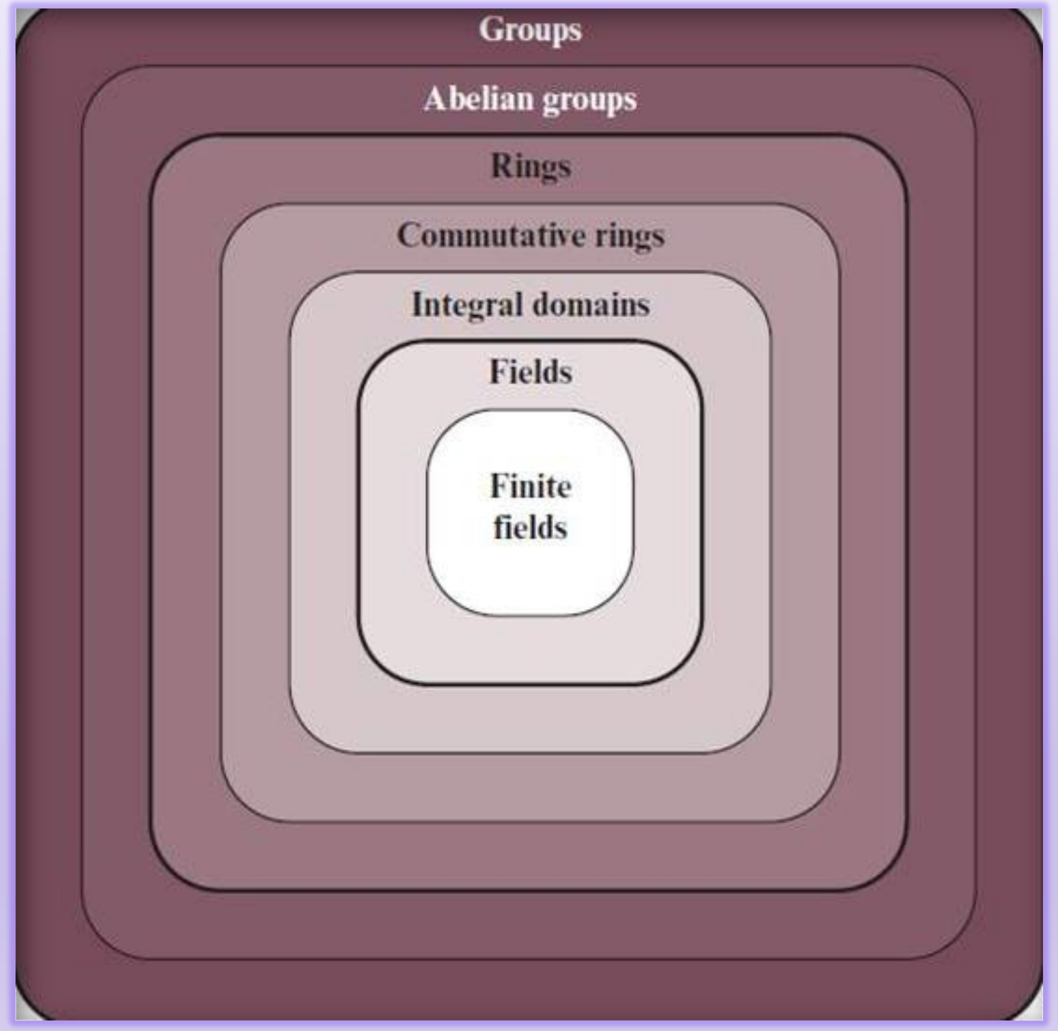
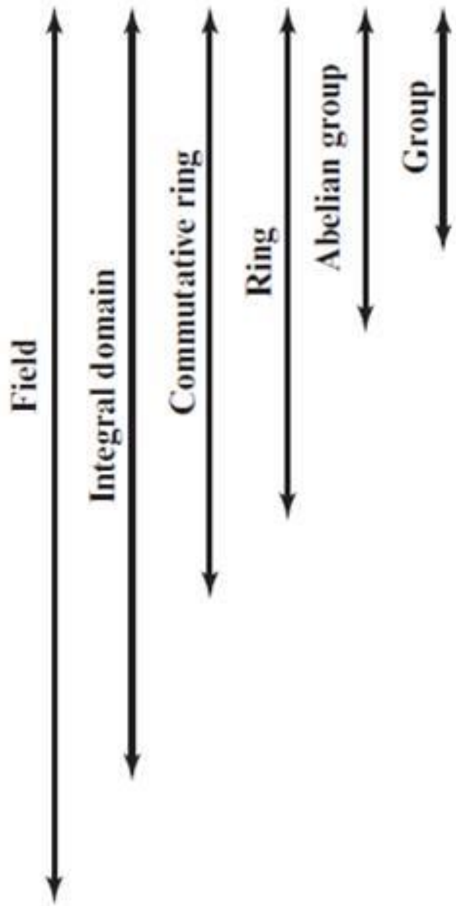
# Chapter 5

