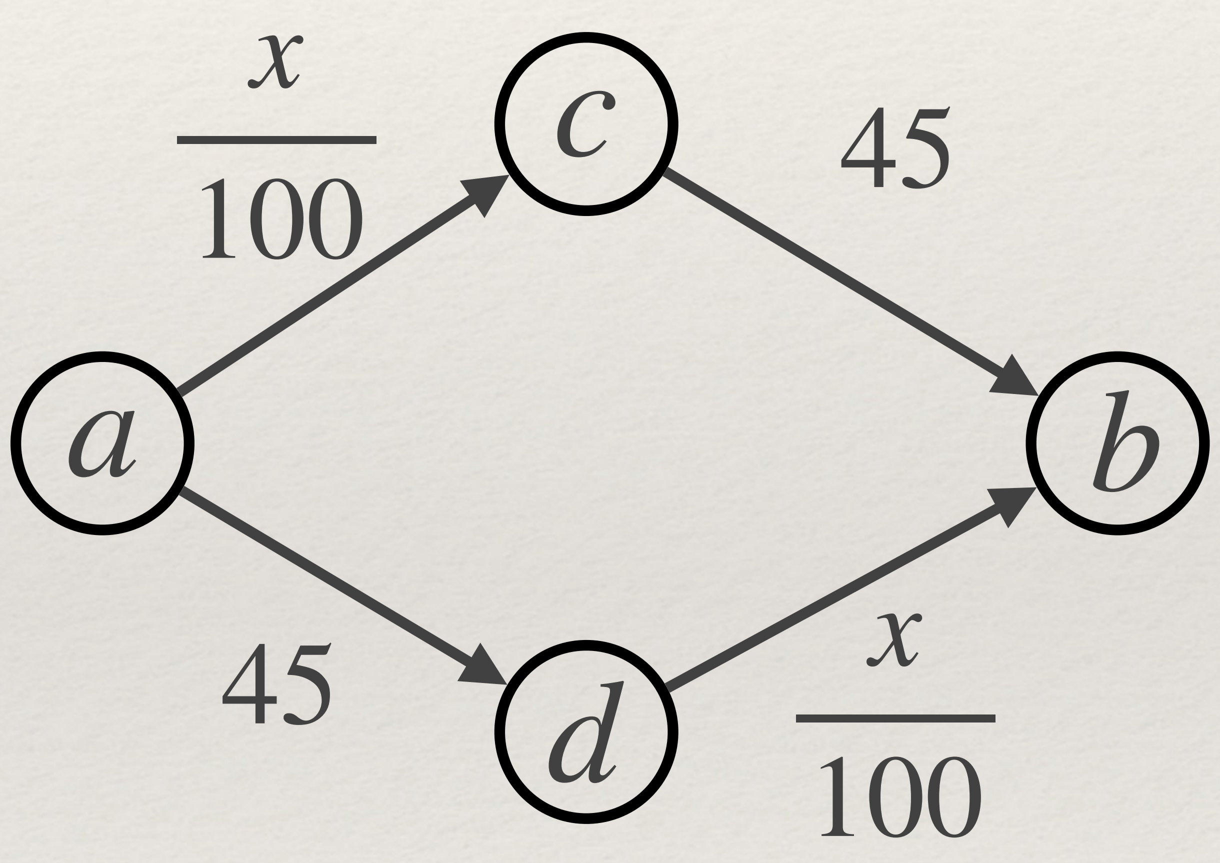
Transportation network model

- Directed graph
  - edges are highways
  - nodes are exits (you can get on or off a particular highway)
- Assumption: everyone wants to drive from a to b
- Weights: travel time
- Suppose we have 4000 cars



Traffic game

- The traffic game:
  - players: drivers
  - each player's has 2 possible strategies: routes from *a* to *b*
  - payoff: the negative of a player's travel time (the faster the better)

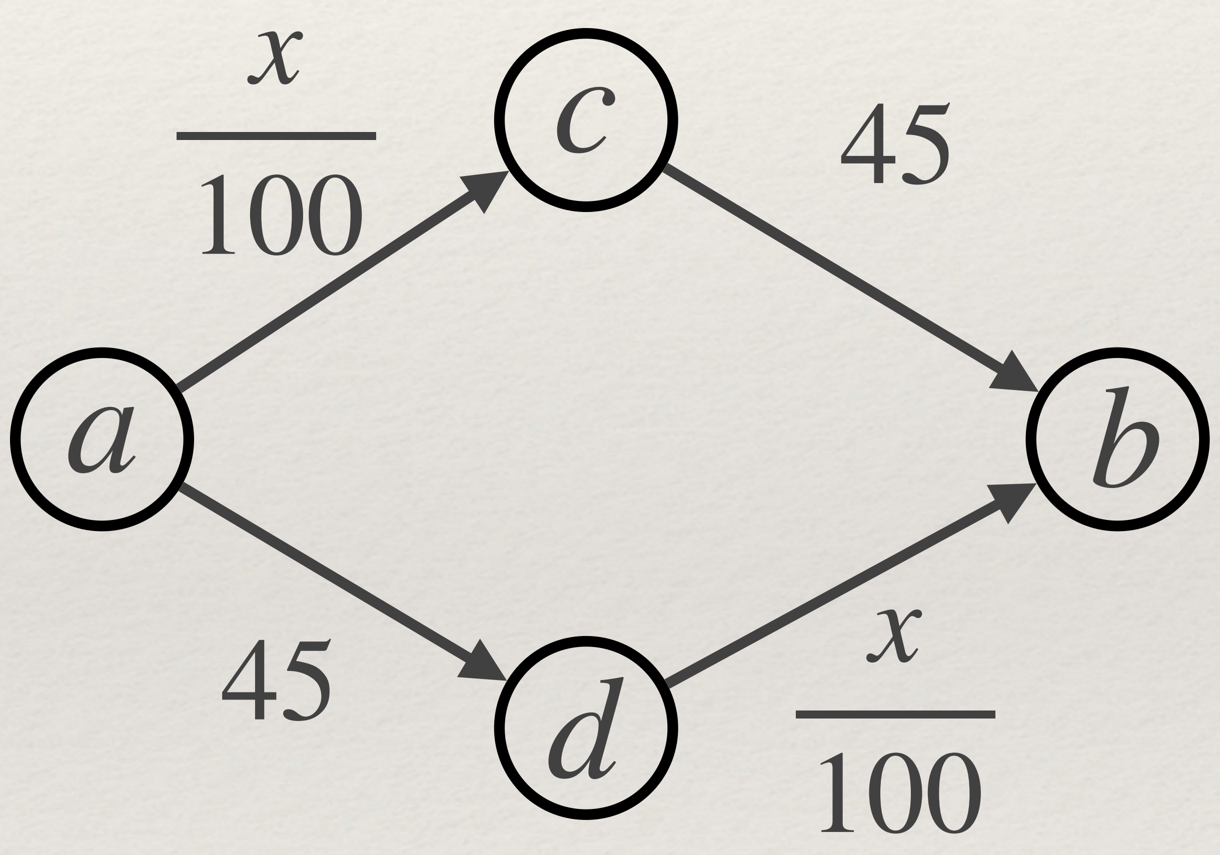


Games with more than 2 players

- As in the 2 players game:
  - the payoff of each player depends on the strategies chosen by all
  - *Nash equilibrium*: a list of strategies (one for each player), so that each one is a best response to all the others
  - Dominant strategies, mixed strategies, Nash equilibrium with mixed strategies: they all have direct parallels

