1 Finite Horizon Optimization

PROBLEM 1

$$\max_{\mathbf{a_t},\mathbf{c_t}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$s.t.\ a_{t+1} = Ra_t - c_t\ (t = 0, 1, \cdots, T), a_{T+1} = 0,$$

where $\mathbf{a_t} = \{a_t\}_{t=1}^{T+1}, \mathbf{c_t} = \{c_t\}_{t=0}^{T}$.

Solution 1 Write down Lagrangian function

$$L = \sum_{t=0}^{T} \beta^t [u(c_t) - \lambda_t(a_{t+1} - Ra_t - c_t)] + \lambda_{T+1}a_{T+1},$$

and we get F.O.C.

$$\frac{\partial L}{\partial a_t} = \beta^t \lambda_t R - \beta^{t-1}\lambda_{t-1} = 0\ (t = 1, 2, \cdots, T),$$

$$\frac{\partial L}{\partial a_{T+1}} = \lambda_{T+1} - \beta^T \lambda_T = 0,$$

$$\frac{\partial L}{\partial c_t} = \beta^t [u'(c_t) - \lambda_t] = 0\ (t = 0, 1, \cdots, T).$$

which gives

$$\beta R = \frac{u'(c_{t-1})}{u'(c_t)} (t = 1, 2, \cdots, T)$$

Solution 2 The constraint conditions $a_{t+1} = Ra_t - c_t\ (t = 0, 1, \cdots, T)$ can be rewritten as

$$\frac{a_{t+1}}{R^{t+1}} = \frac{a_t}{R^t} - \frac{c_t}{R^{t+1}}\ (t = 0, 1, \cdots, T).$$

Adding up the equations above leads to

$$\frac{a_{T+1}}{R^{T+1}} = a_0 - \sum_{t=0}^{T} \frac{c_t}{R^{t+1}}\ (t = 0, 1, \cdots, T).$$

By applying $a_{T+1} = 0$ we get the equivalent optimization problem

$$\max_{\mathbf{c_t}} \sum_{t=0}^{T} \beta^t u(c_t)$$

$$s.t.\ \sum_{t=0}^{T} \frac{c_t}{R^t} = Ra_0,$$

where $\mathbf{c_t} = \{c_t\}_{t=0}^{T}$.

2 Bellman Equation

PROBLEM 2

$$\max_{\mathbf{a_t},\mathbf{c_t}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t.\ k_{t+1} = f(k_t) + k_t - \delta k_t - c_t\ (t = 0, 1, \cdots, T), k_t \geq 0,$$

Bellman equation can be stated as

$$V(k_t) = \max_{c_t}\{u(c_t) + \beta V(k_{t+1})\}$$

By evolope theorem, we have

$$\frac{\partial V(k_t)}{\partial k_t} = V'(k_t) = \beta V'(k_{t+1})[f'(k_t) + 1 - \delta]$$

$$\frac{\partial V(k_t)}{\partial c_t} = 0 = u'(c_t) - \beta V'(k_{t+1})$$

The system of simultineous equations derives

$$u'(c_t) = \beta u'(c_{t+1})[f'(k_{t+1}) + 1 - \delta]$$

Let

$$F(k_t) = f(k_t) + k_t - \delta k_t$$

and we can rewrite it as

$$u'(F(k_t) - k_{t+1}) = \beta u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1}). \tag{1}$$

So as to find a stabel solution k^*, we can solve

$$u'(F(k^*) - k^*) = \beta u'(F(k^*) - k)F'(k^*)$$

namely

$$\beta F'(k^*) = 1.$$

When t is sufficiently large, k_t will be sufficiently close to k^* while c_t approaches c^*. Let

$$g(k_t, k_{t+1}, k_{t+2}) = -u'(F(k_t) - k_{t+1}) + \beta u'(F(k_{t+1}) - k_{t+2})F'(k_{t+1}).$$

We see $g(k_t, k_{t+1}, k_{t+2}) = 0$ at $(k_t, k_{t+1}, k_{t+2}) = (k^*, k^*, k^*)$. Take Taylor expansion of order one at $\mathbf{k}^* = (k^*, k^*, k^*)$ and we get the linearization of g

$$\begin{aligned}
&\widetilde{g}(k_t, k_{t+1}, k_{t+2}) \\
=&\, g(\mathbf{k}^*) + \frac{\partial g(\mathbf{k}^*)}{\partial k_t}(k_t - k^*) + \frac{\partial g(\mathbf{k}^*)}{\partial k_{t+1}}(k_{t+1} - k^*) + \frac{\partial g(\mathbf{k}^*)}{\partial k_{t+2}}(k_{t+2} - k^*) \\
=& - u''(c^*)F'(k^*)(k_t - k^*) + [u''(c^*) + \beta u''(c^*)F'(k^*)^2 + \beta u'(c^*)F''(k^*)](k_{t+1} - k^*) \\
&- \beta u''(c^*)F'(k^*)(k_{t+2} - k^*).
\end{aligned}$$

Indeed, we have also linearized the nonlinear implicit difference equation (1). Let $z_t = k_t - k^*$. The following second order difference equation can describe the behavior of k_t approximately when t is sufficiently large

$$z_{t+2} = \left[\frac{1}{\beta F'(k^*)} + F'(k^*) + \frac{u'(c^*)/u''(c^*)}{F'(k^*)/F''(k^*)} \right] z_{t+1} - \frac{1}{\beta} z_t$$

Since $\beta F'(k^*) = 1$, let $s = \dfrac{u'(c^*)/u''(c^*)}{F'(k^*)/F''(k^*)}$ and we can give the matrix form of the difference equation

$$\begin{bmatrix} z_{t+2} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \beta^{-1} + s & -\beta^{-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix} = A \begin{bmatrix} z_{t+1} \\ z_t \end{bmatrix}.$$

The characteristic polynomial of A is

$$p(\lambda) = |\lambda E - A| = \lambda^2 - (1 + \beta^{-1} + s)\lambda + \beta^{-1}.$$

Note $p(0) = \beta^{-1} > 0$ and $p(1) = -s < 0$. It implies A has two positive real roots $\lambda_1 < 1$ and $\lambda_2 > 1$. Let

$$A = TJT^{-1} = \begin{bmatrix} \beta^{-1} & \lambda_2 \\ \lambda_1 + 1 + \beta^{-1} + s & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \beta^{-1} & \lambda_2 \\ \lambda_1 + 1 + \beta^{-1} + s & 1 \end{bmatrix}^{-1}.$$

We have

$$A^n = T \begin{bmatrix} \lambda_1^n & \\ & \lambda_2^n \end{bmatrix} T^{-1}$$

and

$$\begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix} = T \begin{bmatrix} \lambda_1^{n-1} & \\ & \lambda_2^{n-1} \end{bmatrix} T^{-1} \begin{bmatrix} z_1 \\ z_0 \end{bmatrix}$$