## Finite Fields

# Structural sets






| | |
|---|---|
| (A1) Closure under addition: | If $a$ and $b$ belong to $S$, then $a + b$ is also in $S$ |
| (A2) Associativity of addition: | $a + (b + c) = (a + b) + c$ for all $a, b, c$ in $S$ |
| (A3) Additive identity: | There is an element 0 in $R$ such that $a + 0 = 0 + a = a$ for all $a$ in $S$ |
| (A4) Additive inverse: | For each $a$ in $S$ there is an element $-a$ in $S$ such that $a + (-a) = (-a) + a = 0$ |
| (A5) Commutativity of addition: | $a + b = b + a$ for all $a, b$ in $S$ |
| (M1) Closure under multiplication: | If $a$ and $b$ belong to $S$, then $ab$ is also in $S$ |
| (M2) Associativity of multiplication: | $a(bc) = (ab)c$ for all $a, b, c$ in $S$ |
| (M3) Distributive laws: | $a(b + c) = ab + ac$ for all $a, b, c$ in $S$<br>$(a + b)c = ac + bc$ for all $a, b, c$ in $S$ |
| (M4) Commutativity of multiplication: | $ab = ba$ for all $a, b$ in $S$ |
| (M5) Multiplicative identity: | There is an element 1 in $S$ such that $a1 = 1a = a$ for all $a$ in $S$ |
| (M6) No zero divisors: | If $a, b$ in $S$ and $ab = 0$, then either $a = 0$ or $b = 0$ |
| (M7) Multiplicative inverse: | If $a$ belongs to $S$ and $a \neq 0$, there is an element $a^{-1}$ in $S$ such that $aa^{-1} = a^{-1}a = 1$ |

# Abelian Group

- (G, $\circ$) is a group, where G is a set of elements and the binary operator $\circ$ has the following properties:
  - Closure
    - $a \circ b$, for $a, b\ in\ G$
  - Associativity
    - $a \circ (b \circ c) = (a \circ b) \circ c,$ for $a, b, c\ in\ G$
  - Identity
    - There is an element $e$ in G such that $a \circ e = e \circ a = a$, for $a$ in G
  - Inverse
    - For each $a$ in G, there is an element $a^{-1}\ in\ G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$
  - $\boldsymbol{Commutative}$: $a \circ b = b \circ a$, for $a, b\ in\ G$

# Examples

- Additive groups
  - $(R, +)$
  - $(\mathrm{Z}, +)$
  - $(Z_n, +),\ Z_n = \{0, 1, 2, \dots, n-1\}$, "+" is mod n
- Multiplicative groups
  - $(R - \{0\},\ \times)$,
  - $(Q - \{0\},\ \times)$
  - $(Z_n^*,\ \times),\ Z_n^* = \{a{:}\, 1 \le a < n, \gcd(a, n) = 1\}$, "×" is mod n
- $N_n = \{\pi{:}\, \pi \text{ is a permuation over } \{1, 2, \dots, n\}\}$
  - Not abelian
- The operator is omitted if no misunderstanding occurs