Equilibrium traffic

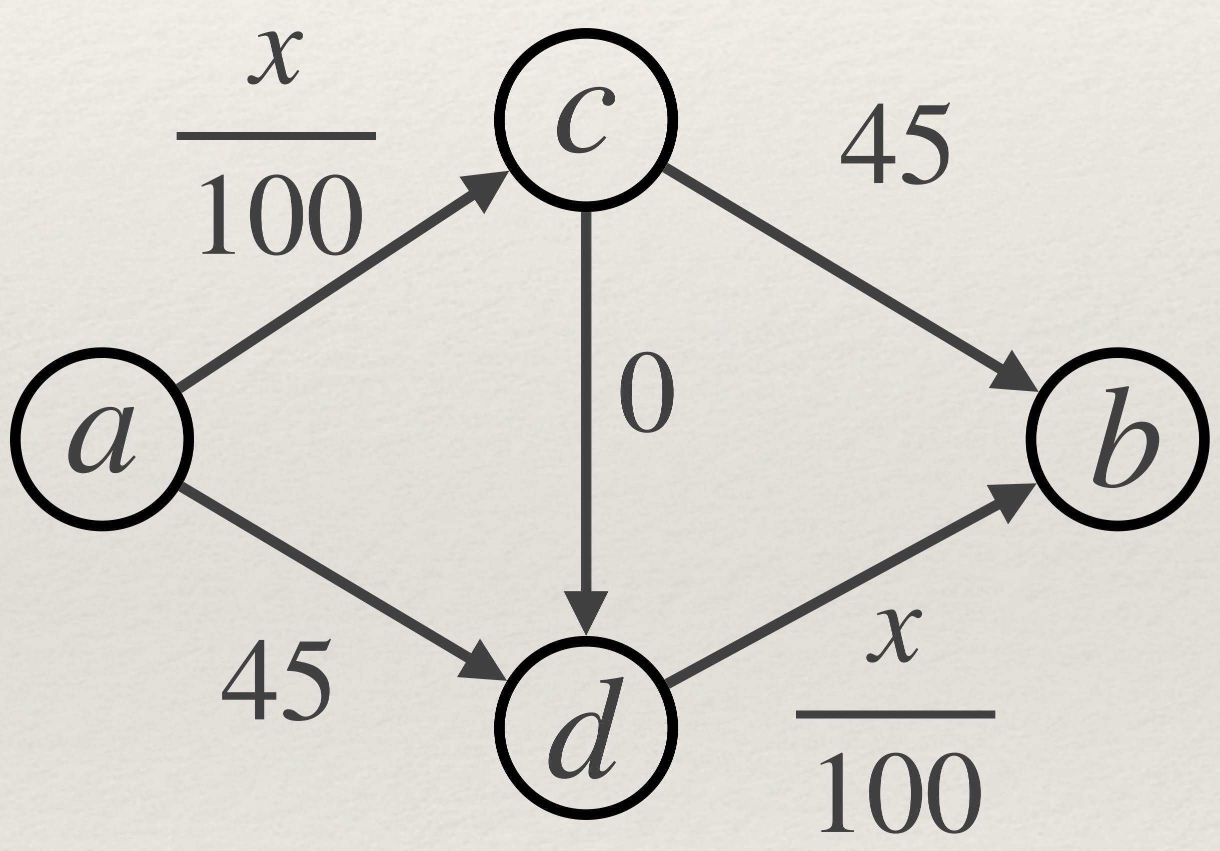
- **No dominant strategy** in a traffic game
  - either route has the potential to be the best choice for the player if all the other players are using the other route!
- **We have Nash equilibria**: any list of strategies in which the drivers balance themselves evenly between the two routes
  - with an even balance, no driver has an incentive to switch over to the other route



Braess′s Paradox

Changing the network

- Small changes can lead to counterintuitive results
- Let's build a new super fast highway from *c* to *d*
  - keep it simple: *(c,d)* travel time is 0
- New (and unique) Nash equilibrium: every driver uses the route through *c* and *d*, leading to worser travel times



The Braess's paradox

- Daniel Braess (1968)
- Even if the NE route takes a longer travel time (80 mins), switching from it will take take 85 mins!
- The new highway acts like a "vortex" that attracts all the drivers into it - to the detriment of all
- There is no way, given self-interested behavior by the driver, to get back to the even balance solution that was better for everyone
- Like many counterintuitive anomalies:
    - it needs the right combination of conditions to actually pop up in real life
    - models ≠ reality!
    - however, it can explain some empirical observation in real transportation networks

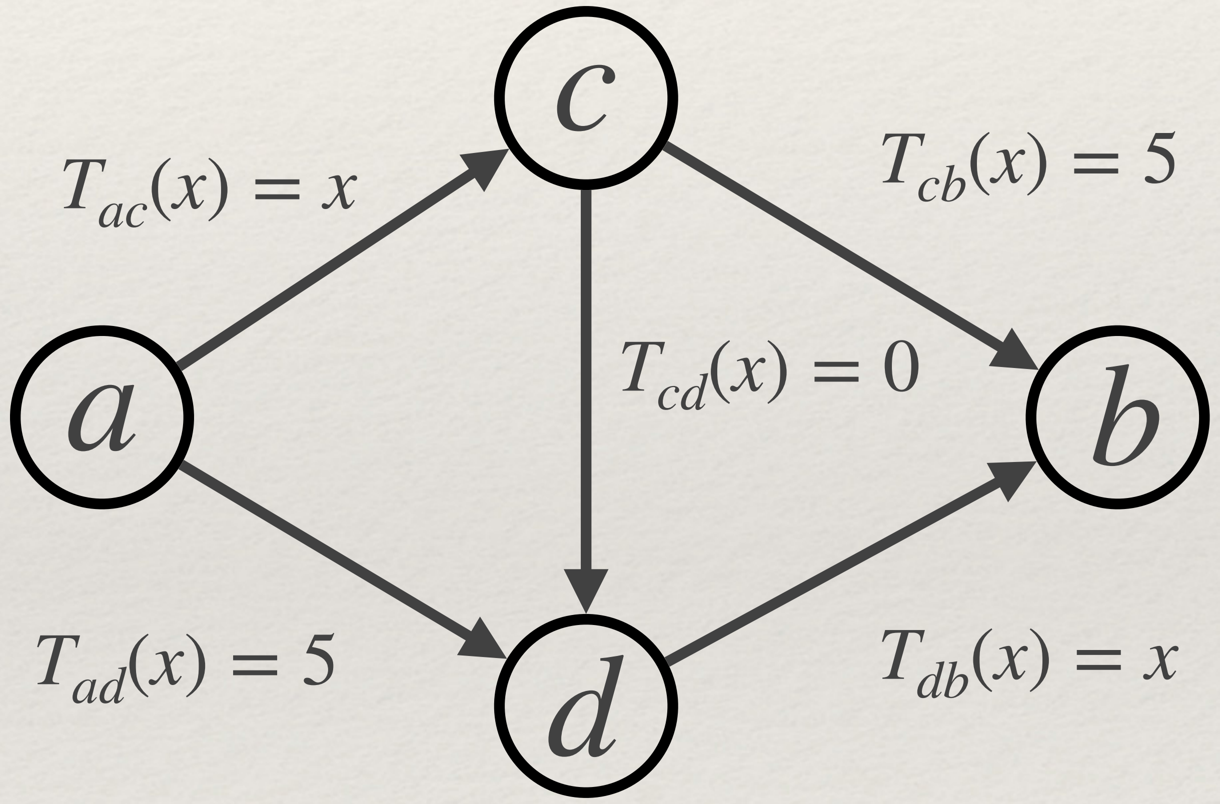
Some observations

- After all, no paradox at all: very similar to the prisoner's dilemma
  - Intuition: upgrading is always a good thing (or also: having more strategies could only improve things)
  - Experience: not always true
- A starting point for a large body of work on game-theoretic analysis of network traffic
  - How much larger can the equilibrium travel time be after the addition of an edge, relative to what it was before? ⇒ more later
  - How can we design networks to prevent bad equilibria from arising? ⇒
    Tim Roughgarden. *Selfish Routing and the Price of Anarchy*. MIT Press, 2005