# Cyclic group

- $G$ is $\boldsymbol{cyclic}$ if there is a generator $g \in G$ such that for any $a \in G$, $a = g^k$ for some k
  - $g$ spans all elements of G, that is, $G = \{g^k | k \geq 0\}$
- Notation
  - $g^k = g \circ g \circ \cdots \circ g$ ($k$ times)
  - $a^0 = e$: identity
  - $a^{-k} = (a^{-1})^k$
- $Z_7^*$ is a cyclic group with generators 3 and 5
  - $3^0 = 1, 3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5$
  - $5^0 = 1, 5^1 = 5, 5^2 = 4, 5^3 = 6, 5^4 = 2, 5^5 = 3$

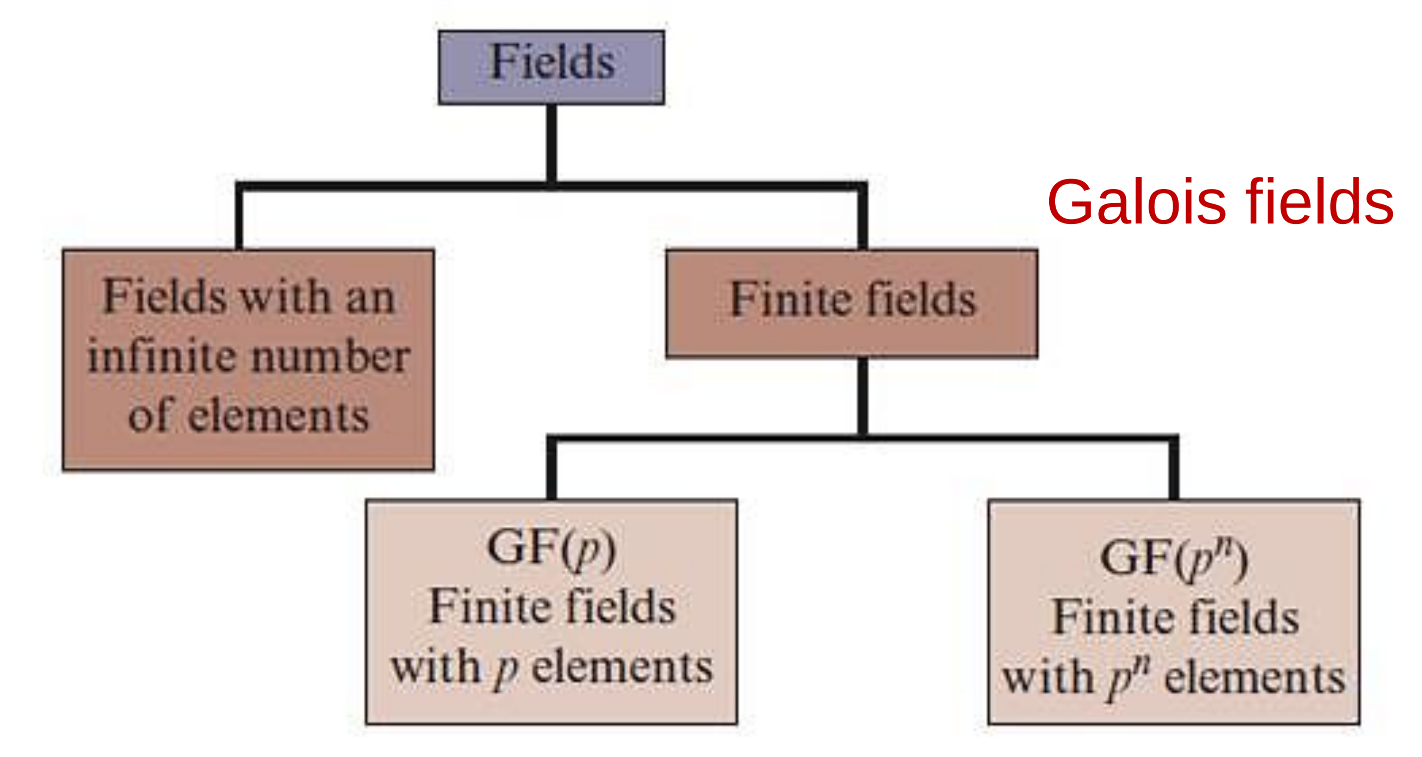
# Field: $\{F, +, \times\}$

- $\{F, +\}$ is an additive abelian group
- $\{F - \{0\}, \times\}$ is a multiplicative abelian group
- Distributive laws
  - $a \times (b + c) = a \times b + a \times c,$ for $a, b, c$ $in\ F$
  - $(a + b) \times c = a \times c + b \times c,$ for $a, b, c$ $in\ F$
- 0: the identity for $+$
- 1: the identity for $\times$
- $-a$: the additive inverse of a
- $a^{-1}$: the multiplicative inverse of a
- $a - b = a + (-b)$
- $a/b = a \times b^{-1}$
- We can do 4 operators $(+,\ -,\ \times,\ /)$ over a field

# Field: examples

- $\{R, +, \times\}$, where R is the set of reals
- $\{Q, +, \times\}$, where Q is the set of rationals
- $\{Z_p, +, \times\}$, where $p$ is prime and operators are mod $p$
  - $-a = p - a$
  - $a^{-1}, 1 \le a < \mathrm{p}$
    - Use extended Euclidean algorithm to compute integral $(x, y)$ for $xa + yp = 1$
    - $a^{-1} = x \bmod p$
  - Additive identity: 0
  - Multiplicative identity: 1

# Field: types




Galois fields

# Finite Field: GF($p^n$)

- Evariste Galois (1811-1832) first studied finite fields
- Finite fields play crucial role in AES and many cryptosystems
- Every finite field F must have $p^n$ elements for some prime $p$ and $n \geq 1$
- For every prime $p$ and $n \geq 1$, there is a finite field of $p^n$ elements
- F of $p^n$ elements may have different forms. Nevertheless, they are all isomorphic
  - Thus, $GF(p^n)$ is the finite field of $p^n$ elements

# Finite Field: $GF(p), n = 1$

- $GF(p) = \{Z_p, +, \times)$, where + and × under "mod p"
  - The finite filed of p elements
- Example
  - $GF(2) = \{Z_2, \text{xor}, \text{and}\}$ : Boolean algebra
  - $GF(7) = Z_7$

| + | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 0 |
| 2 | 2 | 3 | 4 | 5 | 6 | 0 | 1 |
| 3 | 3 | 4 | 5 | 6 | 0 | 1 | 2 |
| 4 | 4 | 5 | 6 | 0 | 1 | 2 | 3 |
| 5 | 5 | 6 | 0 | 1 | 2 | 3 | 4 |
| 6 | 6 | 0 | 1 | 2 | 3 | 4 | 5 |

| × | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 0 | 2 | 4 | 6 | 1 | 3 | 5 |
| 3 | 0 | 3 | 6 | 2 | 5 | 1 | 4 |
| 4 | 0 | 4 | 1 | 5 | 2 | 6 | 3 |
| 5 | 0 | 5 | 3 | 1 | 6 | 4 | 2 |
| 6 | 0 | 6 | 5 | 4 | 3 | 2 | 1 |

# Ordinary polynomials: arithmetic

$$
\begin{array}{r}
x^3 + x^2 \qquad\quad + 2 \\
+ \quad (x^2 - x + 1) \\
\hline
x^3 + 2x^2 - x + 3
\end{array}
$$

**(a) Addition**

$$
\begin{array}{r}
x^3 + x^2 \qquad\quad + 2 \\
- \quad (x^2 - x + 1) \\
\hline
x^3 \qquad\quad + x + 1
\end{array}
$$

**(b) Subtraction**

$$
\begin{array}{r}
x^3 + x^2 \qquad\quad + 2 \\
\times \quad (x^2 - x + 1) \\
\hline
x^3 + x^2 \qquad\quad + 2 \\
-x^4 - x^3 \qquad\quad - 2x \qquad \\
x^5 + x^4 \qquad\quad + 2x^2 \qquad\qquad\quad \\
\hline
x^5 \qquad\qquad\quad + 3x^2 - 2x + 2
\end{array}
$$