The Social Cost of Traffic at Equilibrium

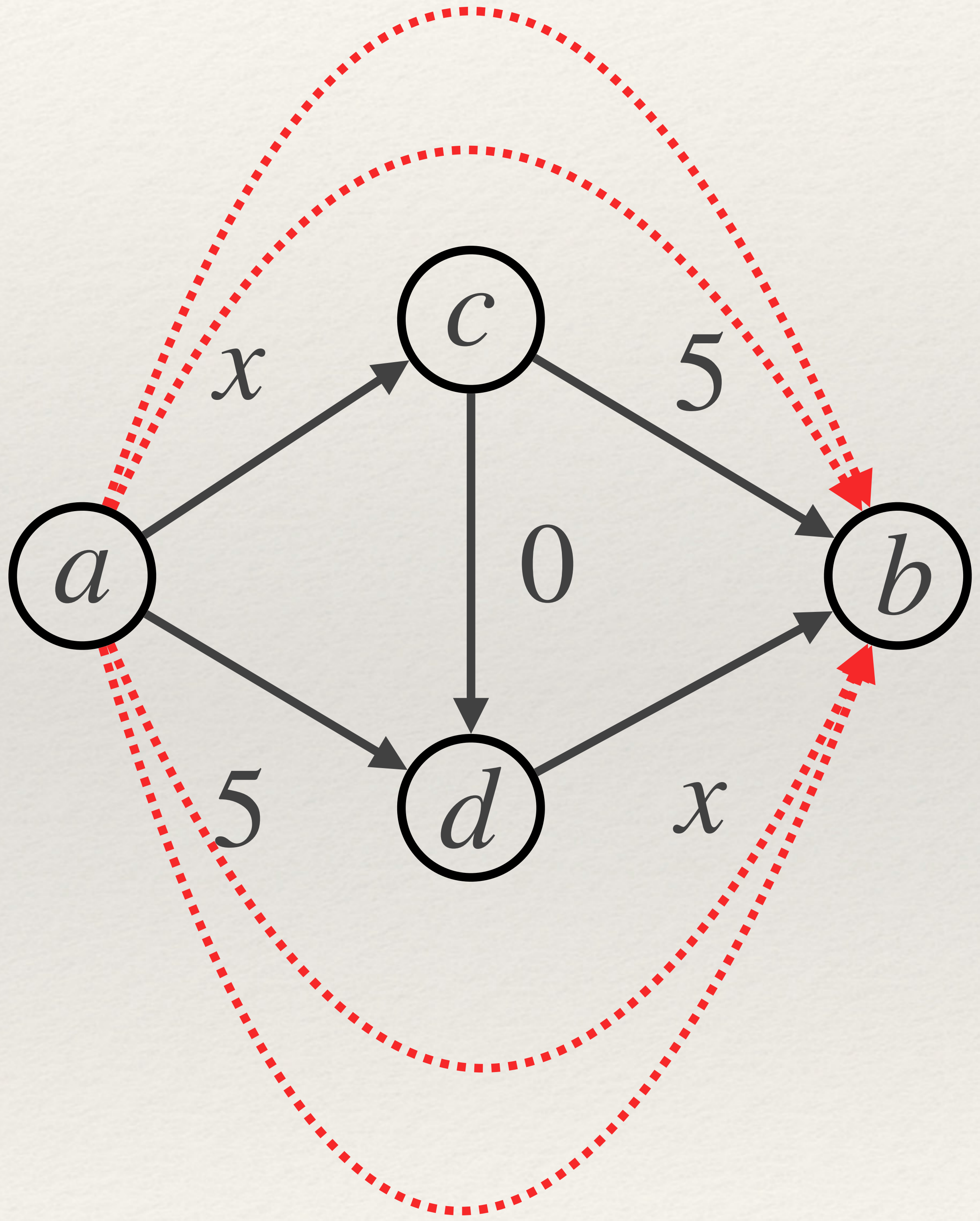
Travel time function

- We want to quantify how far from optimal is traffic at equilibrium
- Each edge *e* has a *Travel-time function* $T_e(x)$
  - Assumption: linear in the amount of traffic $T_e(x) = a_e x + b_e$, with $a_e, b_e \geq 0$