**(c) Multiplication**

$$
\begin{array}{r|l}
 & x + 2 \\
\hline
x^2 - x + 1 & x^3 + x^2 \qquad\quad + 2 \\
 & x^3 - x^2 + x \\
\hline
 & \qquad 2x^2 - x + 2 \\
 & \qquad 2x^2 - 2x + 2 \\
\hline
 & \qquad\qquad\quad x
\end{array}
$$

**(d) Division**

# Polynomials over $GF(p)$

- A polynomial of degree $n$-1 over GF(p) is of form

  $f(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$

  where each $a_i \in GF(p), 0 \leq i < n$

- Polynomials over GF(2)
  - $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1, x^3, \ldots$
- Addition over GF(2)
  - $(x^2+1)+(x^2+x) = 2x^2+x+1 \; = x+1$
- Multiplication over GF(2)
  - $(x^2+1)\times(x^2+1) = x^4+x^2+x^2+1 = x^4+1$

# Polynomials over $GF(p)$ mod $m(x)$

- $f(x) \bmod m(x) = r(x)$, where $\deg(r(x)) < \deg(m(x))$
  - $f(x) = q(x)m(x) + r(x)$
- Example
  - $f(x) = x^4 + x^2 + 1$
  - $m(x) = x^3 + x + 1$
  - $f(x) = x \cdot m(x) + (x + 1)$
- Simple way of computing $f(x) \bmod m(x)$
  - Substitute $m(x) = 0$ to $f(x)$ and get $r(x)$
  - $m(x) = 0 \rightarrow x^3 = x + 1$
  - $f(x) \bmod m(x) = x(x + 1) + x^2 + 1 = x + 1$

# Irreducible polynomial over GF(p)

- $m(x)$ is $\boldsymbol{irreducible}$ if $m(x)$ cannot be factored into a product of two polynomials over $GF(p)$ of degree $\geq 1$
- Example
  - $x^3 + x + 1$ is irreducible over $GF(2)$
  - $x^3 + x^2 + x + 1 = (x+1)^3$ is reducible over $GF(2)$
- Let $m(x)$ be degree-$n$ irreducible polynomial over $GF(p)$
  - $f(x)$ over $GF(p)$ with deg($f(x)$)$\leq n-1$
  - $\gcd(f(x), m(x)) = 1$ for $f(x) \neq 0$
  - $f^{-1}(x) \bmod m(x)$ exists for $f(x) \neq 0$
  - Use extended Euclidean algorithm to find $(a(x), b(x))$ for $a(x)f(x) + b(x)m(x) = 1 = \gcd(f(x), m(x))$
  - $f^{-1}(x) \bmod m(x) = a(x) \bmod m(x)$

# Finite field: $GF(p^n)/m(x), n \geq 2$

- $GF(p^n)$ is the set of polynomials over $GF(p)$ of degree at most $n-1$
- $m(x)$ is an irreducible degree-$n$ monic polynomial over $GF(p)$
- Coefficient operations are over $GF(p)$
- Multiplicative operations are "$\text{mod}\ m(x)$"
- Additive identity: 0
- Additive inverse: -f(x)
- Multiplicative identity: 1
- Multiplicative inverse: $f(x)^{-1} \bmod m(x)$
- Closure, inverse, associative, commutative, and distributive rules are satisfied

# Finite field $GF(2^3)/x^3 + x + 1$

- $S = \{0, 1, x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$
- $m(x) = x^3 + x + 1$: irreducible over $GF(2)$
- Example
  - $(x^2 + 1) + (x + 1) \; = x^2 + x$
  - $(x^2 + 1) \times (x + 1) \; \bmod m(x) = x^2$
  - $-(x^2 + x) \; = x^2 + x$
  - $(x^2 + x)^{-1} \bmod m(x) = x + 1$
  - $(x^2)^{-1} \bmod m(x) = \mathrm{x}^2 + x + 1$
- $GF(2^3)/\, x^3 {+} x + 1$ and $GF(2^3)/\, x^3 {+} 1$ are isomorphic since $x^3 + x^2 + 1$ is also irreducible over $GF(2)$

| | | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|---|
| | + | $0$ | $1$ | $x$ | $x+1$ | $x^2$ | $x^2+1$ | $x^2+x$ | $x^2+x+1$ |
| 000 | $0$ | $0$ | $1$ | $x$ | $x+1$ | $x^2$ | $x^2+1$ | $x^2+x$ | $x^2+x+1$ |
| 001 | $1$ | $1$ | $0$ | $x+1$ | $x$ | $x^2+1$ | $x^2$ | $x^2+x+1$ | $x^2+x$ |
| 010 | $x$ | $x$ | $x+1$ | $0$ | $1$ | $x^2+x$ | $x^2+x+1$ | $x^2$ | $x^2+1$ |
| 011 | $x+1$ | $x+1$ | $x$ | $1$ | $0$ | $x^2+x+1$ | $x^2+x$ | $x^2+1$ | $x^2$ |
| 100 | $x^2$ | $x^2$ | $x^2+1$ | $x^2+x$ | $x^2+x+1$ | $0$ | $1$ | $x$ | $x+1$ |
| 101 | $x^2+1$ | $x^2+1$ | $x^2$ | $x^2+x+1$ | $x^2+x$ | $1$ | $0$ | $x+1$ | $x$ |
| 110 | $x^2+x$ | $x^2+x$ | $x^2+x+1$ | $x^2$ | $x^2+1$ | $x$ | $x+1$ | $0$ | $1$ |
| 111 | $x^2+x+1$ | $x^2+x+1$ | $x^2+x$ | $x^2+1$ | $x^2$ | $x+1$ | $x$ | $1$ | $0$ |

| | | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|---|
| | × | $0$ | $1$ | $x$ | $x+1$ | $x^2$ | $x^2+1$ | $x^2+x$ | $x^2+x+1$ |
| 000 | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| 001 | $1$ | $0$ | $1$ | $x$ | $x+1$ | $x^2$ | $x^2+1$ | $x^2+x$ | $x^2+x+1$ |
| 010 | $x$ | $0$ | $x$ | $x^2$ | $x^2+x$ | $x+1$ | $1$ | $x^2+x+1$ | $x^2+1$ |
| 011 | $x+1$ | $0$ | $x+1$ | $x^2+x$ | $x^2+1$ | $x^2+x+1$ | $x^2$ | $1$ | $x$ |
| 100 | $x^2$ | $0$ | $x^2$ | $x+1$ | $x^2+x+1$ | $x^2+x$ | $x$ | $x^2+1$ | $1$ |
| 101 | $x^2+1$ | $0$ | $x^2+1$ | $1$ | $x^2$ | $x$ | $x^2+x+1$ | $x+1$ | $x^2+x$ |
| 110 | $x^2+x$ | $0$ | $x^2+x$ | $x^2+x+1$ | $1$ | $x^2+1$ | $x+1$ | $x$ | $x^2$ |
| 111 | $x^2+x+1$ | $0$ | $x^2+x+1$ | $x^2+1$ | $x$ | $1$ | $x^2+x$ | $x^2$ | $x+1$ |

# Finite field $GF(3^2)/\ x^2{+}1$

- $S = \{0, 1, 2, x, x + 1, x + 2, 2x, 2x + 1, 2x + 2\}$
- $m(x) = x^2 + 1$: irreducible over $GF(3)$
- Example
  - $(x + 1) + (x + 2) = 2x$
  - $(x + 1) \times (x + 2) \mod m(x) = 1$
  - $-(x + 2) = 2x + 1$
  - $(x + 1)^{-1} \bmod m(x) = x + 2$
  - $(2x)^{-1} \bmod m(x) = x$