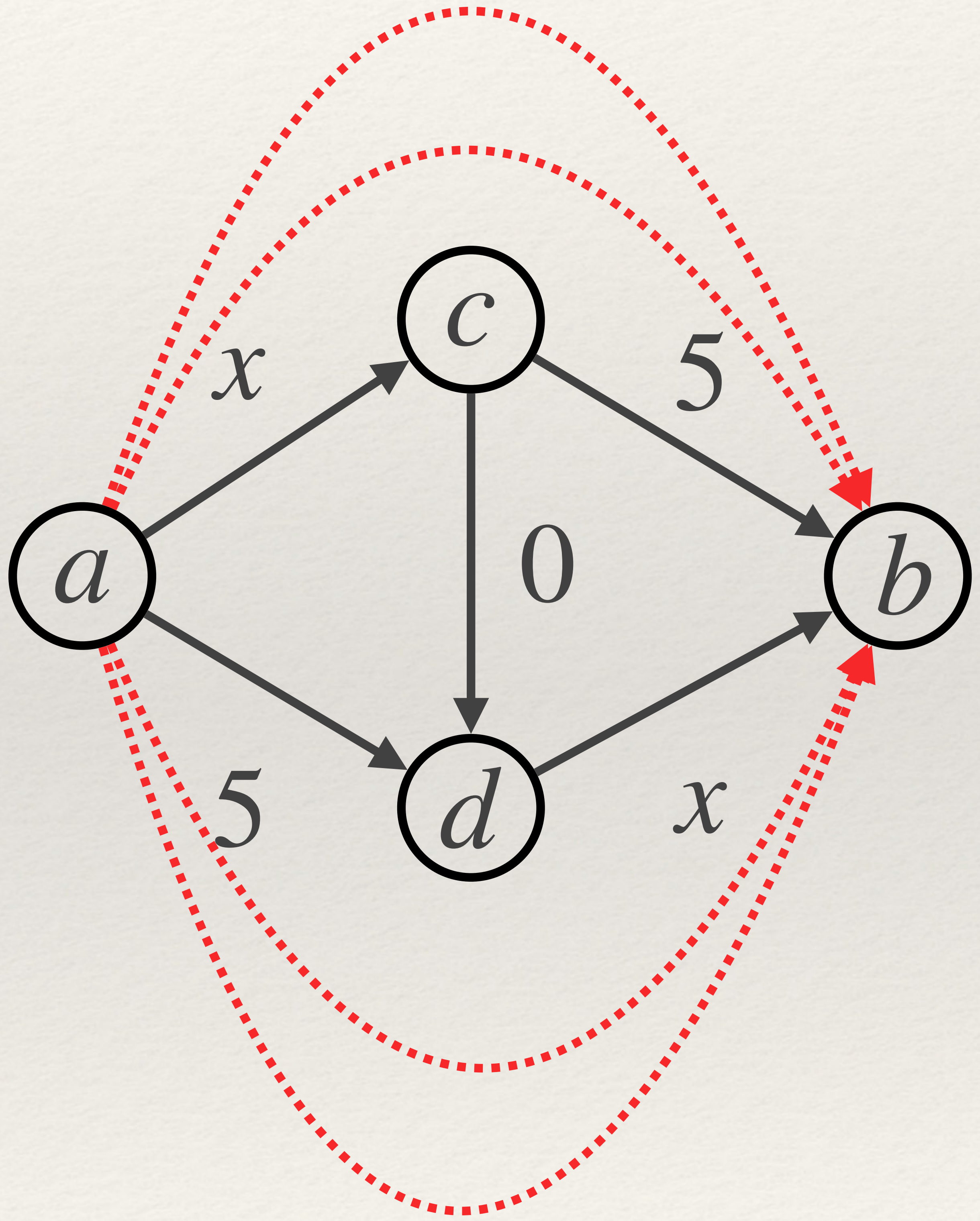
Traffic pattern

- A *traffic pattern*: a choice of a path by each driver
- *Social cost* of a traffic pattern z: the sum of the travel times incurred by all drivers when they use this traffic pattern:
- Ex: 4 drivers, each starting from a and with destination b



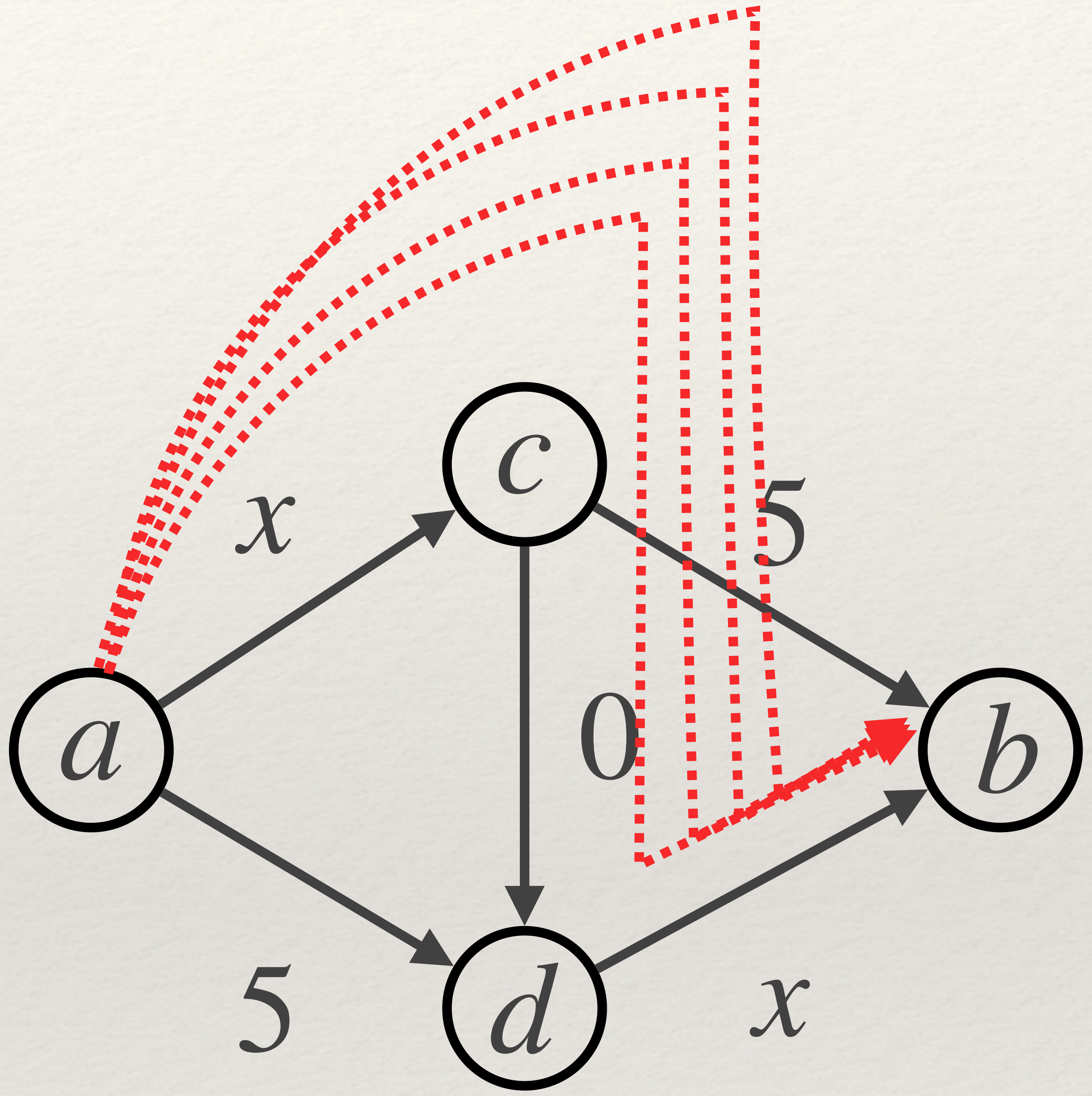
Socially optimal traffic patterns

- When a traffic pattern achieves the minimum possible cost: *socially optimal*
- socially optimal traffic patterns are *social welfare maximizers* in this traffic game



Nash equilibrium

- The unique Nash equilibrium in this game has a larger social cost
- *Is there always an equilibrium traffic pattern?*
- *Does it always exists an equilibrium traffic pattern whose social cost is not much more than the social optimum?*



Finding a traffic pattern at equilibrium

- To prove that an equilibrium exists, let's use a procedure that look for one:
    1. start from any traffic pattern
    2. if it is an equilibrium, stop
    3. else, there is at least one driver whose best response is some alternate path providing a strictly lower travel time
    4. pick one of these drivers, and go to step 2.
- This is called a **best-response dynamics**
- We need to show that best-response dynamics will eventually stop.

Does a best-response dynamics always stop?