| $+$ | $0$ | $1$ | $2$ | $x$ | $x+1$ | $x+2$ | $2x$ | $2x+1$ | $2x+2$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $1$ | $2$ | $x$ | $x+1$ | $x+2$ | $2x$ | $2x+1$ | $2x+2$ |
| $1$ | $1$ | $2$ | $0$ | $x+1$ | $x+2$ | $x$ | $2x+1$ | $2x+2$ | $2x$ |
| $2$ | $2$ | $0$ | $1$ | $x+2$ | $x$ | $x+1$ | $2x+2$ | $2x$ | $2x+1$ |
| $x$ | $x$ | $x+1$ | $x+2$ | $2x$ | $2x+1$ | $2x+2$ | $0$ | $1$ | $2$ |
| $x+1$ | $x+1$ | $x+2$ | $x$ | $2x+1$ | $2x+2$ | $2x$ | $1$ | $2$ | $0$ |
| $x+2$ | $x+2$ | $x$ | $x+1$ | $2x+2$ | $2x$ | $2x+1$ | $2$ | $0$ | $1$ |
| $2x$ | $2x$ | $2x+1$ | $2x+2$ | $0$ | $1$ | $2$ | $x$ | $x+1$ | $x+2$ |
| $2x+1$ | $2x+1$ | $2x+2$ | $2x$ | $1$ | $2$ | $0$ | $x+1$ | $x+2$ | $x$ |
| $2x+2$ | $2x+2$ | $2x$ | $2x+1$ | $2$ | $0$ | $1$ | $x+2$ | $x$ | $x+1$ |

| $\times$ | $0$ | $1$ | $2$ | $x$ | $x+1$ | $x+2$ | $2x$ | $2x+1$ | $2x+2$ |
|---|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $2$ | $x$ | $x+1$ | $x+2$ | $2x$ | $2x+1$ | $2x+2$ |
| $2$ | $0$ | $2$ | $1$ | $2x$ | $2x+2$ | $2x+1$ | $x$ | $x+2$ | $x+1$ |
| $x$ | $0$ | $x$ | $2x$ | $2$ | $x+2$ | $2x+2$ | $1$ | $x+1$ | $2x+1$ |
| $x+1$ | $0$ | $x+1$ | $2x+2$ | $x+2$ | $2x$ | $1$ | $2x+1$ | $2$ | $x$ |
| $x+2$ | $0$ | $x+2$ | $2x+1$ | $2x+2$ | $1$ | $x$ | $x+1$ | $2x$ | $2$ |
| $2x$ | $0$ | $2x$ | $x$ | $1$ | $2x+1$ | $x+1$ | $2$ | $2x+2$ | $x+2$ |
| $2x+1$ | $0$ | $2x+1$ | $x+2$ | $x+1$ | $2$ | $2x$ | $2x+2$ | $x$ | $1$ |
| $2x+2$ | $0$ | $2x+2$ | $x+1$ | $2x+1$ | $x$ | $2$ | $x+2$ | $1$ | $2x$ |

# $GF(2^n)/m(x)$: computaton

- Represent polynomial $f(x) = \sum_{i=0}^{n-1} b_i x^i$ as binary string $b_{n-1}b_{n-2}\cdots b_1 b_0$
- Addition: bitwise XOR (no need to carry)

| | | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|---|
| | + | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 000 | 0 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 001 | 1 | 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 |
| 010 | 2 | 2 | 3 | 0 | 1 | 6 | 7 | 4 | 5 |
| 011 | 3 | 3 | 2 | 1 | 0 | 7 | 6 | 5 | 4 |
| 100 | 4 | 4 | 5 | 6 | 7 | 0 | 1 | 2 | 3 |
| 101 | 5 | 5 | 4 | 7 | 6 | 1 | 0 | 3 | 2 |
| 110 | 6 | 6 | 7 | 4 | 5 | 2 | 3 | 0 | 1 |
| 111 | 7 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