- No. In a zero sum game it will run forever because it lacks of an equilibrium (with pure strategies)
- In principle, even in the traffic game we can have a best-response dynamics that run forever if we do not have an equilibrium
- We will prove that in our traffic game the procedure stop, proving consequently that:
    - equilibria exist
    - an equilibrium can be reached by a simple process in which drivers constantly update what they are doing according to best response

Progress Measure

- To check if the best-response dynamics will eventually stop, we need a *progress measure* to track the process and to assess how far we are from the process to stop
- Is the social cost of the current traffic pattern a good progress measure?
  - Answer: *No* ⇒ Some best-response updates by drivers can make the social cost better, but others can make it worse
  - The social cost of the current traffic pattern can *oscillate*, and the relationship with out progress toward an equilibrium is not clear
- The alternate quantity *must strictly decrease* with each best-response update

Potential Energy

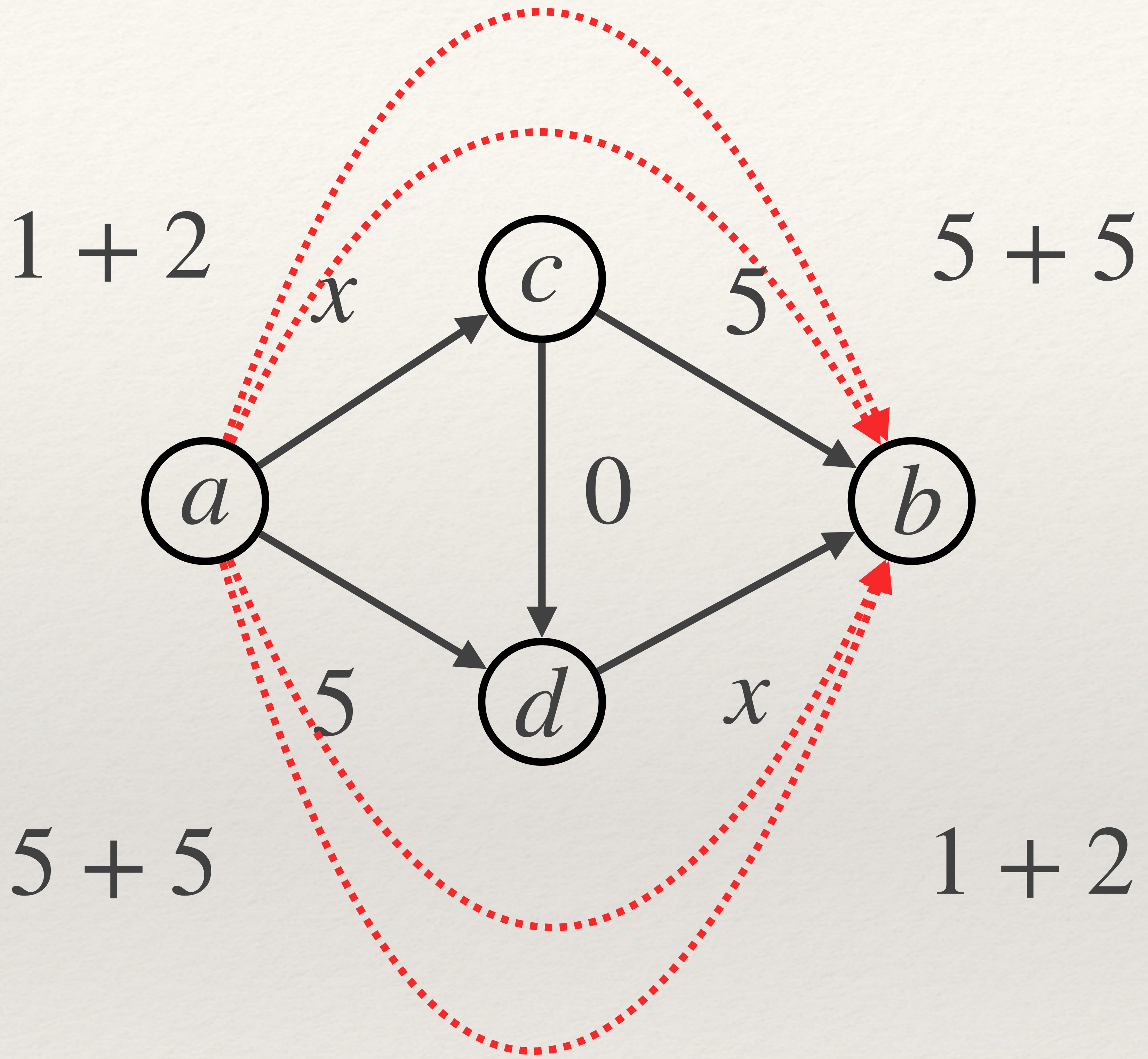
- As a good progress measure, let's introduce the *potential energy* of an edge:

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

- if an edge e has no driver on it: $\text{Energy}(e) = 0$
- The potential energy of a traffic pattern z is the sum of all the the potential energies of all the edges, with the current number of drivers in this traffic pattern:

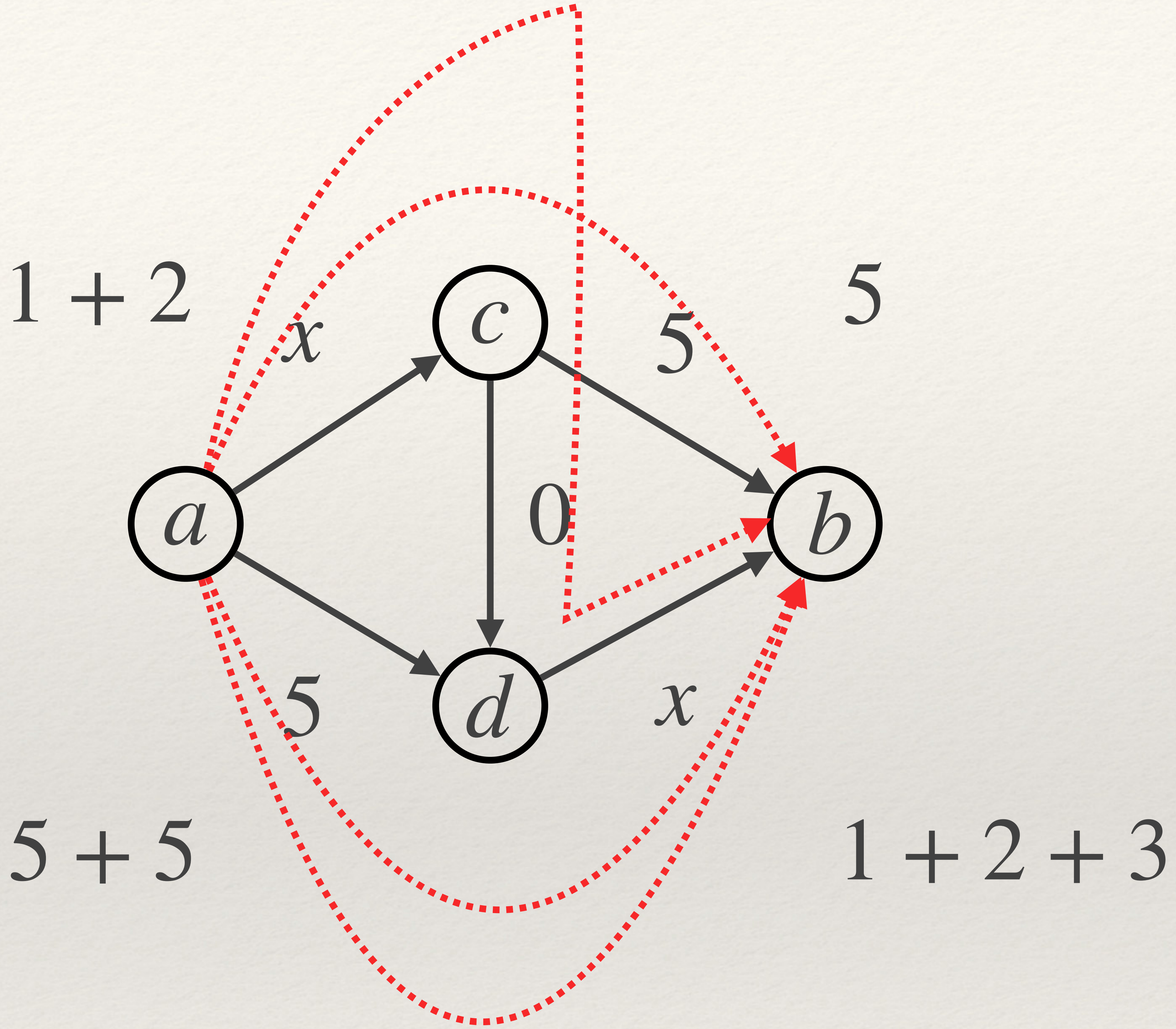
$$\text{Energy}(z) = \sum_{e_i \in z} \text{Energy}(e)$$