- Multiplication: Shift-XOR
  - long-hand multiplication, but from high bit to low bit
  - Example: $1001 \times 1101$
    - Let $1101 = \mathrm{b}_3\mathrm{b}_2\mathrm{b}_1\mathrm{b}_0$
    - Initiation: $f = 0000\ 0000$
    - $b_3 = 1 \rightarrow f = \text{shift-left}(f) \oplus 1001 = 0000\ 1001$
    - $b_2 = 1 \rightarrow f = \text{shift-left}(f) \oplus 1001 = 0001\ 1011$
    - $b_1 = 0 \rightarrow f = \text{shif -left}(f) = 0011\ 0110$
    - $b_0 = 1 \rightarrow f = \text{shift-left}(f) \oplus 1001 = 0110\ 0101$

```
      1 0 0 1  (multiplicand)
    x 1 1 0 1  (multiplier)
-------------
1 0 0 1
  1 0 0 1
    0 0 0 0
      1 0 0 1
-------------
1 1 0 0 1 0 1
```

- Modular multiplication
  - Shift-XOR-Mod: like Shift-XOR, do modulo whenever necessary
  - $m(x) = x^8 + x^4 + x^3 + x + 1$ (binary: 1 0001 1011)
  - $a \times b = 3F \times 86 = 0011\ 1111 \times 1000\ 0110$

| *i* | $b_i$ | f: shift-XOR | f: mod g(x) --> bitwise XOR |
|---|---|---|---|
| Initial | | | 0000 0000 |
| 7 | 1 | 0011 1111 | 0011 1111 |
| 6 | 0 | 0111 1110 | 0111 1110 |
| 5 | 0 | 1111 1100 | 1111 1100 |
| 4 | 0 | 1 1111 1000 | **1110 0011** |
| 3 | 0 | 1 1100 0110 | **1101 1101** |
| 2 | 1 | 1 1000 0101 | **1001 1110** |
| 1 | 1 | 1 0000 0011 | **0001 1000** |
| 0 | 0 | 0011 0000 | 0011 0000 |

# Computation: table lookup

- $GF(2^8)$ / $x^8 + x^4 + x^3 + x + 1$
- Build a table for $a(x)b(x) \bmod x^8 + x^4 + x^3 + x + 1$
- Table size: $2^8 \times 2^8 \times 2^8$ bits = $2^{16}$ bytes = 64K bytes

# Field: $GF(p^{n_1 \times n_2})/m_1(x), m_2(y)$

- Consider degree-($n_2$-1) polynomials of $y$ over $GF(p^{n_1})/m_1(x)$
- Example
  - $\mathrm{p} = 2, n_1 = 3,\ n_2 = 4$
  - $GF(p^{n_1})/m_1(x) = GF(2^3)/x^3 + x + 1$
  - A polynomial of the field is like : $(\mathrm{x} + 1)y^3 + (\mathrm{x}^2)y^2 + 1$
- Let $m_2(y)$ be an irreducible degree-$n_2$ polynomial with coefficients over $GF(p^{n_1})/m_1(x)$
- $GF(p^{n_1 \times n_2})/m_1(x), m_2(y)$
  - The element set consists of all degree-($n_2$-1) polynomials (of y) with coefficients over $G(p^{\mathrm{n}_1})/m_1(x)$
  - Coefficients are operated over$G(p^{n_1})/m_1(x)$

# Example: $GF(2^{3\times 4})/m_1(x), m_2(y)$

- $GF(2^{3\times 4})/x^3 + x + 1, y^4 + (x^2 + 1)y^2 + (x + 1)$
- $m_2(y) = y^4 + (x^2 + 1)y^2 + (x + 1)$ is irreducible over field $GF(2^3)/x^3 + x + 1$
- Multiplication

$$
\begin{aligned}
&[(x+1)y^3 + xy^2 + 1)] \times [y + (x^2 + 1)] \bmod m_2(y) \\
&= (x+1)y^4 + [(x+1)(x^2+1) + x]y^3 + [x(x^2+1)]y^2 \\
&\qquad +y + (x^2+1) \bmod m_2(y) \\
&= (x+1)[(x^2+1)y^2 + (x+1)] + (x^2+x)y^3 + y^2 \\
&\qquad +y + (x^2+1) \bmod m_2(y) \\
&= (x^2+x)y^3 + (x^2+1)y^2 + y
\end{aligned}
$$