Let's label each edge
also with Energy now



Social Optimum

Let's label each edge also with Energy now



Let's label each edge
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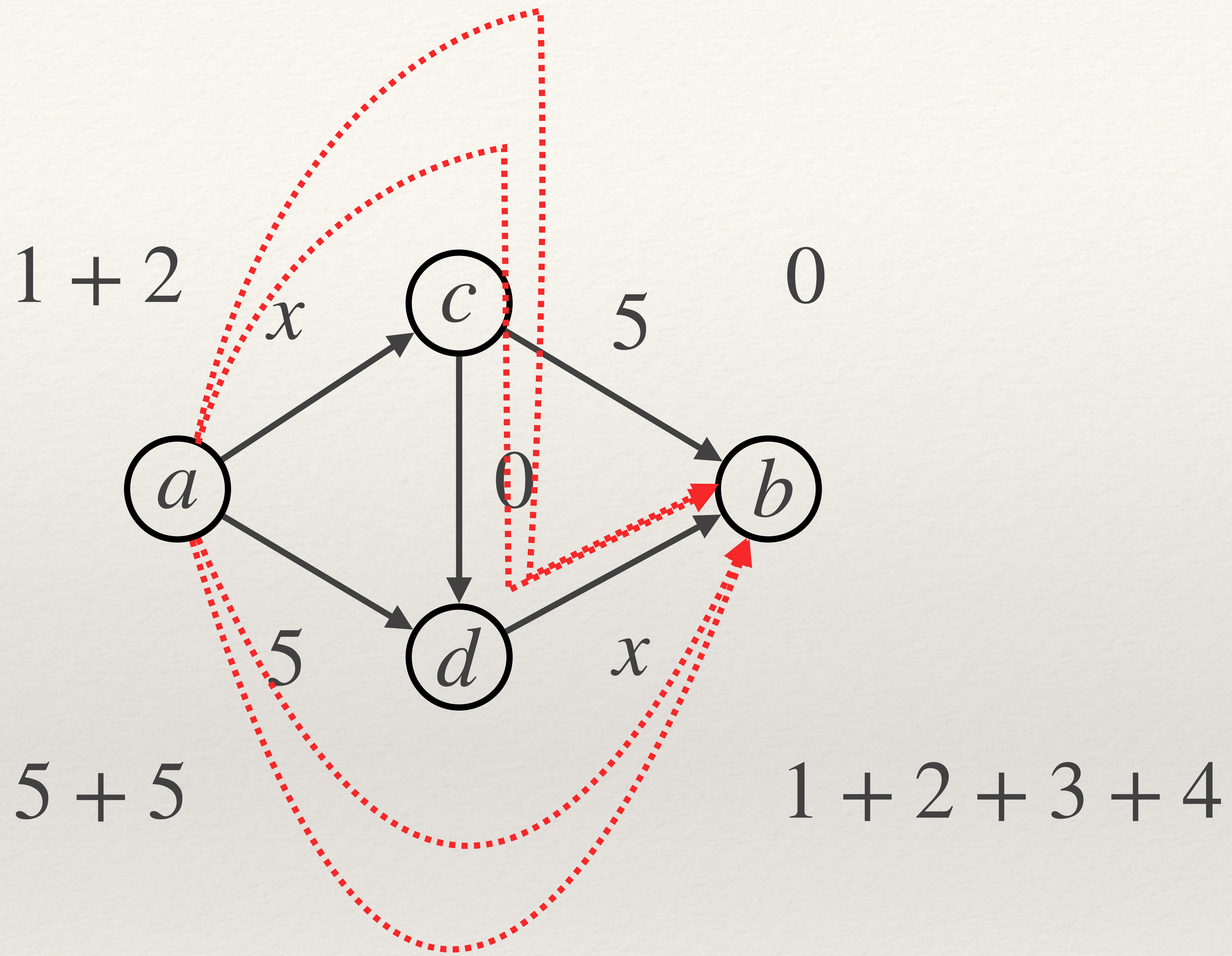

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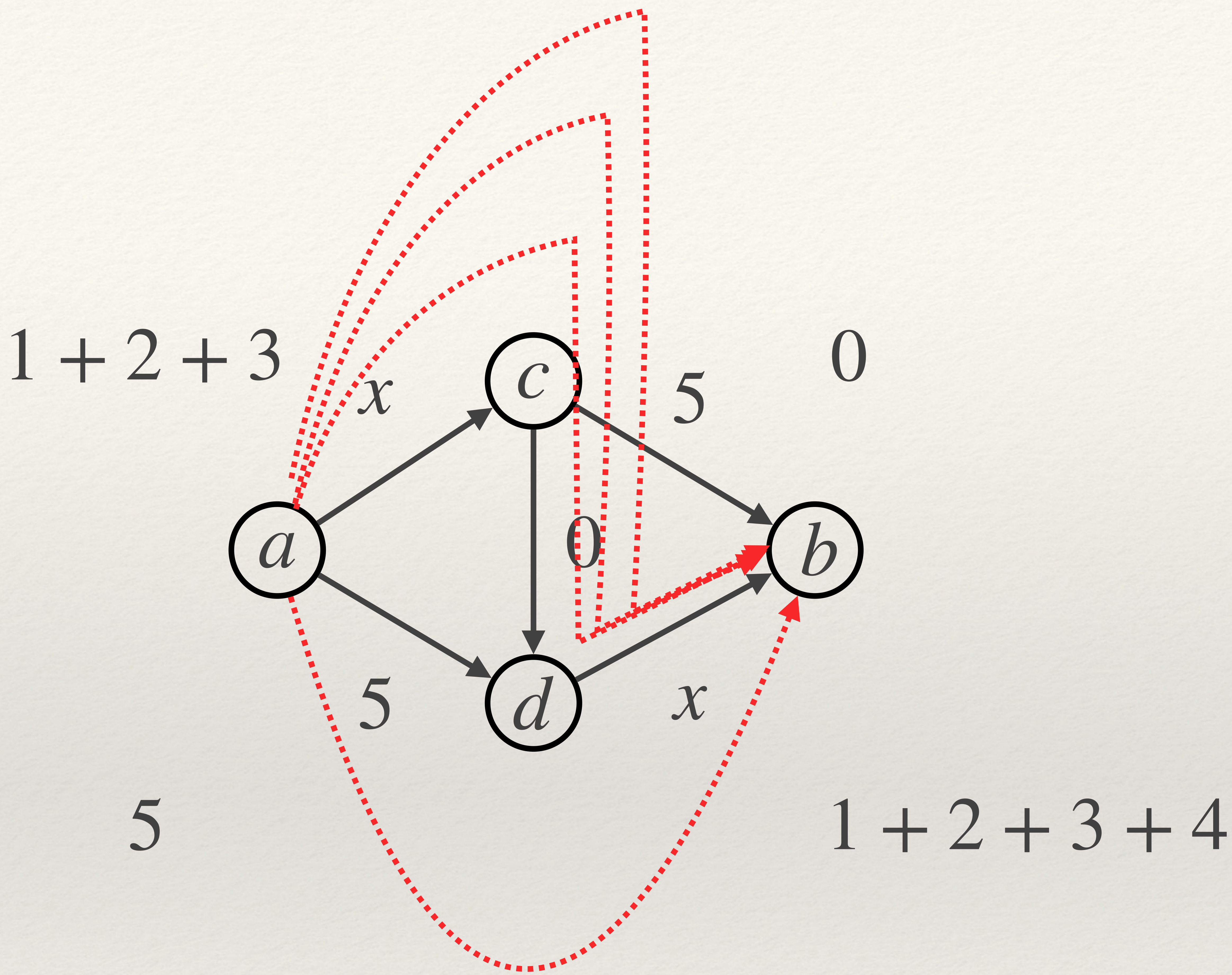
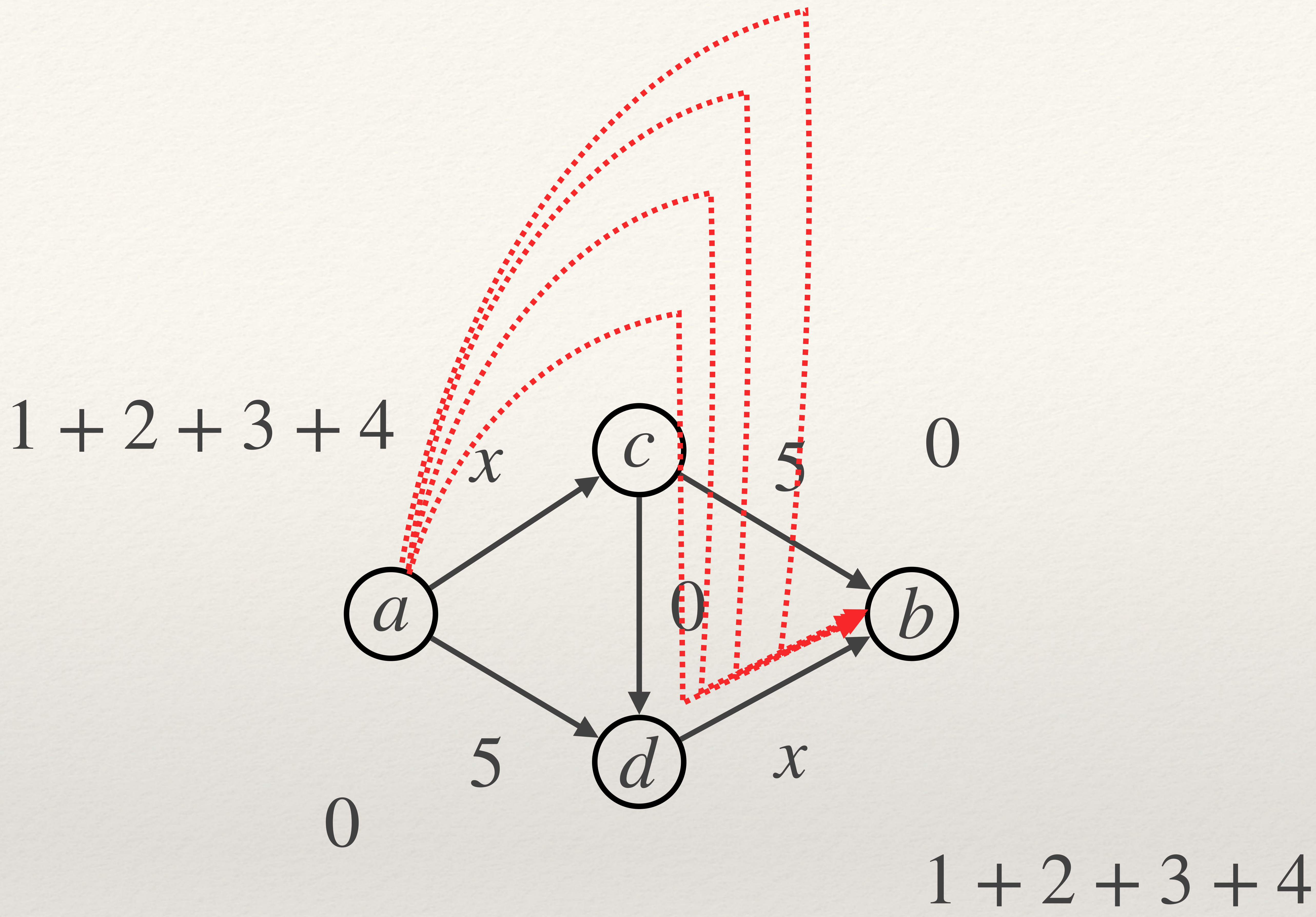


Let's label each edge
also with Energy now



Nash Equilibrium

Does the best-response dynamics stop?

- If we prove that best-response dynamics will stop, then we have proved that an equilibrium always exist
- If we prove that the potential energy strictly decreases at each step, then we have proved that best-response dynamics stops
- Observe in our example that potential energy always decreases at every step:
    - when a driver abandons one path in favor of another, the change in potential energy is exactly the improvement in the driver's travel time
- Is this true for any network and any best-response by a driver?

Let's recall that the potential energy of edge e with x drivers is:

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x-1) + T_e(x)$$

When one of these drivers leaves it drops to:

$$= T_e(1) + T_e(2) + \ldots + T_e(x-1)$$

Summing up, Energy(z) **decreases** accordingly all the travel times that the driver was experiencing on every edges in path z: $\sum_{e \in z} T_e(x)$

It is like that the driver, abandoning path z for the new path z', releases a potential energy that is equal to: $\sum_{e \in z} T_e(x)$

By the same reasoning, for every edge e' in the new path z', before the new driver adopts it, we have this potential energy:

$$\text{Energy}(e') = T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1)$$

When one of the new driver joins it increases to:

$$= T_{e'}(1) + T_{e'}(2) + \ldots + T_{e'}(x-1) + T_{e'}(x)$$

Summing up, Energy(z') **increases** accordingly all the travel times that the new driver is experiencing on every edges in path z': $\sum_{e' \in z'} T_{e'}(x)$

$\Rightarrow$ The *net change* in potential energy is simply the driver new travel time minus their old travel time

$$\Delta E = \sum_{e' \in z'} T_{e'}(x) - \sum_{e \in z} T_e(x)$$

$\Delta(E)$ must be **negative**, because driver must have an incentive to change path (the new strategy must be a best response) ⇒ **the potential energy strictly decreases throughout the process**

☐

Comparing Equilibrium traffic to the Social Optimum

- We proved that an equilibrium traffic pattern always exists.
- How can we compare its travel time to that of a social optimal traffic pattern?
- Let's look for a relationship between the potential energy of an edge and the total travel time of all the drivers crossing the edge
- Then we can sum up these quantities for all the edges in the traffic patterns and compare travel times at equilibrium and at social optimum

Potential energy and total travel time for an edge

$$\text{Energy}(e) = T_e(1) + T_e(2) + \ldots + T_e(x)$$

$$\text{TTT}(e) = xT_e(x)$$

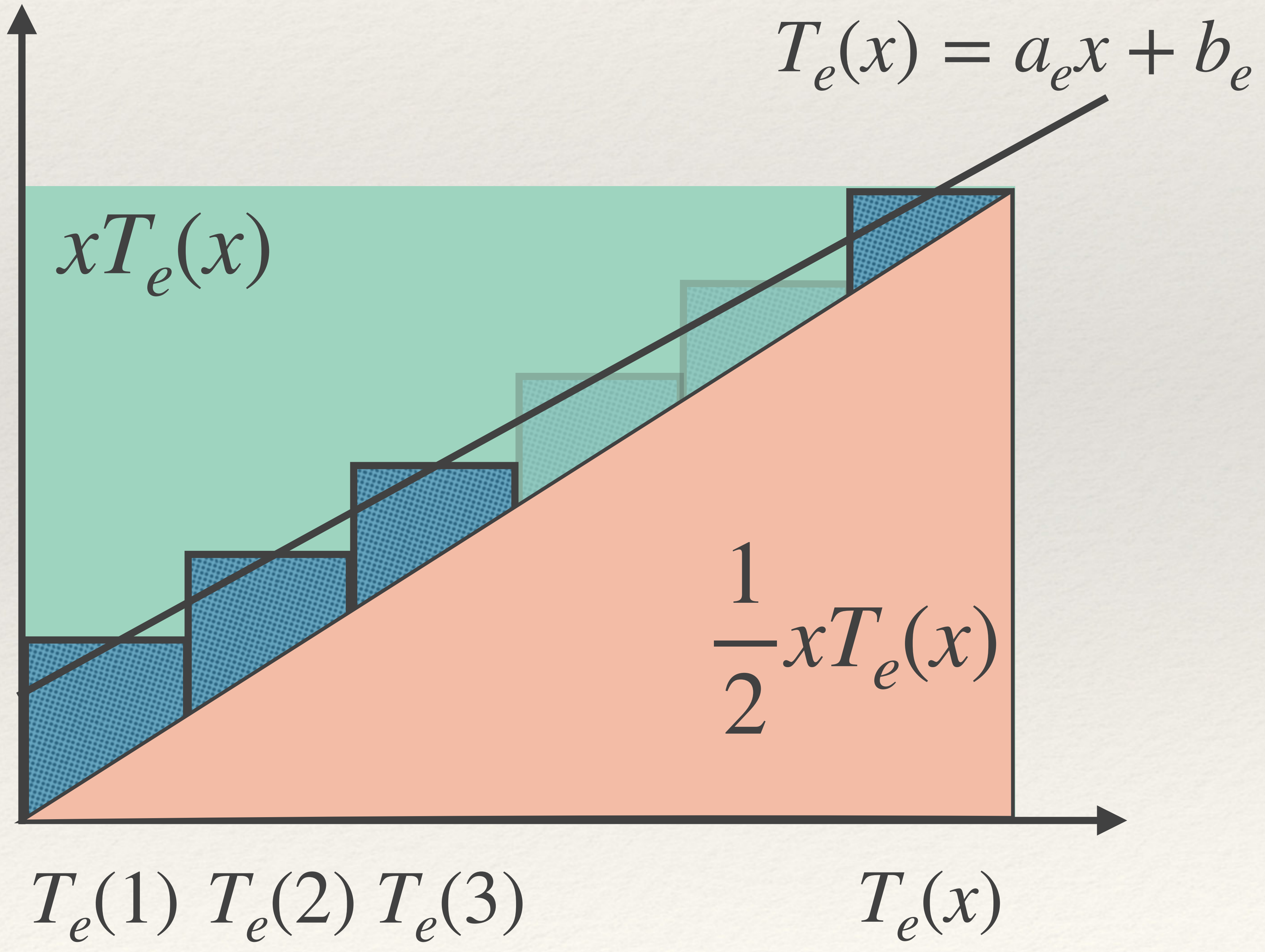
$$= \underbrace{T_e(x) + T_e(x) + \ldots + T_e(x)}_{\text{x times}}$$

$\Rightarrow \text{Energy}(e) \leq \text{TTT}(e)$

Recall that is a linear function: $T_e(x) = a_e x + b_e$

Geometrically, we have that: $\frac{1}{2} xT_e(x) \leq \text{Energy}(e)$

$\Rightarrow \text{Energy}(e) \geq \frac{1}{2}\text{TTT}(e)$



Algebraically:

$$\begin{aligned}
\text{Energy}(e) &= T_e(1) + T_e(2) + \ldots + T_e(x) \\
&= a_e(1 + 2 + \ldots + x) + b_e x \\
&= \frac{a_e x(x+1)}{2} + b_e x \\
&= x\left(\frac{a_e(x+1)}{2} + b_e\right) \\
&\geq \frac{1}{2}x(a_e x + b_e) \\
&= \frac{1}{2}xT_e(x) \\
&= \frac{1}{2}\text{TTT}(e) \qquad \Rightarrow \text{Energy}(e) \geq \frac{1}{2}\text{TTT}(e)
\end{aligned}$$

Wrapping up

We have that Energy(e) $\leq$ TTT(e) and Energy(e) $\geq \frac{1}{2}$TTT(e)

$$\Rightarrow \frac{1}{2}\mathrm{TTT}(e) \leq \mathrm{Energy}(e) \leq \mathrm{TTT}(e)$$

Moreover, if z is a traffic pattern, recall that: Energy(z) $= \sum_{e_i \in z}$ Energy(e)

Recall also that the **social cost** of traffic pattern z is the sum of the travel times incurred by all drivers when they use this traffic pattern: SC(z) $= \sum_{e \in z}$ TTT(e)

Finally, recall that the potential energy decreases as best-response dynamics moves from z to z': $Energy(z') \leq Energy(z)$

Travel time at equilibrium and at social optimality

If z is the traffic pattern at social optimality, and z' is the traffic pattern at the end of the best-response dynamics (i.e., at equilibrium), we have that:

$$Energy(z') \leq Energy(z)$$

Moreover, we have:

$$\mathrm{SC}(z') = \sum_{e' \in z'} \mathrm{TTT}(e') \;\leq \sum_{e' \in z'} 2 \cdot \mathrm{Energy}(e') \;= 2 \cdot \sum_{e' \in z'} \mathrm{Energy}(e') \;= 2 \cdot \mathrm{Energy}(z')$$

and $$Energy(z) = \sum_{e \in z} Energy(e) \;\leq \sum_{e \in z} TTT(e) \;= SC(z)$$

then $$\mathrm{SC}(z') \leq 2 \cdot \mathrm{Energy}(z') \;\leq 2 \cdot \mathrm{Energy}(z) \;\leq 2 \cdot \mathrm{SC}(z)$$

□

Conclusions

- We found that:
  - in the traffic game we can always find a traffic pattern at equilibrium
  - that the social cost of the traffic pattern at equilibrium is at most twice the socially optimal cost (we found a bound!)
- It is also possible to find a "better" bound: traffic pattern social cost at equilibrium is no more than 4/3 times as large than socially optimal traffic pattern (*)

(*) Anshelevich et. al, The price of stability for network design with fair cost allocation, 2004, at Foundations of Computer Science, 1975., 16th Annual Symposium on 38(4):295- 304